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PROEFSCHRIFT

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C O N T E N T S

*This is the name of the coauthor.

INTRODUCTION

Interest in nonlinear dissipative- and dispersive wave equations has been focused in the last twenty years especially on the Burgers respectively Korteweg-de Vries (KdV) equation. The first equation approximately describes the propagation of finite amplitude shock waves in a continuous medium and is given by

$$u_t + uu_x - u_{xx} = 0, \tag{1}$$

where subscripts denote partial differentiations.
Korteweg and de Vries first derived the KdV equation in their study of long water waves in a relatively shallow channel, cf. [1]. Recently, this equation has been derived in plasma physics and the theory of anharmonic lattices, cf. the references in [2]. It is given by

$$u_t + uu_x + u_{xxx} = 0. \tag{2}$$

The Cauchy problem for Burgers' equation is exactly solvable, cf. [3] and [4]. The KdV equation has been extensively investigated analytically as well as numerically, cf. [2] and [5].
(1) and (2) have been derived using a long-wavelength and small-nonlinearity approximation and assuming that the wave-phenomenon is approximately described by waves travelling in one direction only. The original equations, describing the physical situation under consideration, admit solutions travelling in both directions. The right- (towards positive x) and left moving waves interact due to both nonlinear and dissipative (or dispersive) terms in the equations. Moreover, the nonlinearity generates small wavelengths in the Fourier spectra of the solutions as well. Therefore, it seems reasonable to assume that the approximation, which lead to Burgers or KdV equation, only holds for some finite interval of time and it is useful to ask for the range of validity (in some sense) of that approximation.
This thesis mainly deals with that problem. It consists of six papers, listed in the contents as II - VII.
The general problem is quite complicated. Concerning the Burgers equation, we restrict ourselves to the class of equations

$$\alpha_t + [1 + \varepsilon(c\alpha + d\beta)]\alpha_x = \mu(\alpha_{xx} - \beta_{xx}), \tag{3}$$

$$\beta_t - [1 + \varepsilon(c\beta + d\alpha)]\beta_x = \mu(\beta_{xx} - \alpha_{xx}), \tag{4}$$

where c, d, μ and ε are constants chosen such that, if $\mu = 0$, the remaining set is hyperbolic. The positive parameters μ and ε are measures for the dissipation, respectively nonlinearity.

The derivation of the Burgers equation then goes in the following way. Let us consider the special initial conditions

$$\alpha(x,0) = f(x), \tag{5}$$

$$\beta(x,0) = \beta_0 \quad (\beta_0 \text{ constant}), \tag{6}$$

that is we start at $t = 0$ with a wave moving in one direction only.

When $\mu = 0$, equation (4) is satisfied identically by $\beta(x,t) = \beta_0$ and α is a simple wave solution, cf. Lax [6]. When $\mu \neq 0$, but small, the assumption is that, at least for some time, $\beta-\beta_0$ is small and α is approximately described by the solution α_0 of

$$\alpha_t + [1 + \varepsilon(c\alpha + d\beta_0)]\alpha_x = \mu\alpha_{xx}, \tag{7}$$

and (5).

By means of a transformation of scale, (7) may easily be reduced to Burgers' equation. This approximation method, which we shall call the simple wave (sw) approximation method, has been used a.o. by Lighthill [7]. We remark that, recently, Burgers' equation (and the KdV equation also) has been derived by means of the method of stretching coordinates (or singular perturbation as it is also called) from more general systems than (3) and (4), cf. [2] and [8]. When that method is applied to our equations (3) and (4), we also find (7).

In II, equations (3) and (4) where $c = d = 0$ are studied. This is the most simple case. Then the equations describe the longitudinal motion of an elastic bar with some viscous stress in Lagrangian coordinates. For square integrable solutions α and α_0, α_0 is called a good (useful) sw approximation of α in the interval of time $[t_1,t_2]$, if, for every $t \in [t_1,t_2]$:

$$\int_{-\infty}^{\infty} |\alpha - \alpha_0|^2 dx << \int_{-\infty}^{\infty} |\alpha|^2 dx. \tag{8}$$

$\int_{-\infty}^{\infty} |\alpha|^2 dx$ is a quite suitable norm for this problem as it may be seen as a measure for the energy of the α-mode.

It is shown in II that, if the spectral range of the initial value f(x), and thus of α, β and α_0 as it is a linear problem, vanishes identically outside a finite interval $[-\Delta,\Delta]$, (8) holds in the interval of time $[0,T]$, where $\Delta^3\mu^2T << 1$. This demonstrates that, the smaller the spectral range of the initial value f(x) is (long wavelength), the longer the approximation holds. As $t \to \infty$, we find (8) again. This is due to the fact that the dissipation for small wavelengths is much larger than for large ones and therefore, when we wait long enough, only the long wavelengths significantly contribute to the waveform.

In the following paper III, attention is paid to a set of nonlinear equations that, when the diffusion terms are linearized, agrees with (3) and (4) where c has been put equal to zero and d = 2. The set describes the same system as in II, but in Eulerian coordinates. The linear set considered there can be derived from the nonlinear set by means of a simple nonlinear transformation. Using this fact it is shown that, when ε is small enough, the results concerning the range of validity and the behaviour as $t \to \infty$ of the approximation, are quite similar to those we find for the linear equations. The definition of a good sw approximation is similar to that given in II. The linear- as well as the nonlinear set of equations that we treat in II, respectively III, do not admit shock waves for solutions. In the first case this is trivial. In the latter, it may be seen immediately from the fact that the characteristic velocity (put $\mu = 0$) of the α-mode depends on β and of the β-mode on α only. From a physical point of view, nonlinear equations admitting shock waves for solutions, are the most interesting, but undoubtedly the most difficult to study.

In IV a problem of this kind is considered. We have put c = 1 and $d = \Theta = \frac{\gamma-3}{\gamma+1}$, where $\gamma = C_p/C_v$. Then (3) and (4) form an approximation emanating from the Navier-Stokes equations. They are obtained by Lighthill [7] in his paper on viscosity effects in sound waves of finite amplitude and approximately describe the propagation of small amplitude sound waves in a real gas.

Now we shall speak about a good sw approximation in the interval of time $[0,T]$ if, for all $t \in [0,T]$:

$$\int_{-\infty}^{\infty} |\alpha - \alpha_0|^2 dx << \int_{-\infty}^{\infty} |f|^2 dx.$$

Evidently this is a much weaker condition than (8). However, for not too large times and a small dissipation (i.e. $\mu << 1$), $\int_{-\infty}^{\infty} |f|^2 dx$ will not deviate much from $\int_{-\infty}^{\infty} |\alpha|^2 dx$. The condition then is quite useful. Using this definition an upper bound for the range of validity is obtained in terms of f, μ, ε and Θ. This upper bound is "always" smaller than the time a shock wave starts to develop (assuming of course we start from smooth initial conditions). Partly, this is due to the method we followed to derive that bound. An indication is given in what direction future investigations might proceed to derive more accurate results.

Finally, in V, some numerical computations are reported. The results give a graphical impression of the development of the solutions α and β of the equations with $c = d = 0$. At the initial stage of our investigations, they have been used to gain some more insight in the subject studied.

Next we consider a sw approximation to a dispersive set of equations. We have restricted ourselves to the derivation of a linearized KdV equation from a representation of the linear chain problem intermediate between the exact continuum representation and the lowest continuum linear limit, i.e. a, the lattice constant, tends to zero. In VI, we have studied the problem of obtaining a unique continuum representation of the linear chain problem. This has been done by requiring the Fourier transforms of square integrable solutions of the continuum representation to vanish identically outside a finite interval $[-\pi a^{-1}, \pi a^{-1}]$.

The intermediate representation is then given by

$$\alpha_t + \alpha_x = -\mu(\alpha + \beta)_{xxx} \quad (\mu = a^2/24),$$

$$\beta_t - \beta_x = \mu(\alpha + \beta)_{xxx} .$$

The Cauchy problem for these equations is stable in the sense that a positive definite norm for the solutions exists which is, uniformly with respect to time, bounded in terms of the corresponding norm of the initial conditions. When no restriction is put to the spectral range of the solutions, this is not true. The dispersion equation

$$\omega^2 = k^2 - \frac{a^2 k^4}{12}$$

is not positive for all real k.

Starting at $t = 0$ from the initial conditions (5) and (6), the sw approximation of the α-mode is given by the solution α_0 of

$$\alpha_t + \alpha_x + \mu\alpha_{xxx} = 0,$$

and (5).

In VII, it turns out that, in the sense of (8), one can speak about a good sw approximation for $t \in [0,T]$, where $\Delta^5\mu^2T << 1$ (Δ defined as before). As $t \to \infty$, (8) is no longer satisfied. This is due to the oscillatory character of the solutions. The behaviour of α, β and α_0 as $t \to \infty$ is considered in detail. Finally, some remarks concerning the derivation of de KdV equation (2) from a nonlinear set are made.

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*For shortness the full title of the paper is omitted.

ON A SIMPLE WAVE APPROXIMATION

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ABSTRACT.

An approximation (the linear version of Burgers' equation with appropriate initial data) to a simple wave initial value problem for a set of two linear coupled dissipative partial differential equations is discussed. It has been shown that for the class of square integrable initial functions of which the spectra (Fourier transforms) have bounded support 2Δ the approximation is valid for some finite interval of time $[0,T(\Delta)]$. For some finite time $T_1 > T(\Delta)$ the approximation may fail. However, for $t \to \infty$,it is asymptotically valid again. For the class of initial conditions mentioned above expansions in series of the two solutions, which for every finite interval of time $[0,\tau]$ are convergent, may be constructed.

1. INTRODUCTION.

1.1. Statement of the problem.

In physics one occasionally deals with the following simple wave initial value problem

$$\alpha(s,0) = f(s), \tag{1}$$

$$\beta(s,0) = 0 \quad , \tag{2}$$

for the set of nonlinear dissipative partial differential equations

$$\alpha_t + [1+\varepsilon\Phi(\alpha,\beta)]\, \alpha_s = \mu(\alpha_{ss} - \beta_{ss}) \quad , \tag{3}$$

$$\beta_t - [1+\varepsilon\Psi(\alpha,\beta)]\, \beta_s = \mu(\beta_{ss} - \alpha_{ss}) \quad , \tag{4}$$

where s runs through the interval $(-\infty,\infty)$, t through $[0,\infty)$, $\Phi(\alpha,\beta)$ and $\Psi(\alpha,\beta)$ are continuous, often even monotonic functions of α and β, μ and ε are real positive constants and the subscripts s,t denote partial differentiation with respect to s, respectively t. Moreover Φ, Ψ and ε have been chosen such that if $\mu=0$ the remaining set is hyperbolic. A well-known example is found in Lighthill's theory of waves in a real gas (Lighthill [1]).

An exact and complete solution of this initial value problem is at present beyond all possibilities. Therefore Lighthill used an approximation. When $\mu=0$ it is seen that (4) is satisfied identically. (3) then becomes a first order equation in α, which is readily solved. The resultant solution is a simple wave solution (cf. Lax [2]) for the hyperbolic set obtained by putting $\mu=0$. This explains the name we gave to the initial value problem. Lighthill's approximation is based on the assumption that, when μ is small, β will be negligible, at any rate for some finite interval of time. In this way one obtains from (3):

$$\alpha_t + [1+\varepsilon\Phi(\alpha,0)]\alpha_s = \mu\alpha_{ss} \quad , \tag{5}$$

which is an equation of Burgers type. In Lighthill's example Φ is linear in α. The exact solution of the initial value problem is known in that case.

The approximation of the solution α of (1),....,(4) by the solution α_0 of (1) and (5) henceforth will be called the simple wave approximation.

Now some questions that arise are:

1. May, for some finite interval $[0,T]$ of time and some f(x), the simple wave approximation be used indeed? If this is true, what can be said about the dependence of T on the initial data? May T tend to infinity?
2. Frequently in such problems one attempts an expansion in series of α where α_0 is the first term in the expansion. Does such an expansion really exist for some finite interval of time $[0,T]$ and some f(x) and if so, does T depend on f(x) and may T tend to infinity?

In general these questions would present rather formidable difficulties. Therefore we make a simplification by studying the linear system

$$\alpha_t + \alpha_s = \mu(\alpha_{ss} - \beta_{ss}) \quad , \tag{6}$$

$$\beta_t - \beta_s = \mu(\beta_{ss} - \alpha_{ss}) \quad , \tag{7}$$

subject to (1) and (2).

The Burgers approximation equation is given by

$$\alpha_t + \alpha_s - \mu\alpha_{ss} = 0. \tag{8}$$

The solution of (8) subject to (1) will be called α_0 again.
We still did not speak about what precisely we mean with a useful approximation. For solutions which are square integrable (we restrict ourselves to these solutions) we shall call the solution α_0 a useful approximation to α in the interval of time $[t_1,t_2]$ $(t_2>t_1)$ if for every $t \in [t_1,t_2]$

$$\int_{-\infty}^{\infty}|\alpha - \alpha_0|^2\, ds \ll \int_{-\infty}^{\infty}|\alpha|^2\, ds.$$

$\int_{-\infty}^{\infty}|\alpha|^2 ds$ often has the meaning of the energy of the α-mode. It then provides a quite suitable norm for such a problem.
In a forthcoming paper we will treat a physical problem which leads to a special form of (3) and (4). In this case the equations can be transformed into linear equations of the form (6) and (7). Therefore the following considerations have at least some physical meaning.

2. DEFINITIONS AND NOTATIONS.

R: the interval $(-\infty,\infty)$ of the real numbers.

Q: a strip in the s-t plane containing all the points satisfying the inequalities $-\infty < s < \infty$ and $0 < t < T < \infty$.

Consider vector valued functions of n complex-valued components $u = \text{col.}(u_1(s,t),\ldots,u_n(s,t))$ defined on R (t fixed) and Q respectively.
$L_2(R)$ is a Hilbert-space containing all square integrable n component vector valued functions on R, with inner products (,) and norms $||\quad||$ defined by

$$(u,v) = \int_{-\infty}^{\infty} u^{\dagger}(s)\, v(s)\, ds \; ; \; ||u|| = (u,u)^{\frac{1}{2}},$$

$u^{\dagger}$ being the hermitian transpose of u.

The Sobolev-space $W_2^m(R)$ (m a positive natural number) is a Hilbert-space containing all vector valued $L_2(R)$ functions u(s) whose generalized derivatives $D^k u$, (k=1,2,...,m) also are elements of $L_2(R)$ (Smirnow [3]). The inner product and norm are respectively

$$(u,v)_m = \sum_{i=1}^{m} (D^i u, D^i v) + (u,v); \quad ||u||_m = (u,u)_m^{\frac{1}{2}} .$$

C(R) is the set of all continuous-, $C^i(R)$ the set of all i times continuously differentiable functions on R.
$L_2^{\Delta}(R)$ is a Hilbert-space containing all vector valued functions u(s) in $L_2(R)$, of which the Fourier transform $\bar{u}(k)$, defined by

$$\bar{u}(k) = \int_{-\infty}^{\infty} u(s) \exp(-iks).ds, \tag{1}$$

vanishes identically outside a finite interval $[-\Delta,\Delta]$ ($\Delta \in R$), with inner products $(\ ,\ )_{R,\Delta}$ and norms $||\ \ ||_{R,\Delta}$ defined by:

$$(u,v)_{R,\Delta} = \int_{-\infty}^{\infty} u^+(s)\ v(s)\ ds; \quad ||u||_{R,\Delta} = (u,u)_{R,\Delta}^{\frac{1}{2}}.$$

Using Parseval's theorem one easily finds

$$||u||_{R,\Delta}^2 = \frac{1}{2\pi} \int_{-\Delta}^{\Delta} \bar{u}^+(k)\ \bar{u}(k)\ dk .$$

$L_{2,q}^{\Delta}(Q)$ is a Hilbert space containing all vector valued square integrable functions on Q, with the properties: The Fourier transforms defined similar to (1), for almost every $t \in [0,T]$ vanish identically outside a finite interval $[-\Delta,\Delta]$, $\Delta \in R$. The inner products $(\ ,\ )_{Q,q,\Delta}$ and norms $||\ \ ||_{Q,q,\Delta}$ are defined by

$$(u,v)_{Q,q,\Delta} = \frac{1}{2\pi} \int_0^T \int_{-\Delta}^{\Delta} q^2(k,t)\ \bar{u}^+(k,t)\ \bar{v}(k,t)\ dkdt; \quad ||u||_{Q,q,\Delta}^2 = (u,u),$$

where q is a positive continuous function of k and t, which for every $k \in [-\Delta,\Delta]$, $t \in [0,T)$ is bounded from above and from below.
If q = 1, we simply write $L_2^{\Delta}(Q)$, $||\ \ ||_{Q,\Delta}$ and $(\ ,\ )_{Q,\Delta}$.

Finally we quote (for the proof see Smirnow [3], p.486):

LEMMA 1:

Let $u(s) \in W_2^m(R)$ then $D^p u(s) \to 0$ $(1 \leq p < m)$ and $u(s) \to 0$ when $|s| \to \infty$.

Remark: Where not stated otherwise all integrations are in the sense of Lebesque and all differentiations are meant in the generalized sense, although the classical notation will be retained.

3. THE SOLUTION OF THE INITIAL VALUE PROBLEM.

3.1 Existence and uniqueness.

Consider, for vector valued functions of two components $u(s,t)$ defined on Q, the operator equation

$$u_t = \Lambda u, \tag{1}$$

where

$$\Lambda = D \frac{\partial}{\partial s} + A \frac{\partial^2}{\partial s^2} ,$$

$$D = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} , \quad A = \mu \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} ,$$

$$u = \text{col.}(\alpha, \beta).$$

As Λ satisfies: (i) Λ is closed (ii) for every $u \in D_\Lambda$ $\text{Re}(u,\Lambda u) \leq \beta_1(u,u)$ (iii) for every $v \in D_{\Lambda^*}$ $\text{Re}(v,\Lambda^* v) \leq \beta_2(v,v)$ (iv) $D_\Lambda = W_2^2(R)$ is dense in $L_2(R)$, where D_Λ is the domain of Λ, Λ^* the adjoint operator and β_1, β_2 are real positive constants, it can be proved that (de Graaf [4]):

THEOREM 1.

1. The operator equation $u_t = \Lambda u$ is uniquely solvable for every $u(s,0) \in W_2^n(R)$, $n \geq 0$ $(W_2^o(R) \stackrel{\text{def}}{=} L_2(R))$ and for every $0 < t < T < \infty$ the solution is an element of $W_2^n(R)$.
2. $u(s,t) \to u(s,0)$ for $t \to 0$ in the sense of the L_2-norm.

3. For an arbitrary initial condition $u(s,0) \in W_2^n(R)$ $(n \geq 0)$, $u(s,t)$ may be represented by

$$u(s,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp - (Ak^2 + ikD)t.\ \bar{f}(k) \exp (iks).dk, \qquad (2)$$

where $\bar{f}(k)$ is the Fourier transform of the initial value $u(s,0)$ and $i = \sqrt{-1}$.

4. The results of (1), (2) and (3) are true for $L_2^{\Delta}(R)$ instead of $L_2(R)$ as well.

For purely parabolic equations $u(s,0) \in L_2(R)$ implies $u(s,t) \in W_2^n(R)$. The linear Burgers equation, to which the existence theorem applies in the same manner, belongs to this class of equations. The coupled system (1) however is not purely parabolic as A has an eigenvalue zero. This constitutes an essential difference between the coupled system and the linear Burgers equation satisfied by α.

3.2 Stability and a maximum-modulus principle.

THEOREM II.

Let $u(s,0) = f(s) \in L_2(R)$. The solution of the system (1) is stable in the sense that for every $t \geq 0$

$$||u(t)|| \leq K||f||,$$

where K is a positive constant.

Proof:

Let $f(s) \in L_2^{\Delta}(R)$. Premultiply $u_t - Du_s - Au_{ss} = 0$ with u^+, take the complex conjugate of the resulting equation and add. We find

$$\frac{\partial}{\partial t} (u^+u) + \frac{\partial}{\partial s} (-u^+Du - u^+Au_s - u_s{}^+Au) + 2u_s^+ Au_s = 0. \qquad (3)$$

This is essentially the energy balance equation. If we integrate (3) along the entire s-axis and use lemma 1, we get

$$||u(t)|| \leq ||f|| \qquad (t \geq 0) . \qquad (4)$$

The remaining part of the proof depends on closure. As $L_2^{\Delta}(R)$ is dense everywhere in $L_2(R)$ it is possible to find a sequence $\{f_n\} \subset L_2^{\Delta}(R)$ which converges to $f \in L_2(R)$ in the sense of the L_2-norm. Then the solutions $u_n(s,t)$ corresponding to $f_n(s)$ also converge in that norm and according to (4) $\lim_{n\to\infty} u_n(s,t) = u(s,t)$, $u(s,0) = f(s)$, $u(s,t)$ is a solution and $||u(t)|| \leq ||f||$. This proves the theorem.

From a physical point of view it often is desirable or even necessary to have a maximum-modulus principle. It is given by:

THEOREM III.

Let $u(s,0) = f(s) \in W_2^1(R)$ and $||f||_1 \leq \delta\sqrt{2}$ (δ is some real positive constant), then we may define a function $\tilde{u}(s,t) \in C(R)$ such that for $t \geq 0$

$$\tilde{u}(s,t) = u(s,t) \qquad \text{a.e.},$$

$$\sup_{s\in R} |\tilde{u}| \leq \delta \quad ,$$

where $|\tilde{u}| = |\tilde{\alpha}| + |\tilde{\beta}|$.

Proof:

Let $u(s,0) = f(s) \in L_2^{\Delta}(R)$ and $||f||_1 \leq \delta\sqrt{2}$. If u is a solution, u_s is too. In this way we find the balance-equation (3) where u has been replaced by u_s. Adding (3) and the new equation, integrating the result along the entire s-axis and using lemma 1, we obtain

$$||u(t)||_1 \leq ||f||_1 = \delta\sqrt{2} \qquad (t \geq 0). \tag{5}$$

By means of closure we may show that (5) also holds for $f(s) \in W_2^1(R)$. From Sobolev's embedding theorem (Peletier [5]) we deduce the existence of a positive number M and a function $\tilde{u}(s,t) \in C(R)$ such that for $t \geq 0$

$$\tilde{u}(s,t) = u(s,t) \qquad \text{a.e.},$$

$$\sup_{s\in R} |\tilde{u}(t)| \leq M\, ||u(t)||_1 .$$

According to the before mentioned paper [5] the lowest value of M that may be chosen equals $\frac{1}{2}\sqrt{2}$, which completes the proof of the theorem.

3.3 The solution of the simple wave initial value problem.

The solution (2) may be written as

$$\alpha(s,t) = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty}1 + \int_{-\infty}^{\infty}2\right] \; g^{(2)}(z) \exp h(z,\xi)t.dz \,, \tag{6}$$

$$\beta(s,t) = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty}1 + \int_{-\infty}^{\infty}2\right] \; g^{(1)}(z) \exp h(z,\xi)t.dz \,, \tag{7}$$

where

$$\mu k = z \,, \quad \xi = \frac{s}{t} \,,$$

$$g^{(1)}(z) = (2i\mu)^{-1} \, z(1-z^2)^{-\frac{1}{2}} \; \bar{f}(\mu^{-1}z) \,,$$

$$g^{(2)}(z) = (2\mu)^{-1} \left[-1 + (1-z^2)^{\frac{1}{2}}\right](1-z^2)^{-\frac{1}{2}} \quad \bar{f}(\mu^{-1}z),$$

$$h(z,\xi) = \mu^{-1}\left[iz(1-z^2)^{\frac{1}{2}} - z^2 + iz\xi\right].$$

The number 1 respectively 2 through the integration symbol means integration in the first-, respectively second sheet of the complex z-plane. The first sheet is defined by

$$\lim_{|z|\to\infty} \frac{(1-z^2)^{\frac{1}{2}}}{z} = -i \qquad (0 \le \arg z \le \pi).$$

and the second by

$$\lim_{|z|\to\infty} \frac{(1-z^2)^{\frac{1}{2}}}{z} = i \qquad (0 \le \arg z \le \pi).$$

This corresponds to cutting the z-plane from $-\infty$ to -1 and from 1 to ∞.
In the remaining part of this paper we shall confine ourselves almost always to initial data in $\overset{\Delta}{L_2}(R)$, although most of the results also apply to other classes of functions.

4. A SERIES EXPANSION OF THE SOLUTION.

Let f(s) belong to $L_2^{\Delta}(R)$. Defining the operators M and N by

$$M = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} - \mu\frac{\partial^2}{\partial s^2} ,$$

$$N = \frac{\partial}{\partial t} - \frac{\partial}{\partial s} - \mu\frac{\partial^2}{\partial s^2} ,$$

we find by applying the first one to (1.6) and the second to (1.7)

$$L\alpha = \mu^2 \frac{\partial^4 \alpha}{\partial s^4} , \tag{1}$$

$$L\beta = \mu^2 \frac{\partial^4 \beta}{\partial s^4} , \tag{2}$$

where

$$L = MN = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} - 2\mu\frac{\partial^3}{\partial s^2 \partial t} + \mu^2 \frac{\partial^4}{\partial s^4} .$$

The initial conditions for this system are given by

$$\alpha(s,0) = f(s) , \tag{3}$$

$$\alpha_t(s,0) = - \frac{df}{ds} + \mu \frac{d^2 f}{ds^2} , \tag{4}$$

$$\beta(s,0) = 0, \tag{5}$$

$$\beta_t(s,0) = - \mu\frac{d^2 f}{ds^2} . \tag{6}$$

As we assumed α and β to be in $L_2^{\Delta}(R)$, all the operations were allowed indeed. By now it may be seen that (1),...,(6) are equivalent to (1.6) and (1.7) subject to (1.1) and (1.2). Consider, in $L_2^{\Delta}(Q)$, the integral equations

$$\alpha = B\alpha + \alpha_0 , \tag{7}$$

$$\beta = B\beta + \beta_0 ,$$

which hold for every $t \in [0,T]$ and almost every $s \in R$ and where

$$B\alpha = \frac{\mu^2}{2\pi} \int_{-\Delta}^{\Delta} dk \int_0^t d\tau\ k^3 \sin k(t-\tau)\ e^{-\mu k^2(t-\tau)-iks}\ \bar{\alpha}(k,\tau)\ ,$$

$$\alpha_0 = \frac{1}{2\pi} \int_{-\Delta}^{\Delta} \bar{f}(k)\ e^{ik(s-t)\ -\ \mu k^2 t}\ dk\ , \qquad (8)$$

$$\beta_0 = \frac{\mu}{2\pi} \int_{-\Delta}^{\Delta} k\bar{f}(k)\ \sin kt\ e^{iks-\mu k^2 t}\ dk\ .$$

Using these integral equations an expansion in series of the solution of our original system (1.6) and (1.7) will be derived. Of course other integral equations could have been used. However we choose the present ones as α_0 satisfies Burgers' equation and the initial value $\alpha(s,0) = f(s)$ exactly.

THEOREM IV.

For every finite Δ and every finite positive number T, (7) has a solution which for every $t \in [0,T]$ belongs to $L_2^{\Delta}(R)$. It is the limit of the sequence $\sum_{n=o}^{N} \alpha^{(n)}$ for $N \to \infty$, where

$$\alpha^{(0)} = \alpha_0\ ,$$

$$\alpha^{(n+1)} = B\alpha^{(n)} \qquad (n = 0,1,2,\dots.)\ .$$

Proof:

From

$$|\sum_{n=1}^{\infty} \bar{\alpha}^{(n)}\ (k,t)| =$$

$$= |\sum_{n=1}^{\infty} \int_0^t d\tau_1 \dots \int_0^{\tau_{n-1}} d\tau_n\ \mu^{2n} k^{3n} \prod_{j=1}^{n} \sin k(\tau_{j-1}-\tau_j) . e^{-\mu k^2 t - ik\tau_n}\ \bar{f}(k)|$$

$$\leq e^{-\mu k^2 t}\ |\bar{f}(k)|\ \sum_{n=1}^{\infty} \frac{t^n \mu^{2n} |k|^{3n}}{n!} = (e^{t\mu^2|k|^3} -1)\ e^{-\mu k^2 t}\ |\bar{f}(k)|, \qquad (9)$$

where $\qquad \tau_0 = t,$

we immediately deduce

$$|| \sum_{n=1}^{\infty} \alpha^{(n)} \ (t) \ ||_{R,\Delta} \le (e^{t\mu^2\Delta^3}-1) \ . \ || \ \alpha^{(0)} \ (t) \ ||_{R,\Delta} \tag{10}$$

which implies that for all $t \in [0,T]$ $\sum_{n=0}^{N} \alpha^{(n)}$ (t) has a limit for $N \to \infty$ in the sense of the $L_2^{\Delta}(R)$ norm. The limit will be called α. Furthermore we have:

$$|| \ B(\sum_{n=0}^{N} \alpha^{(n)} - \alpha) \ ||_{Q,\Delta} \le \mu^2\Delta^3 T \ \ || \sum_{n=0}^{N} \alpha^{(n)} - \alpha \ ||_{Q,\Delta} \quad (t \in [0,T]).$$

For every $t \in [0,T]$ and almost every s

$$\sum_{n=0}^{N} \alpha^{(n)} = B \sum_{n=0}^{N-1} \alpha^{(n)} + \alpha_0 \quad ,$$

which implies, using $\lim_{N\to\infty} ||\alpha - \sum_{n=0}^{N} \alpha^{(n)} \ ||_{Q,\Delta} = 0$, $\lim_{N\to\infty} ||B(\alpha - \sum_{n=0}^{N} \alpha^{(n)} \ )||_{Q,\Delta} = 0$:

$$|| \ \alpha - B\alpha - \alpha_0 \ ||_{Q,\Delta} = 0 \ .$$

However as, according to (9), α is a continuous function of t, we obtain that for every $t \in [0,T]$ and almost every $s \in R$ α satisfies (7).

THEOREM V.

The solution α(s,t), found in theorem IV, for every $t \in [0,T]$ is unique in the sense of the L_2^{Δ} -norm and

$$|| \ \alpha - \alpha_0 \ ||_{R,\Delta} \le (e^{T\mu^2\Delta^3} - 1) \ ||\alpha||_{R,\Delta} \qquad (t \in [0,T]).$$

Proof:

Let α' be another solution belonging to $L_2^{\Delta}(Q)$ as well.
Introduce the function q(k,t) by

$$q^2(k,t) = \exp - \{\mu^2 k^2 \ (e^{2\mu k^2 t} - 1)\}.$$

Call the difference $\alpha - \alpha' = \hat{\alpha}$. Using Schwarz's inequality we find:

$$||B\hat{\alpha}||^2_{Q,\Delta,q} =$$

$$= \frac{\mu^4}{2\pi} \int_0^T dt \int_{-\Delta}^{\Delta} dk\ k^6 q^2(k,t)\ \left|\int_0^t \sin k(t-\tau).\, e^{-\mu k^2(t-\tau)}\ q(k,\tau)\ \frac{\bar{\hat{\alpha}}(k,\tau)}{q(k,\tau)}\ d\tau\right|^2$$

$$\leq \underset{\{t\in[0,T];k\in[-\Delta,\Delta]\}}{\text{maximum}} \{\mu^4 k^6 \int_0^t dt \int_0^t d\tau\ \frac{q^2(k,t)}{q^2(k,\tau)}\ e^{-2\mu k^2(t-\tau)}\}.\ ||\hat{\alpha}||^2_{Q,\Delta,q}$$

$$\leq \tfrac{1}{2}||\hat{\alpha}||^2_{Q,\Delta,q}$$

As $\hat{\alpha} = B\hat{\alpha}$ for all $t \in [0,T]$ and almost every $s \in R$ and $0 = ||\hat{\alpha}-B\hat{\alpha}||^2_{Q,\Delta,q} \geq \frac{1}{2}||\hat{\alpha}||^2_{Q,\Delta,q}$ we infer that $||\hat{\alpha}||_{Q,\Delta,q} = 0$ and so $||\hat{\alpha}||_{Q,\Delta} = 0$. Using the continuity with respect to t of $\hat{\alpha}$, this proves the first part of the theorem. The second part follows immediately from (10) and the relationship $||\alpha_0||_{R,\Delta} \leq ||\alpha||_{R,\Delta}$ which will be proved in the next section.

Corollary 1.

It is clear that similar results may be proved for the β-mode by using $\sum_{n=1}^{\infty} \beta^{(n)}$, where

$$\beta^{(1)} = \beta_0 ,$$

$$\beta^{(n+1)} = B\beta^{(n)} \qquad (n=1,2,3,\ldots).$$

Corollary 2.

For functions $f(s) \in L_2(R)$, but not in any $L_2^{\Delta}(R)$, similar theorems may be proved (the integration-interval with respect to k then runs from $-\infty$ to ∞) if $\bar{f}(k)$ tends to zero at least as fast as $\exp(-c|k|^3)$ $(c \geq \delta > 0)$ when $|k| \to \infty$. This may be seen from (9).

It remains to be proved that α and β thus found also satisfy the original differential equations (1.6) and (1.7), subject to the initial conditions $\alpha(s,0) = f(s)$, $\beta(s,0) = 0$. Using the formulas of appendix I, it is easily shown that they satisfy (1),....,(6) but then we immediately may deduce that they satisfy the original equations and initial conditions as well.

5. BEHAVIOUR WHEN $t \to \infty$.

5.1 The simple wave approximation when $t \to \infty$.

At first we note an interesting relation between the energy of the α-mode and the energy of the β-mode.

THEOREM VI.

Let $f(s) \in L_2(R)$, then for all $t \geq 0$

$$||\alpha(t)||^2 = ||\beta(t)||^2 + ||\alpha_0(t)||^2, \tag{1}$$

where $||\alpha_0||^2$ is the energy of the solution α_0 of the equivalent Burgers problem.

Proof:

Using (4.8), Parseval's theorem and transformation to the integration variable z by means of $z = \mu k$ gives

$$||\alpha_0(t)||^2 = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} e^{-2z^2\mu^{-1}t} \, |\bar{f}(\mu^{-1}z)|^2 \, dz. \tag{2}$$

As is easily seen from (3.6) and (3.7) the solutions α and β may be represented by one integral (with respect to z) each. Transforming the integration variable z to k, using Parseval's theorem and transforming backwards, we find that

$$||\beta(t)||^2 = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} \frac{z^2}{1-z^2} \sin^2\left(z\sqrt{1-z^2} \, \frac{t}{\mu}\right). \; e^{-2z^2\mu^{-1}t} \, |\bar{f}(\mu^{-1}z)|^2 \, dz, \tag{3}$$

and $||\alpha(t)||^2$ equals the sum of $||\alpha_0(t)||^2$ and $||\beta(t)||^2$.

Remark:

Until now we have not been able to prove this relationship without using the integral-representations of the solutions. An alternative proof is based on the remark that, using the notation of 3.1,

$$||\alpha||^2 - ||\beta||^2 = -(u,Du) .$$

This can be worked out, using the representation (3.2). In this way one is led to (1) provided that the second component of $\bar{f}(k)$ in (3.2) vanishes. The main theorem of this chapter is given by:

THEOREM VII.

Let $f(s) \in L_2^{\Delta}(R)$, $\bar{f}(k)$ analytic in a vicinity of $k = 0$ and $\bar{f}(0) \neq 0$. Then a constant K exists such that for $t \to \infty$

$$||\alpha - \alpha_0||^2 \leq Kt^{-1} \; ||\alpha||^2.$$

Proof:

The proof will be split into some lemmas.

LEMMA 2.

Let $f(s) \in L_2(R)$. For every $t \geq 0$

$$||\alpha - \alpha_0||^2 = 2||\alpha||^2 + ||\beta||^2 + I,$$

where

$$I = \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} 1 + \int_{-\infty}^{\infty} 2 \right] \phi\ (z) \left[e^{\Psi_a(z)t} + e^{\Psi_b(z)t}\right]dz,$$

$$\phi(z) = \tfrac{1}{2}\left[1-(1-z^2)^{\frac{1}{2}}\right] \; (1-z^2)^{-\frac{1}{2}} \; |\bar{f}(\mu^{-1}z)|^2,$$

$$\Psi_a(z) = \mu^{-1}\left[iz + iz(1-z^2)^{\frac{1}{2}} - 2z^2\right],$$

$$\Psi_b(z) = \Psi_a(-z).$$

Proof:

Similar to the proof of theorem VI.

In the three following lemmas we assume $\bar{f}(k)$ to satisfy the conditions of theorem VII.

LEMMA 3.

A real positive constant L exists such that when $t \to \infty$

$$||\alpha_0(t)||^2 \geq Lt^{-\frac{1}{2}} .$$

Proof:

Using the method of saddle-points (de Bruijn [6]) one easily finds

$$||\alpha_0||^2_{R,\Delta} = \frac{|\bar{f}(0)|^2}{2\pi\mu} \{\frac{\pi\mu}{2t}\}^{\frac{1}{2}} + \mathcal{O}(t^{-3/2}) \quad (t \to \infty) ,$$

from which the lemma immediately follows.

LEMMA 4.

Real and positive constants K_1 and K_2 exist such that when $t \to \infty$

$$||\beta||^2 \leq K_1 t^{-3/2}, \tag{4}$$

$$\left| \; ||\alpha||^2 - \frac{|\bar{f}(0)|^2}{2\mu\pi} \{\frac{\pi\mu}{2t}\}^{\frac{1}{2}} \right| \leq K_2 t^{-3/2} .$$

Proof:

At first we remark that if the points -1 and 1 are contained in the integration interval the integrals

$$[\int_{1-\varepsilon}^{1+\delta} + \int_{-1-\delta}^{-1+\varepsilon}] \frac{z^2}{1-z^2} \sin^2 (z\sqrt{1-z^2}\, \frac{t}{\mu}).\ e^{-2z^2\mu^{-1}t} \,|\bar{f}(\mu^{-1}z)|^2 \, dz ,$$

where $\varepsilon > 0$, $\delta > 0$, for $t \to \infty$ are $\mathcal{O}(e^{-t/\mu}\, t^2)$ when ε and δ are chosen small enough. Let ε and δ be chosen in that way then

$$||\beta(t)||^2 = \mathcal{O}(t^2 e^{-t/\mu}) + \frac{1}{2\pi}\left[\int_c^1 + \int_c^2\right] \frac{z^2}{4\mu(z^2-1)} |\bar{f}(\mu^{-1}z)|^2\, e^{2h(z,0)t}\, dz$$

$$- \frac{1}{2\pi} \int_c \frac{z^2}{2\mu(z^2-1)} |\bar{f}(\mu^{-1}z)|^2\, e^{-2z^2\mu^{-1}t}\, dz\ ,$$

where, if $\mu\Delta > 1+\delta$

$$c = [-\mu\Delta, -1-\delta] + [-1+\varepsilon, 1-\varepsilon] + [1+\delta, \mu\Delta]$$

and if $\mu\Delta \leq 1+\delta$

$$c = [-1+\varepsilon, 1-\varepsilon].$$

The real part of $h(z,0)$ is smaller than $-(2\mu)^{-1}$ at the positive side of the cuts in both sheets of the z-plane. This implies that the integrals along $[-\mu\Delta, -1-\delta]$ and $[1+\delta, \mu\Delta]$ are $\mathcal{O}(e^{-t/\mu})$ for $t \to \infty$. As $\frac{\partial h}{\partial z}(z,0) \neq 0$ for $z \in [-1+\rho, 1-\rho]$ we find by using appendix 2 that the part of the first two integrals in the right hand side running from $-1+\varepsilon$ to $1-\varepsilon$ is $\mathcal{O}(t^{-2})$ when $t \to \infty$. The remaining integral from $-1+\varepsilon$ to $1-\varepsilon$ can be approximated by means of the saddle-point techniques which finally results in (4). The second part is also proved by using these techniques.

<u>LEMMA 5.</u>

When $t \to \infty$

$$I = \frac{|\bar{f}(0)|^2}{\pi\mu} \left\{ \frac{\pi\mu}{2t} \right\}^{\frac{1}{2}} + \mathcal{O}(t^{-3/2}).$$

<u>Proof:</u>

As in lemma 4 we may choose $\varepsilon > 0$ and $\delta > 0$ such that

$$\sum_{j=1}^{2} \left[\int_{1-\varepsilon}^{1+\delta} j + \int_{-1-\delta}^{-1+\varepsilon} j\right] \phi(z) \left[e^{\Psi_a(z)t} + e^{\Psi_b(z)t}\right] dz = \mathcal{O}(t\, e^{-t/\mu})$$

when $t \to \infty$. The contributions of $[-\mu\Delta, -1-\delta]$ and $[1+\delta, \mu\Delta]$ (if there are any) are $\mathcal{O}(\exp -(1+\frac{1}{2}\sqrt{3})\mu^{-1}t)$ when $t \to \infty$. Remain two integrals running from $-1+\varepsilon$ to $1-\varepsilon$. Using appendix 2 we find the integral defined in the first sheet to be $\mathcal{O}(t^{-2})$. Application of the method of saddle-points to the second integral then yields the required result. We now return to the proof of our theorem. From lemma 2, 4 and 5 we deduce the existence of a real positive constant K such that when $t \to \infty$

$$||\alpha - \alpha_0||^2 \leq Kt^{-3/2}.$$

From this relationship and lemma 3 the theorem immediately follows.

Remark 1.
The condition $\bar{f}(0) \neq 0$ is not essential to the proof. If $\bar{f}(0) = 0$ the result turns out to be similar.

Remark 2.
The proof may also be given for other classes of functions, for instance the Hermite functions

$$f(s) = (-1)^n e^{\frac{s^2}{2}} \frac{d^n}{ds^n} (e^{-s^2}) \qquad (n = 0,1,2,\ldots) ,$$

the "Laguerre functions"

$$f(s) = \begin{cases} s^n e^s & s \leq 0 \\ 0 & s > 0 \end{cases} \qquad (n = 0,1,2,\ldots) \qquad (5)$$

and modulations of these functions with $\exp(ik_0 s)$. Only slight modifications have to be made.

Remark 3.
Although the spectral range of the initial function $f(s)$ may be very large we see that when $t \to \infty$ Burgers' equation perfectly describes the behaviour of the α-mode. This rather surprising result is essentially due to the fact that the solution when $t \to \infty$ almost only depends on the spectrum $\bar{f}(k)$ of $f(s)$ in a vicinity of $k = 0$.

5.2. The asymptotic behaviour of α and β when $t \to \infty$.

As the results concerning the α and β-mode are a bit surprising it seems worth while to look at the asymptotic behaviour of the solutions α and β itself and to see what actually is going on.
For this purpose we'll use the method of saddle-points again. These are located at the roots of $\frac{\partial h}{\partial z} = 0$ or

$$\xi = -2iz - \frac{(1-2z^2)}{\sqrt{1-z^2}} = G(z).$$

It is clear that with the possible exception of a finite number of values of ξ, $h(z,\xi)$ has three saddle-points. The reflectionprinciple of Schwarz (Bieberbach [7]) shows that $G(iz) = G^*(iz^*)$ where z^* is the complex conjugate of z. If h has a saddle-point in iz then $G^*(iz) = G(iz)$ and so $G(iz) = G(iz^*)$ which implies that the saddle-points are located symmetrically with respect to the imaginary axis of the z-plane. As $G_1(z) = -G_2(-z)$ (the subscript defines the sheet of the z-plane the function is defined in), it follows that if the pair $\{\xi,z\}$ satisfies $\xi = G_1(z)$ then $\{-\xi,-z\}$ is a solution of $\xi = G_2(z)$. Therefore we can confine our investigation to $\xi \geq 0$. By now it may be seen quite easily that if ξ runs from 0 to ∞, h_2 has a saddle-point $z_1(\xi)$ running along the imaginary axis of the z-plane from $-i\infty$ to $i\infty$ and h_1 has two saddle-points $z_2(\xi)$ and $z_3(\xi)$ in the upper half plane. They are situated as shown in fig. 1.

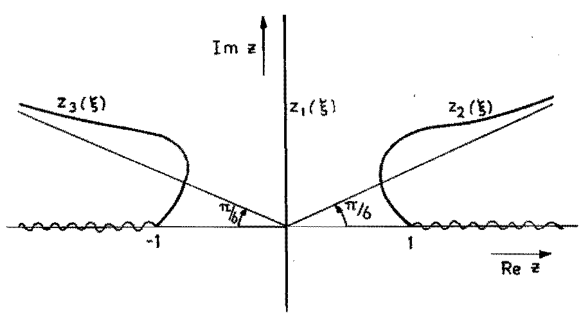

fig. 1 The contours of the saddle-points $z_1(\xi)$, $z_2(\xi)$ and $z_3(\xi)$

By now it is quite standard to derive the asymptotic expansions of the integral-representations. We shall not go into the details of this procedure but merely sketch its result derived for the case we use the continuous functions of (5) as initial data. When $t \to \infty$ the α-mode of the wave-phenomenon consists of a right- and a left travelling wave. It has sharp "peaks" around and maxima along $s = t$ (of $\mathcal{O}(t^{-\frac{1}{2}})$) and $s = -t$ ($\mathcal{O}(t^{-3/2})$). To the left and to the right of these maxima $\alpha = \mathcal{O}(e^{-ct})$, $c > 0$.
The β-mode has the same features as the α-mode, however both maxima are $\mathcal{O}(t^{-3/2})$. The amplitudes at these maxima are opposite in sign. The width at half maximum of the peaks is $\mathcal{O}(t^{-\frac{1}{2}})$. In fig. 2 and 3 the situation, when n is even, is drawn.

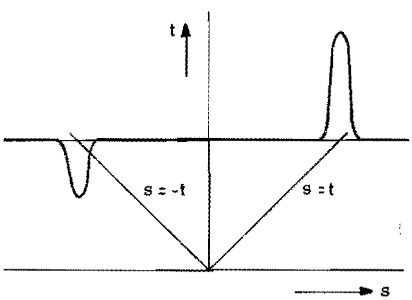

Fig. 2 the α-mode.

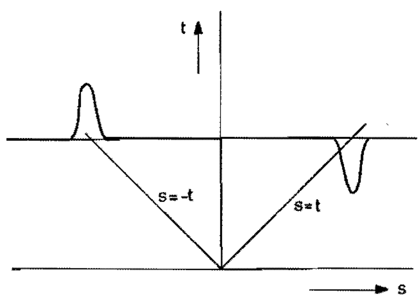

fig. 3 the β-mode.

For $L_2^\Delta(R)$ and also for Hermite functions similar results may be derived.

6. CONCLUSIONS.

For every $f(s) \in L_2^\Delta(R)$ we have shown that for every finite interval of time $[0,T]$ a convergent expansion in series of the solutions α and β does exist indeed. The first term in the expansion of α is given by α_0, the solution of the Burgers approximation problem. Furthermore for every $t \in [0,T]$

$$||\alpha - \alpha_0||_{R,\Delta} \leq (e^{\Delta^3 \mu^2 T} - 1)\ ||\alpha||_{R,\Delta} ,$$

which shows that the approximation in the interval $[0,T]$ may be made as close as one wants to by choosing Δ, μ and T.
From (5.1), (5.2) and (5.3) some further information may be drawn. Let $f(s) = \pi^{-1} \frac{\sin \mu^{-1} s}{s}$, that is $\bar{f}(\mu^{-1}z) = 1$ for $|z| \leq 1$ and $\bar{f}(\mu^{-1}z) = 0$ elsewhere on the real z-axis. Then from (5.3) we easily obtain for $t = \mu$

$$||\beta||^2 > \frac{9}{20\pi\mu} \int_{-1}^{1} z^4 e^{-2z^2} dz > \frac{1}{13} ||\alpha_0||^2 ,$$

or using (5.1), (5.2) and the triangle inequality

$$||\alpha - \alpha_0|| > \frac{1}{2\sqrt{3}} ||\alpha_0|| ,$$

which implies that under circumstances the simple wave approximation may break down.
However, as we have seen before, when $t \to \infty$ a positive constant K exists such that

$$||\alpha - \alpha_0|| \leq Kt^{-\frac{1}{2}} ||\alpha_0|| ,$$

and so the approximation may be used again.

APPENDICES

Appendix 1.

Define

$$P(k,t-\tau,s) = \frac{\mu^2 k^3}{2\pi} \sin k(t-\tau). \exp\left[-\mu k^2(t-\tau)+iks\right].$$

LEMMA 6.
For every $t \in [0,T]$ and almost every $s \in R$:

$$\frac{\partial}{\partial t}(\alpha - \alpha_0) = \int_{-\Delta}^{\Delta} dk \int_0^t d\tau \quad P_t(k,t-\tau,s)\, \bar{\alpha}(k,\tau) ,$$

$$\frac{\partial^2}{\partial t^2}(\alpha - \alpha_0) = \int_{-\Delta}^{\Delta} dk \int_0^t d\tau \quad P_{tt}(k,t-\tau,s)\,\bar{\alpha}(k,\tau) + \int_{-\Delta}^{\Delta} dk\; P_t(k,0,s)\bar{\alpha}(k,t),$$

$$\frac{\partial^{n+1}}{\partial s^n \partial t^j}(\alpha - \alpha_0) = \int_{-\Delta}^{\Delta} dk \int_0^t d\tau\; \frac{\partial^j}{\partial t^j} P(k,t-\tau,s)\;\bar{\alpha}(k,\tau)\;(ik)^n, \; j=0,1;\; n=0,1,2.$$

Proof:

$\int_0^t d\tau \int_{-\Delta}^{\Delta} dk\; \frac{\partial^j}{\partial t^j} P(k,t-\tau,s)\;(ik)^\ell\;\bar{\alpha}(k,\tau)$ $(j=0,1,2;\ \ell=0,1,2,\ldots)$ converges uniformly with respect to $s \in R$, $t \in [0,T]$ for

$$\int_{-\Delta}^{\Delta} dk \int_0^t d\tau \;\left|(ik)^\ell \frac{\partial^j}{\partial t^j} P(k,t-\tau,s)\;\bar{\alpha}(k,\tau)\right| \le \frac{\mu^2}{2\pi}\,\Delta^{\ell+3}(\Delta+\mu\;\Delta^2)^j \int_{-\Delta}^{\Delta} dk \int_0^T d\tau\; |\bar{\alpha}(k,\tau)|$$

$$\le \;\mu^2\;\left(\frac{\Delta T}{\pi}\right)^{\frac{1}{2}}\;\Delta^{\ell+3}(\Delta+\mu\Delta^2)^j\;||\alpha||_{Q,\Delta}$$

and $\alpha \in L_2^\Delta(Q)$ as we have seen before.
Now we have

$$\int_0^t d\tau \int_{-\Delta}^{\Delta} dk\; P_t(k,t-\tau,s)\;\bar{\alpha}(k,\tau) = \frac{\partial}{\partial t}\int_0^t d\theta \int_0^\theta d\tau \int_{-\Delta}^{\Delta} dk\; P_\theta(k,\theta-\tau,s)\;\bar{\alpha}(k,\tau)$$

$$= \frac{\partial}{\partial t}\int_{-\Delta}^{\Delta} dk \int_0^t d\tau \int_\tau^t d\theta\; P_\theta(k,\theta-\tau,s)\;\bar{\alpha}(k,\tau)$$

$$= \frac{\partial}{\partial t}\int_{-\Delta}^{\Delta} dk \int_0^t d\tau\; [P(k,t-\tau,s) - P(k,0,s)]\;\bar{\alpha}(k,\tau) = \frac{\partial(\alpha-\alpha_0)}{\partial t}$$

for $P(k,0,s) = 0$.

The other formulae can be proved in a similar way.

Appendix 2.

Define

$$I = \int_{-N}^{N} g(z)\, e^{h(z)t}\, dz \qquad (t \geq 0,\ N \geq N_0 > 0).$$

g(z) and h(z) are complex valued functions satisfying:

a For every z satisfying $N \geq |z| \geq \delta > 0$ a positive constant p exists such that

$$e^{h(z)t} = \mathcal{O}(e^{-pt}) \qquad (t \to \infty).$$

b For every $|z| \leq N$, $t \geq 0$

$$|e^{h(z)t}| \leq 1.$$

c For every $|z| \leq N$ h(z) is three times continuously differentiable and

$\frac{dh}{dz} \neq 0.$

d $\int_{-N}^{N} |g(z)|\, dz < \infty.$

e g(z) is analytic in a vicinity of $z = 0$.

LEMMA 7.

When $t \to \infty$

$$I = \mathcal{O}(t^{-2}).$$

Proof:

Let g(z) be analytic in an open interval containing $[-\varepsilon,\varepsilon]$ $(\varepsilon \geq \delta > 0)$.
As is easily seen using a and d a positive constant p exists such that

$$\left[\int_{-a}^{-\varepsilon} + \int_{\varepsilon}^{a}\right] g(z)\, e^{h(z)t}\, dz = \mathcal{O}(e^{-pt})$$

when $t \to \infty$.

From partial differentiation of the remaining integral using c, a and the analyticy of g(z) one obtains

$$I = \mathcal{O}(e^{-pt}) + \frac{\mu^2}{t^2} \int_{-\varepsilon}^{\varepsilon} \left\{ \frac{g''h'-gh''}{(h')^3} - \frac{3(g'h'-gh'')}{(h')^4} \right\} e^{h(z)t}\, dz, \qquad (1)$$

where the accent(s) denote differentiation(s).

(1) and b now immediately imply the theorem.

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ON A SIMPLE WAVE APPROXIMATION TO A NON-LINEAR PROBLEM

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1. INTRODUCTION.

1.1 Statement of the problem.

In physics one occasionally has to deal with sets of partial differential equations which are both non-linear and either dissipative or dispersive. An example of such a set might be

$$\alpha_t + \left[1 + \varepsilon\phi(\alpha,\beta)\right]\alpha_x = \mu(\alpha_{xx} - \beta_{xx}) \ , \qquad (1)$$

$$\beta_t - \left[1 + \varepsilon\psi(\alpha,\beta)\right]\beta_x = \mu(\beta_{xx} - \alpha_{xx}) \ , \qquad (2)$$

where x runs through the interval $(-\infty,\infty)$, t through $[0,\infty)$, $\phi(\alpha,\beta)$ and $\psi(\alpha,\beta)$ are continuous, often even monotonic functions of α and β, μ and ε are real positive constants usually much smaller than one and the subscripts x,t denote partial differentiation with respect to x, respectively t. A well known example of equations of this type is found in Lighthill's theory of waves in a real gas (Lighthill [1]).
An exact and complete solution for these equations is, at present, beyond all possibilities. Therefore, various approximations have to be used. In this paper we are concerned with a problem arising in an approximation method used by Lighthill. It applies to a certain class of initial value problems for (1) and (2), viz.:

$$\alpha(x,0) = f(x) \ , \qquad (3)$$

$$\beta(x,0) = \beta_0 \ , \qquad (4)$$

where β_0 is a constant which can be taken equal to zero without any loss of generality.

*Detached by Philips Research Laboratories.

When $\mu = 0$, it is easily seen that (2) is satisfied identically. Then, (1) becomes a first order equation in α, which is easily solved. The resultant solution is a simple wave solution for the hyperbolic set obtained by putting $\mu = 0$.

Now, Lighthill's approximation, which for obvious reasons will be called the simple wave approximation henceforth, is based on the assumption that, when the initial conditions (3) and (4) are prescribed for the equations (1) and (2) with μ small but not zero, β will be negligible, at any rate for some finite interval of time. In this way one obtains from (1):

$$\alpha_t + [1 + \varepsilon\phi(\alpha,0)]\alpha_x = \mu\alpha_{xx}, \tag{5}$$

which is an equation of Burgers type. In Lighthill's example ϕ was linear in α. The exact solution of the initial value problem is known in that case. Now, the problem is that β will grow slowly from zero and therefore it is not at all obvious that α satisfies (5) for longer intervals of time too. In general to answer this question would present rather formidable difficulties. In the present paper these will be circumvented by choosing a special form for (1) and (2) for which the question can be answered. In this example it turns out that α and β may eventually become of the same order.

1.2 Choice of the example and method of solution.

Our problem could be stated in the following terms. When $\mu = 0$ we have as solution a simple wave. That is a pure α-wave running to the right (towards positive x). The dissipative terms in the right hand side provide some coupling. This coupling presumably will cause both the appearance of an α-wave running to the left and β-waves in both directions. A first complication is that, when $\mu = 0$, a shock wave might develop. Undoubtedly this is the most general and physically the most important case. However, to keep things as simple as possible, we will avoid this which can be done by considering only so called totally exceptional equations in the sense of Lax [2]. In these equations ϕ depends on β only, ψ on α. Then, it is easily seen that the characteristic speed in a simple wave is constant, therefore no shock wave develops.

The second point is that it seems plausible to assume that the exact form of the right hand side of (1) and (2) is not of great consequence for our problem as long as the leading terms are of the type indicated. This enables us to choose a form which can be transformed into a linear set of equations by means of a non-linear transformation. These linear equations can be solved formally. Application of the inverse transformation, then supplies the answer to our questions. As a matter of fact most of the information needed can be obtained from the solutions of the linear equations directly. In section 2 the set of non-linear equations having the required properties as well as the linear system obtained by using a non-linear transformation will be given. They admit a conceivable physical interpretation. The linear equations have been treated, in connection with the simple wave approximation, extensively in [3]. Some of the mathematical results obtained in that paper and some new ones will be discussed and interpreted in sections 3, 4, 5 and 6. In section 7 we return to the non-linear equations.

2. THE MODEL EQUATIONS.

A suitable set of equations can be derived from the equations for longitudinal waves in an ideal elastic bar (Broer [4]) by adding a viscous stress term. The coefficient of viscosity is some function of the density. It is possible to choose this function in such a way that the equations become linear upon transformation to moving (Lagrangian) coordinates. We assume therefore the mass and momentum equations in the form

$$\rho_t + v\rho_x + \rho v_x = 0 \ , \tag{1}$$

$$\rho v_t + \rho v v_x = Y_0(\rho^{-1})_x + (\mu\rho_0^{\,2}\rho^{-1}v_x)_x, \tag{2}$$

where ρ is the density, v the velocity, Y_0 a constant viz. Young's modulus. In an ideal elastic bar the specific energy of deformation is $\frac{1}{2}Y_0(\rho^{-1}-\rho_0^{\,-1})^2$, the stress $Y_0(\rho^{-1} - \rho_0^{-1})$. μ is a small positive constant of the dimension of a kinematic viscosity coefficient. The subscript zero refers to the unstrained situation. The sound speed a is given by

$$a^2 = Y_0\rho^{-2}. \tag{3}$$

Its value a_0 for $\rho = \rho_0$ will be useful as a reference speed. When $\mu = 0$ the equations are hyperbolic and the characteristic variables are $\alpha = a - v$ and $\beta = a + v$.

It is easy to write (1) and (2) in terms of these variables. For our purposes it is convenient to make the equations dimensionless by putting:

$$x = Lx', \; t = La_0^{-1}t', \; \mu = 2a_0L\mu',$$

$$\alpha = a_0\left[1 + 2\varepsilon\alpha'\right], \; \beta = a_0\left[1 + 2\varepsilon\beta'\right].$$

In these equations L is some reference length connected with the initial value $\alpha(x,0)$, e.q. the dominant wavelength, ε a dimensionless measure for the strength of the wave that mostly will be chosen such that the absolute maximum of the sum of the solutions α' and β' is equal to or smaller than one.
Performing the indicated substitutions in (1), (2) and (3), and dropping the accents we obtain:

$$\alpha_t + \left[1 + 2\varepsilon\beta\right]\alpha_x = -\mu\left[1 + \varepsilon(\alpha + \beta)\right]\{\left[1 + \varepsilon(\alpha + \beta)\right](\beta - \alpha)_x\}_x, \tag{4}$$

$$\beta_t - \left[1 + 2\varepsilon\alpha\right]\beta_x = \mu\left[1 + \varepsilon(\alpha + \beta)\right]\{\left[1 + \varepsilon(\alpha + \beta)\right](\beta - \alpha)_x\}_x, \tag{5}$$

The equations are of the required form. When terms of $\mathcal{O}(\varepsilon\mu)$ are dropped they reduce to special (and when $\mu = 0$ totally exceptional) cases of (1.1) and (1.2). Now, we transform (4) and (5) to the Lagrangian coordinate $\rho_0 s = m$, where m is the mass coordinate as used in [4]. The details of this transformation will be stripped. We notice only the formulas

$$\left(\frac{\partial x}{\partial t}\right)_s = a_0^{-1}v = \varepsilon(\beta - \alpha), \tag{6}$$

$$\left(\frac{\partial x}{\partial s}\right)_t = \rho_0\rho^{-1} = 1 + \varepsilon(\alpha + \beta), \tag{7}$$

where α, β, x, s and t are dimensionless. For the transformed equations we find

$$\alpha_t + \alpha_s = \mu(\alpha_{ss} - \beta_{ss}), \tag{8}$$

$$\beta_t - \beta_s = \mu(\beta_{ss} - \alpha_{ss}), \tag{9}$$

which are linear indeed.

The initial conditions will be stated in the following way

$$\alpha(s,0) = f(s), \tag{10}$$

$$\beta(s,0) = 0. \tag{11}$$

3. BALANCE EQUATIONS.

Some conservation laws and balance equations will be derived for (2.8) and (2.9). These equations themselves are in the form of a conservation law. Adding and subtracting them gives:

$$\frac{\partial}{\partial t}(\alpha + \beta) + \frac{\partial}{\partial s}(\alpha - \beta) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\alpha - \beta) + \frac{\partial}{\partial s}\left[\alpha + \beta - 2\mu(\alpha_s - \beta_s)\right] = 0,$$

describing conservation of mass, respectively momentum.
For every natural number $n \geq 2$, it is possible to construct two linearly independent balance equations of degree n. They may be written in the form:

$$\frac{\partial}{\partial t}\alpha^n + \frac{\partial}{\partial s}\left[\alpha^n + \mu n\alpha^{n-1}(\beta_s - \alpha_s)\right] + \mu(n-1)n\alpha^{(n-2)}\alpha_s(\alpha_s - \beta_s) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}\beta^n - \frac{\partial}{\partial s}\left[\beta^n + \mu n\beta^{n-1}(\beta_s - \alpha_s)\right] - \mu(n-1)n\beta^{(n-2)}\beta_s(\alpha_s - \beta_s) = 0, \tag{3}$$

and have been found from (2.8) and (2.9) by premultiplying the first one by α^{n-1} and the second one by β^{n-1}. When (2) and (3) are added and n has been put equal to 2 the equation of balance of energy (kinetic- + deformation energy) is found:

$$\frac{\partial}{\partial t}(\alpha^2 + \beta^2) + \frac{\partial}{\partial s}\left[\alpha^2 - \beta^2 + 2\mu(\alpha - \beta)(\beta_s - \alpha_s)\right] + 2\mu(\alpha_s - \beta_s)^2 = 0. \tag{4}$$

From subtracting and putting n = 2 a Bernoulli-like equation (when $\mu = 0$ it is the exact Bernoulli-equation)

$$\frac{\partial}{\partial t}(\alpha^2 - \beta^2) + \frac{\partial}{\partial s}\left[\alpha^2 + \beta^2 + 2\mu(\alpha + \beta)(\beta_s - \alpha_s)\right] + 2\mu(\alpha_s{}^2 - \beta_s{}^2) = 0$$

is found.

It is obvious that $\int_{-\infty}^{\infty} \alpha^2 ds$, if it exists, may be seen as the total energy of the α-mode at time t. $\int_{-\infty}^{\infty} \beta^2 ds$ can be given a similar interpretation. This will be used later on in the paper.

4. SOME MATHEMATICAL AND PHYSICAL ASPECTS OF THE LINEAR EQUATIONS.

4.1 Some notations.

R: the interval $(-\infty,\infty)$ of the real numbers.

Q: a strip in the s-t plane containing all the points satisfying the inequalities $-\infty < s < \infty$ and $0 < t < T < \infty$.

Consider scalar valued functions u(s,t) defined on R (t fixed) and Q respectively.

$L_2(R)$ is a Hilbert-space containing all square integrable functions on R with inner product (,) and norm $||\ ||$ defined by

$$(u,v) = \int_{-\infty}^{\infty} u^*(s)v(s)ds \ ; \ ||u|| = (u,u)^{\frac{1}{2}},$$

u^* being the complex conjugate of u.

The Sobolev-space $W_2^m(R)$ (m a positive natural number) is a Hilbert-space containing all $L_2(R)$ functions u(s) that have generalized derivatives $D^k u \in L_2(R)$, where $k = 1,\ldots,m$ (Smirnow [5]). The inner product $(,)_m$ and norm $||\ ||_m$ are respectively

$$(u,v)_m = \sum_{i=1}^{m} (D^i u, D^i v) + (u,v); \ ||u||_m = (u,u)_m^{\frac{1}{2}}.$$

$L_2^\Delta(R)$ is a Hilbert-space containing all functions $u \in L_2(R)$, of which the Fourier transform $\bar{u}(k)$ defined by

$$\bar{u}(k) = \int_{-\infty}^{\infty} u(s) \exp(-iks)ds \tag{1}$$

vanishes identically outside a finite interval $[-\Delta,\Delta]$ ($\Delta \in R$), with inner product $(,)_{R,\Delta}$ and norm $||\ ||_{R,\Delta}$ defined by

$$(u,v)_{R,\Delta} = \int_{-\infty}^{\infty} u^*(s)v(s)ds \ , \ \ ||u||_{R,\Delta} = (u,u)_{R,\Delta}^{\frac{1}{2}} .$$

Where not stated otherwise all integrations are in the sense of Lebesque and all differentials are meant in the generalized sense, although the classical notation will be retained. The Fourier transform with respect to s of a function u(s,t) will sometimes be called the spectrum of u.

4.2 Existence and uniqueness.

In [3] it has been proved that equations (2.8) and (2.9) are uniquely solvable for every $f \in L_2(R)$ $(W_2^m(R), L_2^\Delta(R))$ and that for every $0 < t < T < \infty$ the solution is an element of $L_2(R)$ $(W_2^m(R), L_2^\Delta(R))$. Furthermore $\alpha \to f$ and $\beta \to 0$ as $t \to 0$ in the sense of the $L_2(R)$ $(L_2^\Delta(R))$ norm. The solutions may be represented by

$$\alpha(s,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}1 + \int_{-\infty}^{\infty}2\right] g^{(2)}(k)\ \exp\left[h(k,\xi)t\right]dk\ , \tag{2}$$

$$\beta(s,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}1 + \int_{-\infty}^{\infty}2\right] g^{(1)}(k)\ \exp\left[h(k,\xi)t\right]dk\ , \tag{3}$$

where

$$g^{(1)}(k) = -\tfrac{1}{2}i\mu k(1-\mu^2k^2)^{-\frac{1}{2}}\bar{f}(k),\ g^{(2)}(k) = \tfrac{1}{2}[-1 + (1-\mu^2k^2)^{\frac{1}{2}}](1-\mu^2k^2)^{-\frac{1}{2}}\bar{f}(k),$$

$$h(k,\xi) = ik(1-\mu^2k^2)^{\frac{1}{2}} - \mu k^2 + ik\xi,$$

$$i = \sqrt{-1},\quad \xi = st^{-1}$$

and $\bar{f}(k)$ is defined similar to (1).
The number 1 respectively 2 through the integration symbol means integration in the first-, respectively second sheet of the complex k-plane. The first sheet is defined by

$$\lim_{|k|\to\infty} \frac{(1-\mu^2k^2)^{\frac{1}{2}}}{\mu k} = -i \qquad (\ 0 \leq \arg k \leq \pi),$$

and the second by

$$\lim_{|k|\to\infty} \frac{(1-\mu^2k^2)^{\frac{1}{2}}}{\mu k} = i \qquad (\ 0 \leq \arg k \leq \pi).$$

This corresponds to cutting the k-plane from $-\infty$ to $-\mu^{-1}$ and from μ^{-1} to ∞.

4.3 Stability and positive definiteness of ρ.

It is shown in [3] that for every $f \in L_2(R)$ the solution of (2.8),...,(2.11) is stable in the sense that for all $t \geq 0$

$$||\alpha(t)||^2 + ||\beta(t)||^2 \leq ||f||^2,$$

which means that the total energy of the system is bounded from above by the initial energy. This result has been derived using the equation of balance of energy (3.4). In a physical problem ρ, the density, must be essentially positive i.e. $\rho \geq \delta > 0$. Now, one may wonder whether it is possible to choose the initial conditions and ε in such a way that this is satisfied. Let $f \in W_2^{\,1}(R)$ and $||f||_1 \leq \sqrt{2}$. From [3] we infer the existence of $\tilde{\alpha}(s,t)$ and $\tilde{\beta}(s,t)$, continuously depending on s, such that for all $t \geq 0$

$$\alpha(s,t) = \tilde{\alpha}(s,t) \quad \text{(a.e.)},$$

$$\beta(s,t) = \tilde{\beta}(s,t) \quad \text{(a.e.)},$$

$$\sup_{s \in R} |\tilde{\alpha} + \tilde{\beta}| \leq 1,$$

the last condition being equivalent to the one posed in section 2 concerning the definition of ε. It is clear that for $\varepsilon \leq 1 - \delta < 1$, $\rho = 1 + \varepsilon(\alpha+\beta) \geq \delta > 0$ indeed.

5. MONOCHROMATIC WAVES.

Following section 1 the simple wave approximation of the set of linear equations under consideration is given by

$$\alpha_{0t} + \alpha_{0s} - \mu\alpha_{0ss} = 0, \tag{1}$$

$$\alpha_0(s,0) = f(s). \tag{2}$$

In this section we shall deal with the particular simple case of solutions $(\alpha, \beta, \alpha_0)$ that are periodic functions with respect to s. This may serve as an introduction to the more difficult problems arising in dealing with a general initial function f(s).

Let

$$f(s) = \exp(ik_1 s) \qquad (k_1 \text{ real}).$$

The solutions α and β may formally be found from (4.2) and (4.3) by substituting $\bar{f}(k) = 2\pi\delta(k-k_1)$, where $\delta(x)$ is the Dirac δ-function. We find for $|k| \le \mu^{-1}$

$$\alpha = \frac{1+c}{2c} \exp(iks - ikct - \mu k^2 t) + \frac{c-1}{2c} \exp(iks + ikct - \mu k^2 t), \tag{3}$$

$$\beta = \frac{i\mu k}{2c} \exp(iks - ikct - \mu k^2 t) - \frac{i\mu k}{2c} \exp(iks + ikct - \mu k^2 t) \tag{4}$$

and for $|k| \ge \mu^{-1}$

$$\alpha = \frac{C-i}{2C} \exp(iks + kCt - \mu k^2 t) + \frac{C+i}{2C} \exp(iks - kCt - \mu k^2 t), \tag{5}$$

$$\beta = \frac{\mu k}{2C} \exp(iks + kCt - \mu k^2 t) - \frac{\mu k}{2C} \exp(iks - kCt - \mu k^2 t), \tag{6}$$

where

$$c(k) = \left|(1 - \mu^2 k^2)^{\frac{1}{2}}\right| \qquad (|k| \le \mu^{-1}),$$

$$C(k) = \left|(\mu^2 k^2 - 1)^{\frac{1}{2}}\right| \qquad (|k| \ge \mu^{-1})$$

and, for convenience, the subscript 1 has been dropped again.
α_0 is given by

$$\alpha_0 = \exp(iks - ikt - \mu k^2 t). \tag{7}$$

(3) and (4) clearly demonstrate the development of right- and left moving waves. $c(k)$ can be seen as a fase velocity. When $|k| \ge \mu^{-1}$, we can speak of travelling waves no longer. Substituting $\sin(kct) = \dfrac{\exp(ikct) - \exp(-ikct)}{2i}$ in (3) and (4), we find for $|k| \le \mu^{-1}$

$$\alpha = \left[\exp(-ikct) + \frac{i(c-1)}{c} \sin(kct)\right] \exp(iks - \mu k^2 t), \tag{8}$$

$$\beta = \frac{\mu k}{c} \sin(kct) \exp(iks - \mu k^2 t),$$

showing that α may also be seen as a superposition of a right moving- and a standing-, β as a pure standing wave.

If $|\mu k| < 1$ we expand $(1-\mu^2k^2)^{\frac{1}{2}}$ around $\mu k = 0$. In this way we find from (8)

$$\alpha = \left[1 + \frac{1}{2}i\mu^2k^3t + \ldots\right] \exp(iks - ikt - \mu k^2t) + \\ +\left[-\frac{1}{2}i\mu^2k^2 \sin(kt) + \ldots\right] \exp(iks - \mu k^2t). \tag{9}$$

However, as α is an analytic function of μk for all finite k (see (3)), this expansion also holds for $|\mu k| \geq 1$ and so, for all finite k.

We have, using (7) and (8)

$$\alpha-\alpha_0 = \frac{1}{2}i\mu^2k^3t \exp(iks - ikt -\mu^2k^2t) - \frac{1}{2}i\mu^2k^2 \sin(kt) \exp(iks - \mu k^2t)+\ldots, \tag{10}$$

from which we infer that the difference between α and α_0 is "small" if $\mu^2|k|^3t \ll 1$.

The expansion (9) may also be found in a different way which will turn out to be succesful for $f \in L_2^{\Delta}(R)$ too.

Write

$$\alpha = \bar{\alpha}(k,t) \exp(iks),$$

$$\beta = \bar{\beta}(k,t) \exp(iks),$$

then $\bar{\alpha}$ and $\bar{\beta}$ satisfy

$$\bar{\alpha}_t + (ik + \mu k^2)\bar{\alpha} = \mu k^2\bar{\beta},$$

$$\bar{\beta}_t + (-ik + \mu k^2)\bar{\beta} = \mu k^2\bar{\alpha},$$

and so

$$\bar{\alpha}_{tt} + 2\mu k^2\bar{\alpha}_t + (\mu^2k^4 + k^2)\bar{\alpha} = \mu^2k^4\bar{\alpha}. \tag{11}$$

The initial data for (11) become

$$\bar{\alpha}(k,0) = 1,$$

$$\bar{\alpha}_t(k,0) = -ik - \mu k^2.$$

Now, it is quite elementary to show that $\bar{\alpha}$ satisfies the integral equation

$$\bar{\alpha}(k,t) = \bar{\alpha}_0(k,t) + \mu^2 k^3 \int_0^t \sin[k(t-\tau)]\ \exp[-\mu k^2(t-\tau)]\ \bar{\alpha}(k,\tau)d\tau,$$

where $\bar{\alpha}_0(k,t) = \alpha_0(s,t)\ \exp(-iks)$.

The solution of this equation may be found by means of iteration:

$$\bar{\alpha}^{(o)}(k,t) = \bar{\alpha}_0(k,t),$$

$$\bar{\alpha}^{(n)}(k,t) = \mu^2 k^3 \int_0^t \bar{\alpha}^{(n-1)}(k,\tau)\ \sin[k(t-\tau)]\ \exp[-\mu k^2(t-\tau)]d\tau \quad (n = 0,1,\ldots),$$

so $\alpha = \sum_{n=0}^{\infty} \bar{\alpha}^{(n)}(k,t)\ \exp(iks)$ (for a proof, see [3]).

Some computations show this expansion to be identical to the one found before.

For β a similar procedure may be followed.

From (10) we see that as $t \to \infty$ the simple wave approximation breaks down. This turns out not to be true when $f \in L_2^{\Delta}(R)$, as will be shown in the next section. Looking at (3),...,(6) we observe that as $t \to \infty$ the dissipation at high frequencies k is much larger than at low frequencies. When $f \in L_2^{\Delta}(R)$ the solution α (see (4.2)) may be seen as a superposition (integral with respect to k) of monochromatic solutions. When $t \to \infty$, only the values in a small $\{\mathcal{O}(t^{-\frac{1}{2}})$ as was proved in [3]$\}$ vicinity of $k = 0$ will contribute significantly to the integral. This "explains" why $f \in L_2^{\Delta}(R)$, as $t \to \infty$, leads to results different from those found for monochromatic waves. In particular it will turn out that, as $t \to \infty$, the simple wave approximation holds again.

6. $L_2^{\Delta}(R)$ solutions.

6.1 An expansion of the solution.

To get some insight in the problem stated by (1.1),...,(1.4), one sometimes uses an expansion in a series of α and β, where the solution of the simple wave approximation (1.3) and (1.5) is used as the first term in the expansion of α (c.f. Lighthill [1]).

The convergence of such an expansion, as far as we know, never has been treated. In general this would be very complicated. However, it has turned out to be possible to show convergence of such an expansion for the simple case treated here.
Let $f \in L_2^{\Delta}(R)$ and $\alpha^{(2n)}$ and $\beta^{(2n-1)}$ satisfy

$$\alpha_t^{(o)} + \alpha_s^{(o)} - \mu\alpha_{ss}^{(o)} = 0,$$

$$\alpha_t^{(2n)} + \alpha_s^{(2n)} - \mu\alpha_{ss}^{(2n)} = -\beta_{ss}^{(2n-1)},$$

$$\beta_t^{(2n-1)} - \beta_s^{(2n-1)} + \mu\beta_{ss}^{(2n-1)} = -\alpha_{ss}^{(2n-2)} \qquad (n = 1,2,\dots)$$

and let $\alpha^{(o)}(s,0) = f(s)$, $\alpha^{(2n)}(s,0) = \beta^{(2n-1)}(s,0) = 0$.
In [3] it has been shown that for all finite $t \geq 0$, $\sum_{n=0}^{N} \alpha^{(2n)}\mu^{2n}$ converges to α, $\sum_{n=0}^{N} \beta^{(2n+1)}\mu^{2n+1}$ converges to β as $N \to \infty$ in the sense of the $L_2^{\Delta}(R)$ norm.
The method used there runs along lines quite similar to those used in section 5 to obtain an expansion in a series of monochromatic solutions.
Other important results, found in [3], are given by

$$||\alpha - \sum_{n=0}^{N} \mu^{2n}\alpha^{(2n)}||_{R,\Delta} \leq (\sum_{n=N+1}^{\infty} t^n \mu^{2n} \Delta^{3n}) \; ||\alpha||_{R,\Delta}, \tag{1}$$

$$||\beta - \sum_{n=0}^{N} \mu^{2n+1}\beta^{(2n+1)}||_{R,\Delta} \leq \mu\Delta \, (\sum_{n=N+1}^{\infty} t^n \mu^{2n} \Delta^{3n}) \; ||\alpha||_{R,\Delta},$$

and will be used repeatedly in the next sections. Finally we add the remark that it is possible to prove convergence for functions of which the spectrum is not of bounded support. However these functions will not be treated here.

6.2 The start of the β-mode and the left running α-mode.

From our considerations in the sections 1.2 and 5 we expect the appearance of a left running α-wave and β-waves in both directions. Let $\Delta^3\mu^2 T << 1$, then for every $t \in [0,T]$:

$$||\alpha - \alpha_0 - \mu^2\alpha^{(2)}||_{R,\Delta} << ||\alpha||_{R,\Delta} \; ,$$

$$||\beta - \mu\beta^{(1)}||_{R,\Delta} << ||\alpha||_{R,\Delta},$$

which implies that for every $t \in [0,T]$, $\alpha_0 + \mu^2\alpha^{(2)}$ is a good approximation to α and so is $\mu\beta^{(1)}$ to β. As is easily seen

$$\alpha_0+\mu^2\alpha^{(2)} = \frac{1}{8}\mu^2(\pi\mu t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{d^2f}{d\xi^2} \exp\left[-\frac{(s-\xi+t)^2}{4\mu t}\right]d\xi +$$

$$+ \frac{1}{2}(\pi\mu t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left[f(\xi) - \frac{1}{2}\mu^2 t\frac{d^3f}{d\xi^3} - \frac{1}{4}\mu^2 \frac{d^2f}{d\xi^2}\right] \exp\left[-\frac{(s-\xi-t)^2}{4\mu t}\right]d\xi,$$

$$\mu\beta^{(1)} = \frac{\mu}{4}(\pi\mu t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{df}{d\xi} \left\{\exp\left[-\frac{(s-\xi-t)^2}{4\mu t}\right] - \exp\left[-\frac{(s-\xi+t)^2}{4\mu t}\right]\right\}d\xi,$$

which confirms our expectations. Some insight in the initial state of the α- and β-mode may be gained by using an asymptotic expansion as $t \to 0$ (appendix 1). We find

$$\alpha_0+\mu^2\alpha^{(2)} = f(s-t) + \frac{\mu^2}{4}\left[\frac{d^2f}{ds^2}(s+t) - \frac{d^2f}{ds^2}(s-t)\right] - \frac{\mu^2 t}{2}\frac{d^3f}{ds^3}(s-t) + \mathcal{O}(t),$$

$$\beta^{(1)} = \frac{1}{2}\left[\frac{df}{ds}(s-t) - \frac{df}{ds}(s+t)\right] + \mathcal{O}(t).$$

6.3 The simple wave approximation.

For solutions α which are square integrable α_0 will be called a useful (good-) approximation to α in the interval of time $[t_1,t_2]$ $(t_2 > t_1)$ if for every $t \in [t_1,t_2]$

$$||\alpha - \alpha_0|| << ||\alpha|| \qquad (2)$$

Of course, if (2) is satisfied α_0 locally still may deviate considerably from α.

Using (2) we immediately find

$$||\alpha - \alpha_0||_{R,\Delta} \leq (e^{\Delta^3\mu^2 t} - 1)\ ||\alpha||_{R,\Delta}$$

and so, for every $t \in [0,T]$, where $\Delta^3\mu^2 T << 1$, α_0 is a good approximation in the sense of (2). This result is entirely similar to the one found in section 5 where we dealt with monochromatic solutions.

In [3] it has been shown that the simple wave approximation may fail for some finite time, but as $t \to \infty$ it holds again as then a positive constant K exists such that

$$||\alpha - \alpha_0||_{R,\Delta} \leq Kt^{-\frac{1}{2}}||\alpha||_{R,\Delta}.$$

This result has already been discussed in section 5. In this context we may notice the following interesting relation, used in some proofs in [3]:

$$||\alpha||^2 = ||\beta||^2 + ||\alpha_0||^2.$$

It holds for every $f \in L_2(R)$.

7. THE NON-LINEAR EQUATIONS.

7.1 The inverse transformation.

Let $f \in W_2^3(R)$ and absolutely integrable on R. Then, α and β are elements of $W_2^3(R)$ too and according to (3.1) $\int_{-\infty}^{s}(\tilde{\alpha} + \tilde{\beta})ds'$ exists in the sense of Riemann. So, by integrating (2.6) and (2.7) we find for every finite $t \geq 0$:

$$s = x - \varepsilon \int_{-\infty}^{s}[\tilde{\alpha}(s',t) + \tilde{\beta}(s',t)]ds'. \tag{1}$$

We are interested in the conditions to be satisfied by f(s) and ε that are sufficient for s to be solvable from (1) as an univalent function of x and t. Let $||f||_1 \leq \sqrt{2}$; $\varepsilon \leq 1-\delta < 1$ again. Define

$$s_0 = x, \tag{2}$$

$$s_n = x - \varepsilon \int_{-\infty}^{s_{n-1}}[\tilde{\alpha}(s',t) + \tilde{\beta}(s',t)]ds'. \tag{3}$$

From section 4.3 we deduce $\sup_{s\in R} |\tilde{\alpha} + \tilde{\beta}| \leq 1$ and $\tilde{\rho} \geq \delta > 0$, so $(\frac{\partial x}{\partial s})_t$ is essentially positive and consequently (1) is uniquely solvable. Furthermore

$$|s_{n+1} - s_n| = \varepsilon\left|\int_{s_{n-1}}^{s_n} (\tilde{\alpha} + \tilde{\beta})ds'\right| \leq (1 - \delta)\ |s_n - s_{n-1}|,$$

which implies that the sequence defined in (2) and (3) converges to s(x,t) for every finite t.

It is easily verified that $s = \sum_{n=0}^{\infty}\varepsilon^n s^{(n)}$, where

$$s^{(0)} = x,$$

$$s^{(1)} = \int_{-\infty}^{x}[\tilde{\alpha}(s',t) + \tilde{\beta}(s',t)]ds',$$

$$\vdots$$

$$s^{(n)} = -\varepsilon^{1-n}\int_{\sum_{j=0}^{n-2}\varepsilon^j s^{(j)}}^{\sum_{j=0}^{n-1}\varepsilon^j s^{(j)}}[\tilde{\alpha}(s',t) + \tilde{\beta}(s',t)]ds' \qquad (n = 2,3,\ldots).$$

7.2 On a simple wave approximation.

Consider (5.1) and (5.2) where s is replaced by x as the simple wave approximation to the non-linear problem. This is not entirely equivalent to section 1, as the initial value $\alpha_0(x,0)$ should have been equal to $f[s(x,0)]$. However, this is just a mathematical difference and is not essential to the problem as all the aspects of approximating a non-linear problem by a linear one are retained.
Besides we will choose ε and $f(s)$ such that at $t = 0$ the simple wave approximation does hold indeed. Let $f \in L_2^{\Delta}(R)$ and absolutely integrable such that a) $||f||_1 \leq \sqrt{2}$ b) $\int_{-\infty}^{\infty}|f(s)|ds = M$ (a positive constant) c) $\bar{f}(k)$ is analytic in a vicinity of $k = 0$ and let $\varepsilon \leq 1-\delta < 1$. Define

$$\phi(s,t) = \int_{-\infty}^{s}[\tilde{\alpha}(s',t) + \tilde{\beta}(s',t)]ds'.$$

Using $|\tilde{\alpha} + \tilde{\beta}| \leq 1$, the mean value theorem of differential calculus and $\int_{-\Delta}^{\Delta}k^2|\bar{g}(k)|^2dk \leq \Delta^2\int_{-\Delta}^{\Delta}|\bar{g}(k)|^2dk$, we find:

$$\int_{-\infty}^{\infty}|\alpha(x,t) - \alpha_0(x,t)|^2dx =$$

$$= \int_{-\infty}^{\infty}|\alpha(s,t) - \alpha_0(s,t) + \alpha_0(s,t) - \alpha_0(s + \varepsilon\phi,t)|^2[1 + \varepsilon(\alpha+\beta)]ds \leq$$

$$\leq 2\int_{-\infty}^{\infty}|\alpha - \alpha_0|^2ds + 2\int_{-\infty}^{\infty}|\alpha_0(s + \varepsilon\phi,t) - \alpha_0(s,t)|^2ds \leq$$

$$\leq 2\int_{-\infty}^{\infty}|\alpha - \alpha_0|^2ds + 2\varepsilon^2[\max_{s\in R}|\phi(s,t)|^2]\int_{\infty}^{\infty}|\frac{\partial\alpha_0}{\partial s}(s + \varepsilon\theta\phi,t)|^2ds \leq$$

$$\leq 2\int_{-\infty}^{\infty}|\alpha - \alpha_0|^2 ds + 2\varepsilon^2[\max_{s\in R}|\phi(s,t)|^2] \int_{-\infty}^{\infty}|\frac{\partial\alpha_0}{\partial s'}(s',t)|^2[1 + \varepsilon(\alpha+\beta)]ds' \leq$$

$$\leq 2\int_{-\infty}^{\infty}|\alpha - \alpha_0|^2 ds + 4\varepsilon^2\Delta^2\{\max_{s\in R}|\phi(s,t)|^2\} \int_{-\infty}^{\infty}|\alpha_0(s',t)|^2 ds', \qquad (4)$$

where $0 \leq \theta \leq 1$.

Using the conservation law of mass (3.1) we find

$$\phi(s,t) = \int_{-\infty}^{s} f(s')ds' + \int_0^t[\tilde{\beta}(s,t') - \tilde{\alpha}(s,t')]dt',$$

which implies that

$$|\phi(s,t)| \leq M + t. \qquad (5)$$

Substituting this in (4) and using (6.1), we obtain that for all $t \in [0,T]$ (T finite)

$$\int_{-\infty}^{\infty}|\alpha - \alpha_0|^2 dx \leq \{2[\exp(T\mu^2\Delta^3) - 1]^2 + 4\varepsilon^2\Delta^2(M + T)^2\} \int_{-\infty}^{\infty}|\alpha_0|^2 dx,$$

which implies that if $M\varepsilon\Delta << 1$, $T << \Delta(\mu^4\Delta^4 + \varepsilon^2)^{\frac{1}{2}}$ the simple wave approximation does hold indeed.

Of course we are also interested in the situtation as $t \to \infty$. Now, a difficulty shows up as an inequality of the form (5) can be used no longer. However, in appendix 2 it has been shown that, as $t \to \infty$, $|\phi(s,t)| \leq 2M$. This implies that given the condition $M\Delta\varepsilon << 1$ the simple wave approximation holds again. Therefore, when ε and/or Δ are chosen small enough the situation is entirely equivalent to the case treated in the former sections.

APPENDICES.

Appendix 1.

Define

$$K_{\pm}(s,t) = (\mu t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g(x) \exp[-\frac{(x - s \pm t)^2}{4\mu t}]dx.$$

LEMMA.

Let $g \in L_2^{\Delta}(R)$. As $t \to 0$, $K_{\pm}(s,t)$ has the following asymptotic expansion

$$K_{\pm}(s,t) \approx 2\sum_{n=0}^{\infty}(4\mu t)^n[(2n)!]^{-1}\Gamma(n + \tfrac{1}{2})\frac{d^{2n}}{dx^{2n}}\tilde{g}(s \mp t), \qquad (1)$$

where

$$\tilde{g}(x) = g(x) \qquad \text{(a.e.)}$$

and $\tilde{g}(x)$ is analytic on the real axis.

Proof.

Define

$$\tilde{g}(x) = \frac{1}{2\pi} \int_{-\Delta}^{\Delta} \bar{g}(k) \exp(ikx)dk,$$

then $g(x) = \tilde{g}(x)$ (a.e.) and $\tilde{g}(x)$ is analytic on the real axis. So $\tilde{g}(x)$ may substitute $g(x)$ in (1) and a positive number ρ exists such that for $|x - s \pm t| \leq \rho$

$$\tilde{g}(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_n = \frac{1}{n!} \frac{d^n}{dx^n} \tilde{g}(s \mp t).$$

Furthermore, using formula 7.1.13 of Abramowitz and Stegun [6]:

$$\left|\left[\int_{-\infty}^{-\rho} + \int_{\rho}^{\infty}\right] (\mu t)^{-\frac{1}{2}} \tilde{g}(z + s \mp t) \exp\left[-z^2(4\mu t)^{-1}\right]dz\right| \leq$$

$$8\rho^{-1}(\mu t)^{\frac{1}{2}}\left[1 + \left(1 + \frac{16\mu t}{\pi\rho^2}\right)^{\frac{1}{2}}\right]^{-1} \exp\left[-\rho^2(4\mu t)^{-1}\right] \left(\max_{z\in R}|\tilde{g}(z)|\right).$$

Now, all conditions required by de Bruijn [7], page 68, to construct an asymptotic expansion of $K_{\pm}(s,t)$ are satisfied. We find (1).

Appendix 2.

LEMMA.

Let $f(s)$ satisfy the conditions required and ϕ be defined as in section 7.2. Then, as $t \to \infty$

$$|\phi(s,t)| \leq 2M.$$

Proof.

As is seen from (4.2) and (4.3)

$$I \stackrel{\text{def}}{=} \int_{-N}^{s}[\tilde{\alpha}(s',t) + \tilde{\beta}(s',t)]ds' =$$

$$[\int_{-\Delta}^{\Delta}1 + \int_{-\Delta}^{\Delta}2]\ (ik)^{-1}\psi(k)\{\ \exp[h(k,st^{-1})t] - \exp[h(k,-Nt^{-1})t]\}dk, \qquad (2)$$

where $N >> 1$ and

$$\psi(k) = (4\pi)^{-1}(1-\mu^2k^2)^{-\frac{1}{2}}[1 - i\mu k + (1-\mu^2k^2)^{\frac{1}{2}}]\bar{f}(k).$$

$\bar{f}(k)$ is analytic in a ρ-vicinity of $k=0$ so we are able to choose a number $0 < \delta < \rho$ such that along $c_{\delta_\pm} := \{k||k| = \delta, -\frac{\pi}{2} \pm \frac{\pi}{2} \leq \arg k \leq \frac{\pi}{2} \pm \frac{\pi}{2}\}$

$$|\psi(k)| \leq \frac{3}{4}\pi^{-1}|\bar{f}(0)|. \qquad (3)$$

(2) may be rewritten in the form

$$I = [\int_{-\Delta}^{-\delta}(1+2) + \int_{\delta}^{\Delta}(1+2) + \int_{c_{\delta+}}(1+2)]\ (ik)^{-1}\psi(k)\{\ \exp[h(k,st^{-1})t] -$$

$$- \exp[h(k, -Nt^{-1})t]\}dk. \qquad (4)$$

As for all $k \in R$, where $|k| \geq \delta$, $\mathrm{Re}\ h(k,st^{-1}) \leq -\mu\delta^2$, the first two integrals in the right hand side of (4) are $\mathcal{O}(t \exp(-\delta^2\mu t))$ as $t \to \infty$. $(ik)^{-1}\psi(k)$ is analytic in a vicinity of $k = 0$ in the first sheet of the complex k-plane. Using the method of saddle-points it is quite standard to derive

$$[\int_{c_{\delta+}}1]\ (ik)^{-1}\psi(k)\{\ \exp[h(k,st^{-1})t] - \exp[h(k,-Nt^{-1})t]\}dk = \mathcal{O}(t^{-\frac{1}{2}}) \quad (t \to \infty).$$

Consider

$$I_1 = [\int_{c_{\delta+}}2]\ (ik)^{-1}\psi(k)\ \exp[h(k,st^{-1})t]dk. \qquad (5)$$

Choose $\delta < \frac{1}{2}\rho\mu^2$ and define $\varepsilon = 2\delta\mu^{-1}$. Let $s \geq (1 + \varepsilon)t$. Then by choosing δ small enough, $\mathrm{Re}\ h \leq 0$ along $c_{\delta+}$. Thus, substituting $k = \delta e^{i\phi}$ in (5) and using (3) we find $|I_1| \leq \frac{3}{4}|\bar{f}(0)|$. If $s \leq (1 - \varepsilon)t$, then, to obtain $\mathrm{Re}\ h \leq 0$, we must shift the path of integration to $c_{\delta-}$. This leads to $|I_1| \leq \frac{7}{4}|\bar{f}(0)|$. Now, let $(1 - \varepsilon)t \leq s \leq (1 + \varepsilon)t$. The saddle-point of $h(k,st^{-1})$ is located on the imaginary axis of the k-plane, inside the circle $|k| = \rho$ (c.f. [3]).

By using the method of saddle-points as developed by van der Waerden [8]. we find that, as $t \to \infty$, $|I_1| \leq \frac{5}{4}|\bar{f}(0)|$. Applying the results concerning I_1 to (4) and using the results obtained earlier in the proof we deduce the lemma.

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ON THE RANGE OF VALIDITY OF A SIMPLE WAVE APPROXIMATION OF A NONLINEAR SET OF DIFFUSIVE WAVE EQUATIONS

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SUMMARY.
The set of wave equations considered is an intermediate approximation of the Navier-Stokes equations. A further approximation leads to Burgers' equation. The range of validity of this simple wave approximation has been studied. The method used is especially useful for small nonlinearity.

1. INTRODUCTION.

In some preceding papers [1, 2], L.J.F. Broer and the present author have paid attention to the validity of an approximation method which applied to a certain class of initial value problems for the set

$$\alpha_t + [1 + \varepsilon\,\phi(\alpha,\beta)]\alpha_x = \mu(\alpha_{xx} - \beta_{xx}),$$

$$\beta_t - [1 + \varepsilon\,\psi(\alpha,\beta)]\beta_x = \mu(\beta_{xx} - \alpha_{xx}),$$

where ε and μ are positive constants, the subscript x (or t) denotes partial differentiation with respect to x (or t) and, if $\mu = 0$, the remaining set is hyperbolic.

In [1], this has been done for a linear set of equations ($\varepsilon = 0$) by making use of the explicit solution and in [2] for a set which is, as $\mu = 0$, totally exceptional in the sense of Lax [3].

The latter equations could be transformed into the linear equations studied in [1]. This was done by means of a nonlinear transformation. In both cases, the solutions of the equations did not contain shock waves. In this paper, we shall deal with equations that do have solutions of that kind.

In the hierarchy of approximations emanating from the Navier-stokes equations, Lighthill [4] finds the set

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$$a_t + va_x + \frac{\gamma-1}{2} av_x = 0, \tag{1}$$

$$v_t + vv_x + \frac{2}{\gamma-1} aa_x = \delta v_{xx}, \tag{2}$$

where a is the sound velocity, v the flow velocity, $\gamma = C_p/C_v$ and δ is the diffusity of sound. We have

$$\delta = \frac{4}{3}\nu + \frac{\mu_v}{\rho_0} + \gamma-1 \; \frac{k}{\rho_0 C_p},$$

where ρ is the density, ν the kinematic-, μ_v the bulk viscosity and k the coefficient of heat conduction. The subscript zero refers to the undisturbed situation.

The left hand side of equations (1) and (2) are the exact forms of the equations for sound waves of finite amplitude under thermodynamically reversible conditions. They form the basis of Riemann's classic analysis [5]. The right hand side consists of a linearized approximation of the diffusion and heat conducting effects, obtained by assuming that δ and the dimensionless velocities $a-a_0/a_0$ and v/a_0 are small.

We shall not concern ourselves with the validity of the approximations Lighthill used to arrive at (1) and (2) but assume that, if δ and the dimensionless velocities are small enough, (1) and (2) describe a real physical situation.

By introducing the Riemann variables

$$\alpha = \frac{1}{2} v + \frac{a}{\gamma-1},$$

$$\beta = -\frac{1}{2} v + \frac{a}{\gamma-1},$$

we find

$$\alpha_t + \left[\frac{\gamma+1}{2}\alpha + \frac{\gamma-3}{2}\beta\right]\alpha_x = \frac{1}{2}\delta(\alpha_{xx} - \beta_{xx}), \tag{3}$$

$$\beta_t - \left[\frac{\gamma+1}{2}\beta + \frac{\gamma-3}{2}\alpha\right]\beta_x = \frac{1}{2}\delta(\beta_{xx} - \alpha_{xx}). \tag{4}$$

These equations are of the required form.

The approximation we shall consider and which is due to Lighthill, has been described extensively in [1] and [2]. Here we shall only give a brief account of the ideas behind it. The approximation applies to the class of initial conditions

$$\alpha(x,0) = f(x),$$

$$\beta(x,0) = \beta_0 .$$

If $\mu = 0$, then (4) is satisfied identically and the solution of (3) is a simple wave. Now, the approximation which, as in [1] and [2],will be called the simple wave (sw) approximation henceforth, is based on the following assumption. When μ is small but not zero, then, for some finite interval of time,β will be negligible and α will be approximately described by the solution of an equation of Burgers type:

$$\alpha_t + \left[\frac{\gamma+1}{2}\,\alpha + \frac{\gamma-3}{2}\,\beta_0\right]\alpha_x = \frac{1}{2}\,\delta\alpha_{xx}. \tag{5}$$

However, the problem is that $\beta-\beta_0$ will grow from zero and therefore, it is not clear a priori that α satisfies (5) for longer intervals of time too. In this paper we shall indicate a range of validity of this sw approximation. The equation of Burgers, which is exactly solvable ([4, 6, 7]), is often used to describe the behaviour of small amplitude shock waves. Therefore, it is especially interesting to know whether or not the sw approximation holds, in a sense yet to be defined, in an interval of time larger than that necessary for a shock wave to develop.

In sections 2 and 3 some mathematical notation needed and the definition of what we shall call a good sw approximation is given. The method we shall follow to deal with the problem is explained in section 4. It is based on a priori estimates. In section 5 local a priori estimates are constructed. From these estimates, we obtain an upper bound for the range of values of t, for which the sw approximation holds. That upper bound is "always" smaller than T_{crit}, the time at which a shock wave starts to develop. Partly this is due to the method followed. In section 6, using global a priori estimates for Burgers' equation, we shall deal with the question whether or not it will be possible to improve the results found in section 5, in this way.

2. MATHEMATICAL NOTATIONS.

R is the interval of the real numbers.

T and N are positive numbers.

Q_T^N is the rectangular domain of points x,t satisfying $0 \le t \le T$, $|x| \le N$.

As a rule the index N will be omitted. If $N = \infty$, Q_T is denoted by H_T.

Γ_T is that part of the boundary of Q_T consisting of the line segments $t = 0$, $x = -N$ and $x = N$.

$L_2(R)$ is the Hilbert-space consisting of all real square (Lebesque) integrable functions. The inner product (,) and norm are defined by

$$(u,v) = \int_{-\infty}^{\infty} u(x)v(x)dx, \quad ||u|| = (u,u)^{\frac{1}{2}}.$$

$W_2^n(R)$ is the Hilbert-space consisting of all elements of $L_2(R)$ having generalized derivatives up to order n inclusively, that are square integrable on R. The inner product $(,)_n$ and norm $|| \ ||_n$ are defined by

$$(u,v)_n = \sum_{i=1}^{n} (D^i u, D^i v) + (u,v), \quad ||u||_n = (u,u)_n^{\frac{1}{2}},$$

where $D^i u$ is the generalized derivative of order i.

Introduce the following distance in Q_T:

$$d(P_1, P_2) = (|x' - x''|^2 + |t' - t''|)^{\frac{1}{2}},$$

where $P_1 = (t', x')$ and $P_2 = (t'', x'')$.

Let

$$|u|_0 = \sup_{Q_T}|u|, \ |u|_\alpha = |u|_0 + \sup_{Q_T} \frac{|u(P_1) - u(P_2)|}{d(P_1, P_2)^\alpha}, \quad 0 < \alpha < 1,$$

$$|u|_{1+\alpha} = |u|_\alpha + |u_x|_\alpha,$$

$$|u|_{2+\alpha} = |u|_{1+\alpha} + |u_t|_\alpha + |u_x|_{1+\alpha}.$$

$C^{2+\nu}(Q_T)$ is the Banach-space consisting of all functions u on Q_T for which $|u|_{2+\alpha} < \infty$. The norm is defined by $|u|_{2+\alpha}$ (cf. Friedman [8]).

Consider in H_T a quasi-linear system of the form

$$\underline{Lu} \equiv \underline{u}_t - A\underline{u}_{xx} + B(\underline{u})\underline{u}_x = \underline{0},$$

in which $\underline{u}(x,t) = (u_1(x,t),\ldots, u_n(x,t))$ is an unknown vector function, A is a constant, nonnegative $n \times n$-matrix and B a $n \times n$-matrix of which the elements depend on $\underline{u}$.

Definition.

In H_T, a classical solution of the Cauchy problem

$$\underline{Lu} = \underline{0}\ , \tag{1}$$

$$\underline{u}(x,0) = \underline{f}(x), \tag{2}$$

is a solution that is continuous in H_T, that has continuous derivatives $\underline{u}_t$, $\underline{u}_x$ and $\underline{u}_{xx}$ and satisfies (1) at all interior points of H_T, that remains bounded as $|x| \to \infty$ and for which (2) is valid (cf. [11]).

Finally we state two lemmas that will be used in the sequel.

LEMMA 1.

Let $u \in W_2^n(R)$, then $D^j u \to 0$ as $|x| \to \infty$, where $j = 0,1,\ldots, n-1$.

A proof may be found in Smirnow [9], p.486. From Peletier and Wessels [10], we infer

LEMMA 2.

Let $u \in W_2^1(R)$, then a continuous function $\tilde{u}$ exists with $\tilde{u} = u$ a.e. and

$$\sup_{x\in R} |\tilde{u}(x)| \le \tfrac{1}{2}\sqrt{2}\ ||u||_1 .$$

The lemma is known as Sobolev's first embedding theorem.

3. THE DEFINITION OF A GOOD SW APPROXIMATION.

First, we shall write equations (1.3) and (1.4) in dimensionless form. Assume $3 \ge \gamma > 1$ (Air $\gamma \approx 1,4$). Introduce

$$x = Lx', \quad t = La_0^{-1}t', \qquad \delta = 2a_0L\mu,$$

$$\alpha = \frac{a_0}{\gamma-1} + \frac{2\varepsilon a_0}{\gamma+1}\alpha' \quad , \qquad \beta = \frac{a_0}{\gamma-1} + \frac{2\varepsilon a_0}{\gamma+1}\beta',$$

where L is some reference length connected with $\alpha'(x',0)$, ε a dimensionless measure for the strength of the wave. ε and L will be chosen such that the absolute maximum of $\alpha'(x',t) + \beta'(x',t')$ is not larger than, and of $\frac{d\alpha'}{dx'}(x',0)$ is equal to one. It is not a priori clear that this is possible for all $t \geq 0$. However, it will turn out to be possible for the range of t-values we are interested in.

Performing the indicated substitutions in (1.3) and (1.4) and dropping the accents, we obtain

$$\alpha_t + [1 + \varepsilon\alpha + \varepsilon\Theta\beta]\alpha_x = \mu(\alpha_{xx} - \beta_{xx}), \tag{1}$$

$$\beta_t - [1 + \varepsilon\beta + \varepsilon\Theta\alpha]\beta_x = \mu(\beta_{xx} - \alpha_{xx}), \tag{2}$$

where $\Theta = \frac{\gamma-3}{\gamma+1}$ $(-1 < \Theta \leq 0)$.

The initial conditions become

$$\alpha(x,0) = f(x), \tag{3}$$

$$\beta(x,0) = 0 . \tag{4}$$

The sw approximation is given by the solution α_0 (here and in the following, the subscript zero no longer denotes the undisturbed value of a quantity) of

$$\alpha_{0t} + [1 + \varepsilon\alpha_0]\alpha_{0x} = \mu\alpha_{0xx}, \tag{5}$$

$$\alpha_0(x,0) = f(x). \tag{6}$$

Definition.

For solutions α and α_0, both belonging to $L_2(R)$ (these are the only ones we consider here), α_0 will be called a good sw approximation of α in the interval of time $[0,T]$ if, for all $t \in [0,T]$:

$$||\alpha - \alpha_0|| \leq \delta||f|| \qquad (0 < \delta < 1). \tag{7}$$

δ is a measure for the deviation of α_0 from α. Let T_m be the largest value of T for which (7) still holds. Our problem will be to find an estimate for T_m in terms of ε, μ, Θ and the initial condition f(x).
Finally we remark that this definition of a good sw approximation is weaker than that used in [1] and [2]. There, (7) has been replaced by $||\alpha - \alpha_0|| << ||\alpha||$.

4. METHOD OF SOLUTION.

From now on, speaking about α, β and α_0, we shall mean the classical solution of (3.1),..., (3.4), respectively (3.5) and (3.6).
Assume that, for all $t \in [0,T]$ (in this and the next sections we assume $T \geq T_m$), α, β and α_0 belong to $W_2^2(R)$. Then according to (3.1), (3.2), (3.5) and lemma 2, for all $t \in [0,T]$, α_t and α_{0t} are in $L_2(R)$ too. Subtracting (3.5) from (3.1), multiplying the resulting equation by $\alpha - \alpha_0$ and integrating with respect to x over the entire x-axis, we find:

$$\frac{d}{dt}||\alpha - \alpha_0||^2 + \int_{-\infty}^{\infty} \frac{\partial}{\partial x}(\alpha - \alpha_0)^2 dx + 2\varepsilon \int_{-\infty}^{\infty} (\alpha - \alpha_0)(\alpha\alpha_x - \alpha_0\alpha_{0x})dx +$$

$$+ 2\varepsilon\Theta \int_{-\infty}^{\infty} (\alpha - \alpha_0)\beta\alpha_x dx = 2\mu \int_{-\infty}^{\infty} (\alpha - \alpha_0)(\alpha - \alpha_0 - \beta)_{xx} dx.$$

The interchangement of differentiation with respect to t and integration with respect to x was allowed as, for all $t \in [0,T]$, $\alpha - \alpha_0$ and $\alpha_t - \alpha_{0t}$ belong to $L_2(R)$ and depend continuously on t. For all $t \in [0,T]$, $\alpha - \alpha_0 \in W_2^2(R)$ so, $\alpha - \alpha_0$ tends to zero as $|x| \to \infty$. Therefore the second integral in the left hand side vanishes. Furthermore, as $(\alpha - \alpha_0)(\alpha\alpha_x - \alpha_0\alpha_{0x}) = \frac{1}{3}\frac{\partial}{\partial x}(\alpha - \alpha_0)^3 + (\alpha - \alpha_0)^2\alpha_{0x} + \frac{1}{2}\alpha_0\frac{\partial}{\partial x}(\alpha - \alpha_0)^2$ and, due to lemma 2 and the continuity in t, $\sup_{x,\tau \in H_t} |\alpha_{0x}(x,\tau)| < \infty$, we have for $t \in [0,T]$:

$$\int_{-\infty}^{\infty} (\alpha - \alpha_0)(\alpha\alpha_x - \alpha_0\alpha_{0x})dx = \frac{1}{2}\int_{-\infty}^{\infty} \alpha_{0x}(\alpha - \alpha_0)^2 dx \leq \frac{1}{2} R_0||\alpha - \alpha_0||^2,$$

where

$$R_0(t) = \sup_{x,\tau \in H_t} |\alpha_{0x}(x,\tau)|.$$

Using Schwarz's inequality we find

$$\int_{-\infty}^{\infty} (\alpha - \alpha_0)\beta\alpha_x dx \leq R||\beta|| \; ||\alpha - \alpha_0||,$$

where

$$R(t) = \sup_{x,\tau \in H_t} \max[|\alpha_x(x,\tau)|, \; |\beta_x(x,\tau)|].$$

Now, we find, for all $t \in [0,T]$:

$$\frac{d}{dt}||\alpha - \alpha_0|| - \frac{1}{2}\varepsilon R_0||\alpha - \alpha_0|| \leq \mu||(\alpha - \beta)_{xx}|| + \mu||\alpha_{0xx}|| - \varepsilon\Theta R||\beta||.$$

Multiplication of this inequality by $\exp[-\frac{1}{2}\varepsilon \int_0^t R_0(\tau)d\tau]$, integration with respect to t and use of: $R_0(t) \leq R_0(t')$ when $t \leq t'$, gives

$$||(\alpha - \alpha_0)(t)|| \leq \mu \int_0^t [||(\alpha_{xx} - \beta_{xx})(\tau)|| + ||\alpha_{0xx}(\tau)|| - \\ -\mu^{-1}\varepsilon\Theta R(\tau)||\beta(\tau)||]e^{\frac{1}{2}\varepsilon R_0(t)(t - \tau)}d\tau \;, \tag{1}$$

where $0 \leq t \leq T$.

By now, our problem is reduced to finding estimates for R, R_0, $||\beta||$, $||(\alpha - \beta)_{xx}||$ and $||\alpha_{0xx}||$.

5. THE RANGE OF VALIDITY OF THE SW APPROXIMATION.

As we already noticed in the introduction, it will be quite interesting to compare T_m with T_{crit}, the time a shock wave starts to develop. It will turn out that T_{crit} for the solution of (3.1),..., (3.4) as well as for the solution of (3.5) and (3.6) is the same. It is defined as the smallest time at which the solution of the hyperbolic equation(s), obtained by putting $\mu = 0$ in (3.5) ((3.1) and (3.2)), has (have) a vertical tangent. For $\mu = 0$, we infer that

$$\alpha = \alpha_0 = f(x - t - \varepsilon\alpha t), \quad \beta = 0.$$

Therefore

$$T_{crit} = \frac{1}{\varepsilon \sup\limits_{x\in R}[-f'(x)]},$$

where the accent denotes differentiation with respect to x.
If f(x) contains a compressive phase, i.e. $\sup\limits_{x\in R}[-f'(x)] > 0$, then T_{crit} is finite.
The main object of this section is the derivation of an expression for T_m in terms of ε, μ, Θ and f(x). We shall use local a priori estimates which also hold for $\mu = 0$. Therefore, those estimates in which derivatives are involved, probably will not hold for times exceeding T_{crit} and with this method, we expect to find $T_m < T_{crit}$. However, the method is of interest as long as ε is so small that the sw approximation breaks down before $t = T_{crit}$.
Let us introduce some additonial notations: $\alpha_x = r$, $\beta_x = s$, $\alpha_{xx} = p$, $\beta_{xx} = q$, $\alpha_{0x} = r_0$ and $\alpha_{0xx} = p_0$. We shall assume that r, s satisfy the ones-, p, q the twice-, with respect to x, differentiated partial differential equations and initial conditions (3.1),..., (3.4), in the classical sense. Furthermore, let r_0 satisfy the ones-, p_0 the twice-, with respect to x, differentiated equations (3.5) and (3.6), in that sense.

THEOREM 1.
Let, for all $t \in [0,T]$, α, β and α_0 belong to $W_2^4(R)$ and $\gamma \geq 0$. Then, for

$$0 \leq t \leq \frac{2\gamma}{5\varepsilon(1 - \Theta)\Lambda(f, \gamma)}, \tag{1}$$

where, for shortness,

$$\Lambda(f, \gamma) = ||f''||\exp(\gamma) + ||f'||\exp(\tfrac{1}{5}\gamma),$$

the following estimates hold

$$||p(t)||^2 + ||q(t)||^2 \leq ||f''||^2\exp(2\gamma),$$

$$||r(t)||^2 + ||s(t)||^2 \leq ||f'||^2\exp(2\gamma/5),$$

$$||\alpha(t)||^2 + ||\beta(t)||^2 \leq ||f||^2\exp[-\tfrac{4}{5}\,\Theta\gamma/(1 - \Theta)],$$

$$R(t) \leq ||f''||\exp(\gamma) + ||f'||\exp(\gamma/5), \tag{2}$$

$$||p_0(t)|| \le ||f''||\exp(\gamma),$$

$$R_0(t) \le ||f''||\exp(\gamma) + ||f'||\exp(\gamma/5).$$

Proof

From (3.1), (3.2) and lemma 2, we easily infer that, for all $t \in [0,T]$, $\alpha_t \in L_2(R)$ and $\beta_t \in L_2(R)$. Upon multiplying (3.1) by α, (3.2) by β, integrating with respect to x from $-\infty$ to ∞ and adding the resulting equations we find:

$$\frac{d}{dt}(||\alpha||^2 + ||\beta||^2) + 2\varepsilon\Theta \int_{-\infty}^{\infty} \alpha\beta(r - s)dx = -2\mu \int_{-\infty}^{\infty} (r - s)^2 dx.$$

The interchangement of differentiation with respect to t and integration with respect to x was allowed as α, β, α_t and β_t belong to $L_2(R)$ and depend continuously on t. Using Cauchy's inequality, we obtain

$$||\alpha(t)||^2 + ||\beta(t)||^2 \le ||f||^2\exp[-2\varepsilon R(t)\Theta t]. \tag{3}$$

In a similar way it is seen that:

$$\frac{d}{dt}(||r||^2 + ||s||^2) + \varepsilon \int_{-\infty}^{\infty} [(r + \Theta s)r^2 - (s + \Theta r)s^2]dx = -2\mu \int_{-\infty}^{\infty} [p - q]^2 dx ,$$

and

$$\frac{d}{dt}(||p||^2 + ||q||^2) + \varepsilon \int_{-\infty}^{\infty} (5r + 3\Theta s)p^2 dx - \varepsilon \int_{-\infty}^{\infty} (5s + 3\Theta r)q^2 dx +$$

$$+ 2\varepsilon\Theta \int_{-\infty}^{\infty} (r - s)pq dx = -2\mu \int_{-\infty}^{\infty} [p_x - q_x]^2 dx.$$

From these relations, we infer:

$$||r(t)||^2 + ||s(t)||^2 \le ||f'||^2\exp[\varepsilon R(t)(1 - \Theta)t], \tag{4}$$

$$||p(t)||^2 + ||q(t)||^2 \le ||f''||^2\exp[5\varepsilon R(t)(1 - \Theta)t]. \tag{5}$$

According to lemma 2 and $\left[\sup_{x\in R} \max(|r|,|s|)\right]^2 \le 2(\sup_{x\in R}|r|)^2 + 2(\sup_{x\in R}|s|)^2$,

we get

$$\sup_{x\in R} \max(|r|,|s|) \le (||r||^2 + ||s||^2)^{\frac{1}{2}} + (||p||^2 + ||q||^2)^{\frac{1}{2}},$$

or combining with (4) and (5)

$$R(t) \le ||f'|| \exp[\tfrac{1}{2}\, \varepsilon R(t)(1 - \Theta)t] + ||f''|| \exp[\tfrac{5}{2}\, \varepsilon R(t)(1 - \Theta)t]. \qquad (6)$$

We have

$$e^{\gamma x} \le 1 + (e^{\gamma} - 1)x \qquad (\gamma > 0), \qquad (7)$$

holding for $0 \le x \le 1$.
Assume that

$$\frac{5}{2}\, \frac{\varepsilon R(t)(1 - \Theta)t}{\gamma} \le 1, \qquad (8)$$

then we infer from (6) and (7)

$$R(t) \le \frac{\gamma(||f'|| + ||f''||)}{\gamma - \frac{5}{2}\, \varepsilon t(1 - \Theta)\{\Lambda(f\,,\gamma) - ||f'|| - ||f''||\}}\,. \qquad (9)$$

Assumption (8) must be satisfied. This implies that (1) must hold. Using (1), we find from (9) the inequality (2). From (8), (3), (4) and (5) and the remark that all estimates already found also hold for $\mu = \Theta = 0$, the remaining part of the theorem follows.

REMARK.
Instead of (7) we could have used: $e^x \le (1 - x)^{-1}$ for $x \in [0,1]$. However, this leads to a quite complicated algebraic equation of order three.

THEOREM 2.
Let α, β and α_0 be defined as in the preceding theorem. If

$$T_m = \max_{\gamma\ge 0} \min[T_0(\gamma)/\varepsilon, \delta T_1(\gamma,\varepsilon)/\mu],$$

where

$$T_0(\gamma) = \frac{2\gamma}{5(1 - \Theta)\Lambda(f\,,\gamma)}\,,$$

$$T_1(\gamma,\ \varepsilon) = \frac{\mu||f||\exp(-\frac{1}{5}\gamma)}{\mu(1+\sqrt{2})||f''||\exp(\gamma) - \varepsilon\Theta||f||\Lambda(f\ ,\gamma)\exp[\frac{2}{5}\Theta\gamma/(\Theta-1)]}$$

then the sw approximation may be called good.

Proof.

As is easily seen, using theorem 1, for $t \le \min[T_0/\varepsilon,\ \delta T_1/\mu]$, (4.1) holds. As $\gamma \ge 0$ is still arbitrary, we may choose this number such that $\min[T_0/\varepsilon$, $\delta T_1/\mu]$ assumes its maximum for some given ε, μ, Θ and f, thus proving the theorem.

Corollary.

We have

$$\frac{T_0(\gamma)}{\varepsilon} \le \frac{1}{\varepsilon}\max_{\gamma\ge 0}\frac{2\gamma}{5(1-\Theta)(||f'|| + ||f''||)\exp(\frac{1}{5}\gamma)},$$

and so, using the triangle inequality, lemma 2 and $\sup_{x\in R}|f'(x)| \ge \sup_{x\in R}[-f'(x)]$, we obtain $T_m \le \frac{\sqrt{2}}{e}T_{crit}$. It is thus seen that $T_m < T_{crit}$ indeed.

Let ε_0 be a special value of ε for which $1+\frac{1}{2}\varepsilon(1+\Theta)(\alpha+\beta)\ge\delta>0$ holds in H_T. Then the sound velocity a is real positive which is a necessary physical condition. As we assumed $\sup_{x,t\in H_T}|\alpha+\beta| \le 1$ (section 3), we may choose $\varepsilon_0 = \frac{2-2\delta}{1+\Theta}$. Let $T_0(\gamma)$ assume its maximum for $\gamma = \gamma_m$. As may be easily verified $\gamma_m = \gamma_m(||f'||/||f''||)$ and $1 \le \gamma_m \le 5$. Define ε_1 by

$$\frac{T_0(\gamma_m)}{\varepsilon_1} = \frac{\delta T_1(\gamma_m\ ,\ \varepsilon_1)}{\mu}\ .$$

ε_1 may be infinite and even negative. Let γ_1 and γ_2 satisfy

$$\frac{T_0(\gamma)}{\varepsilon} = \frac{\delta T_1(\gamma\ ,\ \varepsilon)}{\mu}$$

and put $\gamma_3 = \min(\gamma_1\ ,\gamma_2)$. If $\varepsilon_1 \ge 0$, then define $\varepsilon_2 = \min(\varepsilon_0\ ,\ \varepsilon_1)$, else $\varepsilon_2 = \varepsilon_0$.

THEOREM 3.

Let α, β and α_0 be defined as in theorem 1. The sw approximation is good

(i) for $0 \le \varepsilon \le \varepsilon_2$ if $T_m = \delta T_1(\gamma_3, \varepsilon)/\mu$,

(ii) for $\varepsilon_2 < \varepsilon \le \varepsilon_0$ (assuming this interval is not empty) if $T_m = T_0(\gamma_m)/\varepsilon$.

If $\delta||f||\ ||f'||_1 / ||f''|| \ll 1$ is satisfied, we have $\varepsilon_2 \gg \mu$.*

Proof.

Let $0 \le \varepsilon \le \varepsilon_2$, then $T_0(\gamma)/\varepsilon$ and $\delta T_1(\gamma, \varepsilon)/\mu$, considered as functions of γ, have two points of intersection and

$$\min[T_0/\varepsilon,\ \delta T_1/\mu] = \begin{cases} T_0(\gamma)/\varepsilon & (0 \le \gamma \le \gamma_3), \\ \delta T_1(\gamma, \varepsilon)/\mu & (\gamma_3 \le \gamma \le \max[\gamma_1, \gamma_2]), \\ T_0(\gamma)/\varepsilon & (\gamma \ge \max[\gamma_1, \gamma_2]). \end{cases}$$

For $0 \le \gamma \le \gamma_m$, $T_0(\gamma)$ is a monotonically increasing-, for $\gamma \ge \gamma_m$ a monotonically decreasing function. As $T_1(\gamma, \varepsilon)$ is monotonically decreasing with respect to γ, (i) follows.

Let $\varepsilon_2 < \varepsilon \le \varepsilon_0$ and assume this interval is not empty. Then $T_0(\gamma)/\varepsilon$ and $\delta T_1(\gamma, \varepsilon)/\mu$ do not intersect and for all $\gamma \ge 0$, $T_0/\varepsilon < \delta T_1/\mu$.

This proves (ii).

Finally, using $1 \le \gamma_m \le 5$ and $-1 < \Theta \le 0$, we obtain for $\varepsilon_1 \in [0,\varepsilon_0]$:

$$\varepsilon_2 \ge \frac{\mu||f''||}{5\delta||f||\ ||f'||_1},$$

from which the remaining part of the theorem immediately follows.

For $\varepsilon_2 < \varepsilon \le \varepsilon_0$, the upper bound T_m, [see (ii)], is essentially due to the method followed (see also the corollary after theorem 2). If $0 \le \varepsilon \le \varepsilon_2$ then that bound results from the coupling between the α- and β-mode and T_m is a monotonically decreasing function of ε and μ. This agrees with what we expected from a physical point of view.

In case the equations are linear, i.e. $\varepsilon = 0$, we find:

$$T_m = \frac{\delta||f||}{(1+\sqrt{2})\mu||f''||}.$$

*According to the definition of ε (section 3), we can always choose $\varepsilon \le \varepsilon_0$.

Furthermore, when the ratio $||f''|| \ / \ ||f||$ decreases and ε is small enough, T_m increases. This happens when we start at $t = 0$ with a wave packet with a larger dominant wave length.
If $f(x)$ satisfies $\delta||f|| \ ||f'||_1 \ / \ ||f''|| << 1$, a condition which could be expressed by: "f(x) should not vary too slowly", then $\varepsilon_2 >> \mu$ and the method followed is useful for a large range of ε-values.
One may ask whether it is possible to improve the results found by using other types of estimates. We remark that we have not taken advantage of the dissipative terms in equations (3.1), (3.2) and (3.5). Therefore, especially when $\varepsilon \in [\varepsilon_2,\varepsilon_0]$, it may be even possible to prove that the sw approximation holds for times exceeding T_{crit}.
In the next section, some global a priori estimates have been constructed for the solution of Cauchy's problem for the Burgers equation. Unfortunately, this has not been possible for the solution of (3.1),..., (3.4). However, with help of the global estimates found, we may estimate the second term in (4.1). This gives at least some indication whether or not, in this way it will be possible to improve the results found in this section.

6. A PRIORI ESTIMATES FOR BURGERS' EQUATION.

First, we shall study the mixed problem for the equation of Burgers:

$$\alpha_{0t} + [1 + \varepsilon\alpha_0]\alpha_{0x} = \mu\alpha_{0xx},$$

$$\alpha_0(x,0) = \chi_N(x)f(x) \qquad (|x| \le N),$$

$$\alpha_0(N,t) = \alpha_0(-N,t) = 0 \qquad (0 \le t \le T),$$

where $\chi_N(x)$ is a sufficiently smooth function such that $0 \le \chi_N \le 1$, $\chi_N = 1$ for $|x| \le N - \sqrt{N} - 1$, $\chi_N = 0$ for $|x| \ge N$, $\chi_N'(\pm N) = \chi_N''(\pm N) = 0$ and the derivatives of χ_N are uniformly bounded with respect to N. f(x) is defined as in the preceding sections.
Extend the definition of f(x) to Q_T by putting $f(x,t) = f(x)$ for all $x,t \in Q_T$. Let $f(x,t) \in C^{2+\nu}(Q_T)$, then, by a suitable choice of χ_N, $(\chi_N f)(x,t) \in C^{2+\nu}(Q_T)$ and the compatibility condition $([1 + \varepsilon\alpha_0(x,0)]\alpha_{0x}(x,0) = \mu\alpha_{0xx}(x,0))$ at $x = \pm N$ is satisfied. So according to Oleinik and Kruzhkov [12], a unique solution $\alpha_0(x,t) \in C^{2+\nu}(Q_T)$ exists for all $T > 0$. In the following, we shall denote this solution by α_0^N.

As a direct consequence of the generalized maximum principle for parabolic equations (see ch. I., section 2 of [11]), we have, for all T > 0:

$$\sup_{x,t\in Q_T} |\alpha_0^N(x,t)| \le \sup_{|x|\le N} |f(x)|.$$

Next, we state:

LEMMA 3.
For all T > 0:

$$\sup_{x,t\in Q_T} |\alpha_{0x}^N(x,t)| \le \max\{\frac{9}{2}\, e^2\, \frac{\varepsilon}{\mu}[\sup_{|x|\le N} |f|]^2,\ \sup_{|x|\le N} [(\chi_N f)']\}\cdot$$

The proof may be given by using the method of auxiliary functions due to Bernshtein (cf. [12]). It is postponed to the appendix.
Now, we return to the Cauchy problem.

THEOREM 4.
The solution $\alpha_0(x,t)$ of (3.5) and (3.6) where $f \in C^{2+\nu}(H_T)$, for all T > 0, exists, is unique and belongs to $C^{2+\nu}(H_T)$. Furthermore, for all T > 0:

$$\sup_{x,t\in H_T} |\alpha_0(x,t)| \le \sup_{x\in R}|f(x)| \le 1,$$

$$\sup_{x,t\in H_T} |\alpha_{0x}(x,t)| \le \max\{\sup_{x\in R}|f'(x)|,\ \frac{9}{2}\, e^2\, \frac{\varepsilon}{\mu}[\sup_{x\in R}|f(x)|]^2\} \le \max[1,\ \frac{9}{2}\, e^2\, \frac{\varepsilon}{\mu}]. \quad (1)$$

Proof.
The existence and uniqueness follow immediately from theorem 8.1, p.495 of [11]. Let us consider α_0^N for $N > N_0$ in a fixed cylinder $Q_T^{N_0}$. Then in [11] it is show that a subsequence $\{\alpha_0^{N_k}\}$ exists that converges together with the derivatives $\alpha_{0x}^{N_k}$, $\alpha_{0xx}^{N_k}$ and $\alpha_{0t}^{N_k}$ to the solution α_0 of the Cauchy problem (3.5) and (3.6) and the corresponding derivatives in any fixed $Q_T^{N_0}$. Now, choose $\chi_N(x)$ such that $|\chi_N'(x)| \le c/(\sqrt{N} + 1)$ (c a positive constant). Then, for any positive numbers ε and ε', a number $N_1(\varepsilon,\varepsilon')$ can be found such that for $N_k > N_1(\varepsilon,\varepsilon')$

and $$\sup_{|x|\le N_k} |(f\chi_{N_k})'| \le \sup_{x\in R}|f'| + \varepsilon,$$

$$\sup_{Q_T^{N_o}}|\alpha_{0x}| \le \sup_{Q_T^{N_o}}|\alpha_{0x}^{N_k}| + \varepsilon' \le \sup_{Q_T^{N_k}}|\alpha_{0x}^{N_k}| + \varepsilon' \le \varepsilon + \varepsilon' + $$
$$+ \max\{\sup_{x\in R}|f'(x)|, \frac{9}{2} e^2 \frac{\varepsilon}{\mu}[\sup_{x\in R}|f(x)|]^2\}.$$

As ε, ε' and N_0 are arbitrary we immediately deduce (1). The proof of the remaining inequality runs along similar lines.

REMARK.
To show that the estimate obtained for $|\alpha_{0x}|$ is quite accurate, we remark that the front of a shock wave solution of Burgers' equation, when fully developed, is approximately described by a steady state solution of that equation (Murray [13], Lighthill [4]). That is by

$$\alpha_0 = \frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_1 - a_2)\tanh\{\frac{\varepsilon(a_1 - a_2)[x - t - \frac{1}{2}\varepsilon(a_1 + a_2)t]}{4\mu}\},$$

where a_1 (a_2) is the value of α_0 immediately behind (before) the front of the shock wave. Differentiation of this expression with respect to x shows that the result obtained has the same order of magnitude as the second term between curly brackets in (1).
Finally, we prove

THEOREM 5.
Let $f \in C^{2+\nu}(H_T)$, p_0, defined as in section 5, satisfy the twice with respect to x differentiated equations (3.5) and (3.6) in the classical sense and let, for all $t \in [0,T]$, α_0 belong to $W_2^4(R)$. Then, for $t \in [0,T]$:

$$||p_0(t)|| \le ||f''||\exp(\frac{25}{4}\frac{\varepsilon^2 t}{\mu}).$$

Proof.
According to the preceding theorem $\alpha_0 \in C^{2+\nu}(H_T)$ for every $T > 0$. Differentiate (3.5) with respect to x twice. For all $t \in [0,T]$, $P_{0t} \in L_2(R)$ as may be

seen from the resulting equation easily. Multiply that equation by p_0 and integrate over the entire x-axis. Then, using lemma 1, partial integration with respect to x and interchanging differentiation with respect to t and integration (cf. theorem 1), we find:

$$\frac{d}{dt}||p_0||^2 - 10\varepsilon \int_{-\infty}^{\infty} \alpha_0 p_0 p_{0x} dx = -2\mu \int_{-\infty}^{\infty} p_{0x}{}^2 dx.$$

According to Cauchy's inequality

$$\int_{-\infty}^{\infty} \alpha_0 p_0 p_{0x} dx \le \{\sup_{x\in R}|\alpha|\}\{\frac{\nu}{2}||p_0||^2 + \frac{1}{2\nu}||p_{0x}||^2\} \qquad (\nu > 0).$$

Then, using theorem 4, we obtain

$$\frac{d}{dt}||p_0(t)||^2 - 5\varepsilon\nu||p_0(t)||^2 \le (-2\mu + \frac{5\varepsilon}{\nu})||p_{0x}(t)||^2.$$

Choosing $\nu = 5\varepsilon/2\mu$, we deduce the theorem.

Using the last two theorems, we infer from (4.1) that

$$||(\alpha - \alpha_0)(t)|| \le \int_0^t \mu[||(\alpha_{xx} - \beta_{xx})(\tau)|| - \varepsilon\Theta R(\tau)||\beta(\tau)||]e^{\frac{1}{2}\varepsilon R_0(t)(t-\tau)} d\tau +$$

$$+ \{\mu t||f''||\exp[\frac{1}{2}\varepsilon R_0(t)t + \frac{25}{4}\frac{\varepsilon^2 t}{\mu}]\}.$$

Now, for the terms between curly brackets to be smaller than $\delta||f||$, it is necessary that

$$t \le \min\{\frac{\delta||f||}{\mu||f''||}, \quad [\frac{\delta||f||}{||f''||(\frac{1}{2}\varepsilon\mu R_0(t) + \frac{25}{4}\varepsilon^2)}]^{\frac{1}{2}}\}. \qquad (2)$$

Assume that $\delta||f||\ ||f'||_1 / ||f''|| \ll 1$. Then $\varepsilon_2 \gg \mu$ and according to the former section we do not expect T_m to be larger than T_{crit}. This is confirmed by the method followed in this section as is easily seen from (2). However, if $\delta||f||\ ||f'||_1 / ||f''|| \ll 1$ is violated, it does seem possible that $T_m \ge T_{crit}$ for some initial condition. Therefore, I think future investigations should be concerned with a priori estimates for the set of nonlinear equations. This is not easy, for the nonlinear set (3.1) and (3.2) is not purely parabolic. It might be termed a mixed parabolic-hyperbolic set.

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APPENDIX.

We shall give a proof of lemma 3.

First, we prove that for all $T \geq 0$:

$$\max_{\Gamma_T} |\alpha_{0x}^N(x,t)| \leq \max_{|x|\leq N} |(f(x)\chi_N(x))'|. \tag{1}$$

Define

$$v = \alpha_0^N + M \exp[-K(x + N)], \tag{2}$$

where $K \geq k > 0$, $M \geq m > 0$ but further, as yet, arbitrary. Substitution of (2) in (3.5) and putting $K = 2\mu^{-1}(1 + \varepsilon \max_{|x|\leq N} |f|)$, we obtain: $-v_t - v_x - \varepsilon\alpha_0^N v_x + \mu v_{xx} > 0$. Therefore v cannot assume a positive maximum in Q_T/Γ_T. Since $v(t,x)$ assumes its greatest value for $x = -N$, then, $v_x \leq 0$ for $x = -N$ and therefore $\alpha_{0x}^N\big|_{x=-N} \leq MK$. By considering $\alpha_0^N - M\exp[-K(x + N)]$ we find similarly that $\alpha_{0x}^N\big|_{x=-N} \geq -MK$. Thus, we have an estimate for $|\alpha_{0x}^N|$ at $x = -N$ and similarly for $x = N$. As M is arbitrary, we find (1).

Next, we prove the remaining part of the lemma. Substitute the unknown function $\alpha_0^N = \phi(v)$, $\phi'(v) \geq \phi_0 > 0$ in (3.5). Then, we obtain

$$v_t + [1 + \varepsilon\phi]v_x - \mu v_{xx} - \mu \frac{\phi''}{\phi'} v_x^2 = 0.$$

Differentiation of this equation with respect to x is allowed according to theorem 9 of Oleinik and Kruzhkov [12]. Therefore, the function $p = v_x$ satisfies

$$p_t = -(1 + \varepsilon\phi)p_x - \varepsilon\phi' p^2 + \mu p_{xx} + 2\mu \frac{\phi''}{\phi'} pp_x + \mu\left(\frac{\phi''}{\phi'}\right)' p^3.$$

At a maximum of $|p|$ in Q_T/Γ_T we have $p_x = 0$, $-pp_{xx} \geq 0$ and $pp_t \geq 0$. So we find

$$0 \leq -\varepsilon\phi' p^3 + \mu\left(\frac{\phi''}{\phi'}\right)' p^4 \tag{3}$$

Now, choose

$$\phi(v) = -2M + 3eM \int_0^v \exp(-s^m)ds \qquad (m > 0), \tag{4}$$

where $M = \max_{|x|\le N} |f|$.

If α_0^N varies in the interval $[-M,M]$, v varies over a finite interval $[v_1,v_2]$. Since

$$\int_0^{v_1} e^{-s^m} ds = \frac{1}{3e} > \int_0^{1/3e} e^{-s^m} ds, \quad \int_0^{v_2} e^{-s^m} ds = \frac{1}{e} < \int_0^1 e^{-s^m} ds,$$

we obtain

$$\frac{1}{3e} < v_1 < v_2 < 1. \tag{5}$$

Now, using (3), (4) and (5), it is seen that

$$|p| \le \frac{\varepsilon}{\mu} \frac{3eMe^{-v^m}}{m(m-1)v^{m-2}} .$$

The right hand side of this inequality approximately assumes its smallest value for m = 2. Putting m = 2 and using (5) once again, we find, for all $T \ge 0$

$$\max_{Q_T/\Gamma_T} |\alpha_{0x}^N| \le \frac{9}{2} e^2 \frac{\varepsilon}{\mu} M^2. \tag{6}$$

If $|\alpha_{0x}^N|$ does not assume a maximum in Q_T/Γ_T, then

$$\max_{Q_T/\Gamma_T} |\alpha_{0x}^N| \le \max_{|x|\le N} |(f(x)\chi_N(x))'|. \tag{7}$$

Combining (1), (6) and (7), we find the lemma.

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A NUMERICAL STUDY OF AN INITIAL VALUE PROBLEM FOR A SET OF DIFFUSIVE WAVE EQUATIONS

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1. INTRODUCTION

In this report we study the set of partial differential equations

$$r_t + r_x = \mu(r_{xx} - s_{xx}), \tag{1}$$

$$s_t - s_x = \mu(s_{xx} - r_{xx}), \tag{2}$$

subject to the initial conditions

$$r(x,0) = f(x),$$

$$s(x,0) = 0,$$

where x runs through $(-\infty,\infty)$, $0 \leq t \leq T < \infty$, r and s are real functions of x and t, μ is a (small) positive constant, the subscript x (or t) denotes partial differentiation with respect to x (or t) and f(x) is a sufficiently smooth function. The reasons for our interest are, very briefly stated, the following:
If $\mu \ll 1$ it seems likely that for some small interval of time the behaviour of the r-mode can be described with sufficient accuracy in some sense by the solution $\tilde{r}$ of the so-called Burgers equation

$$\tilde{r}_t + \tilde{r}_x - \mu\tilde{r}_{xx} = 0,$$

subject to the initial condition

$$\tilde{r}(x,0) = f(x).$$

However one might wonder whether this would be true for all $t \geq 0$.
An incomplete answer to this problem was given by L.J.F. Broer and the second author [1]. It turned out that, for an interesting class of initial functions $f(x)$ given by

$$f(x) = \begin{cases} x^n \cos(k_0 x) \exp(\delta x), & x \leq 0, \\ 0, & x > 0 \end{cases}$$

($n = 1,2,\ldots$, k_0 and δ are real positive numbers), the above-mentioned solution $\tilde{r}$, as $t \to \infty$, may be used as a quite satisfactory approximation in the sense that

$$\int_{-\infty}^{\infty} |r - \tilde{r}|^2 dx \leq Kt^{-1} \int_{-\infty}^{\infty} |r|^2 dx, \qquad t \to \infty,$$

where K is a real positive constant.
However, it was not clear at all whether one might speak of an accurate approximation (in some sense) for all times $t \geq 0$. As we only knew the behaviour of r and s for small and large times this was quite a difficult problem. To get some more insight we decided to investigate the behaviour of the solutions r and s by means of a computer. This has been done for the initial-value function

$$f(x) = \begin{cases} x^6 \exp(6x), & x \leq 0, \\ 0, & x > 0. \end{cases}$$

2. THE MIXED INITIAL AND BOUNDARY VALUE PROBLEM

At first sight it seemed useful to start from the integral representation of r and s, found in [1]. However, in this attempt a number of problems were met. Both integrands are strongly oscillating functions with a "period" depending not only on x and t but also on the variable of integration, called z.
The amplitude of this oscillation varies rapidly with z, x and t. However, as from a basic point of view the use of a difference method for the set of partial differential equations seemed to be a more interesting one, we did use the latter method.

For the construction of an unconditionally convergent scheme (a precise definition will follow later on) an implicit difference scheme should be used. The latter in fact implies that the pure initial-value problem should be translated into a mixed initial-boundary-value problem, which must be chosen such that it represents in some (as yet undefined) norm the original problem sufficiently well.
Let D be the rectangular region $|x| < a < \infty$, $0 < t < T < \infty$ and C its boundary. Then consider the following initial and boundary data

$$r(x,0) = \begin{cases} g(x), & -a \le x \le -a + \varepsilon, \\ x^6 \exp(6x), & -a + \varepsilon \le x \le 0, \\ 0, & 0 < x \le a, \end{cases} \tag{1}$$

$$s(x,0) = 0, \quad |x| \le a, \tag{2}$$

$$r(-a,t) = r(a,t) = s(-a,t) = s(a,t) = 0, \quad t \ge 0 \tag{3}$$

in connection with (1.1) and (1.2) and look for the solutions r and s in D. As the problem should be well posed(c.f. Richtmeyer [3]), the function g(x) will be chosen such that r(x,0) is a twice continuously differentiable function that goes to zero together with these derivatives as $|x| \to a$. These conditions are sufficient but certainly not necessary for the well-posedness of the problem. As ε may be chosen arbitrary small and we are going to use a numerical procedure in which only a few significant digits of a result are of interest, the precise choice of g(x) is not of interest at all.
Finally the question remains to what extent the solution of this problem agrees with that of our original one. The answer is given quite easily. In the final computations we have chosen T = 20. It then turns out that a may be chosen equal to 40 because a further increase of a is of no influence on the significant digits of the numerical solution. This has been verified experimentally.

3. THE METHOD OF SOLUTION

Cover the domain D + C by a lattice of discrete points with coordinates (x_m, t_n) given by

$$x_m = -a + m\,h, \qquad m = 0,1,\dots, M+1,$$

$$t_n = n\,k, \qquad n = 0,1,\dots, N+1,$$

where $h = \frac{2a}{M+1}$, $k = \frac{T}{N+1}$ are the net spacings.

We shall introduce the notation

$$u(x_m, t_n) = u_{m,n},$$

and use the following difference approximations:

$$u_t(m,n+\tfrac{1}{2}) = \frac{u_{m,n+1} - u_{m,n}}{k} + \mathcal{O}(k^2),$$

$$u_x(m,n+\tfrac{1}{2}) = \frac{u_{m+1,n} - u_{m-1,n} + u_{m+1,n+1} - u_{m-1,n+1}}{4h} + \mathcal{O}(h^2)\ ,$$

$$u_{xx}(m,n+\tfrac{1}{2}) = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n} + u_{m+1,n+1} - 2u_{m,n+1} + u_{m-1,n+1}}{2h^2} + \mathcal{O}(h^2).$$

Using this so-called Cranck-Nicholson scheme of approximations we find for $m = 1,2,\dots,M;\ \ n = 0,1,\dots,N+1$:

$$a\,r_{m-1,n+1} + d\,r_{m,n+1} + c\,r_{m+1,n+1} + b(s_{m-1,n+1} - 2s_{m,n+1} + s_{m+1,n+1}) =$$

$$= -\{a\,r_{m-1,n} + e\,r_{m,n} + c\,r_{m+1,n} + b(s_{m-1,n} - 2s_{m,n} + s_{m+1,n})\}, \qquad (1)$$

$$b(r_{m-1,n+1} - 2r_{m,n+1} + r_{m+1,n+1}) + c\,s_{m-1,n+1} + d\,s_{m,n+1} + a\,s_{m+1,n+1} =$$

$$= -\{b(r_{m-1,n} - 2r_{m,n} + r_{m+1,n}) + c\,s_{m-1,n} + e\,s_{m,n} + a\,s_{m+1,n}\}, \qquad (2)$$

where a, b, c, d, e are defined in appendix 1.

The initial and boundary data are specified in the obvious way:

$$r_{m,0} = r(x_m,0), \; s_{m,0} = s(x_m,0) = 0, \quad m = 0,1,\ldots,M+1,$$

$$r_{0,n} = s_{0,n} = r_{M+1,n} = s_{M+1,n} = 0, \quad n = 0,1,\ldots,N+1. \qquad (3)$$

For each $t = t_n$, equations (1) and (2) form a system of 2M linear equations in 2M+4 unknowns $r_{n,m}$, $s_{n,m}$. The required additional equations are supplied by (3). So we find

$$A\underline{r}^{(n+1)} + bB\underline{s}^{(n+1)} = - [\, C\underline{r}^{(n)} + bB\underline{s}^{(n)}] , \qquad (4)$$

$$bB\underline{r}^{(n+1)} + A^T\underline{s}^{(n+1)} = - [\, bB\underline{r}^{(n)} + C^T\underline{s}^{(n)}] , \qquad (5)$$

where

$$\underline{r}^{(n)} = \text{col.}\ (r_{1,n},\ldots,r_{M,n}) \quad ,$$

$$\underline{s}^{(n)} = \text{col.}\ (s_{1,n},\ldots,s_{M,n}) \quad ,$$

and the matrices A, B and C are defined in appendix 1. A^T is the transposed of A.

Now, at first sight it seems impossible to avoid using the methods of Crout or Jacobi and Seidel (Isaacson and Keller [2], page 51) for solving (4) and (5). However, a more direct method, requiring a smaller number of operations, has been found.

Multiplication of (4) from the left by A^T and of (5) by bB and subtraction of the resulting equations gives

$$G\underline{r}^{(n+1)} = 2bD_1\underline{s}^{(n)} - \underline{p}^{(n+1)} \quad ,$$

$$\underline{p}^{(n+1)} = b(a-c)\ \text{col.}\ (s_{1,n+1},\, 0,\ldots,0,\, s_{M,n+1}) \; ,$$

where the matrices G, D, and E are defined in appendix 1.

As $s_{1,n+1}$, $s_{M,n+1}$ are very small (a has been chosen such that increasing a has practically no influence which implies that r and s go to zero very smoothly as $x \to \pm a$), we introduce only a very small error by choosing for some fixed n:

$$s_{1,n+1} = s_{1,n} \quad , \tag{6}$$

$$s_{M,n+1} = s_{M,n} \quad . \tag{7}$$

Besides it will turn out that in the neighbourhood of $|x| = a$ the numerical approximation is not very accurate anyhow (see section 5 too). Using (6) and (7) we find

$$G\underline{r}^{(n+1)} = 2bD_1\underline{s}^{(n)} - E\underline{r}^{(n)} - \underline{p}^{(n)}. \tag{8}$$

Multiplication of (4) to the left by bB and of (5) by A, followed by a subtraction of the resulting equations and an approximation similar to (6) and (7) gives

$$H\underline{s}^{(n+1)} = 2bD_2\underline{r}^{(n)} - F\underline{s}^{(n)} + \underline{q}^{(n)} \ , \tag{9}$$

$$q^{(n)} = b(a-c)\ (r_{1,n},0,\ldots,0,\ r_{M,n}),$$

where H, D_2 and F can be found in appendix 1. By using a triangular decomposition of G and H, (8) and (9) can be solved easily. The latter requires only 12M operations consisting of multiplication and division, while the operational count for the Crout method requires $\mathcal{O}(M^3)$ (c.f. [2], page 52).

4. CONSISTENCY, CONVERGENCE AND STABILITY

In this section we shall denote the solution of the difference problem by a capital letter and that of the exact problem by a lower case.
Let us represent the partial differential equations (1.1) and (1.2) and the boundary and initial data (2.1), (2.2) and (2.3) symbolically by

$$L\ \underline{u} = \underline{0} \qquad (x,t) \in D, \tag{1}$$

$$B\underline{u} = \underline{g}(x,t)\ (x,t) \in C, \tag{2}$$

where

$$\underline{u} = \text{col.}\ (r,s).$$

In a similar way the difference problem may be represented by

$$L_\Delta \underline{U} = \underline{0} \qquad (x,t) \in D, \qquad (3)$$

$$B_\Delta \underline{U} = \underline{g}(x,t) \qquad (x,t) \in C, \qquad (4)$$

where $B_\Delta = B$ and

$$k\, L_\Delta \underline{U} = \sum_{j=-1,0,1} B_j \underline{U}(x+jh,\ t+k) - C_j \underline{U}(x+jh,\ t).$$

The matrices B_j and C_j are defined in appendix 1.
For numerical work equations (3) and (4) are used only at the lattice points, but they will be taken to apply equally well to other points of the interval $|x| \le a$ such that if $\underline{U}(x,t)$ is specified for all $|x| \le a$, $\underline{U}(x,t+k)$ is determined for $|x| \le a$ by equations (3) and (4). Starting from this point of view we are able to use the Hilbert-space $L_2([-a,a])$. It contains all square-(Lebesque) integrable two-component vector-valued functions on [-a,a], with inner product (,) and norm $||\ \ ||$ defined by

$$(\underline{u},\underline{v}) = \frac{1}{2a}\int_{-a}^{a} \underline{u}^+(x)\underline{v}(x)dx; \qquad ||\underline{u}|| = (\underline{u},\underline{u})^{\frac{1}{2}},$$

where

$$\underline{u} = \text{col.}\ (u_1(x),\ u_2(x)),$$

and $\underline{u}^+$ is the hermitian transpose of $\underline{u}$.

Def. 1

Let $\phi(t,x)$ be any function with sufficiently many continuous partial derivatives in D+C. For each such function and every point $(x,t) \in D+C$, define the truncation error by

$$\tau(\phi(t,x)) = L(\phi(t,x)) - L_\Delta(\phi(x,t))$$

and for every point $(x,t) \in C$ let the truncation error be

$$\beta(\phi(t,x)) = B(\phi(t,x)) - B_\Delta(\phi(t,x)).$$

Then the difference problem (3), (4) is unconditionally consistent with problem (1), (2) iff

$$\tau(\phi) \to 0, \qquad \beta(\phi) \to 0 ,$$

when $h \to 0$, $k \to 0$ in any manner.
From a Taylor expansion we deduce that $\tau = \mathcal{O}(h^2 + k^2)$. Furthermore $\beta = 0$ and so unconditional consistency is clearly satisfied.

Def. 2

The difference scheme (3), (4) is stable iff there exists a constant K, independent of the net spacing, such that

$$||\underline{U}(t=nk)|| \le K \; ||\underline{U}(0)|| , \qquad n = 0,1,\ldots,N+1,$$

for any $\underline{U}(x,0) \in L_2([-a,a])$.

To prove stability we shall proceed in the following way. As for all $0 \le t \le T$ the solution $\underline{U}(t,x)$ is zero at the boundaries $x = a$ and $x = -a$, we may formally expand $\underline{U}$ in a Fourier series:

$$\underline{U}(t,x) = \sum_{j=-\infty}^{\infty} \underline{V}(t,j) \exp \frac{i\pi jx}{a} , \qquad 0 \le t \le T.$$

Substituting this in (3) and (4) we find

$$A\underline{V}(t+k,j) = B\underline{V}(t,j), \tag{5}$$

$$\underline{V}(j,o) = \frac{1}{2a} \int_{-a}^{a} \underline{U}\,(x,0) \exp\left(-\frac{i\pi jx}{a}\right) dx,$$

where

$$A = \begin{vmatrix} \beta+\alpha & -\beta \\ -\beta & \beta+\bar{\alpha} \end{vmatrix} , \qquad B = \begin{vmatrix} \bar{\alpha}-\beta & \beta \\ \beta & \alpha-\beta \end{vmatrix}$$

$$\beta = -8 \frac{\mu k}{h^2} \sin^2 \frac{\pi jh}{2a} ,$$

$$\alpha = -4-2i \frac{k}{h} \sin \frac{\pi jh}{a} ,$$

and $\bar{\alpha}$ is the complex conjugate of α.
As

$$\det(A) = 16 + 64\,\mu\lambda \sin^2 \frac{\pi jh}{2a} + 4\lambda^2 h^2 \sin^2 \frac{\pi jh}{a} > 0,$$

we may conclude from (5) that

$$\underline{V}(t = nk, j) = G^n \underline{V}(0, j),$$

where

$$G = A^{-1}B = (|\alpha|^2 - 8\beta)^{-1} \begin{vmatrix} \bar{\alpha}^2 & \beta(\alpha + \bar{\alpha}) \\ \beta(\alpha + \bar{\alpha}) & \alpha^2 \end{vmatrix} .$$

Usually, G is called the amplification matrix.
Now using Parseval's theorem we see that

$$||\underline{U}(t)|| \leq \{\max_j ||G(j,h,k)||\}^n \; ||\underline{U}(0)||,$$

where the norm of the matrix G is defined by

$$||G(j,h,k)|| = \sup_{\underline{v} \neq 0} \frac{\underline{v}^+ G^+ G \underline{v}}{\underline{v}^+ \underline{v}}$$

and G^+ is the hermitian transpose of G.
The two eigenvalues λ_1 and λ_2 of G^+G are given by

$$0 \leq \lambda_1 = \frac{(|\alpha|^2 + 8\beta)^2}{(|\alpha|^2 - 8\beta)^2} \leq 1,$$

$$\lambda_2 = 1,$$

and so

$$||G|| \leq 1,$$

from which the stability immediately follows.

Def. 3

The difference solution is unconditionally convergent to the exact solution iff for any $\underline{u}(x,0) \in L_2([-a,a])$

$$||\underline{U}(t,x) - \underline{u}(t,x)|| \to 0$$

as $h \to 0$, $k \to 0$ in any manner.

According to Richtmeyer ([3], page 56), our consistency definition implies consistency in the sense of Lax and Richtmeyer. The definitions 2 and 3 are entirely equivalent to those of Lax and Richtmeyer and, as our continuous problem is well posed, we may conclude from Lax's equivalence theorem to the convergence of the solution of the difference scheme to that of the exact problem.

5. SOME EXPERIMENTAL DATA

The computations were done on the EL-X8 computer of the Technological University of Eindhoven, using an Algol-60 program.
Stability and convergence of the numerical solution in the sense of the definitions given in the preceding section were confirmed experimentally.
To eliminate the influence of the truncation error we used the Romberg-Stiefel extrapolation method. However, as we were limited by the totally "available" computer-time, we could not make both mesh widths as small as we wanted. Some trial runs indicated that the influence of h seems more important than that of k. So we decided to use the Romberg-Stiefel method only with respect to h and to hold k fixed, in fact equal to 0.1.
The solutions r and s consisted of some wave crests separated and surrounded by valleys of very small amplitude. Comparision of the amplitudes in the wave crests for various values of h showed that the relative error made in choosing $h = 0.05$ varied from about 0.1 to a few per cent (the latter of course depending on where one wants to cut off the wave crest(s)). In fact, down from the top of a wave crest of one of the functions the absolute error only slowly decreases while the function itself mostly decreases quite rapidly. So at the top the relative error is much smaller than far below the top. Use of the Romberg-Stiefel procedure in these areas gave a still better result.
In the valleys, however, the relative error could be considerably larger, up to (if the amplitude was <u>very</u> small) 100 per cent. The cause of this large relative error probably must be found in loss of significant digits. Of course the Romberg-Stiefel procedure was of no use in these areas. Fortunately the solution there is of no interest at all.
In drawing the graphs we have not used the Romberg-Stiefel values but the values of r and s obtained with $h = 0.05$. This was done because the difference between the two values was hardly discernable in the graphs. In drawing the graphs we confined ourselves to the relevant part of the wave crests.

6. ON THE GRAPHS

The graphs themselves (which can be found in appendix 2) hardly need any comment. The development of left- and right moving r and s-waves is clearly demonstrated. We have not been able to carry out the computations beyond t = 20, because of practical reasons (e.g. available computer-time). Fortunately this is not necessary. The development of the solution when t >>20 can be accounted for by the asymptotic analysis as given in [1]. First we shall pay attention to the s-mode. From our asymptotic information ([1]) we infer that as $t \to \infty$ there are only two dominant wave crests having sharp peaks around and extrema along x = t and x = -t. The first extremum is a maximum, the second one a minimum.
Looking at the numerical solution at t = 20 we see that two wave crests are situated around x = t and x = -t. They have the expected signs. But there are two additional crests. By comparing the absolute values of the extrema of the latter with those of the first ones, we found that the wave crests situated nearest x = t or x = -t decrease more slowly than the other ones. So we may expect the solution s to go to the asymptotic solution (in this respect) indeed.
The same situation arises for the other mode r. It is easily seen that in the wave running to the left the minimum becomes dominant over the maximum. Therefore we may expect the numerical solution to go to the asymptotic solution again.
For clarity the situation for $t \to \infty$ is sketched in the figures below.

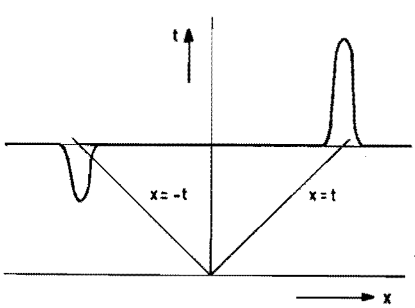

fig. 1: the r-mode

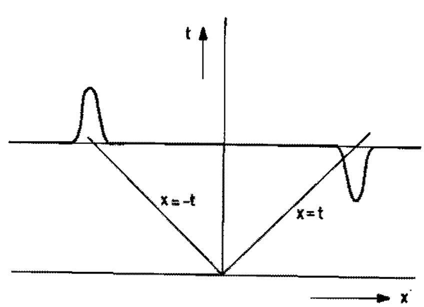

fig. 2: the s-mode

Looking at the graphs in appendix 2 another interesting feature can be noted. For the waves travelling to the right the relation $s = \frac{\mu}{2} r_x$ seems to be satisfied approximately. For the backward running waves the analogous relation would be $r = \frac{\mu}{2} s_x$.

Substituting $s = \frac{\mu}{2} r_x$ in (1.1) and (1.2) gives

$$r_{xxxx} = 0$$

and so

$$r(x,t) = A(t)x^3 + B(t)x^2 + C(t)x + D(t), \qquad (1)$$

$$s(x,t) = \frac{\mu}{2} [3\, A(t)x^2 + 2\, B(t)x + C(t)]. \qquad (2)$$

Using $r = \frac{\mu}{2} s_x$ we find (1) and (2) but with r and s interchanged in the two formulae.Thus in the regions where r (s) can be described (to a certain accuracy) by a polynomial of degree three the relation $s = \frac{\mu}{2} r_x$ ($r = \frac{\mu}{2} s_x$) holds with the same degree of accuracy. Looking more precisely we see that these relations are only valid for small intervals of the x-axis and are not of much practical use. The approximation $s = \frac{\mu}{2} r_x$ has been used for the first time (so far as we know) by Lighthill [4] in his theory of real gases.

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APPENDIX 1.

A number of "constants" and matrices will be given.

$$\lambda = \frac{k}{h^2} \quad ,$$

$$a = \lambda(h+2\mu) \quad ,$$

$$b = -2\mu\lambda \quad ,$$

$$c = \lambda(2\mu - h) \quad ,$$

$$d = -4(1 + \mu\lambda) \quad ,$$

$$e = 4(1 - \mu\lambda) \quad ,$$

$$A = \begin{vmatrix} d & c & & & 0 \\ a & d & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & d & c \\ 0 & & & a & d \end{vmatrix} \qquad B = \begin{vmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{vmatrix} \quad ,$$

$$C = \begin{vmatrix} e & c & & & 0 \\ a & e & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & e & c \\ 0 & & & a & e \end{vmatrix} \qquad D_1 = \begin{vmatrix} -8-4\lambda h & 4 & & & 0 \\ 4 & -8 & 4 & & \\ & \ddots & \ddots & \ddots & \\ & & 4 & -8 & 4 \\ 0 & & & 4 & -8-4\lambda h \end{vmatrix} \quad ,$$

$$D_2 = \begin{vmatrix} -8+4\lambda h & 4 & & & 0 \\ 4 & -8 & 4 & & \\ & \ddots & \ddots & \ddots & \\ & & 4 & -8 & 4 \\ 0 & & & 4 & -8+4\lambda h \end{vmatrix}$$

Putting

$$
\begin{aligned}
u &= ac - b^2 &&, \\
v &= ce + ad + 4b^2 &&, \\
w &= c^2 + a^2 + de - 6b^2 &&, \\
x &= cd + ae + 4b^2 &&, \\
y &= w + b^2 - c^2 &&, \\
z &= w + b^2 - a^2 &&, \\
\alpha &= cd + ad + 4b^2 &&, \\
\beta &= a^2 + d^2 + c^2 - 6b^2 &&, \\
\gamma &= \beta + b^2 - c^2 &&, \\
\delta &= \beta + b^2 - a^2 &&,
\end{aligned}
$$

and

$$
\begin{aligned}
E &= A^T C - b^2 B^2 &&, \\
F &= C A^T - b^2 B^2 &&, \\
G &= A^T A - b^2 B^2 &&, \\
H &= A A^T - b^2 B^2 &&,
\end{aligned}
$$

we find

$$
E = \begin{vmatrix}
y & x & u & & & & & \\
v & w & x & u & & & 0 & \\
u & v & w & x & u & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & 0 & & u & v & w & x & u \\
 & & & & u & v & w & x \\
 & & & & & u & v & z
\end{vmatrix} , \quad
F = \begin{vmatrix}
z & x & u & & & & & \\
v & w & x & u & & & 0 & \\
u & v & w & x & u & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & 0 & & u & v & w & x & u \\
 & & & & u & v & w & x \\
 & & & & & u & v & y
\end{vmatrix}
$$

$$
G = \begin{vmatrix}
\gamma & \alpha & u & & & & & \\
\alpha & \beta & \alpha & u & & & 0 & \\
u & \alpha & \beta & \alpha & u & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & 0 & & u & \alpha & \beta & \alpha & u \\
 & & & & u & \alpha & \beta & \alpha \\
 & & & & & u & \alpha & \delta
\end{vmatrix} , \quad
H = \begin{vmatrix}
\delta & \alpha & u & & & & & \\
\alpha & \beta & \alpha & u & & & 0 & \\
u & \alpha & \beta & \alpha & u & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & 0 & & u & \alpha & \beta & \alpha & u \\
 & & & & u & \alpha & \beta & \alpha \\
 & & & & & u & \alpha & \gamma
\end{vmatrix}
$$

All mentioned matrices are of dimension MxM.

Finally we define

$$B_{-1} = -C_{-1} = \begin{vmatrix} a & b \\ b & c \end{vmatrix},$$

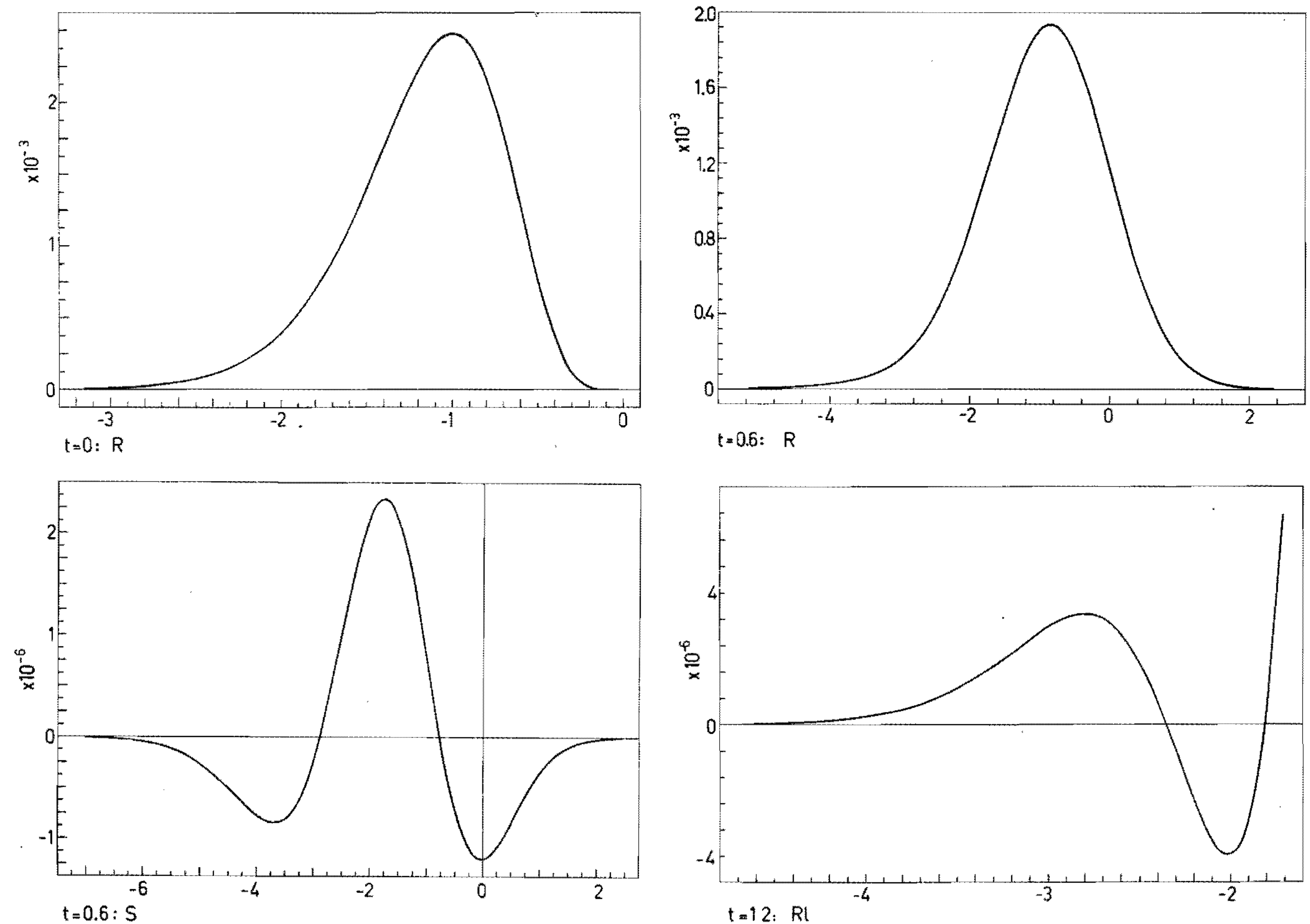
$$B_1 = -C_1 = \begin{vmatrix} c & b \\ b & a \end{vmatrix},$$

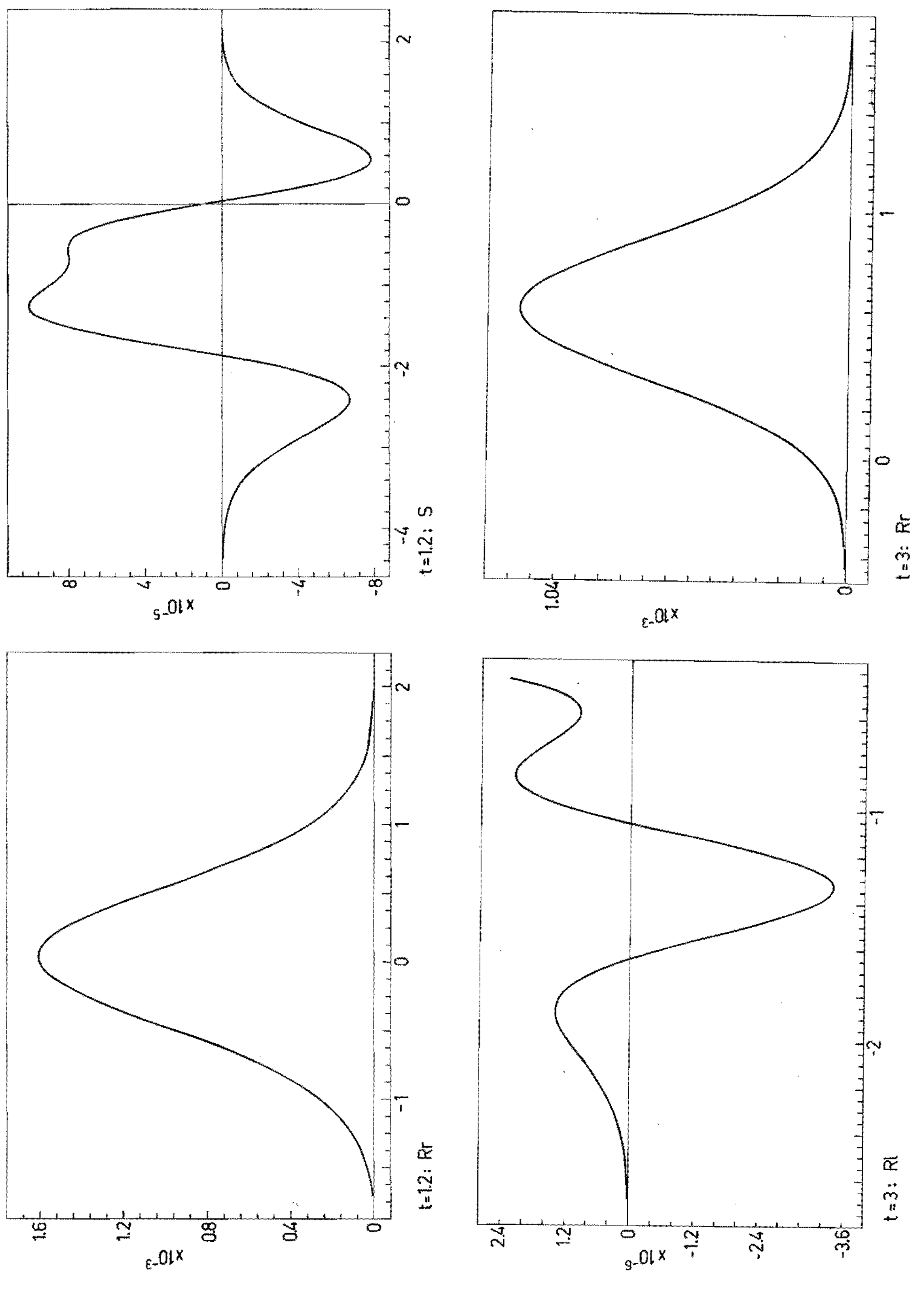
$$B_o = \begin{vmatrix} d & -2b \\ -2b & d \end{vmatrix},$$

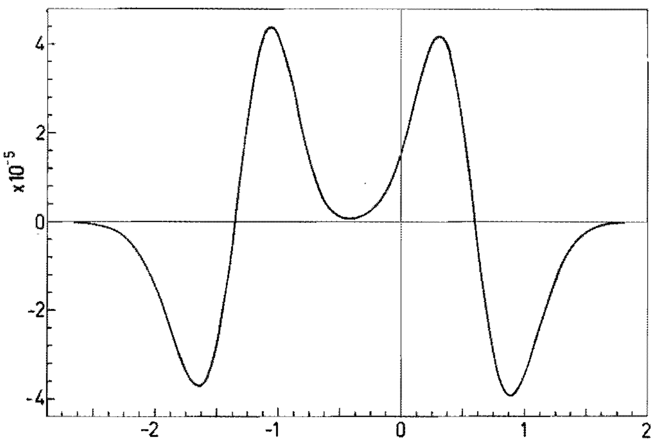
$$C_o = \begin{vmatrix} e & -2b \\ -2b & e \end{vmatrix}.$$

APPENDIX II.

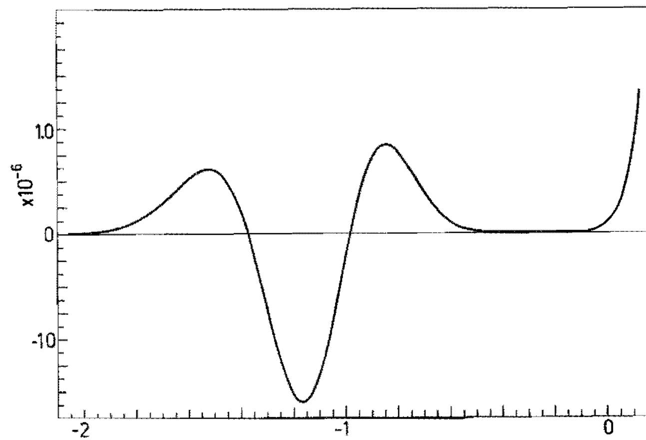
The pictures below still deserve some comment. Along the vertical axis the value of the amplitude of the s-mode (S), the forward-running part - (Rr) or the backward-running part (Rℓ) of the r-mode has been plotted. At $t = 0.6$ these parts can hardly be separated. Therefore we simply wrote R. This has been done for $t = 0$ too. Along the horizontal axis we have plotted at $t = o$ the value of x, at all times $t > 0$ the value of x/t.


x10⁻³
t=0: R
x10⁻³
t=0.6: R
x10⁻⁶
t=0.6: S
x10⁻⁶
t=12: Rl

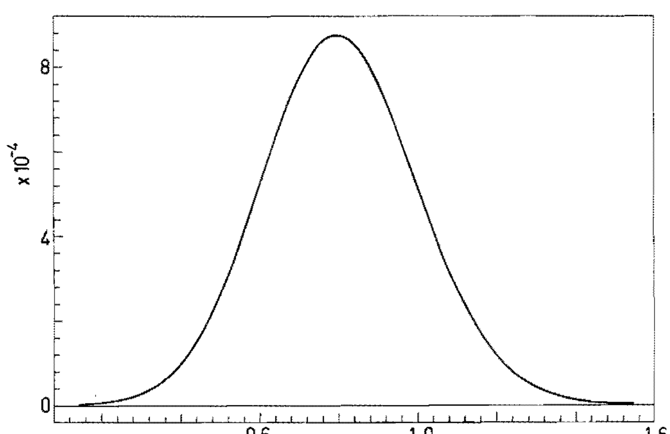

t=1.2: S
x10⁻⁵
t=1.2: Rr
x10⁻³
t=3: Rr
x10⁻³
t=3: Rl
x10⁻⁶



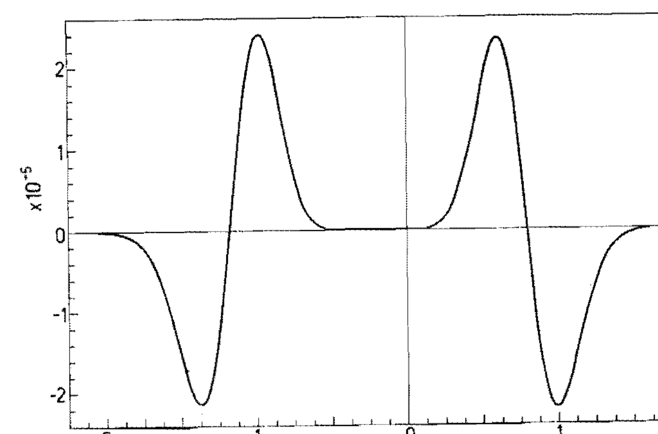
t=3: S



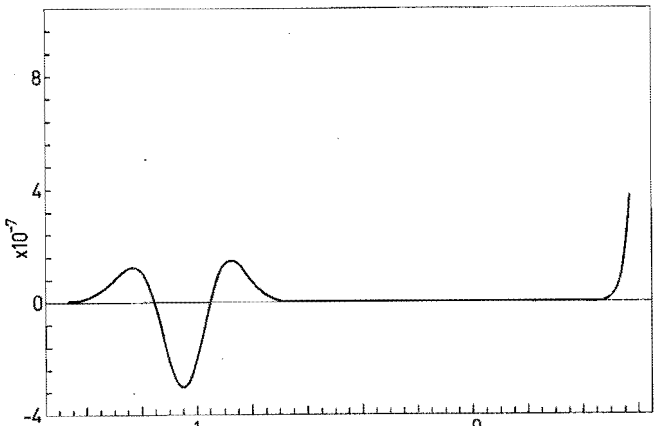
t=6: Rl



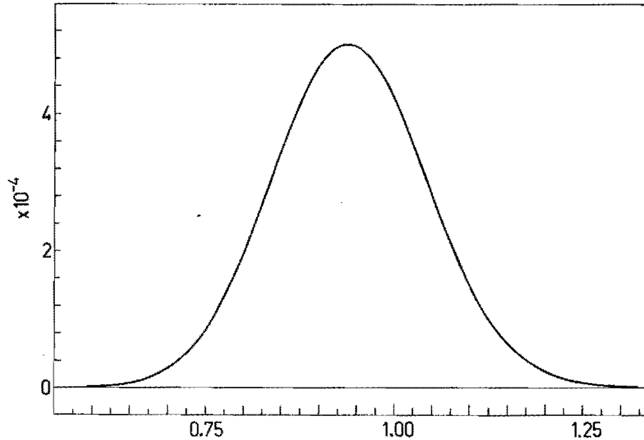
t=6: Rr



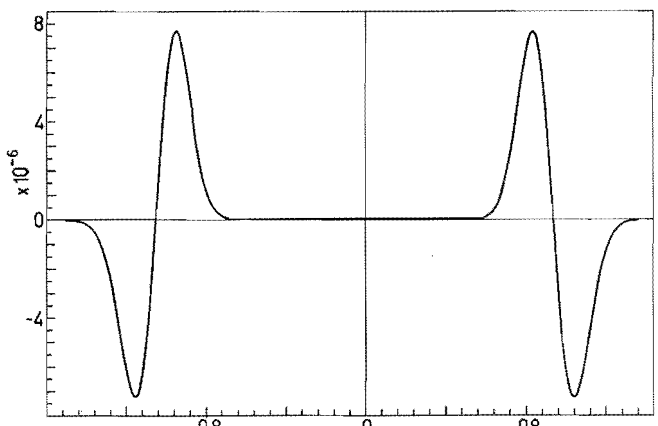
t=6: S



t=20: Rl



t=20: Rr



t=20: S

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ON A UNIQUE CONTINUUM REPRESENTATION FOR THE LINEAR CHAIN PROBLEM

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Summary.

An exact unique continuum representation for the one-dimensional linear chain problem has been constructed. Some aspects of intermediate approximations are discussed.

1. INTRODUCTION.

In physics courses one often studies the longitudinal motion of an infinite chain of identical masses and springs as an one-dimensional model of a crystal. It is well-known that this motion is described by the infinite set of equations:

$$\ddot{u}_n = \omega_0^2(u_{n+1} - 2u_n + u_{n-1}) \tag{1}$$

where $u_n(t)$ is the deviation of the n th particle from the equilibrium position: $u_n = x_n - na$ (a is the lattice constant) and ω_0 is the resonant frequency of a simple linear spring-mass system.
In the standard treatment it is also shown that solutions of the equation:

$$u_{tt} - c^2 u_{xx} = 0 \tag{2}$$

where $c^2 = \omega_0^2 a^2$, provide an approximate continuum representation of the solutions of (1). This means that $|u(na,t) - u_n(t)|$ is small in some sense for small amplitude, long wavelength motions, at least during some finite interval of time. Long wavelength means that $|u_{n+1} - u_n|$ is small, again in "some" sense. The restriction in time is necessary as (1) and (2) have different dispersion laws. These results can be derived e.g. by using Taylor expansions up to the second order.

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All these facts are elementary and well known, apart perhaps from details about the "in some sense" restrictions. In this paper we will dwell upon two questions connected with this treatment which are somewhat less elementary and upon which the available literature gave no clear-cut answers.
The first question is whether an exact and unique continuum representation of the solutions of (1) exists. By this we mean: We require a one-to-one relation between numbers u_n and functions u(x) such that $u_n = u(na)$.
Of course there are infinitely many functions satisfying that requirement. So a suitable choice has to be made. We want to do this in such a way that a linear operator A exists with the property that the, unique, solutions of

$$u_{tt} = -Au \tag{3}$$

correspond to the solutions of (1) in the way mentioned above if and only if the initial conditions upon (1) and (3) correspond in this way.
As our problem is linear we expect that the required mapping of numbers on functions is linear too. As the whole problem is invariant for translations over a multiple of a we only have to choose a suitable representation for $u_n = \delta_{n0}$. It turns out that this is $u = a(\pi x)^{-1} \sin(\pi x a^{-1})$. The operator A then can be constructed by requiring that its spectrum corresponds to that of the right-hand side of (1). In section 3 we will make these statements more precise and supply the proofs. Some mathematics needed in that section will be explained in section 2.
The second problem is that of intermediate representations. By this we mean a representation "between" (2) and (3), containing more information than (2), e.g. some dispersion, but not an exact representation. This looks simple enough. Using e.g. Taylor expansions up to the fourth order we find the equation:

$$u_{tt} = c^2 u_{xx} + \frac{a^2c^2}{12} u_{xxxx}. \tag{4}$$

On closer inspection this is not quite satisfactory as (4) is not stable. The dispersion equation is:

$$\omega^2 = c^2\left[k^2 - \frac{a^2k^4}{12}\right] \tag{5}$$

which is not positive definite. In section 4 we shall return to this question.

2. MATHEMATICAL PRELIMINARIES.

R: the interval $(-\infty,\infty)$ of the real numbers.

Δ: a positive number.

l_2 is a Hilbert-space containing all complex-valued sequences $\underline{a} = \{a_n\}_{n=-\infty}^{\infty}$ where $\sum_{n=-\infty}^{\infty} |a_n|^2$ is finite. The inner product (,) and norm $|| \ ||$ are defined by

$$(\underline{a},\underline{b}) = \sum_{n=-\infty}^{\infty} a_n^* b_n, \quad ||\underline{a}|| = (\underline{a},\underline{a})^{\frac{1}{2}},$$

where a_n^* is the complex conjugate of a_n.

$L_2([-\Delta,\Delta])$ is a Hilbert-space containing all square-integrable complex-valued functions on $[-\Delta,\Delta]$, with inner product $(\ , \ )_\Delta$ and norm $|| \ ||_\Delta$ satisfying

$$(u,v)_\Delta = \int_{-\Delta}^{\Delta} u^*(x)v(x)dx, \quad ||u||_\Delta = (u,u)_\Delta^{\frac{1}{2}}.$$

If $[-\Delta,\Delta] = R$ we shall use the notations $L_2(R)$, (,) and $|| \ ||$.

Of course $L_2([-\Delta,\Delta]) \subseteq L_2(R)$.

For every element $u \in L_2(R)$ define the Fourier transform $\bar{u}(k)$ by

$$\bar{u}(k) = \underset{N\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{(2\pi)}} \int_{-N}^{N} u(x)e^{-ikx}dx,$$

then according to the Fourier-Plancherel theorem (Titchmarsh [1]) $\bar{u} \in L_2(R)$ and

$$u(x) = \underset{N\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{(2\pi)}} \int_{-N}^{N} \bar{u}(k)e^{ikx}dk,$$

$$(u,v) = (\bar{u},\bar{v}). \tag{1}$$

$L_2^\Delta(R)$ contains all square-integrable complex-valued functions $u(x)$ on R of which $\bar{u}(k)$ is identical to zero outside the interval $[-\Delta,\Delta]$. The inner product $(\ , \ )_{R,\Delta}$ and norm $|| \ ||_{R,\Delta}$ are defined by

$$(u,v)_{R,\Delta} = \int_{-\infty}^{\infty} u^*(x)v(x)dx, \quad ||u||_{R,\Delta} = (u,u)_{R,\Delta}^{\frac{1}{2}}.$$

It is easily seen that $L_2^\Delta(R)$ is a Hilbert-space. It is a subspace of $L_2(R)$.

When the Fourier transform is considered as an unitary mapping of $L_2(R)$ in x onto $L_2(R)$ in k then the subspace $L_2^{\Delta}(R)$ is mapped onto $L_2([-\Delta,\Delta])$ in k.

THEOREM I.

Each function $f \in L_2^{\Delta}(R)$ has a continuous representation $\tilde{f}$, where for almost every $x \in R$: $f = \tilde{f}$.

Proof.

According to the definition of $L_2^{\Delta}(R)$ to each function $f \in L_2^{\Delta}(R)$ a function $\bar{f} \in L_2([-\Delta,\Delta])$ exists such that for almost every $x \in R$

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\Delta}^{\Delta} \bar{f}(k) e^{ikx} dk.$$

Using the inequality of Schwarz we see that

$$\left| \int_{-\Delta}^{\Delta} \bar{f}(k) e^{ikx} dk \right| \leq (2\Delta)^{\frac{1}{2}} ||\bar{f}||_{\Delta},$$

and so $\int_{-\Delta}^{\Delta} \bar{f}(k) e^{ikx} dk$ is a continuous function with respect to x.

Putting

$$\tilde{f}(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\Delta}^{\Delta} \bar{f}(k) e^{ikx} dk,$$

the theorem is proved.

Define the operators T_1 and T_2 by

$$(T_1 \underline{a})(k) = \underset{N\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{(2\Delta)}} \sum_{n=-N}^{N} a_n \exp\left(-\frac{in\pi k}{\Delta}\right) \qquad (\underline{a} \in l_2),$$

and

$$(T_2 \bar{a})(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\Delta}^{\Delta} \bar{a}(k) e^{ikx} dk \qquad (\bar{a} \in L_2([-\Delta,\Delta])),$$

the second equation holding for almost every $x \in R$.

THEOREM II.

The operators T_1 and T_2 are isometric mappings (Achieser and Glasmann [2], p.77) from l_2 onto $L_2([-\Delta,\Delta])$ and $L_2([-\Delta,\Delta])$ onto $L_2^{\Delta}(R)$ respectively. Furthermore

$$T_1^{-1}\bar{a} = \frac{1}{\sqrt{(2\Delta)}}\{\int_{-\Delta}^{\Delta}\bar{a}(k)\ \exp(\frac{in\pi k}{\Delta})dk\}_{n=-\infty}^{\infty} \qquad (\bar{a} \in L_2([-\Delta,\Delta])),$$

$$(T_2^{-1}a)(k) = \underset{N\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{(2\pi)}}\int_{-N}^{N}a(x)e^{-ikx}dx \qquad (a \in L_2^{\Delta}(R)).$$

Proof.
The statements simply result from well known Fourier theory (Yosida [3], p. 86-88, Titchmarsh [2], ch.III).

Define on l_2

$$T = T_2T_1.$$

THEOREM III.

The operator T is an isometric mapping from l_2 onto $L_2^{\Delta}(R)$, given by

$$T\underline{f} = \underset{N\to\infty}{\text{l.i.m.}}\ \sum_{n=-N}^{N}\pi\Delta^{-1}f_n\ \text{sinc}(\Delta x - n\pi) \qquad (\underline{f} \in l_2), \tag{2}$$

where the so-called extrapolation- or filter- function sinc(x) is defined by

$$\text{sinc}(x) = \frac{\sin(x)}{x}.$$

The inverse operator may be defined by

$$T^{-1}f = \{\tilde{f}(\frac{n\pi}{\Delta})\}_{n=-\infty}^{\infty} \qquad (f \in L_2^{\Delta}(R)),$$

where $\tilde{f}(x)$ is the continuous representation of f(x).

Proof.
T is isometric. This follows directly from

$$(T_2T_1\underline{f},T_2T_1\underline{g})_{R,\Delta} = (T_1\underline{f},T_1\underline{g})_{\Delta} = (\underline{f},\underline{g}) \qquad (\underline{f} \in l_2, \underline{g} \in l_2).$$

According to theorems II and III

$$\lim_{N\to\infty} ||T_1\underline{f} - \frac{1}{\sqrt{(2\Delta)}} \sum_{n=-N}^{N} f_n \exp(-\frac{in\pi k}{\Delta})||_{\Delta}^{2} =$$

$$\lim_{N\to\infty} ||T\underline{f} - \sum_{n=-N}^{N} (\frac{\Delta}{\pi})^{\frac{1}{2}} f_n \, \mathrm{sinc}(\Delta x - n\pi)||_{R,\Delta}^{2},$$

which proves (2).
For $f \in L_2^{\Delta}(R)$ we have

$$T^{-1}f = T_1^{-1}T_2^{-1}f = T_1^{-1}\bar{f} = \{\frac{1}{\sqrt{(2\pi)}} \int_{-\Delta}^{\Delta} \bar{f}(k) \exp(\frac{ikn\pi}{\Delta}) dk\}_{n=-\infty}^{\infty} \stackrel{\text{def}}{=} \{\check{f}(\frac{n\pi}{\Delta})\}_{n=-\infty}^{\infty},$$

which proves the remaining part of the theorem.

Remark.
T may be seen as an extrapolation -, T^{-1} as a sampling operator.

Corollary 1.
$\{(\frac{\Delta}{\pi})^{\frac{1}{2}} \mathrm{sinc}(\Delta x - \pi l),\ l = 0, \pm 1, \pm 2, \ldots\}$ forms a complete orthonormal set in $L_2^{\Delta}(R)$.

Corollary 2.
A function $f \in L_2^{\Delta}(R)$ is completely determined by the values of its continuous representation $\check{f}$ at the lattice-points $\frac{n\pi}{\Delta}$, $n = 0, \pm 1, \pm 2, \ldots$, i.e., if $\check{f}(\frac{n\pi}{\Delta})$ has been found by sampling $\check{f}(x)$, there is only one function in $L_2^{\Delta}(R)$ that equals $\check{f}(\frac{n\pi}{\Delta})$ at the lattice-points. This function is f. The reverse is also true.

Corollary 2 is the essential tool required in the next section to construct a unique continuum representation.

3. A UNIQUE CONTINUUM REPRESENTATION

First, we write equation (1.1) in the form

$$\ddot{\underline{u}} + A\underline{u} = \underline{0}, \qquad (1)$$

where the sequence $\underline{u}$ is defined by

$$\underline{u} = \{u_n\}_{n=-\infty}^{\infty}$$

and the operator A by

$$A\underline{u} = \{\sum_{m=-\infty}^{\infty} a_{nm} u_m\}_{n=-\infty}^{\infty}$$

where

$$a_{nm} = 0 \qquad (n \neq m, m \pm 1),$$

$$a_{n,n+1} = a_{n,n-1} = -\omega_0^2,$$

$$a_{n,n} = 2\omega_0^2.$$

Consider equation (1) with the initial conditions $\underline{u}(0) = \underline{f} \in l_2$, $\dot{\underline{u}}(0) = \underline{g} \in l_2$. This problem will be called the discrete initial value problem (divp). Next consider the operator equation

$$u_{tt} + \mathcal{A} u = 0,$$

with the conditions $u(x,0) = f(x) \in L_2^{\Delta}(R)$, $u_t(x,0) = g(x) \in L_2^{\Delta}(R)$. $\mathcal{A}$ is defined on $L_2^{\Delta}(R)$ but as yet not further specified. $\Delta = \pi a^{-1}$. This problem will be called the continuous initial value problem (civp). Let T_1, T_2 and T be defined as in the last section.

<u>Definition.</u>

The civp will be called a unique continuum representation of the divp if and only if

$$T\underline{f} = f, \tag{2}$$

$$T\underline{g} = g, \tag{3}$$

imply for all $t \geq 0$

$$T\underline{u} = u. \tag{4}$$

THEOREM IV.

Let

$$A = T_2\omega^2 T_2^{-1} ,$$

where

$$\omega^2(k) = c^2 \frac{\sin^2(\frac{1}{2}ka)}{(\frac{1}{2}a)^2}. \qquad (5)$$

Then the civp is a unique continuum representation of the divp.

Proof:

The square root of A is given by

$$A^{\frac{1}{2}} = T_2\omega T_2^{-1}. \qquad (6)$$

This is a bounded operator as for all $f \in L_2^{\Delta}(R)$

$$||A^{\frac{1}{2}}f||_{R,\Delta}^2 \leq \{\max_{k\in[-\Delta,\Delta]} \omega^2(k)\} ||f||_{R,\Delta}^2 ,$$

Therefore the solution of the civp is given by

$$u(t) = \cos(A^{\frac{1}{2}}t)f + A^{-\frac{1}{2}} \sin(A^{\frac{1}{2}}t)g,$$

where

$$\cos(A^{\frac{1}{2}}t) = \frac{e^{iA^{\frac{1}{2}}t} + e^{-iA^{\frac{1}{2}}t}}{2}$$

$$\sin(A^{\frac{1}{2}}t) = \frac{e^{iA^{\frac{1}{2}}t} - e^{-iA^{\frac{1}{2}}t}}{2i}$$

and

$$e^{iA^{\frac{1}{2}}t} = \sum_{n=0}^{\infty} (itA^{\frac{1}{2}})^n.$$

Using (6) and

$$e^{iA^{\frac{1}{2}}t} = T_2 e^{i\omega t} T_2^{-1}$$

the solution may be written as

$$u(t) = T_2 \cos(\omega t) T_2^{-1} f + T_2 \frac{\sin(\omega t)}{\omega} T_2^{-1} g.$$

By means of substitution in (1.1), it is easily seen that the solution of the divp is given by

$$\underline{u} = \text{col.}\left\{\frac{1}{\sqrt{(2\Delta)}} \int_{-\Delta}^{\Delta} \left(\cos\left[\omega(k)t\right] T_1 \underline{f} + \frac{\sin\left[\omega(k)t\right]}{\omega(k)} T_1 g\right) e^{ikna} dk\right\}_{n=-\infty}^{\infty}$$

or

$$\underline{u} = T_1^{-1} \cos(\omega t) T_1 \underline{f} + T_1^{-1} \frac{\sin(\omega t)}{\omega} T_1 \underline{g}.$$

So

$$T\underline{u} = T_2 \cos(\omega t) T_1 \underline{f} + T_2 \frac{\sin(\omega t)}{\omega} T_1 \underline{g}. \qquad (7)$$

Now, we use (2) and (3) to find

$$T_1 \underline{f} = T_2^{-1} T \underline{f} = T_2^{-1} f$$

and

$$T_1 \underline{g} = T_2^{-1} g.$$

Substitution of these relations in (7) gives (4) and proves the theorem.

Corollary.

Using the interpretation given at the end of section 2 we deduce

a. At the lattice-points na, n = 0, ±1, ±2,..., the continuous representation of the solution u of the civp agrees for all t with the solution $\underline{u}$ of the divp.

b. The solution u is completely determined by the values of its continuous representation $\tilde{u}$ at the lattice-points. The reverse is also true.

The first statement has, for clarity, been given separately. However, it may be deduced from *b*. We observe that the expression "continuum representation" may be misleading in this context. The solution of the civp is not necessarily continuous. However, as it belongs to $L_2^{\Delta}(R)$, it has a continuous representation.

4. SOME REMARKS ON STABILITY.

An initial value problem is called stable when there exists a positive definite norm for the solution which is, uniformely with respect to time, bounded in terms of the corresponding norm of the initial conditions. For the present purpose it is not necessary to go into the details of stability theory. We have to point out however that only equations stable with respect to some norm are in general useful in mathematical physics.
In many cases of physical interest (counterexamples can readily be derived) a suitable norm is provided by the energy equation. For instance the equation

$$\frac{d}{dt}\{\sum_{n=-\infty}^{\infty} \frac{1}{2}\dot{u}_n^2 + \frac{1}{2}\omega_0^2(u_n - u_{n-1})^2\} = 0, \tag{1}$$

which is satisfied by all solutions of (1.1), ensures the stability of (1.1) as the expression between curly brackets is positive definite.
In the same way we deduce from the energy balance equation

$$\frac{\partial}{\partial t}(\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2) + \frac{\partial}{\partial x}(-c^2 u_t u_x) = 0, \tag{2}$$

satisfied by the solutions of (1.2), the relation

$$\frac{d}{dt}\{\frac{1}{2}(u_t,u_t) + \frac{c^2}{2}(u_x,u_x)\} = 0. \tag{3}$$

Therefore (1.2) is stable too, as is well known.
Multiplying (1.3) by u_t and integrating we find:

$$\frac{d}{dt}\{\frac{1}{2}(u_t,u_t)_{R,\Delta} + \frac{1}{2}(u,Au)_{R,\Delta}\} = 0. \tag{4}$$

As A is, according to the preceding section, a positive operator on its entire domain $L_2^{\Delta}(R)$, (1.3) is stable.

We now turn to the intermediate approximation (1.4). The solution of the initial value problem in $L_2(R)$ does indeed exist uniquely for any finite interval of time. To be admissible in mathematical physics a stability proof is the only further requirement.
The energy method is of no help here. There is a balance equation analogeous to (3), viz.:

$$\frac{d}{dt}\{\frac{1}{2}(u_t,u_t) + \frac{c^2}{2}(u_x,u_x) - \frac{a^2c^2}{24}(u_{xx},u_{xx})\} = 0, \qquad (5)$$

As the bracket expression is not definite this does not yield stability. On the other hand this is not a proof of instability, which is often more difficult to achieve. As a matter of fact (1.4) is unstable. This can be inferred from (1.5), showing that ω^2 can be negative for real k. Another argument, using a counterexample runs as follows:
The solution of (1.4) for the initial value problem $u(x,0) = \frac{1}{2\sqrt{\pi}}\exp(-x^2/4a^2)$, $u_t(x,0) = 0$ is:

$$u = \frac{a}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty}\cos\left[\Omega(k)t\right]\ \exp(-ikx - a^2k^2)dk$$

where $\Omega = ck(1 - \frac{a^2k^2}{12})^{\frac{1}{2}}$.

Using Parseval's theorem:

$$||u(t)||^2 = a^2\int_{-\infty}^{\infty} \cos^2[\Omega(k)t]\ \exp(-2a^2k^2)dk \geq$$

$$\geq a^2\int_{L}^{\infty} \cosh^2\{ckt(\frac{a^2k^2}{12} - 1)^{\frac{1}{2}}\}\ \exp(-2a^2k^2)dk$$

where L is some number such that $a^2L^2 > 12$. It is not difficult to see that this expression grows exponentially when t tends to infinity.
We will not dwell further upon the difficult subject of instability proofs. In stead of this we will consider the possibility of repairing the present unsatisfactory state of affairs. This is not without interest for the following reason. The suspect equation (1.4) can also be derived by expansion from (1.3), which is essentially an integral equation. These procedures are not unfamiliar in physics. An example is the derivation of the

Fokker-Planck equation from a master equation. Obviously the resulting differential equation may be useless due to instability even when it has a unique solution.

In general this is a quite difficult problem. In the present case there are however two fairly easy ways out.

The first is, rather obviously, to attempt restriction of the solution to reduced wave numbers. That is, we consider (1.4) as an equation on $L_2^{\Delta}(R)$ in stead of $L_2(R)$, where $\Delta = \pi a^{-1}$ is the maximum of $|k|$. It is easily shown that u(t) stays in $L_2^{\Delta}(R)$ when the initial data belong to this space. In this restricted space (1.5) clearly is positive definite. It is also possible to show that the bracket expression in (5) is positive definite in this space. When this restriction to $L_2^{\Delta}(R)$ would, e.g. for computational reasons, be inconvenient we can proceed in the following way: The physical content of (1.5) is that it approximates the dispersion for not to short waves, that is for small values of ak. For this purpose the relation

$$\omega^2 = c^2k^2\left(1 + \frac{a^2k^2}{12}\right)^{-1} \tag{6}$$

would do about as well. An equation having (5) as dispersion relation would be stable as ω^2 can not be negative for real k. This equation is:

$$u_{tt} - \frac{a^2}{12} u_{ttxx} - c^2u_{xx} = 0. \tag{7}$$

In stead of (2) we now find the balance equation

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2 + \frac{a^2}{24} u_{tx}^2\right\} - \frac{\partial}{\partial x}\left\{c^2u_tu_x + \frac{a^2}{12} u_tu_{ttx}\right\} = 0. \tag{8}$$

The first bracket is positive, therefore (8) provides a stability proof in that domain in $L_2(R)$ where (7) is uniquely solvable. We mention without proof that this domain lies dense in $L_2(R)$. These stable solutions are reasonable approximations only within $L_2^{\Delta}(R)$.

Both methods of obtaining stable approximations to (1.3) can be adapted to higher order equations, obtained by continuing the expansions to a further stage. As this is by now merely a technical matter we will not enter upon this extension here.

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ON A SIMPLE WAVE APPROXIMATION OF A SET OF LINEAR DISPERSIVE WAVE EQUATIONS

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ABSTRACT.

The validity of an approximation α_0 of one of the solutions α of a set of two linear coupled dispersive wave equations has been discussed. α_0 is the solution of a linear Korteweg - de Vries equation and satisfies the same initial condition as α. It is shown that, for square integrable solutions having a spectral range not exceeding $[-\Delta,\Delta]$, the approximation is useful, for $\Delta^5\mu^2 t \ll 1$, in the sense that $||\alpha - \alpha_0|| \ll ||\alpha||$ (L_2 - norm). μ is a measure for the dispersion. The approximation fails in that sense as $t \to \infty$. Some remarks to a similar nonlinear problem are made.

1. INTRODUCTION.

In two papers, [1] and [2], L.J.F. Broer and the present author have considered a set of two linear coupled dissipative wave equations. We were interested especially in the range of validity of an approximation of this set applying to a certain class of initial value problems. The approximation "leads" to a linear Burgers'equation.

In this paper, a similar approximation for a set of linear dispersive wave equations will be treated. This set is given by

$$\alpha_t + \alpha_x = - \mu(\alpha+\beta)_{xxx}, \tag{1}$$

$$\beta_t - \beta_x = \mu(\alpha+\beta)_{xxx}, \tag{2}$$

where μ is a positive constant and the subscript t(or x) denotes partial differentiation with respect to t(x).

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An example of such a set is furnished by an intermediate representation of the equations describing the longitudinal motion of an infinite chain of identical masses and springs. By an intermediate representation we mean a representation "between" the exact continuum representation and the lowest continuum limit (cf. [3]). It is given by

$$u_{tt} = c^2 u_{xx} + \frac{a^2c^2}{12} u_{xxxx}, \tag{3}$$

where a is the lattice constant and c is the propagation speed of waves in the lowest continuum limit, i.e. $a \to 0$.
For reasons of uniqueness of the exact continuum representation (see [3]), it is necessary that u is square integrable and has a spectral range not exceeding $[-\pi a^{-1}, \pi a^{-1}]$. Then, as is shown in [3] too, stability is also assured.
Substituting $\alpha = -u_t + cu_x$, $\beta = u_t + cu_x$ and putting $c = 1$ (which can be done without any loss of generality), from (3) we find (1) and (2) with $\mu = a^2/24$.
The approximation we shall study applies to the class of initial value problems

$$\alpha(x,0) = f(x), \tag{4}$$

$$\beta(x,0) = 0. \tag{5}$$

When $\mu = 0$, it is seen that (2) is satisfied identically. Then (1) becomes a first order equation in α, which is easily solved. The resultant solution is a simple wave solution for the hyperbolic set obtained by putting $\mu = 0$.
Now, the approximation, which as in [1] and [2] will be called the simple wave approximation henceforth, is based on the assumption that, when the initial conditions (4) and (5) are prescribed for the equations (1) and (2) with μ small but not zero, β will be negligible, at any rate for some finite interval of time. In this way one obtains from (1)

$$\alpha_t + \alpha_x = -\mu\alpha_{xxx}. \tag{6}$$

This method of approximation has been used by Zabusky [4] in his theory of wave propagation in a nonlinear one-dimensional lattice. He obtains the Korteweg - de Vries (KdV) equation. (6) is the linearized form of that equation.
Now, the problem is that β will grow slowly from zero and therefore it is not at all obvious that α satisfies (6) for longer intervals of time. In section 4 the range of validity of the simple wave approximation and an expansion of α and β will be considered. Some mathematical notations and the representations of the solutions α, β and α_0 needed there, will be given in sections 2 and 3. α_0 is the solution of (6) subject to (4). The situation as $t \to \infty$ is discussed in section 5 and the last section is devoted to some remarks concerning a similar problem for nonlinear equations.

2. MATHEMATICAL NOTATIONS.

R is the interval $(-\infty,\infty)$ of the real numbers. Consider scalar - valued complex functions $u(x)$ defined on R.
$L_2(R)$ is a Hilbert-space containing all square integrable functions on R with inner product $(\, , \,)$ and norm $|| \; ||$ defined by

$$(u,v) = \int_{-\infty}^{\infty} u^*(x)v^*(x)dx \; ; \; ||u|| = (u,u)^{\frac{1}{2}},$$

where u^* is the complex conjugate of u.
The space $L_2^{\Delta}(R)$ is a Hilbert - space containing all functions $u \in L_2(R)$ of which the Fourier transform $\bar{u}(k)$ defined by

$$\bar{u}(k) = \int_{-\infty}^{\infty} u(x) \exp(-ikx)dx$$

is vanishing identically outside a finite interval $[-\Delta,\Delta]$ $(\Delta \in R)$.
The inner product $(\, , \,)_{R,\Delta}$ and norm $|| \; ||_{R,\Delta}$ are defined by

$$(u,v)_{R,\Delta} = \int_{-\infty}^{\infty} u^*(x)v(x)dx \; ; \; ||u||_{R,\Delta} = (u,u)^{\frac{1}{2}}_{R,\Delta} \; .$$

Where not stated otherwise all integrations are in the sense of Lebesque.

3. $L_2^{\Delta}(R)$ SOLUTIONS.

Let $f \in L_2^{\Delta}(R)$. Assume, for reasons of uniqueness of the exact continuum representation (see section 1 and [3]), $\Delta \leq \frac{1}{2}\pi(6\mu)^{-\frac{1}{2}}$ henceforth. As may be verified now easily, α and β are given by

$$\alpha = \frac{1}{2\pi}\left[\int_{-\Delta}^{\Delta} 1 + \int_{-\Delta}^{\Delta} 2\right] \frac{(k+\omega)^2}{4\omega k} \bar{f}(k) \exp(ikx - i\omega t)dk, \qquad (1)$$

$$\beta = \frac{1}{2\pi}\left[\int_{-\Delta}^{\Delta} 1 + \int_{-\Delta}^{\Delta} 2\right] \frac{k^2 - \omega^2}{4\omega k} \bar{f}(k) \exp(ikx - i\omega t)dk, \qquad (2)$$

where

$$\omega(k) = k(1 - 2\mu k^2)^{\frac{1}{2}}$$

and the numbers 1 and 2 through the integrationsymbol mean integration in the first - respectively second sheet of the complex k - plane. The first sheet is defined by

$$\lim_{|k|\to\infty} \frac{\omega(k)}{k^2} = -i\surd(2\mu) \quad (0 \leq \arg k \leq \pi)$$

and the second by

$$\lim_{|k|\to\infty} \frac{\omega(k)}{k^2} = i\surd(2\mu) \quad (0 \leq \arg k \leq \pi).$$

Finally

$$\alpha_0 = \frac{1}{2\pi} \int_{-\Delta}^{\Delta} \bar{f}(k) \exp(ikx - ikt + i\mu k^3 t)dk \qquad (3)$$

4. THE VALIDITY OF THE SIMPLE WAVE APPROXIMATION.

4.1. Periodic solutions.

Consider the periodic initial condition

$$f(x) = \exp(ik_1 x)$$

where $k_1 \in R$ and $|k_1| \leq \frac{1}{2}\pi(6\mu)^{-\frac{1}{2}}$.

The solutions α, β and α_0 may formally be found by substituting $\overline{f}(k) = 2\pi\delta\,(k - k_1)$ in (3.1), (3.2) and (3.3). Therefore

$$\alpha = \frac{(1 + c)^2}{4c} \exp(ikx - ikct) - \frac{(1 - c)^2}{4c} \exp(ikx + ikct), \qquad (1)$$

$$\beta = \frac{1 - c^2}{4c} \exp(ikx - ikct) + \frac{c^2 - 1}{4c} \exp(ikx + ikct), \qquad (2)$$

$$\alpha_0 = \exp(ikx - ikt + i\mu k^3 t),$$

where

$$c(k) = \left|(1 - 2\mu k^2)^{\frac{1}{2}}\right|$$

and the subscript 1 has been omitted again.
The formulae (1) and (2) clearly demonstrate the development of left - and right moving waves, whereas α_0 consists of a right travelling wave only.
Substitution of $\sin(kct) = \frac{1}{2i}\left[\exp(ikct) - \exp(-ikct)\right]$ in (1) and (2) leads to

$$\alpha = \exp(ikx - ikct) - \tfrac{1}{2}i\,\frac{(1-c)^2}{c} \sin(kct)\exp(ikx),$$

$$\beta = \tfrac{1}{2}i\,\frac{c^2-1}{c} \sin(kct)\exp(ikx),$$

showing that α may also be seen as a superposition of a right moving - and a standing, β as a pure standing wave. Expanding $(1 - 2\mu k^2)^{\frac{1}{2}}$ around $k = 0$ gives

$$\alpha = \left[1 + \tfrac{1}{2}i\mu^2 k^5 t + \ldots\right] \exp\left[ikx - ikt + i\mu k^3 t\right] +$$
$$+ \left[-\tfrac{1}{2}i\mu^2 k^4 \sin(kt) + \ldots\right] \exp(ikx).$$

From this equation we infer that, if $\mu^2|k|^5 t << 1$,

$$|\alpha - \alpha_0| << |\alpha_0| = 1, \qquad (3)$$

so we may speak of a good simple wave approximation.

If $\mu k^2 << 1$, the left travelling part of the α - mode is still small compared with the remaining part as $t \to \infty$. However, the difference between the phases of α's right moving part and α_0 may be large and therefore (3) fails to hold. If one is not interested in the relative phases mentioned, as is often the case in dealing with periodic waves, one may still speak of a useful approximation. A similar problem will arise in dealing with $L_2^\Delta(R)$ solutions.

4.2 Expansion in a series of $L_2^\Delta(R)$ solutions.

To get some more insight in the character of solutions of equations like (1.1) and (1.2), one often uses expansions in a series. We shall construct such an expansion of α, taking as the first term in the series α_0. The method we shall use is entirely similar to that used in [1], therefore all details will be stripped. Let $f \in L_2^\Delta(R)$. Introduce the operators

$$M = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu\frac{\partial^3}{\partial x^3},$$
$$N = \frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \mu\frac{\partial^3}{\partial x^3},$$

Then, (1.1) and (1.2) become

$$M\alpha = -\mu\beta_{xxx},$$

$$N\beta = \mu\alpha_{xxx},$$

so

$$MN\alpha = -\mu\frac{\partial^3}{\partial x^3}N\beta = -\mu^2\frac{\partial^6\alpha}{\partial x^6},$$

which implies that α and β satisfy

$$L\alpha = -\mu^2\frac{\partial^6\alpha}{\partial x^6}, \qquad (4)$$
$$L\beta = -\mu^2\frac{\partial^6\beta}{\partial x^6}$$

respectively.

L is given by

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - 2\mu \frac{\partial^4}{\partial x^4} - \mu^2 \frac{\partial^6}{\partial x^6} .$$

The initial data become

$$\left.\begin{aligned} &\alpha(x,0) = f(x), \\ &\alpha_t(x,0) = -\frac{df(x)}{dx} - \mu\frac{d^3f(x)}{dx^3}, \end{aligned}\right\} \qquad (5)$$

$$\beta(x,0) = 0,$$

$$\beta_t(x,0) = \mu\frac{df(x)}{dx} .$$

The solution of (4) and (5) satisfies

$$\alpha = \alpha_0 + A\alpha, \qquad (6)$$

where

$$A\alpha = \frac{\mu^2}{2\pi}\int_0^t d\tau \int_{-\Delta}^{\Delta} dk\ k^5\ \overline{\alpha}(k,\tau)\ \frac{\sin\left[(k - \mu k^3)(t - \tau)\right]}{1 - \mu k^2}\ \exp(ikx) .$$

In a similar way, we find that β satisfies

$$\beta = \mu\beta_1 + A\beta, \qquad (7)$$

where

$$\beta_1 = \frac{-1}{2\pi}\int_{-\Delta}^{\Delta} ik^3\overline{f}(k)\ \frac{\sin\left[(k - \mu k^3)t\right]}{k - \mu k^3}\ \exp\ (ikx)dk.$$

(6) and (7) may be solved by means of iteration:

$$\alpha^{(0)} = \alpha_0,$$

$$\alpha^{(2n)} = \mu^{-2}A\alpha^{(2n-2)}, \tag{8}$$

$$\beta^{(1)} = \beta_1,$$

$$\beta^{(2n-1)} = \mu^{-2}A\beta^{(2n-3)} \qquad (n = 1,2, \ldots).$$

Now, starting from (8) it may be shown, similar as was done in [1] and choosing the function $q^2(k,t)$ used there equal to $\exp\left[-\frac{\mu^4k^{10}}{(1-\mu k^2)^2}e^t - t\right]$, that, for all finite $t \geq 0$, $\sum_{n=0}^{N} \mu^{2n}\alpha^{(2n)}$ converges to α, $\sum_{n=0}^{N} \mu^{2n+1}\beta^{(2n+1)}$ converges to β as $N \to \infty$ in the sense of the $L_2^\Delta(R)$ norm. Furthermore

$$||\alpha - \sum_{n=0}^{N} \mu^{2n}\alpha^{(2n)}||_{R,\Delta} \leq \{\sum_{n=N+1}^{\infty} \left(\frac{\Delta^5\mu^2 t}{1-\mu\Delta^2}\right)^n \frac{1}{n!}\} \; ||\alpha||_{R,\Delta}. \tag{9}$$

4.3 The simple wave approximation.

We shall call $\alpha_0 \in L_2^\Delta(R)$ a good simple wave approximation to $\alpha \in L_2^\Delta(R)$ in the interval of time $[t_1,t_2]$ if and only if for every $t \in [t_1,t_2]$

$$||\alpha - \alpha_0||_{R,\Delta} \ll ||\alpha||_{R,\Delta}. \tag{10}$$

According to (9), (10) is satisfied for $t \in [0,T]$ where $\mu^2\Delta^5 T \ll 1$. This result is entirely similar to that found in case of periodic solutions discussed in section 4.2. Now, we shall show that (10) is certainly not satisfied for all $L_2^\Delta(R)$ solutions as $t \to \infty$. This is in constrast with the result we found in [1] for the simple wave approximation of the dissipative set of equations.
By using Parseval's theorem we find from (3.1), (3.2) and (3.3)

$$||\alpha - \alpha_0||^2_{R,\Delta} = 2||\alpha_0||^2_{R,\Delta} + ||\beta||^2_{R,\Delta} +$$

$$+ \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \{-(k+\omega)^2\cos[(\omega-\omega_0)t] + (k-\omega)^2\cos[(\omega+\omega_0)t]\}\frac{|\bar{f}|^2 dk}{2k\omega} \quad (11)$$

where

$$\omega_0 = k - \mu k^3,$$

$$||\alpha_0||_{R,\Delta} = ||f||_{R,\Delta},$$

$$||\beta||^2_{R,\Delta} = \frac{1}{2\pi}\int_{-\Delta}^{\Delta}\frac{(k^2-\omega^2)^2}{4k^2\omega^2}\sin^2(\omega t)|\bar{f}|^2 dk.$$

In these formulae ω will be chosen in the first sheet of the complex k - plane. We may also write

$$||\beta||^2_{R,\Delta} = \frac{1}{2\pi}\int_{-\Delta}^{\Delta}\frac{(k^2-\omega^2)^2|\bar{f}|^2 dk}{8k^2\omega^2} - \frac{1}{2\pi}\int_{-\Delta}^{\Delta}\frac{(k^2-\omega^2)^2\cos(2\omega t)|\bar{f}|^2 dk.}{8k^2\omega^2} \quad (12)$$

ω and $\omega + \omega_0$ may have two-, $\omega - \omega_0$ three points of stationairy phase for $k \in [-\Delta,\Delta]$. They are located symmetrically with respect to $k = 0$. Assuming that $\bar{f}(k)$ is of bounded variation in $[-\Delta,\Delta]$, we find by applying the method of stationairy phase to (11) and (12) (Lauwerier [5]) and using the lemma of Riemann - Lebesque

$$\lim_{t\to\infty} ||\alpha-\alpha_0||^2_{R,\Delta} = 2||f||^2_{R,\Delta} + \frac{1}{2\pi}\int_{-\Delta}^{\Delta}\frac{(k^2-\omega^2)^2|\bar{f}|^2 dk.}{8k^2\omega^2}$$

It is thus proved that (10) does not hold for all $L_2^{\Delta}(R)$ solutions as $t \to \infty$. The result is due to the oscillatory character of the solution for large t and will become more clear in the next section.

5. ASYMPTOTIC BEHAVIOUR AS $t \to \infty$.

Let $f \in L_2^{\Delta}(R)$ and $\bar{f}(k)$ analytic in $(-\Delta,\Delta)$. Write

$$\alpha = \alpha_1 + \alpha_2,$$

where

$$\alpha_j = \frac{1}{2\pi}\int_{-\Delta}^{\Delta} j\,\frac{(k+\omega)^2\bar{f}(k)}{4\omega k}\exp\left[ih(k,\xi)t\right]dk \qquad (j = 1,2),$$

$$h(k,\xi) = k\xi - \omega(k),$$

$$\xi = xt^{-1}.$$

We shall study the asymptotic behaviour of α_1 as $t \to \infty$ by means of the method of stationairy phase. The function α_2 may be treated in a similar way.

Let, from now on until stated otherwise, all functions defined in the k-plane, be defined in the first sheet of that plane. Let ε and δ be positive, but arbitrary small, numbers. The points of stationairy phase of $h(k,\xi)$ are solutions of

$$\xi = v(k),$$

where the group velocity $v(k) = \frac{d\omega}{dk}$ is given by

$$v(k) = (1 - 4\mu k^2)\,(1 - 2\mu k^2)^{-\frac{1}{2}}.$$

If $v(\Delta) \leq \xi \leq 1$, two such points exist (say) $\bar{k}(\xi)$ and $-\bar{k}(\xi)(\bar{k} \geq 0)$. Outside that range of ξ - values there is none. So, if $-\infty < \xi \leq v(\Delta) - \delta$ or $1 + \varepsilon \leq \xi < \infty$ we find by means of partial integration:

$$\alpha_1 = \frac{\left[\Delta + \omega(\Delta)\right]^2}{8\pi i\Delta\omega(\Delta)\left[\xi - v(\Delta)\right]t}\left\{\bar{f}(\Delta)e^{i\left[\Delta x - \omega(\Delta)t\right]} + \bar{f}(-\Delta)e^{-i\left[\Delta x - \omega(\Delta)t\right]}\right\} +$$
$$+ \mathcal{O}(t^{-2})\,(t \to \infty). \qquad (1)$$

According to Copson [6], the method of stationary phase yields, if $v(\Delta) + \delta \leq \xi \leq 1 - \varepsilon$,

$$\alpha_1 = \left[-2\pi v'(\bar{k})t\right]^{-\frac{1}{2}}\frac{\left[\bar{k} + \bar{\omega}\right]^2}{4\bar{\omega}\bar{k}}\left\{\bar{f}(\bar{k})e^{i\left[\bar{k}x - \bar{\omega}t + \pi/4\right]} +\right.$$
$$\left. + \bar{f}(-\bar{k})e^{-i\left[\bar{k}x - \bar{\omega}t + \pi/4\right]}\right\} + \mathcal{O}(t^{-1})\,(t \to \infty), \qquad (2)$$

where $\bar{\omega} = \omega(\bar{k})$ and $v'(k)$ is the first derivative of $v(k)$.

When $\xi \to v(\Delta)$, we have $|\bar{k}| \to \Delta$. Therefore, the domain $v(k) - \delta < \xi < v(k)+\delta$ has been omitted from the range of ξ - values. Another method is necessary in that case. However, as these values of ξ are relatively unimportant, we shall not proceed in that direction.

As $\xi \to 1$, $v'(k) \to 0$. So, only if $\bar{f}(0) = 0$ is satisfied, (1) may be used for $v(\Delta) + \delta \leq \xi \leq 1$. Then, thanks to the analyticity of $\bar{f}$ in a vicinity of $k = 0$, $\bar{f}(k) = \mathcal{O}(k)$ as $k \to 0$.

Now, let $\bar{f}(0) \neq 0$.

Theorem 1.

Define $\eta = \xi - 1$. Let $|\eta t| \leq 1$. As $t \to \infty$,

$$\alpha_1 = \bar{f}(0)(3\mu t)^{-\frac{1}{3}} Ai\left[(3\mu t)^{-\frac{1}{3}}\eta t\right] + \mathcal{O}(t^{-\frac{2}{3}}), \qquad (3)$$

where the Airy - function $Ai(x)$ is defined by

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\ (\tfrac{1}{3}y^3 + xy)dy.$$

Proof.

$\bar{f}(k)$ is regular for $|k| < \Delta$. Split the interval of integration in three parts: $[-\Delta, -p)$, $[-p, p]$ and $(p, \Delta]$ where $0 < p < \Delta$. By means of partial integration we see that the contributions of the first- and third interval are $\mathcal{O}(t^{-1})$ as $t \to \infty$.

Introduce a new variable u by means of

$$h(k,\xi) = \eta u + \mu u^3.$$

If $\eta = 0$ this equation has one -, otherwise three solutions

$$k = \sum_{n=1}^{\infty} b_n u^n,$$

all regular in a vicinity of $u = 0$.

Choose that solution for which b_1 is real, so $b_1 = 1$.

It follows that

$$\frac{(k + \omega)^2\overline{f}(k)}{4\omega k} \frac{dk}{du} = \sum_{n=0}^{\infty} c_n u^n,$$

where $c_0 = \overline{f}(0)$. Write

$$\frac{(k + \omega)^2\overline{f}(k)}{4\omega k} \frac{dk}{du} = \overline{f}(0) + u\Psi(u),$$

so, as $t \to \infty$,

$$\alpha_1 = \frac{1}{2\pi}\int_{-q}^{q} \{\overline{f}(0) + u\Psi(u)\} \exp[iu\eta t + i\mu u^3 t]du + \mathcal{O}(t^{-1}), \qquad (4)$$

where

$$q = u(p).$$

We may write (4) in the form

$$\alpha_1 = \frac{\overline{f}(0)}{2\pi}\int_{-\infty}^{\infty} \exp[iu\eta t + i\mu u^3 t]du + I + \mathcal{O}(t^{-1}) \quad (t \to \infty),$$

where

$$I = \frac{1}{2\pi} \int_{-q}^{q} u\Psi(u) \exp[iu\eta t + i\mu u^3 t]du - \frac{\overline{f}(0)}{2\pi}\left[\int_{-\infty}^{-q} + \int_{q}^{\infty}\right] \exp[iu\eta t + i\mu u^3 t]du.$$

In the appendix it is proved that $I = \mathcal{O}(t^{-2/3})$ as $t \to \infty$. This proves the theorem.

The theorem gives information about the asymptotic behaviour in the shaded region of fig.1.

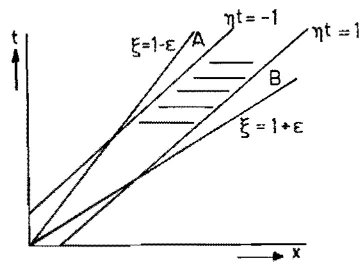


fig. 1

Still, we don't have information about the regions A and B. Most important of course is A.

Theorem2.

As $t \to \infty$, the range of validity of (2) and (3) can be extended to A.

Proof.

We shall demonstrate that formal expansions of (2) and (3) fit together in A. Let $\eta t < 0$. Expand with respect to large η, in particular (t fixed) $-\eta t(3\mu t)^{-1/3} \gg 1$. From Abramowitz and Stegun [7] we obtain

$$Ai(-z) = \pi^{-1/2} z^{-1/4} \sin(\tfrac{2}{3} z^{3/2} + \tfrac{\pi}{4}) + \mathcal{O}(z^{-7/4}),$$

as $z \to \infty$. Using this result we find from (3) as $t \to \infty, -\eta t^{2/3} \to \infty$:

$$\alpha_1 \sim \overline{f}(0)(-3\pi^2 \eta\mu t^2)^{-1/4} \sin[-\tfrac{2}{3}\eta\ (\tfrac{-\eta}{3\mu})^{\frac{1}{2}} t + \tfrac{\pi}{4}]. \qquad (5)$$

Expand all quantities in (2) for small η. We find as $\eta \to 0$

$$\overline{k} \sim (\tfrac{-\eta}{3\mu})^{\frac{1}{2}},\ v'(\overline{k}) \sim -2(-3\mu\eta)^{\frac{1}{2}},\ ht \sim \tfrac{2}{3}\eta t(\tfrac{-\eta}{3\mu})^{\frac{1}{2}}$$

and so, for $\eta \to 0$, $t \to \infty$, (5) may be deduced again. This proves the theorem.

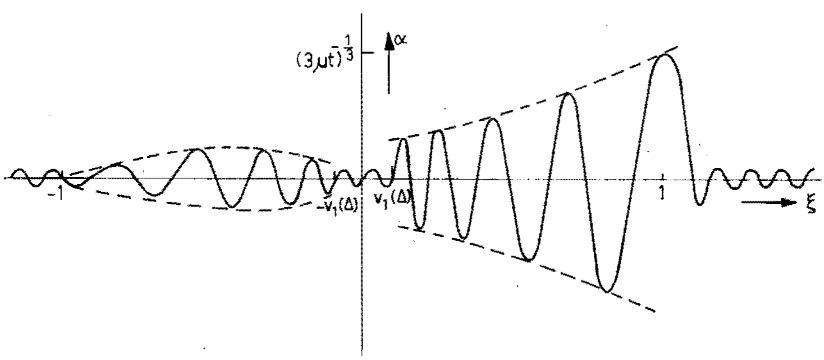
Speaking in terms of singular perturbation theory one may call (2) the outer - and (3) the inner solution. Then, η is interpreted as the large outer - and small inner variable. In fact we have made an outer expansion of the inner solution and the reverse and demonstrated they fit together (Kaplun [8]).

If one is able to construct asymptotic expansions of α_1 for $|\xi - 1| \leq \varepsilon$, $|\eta t| \leq 1$ and for $1 + \varepsilon \leq \xi < \infty$, it must surely be possible to fit them together in B. However, this leads to severe mathematical difficulties lying beyond the scope of this paper.

For α_2 we also find (1) and (2), but now, all functions are defined in the second sheet of the k - plane.† As $\frac{(\omega+k)^2}{\omega k} = \mathcal{O}(k^4)$ as $|k| \to 0$ in that sheet, (1) holds for $v(\Delta) + \delta \leq \xi < \infty$ and $-\infty < \xi \leq -1$, (2) for $-1 \leq \xi \leq v(\Delta) - \delta$. Now, we may sketch the wave phenomenon.

fig.2: $\bar{f}(k) = 1 \quad (|k| \leq \Delta),\ \Delta < 1/2\sqrt{\mu}$

$v_1(\Delta) = (1-4\mu\Delta^2)/\ |(1-2\mu\Delta^2)^{\frac{1}{2}}|$.

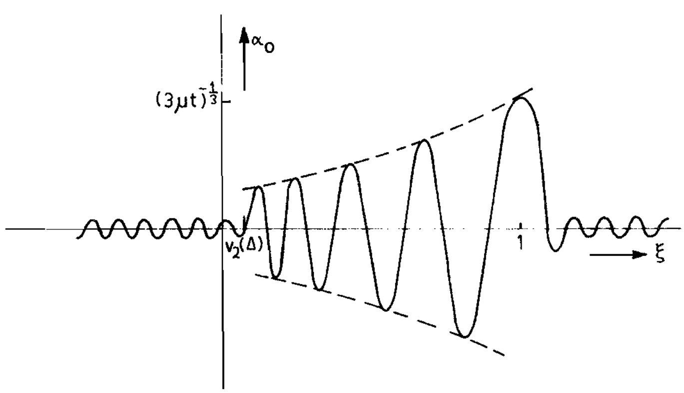


The front of the right travelling part of the α - wave is formed by the "Airy wave". Such a wave may described by a carrier wave which is amplitude modulated. However, the carrier wave has a wavelength infinitely longer than that of the modulation. This kind of waves also constitutes the so called tidal waves. The steepness of the front of the Airy wave and the "wavelength" directly behind the front increase of the cube root of t.

†Of course, $\bar{k}$ is now the negative solution of $\xi = v(k)$.

The asymptotic behaviour of α_0 can be described in a similar way. The result is sketched in fig. 3. It turns out that if $\mu\Delta^2 << 1$, α and α_0 resemble each other for $x \geq 0$ in the sense that local amplitude and wavelength only slightly differ. Then, the left moving part of the α - mode is small (in the maximum norm) compared with the right moving part. Therefore, in that sense, one may call α_0 a good approximation of α as $t \to \infty$. This result resembles that of section 4.2 very much. This section also explains the remark made at the end of section 4.3.

fig.3: $\bar{f}(k) = 1$ $(|k| \leq \Delta)$, $\Delta < (3\mu)^{-\frac{1}{2}}$,
$v_2(\Delta) = 1 - 3\mu\Delta^2$.



6. SOME CRITICAL REMARKS ON A NONLINEAR CASE.

In this section we want to devote some attention to a simple wave approximation of a set of nonlinear equations. Let that set be given by

$$\alpha_t + [1 + \varepsilon(\alpha + \beta)]\alpha_x = -\mu(\alpha + \beta)_{xxx}, \tag{1}$$

$$\beta_t - [1 + \varepsilon(\alpha + \beta)]\beta_x = \mu(\alpha + \beta)_{xxx}, \tag{2}$$

and the initial conditions by (1.4) and (1.5).
Then, by a similar reasoning as used in section 1, we may argue that for some finite interval of time the behaviour of the α - mode approximately is described by the solution α_0 of

$$\alpha_{0t} + \alpha_{0x} + \varepsilon\alpha_0\alpha_{0x} = -\mu\alpha_{0xxx}, \tag{3}$$

$$\alpha_0(x,0) = f(x). \tag{4}$$

We suppose again that $f \in L_2^{\Delta}(R)$.
Equations (1) and (2) are used by Zabusky [4] as an intermediate representation for the equations describing the behaviour of a nonlinear one-dimensional lattice. Then, by the further approximation indicated, he finds (3), which is the KdV equation. This equation has been studied by him and several other authors [9, 10, 11] extensively. It is also used as a long wavelength approximation in various fields of physics such as the theory of cold plasmas and shallow water theory (see [11, 12, 13]).
Here we want to make some remarks on equations (1) and (2) and the corresponding simple wave approximation. First, it is not clear at all whether the equations (1) and (2) subject to (1.4) and (1.5) have stable solutions. The solution α_0 of (3) subject to (4) is stable. By stability we mean that a positive definite norm for the solution exists such that, uniformly with respect to time, it is bounded in terms of the corresponding norms of the initial conditions. (1.1) and (1.2) gave rise to the same problem. However, in that case, stability is assured due to the boundedness of the spectral range of the solutions (see [3]). The solutions of (1) and (2) subject to (1.4) and (1.5) have an unbounded spectral range for each $t > 0$.
This unboundedness also leads to the remark that the simple wave approximation probably will break down even faster than in the linear case. However, one should be very careful stating that conjecture as, especially for large times t, the solution of the KdV equation (3) subject to (4) is of an entirely different character than that of the linearized version (1.6). The nonlinear solution consists of solitons which are steady progressive solutions of (3).

They result from a balance between the dispersive- and nonlinear effects. Nevertheless, when we are far before breakdown time, that is the time at which the solution of (3) where $\mu = 0$ starts developing a shock wave, the solution of (3) and its linear version probably will look very much alike (cf.[4]). In that case, the conjecture made above, presumably is useful.

APPENDIX.

We shall prove that I, defined in section 4, equals $\mathcal{O}(t^{-2/3})$ as $t \to \infty$. Denoting

$$P(u;\eta t,\mu t) = \exp\left[i\eta t u + i\mu t u^3\right]$$

and writing

$$u\Psi(u) = c_1 u + u^2\chi(u), \tag{1}$$

we have

$$I = \frac{c_1}{2\pi}\int_{-\infty}^{\infty} uP\,du - \frac{1}{2\pi}\left[\int_{-\infty}^{-q} + \int_{q}^{\infty}\right]\left[\bar{f}(0)+c_1 u\right] P\,du + \frac{1}{2\pi}\int_{-q}^{q} u^2\chi(u)P\,du.$$

Choose $q \leq 1$. Now, using partial integration twice

$$\left|\int_q^\infty \left[\bar{f}(0) + c_1 u\right]P\,du\right| \leq \left|\int_q^\infty \frac{ic_1\eta t}{u}\,\frac{P}{3i\mu t}\,du\right| +$$

$$+\frac{|\bar{f}(0)| + |c_1|q}{3\mu t q^2} + \left|\int_q^\infty \left(\frac{-2\bar{f}(0)}{u^3} - \frac{c_1}{u^2} + \frac{i\bar{f}(0)\eta t}{u^2}\right)\frac{P}{3i\mu t}\,du\right| \leq$$

$$\leq \frac{|c_1|\,|\eta t|}{3\mu t}\left\{\frac{1}{3\mu t q^3} + \left|\int_q^\infty \left(\frac{i\eta t}{u^3} - \frac{3}{u^4}\right)\frac{P}{3i\mu t}\,du\right|\right\} +$$

$$+ \frac{[3|\overline{f}(0)| + 2|c_1|]}{3q^2\mu t} \leq \frac{[3|\overline{f}(0)| + 2|c_1|]}{3q^2\mu t} + \frac{|c_1|}{3q^4\mu^2 t^2}.$$

In a similar way, we may estimate $\int_{-\infty}^{-q} [\overline{f}(0) + c_1 u]P\, du$.

By partial integration we also find

$$\left|\int_{-q}^{q} u^2\chi(u)P\, du\right| \leq \max_{u=\pm q} \frac{|\chi(q)|}{3\mu t} + \left|\int_{-q}^{q} \left[int\ \chi(u) + \frac{d\chi}{du}\right] \frac{P}{3i\mu t}\, du\right| \leq$$

$$\leq \text{constant.}\ (3\mu t)^{-1}.$$

At this place, the reason for the further splitting (1) becomes clear. Upon partial integration of $\int_{-q}^{q} u\Psi(u)P\, du$, we would have introduced a pole in the new integrand.

Finally

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} uP(u;\eta t;\mu t)du = - i(3\mu t)^{-2/3}\left[\frac{dAi(\xi)}{d\xi}\right]_{\xi = (3\mu t)^{-1/3}\eta t},$$

from which the statement made at the beginning of this appendix immediately follows.

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SAMENVATTING

In de laatste twintig jaren heeft de interesse in niet-lineaire dissipatieve en dispersieve golfvergelijkingen zich vooral toegespitst op de Burgers-, respectievelijk Korteweg-de Vries-vergelijking. De eerste vergelijking beschrijft, bij benadering, de voortplanting van kleine amplitude schokgolven in een continu medium en wordt gegeven door

$$u_t + uu_x - u_{xx} = 0. \qquad (1)$$

De indices geven partiële differentiatie aan.
Korteweg en de Vries leidden de KdV-vergelijking af in hun studie van lange watergolven in een relatief ondiep kanaal, zie [1]*. Recentelijk werd de vergelijking ook afgeleid in de plasmafysica en de theorie van anharmonische roosters (zie de referenties in [2]). Ze wordt gegeven door

$$u_t + uu_x + u_{xxx} = 0. \qquad (2)$$

Het Cauchy probleem voor de Burgers-vergelijking is exact oplosbaar, cf. [3] en [4]. De KdV-vergelijking is intensief bestudeerd zowel m.b.v. analytische als numerieke methoden, cf. [2] en [5].
(1) en (2) zijn afgeleid door gebruik te maken van een lange golflengte en kleine niet-lineariteit benadering en de aanname, dat het golfverschijnsel bij benadering wordt beschreven door golven, die slechts één kant uitlopen. De oorspronkelijke vergelijkingen, welke de beschouwde fysische situatie beschrijven, hebben golven als oplossingen, die beide kanten uitlopen. Als gevolg van de termen van niet-lineaire en/of dissipatieve (of dispersieve) aard in die vergelijkingen, zijn de naar rechts (in positieve x-richting) en de naar links lopende golven in wisselwerking met elkaar. Bovendien zorgt de niet-lineariteit van het probleem ervoor dat ook kleine golflengten in de Fourierspectra van de oplossingen worden gegenereerd. Daarom lijkt het redelijk om te veronderstellen, dat de benadering die leidt tot de Burgers- of de KdV-vergelijking, alleen "geldt" voor een eindig tijdsinterval en is het zinvol om te vragen naar het geldigheidsbereik van die benadering.
Dit proefschrift houdt zich voornamelijk bezig met dat probleem.
Het bestaat, buiten de inleiding, uit zes manuscripten, welke in de inhoudsopgave genummerd zijn als II - VII.

*[a] correspondeert met referentie no. a uit de "INTRODUCTION".

Het algemene probleem is erg gecompliceerd. Wat betreft de Burgers-vergelijking zullen we ons beperken tot de klasse van vergelijkingen

$$\alpha_t + [1 + \varepsilon(c\alpha + d\beta)]\alpha_x = \mu(\alpha_{xx} - \beta_{xx}), \qquad (3)$$

$$\beta_t - [1 + \varepsilon(c\beta + d\alpha)]\alpha_x = \mu(\beta_{xx} - \alpha_{xx}), \qquad (4)$$

waarin c, d, μ en ε constanten zijn die zó gekozen worden dat, voor $\mu = 0$, het resterende stelsel hyperbolisch is. De positieve parameters μ en ε zijn maten voor de dissipatie, respectievelijk niet-lineariteit.
De afleiding van de Burgers-vergelijking verloopt dan als volgt. Beschouw de speciale beginwaarden

$$\alpha(x,0) = f(x), \qquad (5)$$

$$\beta(x,0) = \beta_0 \quad (\beta_0 \text{ constant}). \qquad (6)$$

M.a.w. op t = 0 starten we met een golf, die slechts één kant uitloopt. Als $\mu = 0$ voldoet $\beta(x,t) = \beta_0$ identiek aan vergelijking (4) en is α een enkelvoudige golf oplossing, cf. Lax [6]. Indien μ ongelijk aan nul doch klein is, dan wordt verondersteld dat, tenminste gedurende enige tijd, $\beta-\beta_0$ klein blijft en α bij benadering beschreven wordt door de oplossing α_0 van

$$\alpha_t + [1 + \varepsilon(c\alpha + d\beta)]\alpha_x = \mu\alpha_{xx} \qquad (7)$$

en (5).
(7) kan door middel van een schaaltransformatie gemakkelijk gereduceerd worden tot de Burgers-vergelijking. Deze benaderingsmethode,welke we de enkelvoudige golf (e.g.) benaderingsmethode zullen noemen, is o.a. gebruikt door Lighthill [7]. We merken op dat recentelijk de Burgers-vergelijking (en ook de KdV-vergelijking) afgeleid is voor meer algemene systemen dan (3) en (4) m.b.v. de coördinaten-vervormingsmethode (ook wel singuliere storingsrekening genoemd), zie [2] en [8]. Als die methode wordt toegepast op (3) en (4), vinden we weer (7).
In II worden de vergelijkingen (3) en (4) met c en d beide gelijk aan nul beschouwd. Dit is het eenvoudigste geval. De vergelijkingen beschrijven dan, in Lagrange-coördinaten, de longitudinale beweging van een elastische balk met enige visceuze spanning. Voor kwadratisch integreerbare oplossingen α en α_0, noemen we α_0 een goede e.g. benadering van α in het tijdsinterval

$[t_1,t_2]$, indien, voor elke $t \in [t_1,t_2]$:

$$\int_{-\infty}^{\infty} |\alpha - \alpha_0|^2 dx << \int_{-\infty}^{\infty} |\alpha|^2 dx. \quad (8)$$

Omdat $\int_{-\infty}^{\infty} |\alpha|^2 dx$ kan worden beschouwd als een maat voor de energie in de α-mode, is dit een zeer bruikbare norm.
In II wordt aangetoond dat, als het spectrale bereik van de beginwaarde f(x) en dus, omdat het een lineair probleem betreft, van α, β en α_0, identiek nul is buiten een eindig interval $[-\Delta,\Delta]$, (8) geldt voor $t \in [0,T]$ met $\Delta^3\mu^2 T << 1$. Dit laat zien dat hoe kleiner het spectrale bereik van de beginwaarde f(x) is, des te langer geldt de e.g. benadering. (8) geldt weer als $t \to \infty$. Dit vindt zijn oorzaak in het feit dat de dissipatie voor kleine golflengten veel groter is dan voor grote. Als we dus maar lang genoeg wachten, dan dragen alleen de grote golflengten nog significant bij tot de golfvorm.
In het volgende manuscript III, wordt aandacht geschonken aan een stelsel niet-lineaire vergelijkingen dat, als de diffusietermen ervan gelinearizeerd zijn, overeenstemt met (3) en (4) waarin $c = 0$ en $d = 2$. Het stelsel beschrijft hetzelfde fysische systeem als in II, doch nu in Eulerse coördinaten. Het lineaire stelsel dat in (2) beschouwd wordt kan m.b.v. een eenvoudige niet-lineaire transformatie worden verkregen uit het niet-lineaire stelsel. Gebruik makend van dit feit is aangetoond dat, indien ε klein genoeg is, de resultaten m.b.t. het geldigheidsbereik en het gedrag voor $t \to \infty$ van de benadering, van dezelfde soort zijn als die, welke we voor de lineaire vergelijkingen vinden. De definitie van een goede e.g. benadering is dezelfde als in II.
De lineaire, als ook de niet-lineaire vergelijkingen welke in II, respectievelijk III worden bekeken, laten geen schokgolf-oplossingen toe. In het eerste geval is dit triviaal. In het tweede blijkt dat onmiddelijk uit het feit dat de karakteristieke snelheid van de α-mode alleen van β, van de β-mode alleen van α afhangt.
Uit fysisch oogpunt zijn niet-lineaire vergelijkingen die schokgolf-oplossingen toelaten de meest interessante, echter ongetwijfeld ook de moeilijkste vergelijkingen om te bestuderen.
In III wordt een probleem van deze soort bekeken. We hebben $c = 1$ en $d = \Theta = \frac{\gamma-3}{\gamma+1}$, met $\gamma = C_p/C_v$ gesteld. (3) en (4) vormen dan een benadering

van de Navier-Stokes-vergelijkingen. Ze zijn afgeleid door Lighthill in zijn publikatie [7] over viscositeitseffecten in geluidsgolven van eindige amplitude en beschrijven bij benadering de voortplanting van kleine amplitude geluidsgolven in een echt gas.
Nu zullen we spreken over een goede e.g. benadering in hét tijdsinterval $[0,T]$ indien voor iedere $t \in [0,T]$:

$$\int_{-\infty}^{\infty} |\alpha - \alpha_0|^2 dx << \int_{-\infty}^{\infty} |f|^2 dx.$$

Het is duidelijk dat dit een veel zwakkere conditie is dan (8). Echter voor niet te grote tijden en een kleine dissipatie (d.w.z. $\mu << 1$) zal $\int_{-\infty}^{\infty} |f|^2 dx$ niet veel verschillen van $\int_{-\infty}^{\infty} |\alpha|^2 dx$. De voorwaarde is dan zeker bruikbaar. Uitgaande van deze definitie is een bovengrens voor het geldigheidsbereik verkregen in termen van f, μ, ε en Θ. Deze bovengrens is altijd kleiner dan het tijdstip waarop een schokgolf zich begint te ontwikkelen (er natuurlijk van uit gaande dat de beginwaarden "glad" zijn). Dit vindt gedeeltelijk zijn oorzaak in de methode die we volgen om een uitdrukking voor die bovengrens af te leiden. Er wordt een richting aangegeven waarin toekomstig onderzoek zich zou kunnen begeven om scherpere resultaten te verkrijgen.
In V tenslotte wordt verslag gedaan van enkele numerieke berekeningen. De resultaten geven op grafische wijze een indruk van de ontwikkeling van de oplossingen α en β van de vergelijkingen met $c = d = 0$. Ze zijn, in het beginstadium van het onderzoek, gebruikt om wat meer inzicht te krijgen in het bestudeerde onderwerp.
Vervolgens beschouwen we een e.g. benadering van een stelsel dispersieve golfvergelijkingen. We hebben ons beperkt tot de afleiding van een gelineariseerde KdV-vergelijking uit een intermediaire representatie van de vergelijking welke de longitudinale beweging van een oneindige keten van identieke puntmassa's en veren beschrijft.We bedoelen daarmee een representatie "tussen" de exacte continuum-representatie en de laagste continuum-limiet (d.w.z. $a \to 0$) in. a is de roosterconstante. In VI hebben we een eenduidige continuum-representatie voor het systeem van gekoppelde lineaire oscillatoren geconstrueerd. Dit is gebeurd door te eisen dat de Fourier-getransformeerden van kwadratisch integreerbare oplossingen van de continuum-representatie identiek nul zijn buiten het eindige interval $[-\pi a^{-1}, \pi a^{-1}]$.

De intermediaire representatie wordt dan gegeven door

$$\alpha_t + \alpha_x = -\mu(\alpha+\beta)_{xxx} \qquad (\mu = a^2/24),$$

$$\beta_t - \beta_x = \mu(\alpha+\beta)_{xxx}$$

Het beginwaardeprobleem voor deze vergelijkingen is stabiel in de zin dat een positief definiete norm voor de oplossingen bestaat welke, uniform in de tijd, begrensd is in termen van de corresponderende norm van de beginwaarden. Indien geen beperking wordt opgelegd aan het spectrale bereik van de oplossingen, dan is dit niet waar. De dispersievergelijking

$$\omega^2 = k^2 - \frac{a^2k^4}{12}$$

is nl. niet positief voor elke reële k.
Uitgaande van de beginwaarden (5) en (6), wordt de e.g. benadering van de α-mode gegeven door de oplossing α_0 van

$$\alpha_t + \alpha_x + \mu\alpha_{xxx} = 0$$

en (5).
In VII blijkt dat we, in de zin van (8), over een goede e.g. benadering kunnen spreken voor $t \in [0,T]$ met $\Delta^5\mu^2T << 1$ (Δ is gedefinieerd als voorheen). Voor $t \to \infty$ geldt (8) niet langer. Dit wordt veroorzaakt door het oscillatorisch karakter van de oplossingen. Het gedrag van α, β en α_0 is, voor $t \to \infty$, m.b.v. asymptotiek in detail bekeken. Tot slot worden enkele opmerkingen gemaakt die betrekking hebben op de afleiding van de KdV-vergelijking (2) uit een niet-lineair stelsel.

CURRICULUM VITAE

De auteur van dit proefschrift werd op 17 juli 1946 geboren in Venlo. Van 1958 tot 1963 doorliep hij de H.B.S.-B op het St. Thomascollege te Venlo. Daarna studeerde hij tot 1968 voor natuurkundig ingenieur aan de Technische Hogeschool te Eindhoven. Na beëindiging van deze studie, trad hij in dienst van de N.V. Philips en werd wetenschappelijk medewerker van het Natuurkundig Laboratorium te Waalre. Door de N.V. Philips werd hij in de gelegenheid gesteld in de groep theoretische natuurkunde van de Technische Hogeschool te Eindhoven het promotieonderzoek te verrichten dat resulteerde in dit proefschrift.

S T E L L I N G E N

1. Het argument, dat Zabusky gebruikt om een intermediaire representatie van een vergelijking, die een eendimensionaal anharmonisch rooster beschrijft, te reduceren tot de Korteweg-de Vries vergelijking is niet juist. Bovendien is het niet duidelijk dat die intermediaire representatie stabiel is in de zin van Lyapunov.

   Zabusky, N.J., A Synergetic Approach to Problems of Nonlinear Dispersive Wave Propagation and Interaction, Proceedings of the Symposium on Nonlinear Partial Differential Equations, Academic Press, New York, 1967.

2. De gelinearizeerde Korteweg-de Vries vergelijking is, in de zin van dit proefschrift, voor grote waarden van de tijd t, geen goede enkelvoudige golf benadering meer van de gelinearizeerde, in stelling 1 genoemde intermediaire representatie van Zabusky.

   Dit proefschrift, p. 112.

3. De door Lighthill afgeleide vergelijkingen voor vlakke geluidsgolven van eindige amplitude in een gas

$$v_t + vv_x + \frac{2}{\gamma-1} aa_x = \delta v_{xx} ,$$

$$a_t + va_x + \frac{\gamma-1}{2} av_x = 0 ,$$

   waarin a de geluidssnelheid, v de stroomsnelheid, $\gamma = C_p/C_v$ en δ de "diffusity of sound" is, gelden in feite slechts voor golven die, bij benadering, één kant uit lopen.

   Lighthill, M.J., Viscosity Effects in Sound Waves of Finite Amplitude, Surveys of Mechanics, Cambridge, 1956.

4. De enkelvoudige golf benadering van de gelinearizeerde versie van de in de vorige stelling genoemde vergelijkingen van Lighthill is, in de zin daaraan gegeven in dit proefschrift, niet voor elke tijd t een goede benadering.

   Dit proefschrift, hoofdstuk II.

5. Laat de verstrooiing van een uit negatieve x-richting komend golfpakket aan een deltafunctiepotentiaal beschreven worden door de oplossing van het Cauchy-probleem

$$\underline{u}_{tt} - A^2\underline{u}_{xx} + R^2\underline{u} = S\underline{u}\delta(x) \ ,$$

$$\underline{u}(x,0) = \begin{matrix} \underline{f}(x) & x < 0 \\ \underline{0} & x \geq 0 \end{matrix} \ , \quad \underline{u}_t(x,0) = \begin{matrix} \underline{g}(x) & x < 0 \\ \underline{0} & x \geq 0 \end{matrix} \ ,$$

met $\underline{u}$ = kolom$(u_1(x,t),\ldots, u_n(x,t))$ en A, R en S constante reële n×n-matrices. A^2 is positief en diagonaal, R en S zijn symmetrisch.
Indien $2\sqrt{A^{-1}R^2A^{-1}} - A^{-1}SA^{-1}$ positief is (de wortel is niet negatief en symmetrisch gedefinieerd), dan is de oplossing van het Cauchy probleem stabiel in de zin van Lyapunov.

6. Voor een voldoend "gladde" reële oplossing van het Cauchy probleem

$$\alpha_t + \alpha\alpha_x = \alpha_{xx} \quad \text{(de vergelijking van Burgers)},$$

$$\alpha(x,0) = f(x),$$

geldt voor elke T > 0 de afschatting

$$\int_0^T \int_{-\infty}^{\infty} \alpha_{xx}^2 \, dxdt \leq 2ac + b + 2\sqrt{a^2c^2+abc},$$

$$b = \int_{-\infty}^{\infty} \left(\frac{df}{dx}\right)^2 dx, \quad c = \frac{1}{2} \sup_{x\in(-\infty,\infty)} |f(x)| \quad \text{en} \quad a = c \int_{-\infty}^{\infty} f^2(x)dx.$$

7. De door Sjöberg bewezen stelling over de existentie en eenduidigheid van de oplossing van het Cauchy probleem voor de Korteweg-de Vries vergelijking, met periodieke beginwaarde, is onbevredigend.

Sjöberg, A., On the Korteweg-de Vries Equation, Existence and Uniqueness, Uppsala University, Department of Computer Sciences, Uppsala, Sweden, 1967 (niet gepubliceerd).

8. Het existentie- en eenduidigheidsbewijs, dat Oleinik en Kruzhkov geven voor de klassieke oplossing van het Cauchy probleem voor een klasse van niet lineaire parabolische vergelijkingen, is niet volledig.

Oleinik, O.A. en S.N. Kruzhkov, Quasi Linear Second Order Parabolic Equations with Many Variables, Russ.Math.Surv., Vol. 16, p.p. 105-144, 1961.

9. De definitie van een meting van de eerste soort zoals Jauch die geeft in zijn boek "Foundations of Quantum Mechanics" is geen zinvolle omdat hij niet experimenteel verifieerbaar is.
Bovendien zijn de meeste metingen niet van die soort.

Jauch, J.M., Foundations of Quantum Mechanics, Addison-Wesley, 1968.

10. Het reductiepostulaat uit de quantummechanica, inhoudende dat, als perfecte meting van de observabele *A* (met bijbehorende lineaire zelfgeadjungeerde operator A met een volledig orthonorm stelsel eigenvectoren op een Hilbertruimte) de waarde a_k oplevert, de toestand van het systeem na meting wordt gegeven door α_k met $A\alpha_k = a_k\alpha_k$, is absurd.
Het is betreurenswaardig dat men dit postulaat, ook wel het (sterke) projectiepostulaat genoemd, nog in enkele leerboeken over quantummechanica, zoals de hierna genoemde, aantreft.

Blokhintsev, D.I., Quantum Mechanics, D. Reidel Publ. Co., Dordrecht-Holland, 1964, p.58,
Dirac, P.A.M., The Principles of Quantum Mechanics, Oxford University Press, Oxford, 1949, p.36,
Messiah, A., Quantum Mechanics, Vol.I, North-Holland Publ. Co., Amsterdam, 1967, p.198.

11. In veel leerboeken over quantummechanica wordt te weinig aandacht besteed aan en onkritisch gesproken over het meetproces in de quantummechanica. Met name schenkt men te weinig aandacht aan de experimentele implicaties van die postulaten, welke betrekking hebben op dat meetproces.

12. Het zou wenselijk zijn dat in de toekomst meer aandacht wordt besteed aan gemengd hyperbolisch-parabolische stelsels van partiële differentiaalvergelijkingen, dit met name m.b.t. de (globale) existentie en eenduidigheid van Cauchy en randwaarde problemen voor deze vergelijkingen.