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# CONNECTIVITY THRESHOLD FOR RANDOM SUBGRAPHS OF THE HAMMING GRAPH

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ABSTRACT. We study the connectivity of random subgraphs of the $d$-dimensional Hamming graph $H(d, n)$, which is the Cartesian product of $d$ complete graphs on $n$ vertices. We sample the random subgraph with an i.i.d. Bernoulli bond percolation on $H(d, n)$ with parameter $p$. We identify the window of the transition: when $np - \log n \to -\infty$ the probability that the graph is connected goes to 0, while when $np - \log n \to +\infty$ it converges to 1. We also investigate the connectivity probability inside the critical window, namely when $np - \log n \to t \in \mathbb{R}$. We find that the threshold does not depend on $d$, unlike the phase transition of the giant connected component the Hamming graph (see [1]).

## 1. INTRODUCTION

In this paper we investigate the random edge subgraph of $d-$dimensional Hamming graphs. Hamming graphs are defined as follows:

**Definition 1.1** (Hamming graph). *For integer $n$ write $[n] := \{1, \ldots, n\}$. We define the $d-$dimensional Hamming graph $H(d, n)$ as the graph with vertex set*

$$V = [n]^d, \tag{1.1}$$

*and edge set*

$$E = \{(v, w) : v, w \in V, \ v_j \neq w_j \text{ for exactly one } j\}. \tag{1.2}$$

Now we study a percolation model on the Hamming graph, defining the random subgraph $H_\lambda(d, n)$ as the random edge subgraph with uniform edge retention probability $p = \frac{\lambda}{d(n-1)}$. Since the degree of every vertex in $H(d, n)$ is $d(n - 1)$, the parameter $\lambda$ thus indicates the expected number of outgoing edges from any given vertex.

The phase transition for the existence of a giant component (i.e., when $|\mathcal{C}_{\max}| \approx \zeta|V|$ for $\zeta \in (0, 1)$) was studied in [1] for a larger class of finite transitive graphs that includes $H(d, n)$, while the slightly supercritical behavior was analyzed in [6] and [7] for $d = 2$.

In this work, we move away from the giant component critical point and we aim to determine the asymptotic probability that $H_\lambda(d, n)$ is connected for $d$ fixed and $n \to \infty$. The analogous problem was first solved for the *Erdős-Rényi Random Graph* (ERRG) in [3]. Observe that the ERRG arises as a special case of our problem if we put $d = 1$. We will follow the proof for the Erdős-Rényi Random Graph (see e.g. [5, Section 5.3]), but we find that at places the internal geometry of the Hamming graph plays an important role. To overcome this difficulty we use an induction on the dimension $d$ and an exploration of the graph.

## 2. Main Results

Let $H_n := H_\lambda(d, n)$ be a sequence of random edge subgraphs of $H(d, n)$ with parameter $\lambda = \lambda(n)$. Given $\lambda$ we want to determine the asymptotic probability that $H_n$ is connected.

**Theorem 2.1** (Connectivity threshold for $H_\lambda(d, n)$). *If $\lim_{n\to\infty} \lambda - d \log n = t \in \mathbb{R}$, then*

$$\mathbb{P}_\lambda(H_n \text{ is connected}) \to e^{-e^{-t}}. \tag{2.1}$$

*Consequently,*

$$\mathbb{P}_\lambda(H_n \text{ is connected}) \to \begin{cases} 0 & \text{if } \lambda - d \log n \to -\infty, \\ 1 & \text{if } \lambda - d \log n \to +\infty. \end{cases} \tag{2.2}$$

These results show an interesting difference between the critical values of the giant component threshold and the connectivity threshold. The critical probability of the former, $p_{GC} = \frac{1}{d(n-1)}(1 + o(1))$, depends on $d$, while the latter, $p_{\text{conn}} = \frac{\log n}{n-1}$, does not. This fact provides us with some insight into the structure of $H_\lambda(d, n)$ at the connectivity threshold: Consider the lower-dimensional "hyperplanes" (i.e., the subgraphs of $H(d, n)$ induced by all vertices $(v_1, \dots, v_d)$ that satisfy a set of constraints of the form $v_j = k_j$ for some $j \in [d]$, $k_j \in [n]$, see Definition 4.1 below). Note that these hyperplanes are isomorphic to Hamming graphs. From [1] we know that there exist values of $\lambda$ such that $H_\lambda(d, n)$ has a giant component while the intersections of $H_\lambda(d, n)$ with a hyperplane are subcritical (i.e., the largest components inside a hyperplane are of order $O(\log n)$). But an analogous property does not hold for the connectivity threshold: if $H_\lambda(d, n)$ is connected with probability converging to 1, then the same holds for all its hyperplanar subgraphs.

We believe this phenomenon holds in much greater generality than Hamming graphs: our proof of Theorem 2.1 can easily be modified to show that it also holds for the Cartesian product of $d$ copies of the complete $k$-partite graph, and we believe it to be true for a larger class of powers of high-degree transitive graphs.

2.1. **Related literature.** In [4] Erdős and Spencer studied the connectivity threshold of the *hypercube* $H(d, 2)$, where they found that the connectivity threshold occurs around $p = \frac{1}{2}$ (also independent of $d$). Clark [2] studied the connectivity threshold of $H(d, n)$ for $n$ fixed and $d \to \infty$, showing that if

$$p = 1 - \left(\frac{\xi(d)^{1/d}}{n}\right)^{\frac{1}{n-1}} \tag{2.3}$$

and $\xi(d) \xrightarrow{d\to\infty} a \in (0, \infty)$, then $\lim_{d\to\infty} \mathbb{P}_\lambda(H_\lambda(d, n) \text{ is connected}) = \mathrm{e}^{-a}$. Expansion of the above equation around $n = \infty$ shows that the $d \to \infty$ limit for large values of $n$ has the same behavior as the $n \to \infty$ limit. Moreover, [9] shows that more generally, Cartesian products of fixed graphs have a connectivity threshold that only depends on their degree distribution as $d \to \infty$.

Sivakoff gives a statement analogous to our main theorem for site percolation in [10]. It should be noted that site and edge percolation are very different models on the Hamming graph, as can be seen for instance in the fact that connectivity of site percolation on $K_n$ is trivial, whereas connectivity of edge percolation on $K_n$ (i.e., the ERRG) is not. See also [11].

## 3. Poisson convergence of isolated vertices

We start investigating the number of isolated vertices in the Hamming graph. As in the case of the ERRG, this provides a sharp lower bound on the window of the connectivity threshold. We define the number of isolated vertices

$$Y := \sum_{i \in V} \mathbb{1}_{\{|\mathcal{C}_i|=1\}},$$

where $|\mathcal{C}_i|$ is the number of vertices in the connected component of vertex $i$. We prove that in the critical window (i.e., when $\lambda - d \log n \to t \in \mathbb{R}$) the random variable $Y$ converges in distribution to a Poisson random variable. This proof is standard, and uses the same arguments applied to the proof given for the Erdős-Rényi Random Graph in [5, Section 5.3].

Let $(x)_n$ denote the $n$th *lower factorial* of $x$, i.e., $(x)_n := x(x-1)(x-2)\cdots(x-n+1)$. We will use the following lemmas, whose proofs are given in [5, Section 2.1] (for general versions see [8, Chapter 6]):

**Lemma 3.1.** *A sequence of integer-valued random variables $(X_n)_{n=1}^\infty$ converges in distribution to a Poisson random variable with parameter $k$ when, for all $r = 1, 2, \ldots$*

$$\lim_{n\to\infty} \mathbb{E}[(X_n)_r] = k^r. \tag{3.1}$$

**Lemma 3.2.** *When $X = \sum_{i \in \mathcal{I}} \mathbb{1}_i$ is a sum of at least $r$ indicators then*

$$\mathbb{E}[(X)_r] = \sum_{i_1 \neq i_2 \neq \dots \neq i_r} \mathbb{P}(\mathbb{1}_{i_1} = \mathbb{1}_{i_2} = \dots = \mathbb{1}_{i_r} = 1). \qquad (3.2)$$

Given $H_n = (V_n, E_n)$, we want to prove that (3.1) holds for $Y_n := \sum_{v_i \in V_n} \mathbb{1}_{\{|\mathcal{C}_{v_i}|=1\}}$. We will use Lemma 3.2 with an upper and lower bound on $\mathbb{P}_\lambda(\mathbb{1}_{i_1} = \dots = \mathbb{1}_{i_r} = 1)$ where we take $\mathbb{1}_i$ to be the indicator function of the event that the vertex $v_i$ is isolated. Observe that we have $n^d!/(n^d - r)!$ different sets of distinct vertices of cardinality $r$. We call $m := d(n-1)$ the degree of $H(d,n)$.

The lowest probability comes from sets where none of the $r$ vertices are adjacent, hence we bound

$$\mathbb{P}_\lambda(\mathbb{1}_{i_1} = \mathbb{1}_{i_2} = \dots = \mathbb{1}_{i_r} = 1) \geq \left(1 - \frac{\lambda}{m}\right)^{rm}, \qquad (3.3)$$

while the highest probability comes from sets where all the $r$ vertices belong to the same 1-dimensional subgraph, hence

$$\mathbb{P}_\lambda(\mathbb{1}_{i_1} = \mathbb{1}_{i_2} = \dots = \mathbb{1}_{i_r} = 1) \leq \left(1 - \frac{\lambda}{m}\right)^{rm - \frac{r(r-1)}{2}}. \qquad (3.4)$$

For $n \leq r$ we can find better bounds but we do not mind, since we are interested in the asymptotic behavior when $n \to \infty$ and $r$ is fixed. By the transitivity of the Hamming graph we bound, using $\lambda = d \log n + t(1 + o(1))$,

$$\begin{aligned} \mathbb{E}_\lambda[(Y_n)_r] &\geq \frac{n^d!}{(n^d - r)!}\left(1 - \frac{\lambda}{m}\right)^{rm} \\ &= \frac{n^d!}{(n^d - r)!}e^{-dr \log n - tr(1 + o(1))}. \end{aligned} \qquad (3.5)$$

Since $\dfrac{n^d!}{(n^d - r)!} = n^{dr}(1 - o(1))$, we find

$$\mathbb{E}_\lambda[(Y_n)_r] \geq n^{dr}e^{-dr \log n - tr}(1 - o(1)) = e^{-tr(1 + o(1))}. \qquad (3.6)$$

Similarly

$$\begin{aligned} \mathbb{E}_\lambda[(Y_n)_r] &\leq \frac{n^d!}{(n^d - r)!}\left(1 - \frac{\lambda}{m}\right)^{rm - \frac{r(r-1)}{2}} \\ &= \frac{n^d!}{(n^d - r)!}n^{-dr}e^{-tr(1 + o(1))}\left(1 - \frac{\lambda}{m}\right)^{-\frac{r(r-1)}{2}} \\ &= e^{-tr(1 + o(1))}. \end{aligned} \qquad (3.7)$$

This proves that for each $r$, $\mathbb{E}[(Y_n)_r] \to e^{-tr}$ so that by Lemma 3.1 the distribution of $Y_n$ converges to $\text{Poi}(e^{-t})$ when $\lambda - d \log n \to t$, so that

$$\mathbb{P}_\lambda(Y_n = 0) \to e^{-e^{-t}}. \tag{3.8}$$

Furthermore $\{H_n \text{ connected}\} \subseteq \{Y_n = 0\}$, so we conclude that for $\lambda - \log n \to t$

$$\limsup_{n \to \infty} \mathbb{P}_\lambda(H_\lambda(d, n) \text{ is connected}) \le e^{-e^{-t}}. \tag{3.9}$$

It remains to prove the matching lower bound, i.e., that in the critical window $\mathbb{P}_\lambda(H_\lambda(d, n) \text{ is disconnected } \mid Y = 0) \to 0$.

## 4. CONNECTIVITY CONDITIONED ON NO ISOLATED VERTICES

We prove (2.1) via induction on $d$ (the standard "tree counting" proof for the ERRG given in [5, Section 5.3] is too involved in the presence of geometry). We initialize the induction by noting that that $H(1, n)$ is a complete graph, so the random subgraph $H_\lambda(1, n)$ has the same distribution as an Erdős-Rényi random graph with $p = \frac{\lambda}{n-1}$. For this case it is proved in [3] that (2.1) holds.

The inductive hypothesis is that (2.1) holds for $H(d - 1, n)$.

**Definition 4.1** (Hyperplanes). *Given $H(d, n) = (V, E)$ define the* hyperplanes *$G_{jk} = (V_{jk}, E_{jk})$ for some $j \in [d]$, $k \in [n]$ as*

- *$V_{jk} = \{(i_1, i_2, ...i_d) \in V : i_j = k\}$ ;*
- *$E_{jk} = \{(v, w) \in E : v, w \in V\}$.*

*Note that $H(d, n)$ has exactly $dn$ hyperplanes and that they are all isomorphic to $H(d - 1, n)$.*

*We define $G_{jk}^\lambda$ as the intersection of the Random Edge Subgraph $H_\lambda(d, n)$ with the hyperplane $G_{jk}$, for each couple $j, k$.*

We now state a simple graph theoretic lemma that will be useful in our proof:

**Lemma 4.2.** *If $H_\lambda(d, n)$ has two hyperplanes $G_{jk}$ and $G_{j'k'}$ with $j \ne j'$ such that $G_{jk}^\lambda$ and $G_{j'k'}^\lambda$ are connected, then the union of* all *connected hyperplanes belong to the same component.*

*Proof.* We first observe that $G_{jk} \cap G_{j'k'} \ne \varnothing$.

Suppose we there exists a third connected subgraph $G_{il}^\lambda$ of a hyperplane $G_{il}$. We want to prove that if we choose $v \in G_{jk}$ and $w \in G_{il}$ then $v \leftrightarrow w$ in $H_\lambda(d, n)$.

If $j \ne i$ then $G_{jk} \cap G_{il} \ne \varnothing$, so we can choose $u \in G_{jk} \cap G_{il}$ and then $v \leftrightarrow u$ in $G_{jk}^\lambda$ and $u \leftrightarrow w$ in $G_{il}^\lambda$, and the claim follows.

If $j = i$ then $i \ne j'$ so that $G_{jk} \cap G_{j'k'} \ne \varnothing$ and $G_{j'k'} \cap G_{il} \ne \varnothing$. So we can choose $u \in G_{jk} \cap G_{j'k'}$ and $t \in G_{j'k'} \cap G_{il}$ and then $v \leftrightarrow u$ in $G_{jk}^\lambda$, $u \leftrightarrow t$ in $G_{j'k'}^\lambda$ and $t \leftrightarrow w$ in $G_{il}^\lambda$, and the claim follows. $\square$

We assume that for a fixed $\alpha \in (0, 1)$ in each possible direction $j$ at least $\alpha n$ of the $(d - 1)$-dimensional subgraphs $G_{jk}^\lambda$ are connected.

We will prove the inductive step conditioned on two events:

- $A := \{Y = 0\}$,
- $B := \{\forall j \in [d] \; \exists$ at least $\alpha n$ hyperplanes $G_{jk}$ s.t. $G_{jk}^{\lambda}$ is connected$\}$.

**Proposition 4.3.** *Let* $\lambda - d \log n \to t$, $|t| < \infty$, $d \geq 2$ *then*

$$\lim_{n \to \infty} \mathbb{P}_{\lambda}(H_{\lambda}(d, n) \text{ is disconnected} \mid A \cap B) = 0.$$

*Proof.* We notice that conditioning on $B$ the connected component that contains all the connected hyperplanes contains at least $n^d(1 - (1 - \alpha)^d)$ vertices, thus it is the almost surely unique giant component. We have to prove that with probability converging to 1 all other edges present in the graph are connected to the giant component as well. There are at most $\frac{d}{2}((1-\alpha)n)^{d+1}$ edges such that both their end vertices are not in a connected hyperplane. We call the number of edges that do not connect to the giant component $W$. If $W = 0$ the claim holds, since all points outside the giant component must be isolated and we condition on $A$. To estimate the probability that each edge is connected to the giant component we explore their connected component. We describe the exploration algorithm, starting from the two end vertices of a given edge:

(1) Pick an edge $(v_1, v_2)$ and set as active the two end vertices $v_1$, $v_2$.
(2) Check all the edges $(v_1, w)$, $(v_2, w)$ such that $w$ belongs to a connected hyperplane. If they are all vacant go on to Step 3, else terminate the algorithm.
(3) Check all the edges $(v_1, w)$, $(v_2, w)$ such that $w$ does not belong to a connected hyperplane. If there are at least two neighbors $w_1$, $w_2$ such that either $(v_1, w_i)$ or $(v_2, w_i)$ is occupied go to Step 4 (without considering other possible neighbors of $v_1$, $v_2$), else terminate the algorithm.
(4) Set $w_1$, $w_2$ as the active vertices $v_1$, $v_2$, and return to Step 2.

Activating only two neighbors at each cycle of the algorithm allows some control over the depletion of points outside the connected hyperplanes. This means that the algorithm can terminate before the starting edge has been connected to the giant component or its connected component has been completely explored, but the following calculations show that this algorithm gives a sufficiently sharp result to prove the claim.

We call $P_g = P_g(k)$ the probability that we find a connection to the giant component during the $k$-th cycle of the exploration, conditioned on the event $B$ and the fact that we have not terminated yet the algorithm. We bound

$$1 - P_g \leq \left(1 - \frac{\lambda}{m}\right)^{2\alpha m} \to e^{-2\lambda\alpha} \asymp n^{-2d\alpha}. \tag{4.1}$$

This bound does not depend on $k$, because we terminate the algorithm as soon as the exploration finds a connected hyperplane, so there is no depletion of points inside the connected hyperplanes.

Let $N_k$ denote the number of vertices discovered in Step 3 of the $k$-th cycle of the exploration and let

$$P_{k,2} := \mathbb{P}_\lambda(N_k \geq 2 \mid B) = 1 - \mathbb{P}_\lambda(N_k = 0 \mid B) - \mathbb{P}_\lambda(N_k = 1 \mid B).$$

On the event $B$, $N_k$ is stochastically dominated by $\mathrm{Bin}(2(1-\alpha)m - 2k, \lambda/m)$. We bound

$$\mathbb{P}_\lambda(N_k = 0 \mid B) \leq \left(1 - \frac{\lambda}{m}\right)^{2(1-\alpha)m - 2k}, \tag{4.2}$$

$$\mathbb{P}_\lambda(N_k = 1 \mid B) \leq (1-\alpha)m\frac{\lambda}{m}\left(1 - \frac{\lambda}{m}\right)^{2(1-\alpha)m - 2k - 1}. \tag{4.3}$$

So we obtain for some constant $C$

$$1 - P_{k,2} \leq \left(1 - \frac{\lambda}{m}\right)^{2(1-\alpha)m - 2k}\left(1 + (1-\alpha)\frac{\lambda}{1 - \frac{\lambda}{m}}\right)$$
$$\leq C\lambda n^{-2d(1-\alpha)}\left(1 - \frac{\lambda}{m}\right)^{-2k}. \tag{4.4}$$

Now we want to estimate the probability that the exploration process dies before hitting the giant component, namely that our algorithm terminates during Step 3. We call $T$ the cycle at which this happens, we set $T = \infty$ if the process hits the giant component, namely if the algorithm terminates during Step 2. For each cycle $s$ we have

$$\mathbb{P}_\lambda(T = s \mid B) \leq (1 - P_{s,2})\prod_{k \leq s}(1 - P_g) \leq C\lambda n^{-2d}\left(1 - \frac{\lambda}{m}\right)^{-2s}n^{-2d\alpha s}$$
$$= C\lambda n^{-2d}\left(\left(1 - \frac{\lambda}{m}\right)^2 n^{2d\alpha}\right)^{-s}, \tag{4.5}$$

because for $T = s$ we need that the exploration does not reach a connected hyperplane during the first $s$ cycles and then the algorithm terminates on Step 3 of the $s$-th cycle. So we can estimate

$$\mathbb{P}_\lambda(T < \infty \mid B) = O\left(\lambda n^{-2d}\sum_k\left(\left(1 - \frac{\lambda}{m}\right)^2 n^{2d\alpha}\right)^{-k}\right). \tag{4.6}$$

For sufficiently large $n$ we have $n^{2d\alpha}(1 - \frac{\lambda}{m})^2 > 1$, so the tail of the sum behaves like a convergent geometric series, so that

$$\mathbb{P}_\lambda(T < \infty \mid B) = O(n^{-2d}\log n) \ll \frac{2}{d}((1-\alpha)n)^{-(d+1)}$$

for all $d \geq 2$.

Since on the event $B$ there are at most $\frac{d}{2}((1-\alpha)n)^{d+1}$ edges that do not have an endvertex in a connected hyperplane

$$\mathbb{E}_\lambda[W \mid B] \leq \tfrac{d}{2}((1-\alpha)n)^{d+1}\mathbb{P}_\lambda(T < \infty) \to 0, \tag{4.7}$$

and the claim now follows.                                                                          $\square$

**Completion of the proof: induction on the dimension.** Recall that the case $d = 1$ initiates the induction, since $H_\lambda(1, n)$ is an Erdős-Rényi graph, so (2.1) holds.

For the inductive step we assume that (2.1) holds for $H_\lambda(d - 1, n)$, i.e., that for all $t \in \mathbb{R}$ and all sequences $\lambda = \lambda(n)$ such that $\lim_{n\to\infty} \lambda - (d-1)\log n = t$, we have

$$\mathbb{P}_\lambda(H_\lambda(d - 1, n) \text{ is connected}) = e^{-e^{-t}}.$$

We want to prove that the same holds for $H_\lambda(d, n)$.

Given $H_\lambda(d, n)$, its intersection $G_{jk}^\lambda$ with the hyperplane $G_{jk}$ has the same distribution as $H_{\frac{d-1}{d}\lambda}(d - 1, n)$ since $p = \frac{\lambda}{d(n-1)}$, and each vertex has $(d - 1)(n - 1)$ outgoing edges in $G_{jk}$. We assumed that $\lim_{n\to\infty} \lambda - d\log n = t$, which implies that

$$\lim_{n\to\infty} \frac{d-1}{d}\lambda - (d-1)\log n = \frac{d-1}{d}t.$$

Note moreover that $\mathbb{1}_{\{G_{jk}^\lambda \text{ is connected}\}}$ and $\mathbb{1}_{\{G_{jk'}^\lambda \text{ is connected}\}}$ are i.i.d. random variables when $k \neq k'$ under $\mathbb{P}_\lambda$, so for fixed $j$ all the subgraphs $G_{jk}^\lambda$ are i.i.d. random subgraphs with the same law as $H_{\frac{d-1}{d}\lambda}(d - 1, n)$.

It thus follows by the inductive hypothesis that the asymptotic probability that $G_{jk}^\lambda$ is connected is $e^{-e^{-(d-1)t/d}}$. We define the events

$$B_j := \{H_\lambda(d, n) \text{ contains a connected subgraph } G_{jk}^\lambda \text{ for more than } \alpha n \text{ different } k\},$$

for $j \in [d]$.

If we choose $\alpha = e^{-e^{-(d-1)t/d}} - \varepsilon$, for some $\varepsilon > 0$, for each $j$, we have by the induction hypothesis and the Weak Law of Large Numbers that

$$\begin{aligned}
\mathbb{P}_\lambda(B_j^c) =& \mathbb{P}_\lambda\left(\sum_{k=1}^n \mathbb{1}_{\{G_{jk}^\lambda \text{ is connected}\}} \leq \alpha n\right) \\
\leq& \mathbb{P}_\lambda\left(\left|\frac{1}{n}\sum_{k=1}^n \mathbb{1}_{\{G_{jk}^\lambda \text{ is connected}\}} - e^{-e^{-(d-1)t/d}}\right| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.8}$$

By the FKG-inequality, since $B_j$ is an increasing event for every $j$,

$$\mathbb{P}_\lambda(B) \geq \mathbb{P}_\lambda(B_j)^d \to 1. \tag{4.9}$$

Because

$$\begin{aligned}
\mathbb{P}_\lambda(A^c) \leq& \mathbb{P}_\lambda(H_\lambda(d, n) \text{ is disconnected}) \\
\leq& \mathbb{P}_\lambda(A^c) + \mathbb{P}_\lambda(B^c) + \mathbb{P}_\lambda(H_\lambda(d, n) \text{ is disconnected} \mid A \cap B),
\end{aligned} \tag{4.10}$$

using (3.8), (4.9) and Proposition 4.3 we can now obtain (2.1), completing the proof.  □

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