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Riemann-Finsler Geometry and its Applications to Diffusion Magnetic Resonance Imaging

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Abstract—**Riemannian geometry has become a popular mathematical framework for the analysis of diffusion tensor images (DTI) in diffusion weighted magnetic resonance imaging (DWMRI). If one declines from the a priori constraint to model local anisotropic diffusion in terms of a 6-degrees-of-freedom rank-2 DTI tensor, then Riemann-Finsler geometry appears to be the natural extension. As such it provides an interesting alternative to the Riemannian rationale in the context of the various high angular resolution diffusion imaging (HARDI) schemes proposed in the literature. The main advantages of the proposed Riemann-Finsler paradigm are its manifest incorporation of the DTI model as a limiting case via a "correspondence principle" (operationalized in terms of a vanishing Cartan tensor), and its direct connection to the physics of DWMRI expressed by the (appropriately generalized) Stejskal-Tanner equation and Bloch-Torrey equations furnished with a diffusion term.**

I. INTRODUCTION

Riemann-Finsler geometry, already hinted upon by Riemann in his "Habilitation" [1], is a generalization of Riemannian geometry. The latter has found important applications in Maxwell theory and Einstein's theory of general relativity, contributing greatly to its popularity. The general case was taken up by Finsler [2], Cartan [3] (referring to it as "Finsler geometry"), and others [4], [5], [6].

Despite its great potential, Riemann-Finsler geometry has not become nearly as popular as its Riemannian limit. To some extent this may be explained by its rather mind-boggling technicalities and heavy computational demands (due to the introduction of an additional vectorial dimension extending the base manifold). Another key factor is the still open challenge to find important "natural" application areas for it, and to show its added value in these areas. We conjecture that DWMRI could be one such application area. This imaging modality plays an important role in the unravelment of the human brain connectome, among others.

II. THEORY

The pivot of Riemann-Finsler geometry is a generalised notion of length of a spatial curve C ("Hilbert's invariant integral" [4]):

$$\mathscr{L}(C) = \int_C F(x, dx) \,. \tag{1}$$

The so-called *Finsler function* $F(x, \xi)$ is positive definite for $\xi \neq 0$, and homogeneous of degree one in ξ, i.e. $F(x, \lambda\xi) = |\lambda| F(x, \xi)$ for all λ. In addition, the *Riemann-Finsler metric tensor*, defined as

$$g_{ij}(x, \xi) = \frac{1}{2} \frac{\partial^2 F^2(x, \xi)}{\partial \xi^i \partial \xi^j} \,, \tag{2}$$

is positive definite. It is easy to see that (applying summation convention)

$$F(x, \xi) = \sqrt{g_{ij}(x, \xi)\xi^i \xi^j} \,. \tag{3}$$

Riemann's "quadratic restriction" pertains to the "mildly anisotropic" case $g_{ij}(x, \xi) = g_{ij}(x)$.

The non-Riemannian nature of the Riemann-Finsler manifold is most concisely expressed in terms of the so-called *Cartan tensor*:

$$C_{ijk}(x, \xi) = \frac{1}{4} \frac{\partial^3 F^2(x, \xi)}{\partial \xi^i \partial \xi^j \partial \xi^k} \,. \tag{4}$$

A dual, or Hamiltonian formulation rests upon the identity

$$g^{ik}(x, y)g_{kj}(x, \xi) = \delta^i_j \,, \tag{5}$$

in which the first factor on the l.h.s. defines the *dual Riemann-Finsler metric tensor*, and in which it is understood that

$$y_i = g_{ij}(x, \xi)\xi^j \quad \text{or, equivalently,} \quad \xi^i = g^{ij}(x, y)y_j \,. \tag{6}$$

We stipulate that the *dual Finsler function*, $H(x, y) \equiv F(x, \xi)$, governs signal attenuation in DWMRI if, as with DTI, one relies on the Stejskal-Tanner (mono-exponential decay, Gaussian diffusion) and Bloch-Torrey equations with diffusion term [7], [8], [9], viz.

$$S(x, y) = S(x, 0) \exp\left(-\tau H^2(x, y)\right) \,. \tag{7}$$

Here τ denotes a time constant related to the time Δ between a pair of balanced diffusion-sensitizing gradients G_i and pulse duration δ (in Stejskal-Tanner's scheme we have $\tau = \Delta - \delta/3$), and

$$H(x, y) = \sqrt{g^{ij}(x, y)y_i y_j} \,, \tag{8}$$

with "momentum" $y_i = \gamma\delta G_i$ given in terms of δ, G_i, and hydrogen gyromagnetic ratio γ. The DTI rationale [10], [11] is based on the (strong) simplification that the (y-independent) diffusion tensor image $D^{ij}(x)$ can be identified with $g^{ij}(x, y)$.

Further details can be found in a forthcoming publication [12].

REFERENCES

[1] B. Riemann, "Über die Hypothesen, welche der Geometrie zu Grunde liegen," in *Gesammelte Mathematische Werke*, H. Weber, Ed. Leibzig: Teubner, 1892, pp. 272–287.

[2] P. Finsler, "Ueber kurven und Flächen in allgemeinen Räumen," Ph.D. dissertation, University of Göttingen,, Göttingen, Germany, 1918.

[3] E. Cartan, *Les Espaces de Finsler*. Paris: Hermann, 1934.

[4] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 2000, vol. 2000.

[5] H. Rund, *The Differential Geometry of Finsler Spaces*. Berlin: Springer-Verlag, 1959.

[6] ——, *The Hamilton-Jacobi Theory in the Calculus of Variations*. Huntington, N.Y.: Robert E. Krieger Publishing Company, 1973.

[7] F. Bloch, "Nuclear induction," *Physical Review*, vol. 70, pp. 460–473, 1946.

[8] E. O. Stejskal and J. E. Tanner, "Spin diffusion measurements: Spin echoes in the presence of a time-dependent field gradient," *Journal of Computational Physics*, vol. 42, no. 1, pp. 288–292, 1965.

[9] H. C. Torrey, "Bloch equations with diffusion terms," vol. 104, pp. 563–565, 1956.

[10] P. J. Basser, J. Mattiello, and D. Le Bihan, "MR diffusion tensor spectroscopy and imaging," *Biophysics Journal*, vol. 66, no. 1, pp. 259–267, 1994.

[11] D. Le Bihan, J.-F. Mangin, C. Poupon, C. A. Clark, S. Pappata, N. Molko, and H. Chabriat, "Diffusion tensor imaging: Concepts and applications," *Journal of Magnetic Resonance Imaging*, vol. 13, pp. 534–546, 2001.

[12] L. M. J. Florack and A. Fuster, "Riemann-Finsler geometry for diffusion weighted magnetic resonance imaging," Eindhoven University of Technology, Tech. Rep. CASA-Report 12-23, cf. www.win.tue.nl/analysis/reports/rana12-23.pdf, 2012, to be published in the Lecture Notes in Computer Science.