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The Minkowski–Lyapunov equation for linear dynamics: Theoretical foundations☆

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ABSTRACT

We consider the Lyapunov equation for the linear dynamics, which arises naturally when one seeks for a Lyapunov function with a uniform, exact decrease. In this setting, a solution to the Lyapunov equation has been characterized only for quadratic Lyapunov functions. We demonstrate that the Lyapunov equation is a well-posed equation for strictly stable dynamics and a much more general class of Lyapunov functions specified via Minkowski functions of proper C-sets, which include Euclidean and weighted Euclidean vector norms, polytopic and weighted polytopic $(1, \infty)$-vector norms as well as vector semi-norms induced by the Minkowski functions of proper C-sets. Furthermore, we establish that the Lyapunov equation admits a basic solution, i.e., the unique solution within the class of Minkowski functions associated with proper C-sets. Finally, we provide a characterization of the lower and upper approximations of the basic solution that converge pointwise and compactly to it, while, in addition, the upper approximations satisfy the classical Lyapunov inequality.

1. Introduction

The Lyapunov stability theory, initiated in the seminal work (Lyapunov, 1992), is one of the cornerstones of modern dynamical systems theory, being, *inter alia*, a fundamental tool for the stability analysis of all types of physical systems (Hahn, 1967; Sontag, 1998). Indeed, notions of *Lyapunov functions* and *control Lyapunov functions* offer a general approach for a systematic solution to problems arising in stability analysis and stabilizing controller synthesis (Artstein, 1983; Gurvits, 1995; Hahn, 1967; Lyapunov, 1992; Sontag, 1998; Sontag & Sussmann, 1996). It is well known that the stability of the origin for an autonomous discrete time dynamics $x^+ = f(x)$ can be verified by detecting a Lyapunov function, namely a function $V(\cdot)$ which is lower and upper bounded by Kamke's functions of the norm of the state (i.e., $\kappa_1(|x|) \leq V(x) \leq \kappa_2(|x|)$) and which verifies the decrease condition measured by another Kamke's function of the norm of the state (i.e., $V(f(x)) - V(x) \leq -\kappa_3(|x|)$). A particularly important case, referred to as the *Lyapunov equation*, arises when the inequality in the Lyapunov decrease condition holds with equality for all x. The practical relevance and theoretical importance of the Lyapunov equation originates, *inter-alia*, in the fact that the solution to the Lyapunov equation, apart from verifying asymptotic stability, also provides a uniform, exact characterization of all state trajectories, generated by the dynamics, *via* its sublevel sets. Consequently, a solution to the Lyapunov equation, as opposed to the Lyapunov inequality, offers self-evident benefits within the context of system verification and safety analysis, finite-time reachability to/of a target set and constrained finite-horizon optimal control.

The apparent elegance of the Lyapunov stability theory is, however, accompanied with a highly non-trivial problem of identifying the right class of functions within which the corresponding Lyapunov function candidates should be sought. For linear discrete time dynamics and quadratic Lyapunov functions, the solutions to both the corresponding Lyapunov inequality and equation were characterized. In this case, the problems of solving the Lyapunov inequality and equation fall within the realm of

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linear algebra. In particular, it is well known that the corresponding Lyapunov inequality can be equivalently expressed as a linear matrix inequality (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), while the associated Lyapunov equation takes the form of a linear matrix equation. The importance of the Lyapunov equation is also indicated by the numerous research efforts that have been made to develop algorithms for solving the Lyapunov equation within the class of quadratic functions, see, for instance, Bitmead (1981); Gajić and Qureshi (2008); Lu and Wachspress (1991); Norman (1989); Tian and Gu (2008); Zhou (2011) and the references therein. Another important class of Lyapunov functions is the class of polyhedral Lyapunov functions, e.g., weighted $(1, \infty)$–vector norms, see, for example, Molchanov and Pyatnitskii (1989); Kiendl, Adamy, and Stelzner (1992); Polański (1998); Blanchini and Miani (2008); Lazar (2010) and the references therein. The polyhedral Lyapunov functions are less conservative than the quadratic ones when the system state is subject to polyhedral constraints, which is often the case in practice. In the presence of polytopic constraints, polyhedral Lyapunov functions are a much better tool for approximating the basin of attraction since it is a proper C-set (Blanchini & Miani, 2008) and polytopic sets can approximate arbitrarily well proper C-sets. Additionally, it was recently demonstrated that polyhedral Lyapunov functions can be used to efficiently construct a bisimulation quotient for linear dynamics (Ding, Lazar, & Belta, 2012), which, in turn, can be used for verification and safety analysis.

Motivation: The above introduction suggests the following crucial observations: (i) irrespectively of the type of the Lyapunov decrease condition, i.e., inequality or equality, polyhedral Lyapunov functions are preferable to quadratic Lyapunov functions, see Blanchini and Miani (2008) for a comprehensive analysis, and, (ii) irrespectively of the type of Lyapunov function candidate, i.e., quadratic or polyhedral, solutions to the Lyapunov equation are preferable to solutions to the Lyapunov inequality. Nevertheless, despite numerous efforts on solving the Lyapunov equation within the class of quadratic functions (Bitmead, 1981; Gajić & Qureshi, 2008; Lu & Wachspress, 1991; Norman, 1989; Tian & Gu, 2008; Zhou, 2011) and numerous works on solving the polyhedral Lyapunov inequality (Blanchini & Miani, 2008; Kiendl et al., 1992; Lazar, 2010; Molchanov & Pyatnitskii, 1989; Polański, 1998), which can be traced back to as far as 1963 (Rosenbrock, 1963), to the best of the authors' knowledge, a characterization of the solution to the Lyapunov equation within the class of Minkowski functions is not available.

From a theoretical perspective, characterizing the solution to the Lyapunov equation within the class of Minkowski functions associated with proper C-sets provides a fundamental, missing piece of the polyhedral Lyapunov functions framework (Blanchini & Miani, 2008) and, in addition, offers the first alternative to the Lyapunov equation within the class of quadratic functions. From a practical point of view and as already mentioned, apart from asserting asymptotic stability, a solution to the Lyapunov equation also provides a uniform, exact characterization of all the trajectories, generated by the dynamics, expressed *via* subsets of the state space. Essentially, any state trajectory generated by the dynamics consists of points that lie on the boundaries of nested sublevel sets of a Lyapunov function that satisfies the Lyapunov equation. This fact in conjunction with the computationally beneficial structure of polyhedral Lyapunov functions renders the construction of a bisimulation quotient for linear dynamics (Ding et al., 2012) computationally more tractable. Furthermore, when the exact decrease of the Lyapunov function is specified *via* a polyhedral stage cost function (e.g., weighted $(1, \infty)$–vector norms), to have a solution to the corresponding Lyapunov equation is to solve the stabilizing model predictive control (MPC) synthesis problem based on polyhedral cost functions (Rawlings & Mayne, 2009). As in the case of the Lyapunov function candidates, it is well known that polyhedral costs are very frequently preferred to quadratic costs in MPC and, a terminal cost function that is the solution to the corresponding MPC Lyapunov equation yields a better performance than a terminal cost obtained as a solution to the MPC Lyapunov inequality. Not surprisingly, the problem of characterizing a suitable terminal cost function is still open for stabilizing linear MPC based on polyhedral cost functions (Raković & Lazar, 2012).

Prompted by the theoretical importance and practical relevance of the Lyapunov equation for linear dynamics, we discuss in this paper the Lyapunov equation taking the form, for all $x \in \mathbb{R}^n$,

$$V(x) = V((A + BK)x) + \ell(x)$$

associated with the strictly stable autonomous linear system $x^+ = (A + BK)x$ and stage function $\ell(\cdot)$ given by the sum of Minkowski functions associated with proper C-sets $\mathcal{Q}$ and $\mathcal{R}$ (i.e., $\ell(x) = \mathrm{g}(\mathcal{Q}, x) + \mathrm{g}(\mathcal{R}, Kx)$). Henceforth, the Lyapunov equation within this setting is referred formally to as the *Minkowski–Lyapunov equation*. Likewise, a Minkowski function of a proper C-set that satisfies Lyapunov decrease condition (i.e., $V((A + BK)x) - V(x) \leq -\ell(x)$) is referred, hereafter, to as the *Minkowski–Lyapunov function*.

As already hinted, the dynamics $x^+ = (A+BK)x$ typically arises from linear control systems of the form $x^+ = Ax+Bu(x)$, where the state-feedback control law $u(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $u(x) := Kx$. When the feedback matrix K is known, the closed-loop system takes the form of the autonomous linear dynamics, i.e., $x^+ = \Psi x$ where $\Psi := A + BK$. The reason for explicitly specifying the feedback matrix K, as opposed to working directly with the matrix Ψ, and for considering the associated stage cost $\mathrm{g}(\mathcal{Q}, x)+\mathrm{g}(\mathcal{R}, Kx)$, is motivated by the stabilizing MPC synthesis (Raković & Lazar, 2012; Rawlings & Mayne, 2009). More precisely, given an arbitrary, stabilizing linear control law $u(x) = Kx$, which is referred to as the *terminal control law* in MPC (Rawlings & Mayne, 2009), solving the Minkowski–Lyapunov equation yields directly a tight solution to the corresponding MPC Lyapunov condition.

It is worth pointing out that the considered class of candidate Lyapunov and cost functions is rather broad and is, in fact, equivalent to a class of sublinear functions (Schneider, 1993) and, hence, it encapsulates Euclidean vector norms, polytopic $(1, \infty)$-vector norms and Minkowski functions of proper C-polytopic sets as particular cases. However, in a stark contrast to the Lyapunov equation within the class of quadratic functions, solving the Minkowski–Lyapunov equation poses a significantly more complex problem, whose solvability is not decidable *via* the tools of linear algebra. In particular, the corresponding analysis of the Minkowski–Lyapunov equation requires the employment of more general mathematical techniques from Minkowski theory of convex bodies.

Contributions summary: We establish that the corresponding *Minkowski–Lyapunov equation* is a well-posed equation that admits a special solution referred to, henceforth, as the basic solution. We show that the basic solution is, in fact, the unique solution to the Minkowski–Lyapunov equation over the class of Minkowski functions associated with proper C-sets. (As pointed out by an anonymous referee, these facts alone form an original scientific contribution.) We characterize the basic solution by considering a *polar* system, i.e., $x^+ = (A + BK)^T x + w$, where the artificial additive disturbance satisfies $w \in \mathcal{Q}^* \oplus K^T \mathcal{R}^*$ and $\mathcal{Q}^*$ and $\mathcal{R}^*$ are the polar sets of the sets $\mathcal{Q}$ and $\mathcal{R}$ while $\bigoplus$ denotes the Minkowski set addition. In particular, we establish that the basic solution to the Minkowski–Lyapunov equation is generated, in fact, by the polar set of the minimal robust positively invariant set for the polar system $x^+ = (A + BK)^T x + w,\ w \in \mathcal{Q}^* \oplus K^T \mathcal{R}^*$. (This polarity relation is a highly valuable contribution in its own right.) We employ the polarity relations in conjunction with recent theoretical advances of the theory of minimal invariant sets (Artstein &

Raković, 2008; Raković, 2007) in order to obtain a Riccati-like recursion that yields the consistently improving lower and upper approximations of the basic solution. In addition, the corresponding lower and upper (generic and polytopic) approximations converge pointwise and compactly to the basic solution, while the upper approximations are Minkowski–Lyapunov functions. We also show that the lower and upper approximations can be constructed by employing Minkowski functions associated with suitably specified proper C-polytopic sets.

Paper Structure: Section 2 describes the problem of interest, provides necessary technical preliminaries and outlines the main objectives. Section 3 establishes that the Minkowski–Lyapunov equation is solvable, verifies the existence and provides characterization of its basic solution and discusses lower approximations of the basic solution. Section 4 introduces upper approximations of the basic solution to the Minkowski–Lyapunov equation which satisfy the corresponding Lyapunov inequality. Section 5 detects the corresponding lower and upper approximations via Minkowski functions of proper C-polytopic sets. Conclusions are summarized in Section 6.

Basic nomenclature and definitions: The sets of non-negative and positive integers and non-negative reals are denoted by $\mathbb{N}$, $\mathbb{N}_+$, and $\mathbb{R}_+$, respectively. Given $a \in \mathbb{N}$ and $b \in \mathbb{N}$ such that $a < b$ we denote $\mathbb{N}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$; We write $\mathbb{N}_b$ for $\mathbb{N}_{[0:b]}$. For $M \in \mathbb{R}^{n\times n}$, $\rho(M)$ denotes the spectral radius of M. Given $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^n$, the Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} := \{x + y : x \in \mathcal{X},\ y \in \mathcal{Y}\}$. Given a set $\mathcal{X}$ and a real matrix M of compatible dimensions (possibly a scalar) the image and preimage of $\mathcal{X}$ under M are denoted by $M\mathcal{X} := \{Mx : x \in \mathcal{X}\}$ and $M^{-1}\mathcal{X} := \{x : Mx \in \mathcal{X}\}$, respectively. A function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be positively homogeneous of the first degree if $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$. A function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be subadditive if $f(x_1 + x_2) \le f(x_1) + f(x_2)$ for all $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$. A function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is said to be sublinear if it is both positively homogeneous of the first degree and subadditive. A set $\mathcal{X} \subset \mathbb{R}^n$ is *a C-set* if it is compact, convex, and contains the origin. A set $\mathcal{X} \subset \mathbb{R}^n$ is *a proper C-set, or just a PC-set,* if it is a C-set and contains the origin in its interior. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces. A *polytope* is a closed and bounded polyhedron. Given a non-empty closed convex set $\mathcal{X} \subseteq \mathbb{R}^n$ the function $h(\mathcal{X}, \cdot)$ given by:

$$h(\mathcal{X}, y) := \sup_x \{y^T x : x \in \mathcal{X}\} \quad \text{for } y \in \mathbb{R}^n$$

is called the support function. Given a proper C-set $\mathcal{X} \subset \mathbb{R}^n$ the function $g(\mathcal{X}, \cdot)$ given by:

$$g(\mathcal{X}, x) := \inf_\mu \{\mu : x \in \mu\mathcal{X},\ \mu \ge 0\} \quad \text{for } x \in \mathbb{R}^n$$

is called the Minkowski (gauge) function. For any C-set $\mathcal{S}$ in $\mathbb{R}^n$ we use * to denote its polar set $\mathcal{S}^*$ specified by:

$$\mathcal{S}^* := \{x : \forall y \in \mathcal{S},\ y^T x \le 1\},$$

and we also recall that (Rockafellar, 1970; Schneider, 1993) for any proper C-set $\mathcal{S}$ in $\mathbb{R}^n$ its polar set $\mathcal{S}^*$ is also a proper C-set in $\mathbb{R}^n$ and it holds that $\mathcal{S} = (\mathcal{S}^*)^*$.

For convenience and clarity of presentation, we provide proofs of non-trivial statements in the appendices.

2. Preliminaries

2.1. Setting and problem formulation

We consider discrete time, autonomous, linear time invariant, dynamics described by:

$$x^+ = (A + BK)x, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, $x^+ \in \mathbb{R}^n$ is the successor state and the matrix triplet (A, B, K) is known exactly and is of compatible dimensions, i.e., $(A, B, K) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$. We work under the following necessary assumption.

Assumption 1. (i) The matrix pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is strictly stabilizable; and (ii) the matrix $K \in \mathbb{R}^{m\times n}$ is such that the matrix $(A + BK)$ is strictly stable.

With the dynamics (2.1), we associate a function $\ell(\cdot)$ specified, for all $x \in \mathbb{R}^n$, by:

$$\ell(x) := g(\mathcal{Q}, x) + g(\mathcal{R}, Kx), \tag{2.2}$$

where $\mathcal{Q}$ and $\mathcal{R}$ are proper C-sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The function $\ell(\cdot)$ is referred to as the stage cost function. We focus on the existence and characterization of a solution to the Minkowski–Lyapunov equation:

$$\forall x \in \mathbb{R}^n, \quad V(x) = V((A + BK)x) + \ell(x), \tag{2.3}$$

where a function $V(\cdot)$ is to be determined. In addition, we also investigate the existence and characterization of the arbitrarily close lower and upper approximations of a function $V(\cdot)$, say functions $\underline{V}(\cdot)$ and $\overline{V}(\cdot)$, which satisfy, for a given $\varepsilon > 0$ and any given non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$:

$$\forall x \in \mathbb{R}^n, \quad \underline{V}(x) \le V(x) \le \overline{V}(x), \tag{2.4a}$$

$$\forall x \in \mathbb{R}^n, \quad \overline{V}((A + BK)x) + \ell(x) \le \overline{V}(x), \quad \text{and,} \tag{2.4b}$$

$$\forall x \in \mathbb{X}, \quad |\overline{V}(x) - \underline{V}(x)| \le \varepsilon. \tag{2.4c}$$

2.2. Technical preliminaries

As demonstrated in Raković (2007), Assumption 1(ii) implies directly that there exists a symmetric proper C-set $\mathcal{L}$ in $\mathbb{R}^n$ such that $(A + BK)\mathcal{L} \subseteq \lambda\mathcal{L}$ with $\lambda \in [0, 1)$. (In fact, the corresponding value of λ is necessarily such that $\lambda \in [\rho(A + BK), 1)$.) For convenience and simplicity of our analysis and statements in the remainder of this paper, we invoke the following assumption (which follows directly from Assumption 1 and describes our setting):

Assumption 2. (i) The set–scalar pair $(\mathcal{L}, \lambda)$ satisfies that $\mathcal{L}$ is a symmetric proper C-set in $\mathbb{R}^n$ and:

$$(A + BK)\mathcal{L} \subseteq \lambda\mathcal{L}, \text{ where the scalar} \tag{2.5a}$$

$$\lambda := \min_\eta \{\eta : (A + BK)\mathcal{L} \subseteq \eta\mathcal{L} \text{ and } \eta \ge 0\} \tag{2.5b}$$

is such that $\lambda \in [0, 1)$; (ii) the sets $\mathcal{Q}$ and $\mathcal{R}$ are proper C-sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively; and (iii) the sets $\mathcal{L}$, $\mathcal{Q}$ and $\mathcal{R}$, and the scalar λ are known exactly.

A direct but valuable consequence of Assumption 2(i) is:

Lemma 1. *Suppose Assumption 2(i) holds. Then, for all $\mu \in \mathbb{R}_+$ and all $k \in \mathbb{N}$, it holds that:*

$$(A + BK)^k \mu\mathcal{L} \subseteq \lambda^k \mu\mathcal{L}. \tag{2.6}$$

It is well known (Kolmogorov & Fomin, 1970; Rockafellar, 1970; Schneider, 1993) that the Minkowski function $g(\mathcal{X}, \cdot) : \mathbb{R}^n \to \mathbb{R}$, associated with a proper C-set $\mathcal{X}$ in $\mathbb{R}^n$, is non-negative, finite, sublinear and continuous. We now provide a few less obvious facts concerned with the Minkowski (gauge) functions of proper C-sets.

Lemma 2. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be any three proper C-sets in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^n$ such that $\mathcal{Z} \subseteq \mathcal{X}$ and let $K \in \mathbb{R}^{m\times n}$ be any matrix such that $K\mathcal{Z} \subseteq \mathcal{Y}$. Then, for all $x \in \mathbb{R}^n$, it holds that:*

$$\text{(i)}\quad \forall \alpha \in \mathbb{R}_+ \setminus \{0\},\quad \mathrm{g}(\alpha \mathcal{X}, x) = \alpha^{-1}\,\mathrm{g}(\mathcal{X}, x), \tag{2.7a}$$

$$\text{(ii)}\quad \mathrm{g}(\mathcal{X}, x) \leq \mathrm{g}(\mathcal{Z}, x), \tag{2.7b}$$

$$\text{(iii)}\quad \mathrm{g}(\mathcal{Y}, Kx) \leq \mathrm{g}(\mathcal{Z}, x),\quad \textit{and,} \tag{2.7c}$$

$$\text{(iv)}\quad \mathrm{g}(\mathcal{X}, x) + \mathrm{g}(\mathcal{Y}, Kx) \leq 2\,\mathrm{g}(\mathcal{Z}, x). \tag{2.7d}$$

The proofs of facts (i) , (ii) and (iii) in Lemma 2 are reported in Raković and Lazar (2012), while the claim (iv) is a direct consequence of facts (ii) and (iii) in this Lemma.

We also need to recall the following two fundamental theorems in Minkowski theory of convex bodies (Schneider, 1993, Theorems 1.7.1. and 1.7.6.):

Theorem 1. *If $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there is a unique proper C-set with support function $f(\cdot)$.*

Theorem 2. *For a proper C-set $\mathcal{S}$ in $\mathbb{R}^n$ it holds that:*

$$\forall x \in \mathbb{R}^n,\quad \mathrm{g}(\mathcal{S}, x) = \mathrm{h}(\mathcal{S}^*, x), \tag{2.8}$$

where $\mathcal{S}^*$ is the polar set of $\mathcal{S}$.

2.3. Paper objectives

Our first main objective is to establish that the Minkowski–Lyapunov equation (2.3) is, under Assumption 1(ii) and 2(ii), or equivalently, under Assumption 2, solvable. In this sense, we establish that there is a unique proper C-set in $\mathbb{R}^n$, say $\mathcal{P}$, whose Minkowski function $\mathrm{g}(\mathcal{P}, \cdot)$ forms the basic solution to the Minkowski–Lyapunov equation (2.3).

Our second objective is to characterize two sequences of functions, $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ and $\{\overline{V}_k(\cdot)\}_{k\in\mathbb{N}}$, which are lower and upper approximations of the basic solution to the Minkowski–Lyapunov equation (2.3), i.e., which satisfy (2.4a) and, in addition, (2.4b) for all $k \in \mathbb{N}$. The terms of these sequences also satisfy (2.4c) for all large enough k.

Our third objective is to establish that the lower and upper approximations of the basic solution to the Minkowski–Lyapunov equation satisfying (2.4) can be constructed from the Minkowski functions of suitably specified proper C-polytopic sets.

3. The Minkowski–Lyapunov equation: existence and characterization of basic solution

The stage cost function $\ell(\cdot)$ in conjunction with the dynamics (2.1) induces the function $V_\infty(\cdot)$ specified, for all $x \in \mathbb{R}^n$, by:

$$V_\infty(x) = \sum_{k=0}^{\infty} \ell((A+BK)^k x). \tag{3.1}$$

We aim to establish that the function $V_\infty(\cdot)$ is the basic solution to the Minkowski–Lyapunov equation (2.3). Furthermore, we demonstrate that there exists a unique proper C-set $\mathcal{P}$ in $\mathbb{R}^n$ whose Minkowski function $\mathrm{g}(\mathcal{P}, \cdot)$ satisfies, for all $x \in \mathbb{R}^n$, $V_\infty(x) = \mathrm{g}(\mathcal{P}, x)$. To this end, we consider the sequence of functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ defined, for $k = 0$ and all $x \in \mathbb{R}^n$ by $\underline{V}_0(x) := 0$ and, for all $k \in \mathbb{N}_+$, by:

$$\forall x \in \mathbb{R}^n,\quad \underline{V}_k(x) := \sum_{i=0}^{k-1} \ell((A+BK)^i x). \tag{3.2}$$

By construction we have:

Proposition 1. *Suppose Assumption 2 holds. Then, for all $k \in \mathbb{N}$, the functions $\underline{V}_k(\cdot) : \mathbb{R}^n \to \mathbb{R}$ specified by (3.2) are non-negative, finite, sublinear and continuous. Furthermore, for all $k \in \mathbb{N}$, it holds that:*

$$\forall x \in \mathbb{R}^n,\quad \underline{V}_k(x) \leq \underline{V}_{k+1}(x),\quad \textit{and,} \tag{3.3a}$$

$$\forall x \in \mathbb{R}^n,\quad \underline{V}_{k+1}(x) = \underline{V}_k((A+BK)x) + \ell(x). \tag{3.3b}$$

Before proceeding, let:

$$\alpha := \min_{\eta} \big\{\eta \;:\; \mathcal{L} \subseteq 2^{-1}(1-\lambda)\eta\mathcal{Q},\; K\mathcal{L} \subseteq 2^{-1}(1-\lambda)\eta\mathcal{R} \text{ and } \eta \geq 0\big\}, \tag{3.4}$$

and note that, under Assumption 2, the scalar α is well defined, positive and finite (i.e., $0 < \alpha < \infty$).

Lemmata 1 and 2(i) and 2(iv) in conjunction with (3.4) imply the following observation:

Proposition 2. *Suppose Assumption 2 holds. Then, for all $k \in \mathbb{N}$, it holds that*

$$\forall x \in \mathbb{R}^n,\quad \ell((A+BK)^k x) \leq \lambda^k(1-\lambda)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x). \tag{3.5}$$

Furthermore, for all $k \in \mathbb{N}$, it holds that

$$\forall x \in \mathbb{R}^n,\quad \sum_{i=k}^{\infty} \ell((A+BK)^i x) \leq \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x). \tag{3.6}$$

Propositions 1 and 2 imply directly the following result:

Proposition 3. *Suppose Assumption 2 holds. Then, for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}$, it holds that*

$$\forall x \in \mathbb{R}^n,\quad 0 \leq \underline{V}_{k+j}(x) - \underline{V}_k(x) \leq \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x). \tag{3.7}$$

Clearly, Propositions 1 and 3 imply directly the pointwise convergence (as $k \to \infty$) on $\mathbb{R}^n$ of the sequence of functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified by (3.2) to the function $V_\infty(\cdot)$ given by (3.1); They also imply compact convergence of $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ (as $k \to \infty$) to $V_\infty(\cdot)$ (i.e., the uniform convergence of $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ to $V_\infty(\cdot)$ on any non-empty compact subset of $\mathbb{R}^n$). These facts as well as additional properties of the function $V_\infty(\cdot)$ are summarized by our first main result.

Theorem 3. *Suppose Assumption 2 holds. Then:* (i) *the sequence of functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified by (3.2) converges pointwise on $\mathbb{R}^n$ to the function $V_\infty(\cdot)$ given by (3.1);* (ii) *the sequence of the functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified by (3.2) converges uniformly on any non-empty compact subset of $\mathbb{R}^n$ to the function $V_\infty(\cdot)$ given by (3.1); and* (iii) *the function $V_\infty(\cdot)$ is non-negative, finite, sublinear and continuous.*

A direct inspection of (3.1) reveals that the function $V_\infty(\cdot)$ also satisfies:

$$\forall x \in \mathbb{R}^n,\quad V_\infty(x) = V_\infty((A+BK)x) + \ell(x) \tag{3.8}$$

so that it is a solution to the Minkowski–Lyapunov equation specified in (2.3). Clearly, the Minkowski–Lyapunov equation (2.3) does not admit a unique solution, as, indeed, any function specified, for all $x \in \mathbb{R}^n$, by $V_\infty(x) + \beta$ where β is a scalar is also its solution. Hence, our next task is to demonstrate that the function $V_\infty(\cdot)$ is the basic solution in a well defined sense discussed next.

By Theorem 3(iii), $V_\infty(\cdot)$ is non-negative, finite, sublinear and continuous. Hence, by Theorem 1, there is a unique proper C-set, say $\mathcal{P}^*$, in $\mathbb{R}^n$ whose support function is the function $V_\infty(\cdot)$. But, in view of Theorem 2, this means that there is a unique proper C-set, say $\mathcal{P}$, in $\mathbb{R}^n$ whose Minkowski function $\mathrm{g}(\mathcal{P}, \cdot)$ satisfies:

$$\forall x \in \mathbb{R}^n,\quad V_\infty(x) = \mathrm{g}(\mathcal{P}, x). \tag{3.9}$$

The discussion above yields our second main result:

Theorem 4. *Suppose Assumption 2 holds. Then there is a unique proper C-set, say $\mathcal{P}$, in $\mathbb{R}^n$ such that (3.9) holds and consequently $g(\mathcal{P},\cdot)$ solves the Minkowski–Lyapunov equation (2.3), i.e.,*

$$\forall x \in \mathbb{R}^n,$$
$$g(\mathcal{P},x) = g(\mathcal{P},(A+BK)x) + g(\mathcal{Q},x) + g(\mathcal{R},Kx). \tag{3.10}$$

Furthermore, the function $g(\mathcal{P},\cdot)$ or, equivalently, $V_\infty(\cdot)$ is the unique solution to the Minkowski–Lyapunov equation (2.3) over the class of Minkowski functions associated with C-sets in $\mathbb{R}^n$.

Theorem 4 justifies our use of the term basic solution to the Minkowski–Lyapunov equation. The polarity considerations utilized in the above discussion also provide an alternative equivalent characterization of the sequence of functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified by (3.2) and its limit $V_\infty(\cdot)$ as established by our next main result. To this end, consider the sequence of sets $\{\mathcal{P}_k^*\}_{k\in\mathbb{N}_+}$ specified, for all $k \in \mathbb{N}_+$, by:

$$\mathcal{P}_{k+1}^* = (A+BK)^T\mathcal{P}_k^* \oplus \left(\mathcal{Q}^* \oplus K^T\mathcal{R}^*\right)$$
$$\text{with } \mathcal{P}_1^* := \mathcal{Q}^* \oplus K^T\mathcal{R}^*, \tag{3.11}$$

where $\mathcal{Q}^*$ and $\mathcal{R}^*$ are polar sets of the sets $\mathcal{Q}$ and $\mathcal{R}$.

Theorem 5. *Suppose Assumption 2 holds. Then, for all $k \in \mathbb{N}_+$, it holds that:*

$$\forall x \in \mathbb{R}^n, \quad \underline{V}_k(x) = g(\mathcal{P}_k,x), \tag{3.12}$$

where each $\mathcal{P}_k$ is a proper C-set in $\mathbb{R}^n$ and is the polar set of a proper C-set $\mathcal{P}_k^*$ specified via (3.11), i.e., $\mathcal{P}_k = (\mathcal{P}_k^*)^*$. Furthermore, the function $V_\infty(\cdot)$ is given by:

$$\forall x \in \mathbb{R}^n, \quad V_\infty(x) = g(\mathcal{P},x), \tag{3.13}$$

where the set $\mathcal{P}$ is a unique proper C-set in $\mathbb{R}^n$ whose polar set $\mathcal{P}^*$ is a proper C-set in $\mathbb{R}^n$ and is the unique solution to the set equation:

$$\mathcal{P}^* = (A+BK)^T\mathcal{P}^* \oplus \left(\mathcal{Q}^* \oplus K^T\mathcal{R}^*\right). \tag{3.14}$$

Illustrative example: For a simple, academic illustration of the basic solution to the Minkowski–Lyapunov equation, consider the 1-dimensional linear dynamics:

$$x^+ = (a+bk)x \quad \text{with } u = kx \text{ and } |a+bk| < 1.$$

The stage cost $\ell(\cdot)$ is given by (2.2) with:

$$\mathcal{Q} := [-q,q] \quad \text{and} \quad \mathcal{R} := [-r,r] \text{ with } q > 0 \text{ and } r > 0.$$

Clearly, for all $x \in \mathbb{R}$, we have $g(\mathcal{Q},x) = q^{-1}|x|$ and $g(\mathcal{R},kx) = r^{-1}|k||x|$, and, consequently, $\ell(x) = (rq)^{-1}(r+q|k|)|x|$. In this case, the basic solution to the Minkowski–Lyapunov equation $V_\infty(\cdot)$ takes the form $V_\infty(x) = g(\mathcal{P},x)$ where the set $\mathcal{P}$ is given by:

$$\mathcal{P} = [-p,p] \quad \text{with } p := rq(1-|a+bk|)(r+q|k|)^{-1} > 0.$$

We close this section by pointing out a fundamental correspondence between the developed framework and the theory of the minimal robust positively invariant (mRPI) set (Artstein & Raković, 2008; Kolmanovsky & Gilbert, 1998; Raković, 2007). To this end, we resort to the polar dynamics:

$$x^+ = (A+BK)^Tx + w \quad \text{with } w \in \mathcal{Q}^* \oplus K^T\mathcal{R}^*. \tag{3.15}$$

Under the assumptions invoked in this paper, the mRPI set for the polar dynamics (3.15) is the unique proper C-set that solves the set equation (3.14) over the space of non-empty compact subsets in $\mathbb{R}^n$ (Artstein & Raković, 2008; Kolmanovsky & Gilbert, 1998; Raković, 2007). *Remarkably, the basic solution to the Minkowski–Lyapunov equation (2.3) is uniquely determined by the mRPI set for the polar dynamics (3.15). In particular, the generator of the basic solution to the Minkowski–Lyapunov equation (2.3) and the mRPI set for the polar dynamics (3.15) are polar sets to each other.*

The mRPI set for the polar dynamics (3.15) is explicitly given by Kolmanovsky and Gilbert (1998); Raković (2007); Artstein and Raković (2008):

$$\mathcal{P}^* = \bigoplus_{k=0}^{\infty}((A+BK)^T)^k\left(\mathcal{Q}^* \oplus K^T\mathcal{R}^*\right), \tag{3.16}$$

so that the generator $\mathcal{P} = (\mathcal{P}^*)^*$ of the basic solution to the Minkowski–Lyapunov equation takes the explicit form specified by:

$$\mathcal{P} = \left(\bigoplus_{k=0}^{\infty}((A+BK)^T)^k\left(\mathcal{Q}^* \oplus K^T\mathcal{R}^*\right)\right)^*. \tag{3.17}$$

It is also worth pointing out that, the set sequence $\{\mathcal{P}_k^*\}_{k\in\mathbb{N}_+}$ specified in (3.11) converges, w.r.t. the Hausdorff distance, exponentially fast to the set $\mathcal{P}^*$ (Artstein & Raković, 2008; Kolmanovsky & Gilbert, 1998; Raković, 2007).

4. Consistently improving approximations

Since, by construction, the functions $\underline{V}_k(\cdot)$, $k \in \mathbb{N}$ form consistently improving lower approximations of the basic solution to the Minkowski–Lyapunov equation (2.3), i.e., of the function $V_\infty(\cdot)$, we proceed to obtain the consistently improving upper approximations which in addition satisfy the Lyapunov inequality (2.4b). To this end, we consider the sequence of functions $\{\overline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified, for all $k \in \mathbb{N}$, by:

$$\forall x \in \mathbb{R}^n, \quad \overline{V}_k(x) = \underline{V}_k(x) + \lambda^k g(\alpha^{-1}\mathcal{L},x). \tag{4.1}$$

In analogy with Proposition 1, we have:

Proposition 4. *Suppose Assumption 2 holds. Then, for all $k \in \mathbb{N}$, the functions $\overline{V}_k(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ specified by (4.1) are non-negative, finite, sublinear and continuous. Furthermore, for all $k \in \mathbb{N}$, it holds that:*

$$\forall x \in \mathbb{R}^n, \quad \overline{V}_{k+1}(x) \le \overline{V}_k(x), \quad \text{and,} \tag{4.2a}$$
$$\forall x \in \mathbb{R}^n, \quad \overline{V}_k((A+BK)x) + \ell(x) \le \overline{V}_k(x). \tag{4.2b}$$

To obtain an analogous result to Theorem 5 we consider the sequence of sets $\{\mathcal{S}_k^*\}_{k\in\mathbb{N}}$ specified, for all $k \in \mathbb{N}$, by:

$$\mathcal{S}_k^* = \mathcal{P}_k^* \oplus \lambda^k\alpha\mathcal{L}^* \quad \text{and with } \mathcal{S}_0^* = \alpha\mathcal{L}^*, \tag{4.3}$$

where $\mathcal{L}^*$ is the polar set of the set $\mathcal{L}$ and the sets $\mathcal{P}_k^*$, $k \in \mathbb{N}_+$ are specified via (3.11). We can now establish that:

Proposition 5. *Suppose Assumption 2 holds. Then, for all $k \in \mathbb{N}$, it holds that:*

$$\forall x \in \mathbb{R}^n, \quad \overline{V}_k(x) = g(\mathcal{S}_k,x), \tag{4.4}$$

where each $\mathcal{S}_k$ is a proper C-set in $\mathbb{R}^n$ and is the polar set of a proper C-set $\mathcal{S}_k^*$ specified via (4.3), i.e., $\mathcal{S}_k = (\mathcal{S}_k^*)^*$.

Propositions 1 and 4 in conjunction with Theorem 3 yield our next main result:

Theorem 6. *Suppose Assumption 2 holds. Then, for all $k \in \mathbb{N}$, it holds that:*

$$\forall x \in \mathbb{R}^n, \quad \underline{V}_k(x) \le V_\infty(x) \le \overline{V}_k(x), \tag{4.5a}$$
$$\forall x \in \mathbb{R}^n, \quad \overline{V}_k((A+BK)x) + \ell(x) \le \overline{V}_k(x), \quad \text{and,} \tag{4.5b}$$
$$\forall x \in \mathbb{R}^n, \quad 0 \le \overline{V}_k(x) - \underline{V}_k(x) \le \lambda^k g(\alpha^{-1}\mathcal{L},x). \tag{4.5c}$$

Theorem 6 has a number of direct implications, the first of which is the fact that the sequence of functions $\{\overline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified by (4.1) also converges pointwise on $\mathbb{R}^n$ and uniformly on any non-empty compact subset of $\mathbb{R}^n$ to the function $V_\infty(\cdot)$:

Corollary 1. *Suppose Assumption 2 holds. Then:* (i) *the sequence of functions $\{\overline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified by (4.1) converges pointwise on $\mathbb{R}^n$ to the function $V_\infty(\cdot)$ given by (3.1); and* (ii) *the sequence of functions $\{\overline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ specified by (4.1) converges uniformly on any non-empty compact subset of $\mathbb{R}^n$ to the function $V_\infty(\cdot)$ given by (3.1).*

A more important consequence of Theorem 6 is the fact that the sequences of functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ and $\{\overline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ provide a family of pairs of functions satisfying (2.4) (this fact, in turn, allows us to address our second main objective). To this end, let for any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$:

$$\gamma(\mathbb{X}) := \max_x\{g(\alpha^{-1}\mathcal{L}, x) \ : \ x\in\mathbb{X}\}, \tag{4.6}$$

and note that $\gamma(\mathbb{X})$ is well defined, non-negative and finite for any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$, i.e., $0\le\gamma(\mathbb{X})<\infty$. Given any $\varepsilon>0$ and any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ let:

$$N(\varepsilon,\mathbb{X}) := \{k\in\mathbb{N} \ : \ \lambda^k\gamma(\mathbb{X})\le\varepsilon\}. \tag{4.7}$$

Clearly, for $\mathbb{X}=\{0\}$ we have $N(\varepsilon,\mathbb{X})=\mathbb{N}$, while for any other non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ we have that $N(\varepsilon,\mathbb{X})\neq\emptyset$. We can now verify the following result:

Corollary 2. *Suppose Assumption 2 holds and let $\varepsilon>0$ and a non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ be given. Then, $N(\varepsilon,\mathbb{X})\neq\emptyset$ and for all $k\in N(\varepsilon,\mathbb{X})$, it holds that:*

$$\forall x\in\mathbb{R}^n,\quad \underline{V}_k(x)\le V_\infty(x)\le\overline{V}_k(x), \tag{4.8a}$$

$$\forall x\in\mathbb{R}^n,\quad \overline{V}_k((A+BK)x)+\ell(x)\le\overline{V}_k(x),\ \ \textit{and,} \tag{4.8b}$$

$$\forall x\in\mathbb{X},\quad |\overline{V}_k(x)-\underline{V}_k(x)|\le\varepsilon. \tag{4.8c}$$

We would like to stress that Propositions 4 and 5 provide directly the characterization of a rich family of Minkowski–Lyapunov functions. In particular, the family of functions:

$$\overline{\mathcal{V}} := \{\overline{V}_k(\cdot) \ : \ k\in\mathbb{N}\} = \{g(\mathcal{S}_k,\cdot) \ : \ k\in\mathbb{N}\} \tag{4.9}$$

is a family of standard Lyapunov functions since the sets $\mathcal{S}_k$, $k\in\mathbb{N}$ are proper C-sets. Furthermore, for any given non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$, the members of the family of functions $\overline{\mathcal{V}}$ are arbitrarily close upper approximations of the basic solution $V_\infty(\cdot)$ to the Minkowski–Lyapunov equation (2.3). More precisely, for any given $\varepsilon>0$, the family of functions:

$$\overline{\mathcal{V}}_\varepsilon := \{\overline{V}_k(\cdot) \ : \ k\in N(\varepsilon,\mathbb{X})\} = \{g(\mathcal{S}_k,\cdot) \ : \ k\in N(\varepsilon,\mathbb{X})\}, \tag{4.10}$$

is a family of standard Lyapunov functions that are ε upper approximations of the basic solution $V_\infty(\cdot)$ to the Minkowski–Lyapunov equation (2.3) over the set $\mathbb{X}$.

5. Polytopic lower and upper approximations

Our final main aim is to construct two sequences of functions that in addition to being the lower and upper approximations of the basic solution to the Minkowski–Lyapunov equation (2.3) are Minkowski functions associated with proper *C–polytopic* sets. Throughout this section we invoke for simplicity, but w.l.o.g. (Raković, 2007), an assumption that the set $\mathcal{L}$ is, in addition, a polytope:

Assumption 3. The set $\mathcal{L}$ is a polytope in $\mathbb{R}^n$.

The developments in this section rely essentially on the fact, established in Schneider (1993), that the set of polytopes in $\mathbb{R}^n$ is a dense subset of the space of compact and convex subsets of $\mathbb{R}^n$. In particular, given any $\varphi\in(0,1)$, we employ Schneider (1993, Theorem 1.8.13.) in order to associate proper C-polytopes $\mathcal{Q}_\varphi$ and $\mathcal{R}_\varphi$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with the sets $\mathcal{Q}$ and $\mathcal{R}$, of Assumption 2, via the relations:

$$(1-\varphi)\mathcal{Q}_\varphi\subseteq\mathcal{Q}\subseteq\mathcal{Q}_\varphi,\ \ \text{and,} \tag{5.1a}$$

$$(1-\varphi)\mathcal{R}_\varphi\subseteq\mathcal{R}\subseteq\mathcal{R}_\varphi. \tag{5.1b}$$

Note that Schneider (1993, Theorem 1.8.13.) guarantees the existence of the above postulated proper C-polytopes $\mathcal{Q}_\varphi$ and $\mathcal{R}_\varphi$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, as formalized by:

Lemma 3. *Suppose Assumption 2*(ii) *holds. Then, for all $\varphi\in(0,1)$, the set $\Pi(\mathcal{Q},\mathcal{R},\varphi)$ given by:*

$$\begin{aligned}\Pi(\mathcal{Q},\mathcal{R},\varphi) := \{&(\mathcal{Q}_\varphi,\mathcal{R}_\varphi) \ : \\ &\mathcal{Q}_\varphi \ \textit{is a proper C–polytope in} \ \mathbb{R}^n, \\ &\mathcal{R}_\varphi \ \textit{is a proper C–polytope in} \ \mathbb{R}^m \\ &\textit{and relations (5.1) hold}\}\end{aligned} \tag{5.2}$$

is non-empty.

Now, let, for all $x\in\mathbb{R}^n$,

$$\ell_\varphi(x) := g(\mathcal{Q}_\varphi,x)+g(\mathcal{R}_\varphi,Kx). \tag{5.3}$$

Lemmata 2(i), 2(ii) and 3 imply that, for all $\varphi\in(0,1)$, $\Pi(\mathcal{Q},\mathcal{R},\varphi)\neq\emptyset$ and that for any $(\mathcal{Q}_\varphi,\mathcal{R}_\varphi)\in\Pi(\mathcal{Q},\mathcal{R},\varphi)$, we have, for all $x\in\mathbb{R}^n$,

$$g(\mathcal{Q}_\varphi,x)\le g(\mathcal{Q},x)\le(1-\varphi)^{-1}g(\mathcal{Q}_\varphi,x),\ \ \text{and,} \tag{5.4a}$$

$$g(\mathcal{R}_\varphi,Kx)\le g(\mathcal{R},Kx)\le(1-\varphi)^{-1}g(\mathcal{R}_\varphi,Kx). \tag{5.4b}$$

A direct utilization of Schneider (1993, Theorem 1.8.13.) in conjunction with (5.4) implies directly the following observation:

Proposition 6. *Suppose Assumption 2 holds. Then, for any $\varphi\in(0,1)$ and any pair (of proper C-polytopes) $(\mathcal{Q}_\varphi,\mathcal{R}_\varphi)\in\Pi(\mathcal{Q},\mathcal{R},\varphi)$, it holds that:*

$$\forall x\in\mathbb{R}^n,\quad \ell_\varphi(x)\le\ell(x)\le(1-\varphi)^{-1}\ell_\varphi(x), \tag{5.5a}$$

or, equivalently,

$$\forall x\in\mathbb{R}^n,\quad \ell_\varphi(x)\le\ell(x)\le\ell_\varphi(x)+\varphi(1-\varphi)^{-1}\ell_\varphi(x). \tag{5.5b}$$

Before proceeding, we also note that relations (3.4) and (5.3) in conjunction with Proposition 2 guarantee that, for all $k\in\mathbb{N}$, it holds that:

$$\forall x\in\mathbb{R}^n,\quad \ell_\varphi((A+BK)^kx)\le\lambda^k(1-\lambda)\,g(\alpha^{-1}\mathcal{L},x). \tag{5.6}$$

In the same manner as it was done in Sections 3 and 4, we now introduce the sequences of functions $\{\underline{V}_{\varphi,k}(\cdot)\}_{k\in\mathbb{N}}$ and $\{\overline{V}_{\varphi,k}(\cdot)\}_{k\in\mathbb{N}}$. For any $\varphi\in(0,1)$, the sequence of functions $\{\underline{V}_{\varphi,k}(\cdot)\}_{k\in\mathbb{N}}$ is specified, for $k=0$ and for all $x\in\mathbb{R}^n$ by $\underline{V}_{\varphi,0}(x)=0$, and for all $k\in\mathbb{N}_+$, by

$$\forall x\in\mathbb{R}^n,\quad \underline{V}_{\varphi,k}(x)=\sum_{i=0}^{k-1}\ell_\varphi((A+BK)^ix). \tag{5.7}$$

Similarly, for any $\varphi\in(0,1)$, the sequence of functions $\{\overline{V}_{\varphi,k}(\cdot)\}_{k\in\mathbb{N}}$ is specified, for all $k\in\mathbb{N}$, by

$$\forall x\in\mathbb{R}^n,$$

$$\overline{V}_{\varphi,k}(x)=\underline{V}_{\varphi,k}(x)+(\varphi(1-\varphi)^{-1}+\lambda^k)\,g(\alpha^{-1}\mathcal{L},x). \tag{5.8}$$

In the above equations, we have implicitly assumed that for any $\varphi \in (0,1)$, the corresponding pair of proper C-polytopes $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q},\mathcal{R},\varphi)$ is selected arbitrarily but fixed. (We keep in mind that, our analysis is applicable to any such pair $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q},\mathcal{R},\varphi)$, and hence to all such pairs $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q},\mathcal{R},\varphi)$.) We also note that the analysis provided in Sections 3 and 4 applies in a direct way to the sequences of functions $\{\underline{V}_{\varphi,k}(\cdot)\}_{k\in\mathbb{N}}$ and $\{\overline{V}_{\varphi,k}(\cdot)\}_{k\in\mathbb{N}}$ for any fixed $\varphi \in (0,1)$; we omit the corresponding formalism in order to avoid repetition. However, we do establish the most relevant facts.

Proposition 7. *Suppose Assumptions 2 and 3 hold. Then, for any $\varphi \in (0,1)$ and any pair (of proper C-polytopes) $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q},\mathcal{R},\varphi)$ and all $k \in \mathbb{N}_+$, it holds that:*

$$\forall x \in \mathbb{R}^n, \quad \underline{V}_{\varphi,k}(x) = \mathrm{g}(\mathcal{P}_k, x) \text{ with } \underline{V}_{\varphi,0}(x) = 0, \text{ and,} \tag{5.9a}$$

$$\forall x \in \mathbb{R}^n, \quad \overline{V}_{\varphi,k}(x) = \mathrm{g}(\mathcal{S}_k, x) \text{ with}$$

$$\overline{V}_{\varphi,0}(x) = (1-\varphi)^{-1}\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x), \tag{5.9b}$$

where the sets $\mathcal{P}_k$ and $\mathcal{S}_k$, $k \in \mathbb{N}_+$ are proper C-polytopic sets in $\mathbb{R}^n$ and are the polar sets of the proper C-polytopic sets $\mathcal{P}_k^*$ and $\mathcal{S}_k^*$, $k \in \mathbb{N}_+$ specified via (3.11) and (4.3) with the sets $\mathcal{Q}$ and $\mathcal{R}$ in (3.11) replaced by the proper C-polytopes $\mathcal{Q}_\varphi$ and $\mathcal{R}_\varphi$ and with $\lambda^k\alpha$ in (4.3) replaced, for all $k \in \mathbb{N}$, by $(\varphi(1-\varphi)^{-1} + \lambda^k)\alpha$.

By construction, for any $\varphi \in (0,1)$ and any pair of proper C-polytopes $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q},\mathcal{R},\varphi)$, the functions $\underline{V}_{\varphi,k}(\cdot)$ lower bound the functions $\underline{V}_k(\cdot)$ while the functions $\overline{V}_{\varphi,k}(\cdot)$ upper bound the functions $\overline{V}_k(\cdot)$. Consequently, we can verify the following facts:

Proposition 8. *Suppose Assumptions 2 and 3 hold. Then, for any $\varphi \in (0,1)$ and any pair (of proper C-polytopes) $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q},\mathcal{R},\varphi)$, all $k \in \mathbb{N}_+$, and all $x \in \mathbb{R}^n$, it holds that:*

$$\underline{V}_{\varphi,k}(x) \le \underline{V}_k(x) \le V_\infty(x) \le \overline{V}_k(x) \le \overline{V}_{\varphi,k}(x), \tag{5.10a}$$

$$\overline{V}_{\varphi,k}((A+BK)x) + \ell(x)$$
$$\le \overline{V}_{\varphi,k}((A+BK)x) + (1-\varphi)^{-1}\ell_\varphi(x) \le \overline{V}_{\varphi,k}(x), \tag{5.10b}$$

and,

$$0 \le \overline{V}_{\varphi,k}(x) - \underline{V}_{\varphi,k}(x)$$
$$\le (\varphi(1-\varphi)^{-1} + \lambda^k)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x). \tag{5.10c}$$

Proposition 8 implies that the conditions specified in (2.4) can also be achieved with the functions $\underline{V}_{\varphi,k}(\cdot)$ and $\overline{V}_{\varphi,k}(\cdot)$ for a suitable value of $\varphi \in (0,1)$. To this end we redefine the set $N(\varepsilon,\mathbb{X})$ of (4.7) to account for the utilization of the proper C-polytopes $\mathcal{Q}_\varphi$ and $\mathcal{R}_\varphi$ instead of the proper C-sets $\mathcal{Q}$ and $\mathcal{R}$. Given any $\varepsilon > 0$ and any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ let:

$$\Phi(\varepsilon,\mathbb{X}) := \{\varphi \in (0,1) : \varphi \le \varepsilon(\varepsilon + 2\gamma(\mathbb{X}))^{-1}\}, \quad \text{and,}$$
$$N(\varepsilon,\mathbb{X}) := \{k \in \mathbb{N} : \lambda^k\gamma(\mathbb{X}) \le 2^{-1}\varepsilon\}. \tag{5.11}$$

Note that the set $\Phi(\varepsilon,\mathbb{X})$ is, by construction, non-empty for any $\varepsilon > 0$ and any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ and, for all $\varphi \in \Phi(\varepsilon,\mathbb{X})$, it holds that $\varphi(1-\varphi)^{-1}\gamma(\mathbb{X}) \le 2^{-1}\varepsilon$. Similarly as in (4.7), for $\mathbb{X} = \{0\}$ we have $N(\varepsilon,\mathbb{X}) = \mathbb{N}$ while for any other non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ we have $N(\varepsilon,\mathbb{X}) \ne \emptyset$; consequently, for any $\varepsilon > 0$ and any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ and, for all $k \in N(\varepsilon,\mathbb{X})$, it holds that $\lambda^k\gamma(\mathbb{X}) \le 2^{-1}\varepsilon$. It follows that, for any $\varepsilon > 0$ and any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$, we have, for all $\varphi \in \Phi(\varepsilon,\mathbb{X})$ and all $k \in N(\varepsilon,\mathbb{X})$, that $(\varphi(1-\varphi)^{-1} + \lambda^k)\gamma(\mathbb{X}) \le \varepsilon$. These facts lead to our final formal observation:

Proposition 9. *Suppose Assumptions 2 and 3 hold and let $\varepsilon > 0$ and a non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ be given. Then, $\Phi(\varepsilon,\mathbb{X}) \ne \emptyset$ and for all $\varphi \in \Phi(\varepsilon,\mathbb{X}) \subseteq (0,1)$, the set $\Pi(\mathcal{Q},\mathcal{R},\varphi)$ given by (5.2) is non-empty. Furthermore, for any pair (of proper C-polytopes) $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q},\mathcal{R},\varphi)$, $N(\varepsilon,\mathbb{X}) \ne \emptyset$ and for all $k \in N(\varepsilon,\mathbb{X})$, it holds that:*

$$\forall x \in \mathbb{R}^n, \quad \underline{V}_{\varphi,k}(x) \le V_\infty(x) \le \overline{V}_{\varphi,k}(x), \tag{5.12a}$$

$$\forall x \in \mathbb{R}^n, \quad \overline{V}_{\varphi,k}((A+BK)x) + \ell(x)$$
$$\le \overline{V}_{\varphi,k}((A+BK)x) + (1-\varphi)^{-1}\ell_\varphi(x)$$
$$\le \overline{V}_{\varphi,k}(x), \text{ and,} \tag{5.12b}$$

$$\forall x \in \mathbb{X}, |\overline{V}_{\varphi,k}(x) - \underline{V}_{\varphi,k}(x)| \le \varepsilon. \tag{5.12c}$$

Clearly, Proposition 9 allows for the construction of Minkowski functions $\underline{V}_{\varphi,k}(\cdot)$ and $\overline{V}_{\varphi,k}(\cdot)$ associated with suitably defined proper C-polytopes (as specified in Proposition 7) which provide consistently improving lower and upper approximations of the basic solution to the Minkowski–Lyapunov equation (2.3), and which also satisfy relations (2.4). Furthermore and similarly to closing observations of Section 4, Propositions 8 and 9 provide directly the characterization of a rich family of polyhedral Minkowski–Lyapunov functions. In particular, for any $\varphi \in (0,1)$, a family of functions:

$$\overline{\mathcal{V}}_\varphi := \{\overline{V}_{\varphi,k}(\cdot) : k \in \mathbb{N}\} \tag{5.13}$$

is a family of polyhedral Lyapunov functions. Likewise, for any given non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$, the members of the family of functions $\overline{\mathcal{V}}_\varphi$ are arbitrarily close upper approximations of the basic solution $V_\infty(\cdot)$ to the Minkowski–Lyapunov equation (2.3). More precisely, for any given $\varepsilon > 0$, the family $\overline{\mathcal{V}}_{\varphi,\varepsilon}$ of functions specified, for all $\varphi \in \Phi(\varepsilon,\mathbb{X})$, by:

$$\overline{\mathcal{V}}_{\varphi,\varepsilon} := \{\overline{V}_{\varphi,k}(\cdot) : k \in N(\varepsilon,\mathbb{X})\}, \tag{5.14}$$

is a family of polyhedral Lyapunov functions that are ε upper approximations of the basic solution $V_\infty(\cdot)$ to the Minkowski–Lyapunov equation (2.3) over the set $\mathbb{X}$.

6. Conclusions

It was established that the Minkowski–Lyapunov equation is a well-posed equation for strictly stable linear dynamics and a general class of stage functions specified via Minkowski functions of proper C-sets. Furthermore, it was shown that the Minkowski–Lyapunov equation admits the unique solution within the class of Minkowski functions associated with proper C-sets. A characterization of the lower and upper approximations of the basic solution, which converge pointwise and compactly to it, was also reported. Finally, the theoretical analysis of the Minkowski–Lyapunov equation allows for the development of computationally practicable prototype algorithms, as will be discussed elsewhere.

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Appendix A. Proofs

A.1. Proof of Proposition 1

By Assumption $\ell(\cdot)$ is non-negative, finite, sublinear and continuous. For all $k \in \mathbb{N}$, $x \mapsto (A+BK)^k x$ and $x \mapsto K(A+BK)^k x$ are linear. Hence, for each k, $\underline{V}_k(\cdot)$ is non-negative, finite and continuous being the sum of non-negative, finite and continuous functions. In addition, for each k, $\underline{V}_k(\cdot)$ is sublinear being the sum of compositions of sublinear and linear functions. Now, take any $x \in \mathbb{R}^n$. Clearly,

$$\underline{V}_{k+1}(x) = \underline{V}_k(x) + \ell((A+BK)^k x),$$

but $\ell((A+BK)^k x) \geq 0$ and, in turn, $\underline{V}_k(x) \leq \underline{V}_{k+1}(x)$ as claimed. Furthermore, a direct inspection of (3.2), reveals that, for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\underline{V}_{k+1}(x) &= \sum_{i=0}^{k-1} \ell((A+BK)^i(A+BK)x) + \ell(x)\\ &= \underline{V}_k((A+BK)x) + \ell(x).\end{aligned}$$

A.2. Proof of Proposition 2

Take any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^n$. By (3.4), we have $2(1-\lambda)^{-1}\alpha^{-1}\mathcal{L} \subseteq \mathcal{Q}$ and $2(1-\lambda)^{-1}K\alpha^{-1}\mathcal{L} \subseteq \mathcal{R}$. This fact together with a direct use of Lemma 2(i) and (iv) yields

$$\ell((A+BK)^k x) \leq (1-\lambda)\, \mathrm{g}(\alpha^{-1}\mathcal{L}, (A+BK)^k x).$$

By Lemma 1, $\mathrm{g}(\alpha^{-1}\mathcal{L}, (A+BK)^k x) \leq \lambda^k \mathrm{g}(\alpha^{-1}\mathcal{L}, x)$. Hence,

$$\ell((A+BK)^k x) \leq \lambda^k(1-\lambda)\, \mathrm{g}(\alpha^{-1}\mathcal{L}, x),$$

verifying the first claim. The second claim is true since

$$\begin{aligned}\sum_{i=k}^{\infty} \ell((A+BK)^i x) &\leq \sum_{i=k}^{\infty} \lambda^i(1-\lambda)\, \mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &= \lambda^k\, \mathrm{g}(\alpha^{-1}\mathcal{L}, x).\end{aligned}$$

A.3. Proof of Proposition 3

Take any $k \in \mathbb{N}$, any $j \in \mathbb{N}$ and any $x \in \mathbb{R}^n$. Then, by Proposition 1, $0 \leq \underline{V}_{k+j}(x) - \underline{V}_k(x)$ while, by construction, $\underline{V}_{k+j}(x) - \underline{V}_k(x) = \sum_{i=k}^{k+j-1} \ell((A+BK)^i x) \leq \sum_{i=k}^{\infty} \ell((A+BK)^i x)$. Hence, by Proposition 2, $\underline{V}_{k+j}(x) - \underline{V}_k(x) \leq \lambda^k\, \mathrm{g}(\alpha^{-1}\mathcal{L}, x)$.

A.4. Proof of Theorem 3

(i) By Proposition 3, for all $k \in \mathbb{N}$, all $j \in \mathbb{N}$ and all $x \in \mathbb{R}^n$, $0 \leq \underline{V}_{k+j}(x) - \underline{V}_k(x) \leq \lambda^k\, \mathrm{g}(\alpha^{-1}\mathcal{L}, x)$. Take any $x \in \mathbb{R}^n$. Then, $\mathrm{g}(\alpha^{-1}\mathcal{L}, x)$ is non-negative and finite. Hence, since $\lambda \in [0,1)$, for any $\varepsilon > 0$ there exists a $k^*(\varepsilon, x)$ such that for all $k \geq k^*(\varepsilon, x)$ and all $j \in \mathbb{N}$ it holds that $0 \leq \underline{V}_{k+j}(x) - \underline{V}_k(x) \leq \varepsilon$. Due to Proposition 1 it follows that for all $k \geq k^*(\varepsilon, x)$ and all $j \geq k^*(\varepsilon, x)$ we have $|\underline{V}_k(x) - \underline{V}_j(x)| \leq \varepsilon$. Hence, the sequence of functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ converges pointwise on $\mathbb{R}^n$ to the function $V_\infty(\cdot)$ as claimed. (ii) Now, given any non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ we have that $\max_x\{\mathrm{g}(\alpha^{-1}\mathcal{L}, x) : x \in \mathbb{X}\}$ is non-negative and finite. Hence, similarly as above in (i), Propositions 1 and 3 imply that for any given $\varepsilon > 0$ there exists a $k^*(\varepsilon, \mathbb{X})$ such that for all $k \geq k^*(\varepsilon, \mathbb{X})$, all $j \geq k^*(\varepsilon, \mathbb{X})$ and all $x \in \mathbb{X}$ it holds that $|\underline{V}_k(x) - \underline{V}_j(x)| \leq \varepsilon$. Thus, the sequence of functions $\{\underline{V}_k(\cdot)\}_{k\in\mathbb{N}}$ converges uniformly on any given non-empty compact subset $\mathbb{X}$ of $\mathbb{R}^n$ to the function $V_\infty(\cdot)$ as claimed. (iii) By construction, the function $V_\infty(\cdot)$ is non-negative being the sum of non-negative functions. By Proposition 2, for all $x \in \mathbb{R}^n$ it holds that $V_\infty(x) \leq \mathrm{g}(\alpha^{-1}\mathcal{L}, x)$ and, hence, the function $V_\infty(\cdot)$ is finite. Furthermore, the function $V_\infty(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is pointwise limit of sublinear functions $\underline{V}_k(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and, hence, it is a sublinear function and, in turn, due to Schneider (1993, Theorem 1.5.1.) it is also a continuous function.

A.5. Proof of Theorem 4

By Theorem 3(iii), $V_\infty(\cdot)$ is non-negative, finite, sublinear and continuous. Hence, by Theorem 1 there is a unique proper C-set, say $\mathcal{P}^*$, in $\mathbb{R}^n$ whose support function is the function $V_\infty(\cdot)$. By Theorem 2, the polar set $\mathcal{P} = (\mathcal{P}^*)^*$ is then a unique proper C-set in $\mathbb{R}^n$ whose Minkowski function $\mathrm{g}(\mathcal{P}, \cdot)$ satisfies, for all $x \in \mathbb{R}^n$, $\mathrm{g}(\mathcal{P}, x) = V_\infty(x)$. The uniqueness over the class of Minkowski functions associated with C-sets follows from the uniqueness of the proper C-set $\mathcal{P}^*$ guaranteed by Theorem 1 or, equivalently, from the uniqueness of the polar set $\mathcal{P} = (\mathcal{P}^*)^*$.

A.6. Proof of Theorem 5

Take any $k \in \mathbb{N}_+$ and any $x \in \mathbb{R}^n$. Then

$$\underline{V}_k(x) = \sum_{i=0}^{k-1} \left(\mathrm{g}(\mathcal{Q}, (A+BK)^i x) + \mathrm{g}(\mathcal{R}, K(A+BK)^i x)\right).$$

By Theorem 2, for each $i \in \mathbb{N}_{k-1}$, $\mathrm{g}(\mathcal{Q}, (A+BK)^i x) = \mathrm{h}(\mathcal{Q}^*, (A+BK)^i x)$ and $\mathrm{g}(\mathcal{R}, K(A+BK)^i x) = \mathrm{h}(\mathcal{R}^*, K(A+BK)^i x)$. In addition, for each $i \in \mathbb{N}_{k-1}$, $\mathrm{h}(\mathcal{Q}^*, (A+BK)^i x) = \mathrm{h}(((A+BK)^T)^i \mathcal{Q}^*, x)$ and $\mathrm{h}(\mathcal{R}^*, K(A+BK)^i x) = \mathrm{h}(((A+BK)^T)^i K^T \mathcal{R}^*, x)$. Hence,

$$\underline{V}_k(x) = \sum_{i=0}^{k-1} \left(\mathrm{h}(((A+BK)^T)^i \mathcal{Q}^*, x) + \mathrm{h}(((A+BK)^T)^i K^T \mathcal{R}^*, x)\right).$$

But the support function is additive in the first argument and, hence, for all $i \in \mathbb{N}_{k-1}$, $\mathrm{h}(((A+BK)^T)^i \mathcal{Q}^*, x) + \mathrm{h}(((A+BK)^T)^i K^T \mathcal{R}^*, x) = \mathrm{h}(((A+BK)^T)^i (\mathcal{Q}^* \oplus K^T \mathcal{R}^*), x)$. Consequently,

$$\underline{V}_k(x) = \mathrm{h}\left(\bigoplus_{i=0}^{k-1} ((A+BK)^T)^i (\mathcal{Q}^* \oplus K^T \mathcal{R}^*), x\right).$$

Let, for each $k \in \mathbb{N}_+$, $\mathcal{P}_k^* = \bigoplus_{i=0}^{k-1} ((A+BK)^T)^i (\mathcal{Q}^* \oplus K^T \mathcal{R}^*)$ so that each $\mathcal{P}_k^*$ is a proper C-set (since $\mathcal{Q}^* \oplus K^T \mathcal{R}^*$ is a proper C-set). A direct calculation verifies that the sets $\mathcal{P}_k^*$ satisfy (3.11). Thus, $\underline{V}_k(x) = \mathrm{h}(\mathcal{P}_k^*, x)$ and, by Theorem 2, $\underline{V}_k(x) = \mathrm{g}(\mathcal{P}_k, x)$ where $\mathcal{P}_k = (\mathcal{P}_k^*)^*$ is a proper C-set in $\mathbb{R}^n$.

By Assumption 2, $(A+BK)$ is strictly stable and so is $(A+BK)^T$; in addition, $\mathcal{Q}^* + K^T \mathcal{R}^*$ is a proper C-set in $\mathbb{R}^n$. As established in Raković (2007), there is a unique proper C-set $\mathcal{P}^*$ such that $\mathcal{P}^* = (A+BK)^T \mathcal{P}^* \oplus \left(\mathcal{Q}^* \oplus K^T \mathcal{R}^*\right)$. By (3.14), for all $x \in \mathbb{R}^n$, it holds that $\mathrm{h}(\mathcal{P}^*, x) = \mathrm{h}((A+BK)^T \mathcal{P}^* \oplus \left(\mathcal{Q}^* \oplus K^T \mathcal{R}^*\right), x)$. But,

$$\begin{aligned}&\mathrm{h}((A+BK)^T \mathcal{P}^* \oplus \left(\mathcal{Q}^* \oplus K^T \mathcal{R}^*\right), x)\\ &\quad = \mathrm{h}((A+BK)^T \mathcal{P}^*, x) + \mathrm{h}(\mathcal{Q}^*, x) + \mathrm{h}(K^T \mathcal{R}^*, x)\end{aligned}$$

and $\mathrm{h}((A+BK)^T \mathcal{P}^*, x) = \mathrm{h}(\mathcal{P}^*, (A+BK)x)$ and $\mathrm{h}(K^T \mathcal{R}^*, x) = \mathrm{h}(\mathcal{R}^*, Kx)$. Hence, for all $x \in \mathbb{R}^n$, it holds that $\mathrm{h}(\mathcal{P}^*, x) = \mathrm{h}(\mathcal{P}^*, (A+BK)x) + \mathrm{h}(\mathcal{Q}^*, x) + \mathrm{h}(\mathcal{R}^*, Kx)$. Let $\mathcal{P} = (\mathcal{P}^*)^*$ so that by Theorem 2, for all $x \in \mathbb{R}^n$, it holds that

$$\mathrm{g}(\mathcal{P}, x) = \mathrm{g}(\mathcal{P}, (A+BK)x) + \mathrm{g}(\mathcal{Q}, x) + \mathrm{g}(\mathcal{R}, Kx),$$

which completes our proof.

A.7. Proof of Proposition 4

The proof of the fact that for each $k \in \mathbb{N}$ the functions $\overline{V}_k(\cdot)$ are non-negative, finite, sublinear and continuous follows the ar-

guments used in the proof of Proposition 1 (see Appendix A.1) and it is omitted. Take any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^n$. Then:

$$\begin{aligned}\overline{V}_{k+1}(x) &= \underline{V}_{k+1}(x) + \lambda^{k+1}\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &= \underline{V}_k(x) + \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x) + \ell((A+BK)^k x)\\ &\quad + \lambda^{k+1}\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x) - \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &= \overline{V}_k(x) + \ell((A+BK)^k x) + \lambda^{k+1}\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\quad - \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x).\end{aligned}$$

But, by Proposition 2, $\ell((A+BK)^k x) \le \lambda^k(1-\lambda)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)$ and, hence,

$$\begin{aligned}\overline{V}_{k+1}(x) &\le \overline{V}_k(x) + \lambda^k(1-\lambda)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\quad + \lambda^{k+1}\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x) - \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &= \overline{V}_k(x).\end{aligned}$$

Hence, as claimed, for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$ it holds that $\overline{V}_{k+1}(x) \le \overline{V}_k(x)$. Furthermore,

$$\begin{aligned}&\overline{V}_k((A+BK)x) + \ell(x)\\ &\quad = \underline{V}_k((A+BK)x) + \ell(x) + \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, (A+BK)x)\\ &\quad = \underline{V}_{k+1}(x) + \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, (A+BK)x).\end{aligned}$$

By Lemma 1, $\mathrm{g}(\alpha^{-1}\mathcal{L}, (A+BK)x) \le \lambda\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)$ and, hence,

$$\begin{aligned}\overline{V}_k((A+BK)x) + \ell(x) &\le \underline{V}_{k+1}(x) + \lambda^{k+1}\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &= \overline{V}_{k+1}(x)\\ &\le \overline{V}_k(x).\end{aligned}$$

Thus, as claimed, for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$ it holds that $\overline{V}_k((A+BK)x) + \ell(x) \le \overline{V}_k(x)$.

A.8. Proof of Proposition 5

Take any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^n$. By Theorem 5, $\underline{V}_k(x) = \mathrm{g}(\mathcal{P}_k, x)$ so that $\overline{V}_k(x) = \mathrm{g}(\mathcal{P}_k, x) + \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)$. But, due to Lemma 2(i), it follows that $\overline{V}_k(x) = \mathrm{g}(\mathcal{P}_k, x) + \lambda^k\alpha\,\mathrm{g}(\mathcal{L}, x)$ and, by Theorem 2, $\overline{V}_k(x) = \mathrm{h}(\mathcal{P}_k^*, x) + \lambda^k\alpha\,\mathrm{h}(\mathcal{L}^*, x)$. Hence, since $\lambda^k\alpha\,\mathrm{h}(\mathcal{L}^*, x) = \mathrm{h}(\lambda^k\alpha\mathcal{L}^*, x)$ and the support function is additive in the first argument, we have

$$\overline{V}_k(x) = \mathrm{h}(\mathcal{P}_k^*, x) + \mathrm{h}(\lambda^k\alpha\mathcal{L}^*, x) = \mathrm{h}(\mathcal{P}_k^* \oplus \lambda^k\alpha\mathcal{L}^*, x).$$

Let $\mathcal{S}_k = (\mathcal{S}_k^*)^*$ where $\mathcal{S}_k^* = \mathcal{P}_k^* \oplus \lambda^k\alpha\mathcal{L}^*$ is as specified in (4.3). Then $\overline{V}_k(x) = \mathrm{h}(\mathcal{S}_k^*, x)$ and, by Theorem 2, $\overline{V}_k(x) = \mathrm{h}(\mathcal{S}_k^*, x) = \mathrm{g}(\mathcal{S}_k, x)$. Hence, it suffices to notice that the sets $\mathcal{S}_k^*$ specified by (4.3) are proper C-sets in $\mathbb{R}^n$ and so are their polar sets $\mathcal{S}_k = (\mathcal{S}_k^*)^*$.

A.9. Proof of Theorem 6

Take any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^n$. Let us first establish (4.5a). By construction, $\sum_{i=0}^{k-1} \ell((A+BK)^i x) \le \sum_{i=0}^{\infty} \ell((A+BK)^i x)$ and, due to Proposition 2,

$$\begin{aligned}&\sum_{i=0}^{\infty} \ell((A+BK)^i x)\\ &\quad = \sum_{i=0}^{k-1} \ell((A+BK)^i x) + \sum_{i=k}^{\infty} \ell((A+BK)^i x)\\ &\quad \le \sum_{i=0}^{k-1} \ell((A+BK)^i x) + \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x).\end{aligned}$$

Hence $\underline{V}_k(x) \le V_\infty(x) \le \overline{V}_k(x)$ as claimed. Next, notice that inequality (4.5b) is established in Proposition 4. Finally, let us establish (4.5c). By construction, we have $0 \le \underline{V}_k(x) \le \overline{V}_k(x) = \underline{V}_k(x) + \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)$ and $\overline{V}_k(x) - \underline{V}_k(x) = \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)$. Hence, as claimed,

$$0 \le \overline{V}_k(x) - \underline{V}_k(x) \le \lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x).$$

A.10. Proof of Corollary 1

The claimed facts follow from the inequalities (4.5a) and (4.5c) and the arguments used in the proof of Theorem 3 (see Appendix A.4).

A.11. Proof of Corollary 2

This result follows from Theorem 6 and the fact that $\gamma(\mathbb{X})$ is well defined, non-negative and finite for any given non-empty compact subset $\mathbb{X}$ in $\mathbb{R}^n$. (In turn, the set $N(\varepsilon, \mathbb{X})$ is non-empty and for all $k \in N(\varepsilon, \mathbb{X})$ and all $x \in \mathbb{X}$ it holds that $\lambda^k\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x) \le \lambda^k\gamma(\mathbb{X}) \le \varepsilon$; and, hence, for all $k \in N(\varepsilon, \mathbb{X})$ and all $x \in \mathbb{X}$ it holds that $|\overline{V}_k(x) - \underline{V}_k(x)| \le \varepsilon$.)

A.12. Proof of Proposition 6

The claims follow from (5.4) and the fact that $(1-\varphi)^{-1} = 1 + \varphi(1-\varphi)^{-1}$.

A.13. Proof of Proposition 7

Since the linear transformation of a proper C-polytopic set is a C-polytopic set, and since the Minkowski set addition of a proper C-polytopic set and a C-polytopic set is a proper C-polytopic set, the remaining claims follow by construction, Theorem 5 and Proposition 5.

A.14. Proof of Proposition 8

Fix any $\varphi \in (0, 1)$ and fix any two proper C-polytopes $(\mathcal{Q}_\varphi, \mathcal{R}_\varphi) \in \Pi(\mathcal{Q}, \mathcal{R}, \varphi)$. Take any $k \in \mathbb{N}$ and any $x \in \mathbb{R}^n$. Proposition 6 and relation (5.8) guarantee that relations (5.10a) and (5.10c) hold. We have $\ell(x) \le (1-\varphi)^{-1}\ell_\varphi(x)$ by Proposition 6. To verify the relations (5.10b) we show that $\overline{V}_{\varphi,k}((A+BK)x) + (1-\varphi)^{-1}\ell_\varphi(x) \le \overline{V}_{\varphi,k}(x)$. We have:

$$\begin{aligned}&\overline{V}_{\varphi,k}((A+BK)x) + (1-\varphi)^{-1}\ell_\varphi(x)\\ &\quad = \overline{V}_{\varphi,k}((A+BK)x) + \ell_\varphi(x) + \varphi(1-\varphi)^{-1}\ell_\varphi(x)\\ &\quad = \underline{V}_{\varphi,k}((A+BK)x) + \ell_\varphi(x) + \varphi(1-\varphi)^{-1}\ell_\varphi(x)\\ &\qquad + (\varphi(1-\varphi)^{-1} + \lambda^k)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, (A+BK)x)\\ &\quad \le \underline{V}_{\varphi,k}((A+BK)x) + \ell_\varphi(x)\\ &\qquad + (\lambda\varphi(1-\varphi)^{-1} + \lambda^{k+1})\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\qquad + \varphi(1-\varphi)^{-1}(1-\lambda)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\quad = \underline{V}_{\varphi,k}(x) + \ell_\varphi((A+BK)^k x)\\ &\qquad + (\varphi(1-\varphi)^{-1} + \lambda^{k+1})\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\quad \le \underline{V}_{\varphi,k}(x) + (\varphi(1-\varphi)^{-1} + \lambda^{k+1})\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\qquad + \lambda^k(1-\lambda)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\quad = \underline{V}_{\varphi,k}(x) + (\varphi(1-\varphi)^{-1} + \lambda^k)\,\mathrm{g}(\alpha^{-1}\mathcal{L}, x)\\ &\quad = \overline{V}_{\varphi,k}(x).\end{aligned}$$

Thus, as claimed, for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$ it holds that $\overline{V}_{\varphi,k}((A+BK)x) + \ell(x) \le \overline{V}_{\varphi,k}((A+BK)x) + (1-\varphi)^{-1}\ell_\varphi(x) \le \overline{V}_{\varphi,k}(x)$.

A.15. Proof of Proposition 9

For all $\varphi \in \Phi(\varepsilon, \mathbb{X})$, it holds that $\varphi(1-\varphi)^{-1}\gamma(\mathbb{X}) \leq 2^{-1}\varepsilon$ while, for all $k \in \mathbb{N}(\varepsilon, \mathbb{X})$, it holds that $\lambda^k\gamma(\mathbb{X}) \leq 2^{-1}\varepsilon$; consequently, for all $\varphi \in \Phi(\varepsilon, \mathbb{X})$ and all $k \in \mathbb{N}(\varepsilon, \mathbb{X})$, it holds that $(\varphi(1-\varphi)^{-1} + \lambda^k)\gamma(\mathbb{X}) \leq \varepsilon$ which together with Proposition 8 yields the claimed facts.

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