

# Anomalously slow phase transitions in self-gravitating systems

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## Anomalous slow phase transitions in self-gravitating systems

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The kinetics of collapse and explosion transitions in microcanonical self-gravitating ensembles is analyzed. A system of point particles interacting via an attractive soft Coulomb potential and confined to a spherical container is considered. We observed that for 100–200 particles collapse takes  $10^3$ – $10^4$  particle crossing times to complete; i.e., it is by two to three orders of magnitude slower than the velocity relaxation. In addition, it is found that the collapse time decreases rapidly with an increase of the soft-core radius. We found that such an anomalously long collapse time is caused by the slow energy exchange between a higher-temperature compact core and relatively cold diluted halo. The rate of energy exchange between the faster modes of the core particles and slower-moving particles of the halo is exponentially small in the ratio of the frequencies of these modes. As the soft-core radius increases and the typical core modes become slower, the ratio of core and halo frequencies decreases and the collapse accelerates. Implications for astrophysical systems and phase transition kinetics are discussed.

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### I. INTRODUCTION

Many groups of stellar systems have highly universal structures despite the apparent differences in their history and environment [1,2]. These universal structures are thought to have arisen as a result of relaxation towards equilibrium or to otherwise long-lived states. A comparison between the age of the stellar systems with universal features and corresponding collisional relaxation times reveals, however, that several types of stellar systems, such as elliptical galaxies, have not existed long enough to be collisionally relaxed [1]. Other collisional types of relaxation, such as “violent relaxation” or phase-space mixing caused by strong gravitational field fluctuations [3], have been suggested to explain this apparent contradiction between the time scales. Yet a full understanding of the kinetics of relaxation in naturally occurring self-gravitating systems is still lacking.

A number of fairly idealized models have been analyzed to understand the nature of equilibrium and transitory states of stellar systems. A well-studied example is an ensemble of self-gravitating particles with a sufficiently short-range small-distance regularization confined in a container. That system exhibits a gravitational phase transition between a relatively uniform high-energy state and a low-energy state with a core-halo structure [4–15]. During such a transition in a microcanonical ensemble the system undergoes a discontinuous jump from a state that just ceases to be a local entropy maximum to a global entropy maximum state with the same energy but different temperature. A transition from a high-energy uniform state to a lower-energy core-halo state is usually called collapse. The reverse transition during which the core disappears is often referred to as explosion.

It has been recently observed in molecular dynamics (MD) simulations [15] that a typical time scale for such gravitational transitions is paradoxically large, for a system of 125–250 particles being of the order of  $10^3$  relaxation times for a collapse and  $10^2$  relaxation times for an explo-

sion. The relaxation time  $t_r = R^{3/2} N^{1/2} / \ln N$  is the time scale of typical particle velocity thermalization which proceeds mostly via soft Coulomb collisions [1]. It was also observed in Ref. [15] that the density relaxation, such as the formation of the core, advances relatively fast, while the evolution of the kinetic energy or temperature proceeds noticeably slower [15].

In this paper we undertake a more detailed study of the kinetics of collapses in self-gravitating systems, in some way completing the investigation initiated in [15]. The structure of the paper is the following: After this introduction we briefly outline the simulation setup and present the results for collapse kinetics. A section analyzing a slow core-halo energy transfer as the bottleneck of system relaxation follows. Conclusions and discussion of the results completes the paper.

### II. SIMULATIONS

We consider systems consisting of  $N=125$ – $250$  identical particles of unit mass confined in a spherical container of radius  $R$  with reflecting walls. The Hamiltonian of the system reads as

$$H = \sum_{i=1}^N \frac{p_i^2}{2} - \sum_{i<j}^N \frac{1}{\sqrt{r_{ij}^2 + r_0^2}}, \quad (1)$$

where  $r_0$  is the soft-core radius. Along with the physical units, we use the standard rescaled units (as discussed, e.g., in Ref. [4]). For energy  $\epsilon$ , temperature  $\Theta$ , distance  $x$ , and time  $\tau$  they read as

$$\epsilon \equiv E \frac{R}{N^2},$$

$$\Theta \equiv T \frac{R}{N},$$

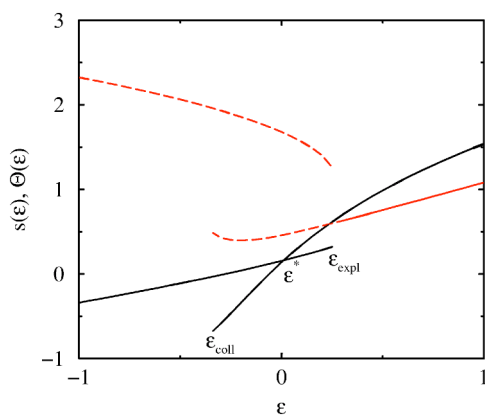


FIG. 1. Plots of entropy  $s(\epsilon)$  (solid line) and temperature  $\tau(\epsilon) = d\epsilon/ds$  (dashed line) vs energy  $\epsilon$  for a system with a gravitational phase transition and a soft-core radius  $x_0=0.005$ .

$$x \equiv \frac{r}{R},$$

$$\tau \equiv t \sqrt{\frac{N}{R^3}}. \quad (2)$$

Expressed in these rescaled units, the equilibrium properties of self-gravitating systems become universal. The velocity relaxation, assuming that it is caused mostly by soft collisions, is expected to be universal in terms of time  $\tau_{vel} = \tau \ln N / N$  [1], where the factor  $N / \ln N$  is proportional to the number of crossings a particle needs to change its velocity by a factor of 2.

The phase diagram of the system is presented in Fig. 1; see also Refs. [8,12,15]. High- and low-energy branches terminating at the energies  $\epsilon_{coll}$  and  $\epsilon_{expl}$  correspond to the uniform and core-halo states. The collapse and explosion energies are  $\epsilon_{coll} \approx -0.339$  and  $\epsilon_{expl} \approx 0.267$  for  $x_0 = 5 \times 10^{-3}$ .

Each MD run was initiated with a configuration in which the particles were seeded according to the corresponding  $\epsilon_{coll}$  or  $\epsilon_{expl}$  equilibrium (metastable) density profiles, and the velocities were assigned according to the Maxwell distribution. A more detailed description of the MD simulation procedure is presented in [15].

To reveal all facets of gravitational phase transitions in the most informative way, we consider the following parameters.

(i) Temperature  $\Theta$  as the indicator of the advancement of a phase transition as a whole. When  $\Theta$  reaches the target phase equilibrium value, all other system parameters come to equilibrium as well and the phase transition is complete.

(ii) Number of core particles,  $N_c$ , as the measure of density relaxation.

(iii) Temperature of the core,  $\Theta_c$ , which is proportional to the average kinetic energy of the core particles. Deviations of  $\Theta_c$  from  $\Theta$  quantify the temperature gradients occurring during a phase transition.

We do not list here any parameters which characterize the velocity relaxation: As follows from the definition of  $\tau_{vel}$  [1] and as observed in simulations [15], the velocity distributions in both the core and halo become thermalized within  $\tau_{vel} \sim 1$ .

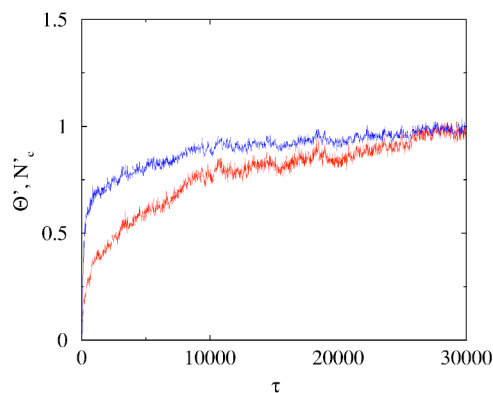


FIG. 2. Plots of the relative values of (from top to the bottom) the number of core particles,  $N'_c(\tau)$  (blue), and total temperature  $\Theta'(\tau)$  (red) vs  $\tau$  for a collapse in system with  $\epsilon=-0.5$ ,  $N=125$ , and  $x_0=0.005$ . The time evolution of the core temperature  $\Theta'_c(\tau)$  is practically indistinguishable from  $N'_c(\tau)$  and cannot be seen in the plot.

Results, averaged over four runs, for the temperature, core temperature, and the number of core particles for a collapsing system are presented in Fig. 2. The time evolution of these parameters is described in terms of the relative variables  $\Theta'(\tau)$ ,  $\Theta'_c(\tau)$ , and  $N'_c(\tau)$  which are defined as  $\Theta'(\tau) \equiv [\Theta(\tau) - \Theta(u)] / [\Theta(c-h) - \Theta(u)]$ . The values  $\Theta(u)$  and  $\Theta(c-h)$  correspond to the uniform and core-halo states in equilibrium.

As in Ref. [15], we observe that a collapse in a system with  $N=125-250$  particles and  $x_0=0.005$  takes about  $10^3$  velocity relaxation times to complete. It also follows from Fig. 2 that the growth of the core is significantly faster than the relaxation of the average kinetic energy: The core reaches half of its equilibrium size in only about 5 velocity relaxation times, while temperature relaxes to halfway in only 110 velocity relaxation times. In addition, we conclude that the core temperature is evolving synchronously with the number of core particles—i.e., noticeably faster than the total temperature of the system.

The results for the collapse in an otherwise identical system but with soft-core radius twice larger,  $x_0=0.01$ , are presented in Fig. 3. It follows from a comparison between Figs. 2 and 3 that while the initial stages of relaxation are not affected by the change of short-range potential, the overall collapse proceeds much faster for larger  $x_0$ .

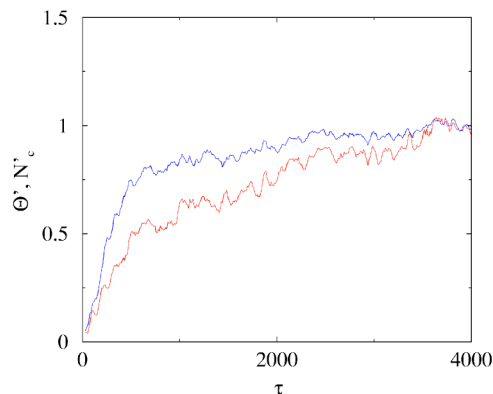


FIG. 3. Same as in Fig. 2 but for  $x_0=0.01$ .

The above numerical results suggest that after a rapid initial core growth, which takes a few velocity relaxation times, further evolution is hindered by some slow process that is essential for the completion of the phase transition. The kinetics of this slow process strongly depends on the short-range part of the interparticle potential. It follows from Figs. 2 and 3 that a collapse comes to its completion only when the temperatures of the core and halo become equal. Hence, it seems natural to assume that the bottleneck process is the energy, or heat, exchange between the core and the rest of the system. The rate of this heat exchange depends on the structure of the core which in turn is determined by the potential softening. In the next section we consider the heat exchange between the core and halo in more detail.

### III. CORE-HALO ENERGY EXCHANGE

To analyze the energy exchange between the core and halo, let us first examine the motion of a core particle. In a system of reference of the center of mass (c.m.) of the core we expand the potential energy terms in the Hamiltonian, Eq. (1), in powers of  $r_{ij}/r_0$  and arrive at the harmonic oscillator Hamiltonian

$$H \approx -\frac{N_c}{2r_0^3} + \frac{1}{2} \sum_{i=1}^{N_c} \left( p_i^2 + \frac{N_c}{r_0^3} r_i^2 \right). \quad (3)$$

Hence the motion of core particles relative to the core c.m. is characterized by harmonic oscillations with frequency

$$\omega_c = \sqrt{\frac{N_c}{r_0^3}}. \quad (4)$$

It is interesting to note that the frequency of the motion of a particle in a uniform self-gravitating sphere of  $N_c$  particles and radius  $r_0$  with the bare gravitational ( $-1/r$ ) interaction is also given by Eq. (4). Since the core radius is roughly equal to  $r_0$  [15], both the bare interaction and the “very soft” potential frequencies are essentially the same.

While the motion of the core particles is bound by gravity, the higher-energy halo particles can be viewed as free and being confined by the container walls only. Hence, the interaction between a core and a halo particle can be approximated as an interaction between a point mass on a rectilinear trajectory and a harmonic oscillator. The first relevant term in the multipole expansion of this interaction is the monopole-dipole term

$$V(t) = \sum_{i=1}^{N_c} \frac{x_i v t + z_i \rho}{[(vt)^2 + \rho^2]^{3/2}}. \quad (5)$$

Here  $x_i$  and  $z_i$  are the coordinates of core particles, and  $v$  and  $\rho$  are the velocity and impact parameter of the halo particle, respectively (Fig. 4). The monopole-monopole term, corresponding to the interaction between the halo particle and the core c.m., is irrelevant to the internal motion of the core particles.

With the introduced approximations the core-halo energy exchange becomes physically identical to the well-studied process of energy exchange between the vibrational degrees

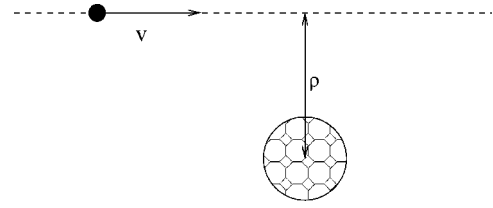


FIG. 4. Sketch of a core-halo particle scattering event.

of freedom of bound states and fast free particles in a plasma. Naturally, such molecular processes are usually considered in quantum mechanical terms. Following a standard textbook [16] we start with the first-order term of the interaction representation expansion of the perturbation of the oscillator wave function during a complete single collision,

$$C = -\frac{i}{\hbar} \int_{-\infty}^{+\infty} V(t) \exp(i\omega_c t) dt, \quad (6)$$

and arrive at the following expression for the probability  $|C|^2$  of the excitation from (or de-excitation to) the ground state of a harmonic oscillator:

$$|C|^2 = \frac{2\omega_c^2 N_c^2}{\hbar v^4} \left[ K_0^2 \left( \frac{\omega_c \rho}{v} \right) + K_1^2 \left( \frac{\omega_c \rho}{v} \right) \right]. \quad (7)$$

Here  $K_j$  are the McDonald functions (modified Bessel functions of the second kind), the factor  $N_c^2$  appears since the interaction potential in Eq. (7) is the sum of  $N_c$  identical terms, and the probability is quadratic in  $V(t)$ . Multiplying Eq. (7) by the transferred energy  $\hbar\omega_c$ , we get rid of the quantum constants and obtain an expression for the typical energy exchange during a collision between a core and a halo particle:

$$\begin{aligned} \delta E(\rho, v) &= \frac{2\omega_c^2 N_c^2}{v^4} \left[ K_0^2 \left( \frac{\omega_c \rho}{v} \right) + K_1^2 \left( \frac{\omega_c \rho}{v} \right) \right] \\ &\xrightarrow{\omega_c/v \rightarrow \infty} \frac{2\pi\omega_c N_c^2}{v^3 \rho} \exp\left(-\frac{2\omega_c \rho}{v}\right). \end{aligned} \quad (8)$$

The same expression can be obtained by a completely classical analysis considering the energy transfer during forced oscillation [17]. The last limit in Eq. (8) is taken since  $\rho/v$ , which is of order of a halo particle crossing time  $R/v$ , is much larger than a period of oscillation of the core particle  $1/\omega_c$ . It follows from Eq. (8) that the rate of energy exchange between the core and halo particles is exponentially small in the ratio of typical frequencies of their motion.

To obtain the rate of energy transfer between the core and all halo particles per unit time, one needs to average Eq. (8) over impact parameters and velocities:

$$\frac{\Delta E}{\Delta t} = \int_{r_0}^R 2\pi\rho d\rho \int v W_M(v) n \delta E(\rho, v) d^3v. \quad (9)$$

Here the lower limit of integration for the impact parameter is set equal to the core radius for the dipole approximation to be correct,  $W_M(v)$  is the Maxwell distribution, and  $n = 3N/(4\pi R^3)$  is the particle density. Using that  $r_0 \ll R$  and evaluating the velocity integral in the steepest descent ap-

proximation, the following expression is obtained:

$$\frac{\Delta E}{\Delta t} \approx \frac{\pi\sqrt{3}N_c^2N(2T\omega_c r_0)^{1/3}}{TR^3} \exp\left[-3\left(\frac{\omega^2 r_0^2}{2T}\right)^{1/3}\right]. \quad (10)$$

Assuming that the typical time of core-halo relaxation  $t_{c-h}$  is roughly equal to the ratio of the total transferred energy to the rate of the transfer and using Eq. (4), we arrive at the following expression for  $t_{c-h}$ :

$$t_{c-h} = \sqrt{\frac{R^3}{N}} \Delta\epsilon \frac{\Theta^{2/3}(x_0/n_c)^{1/6}}{2^{1/3}\sqrt{3}\pi n_c^2} \exp\left[3\left(\frac{n_c}{2x_0\Theta}\right)^{1/3}\right]. \quad (11)$$

Here  $\Delta\epsilon$  is the total transferred energy in rescaled units and  $n_c = N_c/N$ .

To evaluate the numerical value of  $t_{c-h}$  we consider the example from Fig. 2 with the following parameters:  $\epsilon = -0.5$ ,  $x_0 = 0.005$ ,  $n_c \approx 0.2$ , and  $\Theta \approx (\epsilon_{kin}^{c-h} + \epsilon_{kin}^u)/3 \approx 1.2$  (which is the average between the initial and final halo temperatures) [15]. Since the total energy during a microcanonical collapse is conserved, a change in kinetic energy of the system must be compensated for by a simultaneous change in the total potential energy which is roughly equal to the potential energy of the core. Given that most (80%) of the system particles are in the halo, the energy transfer from the core to the halo can be estimated as  $\Delta\epsilon \approx \epsilon_{kin}^{c-h} - \epsilon_{kin}^u \approx 2.6$ . With these parameters we obtain

$$\tau_{c-h} \equiv t_{c-h} \sqrt{\frac{N}{R^3}} \approx 12\,300 \quad (12)$$

for the relaxation time. Given the number of approximations used in obtaining Eq. (11), the agreement with the simulation result for the complete phase transition time  $\tau_{c-h}^{MD} \approx 27\,000$  (see Fig. 2) is surprisingly good. The agreement is even better for systems with the same energy  $\epsilon = -0.5$  but larger soft-core radius,  $x_0 = 0.01$ , where  $n_c \approx 0.22$ ,  $\Theta \approx 0.83$ , and  $\Delta\epsilon \approx 1.5$ . The theoretical estimate, Eq. (11), yields  $\tau_{c-h} \approx 3600$  which is only very little below the simulation result  $\tau_{c-h}^{MD} \approx 3800$ .

The most significant contributions to underestimating the value of  $\tau_{c-h}$  are the following.

(i) All collision were considered complete—i.e., the integral in Eq. (6) had infinite limits. This is certainly not true for a confined system especially for large impact parameters.

(ii) The energy exchange between the core oscillations and halo particles was always considered in one direction—i.e., from core to halo. In reality, this is only true for slow halo particles. Close to equilibrium, the exchange becomes progressively mutual with increasing number of fast halo particles losing their energy to the core vibrations.

Other factors not taken into account in our estimate, but possibly affecting the core-halo energy relaxation are quadrupole and higher-order terms in the potential expansion, overlapping collision, more complex dynamics than oscillation and rectilinear motion of the core and halo particles, higher than one-photon processes, or higher-order terms in the perturbation expansion to mention some. Yet we believe

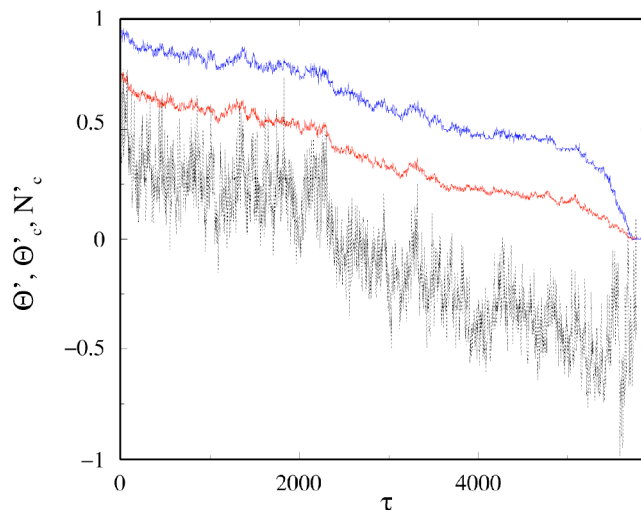


FIG. 5. Plots of the relative values of the (from top to the bottom) number of core particles,  $N'_c(\tau)$  (blue), total temperature  $\Theta'(\tau)$  (red), and core temperature  $\Theta'_c(\tau)$  (black) vs  $\tau$  for the explosion in system with  $\epsilon = 0.5$ ,  $N = 125$ , and  $x_0 = 0.005$ .

that our relatively simple approach presented above captures the essence of the core-halo energy relaxation and has semi-quantitative predictive power.

#### IV. DISCUSSION AND CONCLUSION

In the previous two sections we obtained the following results for the kinetics of collapse from the uniform to the core-halo state in self-gravitating systems.

(i) The molecular dynamics simulations revealed that in a system of 100–200 particles the collapse time is by two to three orders of magnitude longer than the velocity relaxation and strongly depends on the short-range part of the interaction potential.

(ii) It was found that the nonequilibrium feature with the slowest relaxation time is the temperature difference between the core and halo. In contrast, such parameters as the number of core particles and core kinetic energy evolve relatively fast.

(iii) A mechanism similar to the vibrational-translational relaxation in plasmas was suggested for the core-halo energy exchange. For this mechanism we show that the core-halo thermalization is exponentially slow in the ratio of typical frequencies of the motion of core and halo particles. Despite several rather strong approximations used in our analysis, a theoretical estimate for the relaxation time is in a good agreement (not more than by a factor of 2 off) with the simulation results.

So far nothing has been said about the reverse to collapse transition—i.e., explosion—illustrated in Fig. 5. It follows from this figure that similarly to the collapse, the kinetic energy is the slowest-evolving quantity while the number of core particles and core kinetic energy lead the explosion. Since the fastest particles leave the core first, the core is always colder than the rest of the system. At the end of explosion the core becomes just a single cold particle indistinguishable from any other particle of the system. This ex-



plains the sudden jump of the core temperature at the very end of explosion. Using Eq. (11) it is straightforward to explain why an explosion is faster than a collapse: Since the explosion starts at higher energy than collapse, the corresponding “explosion core-halo state” has noticeably fewer core particles (twice for the considered here case) than the final collapse state. This reduces the exponential term in Eq. (11). In addition, the total amount of energy which needs to be transferred between core and halo is smaller in the case of an explosion than in the case of collapse. It follows from Fig. 1 where the temperature jumps at collapse and explosion points are proportional to the energy exchanged between core and halo.

Are these results applicable to self-gravitating systems with other types of short-range regularization? For systems with continuous potentials at low energies the particle motions near the equilibrium position are harmonic oscillations with a frequency roughly given by Eq. (4) where  $r_0$  is of the order of the core radius. Examples include ensembles with Fourier-truncated Coulomb potentials [14], truncation of the expansion of the potential in spherical Bessel functions [18], and exchange interaction in systems with phase-space exclusion [6]. For potentials with a singular short-range part such as a hard-core repulsion, the motion of core particles is discontinuous, yet a typical inverse time scale of such motion is given by Eq. (4) as well. This follows from the fact that the expression, Eq. (4), can be obtained by dividing a typical core particle velocity  $\sqrt{N_c}/r_0$  by the core radius  $r_0$ . The analysis and conclusion made in Sec. III are based on a general principle that for a perturbation of a fast system by a slow one, the rate of energy transfer between these systems is exponentially small in the ratio of their frequencies. And since the motion of core particles is always faster than the motion of the halo ones, we conclude that the results obtained here are qualitatively applicable to all collapsing self-gravitating systems independent of the form of the short-range cutoff.

An important implication of the essential nonisothermicity of collapsing system demonstrated here is about the ap-

plicability of the Smoluchowski equation description to these systems [11]. Compared to the more complete Boltzmann equation with a Landau collision term (often called in case of the self-gravitating system Fokker-Plank-Vlasov equation [19]), the Smoluchowski equation offers a significant simplification: In the physically relevant case of spherically symmetric systems the solution depends only on one radial coordinate  $r$ . To make such a description more realistic, one needs to take into account the nonuniform temperature field. This can be achieved by coupling the Smoluchowski and heat conduction equations. The nonisothermicity of the evolution also indicates that the collapse in the canonical ensemble must be radically different from its microcanonical counterpart.

Finally a few words about the astrophysical relevance of the obtained results. The structure and mere existence of the equilibrium core is a consequence of the short-range regularization of the gravitational potential and the confining container which are the artifacts of the model. Hence true equilibrium core-halo states never occur in stellar systems. However, the observation that the collapse progress is hindered by the core-halo energy exchange which is exponentially slow in the ratio of the typical frequencies of motion of the core and halo particles remains applicable. Hence it is possible to interpret the states of the systems such as globular clusters whose age is noticeably larger than the corresponding velocity relaxation (and, therefore, initial core formation) time [1] as the transitory long-living states similar to core-halo thermalization states observed in this study.

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