A complex-like calculus for spherical vectorfields

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Dedicated to Professor Bob Mattheij at his retirement

Abstract

First, $\mathbb{R}^{1+\mathfrak{d}}$, $\mathfrak{d} \in \mathbb{N}$, is turned into an algebra by mimicing the usual complex multiplication. Indeed the special case $\mathfrak{d} = 1$ reproduces $\mathbb{C}$. For $\mathfrak{d} > 1$ the considered algebra is commutative, but non-associative and even non-alternative.
Next, the Dijkhuis class of mappings ('vectorfields') $\mathbb{R}^{1+\mathfrak{d}} \to \mathbb{R}^{1+\mathfrak{d}}$, suggested by C.G. Dijkhuis for $\mathfrak{d} = 3$, $\mathfrak{d} = 7$, is introduced. This special class is then fully characterized in terms of analytic functions of **one** complex variable.
Finally, this characterization enables to show easily that the Dijkhuis-class is closed under *pointwise* $\mathbb{R}^{\mathfrak{d}+1}$-multiplication: It is a commutative and associative algebra of vector fields.
Previously it had not been observed that the Dijkhuis-class **only** contains vectorfields with a 'time-dependent' spherical symmetry. Such disappointment was to be expected!
The class of functions which are differentiable with respect to the algebraic structure, that we impose on $\mathbb{R}^{1+\mathfrak{d}}$, contains **only linear** functions if $\mathfrak{d} > 1$. The Dijkhuis-class does not appear this way either!
In our treatment neither quaternions nor octonions play a role.

1 Imitation of complex calculus in higher dimensions

On $\mathbb{R}^{1+\mathfrak{d}}$, with $\mathfrak{d} \in \mathbb{N}$, a commutative multiplication structure is introduced by

$$(\alpha\,;\,\underline{a}) \cdot (\beta\,;\,\underline{b}) = (\alpha\beta - \underline{a}^\top \underline{b}\,;\,\alpha\underline{b} + \beta\underline{a}), \qquad \alpha, \beta \in \mathbb{R}\,, \quad \underline{a}, \underline{b} \in \mathbb{R}^{\mathfrak{d}}. \tag{1.1}$$

Note 1. This multiplication structure is non-associative (non-alternative) if $\mathfrak{d} > 1$. Indeed

$$\Big((\alpha\,;\,\underline{a}) \cdot (\beta\,;\,\underline{b})\Big) \cdot (\gamma\,;\,\underline{c}) - (\alpha\,;\,\underline{a}) \cdot \Big((\beta\,;\,\underline{b}) \cdot (\gamma\,;\,\underline{c})\Big) = (0\,;\,\underline{b}^\top \underline{ca} - \underline{a}^\top \underline{bc}),$$

which may not vanishe if for $(\lambda\,,\mu) \neq (0,0)$ one has $\lambda\underline{a} + \mu\underline{c} \neq \underline{0}$.
Clearly, with suitable interpretation, $\underline{b}^\top \underline{ca} - \underline{a}^\top \underline{bc} = -\underline{b} \times (\underline{c} \times \underline{a})$. **Note 2.** If $\mathfrak{d} = 3$ or $\mathfrak{d} = 7$, the product $(\alpha\,;\,\underline{a}) \cdot (\alpha\,;\,\underline{a})$ of ***equal*** elements corresponds, respectively, to the quaternion product and the octonion product.

Note 3. Symbolically, and sometimes conveniently, (1.1) can be written

$$(\alpha + i\underline{a}) \cdot (\beta + i\underline{b}) = (\alpha\beta - \underline{a}^\top \underline{b}) + i(\alpha\underline{b} + \beta\underline{a}).$$

Note 4. If for $\mathsf{v} = v_1 + iv_2 \in \mathbb{C}$ and $\underline{\xi} \in \mathbb{R}^{\mathfrak{d}}$ we introduce $\mathsf{v}\underline{\xi} \in \mathbb{R}^{1+\mathfrak{d}}$ by

$$\mathsf{v}\underline{\xi} = (v_1\,;\,\frac{v_2}{|\underline{\xi}|}\underline{\xi})\,,$$

we have the multiplication rule

$$\mathsf{v}\underline{\xi} \cdot \mathsf{w}\underline{\xi} = (\mathsf{vw})\underline{\xi}.$$

Here vw is the usual product of complex numbers.

Note 5. By induction one easily shows that, with $r = |\underline{x}|$, one has for $n = 1\,,\,2\,,\ldots$

$$\big(t\,;\,\underline{x}\big)^n = \big(\,\mathrm{Re}\,(t+ir)^n\,;\,\frac{\mathrm{Im}\,(t+ir)^n}{r}\,\underline{x}\,\big).$$

Some calculations

- $(\alpha\,;\,\underline{a}) \cdot (t\,;\,\underline{x})^n = \Big(\,\alpha\,\mathrm{Re}\,(t+ir)^n - \dfrac{\mathrm{Im}\,(t+ir)^n}{r}\,\underline{x}^\top \underline{a}\,;\,\alpha\dfrac{\mathrm{Im}\,(t+ir)^n}{r}\,\underline{x} + \mathrm{Re}\,(t+ir)^n\,\underline{a}\Big)$
- $(t\,;\,\underline{x})^m \cdot \big((\alpha\,;\,\underline{a}) \cdot (t\,;\,\underline{x})^n\big) = \big((\alpha\,;\,\underline{a}) \cdot (t\,;\,\underline{x})^n\big) \cdot (t\,;\,\underline{x})^m = (t\,;\,\underline{x})^n \cdot \big((\alpha\,;\,\underline{a}) \cdot (t\,;\,\underline{x})^m\big) =$

$$= \Big(\alpha\,\mathrm{Re}\,(t+ir)^{m+n} - \frac{\mathrm{Im}\,(t+ir)^{m+n}}{r}\,\underline{x}^\top \underline{a}\,;$$

$$;\,\Big\{\alpha\frac{\mathrm{Im}\,(t+ir)^{m+n}}{r} - \frac{\mathrm{Im}\,(t+ir)^m}{r}\,\frac{\mathrm{Im}\,(t+ir)^n}{r}\,\underline{x}^\top \underline{a}\Big\}\underline{x} + \Big\{\,\mathrm{Re}\,(t+ir)^m\,\mathrm{Re}\,(t+ir)^n\Big\}\,\underline{a}\Big).$$

Definition 1.1 [1](**Dijkhuis: A special class of functions**)
On open sets in $\mathbb{R}^{1+\mathfrak{d}}$ we introduce the class of functions

$$(t\,;\,\underline{x}) \;\mapsto\; (T(t\,;\,\underline{x})\,;\,\underline{X}(t\,;\,\underline{x})) \;\in \mathbb{R}^{1+\mathfrak{d}}, \tag{1.2}$$

where T and $\underline{X} = \text{column}[X_1, \ldots, X_{\mathfrak{d}}]$ are supposed to satisfy

$$\begin{aligned} \nabla T &= -\frac{\partial \underline{X}}{\partial t} \\ \nabla \times \underline{X} &= \underline{0} \\ \underline{x} \times \underline{X} &= \underline{0} \\ (\underline{x}\cdot\nabla)\underline{X} &= \frac{\partial T}{\partial t}\,\underline{x}. \end{aligned} \tag{1.3}$$

Here we denote

$$\nabla T = \text{column}[\,\partial_1 T, \ldots, \partial_{\mathfrak{d}} T\,]\,,$$

$\underline{x} \times \underline{X}$ stands for the anti-symmetric matrix $[\underline{x}\,\underline{X}^T - \underline{X}\,\underline{x}^T]_{k\ell} = [x_k X_\ell - x_\ell X_k] \in \mathbb{R}^{\mathfrak{d}\times\mathfrak{d}}$,

$\nabla \times \underline{X}$ stands for the anti-symmetric matrix $[(\mathcal{D}\underline{X})^T - \mathcal{D}\underline{X}]_{k\ell} = [\partial_k X_\ell - \partial_\ell X_k] \in \mathbb{R}^{\mathfrak{d}\times\mathfrak{d}}$.

If $\mathfrak{d} = 3$ the identities in (1.3) correspond with the usual interpretation!

Note 6. From $[\underline{x}\underline{X}^\top - \underline{X}\,\underline{x}^\top] = [0]$ it immediately follows that $\underline{X}$ can only be a multiple of $\underline{x}$.

Theorem 1.2 *Suppose (1.3). Then the function*

$$(t\,;\,\underline{x}) \;\mapsto\; (t\,;\,\underline{x}) \cdot \big(T(t\,;\,\underline{x})\,;\,\underline{X}(t\,;\,\underline{x})\big) \tag{1.4}$$

also satisfies (1.3).

Proof In index notation the conditions (1.3) read

$$\partial_k T = -\partial_0 X_k\,, \quad \partial_i X_j - \partial_j X_i = 0\,, \quad x_i(\partial_i X_k) = (\partial_0 T)x_k, \qquad 1 \le i,j,k \le \mathfrak{d}.$$

The product (1.4) reads $(tT - \underline{x}^\top\underline{X}\,;\,t\underline{X} + T\underline{x})$. We list the components of all derivatives needed. Summation over repeated indices.

$$\begin{aligned}
\nabla(tT - \underline{x}^\top\underline{X}) \quad &: \quad \partial_k(tT - x_iX_i) = t(\partial_k T) - \delta_{ki}X_i - x_i(\partial_k X_i) = \\
&\quad\;\; = t(\partial_k T) - X_k - x_i(\partial_i X_k) + x_i(\partial_i X_k - \partial_k X_i) \\
\partial_t(tT - \underline{x}^\top\underline{X}) \quad &: \quad T + t\partial_0 T - x_i\partial_0 X_i = T + t\partial_0 T + x_i\partial_i T \\
\nabla \times (t\underline{X} + T\underline{x}) \quad &: \quad \partial_k(tX_\ell + Tx_\ell) - \partial_\ell(tX_k + Tx_k) = \\
&\quad\;\; = t(\partial_k X_\ell - \partial_\ell X_k) + (\partial_k T)x_\ell - (\partial_\ell T)x_k = \\
&\quad\;\; = t(\partial_k X_\ell - \partial_\ell X_k) + \partial_0(X_\ell x_k - X_k x_\ell) \\
\partial_t(t\underline{X} + T\underline{x}) \quad &: \quad X_k + t(\partial_0 X_k) + (\partial_0 T)x_k \\
(\underline{x}\cdot\nabla)(t\underline{X} + T\underline{x}) \quad &: \quad x_i\partial_i(tX_k + Tx_k) = tx_i(\partial_i X_k) + x_i(\partial_i T)x_k + x_iT\delta_{ik}
\end{aligned}$$

[1]Introduced by G.C. Dijkhuis for $\mathbb{R}^{1+3}$ and $\mathbb{R}^{1+7}$. Private communication.

Taking into account (1.3) leads to the desired result. ■

Corollary 1.3 *Convergent power series with* ***real*** *coefficients* c_n

$$(T(t\,;\,\underline{x})\,;\,\underline{X}(t\,;\,\underline{x}) = \sum_{m=0}^{\infty} c_n(t\,;\,\underline{x})^m\,, \tag{1.5}$$

all lead to functions which satisfy (1.3).

Note 7. The 'vectorial part' of the sum of such power series is always a multiple of $\underline{x}$.

Note 8. If $\mathfrak{d} = 3$ or $\mathfrak{d} = 7$ these series correspond to *quaternion* and *octonion* power series, respectively. It is emphasized again that the coefficients are real!

Inspired by Note 5. we come to a full description of functions (1.2) that satisfy (1.3).

Theorem 1.4 *The functions (1.2) satisfy (1.3) if and only if, locally, there exists an analytic function* $t + ir \mapsto \mathsf{F}(t + ir) = \operatorname{Re}\mathsf{F}(t, r) + i\operatorname{Im}\mathsf{F}(t, r)$*, such that*

$$(t\,;\,\underline{x}) \;\mapsto\; \big(T(t\,;\,\underline{x})\,;\,\underline{X}(t\,;\,\underline{x})\big) = \Big(\operatorname{Re}\mathsf{F}(t, r)\,;\,\frac{\operatorname{Im}\mathsf{F}(t, r)}{r}\,\underline{x}\Big).$$

For convenience in the proof I first summarize

Some properties of analytic functions

- A function $f : \mathbb{C} \to \mathbb{C}$ ia analytic iff $f(z) = f(x + iy) = \operatorname{Re} f(x, y) + i\operatorname{Im} f(x, y)$ satisfies the Cauchy-Riemann identities

$$\frac{\partial}{\partial \bar{z}}\,f(z) = \frac{1}{2}(\partial_x + i\partial_y)f(x + iy) = \frac{1}{2}(\partial_x + i\partial_y)\big(\operatorname{Re} f(x, y) + i\operatorname{Im} f(x, y)\big) = 0,$$

which corresponds to

$$\partial_x \operatorname{Re} f - \partial_y \operatorname{Im} f = 0\,, \qquad \partial_y \operatorname{Re} f + \partial_x \operatorname{Im} f = 0.$$

- For the 'complex' derivative we have

$$\frac{\partial}{\partial z}\,f(z) = f'(z) = \frac{1}{2}(\partial_x - i\partial_y)f(x + iy) = \frac{1}{2}(\partial_x - i\partial_y)\big(\operatorname{Re} f(x, y) + i\operatorname{Im} f(x, y)\big) =$$

$$= \frac{1}{2}\{\partial_x \operatorname{Re} f + \partial_y \operatorname{Im} f\} + \frac{i}{2}\{\partial_x \operatorname{Im} f - \partial_y \operatorname{Re} f\} = \partial_x \operatorname{Re} f - i\,\partial_y\,\mathbf{Re}\,f.$$

- Analytic functions are harmonic, indeed

$$\Delta(\operatorname{Re} f(x, y) + i\operatorname{Im} f(x, y)) = 4\frac{1}{2}(\partial_x - i\partial_y)\frac{1}{2}(\partial_x + i\partial_y)(\operatorname{Re} f(x, y) + i\operatorname{Im} f(x, y)) = 0$$

$$= \Delta \operatorname{Re} f(x,y) + i\Delta \operatorname{Im} f(x,y) = 0.$$

- $$z\frac{\mathrm{d}}{\mathrm{d}\, z} f = zf'(z) = (x\partial_x + y\partial_y)\operatorname{Re} f - i\,(x\partial_y - y\partial_x)\,\mathbf{Re}\, f$$
- If $(x,y) \mapsto h(x,y)$ is harmonic, that means $\Delta h(x,y) = 0$, then the function

$$z = x + iy \quad \mapsto \quad \partial_x h(x,y) - i\partial_y(x,y),$$

is analytic.

Proof of Theorem 1.4 ($\Leftarrow$) If $T = \operatorname{Re}\mathsf{F}$ and $\underline{X} = \frac{\operatorname{Im}\mathsf{F}}{r}\underline{x}$, the 2nd and 3rd property in (1.3) follow from the symmetry of

$$x_i x_j = \frac{x_i x_j}{r}\mathsf{F} \qquad \text{and} \qquad \partial_i X_j = \frac{x_i x_j}{r^2}(\partial_r \mathsf{F}) + \big(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\big)\mathsf{F}\,.$$

Substitution in the 1st condition leads to

$$\partial_r(\operatorname{Re}\mathsf{F})\,\frac{1}{r}\,\underline{x} = -\partial_t \operatorname{Im}\mathsf{F}\,\frac{1}{r}\,\underline{x},$$

which is OK because of one of the Cauchy-Riemann properties. Substitution in the 4th condition, because of $(\underline{x}\cdot\nabla)(\frac{1}{r}\,\underline{x}) = \underline{0}$, leads to

$$r(\partial_r \operatorname{Im}\mathsf{F})\,\frac{1}{r}\,\underline{x} = (\partial_t \operatorname{Re}\mathsf{F})\underline{x},$$

which is also OK because of the other Cauchy-Riemann property.

($\Rightarrow$) Since $\underline{X}$ has rotation $\underline{0}$ it has a potential. Write $\underline{X}(t;\underline{x}) = -\nabla G(t,\underline{x})$. Further, from $[\underline{x}\underline{X}^\top - \underline{X}\underline{x}^\top] = [0]$ it follows that $\underline{X}$ can only be a multiple of $\underline{x}$. It follows that there exists a scalar function $(t\,;\,\underline{x}) \;\mapsto\; \alpha(t\,;\,\underline{x})$, such that $\nabla G(t,\underline{x}) = \alpha(t\,;\,\underline{x})\underline{x}$. We want to show that, for all fixed t, the function $\underline{x} \mapsto G(t,\underline{x})$ is constant on spheres $|\underline{x}| = r$. Take $\underline{a}\,,\underline{b}$ with $|\underline{a}| = |\underline{b}| = r$. Let $\mathscr{C}$ be an oriented curve $s \to \underline{x}(s)$ which runs from $\underline{a}$ to $\underline{b}$ and which lies entirely on the sphere $|\underline{x}| = r$. Then $G(t,\underline{b}) - G(t,\underline{a}) = \int_{\mathscr{C}} \nabla G(t,\underline{x}(s))\cdot\dot{\underline{x}}(s)\,\mathrm{d}s$. The integrand vanishes at all points of the curve because ∇G is orthogonal to the sphere at all points of it. From now on we write $G(t,\underline{x}) = G(t,r)$. Therefore $\underline{X}(t;\underline{x}) = -(\partial_r G(t,r))\,\frac{1}{r}\,\underline{x}$. Put $T(t,r) = \partial_t G(t,r)$ and we only have to satisfy the final condition in (1.3). Substitute our T and G. The condition reads

$$-r(\partial_r\partial_r G)\,\frac{1}{r}\,\underline{x} = \partial_t\partial_t G\underline{x}.$$

It follows that G has to be harmonic: $\Delta G = 0$. We now define

$$\mathsf{F}(t + ir) = \partial_t G(t,r) - i\partial_r G(t,r),$$

and we are done. ■

Examples The analytic functions $\mathsf{F}(t,r) = (t + \mathrm{i}r)^m$, $m \in \mathbb{N}$, represent the polynomial vectorfields $(t\,;\,\underline{x})^m$.

Theorem 1.5 *Endowed with pointwise multiplication the Dijkhuis class of vectorfields, defined by (1.3), is a commutative and associative algebra.*

Proof For analytic F, G we only have to check the multiplication

$$\big(\operatorname{Re}\mathsf{F}\,;\ \frac{\operatorname{Im}\mathsf{F}}{r}\underline{x}\big)\cdot\big(\operatorname{Re}\mathsf{G}\,;\ \frac{\operatorname{Im}\mathsf{G}}{r}\underline{x}\big) = \big(\operatorname{Re}\mathsf{FG}\,;\ \frac{\operatorname{Im}\mathsf{FG}}{r}\underline{x}\big)\,.$$

Associativity follows because all vectorial parts are multiples of $\underline{x}$. ■

Further Consequences
It will be clear by now that operations on the Dijkhuis class can be represented fully by operations on analytic functions. We mention some examples

- Multiplication by $(t\,;\,\underline{x})$ corresponds to $\mathsf{F} \ \mapsto\ \ \{z \mapsto z\mathsf{F}(z)\}$.
- The Kelvin transform corresponds to $\mathsf{F} \ \mapsto\ \ \{z \mapsto \mathsf{F}(\frac{1}{z})\}$.
- The harmonic conjugate corresponds to $\mathsf{F} \ \mapsto\ \ \{z \mapsto \mathsf{i}\mathsf{F}(z)\}$.
- The Euler operator corresponds to $\mathsf{F} \ \mapsto\ \ \{z \mapsto z\frac{\mathrm{d}}{\mathrm{d}z}\mathsf{F}(z)\}$.
- Meaningful derivatives are given by $\mathsf{F} \ \mapsto\ \ \{\mapsto z\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathsf{F}(z)\}$.

2 Differentiability with respect to the algebra

A mapping

$$\mathbb{R}^{1+\mathfrak{d}} \to \mathbb{R}^{1+\mathfrak{d}}\ :\quad \begin{bmatrix} t \\ \underline{x} \end{bmatrix} \ \mapsto\ \begin{bmatrix} T(t\,;\,\underline{x}) \\ \underline{X}(t\,;\,\underline{x}) \end{bmatrix}\,, \tag{2.1}$$

is differentiable (in the usual sense) at $\begin{bmatrix} t \\ \underline{x} \end{bmatrix} \in \mathbb{R}^{1+\mathfrak{d}}$, if for any $\begin{bmatrix} h \\ \underline{k} \end{bmatrix} \in \mathbb{R}^{1+\mathfrak{d}}$, we have

$$\begin{bmatrix} T(t+h\,;\,\underline{x}+\underline{k}) \\ \underline{X}(t+h\,;\,\underline{x}+\underline{k}) \end{bmatrix} = \begin{bmatrix} T(t\,;\,\underline{x}) \\ \underline{X}(t\,;\,\underline{x}) \end{bmatrix} + \begin{bmatrix} \partial_t T(t;\underline{x}) & \nabla T(t;\underline{x}) \\ \partial_t \underline{X}(t;\underline{x}) & \mathcal{D}\underline{X}(t;\underline{x}) \end{bmatrix} \begin{bmatrix} h \\ \underline{k} \end{bmatrix} + o(\sqrt{h^2+|\underline{k}|^2})\,. \tag{2.2}$$

Here, $\nabla T = \operatorname{row}(\partial_1 T\,, \ldots, \partial_{\mathfrak{d}} T)$ and $\mathcal{D}\underline{X} = \operatorname{matrix}[\partial_j X_\ell]\,,\ 1 \le j, \ell \le \mathfrak{d}$.

For *left/right differentiability with respect to the algebraic structure imposed on* $\mathbb{R}^{1+\mathfrak{d}}$, it is required that the linearization term in (2.2) has the form

$$\begin{bmatrix} \partial_t T(t;\underline{x}) & \nabla T(t;\underline{x}) \\ \partial_t \underline{X}(t;\underline{x}) & \mathcal{D}\underline{X}(t;\underline{x}) \end{bmatrix} \begin{bmatrix} h \\ \underline{k} \end{bmatrix} = \begin{bmatrix} \alpha(t;\underline{x}) & -\underline{a}^\top(t;\underline{x}) \\ \underline{a}(t;\underline{x}) & \alpha(t;\underline{x})\mathcal{I} \end{bmatrix} \begin{bmatrix} h \\ \underline{k} \end{bmatrix}. \tag{2.3}$$

As a consequence the conditions for differentiability, with respect to the algebra, are

$$\partial_j X_\ell = 0, \text{ if } j \neq \ell\,, \qquad \alpha = \partial_t T = \partial_j X_j\,,\ 1 \leq j, \ell \leq \mathfrak{d}\,, \qquad \underline{a} = -\nabla T = \partial_t \underline{X}\,. \tag{2.4}$$

It follows that, for $\mathfrak{d} > 1$, the only differentiable functions are

$$T = Bt - \underline{A}\cdot\underline{x} + D\,, \qquad \underline{X} = B\underline{x} + t\underline{A}\,, \qquad B, D \in \mathbb{R}\,,\ \underline{A} \in \mathbb{R}^{\mathfrak{d}}\,, \tag{2.5}$$

which does not look very exciting.

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J. de Graaf, May 2011.

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