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BASED ON THE WKB APPROXIMATIONS

THEO VERHEGGEN

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$$\Phi_{xx} - c^{-2}(x)\,\Phi_{tt} = 0$$

BASED ON THE WKB APPROXIMATIONS

PROEFSCHRIFT

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GENERAL INTRODUCTION

In many branches of physics one has to deal with the equation

$$\phi_{xx} - c^{-2}(x)\phi_{tt} = 0 \tag{1}$$

or with equations that can be reduced to this form by means of a simple transformation.
We mention some.

Acoustics. The pressure equation for acoustic waves in an inhomogeneous fluid at rest reads

$$\rho(x,y,z) \operatorname{div} \frac{1}{\rho(x,y,z)} \operatorname{grad} p - c^{-2}(x,y,z)\frac{\partial^2 p}{\partial t^2} = 0.$$

Here p is the difference between the instantaneous pressure and the equilibrium pressure, $\rho(x,y,z)$ is the equilibrium density, and $c(x,y,z)$ denotes the local velocity of sound. In one dimension this equation reads

$$\rho(x)\frac{\partial}{\partial x}\frac{1}{\rho(x)}\frac{\partial p}{\partial x} - c^{-2}(x)\frac{\partial^2 p}{\partial t^2} = 0.$$

Then it can be reduced to (1) by a transformation of the space coordinate:

$$z = \int^x \rho(x_1)dx_1.$$

Transmission lines. If I and V denote the current and the voltage in the line and if $C(x)$ and $L(x)$ are the capacitance and inductance per unit length, then the equations satisfied by I and V are

$$L(x)\frac{\partial I}{\partial t} + \frac{\partial V}{\partial x} = 0,$$
$$C(x)\frac{\partial V}{\partial t} + \frac{\partial I}{\partial x} = 0.$$

This leads to the following second-order equations for I and V:

$$L(x)\frac{\partial^2 I}{\partial t^2} - \frac{\partial}{\partial x}\frac{1}{C(x)}\frac{\partial I}{\partial x} = 0$$

and

$$C(x)\frac{\partial^2 V}{\partial t^2} - \frac{\partial}{\partial x}\frac{1}{L(x)}\frac{\partial V}{\partial x} = 0.$$

These equations can be reduced again to the form (1) by means of a transformation of the space coordinate.

Electromagnetics. We consider the propagation of a plane wave in the direction of the x-axis in an isotropic dielectric medium with dielectric permittivity $\varepsilon(x)$ and magnetic permeability $\mu(x)$. If the electric field is $\vec{E} = (0,E(x),0)$ and if the magnetic induction is $\vec{B} = (0,0,B(x))$, then the Maxwell equations are

$$\frac{\partial}{\partial x}\frac{1}{\mu(x)} B + \varepsilon(x)\frac{\partial E}{\partial t} = 0,$$

$$\frac{\partial E}{\partial x} + \frac{\partial B}{\partial t} = 0.$$

This leads to

$$\varepsilon(x)\frac{\partial^2 E}{\partial t^2} - \frac{\partial}{\partial x}\frac{1}{\mu(x)}\frac{\partial E}{\partial x} = 0$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{\partial}{\partial x}\frac{1}{\varepsilon(x)}\frac{\partial}{\partial x}\frac{1}{\mu(x)} B = 0.$$

We are interested in two types of solutions of (1) for the infinite interval $-\infty < x < +\infty$. In our first chapter we shall be concerned with *monochromatic solutions*. That is, with solutions of the form

$$\Phi = \phi(x)\exp -i\omega t, \ \omega > 0,$$

where ϕ satisfies

$$\phi_{xx} + k^2(x)\phi = 0. \tag{2}$$

Here

$$k^2(x) = \omega^2 c^{-2}(x).$$

In the second chapter we shall consider the *Cauchy* or *initial-value problem* for (1).

We notice that the equation (2) need not have its origin in (1). It appears, for example, in quantum mechanics. The Schrödinger equation for a spinless particle in one dimension is

$$\phi_{xx} + \frac{2m}{\hbar^2}\{E - V(x)\}\phi = 0.$$

This equation is of the form (2) if for all x $E > V(x)$. Equation (2) also describes the motion of a simple pendulum when the length of the suspending thread is changed, and the motion of a charged particle in a time-dependent magnetic field. However, we shall always talk about equation (2) as stemming from a wave phenomenon governed by (1).

CHAPTER I

MONOCHROMATIC SOLUTIONS

§1.1. INTRODUCTION

The first chapter of this thesis is devoted to the study of monochromatic solutions of the wave equation

$$\Phi_{xx} - c^{-2}(x)\Phi_{tt} = 0. \tag{1}$$

That is, the study of solutions of the type

$$\Phi = \phi(x)\exp -i\omega t, \ \omega > 0, \tag{2}$$

where ϕ obeys the equation

$$\phi_{xx} + k^2(x)\phi = 0, \quad k^2(x) = \omega^2 c^{-2}(x). \tag{3}$$

We assume that $c(x)$, and hence $k(x)$, has a positive upper and lower bound. In particular we are interested in situations where $c(x)$ tends to finite limits as x tends to $\pm\infty$, such that for large $|x|$ the solutions of (3) consist of two uncoupled waves travelling to the left and to the right, respectively.

It is well-known that the solutions of (3) can be approximated by linear combinations of the WKB approximations

$$\phi_{\pm} = k^{-\frac{1}{2}}(x)\exp \pm i\int^{x} k(x_1)dx_1.$$

This is possible when $k(x)$ is a "slowly varying" function of x. One usually finds conditions of the type

$$k^{-2}|k'| = \omega^{-1}|c'| << 1$$

or

$$k^{-3}|k''| = \omega^{-2}c^3|(c^{-1})''| << 1.$$

(The prime denotes differentiation with respect to x.)

Several methods exist that provide corrections to these WKB approximations. Especially we are interested in sequences that have these WKB approximations as their first term and that converge to a solution of (3). The first one to derive such a series was Liouville in 1837; see Liouville [26]. In fact Liouville was also the first to derive the WKB approximations themselves, simultaneously with Green,

who published independently an approximation of (1) in the same year; see Green [17].
(The WKB approximations are named after Wentzel, Kramers, and Brillouin, who attained them in the twenties and who studied their continuation over a turning point, i.e., a point where $k^2(x)$ changes sign.)
Lateron Liouville's method has been used or rediscovered by many authors; cf, e.g., Broer [9], Cavaliere et alii [11], Erdélyi [15], Knorr and Pfirsch [24], and Olver [28]. We shall describe how Liouville's method applies to our purposes.
Liouville observed that the "flattening" transformation of variables

$$z = \int_{x_0}^{x} c^{-1}(x_1)dx_1, \tag{4}$$

$$\psi = c^{-\frac{1}{2}}\phi \tag{5}$$

transforms (3) into

$$\psi_{zz} + \omega^2\psi = (b^2-\dot{b})\psi, \tag{6}$$

$$b = \frac{1}{2c}\frac{dc}{dz}. \tag{7}$$

(Here the dot denotes differentiation with respect to z. With expressions like $\frac{dc}{dz}$ we mean $\frac{d}{dz}$ c(x(z)).)
The transformation (4), (5) will henceforth be called the Liouville transformation. Then the right-hand side of (6) is treated as a perturbation, and (6) is solved by successive approximation using a Green function adapted to the boundary conditions that are imposed. The first approximation is a solution of

$$\psi_{zz} + \omega^2\psi = 0.$$

This leads after inverse Liouville transformation to all linear combinations of the WKB approximations.
We are interested in two types of solutions. First, the solution that as $z \to -\infty$ consists of an incoming wave of unit amplitude and

an outgoing wave of unknown amplitude; as $z \to \infty$ it consists of an outgoing wave only. Using the proper Green function we find by successive approximation

$$\psi = \sum_{n=0}^{\infty} \psi^{(n)}, \tag{8}$$

with

$$\psi^{(0)} = \exp i\omega z \tag{9}$$

and

$$\psi^{(n)}(z) = \frac{1}{2i\omega} \int_{-\infty}^{+\infty} \exp i\omega|z-z_1|(b^2-\dot{b})\psi^{(n-1)} dz_1 . \tag{10}$$

The series (8) converges absolutely and uniformly for all z if

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}| dz < 2\omega; \tag{11}$$

that is, if

$$\int_{-\infty}^{+\infty} c^{\frac{1}{2}} |(c^{\frac{1}{2}})''| dx < 2\omega. \tag{12}$$

The second solution has a different normalization. As $z \to \infty$ it consists of an outgoing wave of unit amplitude only. Successive approximation now leads to

$$\psi = \sum_{n=0}^{\infty} \psi^{(n)}, \tag{13}$$

with

$$\psi^{(0)} = \exp i\omega z \tag{14}$$

and

$$\psi^{(n)}(z) = -\omega^{-1} \int_{z}^{\infty} \sin \omega(z-z_1)(b^2-\dot{b})\psi^{(n-1)} dz_1 . \tag{15}$$

A weaker condition than (11) now assures that the series converges absolutely and uniformly for all z:

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}| dz < \infty, \tag{16}$$

as has been pointed out by many authors; e.g., by Berk, Book, and Pfirsch [5].

Because of their asymptotic behaviour as $z \to \infty$ these two solutions differ from each other only by a constant factor whereas the convergence of their series differs widely.

Investigations of another type of solutions of (3) have been initiated, for example, by Bremmer. He approximated the profile $k(x)$ by a number of thin homogeneous layers. Then the reflection of a wave incoming from $x \to -\infty$ is taken into account by successively adding the contributions of the waves that have undergone a fixed number of reflections, and then passing to the limit of the thickness of the layers tending to zero. The details may be found in Bremmer's papers; see, for example, Bremmer [6], [7], and [8]. We only cite his results. The contribution to the solution of the waves that have undergone no reflection is nothing but the WKB approximation

$$\phi_{\uparrow}^{(0)} = k^{-\frac{1}{2}}(x)\exp i\int^{x} k(x_1)dx_1. \tag{17}$$

If we define $\phi_{\downarrow}^{(2n+1)}$ to denote the contribution to the solution that is the result of 2n+1 reflections and if $\phi_{\uparrow}^{(2n)}$ stands for the contribution due to 2n reflections, then the following relations are found:

$$\phi_{\downarrow}^{(2n+1)}(x) = c^{\frac{1}{2}}(x)\int_{x}^{\infty} \exp -i\omega\int_{x_1}^{x} c^{-1}(x_2)dx_2 \ \ \frac{1}{2}\, c^{-3/2}c'\phi_{\uparrow}^{(2n)}dx_1, \tag{18}$$

$$\phi_{\uparrow}^{(2n)}(x) = -c^{\frac{1}{2}}(x)\int_{-\infty}^{x} \exp i\omega\int_{x_1}^{x} c^{-1}(x_2)dx_2 \ \ \frac{1}{2}\, c^{-3/2}c'\phi_{\downarrow}^{(2n-1)}dx_1. \tag{19}$$

Then the contributions of an even number of reflections are added, and so are the contributions of an odd number of reflections. This leads to the total waves "travelling to the right" and "to the left", respectively:

$$\phi_{\uparrow} = \sum_{n=0}^{\infty} \phi_{\uparrow}^{(2n)} \tag{20}$$

and

$$\phi_{\downarrow} = \sum_{n=0}^{\infty} \phi_{\downarrow}^{(2n+1)}. \tag{21}$$

The convergence of these series has been studied thoroughly especially by Atkinson [2]; he proved that they converge absolutely and uniformly for all z if

$$\int_{-\infty}^{+\infty} \frac{1}{2c} |c'| dx \le \pi/2 . \tag{22}$$

He even proved that in some sense this is the best possible result. If the series (20) and (21) converge, $\phi = \phi_{\downarrow}+\phi_{\uparrow}$ constitutes a solution of (3). This can be seen in the following way. From the recurrence relations (18) and (19) it follows that $\phi_{\downarrow}$ and $\phi_{\uparrow}$ obey the set of differential equations

$$\begin{aligned} \phi_{\downarrow x} - \frac{1}{2} c^{-1} c' \phi_{\downarrow} + i\omega c^{-1} \phi_{\downarrow} &= - \frac{1}{2} c^{-1} c' \phi_{\uparrow}, \\ \phi_{\uparrow x} - \frac{1}{2} c^{-1} c' \phi_{\uparrow} - i\omega c^{-1} \phi_{\uparrow} &= - \frac{1}{2} c^{-1} c' \phi_{\downarrow}. \end{aligned} \tag{23}$$

Differentiation leads to the result that $\phi_{\downarrow}+\phi_{\uparrow}$ satisfies (3). The solution (20) and (21) of (23) can be seen to originate from an approach of (23) by successive approximation in which the right-hand side is treated as a perturbation. Relations (18) and (19) and equations (23) can be simplified somewhat by means of the Liouville transformation (4), (5):

$$\psi_{\downarrow}^{(2n+1)}(z) = \int_z^{\infty} \exp -i\omega(z-z_1) b \psi_{\uparrow}^{(2n)} dz_1,$$

$$\psi_{\uparrow}^{(2n)}(z) = -\int_{-\infty}^{z} \exp i\omega(z-z_1) b \psi_{\downarrow}^{(2n-1)} dz_1,$$

and

$$\begin{aligned} \psi_{\downarrow z} + i\omega\, \psi_{\downarrow} &= -b\psi_{\uparrow}, \\ \psi_{\uparrow z} - i\omega\, \psi_{\uparrow} &= -b\psi_{\downarrow}. \end{aligned} \tag{24}$$

The convergence condition (22) now reads

$$\int_{-\infty}^{+\infty} |b| dz \le \frac{\pi}{2} . \tag{25}$$

Once all this is known, a solution of Bremmer type can be constructed converging whenever

$$\int_{-\infty}^{+\infty} |b| dz < \infty .$$

This has been done, for example, by Kay [21]. The gain in convergence can be achieved again by constructing a series solution of (3) that has a different normalization. Using the appropriate Green function we find the solution of (24) that merely consists of an outgoing wave of unit amplitude as $z \to \infty$:

$$\psi_{\downarrow} = \sum_{n=0}^{\infty} \psi_{\downarrow}^{(2n+1)} \tag{26}$$

and

$$\psi_{\uparrow} = \sum_{n=0}^{\infty} \psi_{\uparrow}^{(2n)}, \tag{27}$$

with

$$\psi_{\uparrow}^{(0)} = \exp i\omega z \tag{28}$$

and with the recurrence formulas

$$\psi_{\downarrow}^{(2n+1)}(z) = \int_{z}^{\infty} \exp -i\omega(z-z_1) b\psi_{\uparrow}^{(2n)} dz_1, \tag{29}$$

$$\psi_{\uparrow}^{(2n)}(z) = \int_{z}^{\infty} \exp i\omega(z-z_1) b\psi_{\downarrow}^{(2n-1)} dz_1. \tag{30}$$

Bremmer-type equations and the corresponding series have also been obtained by other authors using different techniques; see, for example, Baird [4], Kemble [23], and Landauer [25].

There exists a vast literature dealing with the above series and with their convergence. We intend to give a unification and a generalization of what has been published. In particular we are interested in the frequency dependence of the convergence, which has been neglected in most of the studies. This is the more remarkable because one might expect the convergence to be faster for high frequencies.

It is even possible that a series that does not converge for some frequency does converge for another, sufficiently high frequency. This is so because of the oscillating factors appearing in the integrands. Besides it is well-known that the WKB approximation is an asymptotic solution of (3) as $\omega \to \infty$.

We shall also be concerned with how a method that leads to an asymptotic solution of (3) as $\omega \to \infty$ might be changed so as to lead to a convergent sequence related to the above series.

We shall not study thoroughly other series solutions of (3), which have been constructed, for example, by Bahar [3], Van Kampen [19], Sluijter [31], and Schep [30]. We shall not go into them although some of the methods to be described also apply to these series. Something about the concept that underlies the method of Van Kampen can be found in an appendix. Another subject we shall not occupy ourselves with is the problem of series solutions of (3) when $k^2(x)$ is not strictly positive. This has been studied, for example, by Fröman and Fröman [16], using methods akin to ours. Also Bahar's method applies to this case.

§1.2. ASYMPTOTIC BEHAVIOUR AS $|x| \to \infty$ OF THE SOLUTIONS OF $\phi_{xx} + k^2(x)\phi = 0$

In this section we study the conditions that have to be imposed on the profile $k(x)$ to assure that the solutions of

$$\phi_{xx} + k^2(x)\phi = 0, \ k^2(x) = \omega^2 c^{-2}(x), \tag{1}$$

behave like the WKB approximations as $|x| \to \infty$. Furthermore we will prove that the conditions we will find admit an interpretation of these approximations as waves propagating in the direction of the positive and negative x-axis, respectively. This will enable us in the next section to formulate the reflection problems to be studied, in terms of the asymptotic behaviour of the solutions of (1).

Most of the results presented here are not entirely new; see, for example, Atkinson [1] and Coppel [12], Chapter 4.

Throughout this chapter we shall assume that $c(x)$ is subjected to the following restrictions:
(i) $c(x)$ is twice continuously differentiable;
(ii) $c(x)$ satisfies $0 < c_1 \le c(x) \le c_2$.
These conditions are imposed for simplicity even if weaker conditions would do. The main results to be proved in this section are stated in the following theorem.

Theorem 1

(i) Let $k(x)$ satisfy

$$\int_{-\infty}^{+\infty} k^{-\frac{1}{2}}|(k^{-\frac{1}{2}})''|dx = \omega^{-1}\int_{-\infty}^{+\infty} c^{\frac{1}{2}}|(c^{\frac{1}{2}})''|dx < \infty. \tag{2}$$

That is, $$\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < \infty. \tag{2'}$$

Then all solutions of (1) are bounded, and two linearly independent solutions exist satisfying as $x \to \infty$

$$\left.\begin{aligned} \phi &= k^{-\frac{1}{2}}(x)\{\exp i\int_{x_0}^{x} k(x_1)dx_1 + o(1)\}, \\ \phi_x &= ik^{\frac{1}{2}}(x)\{\exp i\int_{x_0}^{x} k(x_1)dx_1 + o(1)\}, \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} \phi &= k^{-\frac{1}{2}}(x)\{\exp -i\int_{x_0}^{x} k(x_1)dx_1 + o(1)\}, \\ \phi_x &= -ik^{\frac{1}{2}}(x)\{\exp - i\int_{x_0}^{x} k(x_1)dx_1 + o(1)\}. \end{aligned}\right\} \tag{4}$$

A similar result holds as $x \to -\infty$.

(ii) A sufficient condition for these results is also

$$\int_{-\infty}^{+\infty} |k^{-1}k'|dx = \int_{-\infty}^{+\infty} |c^{-1}c'|dx < \infty. \tag{5}$$

This is

$$\int_{-\infty}^{+\infty} |b|dz < \infty. \tag{5'}$$

The conditions (2) and (5) are basic in the remainder of this work. Before we prove Theorem 1, we state some consequences of these conditions in the following lemma; cf Coppel [12], pg 121.

Lemma 1

(i) If

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < \infty, \tag{2'}$$

then b tends to zero as $z \to \infty$ and $z \to -\infty$, and

$$\int_{-\infty}^{+\infty} b^2 dz < \infty,$$

$$\int_{-\infty}^{+\infty} |\dot{b}|dz < \infty.$$

(ii) If

$$\int_{-\infty}^{+\infty} |b|dz < \infty, \tag{5'}$$

then c has a finite limit c_+ as $z \to \infty$ and a finite limit c_- as $z \to -\infty$.

Proof

We only consider the first part of the lemma because the second part is evident. From (2') the existence follows of the limit

$$\lim_{z\to\infty}\{\int_{z_0}^{z} b^2 dz_1 - b(z)\}.$$

Now suppose, contrary to the lemma, that

$$\int_{z_0}^{\infty} b^2 dz = \infty.$$

Then it follows from the existence of the above limit that $b(z) \to \infty$ as $z \to \infty$. With the definition of b this is easily seen to contradict the boundedness condition to which $c(x)$ is submitted.

Hence

$$\int_{z_0}^{\infty} b^2 dz < \infty.$$

But then b has a finite limit as $z \to \infty$; of course this limit is zero. Further this leads to

$$\int_{z_0}^{\infty} |\dot{b}| dz < \infty$$

because $|\dot{b}| \le |b^2 - \dot{b}| + b^2$. Likewise it can be proved that

$$\int_{-\infty}^{z_0} b^2 dz < \infty \quad \text{and} \quad \int_{-\infty}^{z_0} |\dot{b}| dz < \infty.$$

This completes the proof of the lemma.† □

Although we are mainly interested in cases where k tends to a finite limit as x tends to $\pm\infty$, we should point out that this is not implied by the condition (2). The profile $k(x)$ even need not be bounded from above by (2) alone, as can be seen from the example $k^2(x) = x$ for x sufficiently large; we remark that the results of Theorem 1 (i) remain valid if the condition that $k(x)$ be bounded is dropped. We also notice that it is not difficult to construct an example where condition (2) is violated and condition (5) holds; the converse is also true even when $k(x)$ has finite limits as $x \to \pm\infty$.

† With some additional effort Lemma 1 (i) can be proved to apply for any continuously differentiable function $b(z)$.

As we saw in the introduction, the Liouville transformation

$$\psi = c^{-\frac{1}{2}}\phi, \qquad z = \int_{x_0}^{x} c^{-1}(x_I)dx_I \tag{6}$$

transforms (1) into

$$\psi_{zz} + \omega^2\psi = (b^2-\dot{b})\psi. \tag{7}$$

This equation is the starting point for the proof of Theorem 1 (i) and plays an important role in a major part of this work. In the proof of the second part of Theorem 1 we will use the set of first-order equations

$$\begin{aligned} \psi_{\downarrow z} + i\omega\psi_{\downarrow} &= -b\psi_{\uparrow}, \\ \psi_{\uparrow z} - i\omega\psi_{\uparrow} &= -b\psi_{\downarrow}. \end{aligned} \tag{8}$$

This system is equivalent to equation (7) to the effect that it is an "amplitude splitting". This means:
for any solution $(\psi_{\downarrow},\psi_{\uparrow})$ of (8) $\psi = \psi_{\downarrow} + \psi_{\uparrow}$ is a solution of (7); conversely, for any solution ψ of (7) a unique $\psi_{\downarrow}$ and a unique $\psi_{\uparrow}$ exist such that $\psi = \psi_{\downarrow} + \psi_{\uparrow}$ and such that $(\psi_{\downarrow},\psi_{\uparrow})$ constitutes a solution of (8); these unique $\psi_{\downarrow}$ and $\psi_{\uparrow}$ are

$$\begin{aligned} \psi_{\downarrow} &= \frac{1}{2}\psi + \frac{i}{2\omega}\psi_z + \frac{ib}{2\omega}\psi, \\ \psi_{\uparrow} &= \frac{1}{2}\psi - \frac{i}{2\omega}\psi_z - \frac{ib}{2\omega}\psi. \end{aligned} \tag{9}$$

The verification of these statements is immediate.
The set of equations (8) often occurs in literature, and Bremmer's concept need not serve as a base for its derivation; see the appendix attached at the end of this chapter. In the proofs we shall use Gronwall's Lemma. We shall also use it in a form different from the one stated below, but in these cases it will be clear how the lemma reads. We omit the proof; it can be found in many textbooks; see, for example, Coppel [12], pg 19.

Gronwall's Lemma

Let λ be a real constant and $\mu(z)$ a nonnegative, continuous function for $z \geq z_0$. If $y(z)$ is a continuous function for $z \geq z_0$ that has the property

$$y(z) \leq \lambda + \int_{z_0}^{z} \mu(z_1)y(z_1)dz_1,$$

then the following inequality holds for $z \geq z_0$:

$$y(z) \leq \lambda \exp \int_{z_0}^{z} \mu(z_1)dz_1.$$

Proof of Theorem 1 (i)

Each solution of (7) satisfies an integral equation of the type

$$\psi(z) = \alpha \exp i\omega(z-z_0) + \beta \exp -i\omega(z-z_0) + \\ + \omega^{-1}\int_{z_0}^{z} \sin \omega(z-z_1)(b^2-\dot{b})\psi dz_1. \tag{10}$$

This is readily seen with the aid of the method of variation of constants. This technique brings about the choice of a proper Green function. (Conversely, any continuous solution of (10) is a solution of (7).)

Thus $\quad |\psi(z)| \leq |\alpha| + |\beta| + \omega^{-1}\int_{z_0}^{z} |b^2-\dot{b}||\psi|dz_1$ for $z \geq z_0$.

A similar inequality holds for $z \leq z_0$.

Using Gronwall's Lemma and condition (2') we see that all solutions of (7) are bounded. Returning to the original variables we obtain the result that all solutions of (1) are bounded. To prove the existence of the required solutions, we need another integral equation:

$$\psi(z) = \omega^{-\frac{1}{2}} \exp i\omega z - \omega^{-1}\int_{z}^{\infty} \sin \omega(z-z_1)(b^2-\dot{b})\psi dz_1. \tag{11}$$

Any bounded, continuous solution of (11) is again a solution of (7). We will come across this integral equation in some subsequent sections. For the time being, and we will prove it lateron, we suppose it to have a unique, bounded, and continuous solution. From (11) we see that this solution has the asymptotic behaviour

$$\psi = \omega^{-\frac{1}{2}}\exp i\omega z + o(1),\ \psi_z = i\omega^{\frac{1}{2}}\exp i\omega z + o(1) \quad \text{as } z \to \infty.$$

Returning to the original variables we obtain

$$\phi = k^{-\frac{1}{2}}(x)\{\exp i\int_{x_0}^{x} k(x_1)dx_1 + o(1)\} \quad \text{as } x \to \infty,$$

and because $b \to 0$ as $z \to \infty$ (Lemma 1),

$$\phi_x = ik^{\frac{1}{2}}(x)\{\exp i\int_{x_0}^{x} k(x_1)dx_1 + o(1)\} \quad \text{as } x \to \infty.$$

The truth of the other statements can be proved in a similar way. The integral equations that have to be used then suggest themselves.

□

Proof of Theorem 1 (ii)

In proving this part we proceed on the lines of the foregoing proof. Each solution of (8) satisfies a set of integral equations of the type

$$\begin{aligned} \psi_{\downarrow}(z) &= \alpha \exp -i\omega(z-z_0) - \int_{z_0}^{z} \exp -i\omega(z-z_1)b\psi_{\uparrow}dz_1, \\ \psi_{\uparrow}(z) &= \beta \exp i\omega(z-z_0) - \int_{z_0}^{z} \exp i\omega(z-z_1)b\psi_{\downarrow}dz_1. \end{aligned} \tag{12}$$

(Conversely, any continuous solution of (12) is a solution of (8).)

Thus $|\psi_{\downarrow}(z)| + |\psi_{\uparrow}(z)| \le |\alpha|+|\beta| + \int_{z_0}^{z} |b|(|\psi_{\downarrow}|+|\psi_{\uparrow}|)dz_1$ for $z \ge z_0$.

A similar inequality holds for $z \le z_0$. Using Gronwall's Lemma and condition (5') we see that both $\psi_\downarrow$ and $\psi_\uparrow$ are bounded; therefore all solutions of (7) are bounded, and returning to the original variables we see that all solutions of (1) are bounded. To prove the existence of one of the required solutions, we use another integral equation:

$$\begin{aligned} \psi_\downarrow(z) &= \int_z^\infty \exp -i\omega(z-z_1)b\psi_\uparrow dz_1, \\ \psi_\uparrow(z) &= \omega^{-\frac{1}{2}}\exp i\omega z + \int_z^\infty \exp i\omega(z-z_1)b\psi_\downarrow dz_1. \end{aligned} \tag{13}$$

Any bounded, continuous solution of (13) is a solution of (8) and hence gives rise to a solution of (7). Once again we postpone proving the existence of a unique, bounded $\psi_\downarrow$ and $\psi_\uparrow$ that are a solution of (13). The asymptotic behaviour of this solution clearly is

$$\psi_\downarrow = o(1), \ \psi_\uparrow = \omega^{-\frac{1}{2}}\exp i\omega z + o(1) \quad \text{as } z \to \infty.$$

In terms of the original variables we obtain again the existence of a solution of (1) such that

$$\phi = k^{-\frac{1}{2}}(x)\{\exp i\int_{x_0}^x k(x_1)dx_1 + o(1)\} \quad \text{as } x \to \infty;$$

because $\phi_x = c^{-\frac{1}{2}}(x)(\psi_z+b\psi)$ and on account of (8), the derivative of this solution satisfies

$$\phi_x = ik^{\frac{1}{2}}(x)\{\exp i\int_{x_0}^x k(x_1)dx_1 + o(1)\} \quad \text{as } x \to \infty. \qquad \square$$

How far these results are from being the best possible may be seen from the example $k^2(x) = 1 + 4x^{-1}\sin 2x$. Equation (1) then even has unbounded solutions; two linearly independent solutions exist having the asymptotic behaviour as $x \to \infty$

$$\phi_1 = x[\cos x + o(1)],$$
$$\phi_2 = x^{-1}[\sin x + o(1)].$$

This case of resonance can be found in many textbooks; for example, Coppel [12], pg 128.

Thus far we have proved that the solutions of (1) behave, as $|x| \to \infty$, like a linear combination of the two WKB approximations provided condition 2 or 5 holds. These, however, do not represent waves propagating without being disturbed in the direction of the positive and negative x-axis, respectively. Nonetheless they represent waves of which the energy propagates in these directions. This can be seen in the following way. The equation (1.1.1) admits a conservation law

$$E_t + F_x = 0, \tag{14}$$

where E, the density, and F, the intensity, are defined by

$$E = \frac{1}{2}\,(\Phi_x^*\Phi_x + c^{-2}(x)\Phi_t^*\Phi_t),$$

$$F = -\,\frac{1}{2}\,(\Phi_t^*\Phi_x + \Phi_x^*\Phi_t).$$

In many cases this represents conservation of energy, and F/E may then be interpreted as the propagation velocity of energy. This quotient satisfies

$$-c(x) \le F/E \le c(x).$$

(Cf Broer and Van Vroonhoven [10].)

In passing we notice that not in all the examples of the general introduction (14) does represent conservation of energy. However, in these cases the proper choice of E and F requires only minor changes in our reasoning. Now, for a wave having (apart from the factor exp $-i\omega t$) the asymptotic behaviour as $x \to \infty$

$$\begin{aligned}\phi &= \alpha k^{-\frac{1}{2}}(x)\exp i\int_{x_0}^{x} k(x_1)dx_1 + \beta k^{-\frac{1}{2}}(x)\exp -i\int_{x_0}^{x} k(x_1)dx_1 + o(1),\\ \phi_x &= i\alpha k^{\frac{1}{2}}(x)\exp i\int_{x_0}^{x} k(x_1)dx_1 - i\beta k^{\frac{1}{2}}(x)\exp -i\int_{x_0}^{x} k(x_1)dx_1 + o(1),\end{aligned} \tag{15}$$

the behaviour of E and F is

$$E = k(x)(|\alpha|^2+|\beta|^2 + o(1)),$$
$$F = \omega(|\alpha|^2-|\beta|^2 + o(1)). \tag{16}$$

Parenthetically we remark that F = constant for monochromatic solutions because of (14), as we shall frequently use lateron; therefore for all x $\quad F = \omega(|\alpha|^2-|\beta|^2).$

Because of (16) E and F consist of separated contributions of the two WKB approximations as $x \to \infty$; the propagation velocity of energy satisfies at ∞

$$F/E = c(x)\left(\frac{|\alpha|^2-|\beta|^2}{|\alpha|^2+|\beta|^2} + o(1)\right).$$

Hence $F/E = c(x) + o(1)$ if and only if $\beta = 0,\ \alpha \neq 0,$

$F/E = -c(x) + o(1)$ if and only if $\beta \neq 0,\ \alpha = 0.$

So a solution for which $\beta = 0$ and $\alpha \neq 0$ amounts to a single "outgoing" wave at $x = \infty$ propagating with the local velocity and with an "amplitude" α. Similarly a solution for which $\beta \neq 0$ and $\alpha = 0$ amounts to a single "incoming" wave at $x = \infty$ propagating with the local velocity and with an "amplitude" β. Of course, these identifications are valid only at $x = +\infty$.

§1.3. THE SCATTERING PROBLEMS AND THEIR UNIQUE SOLVABILITY

In the preceding section we have established the conditions which guarantee that all solutions of (1.2.1) behave like a linear combination of the WKB approximations as $|x| \to \infty$ and, moreover, that these may be interpreted as incoming and outgoing waves. Knowing this we are able to state the scattering problems that form our main subject in this chapter. We formulate them in terms of the asymptotic behaviour of the solution we look for. (Always at least one of the conditions (1.2.2) and (1.2.5) is supposed to hold. So we can use the results of §1.2.)

(i) The first solution is a solution with *normalization of the transmitted wave* at $x = \infty$. This means that it consists merely of an outgoing wave of unit amplitude as $x \to \infty$; no boundary conditions are imposed at $x = -\infty$.

(ii) The second solution has a *normalization of the incident wave* at $x = -\infty$. That is, it consists of an incoming wave of unit amplitude and of an outgoing wave of unknown amplitude as $x \to -\infty$; as $x \to \infty$ it consists of an outgoing wave only.

We shall deal with the existence and uniqueness of these solutions in this section. Later we shall construct series for them; especially we shall study the convergence of these series. From a comparison of the asymptotic behaviour of the solutions as $x \to \infty$ we see that they differ from each other only by a constant factor. However, the convergence of the corresponding series differs widely. This is a main reason for our study of both solutions. Of course, other solutions might also be studied, but these are characteristic of the problems that emerge.

First we state some terminology.
$C(R)$ denotes the space of all complex-valued, bounded, and continuous functions defined on the real axis R. If we define a norm of $\zeta \in C(R)$ according to

$$\|\zeta\| = \sup_{z \in R} |\zeta(z)|,$$

this space becomes a Banach space. $C(R) \times C(R)$ stands for the Cartesian product of two such spaces; this space is also a Banach space if we provide $(\zeta_\downarrow, \zeta_\uparrow) \in C(R) \times C(R)$ with a norm

$$\|(\zeta_\downarrow, \zeta_\uparrow)\| = \max(\|\zeta_\downarrow\|, \|\zeta_\uparrow\|).$$

We convert the above problems into integral equations in these Banach spaces using the standard technique of variation of constants. That is, for each problem we choose the proper Green function. If condition (1.2.2) applies, we use equation (1.2.7); if condition (1.2.5) holds, we use the system (1.2.8).

(i) If condition (1.2.2) holds, ψ is, apart from the Liouville transformation, a solution with normalization of the transmitted wave if and only if ψ satisfies the integral equation in C(R)

$$\psi(z) = \exp i\omega z - \omega^{-1}\int_z^\infty \sin \omega(z-z_1)(b^2-\dot{b})\psi dz_1. \tag{1}$$

(ii) Suppose condition (1.2.5) is satisfied. Then ψ is, apart from the Liouville transformation, a solution with normalization of the transmitted wave if and only if $\psi_\downarrow$ and $\psi_\uparrow$ exist satisfying the integral equation in C(R) × C(R)

$$\begin{aligned} \psi_\downarrow(z) &= \int_z^\infty \exp -i\omega(z-z_1)b\psi_\uparrow dz_1, \\ \psi_\uparrow(z) &= \exp i\omega z + \int_z^\infty \exp i\omega(z-z_1)b\psi_\downarrow dz_1. \end{aligned} \tag{2}$$

Then $\psi = \psi_\downarrow + \psi_\uparrow$.

(iii) Suppose that condition (1.2.2) holds. Then ψ is, apart from the Liouville transformation, a solution with normalization of the incident wave if and only if ψ satisfies the integral equation in C(R)

$$\psi(z) = \exp i\omega z + \frac{1}{2i\omega}\int_{-\infty}^{+\infty} \exp i\omega|z-z_1|(b^2-\dot{b})\psi dz_1. \tag{3}$$

(iv) If (1.2.5) holds, ψ is, apart from the Liouville transformation, a solution with normalization of the incident wave if and only if $\psi_\downarrow$ and $\psi_\uparrow$ exist satisfying the integral equation in C(R) × C(R)

$$\psi_{\downarrow}(z) = \int_{z}^{\infty} \exp -i\omega(z-z_1)b\psi_{\uparrow}dz_1,$$
$$\psi_{\uparrow}(z) = \exp i\omega z - \int_{-\infty}^{z} \exp i\omega(z-z_1)b\psi_{\downarrow}dz_1. \tag{4}$$

Then $\psi = \psi_{\downarrow} + \psi_{\uparrow}$.

The verification of this is straightforward, and we will not go through it. We notice that equations (1) and (2) are the very same ones the unique solvability of which we still have to prove in order to complete the proof of Theorem 1.
For the j th integral equation we write formally

$$\chi = \chi_0 + T_j\chi, \quad j = 1,2,3,4, \tag{5}$$

where T_j stands for the integral operator at the right-hand side of the j th equation.
Unless stated otherwise, we will always assume, without mentioning it explicitly, that if we deal with T_1 *and* T_3, *the condition* (1.2.2) *is satisfied, and that if we are concerned with* T_2 *and* T_4, (1.2.5) *holds.*
Instead of (5) we shall study the extended problem of the solvability of

$$\chi = \zeta + \lambda T_j\chi, \quad j = 1,2,3,4, \tag{6}$$

for any complex λ and any ζ in the space in question. The operators T_j are easily seen to be bounded. But they are even compact, which is readily obtained from the following lemma. The proof may be found, for example, in Atkinson [2].

<u>Lemma 2</u>
Let f(z) be continuous and satisfy

$$\int_{-\infty}^{+\infty} |f|dz < \infty.$$

Then the integral operators on C(R) into itself defined by

$$T\zeta(z) = \int_{\pm\infty}^{z} \exp \pm i\omega(z-z_1) f\zeta dz_1$$

are compact.

This enables us to use the spectral theory of compact operators; see, for example, Taylor [33] or Dunford and Schwartz [14]. From this theory we know that the equation

$$\chi = \zeta + \lambda T_j \chi \tag{7}$$

is uniquely soluble for any ζ unless λ is an "eigenvalue" of T_j. That is, unless there exists a $\chi \neq 0$ such that

$$\chi = \lambda T_j \chi.$$

At most a countable set of such eigenvalues exists having no point of accumulation. The smallest absolute value of the eigenvalues is called the spectral radius. If no eigenvalues exist, it is defined to be ∞. The spectral radius equals

$$\{\lim_{n\to\infty} \|T_j{}^n\|^{1/n}\}^{-1}. \tag{8}$$

Further, and we state it now for later reference, the solution of (7) is an analytic function of λ with values in the Banach space in question if λ is not an eigenvalue; in general an eigenvalue need not be a singular point of the solution.

Theorem 2 (i)

T_1 has no eigenvalues at all. Therefore the equation

$$\chi = \zeta + \lambda T_1 \chi$$

is uniquely soluble for any λ and any $\zeta \in C(R)$, and in particular for $\lambda = 1$ and $\zeta = \chi_0$.

Proof

Suppose that λ is an eigenvalue and $\psi \in C(R)$ is a corresponding eigenvector:

$$\psi(z) = -\lambda\omega^{-1}\int_z^\infty \sin\omega(z-z_1)(b^2-\dot{b})\psi dz_1.$$

Hence

$$|\psi(z)| \le |\lambda|\omega^{-1}\int_z^\infty |b^2-\dot{b}||\psi|dz_1,$$

and Gronwall's Lemma immediately leads to the contradiction

$$\psi(z) \equiv 0.$$

For another proof we refer the reader to the proof of Theorem 3 (i), pg 30. □

In the same way we can prove the following theorem.

Theorem 2 (ii)

T_2 has no eigenvalues at all. The equation

$$\chi = \zeta + \lambda T_2\chi$$

is uniquely soluble for any λ and any $\zeta \in C(R)\times C(R)$, and in particular for $\lambda = 1$ and $\zeta = \chi_0$.

Theorem 2 (iii)

T_3 has no real eigenvalues. Hence the equation

$$\chi = \zeta + \lambda T_3\chi$$

is uniquely soluble for any $\zeta \in C(R)$ if λ is real, and in particular for $\lambda = 1$ and $\zeta = \chi_0$.

Proof

Suppose λ is a real eigenvalue and ψ is a corresponding eigenvector:

$$\psi(z) = \frac{\lambda}{2i\omega}\int_{-\infty}^{+\infty} \exp i\omega|z-z_1|(b^2-\dot{b})\psi dz_1.$$

By differentiation we have

$$\psi_{zz} + \omega^2\psi = \lambda(b^2-\dot{b})\psi. \tag{9}$$

Besides this we see that constants δ_- and γ_+ exist such that

$$\psi = \delta_-\exp -i\omega z + o(1),\ \psi_z = -i\omega\delta_-\exp -i\omega z + o(1) \quad \text{as } z \to -\infty, \tag{10}$$

$$\psi = \gamma_+\exp\ i\omega z + o(1),\ \psi_z = i\omega\gamma_+\exp\ i\omega z + o(1) \quad \text{as } z \to \infty. \tag{11}$$

For real λ any solution of (9) satisfies

$$\psi^*\psi_z - \psi\psi_z^* = \text{const.}$$

(This corresponds to the energy conservation law (1.2.14) for a monochromatic wave or to the Wronskian of the solutions ψ and ψ^* of (9).) On account of (10) this constant must be $-2i\omega|\delta_-|^2$, and because of (11) it must be equal to $2i\omega|\gamma_+|^2$; therefore it must be zero, and both δ_- and γ_+ are zero. Hence ψ is a solution of (9) satisfying the boundary condition

$$\psi = o(1) \text{ as } z \to \infty.$$

But then ψ is a nonzero solution of

$$\psi(z) = -\lambda\omega^{-1}\int_z^\infty \sin\omega(z-z_1)(b^2-\dot{b})\psi dz_1,$$

which contradicts Theorem 2 (i). □

Theorem 2 (iv)

T_4 has no real eigenvalues. Thus the equation

$$\chi = \zeta + \lambda T_4\chi$$

is uniquely soluble for any $\zeta \in C(R)\times C(R)$ if λ is real, and in particular for $\lambda = 1$ and $\zeta = \chi_0$.
The proof can be found in Atkinson [2]; it proceeds on the lines of the foregoing one.

The fact that T_1 and T_2 have no eigenvalues may be seen to stem from their Volterra character whereas T_3 and T_4 are of Fredholm type.

§1.4. NEUMANN EXPANSIONS AND THEIR CONVERGENCE

After having proved the unique solvability of the scattering problems, we try to solve them now successively by applying the method of successive approximation to the integral equations

$$\chi = \zeta + \lambda T_j \chi, \quad j = 1,2,3,4. \tag{1}$$

That is, we study the conditions which assure that the equations (1) can be solved by the Neumann expansions

$$\chi = \sum_{n=0}^{\infty} \lambda^n T_j^{\,n} \zeta, \qquad j = 1,2,3,4. \tag{2}$$

In particular we examine whether these expansions converge for $\lambda = 1$ and $\zeta = \chi_0$. If they do, we obtain series solutions of the scattering problems, of which the first term is the WKB approximation. It can be seen simply that these series are precisely the series that we already found in the introduction. We notice that the Liouville transformation does not affect the convergence properties of the series we study. This is so because of the boundedness conditions we imposed on pg 14.

The convergence of the Neumann expansions is closely related to the theory of analytic functions of a complex variable with values in a Banach space. Indeed, a function that is analytic on the set $\{\lambda \mid |\lambda| < r\}$ has a unique Taylor expansion in positive powers of λ, which converges absolutely when $|\lambda| < r$ and uniformly on each closed subset of the disk $|\lambda| < r$. The relationship between the analyticity of the solution of (1) and spectral theory has already been noticed in the preceding section, pg 26. For $|\lambda|$ smaller than the spectral radius of the operator in question there exists a convergent Taylor expansion of the solution of (1). This Taylor expansion must be the Neumann expansion (2). To study the convergence of (2) one obviously should estimate the spectral radius of the operators. This has been done already for T_1 and T_2 in the Theorems 2 (i) and (ii), pp 26 and 27, which leads to the following theorems.

Theorem 3 (i)

The Neumann expansion

$$\sum_{n=0}^{\infty} \lambda^n T_1{}^n \zeta \qquad (3)$$

converges in C(R) for any λ and any ζ. Therefore it represents for $\lambda = 1$ and $\zeta = \chi_0$ the unique solution of the scattering problem with normalization of the transmitted wave.

Proof

The assertions are an immediate consequence of Theorem 2 (i), pg 26. However, the proof may be carried out also in another way. From the theory of Taylor expansions it is well-known that the expansion (3) converges if

$$|\lambda| < \{\limsup_{n\to\infty} \|T_1{}^n\zeta\|^{1/n}\}^{-1},$$

and diverges if

$$|\lambda| > \{\limsup_{n\to\infty} \|T_1{}^n\zeta\|^{1/n}\}^{-1}.$$

We will show inductively that for any $\zeta \in C(R)$

$$|T_1{}^n\zeta(z)| \le \frac{1}{n!} \{\omega^{-1}\int_z^\infty |b^2-\dot{b}|dz_1\}^n \|\zeta\| , \qquad (4)$$

and this obviously leads to

$$\limsup_{n\to\infty} \|T_1{}^n\zeta\|^{1/n} = 0.$$

For n = 0 (4) is self-evident.

Now suppose that (4) holds for n = m. Then because of the definition of T_1 we have

$$T_1{}^{m+1}\zeta(z) = -\omega^{-1}\int_z^\infty \sin \omega(z-z_1)(b^2-\dot{b})T_1{}^m\zeta dz_1$$

and

$$|T_1^{m+1}\zeta(z)| \le \frac{1}{m!}\,\omega^{-m-1}\int_z^\infty |b^2-\dot{b}|dz_2\{\int_{z_2}^\infty |b^2-\dot{b}|dz_1\}^m \;\|\zeta\| =$$

$$= \frac{1}{(m+1)!}\{\omega^{-1}\int_z^\infty |b^2-\dot{b}|dz_1\}^{m+1} \;\|\zeta\| .$$

We notice that from the estimate of $\|T_1^n\|$ and using (1.3.8) we obtain again the result of Theorem 2 (i) that T_1 has no eigenvalues.

□

<u>Theorem 3 (ii)</u>

The Neumann expansion

$$\sum_{n=0}^{\infty} \lambda^n T_2^n \zeta \tag{5}$$

converges in C(R)×C(R) for any λ and any $\zeta \in C(R)\times C(R)$. For $\lambda = 1$ and $\zeta = \chi_0$ it represents the unique solution of the scattering problem with normalization of the transmitted wave.

<u>Proof</u>

This theorem is a consequence of Theorem 2 (ii), pg 27. It can also be proved in the same way as we proved Theorem 3 (i). Inductively it can be shown that for any $\zeta \in C(R)\times C(R)$

$$|(T_2^n\zeta)_\downarrow(z)| \le \frac{1}{n!}\{\int_z^\infty |b|dz_1\}^n \;\|\zeta\|$$

and

$$|(T_2^n\zeta)_\uparrow(z)| \le \frac{1}{n!}\{\int_z^\infty |b|dz_1\}^n \;\|\zeta\| .$$

Thus $$\limsup_{n\to\infty} \|T_2^n\zeta\|^{1/n} = 0.$$

□

Before we estimate the spectral radius of T_3 and T_4, we want to show that the Neumann series

$$\sum_{n=0}^{\infty} \lambda^n T_3{}^n \chi_0 \quad \text{and} \quad \sum_{n=0}^{\infty} \lambda^n T_4{}^n \chi_0$$

cannot converge when $|\lambda|$ exceeds the spectral radius of T_3 and T_4, respectively. If they would,there would exist a solution of the equations $\chi = \chi_0 + \lambda T_j \chi$ for at least one eigenvalue of T_j, $j = 3,4$. This contradicts the following lemmas. Moreover, it will be shown that eigenvalues of T_3 and T_4 are singular points of the respective solutions.

<u>Lemma 3</u>

Suppose λ_0 is an eigenvalue of T_3. Then the equation

$$\chi = \chi_0 + \lambda T_3 \chi \tag{6}$$

is not soluble for $\lambda = \lambda_0$, and λ_0 is a (isolated) singular point of the solution of (6).

<u>Proof</u>

First we reduce the proof of the second assertion to the proof of the first one. If an eigenvalue of T_3 is a singular point, it is isolated because the eigenvalues of T_3 have no points of accumulation. (See pg 26.) Now suppose λ_0 is a removable singularity. Then the solution of (6) has a Taylor expansion about λ_0 having a positive radius of convergence. For $\lambda \neq \lambda_0$ this series represents the solution of (6). However, it is easily seen also to satisfy (6) for $\lambda = \lambda_0$. So a solution of (6) exists for $\lambda = \lambda_0$, whereas we will presently prove that this is impossible.

Suppose ψ_1 is a solution of (6) for $\lambda = \lambda_0$, and let ψ_0 denote an eigenvector of T_3 corresponding to λ_0. (Then $\psi_0+\psi_1$ solves (6) too.) By differentiation we see that both ψ_1 and ψ_0 satisfy

$$\psi_{zz} + \omega^2\psi = \lambda_0(b^2-\dot{b})\psi.$$

ψ_1 satisfies the boundary conditions

$$\psi_1 = \gamma_{1+}\exp i\omega z + o(1) \qquad \text{as } z \to \infty,$$

$$\psi_1 = \exp i\omega z + \delta_{1-}\exp -i\omega z + o(1) \quad \text{as } z \to -\infty.$$

ψ_o satisfies

$$\psi_o = \gamma_{2+}\exp i\omega z + o(1) \qquad \text{as } z \to \infty,$$

$$\psi_o = \delta_{2-}\exp -i\omega z + o(1) \qquad \text{as } z \to -\infty.$$

Therefore they both satisfy an integral equation of the type

$$\psi(z) = \gamma \exp i\omega z - \lambda_0\omega^{-1}\int_z^{\infty} \sin \omega(z-z_1)(b^2-\dot{b})\psi dz_1.$$

Because of the unique solvability of this equation (cf Theorem 2 (i), pg 26) ψ_1 is a multiple of ψ_o. This contradicts the behaviour of ψ_1 and ψ_o as $z \to -\infty$. □

Likewise we obtain the following lemma.

Lemma 4

Suppose λ_o is an eigenvalue of T_4. Then the equation

$$x = x_o + \lambda T_4 x$$

is not soluble for $\lambda = \lambda_o$; λ_o is an isolated singular point of the solution of this equation.

Theorem 3 (iii)

No eigenvalues of T_3 exist satisfying

$$|\lambda|\int_{-\infty}^{+\infty}|b^2-\dot{b}|dz \le 2\omega. \dagger \tag{7}$$

The spectral radius of T_3 tends to zero as $\omega \to 0$.

....

† It can be shown that the number 2 in the right-hand side of (7) cannot be replaced by a larger one such that the assertion remains valid for all frequencies and all profiles c(x).

Therefore the Neumann expansion

$$\sum_{n=0}^{\infty} \lambda^n T_3{}^n \zeta \tag{8}$$

converges for $\lambda = 1$ and $\zeta = \chi_0$ if

$$\int_{-\infty}^{+\infty} |b^2 - \dot{b}|dz \le 2\omega. \tag{9}$$

Then it represents the solution of the scattering problem with normalization of the incident wave. For these λ and ζ it certainly diverges for sufficiently low frequencies.

Proof

We only have to prove the assertions concerning the spectrum of T_3; from this the other assertions follow readily. Because of the definition of T_3:

$$T_3\psi(z) = \frac{1}{2i\omega}\int_{-\infty}^{+\infty} \exp\, i\omega|z-z_1|(b^2-\dot{b})\psi dz_1,$$

we have

$$||T_3|| \le \frac{1}{2\omega}\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz,$$

and it follows from the contraction principle that no eigenvalues exist satisfying

$$|\lambda|\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < 2\omega.$$

It remains to exclude the equal-sign in (7). Let λ be an eigenvalue satisfying

$$|\lambda|\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz = 2\omega$$

and ψ a corresponding eigenvector:

$$\psi = \lambda T_3\psi.$$

Then

$$\|\psi\| = |\lambda|\ \|T_3\psi\| = \sup_z \frac{|\lambda|}{2\omega}\left|\int_{-\infty}^{+\infty} \exp\ i\omega|z-z_1|(b^2-\dot{b})\psi dz_1\right|. \qquad (10)$$

Now suppose that the right-hand side of (10) attains its maximal value as z tends to ∞. Then

$$\|\psi\| = \frac{|\lambda|}{2\omega}\left|\int_{-\infty}^{+\infty} \exp\ -i\omega z_1(b^2-\dot{b})\psi dz_1\right| \le \frac{|\lambda|}{2\omega}\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz_1\ \|\psi\| = \|\psi\|. \qquad (11)$$

Equality in (11) can only hold if throughout an interval where $b^2-\dot{b} \neq 0$

$$\exp\ -i\omega z_1\ (b^2-\dot{b})\psi(z_1) = |b^2-\dot{b}|\ \|\psi\|\ \exp\ i\alpha,$$

where α is a real constant. This implies that throughout this interval

$$\psi(z_1) = \|\psi\|\ \exp\ i\omega(z_1-z_0). \qquad (12)$$

Because ψ is an eigenvector, it satisfies

$$\psi_{zz} + \omega^2\psi = \lambda(b^2-\dot{b})\psi,$$

and this contradicts (12). Similarly the possibilities can be treated that the right-hand side of (10) attains its maximal value as $z \to -\infty$ or at a finite value. This completes the proof of (7).
To study the behaviour of the spectral radius of T_3 as $\omega \to 0$, we study the eigenvalues of ωT_3; this amounts to a division of all eigenvalues of T_3 by a factor ω. A difficulty in proving the statement that we want to prove is that ωT_3 is not a continuous function of ω. We overcome this difficulty by constructing a compact operator $O_1(\omega)$ on C(R)×C(R) that has the same spectrum and that is a continuous function of ω. Let λ be an eigenvalue of ωT_3 and ψ a corresponding eigenvector:

$$\psi(z) = \frac{\lambda}{2i}\int_{-\infty}^{z} \exp\ i\omega(z-z_1)(b^2-\dot{b})\psi dz_1 + \frac{\lambda}{2i}\int_{z}^{\infty} \exp\ -i\omega(z-z_1)(b^2-\dot{b})\psi dz_1. \qquad (13)$$

Then we define

$$\psi_{\downarrow}(z) = \frac{1}{2i}\int_{z}^{\infty} \exp\, i\omega z_1 (b^2-\dot{b})\psi dz_1,$$

$$\psi_{\uparrow}(z) = \frac{1}{2i}\int_{-\infty}^{z} \exp\, -i\omega z_1 (b^2-\dot{b})\psi dz_1.$$

From (13) we see that $(\psi_{\downarrow},\psi_{\uparrow}) \in C(R)\times C(R)$ is an eigenvector corresponding to the eigenvalue λ of an operator $O_1(\omega)$ on $C(R)\times C(R)$:

$$\psi_{\downarrow}(z) = \frac{\lambda}{2i}\int_{z}^{\infty} (b^2-\dot{b})(\psi_{\downarrow} + \exp\, 2i\omega z_1 \psi_{\uparrow})dz_1,$$

$$\psi_{\uparrow}(z) = \frac{\lambda}{2i}\int_{-\infty}^{z} (b^2-\dot{b})(\exp\, -2i\omega z_1 \psi_{\downarrow} + \psi_{\uparrow})dz_1.$$

Conversely, an eigenvalue λ and a corresponding eigenvector $(\psi_{\downarrow},\psi_{\uparrow})$ of $O_1(\omega)$ yield an eigenvalue λ and an eigenvector $\psi = \psi_{\downarrow}\exp -i\omega z + \psi_{\uparrow}\exp i\omega z$ of ωT_3. The operator $O_1(\omega)$ is easily seen to be compact; cf Lemma 2, pg 25. $O_1(\omega)$ is also simply proved to be a continuous function of ω. So we know that the spectrum of $O_1(\omega)$ also depends continuously on ω in the following sense (see Kato [20], pg 213): if λ_0 is an eigenvalue of $O_1(\omega_0)$, then in any neighbourhood of λ_0 there is an eigenvalue of $O_1(\omega)$ for $|\omega-\omega_0|$ sufficiently small. The spectrum of $O_1(0)$ can be found simply. Let λ_0 be an eigenvalue of $O_1(0)$ and $(\psi_{\downarrow},\psi_{\uparrow})$ a corresponding eigenvector:

$$\psi_{\downarrow}(z) = \frac{\lambda_0}{2i}\int_{z}^{\infty} (b^2-\dot{b})(\psi_{\downarrow}+\psi_{\uparrow})dz_1,$$

$$\psi_{\uparrow}(z) = \frac{\lambda_0}{2i}\int_{-\infty}^{z} (b^2-\dot{b})(\psi_{\downarrow}+\psi_{\uparrow})dz_1.$$

By addition we find immediately that $O_1(0)$ has only one eigenvalue λ_0 , and this eigenvalue satisfies

$$\frac{\lambda_0}{2i}\int_{-\infty}^{+\infty} (b^2-\dot{b})dz_1 = \frac{\lambda_0}{2i}\int_{-\infty}^{+\infty} b^2 dz_1 = 1. \tag{14}$$

Because of the continuity of the spectrum there is in any neighbourhood of λ_0 an eigenvalue of $O_1(\omega)$ and hence also an eigenvalue of $\omega T_3(\omega)$ for sufficiently small $|\omega|$. By multiplication by ω we return to the eigenvalues of T_3, and this immediately leads to the required result.
This completes the proof of Theorem 3 (iii). □

Our next theorem essentially restates Atkinson's proof concerning the convergence of the Bremmer series; see Atkinson [2].

Theorem 3 (iv)
All eigenvalues of T_4 satisfy

$$|\lambda|\int_{-\infty}^{+\infty}|b|dz > \frac{\pi}{2} . \qquad (15)$$

The Neumann expansion

$$\sum_{n=0}^{\infty} \lambda^n T_4{}^n \zeta \qquad (16)$$

converges for $\lambda = 1$ and $\zeta = \chi_0$ if

$$\int_{-\infty}^{+\infty}|b|dz \leq \frac{\pi}{2} , \qquad (17)$$

and then it represents the solution of the scattering problem with normalization of the incident wave. The estimate (15) and consequently the criterion (17) are the best possible in the sense that $\frac{\pi}{2}$ cannot be replaced by a larger number such that (15) and (17) remain valid for all frequencies and all profiles $c(x)$. This is so because for a profile $c(x)$ that is nonincreasing or nondecreasing there exists an eigenvalue that tends to λ_0 as $\omega \to 0$; λ_0 satisfies

$$|\lambda_0|\int_{-\infty}^{+\infty}|b|dz = \frac{\pi}{2} .$$

It is worth noting that although the criteria stated in Theorem 3 (iii) and (iv) are the best possible in the specified sense, this does not entail the impossibility of finding other criteria. We shall investigate this in the next section. Further we remark that whereas the series (3) converges whenever (1.2.2) is satisfied, the series (8) does not. A similar remark can be made with regard to the convergence of the series (5) and (16). These differences have already been observed by several authors; see, for example, Broer and Van Vroonhoven [10], Berk, Book, and Pfirsch [5], and Kay [21]. They are bound up with the different normalization. Let us examine, for example, the case of (3) and (8), and we restrict ourselves to $\zeta = \chi_0$.
We derive the solution with normalization of the incident wave from the one with normalization of the transmitted wave. If the solutions of $\chi = \chi_0 + \lambda T_j \chi$, $j = 1$ and 3, both exist, they both satisfy

$$\psi_{zz} + \omega^2 \psi = \lambda(b^2 - \dot{b})\psi.$$

Because they consist only of an outgoing wave as $z \to \infty$, they differ by a constant factor. The solution with normalization of the transmitted wave satisfies as $z \to -\infty$

$$\psi = \gamma_-(\lambda) \exp i\omega z + \delta_-(\lambda) \exp -i\omega z + o(1);$$

$\gamma_-(\lambda)$ readily follows from the expansion (3):

$$\gamma_-(\lambda) = 1 - \frac{\lambda}{2i\omega} \int_{-\infty}^{+\infty} (b^2 - \dot{b}) dz_1 +$$

$$+ \frac{\lambda^2}{2i\omega^2} \int_{-\infty}^{+\infty} \exp -i\omega z_2 (b^2 - \dot{b}) dz_2 \int_{z_2}^{\infty} \sin \omega(z_2 - z_1)(b^2 - \dot{b}) \exp i\omega z_1 dz_1 + \ldots$$

If this solution is divided by $\gamma_-(\lambda)$, the normalization is changed from the transmitted wave to the incident wave. This can be done only if $\gamma_-(\lambda) \neq 0$. Zeros of $\gamma_-(\lambda)$ are singular points of the solution with normalization of the incident wave, these zeros are the eigenvalues of T_4; the Taylor expansion of the latter solution can only converge if $|\lambda|$ does not exceed the smallest absolute value of the zeros of $\gamma_-(\lambda)$.

§1.5. NEUMANN EXPANSIONS AND THEIR CONVERGENCE. CONTINUATION

While Theorem 3 (iii), pg 33, states that the expansion

$$\sum_{n=0}^{\infty} T_3^{\,n} \chi_0 \tag{1}$$

converges for sufficiently high frequencies if

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < \infty,$$

Theorem 3 (iv), pg 37, does not imply that the Bremmer series

$$\sum_{n=0}^{\infty} T_4^{\,n} \chi_0 \tag{2}$$

converges for sufficiently high frequencies if

$$\int_{-\infty}^{+\infty} |b|dz < \infty.$$

However, one might expect this to be true because the integrands of the terms of the Bremmer series oscillate rapidly for high frequencies. Besides, from asymptotics it is known that the WKB approximation is an asymptotic solution of the differential equation (1.2.1) as $\omega \to \infty$; see, for example, Erdélyi [15], Chapter 4. The next theorems express how the convergence of the Bremmer series depends on the frequency. We will indicate how the methods used may also lead to other criteria for (1).

Theorem 4

All eigenvalues of T_4 satisfy simultaneously the inequalities

$$|\lambda|\{\sup_z \int_{-\infty}^{+\infty} |b(z_1)||F_i(z,z_1)|dz_1\}^{\frac{1}{2}} \geq 1 \quad i = 1,2. \tag{3,4}$$

Here

$$\begin{aligned} F_1(z,z_1) &= \int_z^{\infty} \exp 2i\omega z_2 b(z_2)dz_2 \qquad \text{for } z_1 \leq z, \\ &= \int_{z_1}^{\infty} \exp 2i\omega z_2 b(z_2)dz_2 \qquad \text{for } z_1 \geq z, \end{aligned} \tag{5}$$

$$\begin{aligned} F_2(z,z_1) &= \int_{-\infty}^{z_1} \exp -2i\omega z_2 b(z_2)dz_2 \qquad \text{for } z_1 \leq z, \\ &= \int_{-\infty}^{z} \exp -2i\omega z_2 b(z_2)dz_2 \qquad \text{for } z_1 \geq z\ . \end{aligned} \tag{6}$$

The terms within braces in (3) and (4) tend to zero as ω tends to ∞. Hence the spectral radius of T_4 tends to ∞ as ω tends to ∞.
The Bremmer series converges if one of the conditions

$$\sup_z \int_{-\infty}^{+\infty} |b(z_1)||F_i(z,z_1)|dz_1 < 1, \qquad i = 1,2, \tag{7,8}$$

is satisfied. It converges for sufficiently high frequencies.

Proof

Suppose that λ is an eigenvalue of T_4 and $\psi = (\psi_\downarrow,\psi_\uparrow)$ a corresponding eigenvector. Then ψ is also an eigenvector of T_4^2 corresponding to the eigenvalue λ^2:

$$\psi = \lambda^2 T_4^2 \psi. \tag{9}$$

Equation (9) splits into two eigenvalue equations in $C(R)$:

$$\psi_\downarrow = \lambda^2 T_{4\downarrow}^2 \psi_\downarrow,$$

$$\psi_\uparrow = \lambda^2 T_{4\uparrow}^2 \psi_\uparrow.$$

Here $T^2_{4\downarrow}$ and $T^2_{4\uparrow}$ are defined in an obvious way; for example,

$$T^2_{4\downarrow}\zeta(z) = -\int_z^{\infty} \exp -i\omega(z-z_2)b(z_2)dz_2 \int_{-\infty}^{z_2} \exp i\omega(z_2-z_1)b(z_1)\zeta(z_1)dz_1.$$

Let us consider the first of these two eigenvalue problems. From the contraction principle it follows that all eigenvalues satisfy

$$|\lambda|\, \|T^2_{4\downarrow}\|^{\frac{1}{2}} \geq 1.$$

By an interchange of the order of integration and by the definition of F_1 we obtain

$$T^2_{4\downarrow}\zeta(z) = -\int_{-\infty}^{+\infty} \exp -i\omega(z+z_1)b(z_1)F_1(z,z_1)\zeta(z_1)dz_1.$$

Hence

$$|T^2_{4\downarrow}\zeta(z)| \leq \sup_z \int_{-\infty}^{+\infty} |b(z_1)|\,|F_1(z,z_1)|dz_1\, \|\zeta\|.$$

This leads to (3). In the same manner we find (4).
The terms within braces in (3) and (4) tend to zero as $\omega \to \infty$ because according to the Riemann-Lebesgue Lemma

$$\int_\alpha^\beta b(z)\exp 2i\omega z\, dz \to 0 \text{ as } \omega \to \infty$$

uniformly for all α and β if

$$\int_{-\infty}^{+\infty} |b|dz < \infty.$$

The other assertions are an immediate consequence of what we proved.
□

We remark that an infinite number of conditions may be derived in almost the same manner which all assure the convergence of the Bremmer series. To that end we should use the fact that no eigenvalues of T_4 exist satisfying

$$|\lambda|\ \|T_4^m\|^{1/m} < 1$$

for any integer m. This is a consequence of (1.3.8).
Further we remark that the same reasoning should also yield other criteria for the series (1). We do not work this out, and we proceed with the proof of a sufficient condition for the convergence of the Bremmer series that is somewhat similar to the one we proved in Theorem 3 (iii), pg 33, for the series (1).

Theorem 5

Suppose that

$$\int_{-\infty}^{+\infty} |b|dz < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < \infty.$$

There are no eigenvalues of T_4 satisfying

$$\int_{-\infty}^{+\infty} (|\lambda|^2 b^2+|\lambda||\dot{b}|)dz \leq 2\omega. \tag{10}$$

The Bremmer series (2) converges if

$$\int_{-\infty}^{+\infty} (b^2+|\dot{b}|)dz \leq 2\omega. \tag{11}$$

Proof

Let λ be an eigenvalue of T_4 and $(\psi_\downarrow, \psi_\uparrow)$ a corresponding eigenvector. By differentiation we find

$$\begin{aligned} \psi_{\downarrow z} + i\omega\psi_\downarrow &= -\lambda b\psi_\uparrow, \\ \psi_{\uparrow z} - i\omega\psi_\uparrow &= -\lambda b\psi_\downarrow. \end{aligned} \tag{12}$$

This involves that $\psi = \psi_\downarrow+\psi_\uparrow$ satisfies

$$\psi_{zz} + \omega^2\psi = (\lambda^2 b^2-\lambda\dot{b})\psi. \tag{13}$$

Further $\psi_\downarrow$ and $\psi_\uparrow$ satisfy

$$\psi_\downarrow = o(1) \text{ and } \psi_\uparrow = \gamma_+ \exp i\omega z + o(1) \text{ as } z \to \infty,$$
$$\psi_\downarrow = \delta_- \exp -i\omega z + o(1) \text{ and } \psi_\uparrow = o(1) \text{ as } z \to -\infty.$$

Both γ_+ and δ_- are not zero; if they were, this would imply that $(\psi_\downarrow,\psi_\uparrow) = 0$. For example, suppose that $\gamma_+ = 0$. On account of (12) and the boundary conditions, $(\psi_\downarrow,\psi_\uparrow)$ must be a nonzero solution of

$$\psi_\downarrow(z) = \lambda\int_z^\infty \exp -i\omega(z-z_1)b\psi_\uparrow dz_1,$$
$$\psi_\uparrow(z) = \lambda\int_z^\infty \exp \;\; i\omega(z-z_1)b\psi_\downarrow dz_1.$$

But by Theorem 2 (ii), pg 27, this equation has no nonzero solutions. So ψ satisfies

$$\psi = \gamma_+ \exp i\omega z + o(1) \quad \text{as } z \to \infty,$$
$$\psi = \delta_- \exp -i\omega z + o(1) \quad \text{as } z \to -\infty,$$

with $\gamma_+ \neq 0$ and $\delta_- \neq 0$. A solution of (13) that satisfies these boundary conditions must be an eigenvector of the following nonlinear eigenvalue problem in $C(R)$:

$$\psi(z) = \frac{1}{2i\omega}\int_{-\infty}^{+\infty} \exp i\omega|z-z_1|(\lambda^2 b^2 - \lambda\dot{b})\psi dz_1.$$

(On account of Lemma 1, pg 15, the integral converges.)
From the contraction principle it follows that no eigenvalues exist satisfying

$$\int_{-\infty}^{+\infty}(|\lambda|^2 b^2 + |\lambda||\dot{b}|)dz < 2\omega. \qquad (14)$$

By arguing in a similar way as we did in Theorem 3 (iii), pg 33, we can exclude the equal-sign in (10). Therefore the spectral radius is larger than 1 if

$$\int_{-\infty}^{+\infty}(b^2 + |\dot{b}|)dz \le 2\omega. \qquad \square$$

In order to give a more comprehensive account of published results concerning the convergence of the Bremmer series, we shall now derive a condition that is essentially the one Berk, Book, and Pfirsch [5] used, but which they derived in an awkward way. It is based on the observation that to assure the convergence of

$$\sum_{n=0}^{\infty} \lambda^n T_4^{\,n} \zeta \tag{15}$$

it is sufficient that the inequalities

$$|\lambda| \limsup_{n\to\infty} \{\int_{-\infty}^{+\infty} |b(z)|\,|(T_4^{\,n}\zeta)_{\downarrow,\uparrow}(z)|^2 dz\}^{1/2n} < 1 \tag{16}$$

are satisfied. This can be understood in the following way. (See also Riesz and Nagy [29], pg 151.) From Cauchy's convergence test we know that (15) converges if

$$|\lambda| < \{\limsup_{n\to\infty} \|T_4^{\,n}\zeta\|^{1/n}\}^{-1}. \tag{17}$$

Using the Schwarz inequality we can obtain an estimate of the right-hand side of (17):

$$|(T_4^{\,n}\zeta)_{\downarrow}(z)| = |\int_z^{\infty} \exp -i\omega(z-z_1) b (T_4^{\,n-1}\zeta)_{\uparrow} dz_1| \le$$

$$\le \{\int_z^{\infty} |b| dz_1\}^{\frac{1}{2}} \{\int_z^{\infty} |b|\,|(T_4^{\,n-1}\zeta)_{\uparrow}|^2 dz_1\}^{\frac{1}{2}} \le$$

$$\le \{\int_{-\infty}^{+\infty} |b| dz_1\}^{\frac{1}{2}} \{\int_{-\infty}^{+\infty} |b|\,|(T_4^{\,n-1}\zeta)_{\uparrow}|^2 dz_1\}^{\frac{1}{2}}.$$

Therefore

$$\limsup_{n\to\infty} \|(T_4^{\,n}\zeta)_{\downarrow}\|^{1/n} \le \limsup_{n\to\infty} \{\int_{-\infty}^{+\infty} |b|\,|(T_4^{\,n}\zeta)_{\uparrow}|^2 dz\}^{1/2n}.$$

In the same way we obtain

$$\limsup_{n\to\infty} \|(T_4^{\,n}\zeta)_{\uparrow}\|^{1/n} \le \limsup_{n\to\infty} \{\int_{-\infty}^{+\infty} |b|\,|(T_4^{\,n}\zeta)_{\downarrow}|^2 dz\}^{1/2n}.$$

Together with (17) this leads to (16).
In our next theorem we obtain an upper bound of the limsup in (16). This leads to another sufficient criterion for the convergence of the Bremmer series. Again many estimates can be made, leading to different criteria. We restrict ourselves to the following theorem. (We notice that the technique we described can also be applied to obtain new criteria for (1).)

Theorem 6

A sufficient condition for the convergence of (15) is that both inequalities

$$|\lambda|\{\iint_{-\infty}^{+\infty} |b(z_1)||b(z_2)||F_i(z_1,z_2)|^2 dz_1 dz_2\}^{\frac{1}{2}} < 1, \quad i = 1,2, \tag{18}$$

are satisfied. The Bremmer series converges if

$$\iint_{-\infty}^{+\infty} |b(z_1)||b(z_2)||F_i(z_1,z_2)|^2 dz_1 dz_2 < 1, \ i = 1 \text{ and } 2. \tag{19}$$

Here F_1 and F_2 are defined by (5) and (6).

Proof

With the definitions of T_4 and F_1 we find for $m \geq 2$

$$\int_{-\infty}^{+\infty} |b||(T_4{}^m\zeta)_{\downarrow}|^2 dz =$$

$$= \int_{-\infty}^{+\infty} |b(z_1)|dz_1|\int_{-\infty}^{+\infty} b(z_2)F_1(z_1,z_2)(T_4{}^{m-2}\zeta)_{\downarrow}(z_2)dz_2|^2 \leq$$

$$\leq \iint_{-\infty}^{+\infty} |b(z_1)||b(z_2)||F_1(z_1,z_2)|^2 dz_1 dz_2 \int_{-\infty}^{+\infty} |b||(T_4{}^{m-2}\zeta)_{\downarrow}|^2 dz.$$

The same inequality holds for the "upward" terms with F_2 for F_1. By induction this leads to

$$\int_{-\infty}^{+\infty} |b||(T_4^{2m}\zeta)_{\downarrow,\uparrow}|^2 dz \leq$$

$$\leq \{\iint_{-\infty}^{+\infty} |b(z_1)||b(z_2)||F_{1,2}(z_1,z_2)|^2 dz_1 dz_2\}^m \int_{-\infty}^{+\infty} |b||\zeta_{\downarrow,\uparrow}|^2 dz$$

and

$$\int_{-\infty}^{+\infty} |b||(T_4^{2m+1}\zeta)_{\downarrow,\uparrow}|^2 dz \leq$$

$$\leq \{\iint_{-\infty}^{+\infty} |b(z_1)||b(z_2)||F_{1,2}(z_1,z_2)|^2 dz_1 dz_2\}^m \int_{-\infty}^{+\infty} |b||(T_4\zeta)_{\downarrow,\uparrow}|^2 dz.$$

Therefore

$$\limsup_{n\to\infty}\{\int_{-\infty}^{+\infty} |b||(T_4^{n}\zeta)_{\downarrow,\uparrow}|^2 dz\}^{1/2n} \leq$$

$$\leq \quad \{\iint_{-\infty}^{+\infty} |b(z_1)||b(z_2)||F_{1,2}(z_1,z_2)|^2 dz_1 dz_2\}^{\frac{1}{4}}.$$

Putting this in (16) we obtain (18). □

Up to now we assumed in discussing the convergence of

$$\sum_{n=0}^{\infty} T_2^{\,n} x_0 \quad \text{and} \quad \sum_{n=0}^{\infty} T_4^{\,n} x_0 \tag{20,21}$$

that

$$\int_{-\infty}^{+\infty} |b| dz < \infty.$$

To conclude this section we shall examine whether we can dispense with this condition. Of course, we can no longer be sure that the terms of the series make sense. We shall use a method due to Broer and Van Vroonhoven [10], who established the interrelationship of the terms of (20,21) and those of

$$\sum_{n=0}^{\infty} T_1^{\,n} x_0 \quad \text{and} \quad \sum_{n=0}^{\infty} T_3^{\,n} x_0, \tag{22,23}$$

respectively. But they did not realize that from this a new criterion for the convergence of the series (20) and (21) can be derived.

Theorem 7 (i)

A sufficient condition for the convergence of (20) in C(R)×C(R) is

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < \infty. \qquad (24)$$

If we denote the sum of (20) by $(\psi_\downarrow,\psi_\uparrow)$, then $\psi = \psi_\downarrow+\psi_\uparrow$ is the unique solution of the scattering problem with normalization of the transmitted wave.

Theorem 7 (ii)

The terms of (21) belong to C(R)×C(R) if (24) is satisfied.
The series (21) converges in C(R)×C(R) if

$$\int_{-\infty}^{+\infty} (b^2+|\dot{b}|)dz < 2\omega. \qquad (25)$$

If we denote the sum of (21) by $(\psi_\downarrow,\psi_\uparrow)$, then $\psi = \psi_\downarrow+\psi_\uparrow$ is the unique solution of the scattering problem with normalization of the incident wave. (Cf Theorem 5, pg 42.)

Because in essence the proofs are the same, we shall only sketch the proof of the first part of Theorem 7.

Proof of Theorem 7 (i)

Let $(\psi_\downarrow{}^{(n)},\psi_\uparrow{}^{(n)})$ denote $T_2{}^n\chi_0$ (we shall successively prove that these terms make sense); and let $\psi^{(n)}$ denote $T_1{}^n\chi_0$. We shall expose a simple relation between $\psi^{(n)}$ and $(\psi_\downarrow{}^{(m)},\psi_\uparrow{}^{(m)})$. First we remark that $\psi_\uparrow{}^{(2n+1)} = 0$ and $\psi_\downarrow{}^{(2n)} = 0$ because $\psi_\downarrow{}^{(0)} = 0$. By the definition $\psi_\uparrow{}^{(0)}$ is equal to $\psi^{(0)}$. Then we prove that the integral defining $\psi_\downarrow{}^{(1)}$:

$$\psi_\downarrow{}^{(1)}(z) = \int_z^{\infty} \exp -i\omega(z-z_1)b \exp i\omega z_1 dz_1,$$

converges, and that $\psi_{\downarrow}^{(1)}$ is uniformly bounded and is a continuous function of z. This can be proved with the aid of integration by parts if we use the results of Lemma 1, pg 15, to the effect that $\dot{b}$ is absolutely integrable and that b approaches zero as $|z| \to \infty$. We obtain

$$\psi_{\downarrow}^{(1)}(z) = \omega^{-1}\int_{z}^{\infty}\sin\omega(z-z_1)\dot{b}\exp i\omega z_1 dz_1.$$

So $\psi_{\downarrow}^{(1)}$ is a part of the integral defining $\psi^{(1)}$:

$$\psi^{(1)}(z) = -\omega^{-1}\int_{z}^{\infty}\sin\omega(z-z_1)(b^2-\dot{b})\exp i\omega z_1 dz_1.$$

The remaining part of $\psi^{(1)}$ emerges as a part of $\psi_{\uparrow}^{(2)}$. How in general the terms $\psi_{\downarrow}^{(2n+1)}$ and $\psi_{\uparrow}^{(2n)}$ are related to $\psi^{(m)}$ is expressed by the following assertions. Suppose that $\psi_{\uparrow}^{(2n)}$ and $\psi_{\downarrow}^{(2n-1)}$ are uniformly bounded and that they satisfy

$$\psi_{\uparrow z}^{(2n)} - i\omega\psi_{\uparrow}^{(2n)} = -b\psi_{\downarrow}^{(2n-1)};$$

then the integral defining $\psi_{\downarrow}^{(2n+1)}$:

$$\psi_{\downarrow}^{(2n+1)}(z) = \int_{z}^{\infty}\exp -i\omega(z-z_1)b\psi_{\uparrow}^{(2n)}dz_1,$$

converges, is uniformly bounded, and is a continuous function of z. Further $\psi_{\downarrow}^{(2n+1)}$ satisfies

$$\psi_{\downarrow}^{(2n+1)}(z) = -\omega^{-1}\int_{z}^{\infty}\sin\omega(z-z_1)b^2\psi_{\downarrow}^{(2n-1)}dz_1 + \omega^{-1}\int_{z}^{\infty}\sin\omega(z-z_1)\dot{b}\psi_{\uparrow}^{(2n)}dz_1 \tag{26}$$

and satisfies

$$\psi_{\downarrow z}^{(2n+1)} + i\omega\psi_{\downarrow}^{(2n+1)} = -b\psi_{\uparrow}^{(2n)}. \tag{27}$$

This assertion can be proved simply with the aid of integration by parts and Lemma 1. Similarly the following assertion can be proved to hold. If $\psi_{\downarrow}^{(2n+1)}$ and $\psi_{\uparrow}^{(2n)}$ are uniformly bounded and if they satisfy

$$\psi_{\downarrow z}^{(2n+1)} + i\omega\psi_{\downarrow}^{(2n+1)} = -b\psi_{\uparrow}^{(2n)},$$

then the integral defining $\psi_{\uparrow}^{(2n+2)}$:

$$\psi_{\uparrow}^{(2n+2)}(z) = \int_z^{\infty} \exp i\omega(z-z_1) b\psi_{\downarrow}^{(2n+1)} dz_1,$$

converges, is uniformly bounded, and depends continuously on z. Further $\psi_{\uparrow}^{(2n+2)}$ satisfies

$$\psi_{\uparrow}^{(2n+2)}(z) = -\omega^{-1}\int_z^{\infty} \sin \omega(z-z_1) b^2\psi_{\uparrow}^{(2n)} dz_1 + \\ + \omega^{-1}\int_z^{\infty} \sin \omega(z-z_1)\dot{b}\psi_{\downarrow}^{(2n+1)} dz_1 \tag{28}$$

and satisfies

$$\psi_{\uparrow z}^{(2n+2)} - i\omega\psi_{\uparrow}^{(2n+2)} = -b\psi_{\downarrow}^{(2n+1)}. \tag{29}$$

In this manner we find successively and piecewise the terms $\psi^{(n)}$. It turns out that the series (20) results from a rearrangement of the terms of the series (22) such that the resulting series is arranged according to increasing powers of b. The terms of order n can be found by premultiplication of the terms of order n-2 by

$$-\omega^{-1}\int_z^{\infty} \sin \omega(z-z_1) b^2 \;\ldots.\; dz_1$$

and by premultiplication of the terms of order n-1 by

$$\omega^{-1}\int_z^{\infty} \sin \omega(z-z_1)\dot{b} \;\ldots.\; dz_1,$$

as can be seen from (26) and (28). Because we find successively all the pieces that build up $\psi^{(n)}$, the following estimate must be valid. (See also the proof of Theorem 3 (i), pg 30.)

$$\sum_{n=0}^{\infty}\{|\psi_{\downarrow}{}^{(n)}(z)| + |\psi_{\uparrow}{}^{(n)}(z)|\} \leq$$

$$\leq \quad 1 + \omega^{-1}\int_{z}^{\infty}(b^2+|\dot{b}|)dz_1 + \omega^{-2}\int_{z}^{\infty}(b^2+|\dot{b}|)dz_2\int_{z_2}^{\infty}(b^2+|\dot{b}|)dz_1 + \ldots =$$

$$= \quad \exp\ \omega^{-1}\int_{z}^{\infty}(b^2+|\dot{b}|)dz_1 \leq \exp\ \omega^{-1}\int_{-\infty}^{+\infty}(b^2+|\dot{b}|)dz_1.$$

So the condition (24) is a sufficient condition for the convergence of (20) in $C(R)\times C(R)$. Because of the differential equations (27) and (29) $\psi_{\downarrow}$ and $\psi_{\uparrow}$ satisfy Bremmer's equation (1.2.8); they also satisfy the required boundary conditions. Therefore $\psi = \psi_{\downarrow} + \psi_{\uparrow}$ is the unique solution with normalization of the transmitted wave. □

§1.6. ANOTHER ITERATIVE PROCEDURE
APPLICATION OF THE FOREGOING THEORY

Several methods exist that all provide an asymptotic solution of (1.2.1) as $\omega \to \infty$; see, for example, Erdélyi [15]. We describe one of these only and show how we can replace this asymptotic method by an iterative procedure that leads to a sequence converging to a solution of (1.2.1). However, some other asymptotic methods may be handled as well in essentially the same manner.

The solutions of (1.2.1) that have no zeros can, as is well-known, be represented in one of the forms

$$\phi_{\pm} = \gamma F^{-\frac{1}{2}}(x)\exp \pm i\int^{x} F(x_1)dx_1, \quad F(x) > 0, \qquad (1)$$

where F(x) obeys the nonlinear differential equation

$$F^2 = k^2(x) + F^{\frac{1}{2}} \frac{d^2}{dx^2} F^{-\frac{1}{2}}, \quad k^2 = \omega^2 c^{-2}(x). \tag{2}$$

The truth of this can be seen in the following way. For every solution of (1.2.1) having no zeros a real $A(x) > 0$ and a real $B(x)$, both twice continuously differentiable, exist such that

$$\phi = A^{-\frac{1}{2}}(x) \exp iB(x).$$

Now, for any solution of (1.2.1) we have

$$\phi\phi_x^* - \phi^*\phi_x = -2i\alpha,$$

where α is a real constant. This implies $B_x = \alpha A(x)$. The constant α cannot be zero because this would imply that ϕ is a multiple of a real solution of (1.2.1), and real solutions do have zeros. This immediately leads to the representation (1); the sign in (1) corresponds with the sign of the intensity. (A solution that has no zeros is a solution with nonzero intensity, and conversely.)

One might try to solve equation (2) by an expansion in powers of ω. The first terms of this expansion are easily found:

$$F = \omega c^{-1}(x) + \frac{1}{2}\,\omega^{-1} c^{\frac{1}{2}} \frac{d^2}{dx^2} c^{\frac{1}{2}} + \ldots \tag{3}$$

Insertion of the first term of (3) in (1) leads to the two first-order WKB approximations. However, the expansion (3), if possible at all, will in general not converge as is commonly known; it constitutes an asymptotic sequence.

In this section we shall be concerned with an iterative method that provides a solution of (2) in the form of a convergent series. The methods used will turn out to be intimately related to those of the preceding sections. We start with "flattening" the equation (2) by means of a Liouville-type transformation:

$$G = cF, \quad z = \int_{x_0}^{x} c^{-1}(x_1)dx_1. \tag{4}$$

This transforms (2) into

$$G^2 = \omega^2 + G^{\frac{1}{2}} \frac{d^2}{dz^2} G^{-\frac{1}{2}} - (b^2 - \dot{b}) \tag{5}$$

or

$$G^2 = \omega^2 + \frac{3}{4} G^{-2}\left(\frac{dG}{dz}\right)^2 - \frac{1}{2} G^{-1} \frac{d^2G}{dz^2} - (b^2 - \dot{b}). \tag{6}$$

Instead of (6) we study the extended problem of finding the solutions of

$$G^2 = \omega^2 + \frac{3}{4} G^{-2}\left(\frac{dG}{dz}\right)^2 - \frac{1}{2} G^{-1} \frac{d^2G}{dz^2} - \lambda(b^2 - \dot{b}) \tag{7}$$

for arbitrary complex λ. We try to find a solution of (7) in the form of a power series in λ. This expansion can be carried out formally for arbitrary λ. In particular we are interested in the convergence for $\lambda = 1$. Its first term $G^{(0)}$ is a solution of the equation

$$G^2 = \omega^2 + \frac{3}{4} G^{-2}\left(\frac{dG}{dz}\right)^2 - \frac{1}{2} G^{-1} \frac{d^2G}{dz^2}.$$

The real, positive solutions of this equation are

$$G^{(0)} = \omega\{\beta + (\beta^2 - 1)^{\frac{1}{2}} \sin 2\omega(z - z_0)\}^{-1}, \quad \beta \geq 1.$$

One way to prove this is the following. From the differential equation that is satisfied by $G^{(0)}$ it follows that $H^{(0)} = G^{(0)-1}$ satisfies a linear differential equation of the third order which can be solved readily:

$$H_{zzz} + 4\omega^2 H_z = 0.$$

Insertion of the above solutions in (1) leads to all linear combinations of the WKB approximations:

$$\phi = \alpha_1 k^{-\frac{1}{2}}(x) \exp i\int^{x} k(x_1)dx_1 + \alpha_2 k^{-\frac{1}{2}}(x) \exp -i\int^{x} k(x_1)dx_1, \quad |\alpha_1| \neq |\alpha_2|,$$

as can be verified by simple calculations. If the plus sign is chosen in (1), we obtain the combinations with $|\alpha_1| > |\alpha_2|$; the minus sign in (1) leads to the combinations with $|\alpha_1| < |\alpha_2|$. So, the fact that $G^{(0)}$ is periodic with the double frequency 2ω is essential for the presence of both WKB approximations at the same time. The second term is a solution of

$$\frac{d^2}{dz^2} G^{(1)} + G^{(1)}\{3G^{(0)-2}(\frac{dG^{(0)}}{dz})^2 - G^{(0)-1}\frac{d^2G^{(0)}}{dz^2} + 4G^{(0)2}\} -$$
$$- 3\frac{dG^{(1)}}{dz} G^{(0)-1}\frac{dG^{(0)}}{dz} + 2G^{(0)}(b^2-\dot{b}) = 0.$$

Etc.

How the expansion has to be performed and how the convergence of the resultant series can be proved will be illustrated by the expansion that leads to the solution with normalization of the transmitted wave. We assume that

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < \infty.$$

It will be shown how the proposed expansion is related to the series

$$\sum_{n=0}^{\infty} \lambda^n T_1{}^n \chi_o, \tag{8}$$

that for $\lambda = 1$ also represents the solution with normalization of the transmitted wave. First we notice that the change of variables (4) corresponds with the Liouville transformation (1.2.6). The Liouville transformation transforms the solutions (1) of (1.2.1) into the solutions

$$\psi_\pm = \gamma G^{-\frac{1}{2}}(z)\exp \pm i\int^z G(z_1)dz_1 \tag{9}$$

of

$$\psi_{zz} + \omega^2\psi = (b^2-\dot{b})\psi, \tag{10}$$

where G is related to F by means of (4) and G satisfies (5). The solution of (10) with normalization of the transmitted wave satisfies

$$\psi = \exp i\omega z + o(1) \tag{11}$$

and

$$\psi_z = i\omega \exp i\omega z + o(1). \tag{12}$$

In view of this it is convenient to rewrite the representation (9) in a slightly different form. The solution of (10) that satisfies (11) and (12) can be represented by

$$\psi = \omega^{\frac{1}{2}}G^{-\frac{1}{2}}(z)\exp\{i\omega z - i\int_z^\infty (G-\omega)dz_1\}, \qquad G > 0, \tag{13}$$

where G satisfies (5) and G satisfies as $z \to \infty$

$$G = \omega + o(1) \tag{14}$$

such that the integral in (13) converges. This can be proved in the same manner as we proved the validity of the representation (1). We notice that the solution of (10) that satisfies (11) and (12) cannot have zeros. This is so because any solution of (10) satisfies

$$\psi\psi_z^* - \psi^*\psi_z = \text{const.}$$

For the solution with normalization of the transmitted wave this constant must be $-2i\omega$ because of (11) and (12). Therefore this solution cannot have zeros.

If we impose the boundary condition (14), the first terms of the expansion of the solution of (7) are readily found:

$$G(z,\lambda) = \omega + \lambda\int_z^\infty \sin 2\omega(z-z_1)(b^2-\dot{b})dz_1 + \lambda^2\int_z^\infty \sin 2\omega(z-z_1)\ldots \tag{15}$$

We observe that at $z = -\infty$ the first terms of the expansion are periodic with the frequency 2ω,representing the fact that at $z = -\infty$ both WKB approximations are present. So we take reflection into account in the expansion. (We denote a function f of the variables z and λ by $f(z,\lambda)$. But we do not scruple about using the same notation lateron for a C(R)-valued function of the complex variable λ.)

To study the possibility of a formal expansion and its convergence, we want to expose the relationship between the parameter λ in (7) and the parameter in (8). The expansion (8) amounts to an expansion of the solution of

$$\psi_{zz} + \omega^2\psi = \lambda(b^2-\dot{b})\psi \tag{16}$$

that satisfies (11) and (12) in powers of λ for arbitrary complex λ. On the other hand, suppose that for real λ $G > 0$ is a solution of (7) that satisfies (14) such that the integral in (13) converges. Then (13) is a solution of (16) that satisfies the boundary condition (11). So, by (13) the expansion of G amounts for real λ to an expansion of

$$\omega\{\psi^*(z,\lambda)\psi(z,\lambda)\}^{-1}, \ \lambda \text{ real}, \tag{17}$$

in powers of λ. Here ψ denotes the solution (8) of (16). This links the expansion (8) that we studied in the preceding sections to the one of this section.

Let $\psi(z,\lambda)$ denote the solution with normalization of the transmitted wave. We know that for real λ

$$G(z,\lambda) = \omega\{\psi^*(z,\lambda)\psi(z,\lambda)\}^{-1}. \tag{18}$$

From the preceding sections we know already that $\psi(z,\lambda)$ is a C(R)-valued analytic function of λ on all of the complex plane. Then, of course, in studying the existence of an expansion of G in powers of λ, one is naturally led to the examination of the existence of an analytic continuation of the right-hand side of (18). We shall see that it is possible to investigate this within the concept of C(R)-valued analytic functions. (As we will prove lateron,

$$\{\psi^*(z,\lambda)\psi(z,\lambda)\}^{-1}$$

belongs to C(R) for real λ. See the corollary, pg 58.) In examining this analytic continuation we need the following lemma. It is an analogue of a simple assertion in ordinary function theory.

Lemma 5

Let U be an open subset of the complex plane, and suppose that $f(z,\lambda)$ is a C(R)-valued function of $\lambda \in U$ that is analytic on U. Then $\frac{1}{f(z,\lambda)}$ is also a C(R)-valued analytic function on U if and only if for all $\lambda \in U$ $\inf_z |f(z,\lambda)| > 0$.

A C(R)-valued function exists, analytic everywhere on the complex plane, such that for real λ it equals $\psi^*(z,\lambda)\psi(z,\lambda)$. This function is

$$\psi^*(z,\lambda^*)\psi(z,\lambda).$$

The analyticity is a direct consequence of the analyticity of $\psi(z,\lambda)$. By Lemma 5 $\omega\{\psi^*(z,\lambda^*)\psi(z,\lambda)\}^{-1}$ is a C(R)-valued analytic function on the subset of the complex plane consisting of all λ for which

$$\inf_z|\psi^*(z,\lambda^*)\psi(z,\lambda)| > 0.$$

The complement of this set consists of all λ for which

$$\inf_z|\psi(z,\lambda)| = 0 \tag{19}$$

and their complex conjugates. Because $\inf_z|\psi(z,\lambda)|$ is a continuous function of λ, the latter set is closed. It remains to be examined for which λ (19) holds. Suppose λ to be so. Because of the asymptotic behaviour (11) we know that for sufficiently large z $|\psi(z,\lambda)| > \frac{1}{2}$. Therefore one of the following situations must occur.

(i) ψ has a finite zero at $z = z_0$.

(ii) Because we know already that constants γ_- and δ_- exist such that

$$\psi = \gamma_-\exp i\omega z + \delta_-\exp -i\omega z + o(1) \text{ as } z \to -\infty,$$

(19) also holds if $|\gamma_-| = |\delta_-|$. This implies the existence of a real constant z_0 and a complex constant γ such that

$$\psi = \gamma \sin \omega(z-z_0) + o(1) \text{ as } z \to -\infty. \tag{20}$$

The examination of both possibilities can be reduced to the study of an eigenvalue problem. Suppose, for example, ψ is a solution of (16) satisfying both (11) and (20). Then by the method of variation of constants we see that ψ must be an eigenvector corresponding to the eigenvalue λ of an operator O_2 defined on all of C(R):

$$\begin{aligned}\psi(z) = &-\frac{\lambda}{\omega}\exp i\omega(z-z_0)\int_{-\infty}^{z}\sin \omega(z_1-z_0)(b^2-\dot{b})\psi\, dz_1 - \\ &-\frac{\lambda}{\omega}\sin \omega(z-z_0)\int_{z}^{\infty}\exp i\omega(z_1-z_0)(b^2-\dot{b})\psi\, dz_1.\end{aligned} \tag{21}$$

This eigenvalue problem can be handled in the same way as we treated the eigenvalue problems of §1.3-5. We prove a lemma, the proof of which resembles that of Theorem 2 (iii), pg 27, and that of Theorem 3 (iii), pg 33. In passing we notice that the methods developed in §1.5 may provide other estimates of the eigenvalues, but we do not go into them.

Lemma 6

The operator O_2 in (21) has no real eigenvalues for any z_0, and for any z_0 it has no eigenvalues satisfying

$$|\lambda|\int_{-\infty}^{+\infty}|b^2-\dot{b}|dz \le \omega. \tag{22}$$

Proof

Suppose λ is a real eigenvalue and ψ a corresponding eigenvector. Then ψ satisfies $\psi_{zz} + \omega^2\psi = \lambda(b^2-\dot{b})\psi$ and therefore $\psi\psi_z^* - \psi^*\psi_z$ = const. Because of the asymptotic behaviour of ψ as $z \to -\infty$, this constant must be zero. Besides we know that a constant γ exists such that

$$\psi = \gamma \exp i\omega z + o(1) \text{ and } \psi_z = i\omega\gamma \exp i\omega z + o(1) \text{ as } z \to \infty. \tag{23}$$

But then γ must be zero, and a solution satisfying (23) with $\gamma = 0$ must also be zero, as we have already seen several times. In the proof of the second part the arguments of Theorem 3 (iii), pg 33, can be repeated almost verbatim. That is, from the contraction principle we obtain that no eigenvalues exist satisfying

$$|\lambda|\int_{-\infty}^{+\infty}|b^2-\dot{b}|dz < \omega;$$

there can also be no eigenvalue satisfying

$$|\lambda|\int_{-\infty}^{+\infty}|b^2-\dot{b}|dz = \omega$$

because a corresponding eigenvector cannot be a solution of the differential equation. □

In the same way we can examine the possibility that ψ has a finite zero z_0. Then ψ is an eigenvector of an operator 0_3 in $C(R)$:

$$\psi(z) = -\frac{\lambda}{\omega} \exp i\omega(z-z_0)\int_{z_0}^{z} \sin \omega(z_1-z_0)(b^2-\dot{b})\psi \, dz_1 - \tag{24}$$

$$-\frac{\lambda}{\omega} \sin \omega(z-z_0)\int_{z}^{\infty} \exp i\omega(z_1-z_0)(b^2-\dot{b})\psi \, dz_1.$$

For this eigenvalue problem the following counterpart of Lemma 6 can be proved.

Lemma 7

The operator 0_3 in (24) has no real eigenvalues for any z_0, and for any z_0 it has no eigenvalues satisfying

$$|\lambda|\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz \le \omega. \tag{25}$$

From the foregoing we immediately obtain the following corollary.

Corollary

An open set U in the complex plane exists containing the real axis and containing the set

$$|\lambda|\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz \le \omega \, ,$$

such that

$$\omega\{\psi^*(z,\lambda^*)\psi(z,\lambda)\}^{-1}$$

is a C(R)-valued analytic function on U.

It is not difficult to prove that in this way we have found a solution of (7) for $\lambda \in U$. This can be seen in the following way. Both $\psi_1 = \psi(z,\lambda)$ and $\psi_2 = \psi^*(z,\lambda^*)$ are solutions of

$$\psi_{zz} + \omega^2\psi = \lambda(b^2-\dot{b})\psi.$$

The Wronskian of any two solutions of this equation is constant. Hence

$$\psi_1\psi_{2z} - \psi_2\psi_{1z} = \text{const.}$$

This constant must be $-2i\omega$ because of the asymptotic behaviour of ψ_1 and ψ_2 as $z \to \infty$. If we use this, then substitution of

$$G(z,\lambda) = \omega\{\psi^*(z,\lambda^*)\psi(z,\lambda)\}^{-1} \tag{26}$$

immediately leads to the result that $G(z,\lambda)$ defined in this way is a solution of (7) for $\lambda \in U$. Because of the corollary this $G(z,\lambda)$ has a Taylor expansion in powers of λ convergent in $C(R)$ if

$$|\lambda|\int_{-\infty}^{+\infty}|b^2-\dot{b}|dz \leq \omega. \tag{27}$$

This Taylor expansion must be the expansion (15), of which we studied the convergence; up to now we even were not sure that an expansion in $C(R)$ was possible. In particular the expansion (15) definitely converges for $\lambda = 1$ if

$$\int_{-\infty}^{+\infty}|b^2-\dot{b}|dz \leq \omega. \tag{28}$$

And then substitution in the representation (13) leads to the solution with normalization of the transmitted wave. As a matter of fact the terms of the expansion can also be found from the known Taylor expansion of ψ. If we denote the expansions by

$$G(z,\lambda) = \sum_{n=0}^{\infty}\lambda^n G^{(n)}(z) \text{ and } \psi(z,\lambda) = \sum_{n=0}^{\infty}\lambda^n\psi^{(n)}(z),$$

then we have by (26)

$$\{\sum_{n=0}^{\infty}\lambda^n G^{(n)}(z)\}\{\sum_{n=0}^{\infty}\lambda^n\psi^{(n)}(z)\}\{\sum_{n=0}^{\infty}\lambda^n\psi^{(n)*}(z)\} = \omega.$$

This implies

$$G^{(0)}(z) = \omega\{\psi^{(0)*}(z)\psi^{(0)}(z)\}^{-1} = \omega,$$

$$G^{(1)}(z) = -G^{(0)}\{\psi^{(0)}(z)\psi^{(1)*}(z) + \psi^{(1)}(z)\psi^{(0)*}(z)\} =$$

$$= \int_z^\infty \sin\omega(z-z_1)(b^2-\dot{b})dz_1,$$

etc.

Up to now we proved that the proposed expansion of G converges if the condition (27) is satisfied. But we are not sure that $G-\omega$ can be integrated term by term such that we obtain an expansion of the argument

$$\omega z - \int_z^\infty (G-\omega)dz_1 \tag{29}$$

of the solution with normalization of the transmitted wave. This can be shown to be possible in the following way. We know that an open, simply connected set V in the complex plane exists containing the real axis and also containing the set

$$|\lambda|\int_{-\infty}^{+\infty}|b^2-\dot{b}|dz \le \omega$$

such that $\psi(z,\lambda)^{-1}$ and $\{\psi^*(z,\lambda^*)\psi(z,\lambda)\}^{-1}$ are C(R)-valued analytic functions on V. (The set U of the corollary need not be simply connected.) Then

$$\begin{aligned}&\frac{i}{2}\int_0^\lambda \{\psi^*(z,\lambda_1^*)\psi(z,\lambda_1)\}^{-1}\frac{d}{d\lambda_1}\{\psi^*(z,\lambda_1^*)\psi(z,\lambda_1)\}d\lambda_1 -\\ &-i\int_0^\lambda \psi(z,\lambda_1)^{-1}\frac{d}{d\lambda_1}\psi(z,\lambda_1)d\lambda_1\end{aligned} \tag{30}$$

is also a C(R)-valued analytic function on V; the path of integration is arbitrary, contained in V. Now, from the connection between the integrals in (30) and logarithms one may infer that for real λ ωz plus the expression (30) is an argument of $\psi(z,\lambda)$ continuous with respect to both λ and z. We know that for real λ this continuous argument is

$$\omega z - \int_z^\infty (G-\omega)dz_1,$$

apart from a multiple of 2π. Because for $\lambda = 0$ we have $G = \omega$, (30) must reduce to

$$-\int_{z}^{\infty} (G-\omega)dz_1$$

for real λ. But because (30) is a $C(R)$-valued analytic function on V, as is G, they must be identical for all $\lambda \in V$, and the Taylor expansion of $G-\omega$ can be integrated term by term such that the resultant series certainly converges in $C(R)$ if

$$|\lambda|\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz \leq \omega,$$

and for $\lambda = 1$ if

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz \leq \omega.$$

Once we know how the expansion (15) is related to the series solution (8), another iterative procedure suggests itself. One might expect that an expansion of the solutions of (6) exists that is related to the solution

$$(\psi_{\downarrow},\psi_{\uparrow}) = \sum_{n=0}^{\infty} \lambda^n T_2{}^n \chi_0 ,$$

which also has a normalization of the transmitted wave. To that end we introduce a complex parameter λ in (6) according to

$$G^2 = \omega^2 + \frac{3}{4}\, G^{-2}\left(\frac{dG}{dz}\right)^2 - \frac{1}{2}\, G^{-1}\, \frac{d^2G}{dz^2} - (\lambda^2 b^2 - \lambda\dot{b}), \tag{31}$$

and we try to find a solution of (31) in powers of λ. We will not pursue this any further; we only remark that it can be done.

APPENDIX

Diagonalization

Motion of a spin in a time-dependent magnetic field

There exists another natural way to obtain Bremmer's equations (1.1.23), different from the method described in the introduction to this chapter. The method to be described even leads to other series solutions of the equation

$$\phi_{xx} + k^2(x)\phi = 0, \quad k^2(x) = \omega^2 c^{-2}(x). \tag{1}$$

The method is also suitable to obtain asymptotic solutions of (1) as $\omega \to \infty$, as has been done by Van Kampen [19]. It has also been described by Keller and Keller [22].

We define

$$u_1(x) = \phi(x), \quad u_2(x) = \phi_x(x)$$

and rewrite the equation (1) as a first-order system:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ -\omega^2 c^{-2}(x) & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{2}$$

When new variables are introduced according to $u = Pw$, where P is a nonsingular, differentiable matrix, the new variables obey

$$w_x = (P^{-1}AP - P^{-1}P')w. \tag{3}$$

If now it were possible to choose the matrix P in (3) such that

$$P^{-1}AP - P^{-1}P'$$

would be a diagonal matrix, the system (3) could be solved directly by exponentials. However, no general method exists that enables us to find such a matrix. What can be done is the following. The matrix P is chosen in such a way that $P^{-1}AP$ is diagonal. If A is "slowly varying", then P can possibly be chosen such that the coupling caused by $P^{-1}P'$ is weak. A matrix that meets the requirement of diagonalizing A is

$$P = \begin{pmatrix} 1 & 1 \\ -ik & +ik \end{pmatrix}. \tag{4}$$

The matrix P is determined to within an arbitrary diagonal matrix D(x) to the right; however, the matrix (4) is the only one that establishes an "amplitude splitting":

$$\phi = u_1 = w_1 + w_2.$$

(Of course, this is only true apart from a trivial interchange of the variables.) With P chosen according to (4), the set of differential equations (3) is exactly the set Bremmer found:

$$\begin{aligned} w_{1x} &= \tfrac{1}{2}\, c^{-1}c'w_1 - i\omega c^{-1}w_1 - \tfrac{1}{2}\, c^{-1}c'w_2, \\ w_{2x} &= \tfrac{1}{2}\, c^{-1}c'w_2 + i\omega c^{-1}w_2 - \tfrac{1}{2}\, c^{-1}c'w_1. \end{aligned} \tag{5}$$

(The amplitude splitting we found in this way need not be the only one; a splitting not necessarily results from a diagonalization. An example is Sluijter's splitting; see Sluijter [31] and [32].)
If we drop the condition that the transformation should bring about an amplitude splitting, we may choose D(x) so that the diagonal part of $P^{-1}P'$ is zero. Then D(x) is apart from a constant diagonal matrix

$$D(x) = \begin{pmatrix} c^{\frac{1}{2}}(x) & 0 \\ 0 & c^{\frac{1}{2}}(x) \end{pmatrix},$$

which leads to a transformation of the variables as described by the Liouville transformation. If certain conditions are fulfilled, we can diagonalize once more in order to take into account the coupling to "higher order". The resulting equations may serve then as a base for an iterative procedure as we developed for Bremmer's equations (5). If this procedure is repeated and if the resulting series are curtailed after the first term, the resulting sequence is a sequence asymptotic to a solution of (1) as $\omega \to \infty$. But rather then pursue this method any further, we apply the method of diagonalization to another problem, viz the problem of the motion of a spin in a time-dependent magnetic field.

For the sake of simplicity we shall restrict the discussion to spin $\frac{1}{2}$ particles; however this is not essential. The Schrödinger equation for a spin $\frac{1}{2}$ particle in a field $\vec{B} = (B_x, B_y, B_z)$ reads (see, for example, Merzbacher [27], pg 285)

$$i\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \frac{\gamma}{2}\begin{pmatrix} -B_z & iB_y - B_x \\ -iB_y - B_x & B_z \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{\gamma}{2} B\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{6}$$

This system of equations with initial conditions $u_1(t=0) = u_{10}$ and $u_2(t=0) = u_{20}$ can be solved, of course, directly by an iterative procedure in which the offdiagonal terms of B are treated as a perturbation in the same way as we did for Bremmer's equations:

$$\begin{aligned} u_1 &= u_{10}\exp(i\,\frac{\gamma}{2}\int_0^t B_z dt_1) + u_{20}\,\frac{\gamma}{2}\int_0^t \exp(i\,\frac{\gamma}{2}\int_{t_1}^t B_z dt_2)(iB_x + B_y)dt_1 + \ldots \\ u_2 &= u_{20}\exp(-i\,\frac{\gamma}{2}\int_0^t B_z dt_1) + u_{10}\,\frac{\gamma}{2}\int_0^t \exp(-i\,\frac{\gamma}{2}\int_{t_1}^t B_z dt_2)(iB_x - B_y)dt_1 + \ldots \end{aligned} \tag{7}$$

A majorant of both series in (7) is

$$\sum_{n=0}^{\infty} \frac{1}{n!}\{\frac{\gamma}{2}\int_0^t |iB_y - B_x| dt_1\}^n \max(|u_{10}|, |u_{20}|). \tag{8}$$

Hence they converge certainly for all t, and they converge uniformly for all $t \geq 0$ when

$$\int_0^{\infty} |iB_y - B_x| dt_1 < \infty. \tag{9}$$

Now, the method that we described above may provide series that converge faster than the series (7) and that, in addition, converge uniformly for all $t \geq 0$ although the condition (9) is violated. For this purpose we introduce again new variables according to $u = Pw$. Then w satisfies

$$iw_t = \frac{\gamma}{2} P^{-1}BPw - iP^{-1}\dot{P}w = \frac{\gamma}{2}\tilde{B}w. \tag{10}$$

(Here the dot denotes differentiation with respect to t.)

A matrix P such that $P^{-1}BP$ is diagonal and such that diag $P^{-1}\dot{P} = 0$, is, provided $b_z \neq -1$, a matrix of which the elements are

$$P_{11} = \left(\frac{1+b_z}{2}\right)^{\frac{1}{2}} \exp \frac{i}{2} \int_0^t \frac{-b_x\dot{b}_y + b_y\dot{b}_x}{1+b_z}\, dt_1,$$

$$P_{12} = \left(\frac{1+b_z}{2}\right)^{\frac{1}{2}} \left(\frac{-b_x+ib_y}{1+b_z}\right) \exp -\frac{i}{2} \int_0^t \frac{-b_x\dot{b}_y + b_y\dot{b}_x}{1+b_z}\, dt_1, \tag{11}$$

$$P_{22} = P_{11}^{*}, \text{ and } P_{21} = -P_{12}^{*},$$

where

$$b_x = \frac{B_x}{|\vec{B}|},\quad b_y = \frac{B_y}{|\vec{B}|},\quad b_z = \frac{B_z}{|\vec{B}|}\ .$$

This transformation corresponds to the introduction of a coordinate system with the z-axis along $\vec{B}$. Then the elements of $\tilde{B}$ are

$$\tilde{B}_{12} = -\frac{1}{\gamma}\left\{\frac{ib_x\dot{b}_z + b_y\dot{b}_z}{1+b_z} - i\dot{b}_x - \dot{b}_y\right\}\exp -i \int_0^t \frac{-b_x\dot{b}_y + b_y\dot{b}_x}{1+b_z}\, dt_1,$$

$$\tilde{B}_{11} = -|\vec{B}|,\quad \tilde{B}_{22} = |\vec{B}|,\quad \tilde{B}_{21} = \tilde{B}_{12}^{*}. \tag{12}$$

The set of equations (10) with initial conditions $w_1(t=0) = w_{10}$ and $w_2(t=0) = w_{20}$ is again a starting point for an iterative procedure. Simple calculations now show that the resulting series are majorized by

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left\{\frac{1}{2} \int_0^t |\dot{\vec{b}}|\, dt_1\right\}^n \max(|w_{10}|, |w_{20}|). \tag{13}$$

So they converge uniformly for all t if

$$\int_0^{\infty} |\dot{\vec{b}}|\, dt < \infty . \tag{14}$$

From a comparison of (13) and (8) we see that, in general, the new series will converge faster than the series in (7) when

$$\frac{1}{2} \int_0^t |\dot{\vec{b}}|\, dt_1 \ll \frac{\gamma}{2} \int_0^t |iB_y + B_x|\, dt_1. \tag{15}$$

We note that $\frac{\gamma}{2}\,|iB_y+B_x| = \omega_L \sin\theta$, where ω_L is the Larmor frequency and θ is the angle between the field direction and the positive z-axis; $|\dot{\vec{b}}|$ is the rate of change of the direction of the field. Therefore (15) certainly applies when the rate of change of the field direction is small compared to the Larmor frequency calculated from the transverse part of the field. The process of diagonalization we have described may be repeated as long as the derivatives exist. This possibly leads to better converging series, but it also causes more and more bulky expressions.

CHAPTER II

THE CAUCHY PROBLEM

§2.1 INTRODUCTION

In the first chapter we were concerned with monochromatic solutions. These solutions describe the steady state that is supposed to result from the reflection of a monochromatic wave incident on an inhomogeneous medium. We found that in the case of the solutions with normalization of the incident wave the conditions that guarantee the desired asymptotic behaviour do not assure the convergence of their series representations.

In this chapter we shall study the Cauchy problem for the wave equation

$$\Phi_{xx} - c^{-2}(x)\Phi_{tt} = 0. \tag{1}$$

It will turn out that the concepts that underly the first chapter also provide series solutions of the Cauchy problem. For the convergence of these series less severe restrictions are required.

In this introduction we shall sketch roughly these concepts and work them out in more detail later.

First we describe the counterpart of Liouville's method. In order to "flatten" (1) we apply the same transformation we used for the Helmholtz equation:

$$\Psi = c^{-\frac{1}{2}}\Phi, \tag{2}$$

$$z = \int_{x_0}^{x} c^{-1}(x_1)dx_1. \tag{3}$$

By this transformation the wave equation (1) is transformed into

$$\Psi_{zz} - \Psi_{tt} = (b^2-\dot{b})\Psi. \tag{4}$$

As we did for the monochromatic solutions, we can treat the right-hand side of (4) as a perturbation to be taken into account by successive approximation. The first approximation then takes the form

$$\Psi_1(z+t) + \Psi_2(z-t),$$

and inverse Liouville transformation leads to

$$c^{\frac{1}{2}}(x)\Psi_1\left(\int_{x_0}^{x} c^{-1}(x_1)dx_1 + t\right) + c^{\frac{1}{2}}(x)\Psi_2\left(\int_{x_0}^{x} c^{-1}(x_1)dx_1 - t\right), \tag{5}$$

the counterpart in the nonmonochromatic case of the WKB approximations of the first chapter.

There also exists an equivalent of Bremmer's method that leads to a series solution of the initial-value problem. This solution is akin to the nonmonochromatic solution that Berk, Book, and Pfirsch [5] found. In order to find this series, one has to approximate the profile c(x) by a number of homogeneous layers; next the contributions to the solution that have undergone a fixed number of reflections are summed up, and ultimately one passes to the limit of the thickness of the layers tending to zero. In this way we find the counterpart of (1.1.23):

$$\begin{aligned} \Phi_{\downarrow x} - \tfrac{1}{2}\, c^{-1}c'\Phi_{\downarrow} - c^{-1}\Phi_{\downarrow t} &= -\tfrac{1}{2}\, c^{-1}c'\Phi_{\uparrow}, \\ \Phi_{\uparrow x} - \tfrac{1}{2}\, c^{-1}c'\Phi_{\uparrow} + c^{-1}\Phi_{\uparrow t} &= -\tfrac{1}{2}\, c^{-1}c'\Phi_{\downarrow}. \end{aligned} \tag{6}$$

Here $\Phi_{\downarrow}$ stands for the total contribution to the solution of the partial waves that have undergone an odd number of reflections; $\Phi_{\uparrow}$ consists of the waves that have undergone an even number. By differentiation it can be seen that $\Phi_{\downarrow}+\Phi_{\uparrow}$ is a solution of (1) if $\Phi_{\downarrow}$ and $\Phi_{\uparrow}$ are a solution of (6). The series solution that can be found using Bremmer's concept can be seen to originate from an approach of the equations (6) by successive approximation in which the right-hand side of (6) is treated as a perturbation. The first approximation of $\Phi_{\downarrow}$ has the form

$$\Phi_{\downarrow} = c^{\frac{1}{2}}(x)\Psi_1\left(\int_{x_0}^{x} c^{-1}(x_1)dx_1 + t\right),$$

that of $\Phi_{\uparrow}$

$$\Phi_{\uparrow} = c^{\frac{1}{2}}(x)\Psi_2\left(\int_{x_0}^{x} c^{-1}(x_1)dx_1 - t\right).$$

In this manner we find the two WKB approximations again. By the Liouville transformation (2), (3) the equations (6) can be simplified. So we introduce

$$\Psi_{\downarrow} = c^{-\frac{1}{2}}\Phi_{\downarrow},\ \Psi_{\uparrow} = c^{-\frac{1}{2}}\Phi_{\uparrow},\ z = \int_{x_0}^{x} c^{-1}(x_1)dx_1.$$

This leads to the equations

$$\begin{aligned} \Psi_{\downarrow z} - \Psi_{\downarrow t} &= -b\Psi_{\uparrow}, \\ \Psi_{\uparrow z} + \Psi_{\uparrow t} &= -b\Psi_{\downarrow}, \end{aligned} \tag{7}$$

the equivalent of (1.1.24).

In the appendix attached at the end of the first chapter, pg 62, we described how one can attain Bremmer's equations by diagonalization. Using a similar procedure we can also obtain the equations (6). If we introduce a new variable $\overline{\Phi}$, the second-order partial differential equation (1) can be written in the form of a first-order system:

$$\begin{pmatrix}\Phi \\ \overline{\Phi}\end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ c^{-2}(x) & 0\end{pmatrix}\begin{pmatrix}\Phi \\ \overline{\Phi}\end{pmatrix}_t = A\begin{pmatrix}\Phi \\ \overline{\Phi}\end{pmatrix}_t. \tag{8}$$

Indeed, eliminating $\overline{\Phi}$ we see that Φ satisfies (1). (For those examples as given in the general introduction where the equations were originally presented in the form of a first-order system, the interpretation of $\overline{\Phi}$ is clear.) Just like we did in the appendix of Chapter 1,we introduce new variables according to

$$\begin{pmatrix}\Phi \\ \overline{\Phi}\end{pmatrix} = P(x)\begin{pmatrix}\Phi_{\downarrow} \\ \Phi_{\uparrow}\end{pmatrix},$$

where P is a nonsingular, differentiable matrix independent of t.

Then the new variables obey

$$\begin{pmatrix}\Phi_{\downarrow}\\ \Phi_{\uparrow}\end{pmatrix}_x = P^{-1}AP\begin{pmatrix}\Phi_{\downarrow}\\ \Phi_{\uparrow}\end{pmatrix}_t - P^{-1}P'\begin{pmatrix}\Phi_{\downarrow}\\ \Phi_{\uparrow}\end{pmatrix}. \tag{9}$$

Again we choose P so that $P^{-1}AP$ is diagonal, and we demand that P should bring about an amplitude splitting $\Phi = \Phi_{\downarrow}+\Phi_{\uparrow}$. Then P is unique again:

$$P = \begin{pmatrix} 1 & 1\\ c^{-1}(x) & -c^{-1}(x)\end{pmatrix}.$$

Substituting this P in (9) we obtain precisely the equations (6).

§2.2 SERIES SOLUTIONS OF THE CAUCHY PROBLEM

In this section we shall, using the methods indicated in the introduction, construct series solutions of the Cauchy problem

$$\Phi_{xx} - c^{-2}(x)\Phi_{tt} = 0, \tag{1}$$

$$\Phi(x,0) = \Phi^{o}(x), \tag{2}$$

$$\Phi_t(x,0) = \Phi^{o}_t(x), \tag{3}$$

where $-\infty < x < +\infty$ and $t \geq 0$.

We shall restrict ourselves to classical solutions, i.e., to solutions $\Phi \in C^2$. Here C^m denotes the set of complex-valued functions that are m times continuously differentiable with respect to both x and t for $-\infty < x < +\infty$ and $t \geq 0$ (including all mixed derivatives up to the m th order). C is the set of functions continuous for $-\infty < x < +\infty$ and $t \geq 0$. We assume that Φ^o is twice continuously differentiable and Φ^o_t once. In this section we shall not yet be concerned with series solutions uniformly convergent for all x and all $t \geq 0$. Therefore c(x) is subjected to less restrictive conditions than in the first chapter (cf pag 14):

c(x) is twice continuously differentiable;

$c(x) > 0$;

$c(x)$ satisfies $\displaystyle\int_{x_0}^{\infty} c^{-1}(x_1)dx_1 = \infty$

and $\displaystyle\int_{-\infty}^{x_0} c^{-1}(x_1)dx_1 = \infty.$

These conditions admit the Liouville transformation

$$\Psi = c^{-\frac{1}{2}}\Phi, \tag{4}$$

$$z = \int_{x_0}^{x} c^{-1}(x_1)dx_1. \tag{5}$$

Then the Cauchy problem (1), (2), (3) is transformed into

$$\Psi_{zz} - \Psi_{tt} = (b^2-\dot{b})\Psi, \tag{6}$$

$$\Psi(z,0) = \Psi^0(z), \tag{7}$$

$$\Psi_t(z,0) = \Psi_t^0(z), \tag{8}$$

where $-\infty < z < +\infty$ and $t \geq 0$.

The new initial values can be expressed readily in terms of the old ones. This initial-value problem is equivalent to an initial-value problem for the system

$$\begin{aligned} \Psi_{\downarrow z} - \Psi_{\downarrow t} &= -b\Psi_{\uparrow}, \\ \Psi_{\uparrow z} + \Psi_{\uparrow t} &= -b\Psi_{\downarrow}. \end{aligned} \tag{9}$$

The initial values that have to be imposed are

$$\Psi_{\downarrow}(z,0) = \Psi_{\downarrow}^0(z) = \frac{1}{2}\Psi^0(z) + \frac{1}{2}c^{\frac{1}{2}}(z)\int_0^z \Psi_t^0(z_1)c^{-\frac{1}{2}}(z_1)dz_1 + ac^{\frac{1}{2}}(z). \tag{10}$$

$$\Psi_{\uparrow}(z,0) = \Psi_{\uparrow}^0(z) = \frac{1}{2}\Psi^0(z) - \frac{1}{2}c^{\frac{1}{2}}(z)\int_0^z \Psi_t^0(z_1)c^{-\frac{1}{2}}(z_1)dz_1 - ac^{\frac{1}{2}}(z). \tag{11}$$

Here a may be assigned arbitrarily. By inspection the following assertions can be verified immediately:

The sum of $\Psi_\downarrow$ and $\Psi_\uparrow$ is a solution of (6) if $\Psi_\downarrow$ and $\Psi_\uparrow$ constitute a twice continuously differentiable solution of (9). If $\Psi_\downarrow$ and $\Psi_\uparrow$ also satisfy the initial conditions (10) and (11), then $\Psi = \Psi_\downarrow + \Psi_\uparrow$ satisfies (7) and (8).
Conversely, if Ψ is a solution of the Cauchy problem (6), (7), (8), then twice continuously differentiable $\Psi_\downarrow$ and $\Psi_\uparrow$ exist such that $\Psi = \Psi_\downarrow + \Psi_\uparrow$, and such that $\psi_\downarrow$ and $\Psi_\uparrow$ are a solution of (9). These $\Psi_\downarrow$ and $\Psi_\uparrow$ are unique apart from an arbitrary constant a:

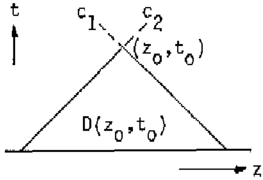
$$\begin{aligned}\Psi_\downarrow(z,t) &= \frac{1}{2}\Psi(z,t) + \frac{1}{2}\int_0^t \{b(z)\Psi(z,t_1) + \Psi_z(z,t_1)\}dt_1 + \\ &\quad + \frac{1}{2}c^{\frac{1}{2}}(z)\int_0^z \Psi_t^0(z_1)c^{-\frac{1}{2}}(z_1)dz_1 + ac^{\frac{1}{2}}(z),\\ \Psi_\uparrow(z,t) &= \frac{1}{2}\Psi(z,t) - \frac{1}{2}\int_0^t \{b(z)\Psi(z,t_1) + \Psi_z(z,t_1)\}dt_1 - \\ &\quad - \frac{1}{2}c^{\frac{1}{2}}(z)\int_0^z \Psi_t^0(z_1)c^{-\frac{1}{2}}(z_1)dz_1 - ac^{\frac{1}{2}}(z).\end{aligned} \tag{12}$$

They satisfy (10) and (11).

We transform the Cauchy problems for (6) and for (9) into integral equations in an appropriate function space. Thereupon we shall prove that these integral equations are uniquely soluble by successive approximation. This method is exactly the method used in many textbooks to prove the unique solvability of such Cauchy problems. See, for example, Courant and Hilbert [13] or Hellwig [18].

We define $D(z_0,t_0)$ to be the closed triangle in the (z,t) plane bounded by the z-axis and by the characteristics through (z_0,t_0):

$$c_1 = \{(z,t)\,|\,z+t = z_0+t_0\},$$
$$c_2 = \{(z,t)\,|\,z-t = z_0-t_0\}.$$



The set $D(z_0,t_0)$ is the domain of dependence belonging to the point (z_0,t_0) for the Cauchy problem (6), (7), (8). If the equation (6) is integrated over the triangle $D(z,t)$ and if we use the initial conditions, we obtain the equivalent integral equation

$$\Psi \in C,$$
$$\Psi = \Lambda(z,t) - \frac{1}{2} \iint_{D(z,t)} (b^2-\dot{b})\Psi(z_1,t_1)dz_1dt_1. \qquad (13)$$

Here $\Lambda(z,t)$ is defined by

$$\Lambda(z,t) = \frac{1}{2}\Psi^o(z-t) + \frac{1}{2}\Psi^o(z+t) + \frac{1}{2}\int_{z-t}^{z+t} \Psi_t^o(z_1)dz_1. \qquad (14)$$

It is easily proved that if Φ^o and Φ_t^o satisfy the differentiability conditions that we imposed, any solution $\in C$ of (13) is a twice continuously differentiable solution of (6) that satisfies the initial conditions (7) and (8).

In a similar way the system (9) with initial conditions (10) and (11) can be transformed into a set of integral equations. The first equation of (9) is integrated along the characteristic curve c_1 and the second along c_2. If we use the initial conditions (10) and (11), we obtain the equivalent set of integral equations

$$(\Psi_\downarrow,\Psi_\uparrow) \in C\times C,$$
$$\left.\begin{aligned} \Psi_\downarrow(z,t) &= \Psi_\downarrow^o(z+t) + \int_0^t b(-s+z+t)\Psi_\uparrow(-s+z+t,s)ds, \\ \Psi_\uparrow(z,t) &= \Psi_\uparrow^o(z-t) - \int_0^t b(s+z-t)\Psi_\downarrow(s+z-t,s)ds. \end{aligned}\right. \qquad (15)$$

Any solution $(\Psi_\downarrow,\Psi_\uparrow) \in C\times C$ of (15) is a twice continuously differentiable solution of (9) that automatically satisfies the initial conditions (10) and (11).

Now we shall prove that the integral equations (13) and (15) are uniquely soluble in their respective spaces and that they can be solved by the conventional Neumann iteration. For the integral equations (13) and (15) we write formally

$$X = X_0 + S_i X, \quad i = 1,2, \tag{16}$$

where S_1 and S_2 stand for the integral operators occurring in the right-hand side of (13) and (15), respectively. We study again the extended problem of the solvability of the equations

$$X = Z + \lambda S_i X, \quad i = 1,2, \tag{17}$$

for any given complex λ and any given Z in the space in question. First we consider the equations (17) in an arbitrary triangle $D(z_0,t_0)$. We denote by $C(z_0,t_0)$ the Banach space of all complex-valued continuous functions on $D(z_0,t_0)$ with the maximum-modulus norm; in $C(z_0,t_0) \times C(z_0,t_0)$ we define the norm of $X = (X_\downarrow,X_\uparrow)$ by $\|X\| = \max(\|X_\downarrow\|, \|X_\uparrow\|)$. Further b_D is the maximal value of $|b(z)|$ for $(z,t) \in D(z_0,t_0)$, and g_D is the maximal value of $|b^2-\dot{b}|$ for $(z,t) \in D(z_0,t_0)$. Then S_1 is a bounded operator on $C(z_0,t_0)$, and S_2 is a bounded operator on $C(z_0,t_0) \times C(z_0,t_0)$; of course the bounds depend on z_0 and t_0. Moreover S_1 and S_2^2 are compact as a consequence of the Arzelà-Ascoli Theorem. (See Dunford and Schwartz [14], pg 266.) S_2 is not compact. We do not need the compactness; the boundedness suffices our needs. From the spectral theory of bounded operators on a Banach space it is known that the equation

$$X = Z + \lambda S_i X$$

is uniquely soluble for any Z if λ does not belong to the spectrum of S_i. As contrasted with the spectrum of a compact operator, the spectrum of a bounded operator does, in general, not consist of eigenvalues only. There is a continuous and residual spectrum. The infimum of all absolute values of the points in the spectrum is called the spectral radius. Just like in the case of a compact operator (see pg 26) the spectral radius equals

$$\{\lim_{n\to\infty} \|S_i{}^n\|^{1/n}\}^{-1}. \tag{18}$$

If λ does not belong to the spectrum of S_i, the solution of (17) is an analytic function of λ. The solution has a Taylor expansion

$$\sum_{n=0}^{\infty} \lambda^n S_i{}^n Z, \tag{19}$$

that certainly converges absolutely if $|\lambda|$ is less than the spectral radius. Because we are interested mainly in the equations (16), we want to know whether the spectral radius exceeds the value 1. If it does, (19) represents the unique solution of the Cauchy problems if λ is chosen to be 1 and $Z = X_0$.

Theorem 8 (i)

The integral equation $X = Z + \lambda S_1 X$ is uniquely soluble in $C(z_0,t_0)$ for any λ and for any $Z \in C(z_0,t_0)$. The solution can be represented by

$$\sum_{n=0}^{\infty} \lambda^n S_1{}^n Z.$$

For $\lambda = 1$ and $Z = X_0$ we obtain the unique solution of (13) in $C(z_0,t_0)$:

$$\sum_{n=0}^{\infty} S_1{}^n X_0. \tag{20}$$

Proof

It is sufficient to prove

$$\lim_{n\to\infty} \|S_1{}^n\|^{1/n} = 0. \tag{21}$$

By induction we shall show that for all $n \geq 0$ and for all $(z,t) \in D(z_0,t_0)$

$$|S_1{}^n Z(z,t)| \le \frac{1}{(2n)!} g_D{}^n t^{2n} \|Z\|. \tag{22}$$

From this (21) is readily obtained.

For n = 0 (22) is self-evident. Suppose (22) holds for n = m.

Because

$$S_1{}^{m+1} Z(z,t) = -\frac{1}{2}\int_0^t dt_1 \int_{z-t+t_1}^{z+t-t_1} dz_1 (b^2-\dot{b})(S_1{}^m Z)(z_1,t_1),$$

we have

$$|S_1{}^{m+1} Z(z,t)| \le \frac{1}{2}\int_0^t dt_1 \int_{z-t+t_1}^{z+t-t_1} dz_1 |b^2-\dot{b}|\,|S_1{}^m Z(z_1,t_1)| \le$$

$$\le \frac{1}{2}\int_0^t dt_1 \int_{z-t+t_1}^{z+t-t_1} dz_1 g_D \frac{1}{(2m)!} g_D{}^m t_1{}^{2m} \|Z\| = \frac{1}{(2m+2)!} g_D{}^{m+1} t^{2m+2} \|Z\|.$$

□

Theorem 8 (ii)

The integral equation $X = Z + \lambda S_2 X$ is uniquely soluble in $C(z_0,t_0) \times C(z_0,t_0)$ for any λ and for any $Z \in C(z_0,t_0) \times C(z_0,t_0)$. The solution can be represented by

$$\sum_{n=0}^{\infty} \lambda^n S_2{}^n Z.$$

For $\lambda = 1$ and $Z = X_0$ we obtain the unique solution of (15) in $C(z_0,t_0) \times C(z_0,t_0)$:

$$\sum_{n=0}^{\infty} S_2{}^n X_0. \tag{23}$$

Proof

It is sufficient to prove

$$\lim_{n\to\infty} \|S_2{}^n\|^{1/n} = 0. \tag{24}$$

By induction we shall show that for all $n \ge 0$ and for all $(z,t) \in D(z_0,t_0)$

$$|(S_2^{\,n}z)_{+,+}(z,t)| \le \frac{1}{n!}\, b_D^{\,n} t^n \, \|z\| \, . \tag{25}$$

From this (24) follows.

For n = 0 (25) is self-evident. Suppose (25) holds for n = m. Then

$$(S_2^{\,m+1}z)_+(z,t) = \int_0^t b(-s+z+t)(S_2^{\,m}z)_+(-s+z+t,s)ds \, ,$$

and hence

$$|(S_2^{\,m+1}z)_+(z,t)| \le \int_0^t |b(-s+z+t)|\,|(S_2^{\,m}z)_+(-s+z+t,s)|ds \le$$

$$\le \frac{1}{m!}\, b_D^{\,m+1}\, \|z\| \int_0^t s^m ds = \frac{1}{(m+1)!}\, b_D^{\,m+1} t^{m+1}\, \|z\| \, .$$

In the same way it can be proved that

$$|(S_2^{\,m+1}z)_+(z,t)| \le \frac{1}{(m+1)!}\, b_D^{\,m+1} t^{m+1}\, \|z\| \, .$$

□

In order to obtain "global" results, it suffices to notice that Theorem 8 is valid for any triangle $D(z_0,t_0)$. Therefore the integral equations (13) and (15) are uniquely soluble in their respective spaces. They can be solved by their Neumann expansions. In the case of (13) the Neumann expansion converges absolutely for all z and t; it converges uniformly in any finite domain. For (15) this is true for both series into which the Neumann expansion splits.

Before we proceed with a more close examination of the above solutions, we notice that both series solutions clearly represent the same solution of the Cauchy problem. However, their first terms do not correspond. While the first term of (20) is

$$\frac{1}{2}\,\Psi^o(z+t) + \frac{1}{2}\,\Psi^o(z-t) + \frac{1}{2}\int_{z-t}^{z+t} \Psi_t^o(z_1)dz_1,$$

the first term of (23) is

$$\frac{1}{2}\,\Psi^{0}(z+t) + \frac{1}{2}\,\Psi^{0}(z-t) + \frac{1}{2}\,c^{\frac{1}{2}}(z+t)\int_{0}^{z+t}\Psi_t^{0}(z_1)c^{-\frac{1}{2}}(z_1)dz_1 + ac^{\frac{1}{2}}(z+t) -$$

$$- \frac{1}{2}\,c^{\frac{1}{2}}(z-t)\int_{0}^{z-t}\Psi_t^{0}(z_1)c^{-\frac{1}{2}}(z_1)dz_1 - ac^{\frac{1}{2}}(z-t).$$

The relations between the other terms are similar to the relations between the terms of the series solutions in the monochromatic case. We examined these in Theorem 7, pg 47.

§2.3 UNIFORM CONVERGENCE OF THE SERIES SOLUTIONS OF THE CAUCHY PROBLEM

In the preceding section we proved that the series (2.2.20) and both series into which (2.2.23) splits converge for all z and t. In this section we shall examine whether we can find conditions that guarantee the uniform convergence of these series for all z and for all $t \geq 0$. This will prove possible for the series (2.2.23); we shall not succeed in this objective with respect to (2.2.20). First we shall treat the case of the series (2.2.23).

Suppose that both $\Psi_{\downarrow}^{0}$ and $\Psi_{\uparrow}^{0}$ are uniformly bounded. Then both series in (2.2.23) are uniformly convergent for all z and all $t \geq 0$ if

$$\int_{-\infty}^{+\infty}|b|dz < 1. \tag{1}$$

This is readily verified because the series

$$\Psi_{\downarrow}(z,t) = \Psi_{\downarrow}^{0}(z+t) + \int_{0}^{t} b(-s+z+t)\Psi_{\uparrow}^{0}(-2s+z+t)ds + \ldots$$

$$\Psi_{\uparrow}(z,t) = \Psi_{\uparrow}^{0}(z-t) - \int_{0}^{t} b(s+z-t)\Psi_{\downarrow}^{0}(2s+z-t)ds + \ldots$$

are both majorized by

$$\sum_{n=0}^{\infty}\{\int_{-\infty}^{+\infty}|b|dz\}^{n}M.$$

However, by means of a more sophisticated reasoning the criterion (1) can be replaced by a less restrictive one:

$$\int_{-\infty}^{+\infty} |b|dz < \frac{\pi}{2} . \tag{2}$$

This is what we shall do now.

It is rather obvious that the appropriate space to study the problem of the uniform convergence is the Banach space $C_B \times C_B$. C_B is the set of complex-valued, bounded, and continuous functions defined on the half plane $t \geq 0$. The norm in C_B is given by

$$\|\Psi\| = \sup_{z,t} |\Psi(z,t)|,$$

the norm of $(\Psi_{\downarrow}, \Psi_{\uparrow}) \in C_B \times C_B$ by

$$\max(\|\Psi_{\downarrow}\| , \|\Psi_{\uparrow}\|).$$

We suppose that

$$\int_{-\infty}^{+\infty} |b|dz < \infty. \tag{3}$$

Then the operator S_2 defined on pg 76 is a bounded operator on $C_B \times C_B$; S_2 is not compact, cf pg 76. We consider the integral equation

$$X = Z + \lambda S_2 X \tag{4}$$

in this space. In order to study the convergence of the series

$$\sum_{n=0}^{\infty} \lambda^n S_2{}^n Z \tag{5}$$

in $C_B \times C_B$, we have to examine the spectrum of S_2 as an operator on $C_B \times C_B$. (See pg 76 and 77.) For $|\lambda|$ less than the spectral radius of S_2 the series (5) converges in $C_B \times C_B$. As we have already seen several times, the spectral radius equals

$$\{\lim_{n\to\infty} \|S_2{}^n\|^{1/n}\}^{-1}. \tag{6}$$

To obtain an estimate of the spectral radius of S_2, it is convenient to introduce an operator O_4 in $C(R)\times C(R)$. (See pg 23 for the definition of $C(R)\times C(R)$.) The operator O_4 is for all $\chi = (\chi_\downarrow, \chi_\uparrow) \in C(R)\times C(R)$ defined by

$$\begin{aligned}(O_4\chi)_\downarrow(z) &= \int_z^\infty |b|\chi_\uparrow dz_1,\\ (O_4\chi)_\uparrow(z) &= \int_{-\infty}^z |b|\chi_\downarrow dz_1.\end{aligned} \tag{7}$$

It is not difficult to prove that O_4 is a compact operator. (Cf Lemma 2, pg 25.) The spectral radius of O_4 is

$$\{\lim_{n\to\infty} \|O_4{}^n\|^{1/n}\}^{-1} \tag{8}$$

and can be determined explicitly because the eigenvectors and the corresponding eigenvalues can be calculated.

<u>Lemma 8</u>

The spectral radius of O_4 is

$$\frac{\pi}{2}\{\int_{-\infty}^{+\infty} |b|dz\}^{-1}. \tag{9}$$

<u>Proof</u>

Suppose $\chi = (\chi_\downarrow, \chi_\uparrow)$ is an eigenvector of O_4 corresponding to the eigenvalue λ:

$$\begin{aligned}\chi_\downarrow(z) &= \lambda\int_z^\infty |b|\chi_\uparrow dz_1,\\ \chi_\uparrow(z) &= \lambda\int_{-\infty}^z |b|\chi_\downarrow dz_1.\end{aligned} \tag{10}$$

By differentiation we find

$$\begin{aligned}\chi_{\downarrow z} &= -\lambda|b|\chi_\uparrow,\\ \chi_{\uparrow z} &= \lambda|b|\chi_\downarrow.\end{aligned}$$

This set of differential equations can be solved explicitly:

$$\chi_{\downarrow}(z) = \alpha_1 \cos \lambda \int_z^{\infty} |b| dz_1 + \alpha_2 \sin \lambda \int_z^{\infty} |b| dz_1,$$

$$\chi_{\uparrow}(z) = -\alpha_1 \sin \lambda \int_z^{\infty} |b| dz_1 + \alpha_2 \cos \lambda \int_z^{\infty} |b| dz_1.$$

The solution has to satisfy boundary conditions that follow from (10):

$$\chi_{\downarrow} = o(1) \text{ as } z \to \infty,$$

$$\chi_{\uparrow} = o(1) \text{ as } z \to -\infty.$$

Then we find $\alpha_1 = 0$, and the eigenvalues are determined by

$$\lambda \int_{-\infty}^{+\infty} |b| dz = \frac{\pi}{2} + k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

Therefore the spectral radius of 0_4 is equal to (9). □

In principle it should also be possible to determine the spectral radius of 0_4 from (8). From the definition of 0_4 we infer that for $n \geq 1$

$$\|0_4{}^n\| = \max\{\int_{-\infty}^{+\infty} |b(z_n)| dz_n \int_{-\infty}^{z_n} |b(z_{n-1})| dz_{n-1} \int_{z_{n-1}}^{\infty} \ldots \int |b(z_1)| dz_1, \quad (11)$$
$$\int_{-\infty}^{+\infty} |b(z_n)| dz_n \int_{z_n}^{\infty} |b(z_{n-1})| dz_{n-1} \int_{-\infty}^{z_{n-1}} \ldots \int |b(z_1)| dz_1\}.$$

The two expressions on the right of (11) are equal, as follows by induction. Integration by parts leads to

$$\|0_4{}^n\| = \alpha_n \{\int_{-\infty}^{+\infty} |b| dz\}^n, \quad n \geq 0,$$

where the α_n are numbers independent of the profile $c(x)$. The first α_n can be calculated simply:

$$1, 1, \frac{1}{2}, \frac{1}{3}, \frac{5}{24}, \frac{2}{15}, \ldots$$

However, we do not see a direct way of determining

$$\lim_{n\to\infty} \alpha_n^{1/n}$$

that according to Lemma 8 is known to be $\frac{2}{\pi}$.

From Lemma 8 we obtain an estimate of the spectral radius of S_2 because the spectral radius of 0_4 does not exceed that of S_2 according to the following lemma.

Lemma 9

For all $n \geq 0$ the following inequality holds:

$$\|0_4{}^n\| \geq \|S_2{}^n\| . \tag{12}$$

So the spectral radius of 0_4 does not exceed that of S_2.

Proof

For $n = 0$ (12) is self-evident.

According to (11) we have for $n \geq 1$

$$\|0_4{}^n\| = \max\{\int_{-\infty}^{+\infty} |b(z_n)|dz_n \int_{-\infty}^{z_n} |b(z_{n-1})|dz_{n-1} \int_{z_{n-1}}^{\infty} \ldots \int |b(z_1)|dz_1,$$

$$\int_{-\infty}^{+\infty} |b(z_n)|dz_n \int_{z_n}^{\infty} |b(z_{n-1})|dz_{n-1} \int_{-\infty}^{z_{n-1}} \ldots \int |b(z_1)|dz_1\}.$$

By induction we shall prove that for all $n \geq 1$ and for all $X = (X_\downarrow, X_\uparrow) \in C_B \times C_B$

$$|(S_2{}^n X)_\downarrow(z,t)| \leq \int_{z}^{\infty} |b(z_n)|\, dz_n \int_{-\infty}^{z_n} |b(z_{n-1})|dz_{n-1} \int_{z_{n-1}}^{\infty} \ldots \int |b(z_1)|dz_1 \|X\| \tag{13}$$

and

$$|(S_2^{\,n}X)_{\uparrow}(z,t)| \le \int_{-\infty}^{z}|b(z_n)|dz_n\int_{z_n}^{\infty}|b(z_{n-1})|dz_{n-1}\int_{-\infty}^{z_{n-1}}\cdots\int|b(z_1)|dz_1\,\|X\| . \tag{14}$$

This implies

$$\|S_2^{\,n}\| \le \|O_4^{\,n}\| .$$

The inequalities (13) and (14) hold for n = 1:

$$(S_2X)_{\downarrow}(z,t) = \int_0^t b(-s+z+t)X_{\uparrow}(-s+z+t,s)ds = \int_z^{z+t} b(z_1)X_{\uparrow}(z_1,-z_1+z+t)dz_1;$$

hence

$$|(S_2X)_{\downarrow}(z,t)| \le \int_z^{z+t}|b(z_1)|dz_1\,\|X\| \le \int_z^{\infty}|b(z_1)|dz_1\,\|X\| ;$$

similarly the other inequality can be proved for n = 1.
Suppose (13) and (14) hold for n = m. Then

$$(S_2^{\,m+1}X)_{\uparrow}(z,t) = -\int_0^t b(s+z-t)(S_2^{\,m}X)_{\downarrow}(s+z-t,s)ds =$$

$$= -\int_{z-t}^{z} b(z_1)(S_2^{\,m}X)_{\downarrow}(z_1,z_1-z+t)dz_1;$$

hence

$$|(S_2^{\,m+1}X)_{\uparrow}(z,t)| \le \int_{z-t}^{z}|b(z_{m+1})|dz_{m+1}\int_{z_{m+1}}^{\infty}|b(z_m)|dz_m\int\cdots\int|b(z_1)|dz_1\,\|X\| \le$$

$$\le \int_{-\infty}^{z}|b(z_{m+1})|dz_{m+1}\int_{z_{m+1}}^{\infty}|b(z_m)|dz_m\int\cdots\int|b(z_1)|dz_1\,\|X\| ;$$

in the same manner the other inequality can be proved for n = m+1.
□

The Lemmas 8 en 9 lead to the criterion (2) for the convergence of (2.2.23) in $C_B\times C_B$, that we stated previously.

Corollary

The equation $X = Z + \lambda S_2 X$ has a unique solution in $C_B \times C_B$ if

$$|\lambda| < \frac{\pi}{2} \{\int_{-\infty}^{+\infty} |b| dz\}^{-1}.$$

Then the series (5) converges in $C_B \times C_B$ and represents the solution. For $\lambda = 1$ and $Z = X_0$ (5) converges in $C_B \times C_B$ definitely if

$$\int_{-\infty}^{+\infty} |b| dz < \frac{\pi}{2} . \tag{15}$$

The convergence criterion that we have obtained in this way is the best possible in the sense specified by the following theorem.

Theorem 9

The number $\frac{\pi}{2}$ in (15) cannot be replaced by a larger one such that the corollary remains valid for all profiles c(x) and all possible initial values.

Proof

We shall prove this theorem by giving an example for which the series (2.2.23) diverges in $C_B \times C_B$ if

$$\int_{-\infty}^{+\infty} |b| dz > \frac{\pi}{2} .$$

Suppose that b(z) is nonnegative and such that this inequality holds. (If b(z) is nonpositive, the proof goes through in the same way.) Suppose $\Psi^0_{\downarrow}(z) \equiv 0$ and $\Psi^0_{\uparrow}(z) \equiv 1$. We shall show that in this example

$$\| S_2{}^n X_0 \| = \| 0_4{}^n \| . \tag{16}$$

Therefore we have in this example

$$\lim_{n\to\infty} \| S_2{}^n X_0 \|^{1/n} = \lim_{n\to\infty} \| 0_4{}^n \|^{1/n} = \frac{2}{\pi} \int_{-\infty}^{+\infty} |b| dz > 1.$$

(See Lemma 8, pg 82.) By Cauchy's convergence test the series (2.2.23) diverges in $C_B \times C_B$, and we have proved the theorem.
We shall prove (16) in two parts.

(i) $\|S_2{}^n \chi_0\| \le \|O_4{}^n\|$.

This is true by Lemma 9, pg 84, because $\|\chi_0\| = 1$.

(ii) $\|S_2{}^n \chi_0\| \ge \|O_4{}^n\|$.

This is so because

$$\sup_{z,t}|(S_2{}^n\chi_0)_\downarrow(z,t)| \ge \sup_z\{\lim_{t\to\infty}|(S_2{}^n\chi_0)_\downarrow(z,t)|\} =^*$$

$$\sup_z\{\int_z^\infty |b(z_n)|dz_n \int_{-\infty}^{z_n} |b(z_{n-1})|dz_{n-1} \int \cdots \int |b(z_1)|dz_1\} =$$

$$\int_{-\infty}^{+\infty} |b(z_n)|dz_n \int_{-\infty}^{z_n} |b(z_{n-1})|dz_{n-1} \int \cdots \int |b(z_1)|dz_1$$

and because

$$\sup_{z,t}|(S_2{}^n\chi_0)_\uparrow(z,t)| \ge \sup_z\{\lim_{t\to\infty}|(S_2{}^n\chi_0)_\uparrow(z,t)|\} =^*$$

$$\sup_z\{\int_{-\infty}^z |b(z_n)|dz_n \int_{z_n}^\infty |b(z_{n-1})|dz_{n-1} \int \cdots \int |b(z_1)|dz_1\} =$$

$$\int_{-\infty}^{+\infty} |b(z_n)|dz_n \int_{z_n}^\infty |b(z_{n-1})|dz_{n-1} \int \cdots \int |b(z_1)|dz_1.$$

Here the only steps that are not self-evident are marked with an asterisk. These can be proved by induction. □

Up to now we examined the spectrum of S_2 as an operator in $C_B \times C_B$ independent of our knowledge of the spectrum of the operator T_4 in the first chapter. However, one might expect that a relation exists between these spectra. We shall study this now, but we shall not succeed in exposing this relationship completely.

First we make some general remarks concerning the structure of the spectrum of S_2. From the preceding section we know that the equations (2.2.17) are both uniquely soluble. Therefore eigenvalues of S_2 as an operator on $C_B \times C_B$ do not exist. The spectrum of S_2 consists of a continuous and a residual part. It consists of those λ's for which a $\mathcal{Z} \in C_B \times C_B$ exists such that the equation (2.2.17), i = 2, is not soluble in $C_B \times C_B$, i.e., such that the solution in $C \times C$ for $\mathcal{Z} \in C_B \times C_B$ is not bounded. (See Taylor [33], pg 274, problem 3.) Now we shall expose the relationship of S_2 and T_4 by means of a Fourier transform. Suppose that the solution of the set of integral equations

$$\begin{aligned} \mathcal{X}_\downarrow(z,t) &= \mathcal{Z}_\downarrow(z,t) + \lambda\int_0^t b(-s+z+t)\mathcal{X}_\uparrow(-s+z+t,s)ds, \\ \mathcal{X}_\uparrow(z,t) &= \mathcal{Z}_\uparrow(z,t) - \lambda\int_0^t b(s+z-t)\mathcal{X}_\downarrow(s+z-t,s)ds \end{aligned} \tag{17}$$

is uniformly bounded for some fixed $(\mathcal{Z}_\downarrow,\mathcal{Z}_\uparrow) \in C_B \times C_B$. Then the Fourier transform $\chi_\downarrow(z,\omega)$ of $\mathcal{X}_\downarrow(z,t)$ can be defined by

$$\chi_\downarrow(z,\omega) = \int_0^\infty \mathcal{X}(z,t)\exp i\omega t\; dt, \text{ Im } \omega > 0.$$

This transform is uniformly bounded and continuous with respect to z. In the same manner the Fourier transforms of $\mathcal{X}_\uparrow$, $\mathcal{Z}_\downarrow$, and $\mathcal{Z}_\uparrow$ can be defined. By taking the Fourier transform of (17) we see that $\chi_\downarrow$ and $\chi_\uparrow$ satisfy

$$\begin{aligned} \chi_\downarrow(z,\omega) &= \zeta_\downarrow(z,\omega) + \lambda\int_z^\infty \exp -i\omega(z-z_1)b(z_1)\chi_\uparrow(z_1,\omega)dz_1, \\ \chi_\uparrow(z,\omega) &= \zeta_\uparrow(z,\omega) - \lambda\int_{-\infty}^z \exp i\omega(z-z_1)b(z_1)\chi_\downarrow(z_1,\omega)dz_1. \end{aligned} \tag{18}$$

We consider this set of integral equations for fixed ω, Im $\omega > 0$. The operator in the right-hand side of (18) is the same operator we met in the first chapter on pg 25, now for complex ω, Im $\omega > 0$.

Therefore we shall also designate it by $T_4(\omega)$. $T_4(\omega)$ is compact for $\text{Im}\ \omega > 0$ (cf Lemma 2, pg 25), and its spectrum consists of eigenvalues only. Because of the above mentioned relation between S_2 and $T_4(\omega)$, we expect that also a relation exists between their spectra. We shall prove that the eigenvalues of $T_4(\omega)$ for $\text{Im}\ \omega \geq 0$ belong to the spectrum of S_2. First the case that $\text{Im}\ \omega > 0$.

Let λ be an eigenvalue of $T_4(\omega)$, $\text{Im}\ \omega > 0$, and $(\chi_\downarrow, \chi_\uparrow)$ a corresponding eigenvector:

$$\chi_\downarrow(z) = \lambda \int_z^\infty \exp -i\omega(z-z_1) b \chi_\uparrow dz_1,$$

$$\chi_\uparrow(z) = -\lambda \int_{-\infty}^z \exp i\omega(z-z_1) b \chi_\downarrow dz_1.$$

By differentiation we find

$$\chi_{\downarrow z} + i\omega\chi_\downarrow = -\lambda b \chi_\uparrow,$$

$$\chi_{\uparrow z} - i\omega\chi_\uparrow = -\lambda b \chi_\downarrow.$$

Then $X_\downarrow = \chi_\downarrow \exp -i\omega t$ and $X_\uparrow = \chi_\uparrow \exp -i\omega t$ are an exponentially increasing solution of

$$X_{\downarrow z} - X_{\downarrow t} = -\lambda b X_\uparrow,$$

$$X_{\uparrow z} + X_{\uparrow t} = -\lambda b X_\downarrow.$$

They must also constitute an unbounded solution of (17) for $Z_\downarrow(z,t) = \chi_\downarrow(z+t)$ and $Z_\uparrow(z,t) = \chi_\uparrow(z-t)$, which are both bounded. Therefore λ must belong to the spectrum of S_2.

Also the spectrum of $T_4(\omega)$, $\text{Im}\ \omega = 0$, is contained in the spectrum of S_2. The proof is based on the continuous change of the spectrum of $T_4(\omega)$ as ω tends to ω_0, $\text{Im}\ \omega_0 = 0$. The continuity is meant in the following sense (cf pg 36): let λ be an eigenvalue of $T_4(\omega_0)$; then in any neighbourhood of λ there must be an eigenvalue of $T_4(\omega)$, $\text{Im}\ \omega > 0$, for sufficiently small $|\omega-\omega_0|$. The proof can be carried out in a manner closely related to the proof of Theorem 3 (iii), pg 33; however, some care has to be taken because of the unbounded factors exp $\pm i\omega z$. The spectrum of S_2 is closed. (See Taylor [33], pg 257.) We know already that the eigenvalues of $T_4(\omega)$, Im $\omega > 0$, belong to the spectrum. Therefore the eigenvalues of $T_4(\omega_0)$ must also belong to it.

We did not succeed in proving the converse, viz that the spectrum of S_2 only consists of the eigenvalues of $T_4(\omega)$, Im $\omega \geq 0$. This frustrates our attempts to acquire information about the spectrum of S_2 from our knowledge of the spectrum of $T_4(\omega)$. For example, in essentially the same way as we proved Theorem 2 (iii) and (iv) on pg 27 and 28, it can be proved that $T_4(\omega)$, Im $\omega \geq 0$, has no real eigenvalues. But from this we cannot infer that the spectrum of S_2 does not contain any real point either.

Now let us pay attention to the uniform convergence of the series (2.2.20). Reexamining the arguments which led to the criterion (15) for the uniform convergence of (2.2.23), we see that these arguments do not provide sufficient conditions for the uniform convergence of (2.2.20). S_1 is not a bounded operator on C_B, even not if we impose the condition

$$\int_{-\infty}^{+\infty} |b^2-\dot{b}|dz < \infty.$$

Also no bound of $|\lambda|$ can assure the boundedness of the solution of

$$X = Z + \lambda S_1 X \qquad (Z \in C_B). \tag{19}$$

This follows from the fact that every eigenvector of $T_3(\omega)$, Im $\omega > 0$, leads to an unbounded solution of (19); see also pg 89. In the same way as we proved Theorem 3 (iii), pg 33, we can prove that in any neighbourhood of $\lambda = 0$ there is an eigenvalue of $T_3(\omega)$, Im $\omega > 0$, for sufficiently small $|\omega|$.

To conclude this section we return to Bremmer's series, and we show that indeed Bremmer's series is the steady state resulting from a monochromatic wave incident on an inhomogeneous medium. To that end we consider an initial-value problem describing the propagation of a time-harmonic wave incident from a homogeneous region $x < 0$ that at $t = 0$ arrives at the inhomogeneous region $x > 0$. Therefore we choose the following initial values:

$$\Psi_{\downarrow}^{0} \equiv 0;$$

$$\begin{aligned} \Psi_{\uparrow}^{0} &= \exp i\omega z && \text{if } z < -\Delta,\ \Delta > 0 \quad (\omega \text{ real}); \\ &= f(z) && \text{if } -\Delta < z < 0; \\ &= 0 && \text{if } z \geq 0. \end{aligned}$$

Here $f(z)$ is chosen as to make $\Psi_{\uparrow}^{0}$ twice continuously differentiable. For this initial-value problem the following theorem can be proved by induction. We omit the proof.

Theorem 10

We denote the terms of the series (2.2.23) by $\Psi_{\downarrow}^{(n)}$ and $\Psi_{\uparrow}^{(n)}$ and the terms of Bremmer's series by $\psi_{\downarrow}^{(n)}$ and $\psi_{\uparrow}^{(n)}$. The sum of these series is denoted by $\Psi_{\downarrow}$, $\Psi_{\uparrow}$, $\psi_{\downarrow}$, and $\psi_{\uparrow}$, respectively.

(i) If $\displaystyle\int_{-\infty}^{+\infty} |b| dz < \infty$,

then for all $n \geq 0$ and for all z

$$\Psi_{\downarrow}^{(n)}(z,t) = \psi_{\downarrow}^{(n)}(z) \exp -i\omega t + o(1) \quad \text{as} \quad t \to \infty,$$

$$\Psi_{\uparrow}^{(n)}(z,t) = \psi_{\uparrow}^{(n)}(z) \exp -i\omega t + o(1) \quad \text{as} \quad t \to \infty.$$

(ii) If $\displaystyle\int_{-\infty}^{+\infty} |b| dz < \frac{\pi}{2}$,

then for all z

$$\Psi_{\downarrow}(z,t) = \psi_{\downarrow}(z) \exp -i\omega t + o(1) \qquad \text{as} \quad t \to \infty,$$

$$\Psi_{\uparrow}(z,t) = \psi_{\uparrow}(z) \exp -i\omega t + o(1) \qquad \text{as} \quad t \to \infty.$$

This theorem expresses that termwise Bremmer's series indeed represents the steady state of the series solution of an initial-value problem. However, a natural condition is that the series solution of the initial-value problem converges uniformly.

§2.4 PROPAGATION OF ENERGY

Thus far we studied in this chapter the actual solution of the Cauchy problem. Of course, the series solutions we found yield some information about the propagation of the energy involved. However, there is another way open to us. We shall describe it now briefly.

Let us observe that the system (2.2.9)

$$\begin{aligned} \Psi_{\downarrow z} - \Psi_{\downarrow t} &= -b\Psi_{\uparrow}, \\ \Psi_{\uparrow z} + \Psi_{\uparrow t} &= -b\Psi_{\downarrow} \end{aligned} \tag{1}$$

need not be interpreted as an amplitude splitting of the wave equation. Suppose that Φ is a solution of (2.2.1) and that Ψ is the corresponding solution of (2.2.6), then

$$\Psi_{\downarrow} = \frac{1}{2}\, c^{-\frac{1}{2}}\Phi_t + \frac{1}{2}\, c^{\frac{1}{2}}\Phi_x = \frac{1}{2}\,\Psi_t + \frac{1}{2}\,\Psi_z + \frac{1}{2}\, b\Psi, \tag{2}$$

and

$$\Psi_{\uparrow} = \frac{1}{2}\, c^{-\frac{1}{2}}\Phi_t - \frac{1}{2}\, c^{\frac{1}{2}}\Phi_x = \frac{1}{2}\,\Psi_t - \frac{1}{2}\,\Psi_z - \frac{1}{2}\, b\Psi \tag{3}$$

satisfy (1). These $\Psi_{\downarrow}$ and $\Psi_{\uparrow}$ are the derivatives with respect to t of the $\Psi_{\downarrow}$ and $\Psi_{\uparrow}$ defined by (2.2.12); of course they satisfy the same set of equations. The advantage of this interpretation is that it yields particularly simple expressions for the density and the intensity of the energy:

$$E = \frac{1}{2}\,(\Phi_x^*\Phi_x + c^{-2}(x)\Phi_t^*\Phi_t) = c^{-1}(\Psi_{\downarrow}^*\Psi_{\downarrow} + \Psi_{\uparrow}^*\Psi_{\uparrow}), \tag{4}$$

$$F = -\frac{1}{2}\,(\Phi_t^*\Phi_x + \Phi_x^*\Phi_t) = (-\Psi_{\downarrow}^*\Psi_{\downarrow} + \Psi_{\uparrow}^*\Psi_{\uparrow}). \tag{5}$$

(Again we assume that (1.2.14) represents conservation of energy.)

The appropriate initial values, expressed in terms of the initial values (2.2.7) and (2.2.8), are

$$\Psi_{\downarrow}(z,0) = \Psi^{o}_{\downarrow}(z) = \frac{1}{2}\,\Psi^{o}_{t}(z) + \frac{1}{2}\,\Psi^{o}_{z}(z) + \frac{1}{2}\,b(z)\Psi^{o}(z), \tag{6}$$

$$\Psi_{\uparrow}(z,0) = \Psi^{o}_{\uparrow}(z) = \frac{1}{2}\,\Psi^{o}_{t}(z) - \frac{1}{2}\,\Psi^{o}_{z}(z) - \frac{1}{2}\,b(z)\Psi^{o}(z). \tag{7}$$

Using the techniques we applied before we can convert the system (1) with these initial values into a set of integral equations in the space C×C. This set of integral equations can be solved by successive approximation. If we denote $(\Psi^{o}_{\downarrow}(z+t),\Psi^{o}_{\uparrow}(z-t)) \in C\times C$ by $\mathcal{X}_{o}$, then the solution $(\Psi_{\downarrow},\Psi_{\uparrow})$ can be represented by

$$\sum_{n=0}^{\infty} S_{2}{}^{n}\mathcal{X}_{o}. \tag{8}$$

The two scalar series into which (8) splits converge absolutely for all z and t, and absolutely and uniformly on any finite domain. If $\Psi^{o}_{\downarrow}$ and $\Psi^{o}_{\uparrow}$ are uniformly bounded and if

$$\int_{-\infty}^{+\infty} |b|\,dz < \frac{\pi}{2}\,, \tag{9}$$

the series (8) converges in $C_{B}\times C_{B}$.
This criterion will also turn out to be of particular importance to obtain some results concerning the "global" propagation of energy. (We remark that the series (8) cannot be obtained by term-by-term differentiation of the series (2.2.23).)

Because of the absolute convergence of the two scalar series into which (8) splits, we can carry out the multiplication

$$\Psi^{*}_{\downarrow}\Psi_{\downarrow} = \Big(\sum_{n=0}^{\infty}\Psi^{(n)*}_{\downarrow}\Big)\Big(\sum_{n=0}^{\infty}\Psi^{(n)}_{\downarrow}\Big) \text{ and } \Psi^{*}_{\uparrow}\Psi_{\uparrow} = \Big(\sum_{n=0}^{\infty}\Psi^{(n)*}_{\uparrow}\Big)\Big(\sum_{n=0}^{\infty}\Psi^{(n)}_{\uparrow}\Big)$$

in an arbitrary way. For example:

$$\Psi^{*}_{\downarrow}\Psi_{\downarrow} = \Psi^{(0)*}_{\downarrow}\Psi^{(0)}_{\downarrow} + \Psi^{(0)*}_{\downarrow}\Psi^{(1)}_{\downarrow} + \Psi^{(1)*}_{\downarrow}\Psi^{(0)}_{\downarrow} + \Psi^{(0)*}_{\downarrow}\Psi^{(2)}_{\downarrow} + \ldots \tag{10}$$

$$\Psi^{*}_{\uparrow}\Psi_{\uparrow} = \Psi^{(0)*}_{\uparrow}\Psi^{(0)}_{\uparrow} + \Psi^{(0)*}_{\uparrow}\Psi^{(1)}_{\uparrow} + \Psi^{(1)*}_{\uparrow}\Psi^{(0)}_{\uparrow} + \Psi^{(0)*}_{\uparrow}\Psi^{(2)}_{\uparrow} + \ldots \tag{11}$$

These series also have the convergence properties that we mentioned for the original series. They are suitable for the study of the propagation of energy as expressed by the following theorem.

Theorem 11

Suppose that the total amount of energy is finite at $t = 0$:

$$\int_{-\infty}^{+\infty} |\Psi_{\downarrow}^{0}|^2 dz \leq M < \infty$$

and
$$\int_{-\infty}^{+\infty} |\Psi_{\uparrow}^{0}|^2 dz \leq M < \infty.$$

Also assume that (9) holds. Then the integral

$$\int_{0}^{\infty} F(z_0,t)dt$$

converges for all z_0. It represents the total amount of energy passing at $z = z_0$ for $t \geq 0$. It can be obtained by term-by-term integration of the series (10) and (11). The resulting series converge uniformly for all z_0.

The proof of this theorem is readily obtained from the following lemma.

Lemma 10

For all $n \geq 0$ and for all z the following inequalities hold:

$$\int_{0}^{\infty} |\Psi_{\downarrow}^{(n)}(z,t)|^2 dt \leq M\{\int_{z}^{\infty} |b(z_n)|dz_n \int_{-\infty}^{z_n} |b(z_{n-1})|dz_{n-1}\int \cdots \int |b(z_1)|dz_1\}^2, \tag{12}$$

$$\int_{0}^{\infty} |\Psi_{\uparrow}^{(n)}(z,t)|^2 dt \leq M\{\int_{-\infty}^{z} |b(z_n)|dz_n \int_{z_n}^{\infty} |b(z_{n-1})|dz_{n-1}\int \cdots \int |b(z_1)|dz_1\}^2. \tag{13}$$

(For $n = 0$ the right-hand sides of (12) and (13) are meant to be M.)

Proof

The lemma can be proved simply by induction.

For n = 0 the assertions are evident. Suppose that (12) holds for n = m. Because

$$\Psi_{\uparrow}^{(m+1)}(z,t) = -\int_0^t b(s+z-t)\Psi_{\downarrow}^{(m)}(s+z-t,s)ds = -\int_{z-t}^{z} b(z_1)\Psi_{\downarrow}^{(m)}(z_1,z_1-z+t)dz_1,$$

we have

$$\int_0^{t_1} |\Psi_{\uparrow}^{(m+1)}(z,t)|^2 dt \leq \int_0^{t_1} \{\int_{z-t}^{z} |b(z_1)||\Psi_{\downarrow}^{(m)}(z_1,z_1-z+t)|dz_1\}^2 dt.$$

If we interpret the right-hand side as a repeated integral and if we change the order of integration, we obtain

$$= \int_{z-t_1}^{z} |b(z_1)|dz_1 \int_{z-t_1}^{z} |b(z_2)|dz_2 \int_{\xi}^{t_1} |\Psi_{\downarrow}^{(m)}(z_1,z_1-z+t)||\Psi_{\downarrow}^{(m)}(z_2,z_2-z+t)|dt,$$

where

$$\begin{aligned} \xi &= z-z_1 \quad \text{if} \quad z_1 \geq z_2, \\ &= z-z_2 \quad \text{if} \quad z_1 \leq z_2. \end{aligned}$$

By the Cauchy-Schwarz inequality and by substitution of our assumption this leads to

$$\leq M\{\int_{z-t_1}^{z} |b(z_{m+1})|dz_{m+1}\int_{z_{m+1}}^{\infty} |b(z_m)|dz_m\int\ldots\int|b(z_1)|dz_1\}^2 \leq$$

$$\leq M\{\int_{-\infty}^{z} |b(z_{m+1})|dz_{m+1}\int_{z_{m+1}}^{\infty} |b(z_m)|dz_m\int\ldots\int|b(z_1)|dz_1\}^2.$$

In the same manner the proof can be finished. □

Another consequence of the assumptions made in Theorem 11 is that the energy in any finite interval approaches zero as t tends to ∞. This can also be proved by term-by-term integration.

APPENDIX

The inhomogeneous wave equation

Thus far we were only concerned with the homogeneous wave equation. However, the methods we used also apply to the inhomogeneous case:

$$\Phi_{xx} - c^{-2}(x)\Phi_{tt} = f(x,t). \tag{1}$$

We shall indicate how we can handle the Cauchy problem for (1). The following initial values are imposed:

$$\Phi(x,0) = 0, \tag{2}$$

$$\Phi_t(x,0) = 0. \tag{3}$$

The Liouville transformation

$$\Psi = c^{-\frac{1}{2}}\Phi,$$

$$z = \int_{x_0}^{x} c^{-1}(x_1)dx_1$$

transforms the Cauchy problem (1), (2), (3) into

$$\Psi_{zz} - \Psi_{tt} - (b^2-\dot{b})\Psi = h(z,t), \tag{4}$$

$$\Psi(z,0) = \Psi_t(z,0) = 0, \tag{5}$$

where $\qquad h(z,t) = c^{3/2}(z)f(x(z),t).$

By integration over the domain of dependence of a point (z,t) this Cauchy problem is transformed into the equivalent integral equation

$$\Psi(z,t) = -\frac{1}{2}\iint\limits_{D(z,t)} h(z_1,t_1)dz_1dt_1 - \frac{1}{2}\iint\limits_{D(z,t)} (b^2-\dot{b})\Psi(z_1,t_1)dz_1dt_1. \tag{6}$$

This integral equation is of the same type as the one for the Cauchy problem for the homogeneous equation.
In the same way as we did in §2.2, it can be solved by successive approximation leading to the same convergence properties.

In order to find other series solutions, we can try to find an amplitude splitting of (4). There exist an infinite number of them all having the set (2.1.7) as their homogeneous part. We mention two. The first is

$$\begin{aligned} \Psi_{\downarrow z} - \Psi_{\downarrow t} &= -b\Psi_{\uparrow} + \frac{1}{2}\int_0^t h(z,t_1)dt_1, \\ \Psi_{\uparrow z} + \Psi_{\uparrow t} &= -b\Psi_{\downarrow} - \frac{1}{2}\int_0^t h(z,t_1)dt_1, \end{aligned} \tag{7}$$

with initial conditions

$$\Psi_{\downarrow}(z,0) = 0, \quad \Psi_{\uparrow}(z,0) = 0. \tag{8}$$

The second is

$$\begin{aligned} \Psi_{\downarrow z} - \Psi_{\downarrow t} &= -b\Psi_{\uparrow} + \frac{1}{2}\, c^{\frac{1}{2}}(z)\int_{z_0}^{z} h(z_1,t)c^{-\frac{1}{2}}(z_1)dz_1, \\ \Psi_{\uparrow z} + \Psi_{\uparrow t} &= -b\Psi_{\downarrow} + \frac{1}{2}\, c^{\frac{1}{2}}(z)\int_{z_0}^{z} h(z_1,t)c^{-\frac{1}{2}}(z_1)dz_1, \end{aligned} \tag{9}$$

with initial conditions

$$\Psi_{\downarrow}(z,0) = 0, \quad \Psi_{\uparrow}(z,0) = 0. \tag{10}$$

By integration along the characteristic curves these Cauchy problems can be transformed into integral equations of the type (2.2.15). They can be solved by successive approximation, and convergence criteria can be proved that are similar to the criteria we found for the homogeneous equation.

There also exists a counterpart of what we did in §2.4. Equation (1) admits a balance equation for the energy:

$$E_t + F_x = K,$$

where $$E = \frac{1}{2}\,(\Phi_x^*\Phi_x + c^{-2}(x)\Phi_t^*\Phi_t),$$

$$F = -\frac{1}{2}(\phi_t^*\phi_x + \phi_x^*\phi_t),$$

$$K = -\frac{1}{2}(f^*\phi_t + f\phi_t^*).$$

In the same manner as we did in §2.4, we define

$$\Psi_\downarrow = \frac{1}{2}c^{-\frac{1}{2}}\phi_t + \frac{1}{2}c^{\frac{1}{2}}\phi_x = \frac{1}{2}\Psi_t + \frac{1}{2}\Psi_z + \frac{1}{2}b\Psi,$$

$$\Psi_\uparrow = \frac{1}{2}c^{-\frac{1}{2}}\phi_t - \frac{1}{2}c^{\frac{1}{2}}\phi_x = \frac{1}{2}\Psi_t - \frac{1}{2}\Psi_z - \frac{1}{2}b\Psi,$$

which leads again to the expressions (2.4.4) and (2.4.5) for E and F. These $\Psi_\downarrow$ and $\Psi_\uparrow$ obey the set of equations

$$\begin{aligned} \Psi_{\downarrow z} - \Psi_{\downarrow t} &= -b\Psi_\uparrow + \frac{1}{2}h, \\ \Psi_{\uparrow z} + \Psi_{\uparrow t} &= -b\Psi_\downarrow - \frac{1}{2}h. \end{aligned} \tag{11}$$

Integrating (11) along the characteristics we obtain the equivalent set of integral equations, which can be solved by successive approximation. The following counterpart of Theorem 11 can be proved.

<u>Theorem 12</u>

Suppose that $h(z,t)$ satisfies

$$\int_0^\infty \{\int_{-\infty}^{+\infty} |h(z,t)|^2 dz\}^{\frac{1}{2}} dt < \infty .$$

Also suppose that (2.4.9) holds. Then the integral

$$\int_0^\infty F(z_0,t)dt$$

converges for all z_0. It can be obtained by term-by-term integration of the series (2.4.10) and (2.4.11). The resulting series converge uniformly for all z_0.

REFERENCES

[1] Atkinson, F.V. Proc.Glasgow Math.Assoc. 3 (1956-8) 105-111

[2] Atkinson, F.V. J.Math.Anal.Appl. 1 (1960) 255-276

[3] Bahar, E. J.Math.Phys. 8 (1967) 1735-1746

[4] Baird, L.C. J.Math.Phys. 11 (1970) 2235-2242

[5] Berk, H.L., D.L.Book, and D.Pfirsch J.Math.Phys. 8 (1967) 1611-1619

[6] Bremmer, H. "Handelingen Natuur- en Geneeskundig Congres, Nijmegen", Ruygrok, Haarlem, 1939, pg 88-91

[7] Bremmer, H. Physica 15 (1949) 593-608

[8] Bremmer, H. "The Theory of Electromagnetic Waves", Interscience, New York, 1951, pg 169-179

[9] Broer, L.J.F. Appl.Sci.Res. 10 (1962) 110-118

[10] Broer, L.J.F. and J.B.van Vroonhoven Physica 52 (1971) 441-453

[11] Cavaliere, A., B.Crosignani, and F.Gratton Nuovo Cimento 23 (1964) 1338-1355

[12] Coppel, W.A. "Stability and Asymptotic Behavior of Differential Equations", D.C.Heath, Boston, 1965

[13] Courant, R. and D.Hilbert "Methods of Mathematical Physics", Interscience, New York, 1962

[14] Dunford, N. and J.T.Schwartz "Linear Operators", Interscience, New York, 1957

[15] Erdélyi, A. "Asymptotic Expansions", Dover, New York, 1956

[16] Fröman, N. and P.O.Fröman "JWKB Approximation. Contributions to the Theory", North Holland, Amsterdam, 1965

[17] Green, G. Trans.Cambridge Philos.Soc. 6 (1837) 457-462

[18] Hellwig, G. "Partial Differential Equations", Blaisdell, New York, 1964

[19] Van Kampen, N.G. Physica 35 (1967) 70-79

[20] Kato, T. "Perturbation theory for linear operators", Springer, Berlin, 1966

[21] Kay, I. J.Math.Anal.Appl. 3 (1961) 40-49

[22] Keller, H.B. and J.B.Keller J.Soc.Ind.Appl.Math. 10 (1962) 246-259

[23] Kemble, E.C. Phys.Rev. 48 (1935) 549-561

[24] Knorr, G. and D.Pfirsch Z.Naturforsch. 21a (1966) 688-693

[25] Landauer, R. Phys.Rev. 82 (1951) 80-83

[26] Liouville, J. Journal de Mathématiques 2 (1837) 16-35

[27] Merzbacher, E. "Quantum Mechanics", J.Wiley, New York, 1961

[28] Olver, F.W.J. Proc.Cambridge Philos.Soc. 57 (1961) 790-810

[29] Riesz, F. and B.Sz.-Nagy "Functional Analysis", Frederic Ungar, New York, 1955

[30] Schep, T.J. "Charged Particle Motion Near Cyclotron Resonance", Thesis, Eindhoven, 1973

[31] Sluijter, F.W. J.Math.Anal.Appl. 27 (1969) 282-302

[32] Sluijter, F.W. J.Opt.Soc.Am. 60 (1970) 8-10

[33] Taylor, A.E. "Introduction to Functional Analysis", J.Wiley, New York, 1958

LIST OF SYMBOLS

$b(z)$	pg 8	
C	72	
C^m	72	
$C(R)$	23	
$C(R)\times C(R)$	23	
$C(z_o,t_o)$	76	
$C(z_o,t_o)\times C(z_o,t_o)$	76	
C_B	81	
$C_B\times C_B$	81	
$D(z_o,t_o)$	74	
E	21	
F	21	
$F_1(z,z_1)$, $F_2(z,z_1)$	40	
O_4	82	
S_1, S_2	76	
T_1, T_2	25	
T_3	25	90
T_4	25	89

An asterisk denotes complex conjugation.
Differentiation with respect to x is sometimes designated by a prime; in general, a dot denotes differentiation with respect to the Liouville-transformed space coordinate z.
To formulas is outside their section referred by their number preceded by their section number.
The end of a proof is denoted by □.

ABSTRACT

This work deals with the study of series solutions of the one-dimensional wave equation

$$\Phi_{xx} - c^{-2}(x)\Phi_{tt} = 0$$

that have the WKB approximations as their first term. These approximations have been studied already since Liouville and Green. The thesis is divided into two parts.
In the first part we study monochromatic solutions. We give a unification and a generalization of much that has been done. Particular attention is paid to the frequency dependence of the convergence of the series in question.
The second part is devoted to the study of the Cauchy problem. Using the methods developed in the first part we construct series solutions of this Cauchy problem and study their convergence.

SAMENVATTING

In dit proefschrift worden oplossingen bestudeerd van de een-dimensionale golfvergelijking

$$\Phi_{xx} - c^{-2}(x)\Phi_{tt} = 0$$

in de vorm van convergente reeksen waarvan de eerste term de WKB benadering is. Deze benadering wordt al onderzocht sinds Liouville en Green.

Het proefschrift bestaat uit twee delen.

In het eerste deel worden monochromatische oplossingen onderzocht. Er wordt een unificatie en een generalisatie gegeven van veel dat door anderen gedaan is. Bijzondere aandacht wordt gewijd aan de manier waarop de convergentie van de beschouwde reeksen afhangt van de frequentie.

In het tweede deel wordt het Cauchy probleem bestudeerd. Met behulp van de methodes van het eerste deel worden oplossingen geconstrueerd in de vorm van reeksen en wordt hun convergentie onderzocht.

CURRICULUM VITAE

De auteur van dit proefschrift werd geboren op 7 juli 1948 te Stevensbeek.
Het diploma van natuurkundig ingenieur werd door hem behaald op 26 november 1971 aan de Technische Hogeschool te Eindhoven.
Daarna trad hij in dienst van de Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek om een promotieonderzoek te verrichten.

S T E L L I N G E N

1. Met de methoden van dit proefschrift kan, onder zekere voorwaarden, een oplossing van de Ricattivergelijking voor de reflectiecoëfficient, zoals die bijvoorbeeld is afgeleid door Bellman en Kalaba, worden geconstrueerd in de vorm van een convergente reeks.

   R.Bellman en R.Kalaba, "Functional Equations, Wave Equations and Invariant Imbedding", J.Math.Mech. 8 (1959) 683-704.

2. De suggestie van Brekhovskikh dat een door hem geconstrueerde iteratieprocedure voor de Ricattivergelijking zoals in stelling 1 genoemd, in het bijzonder geschikt zou zijn voor het geval van een dunne laag, is onjuist.

   L.M.Brekhovskikh, "Waves in Layered Media", Academic Press, New York, 1960, blz. 220-222.

3. De benaderde oplossing van de radiale Schrödingervergelijking die Burt en Woods geconstrueerd hebben, berust op een onjuiste toepassing van een methode van Bremmer.
   Met deze methode is het, onder zekere voorwaarden, mogelijk een reguliere oplossing te construeren in de vorm van een convergente reeks.

   P.B.Burt en F.J.Woods, "Integral Approximations in Potential Scattering", J.Math.Phys. 12 (1971) 2275-2280.

   H.Bremmer in "The Theory of Electromagnetic Waves", Interscience, New York, 1951, blz. 169-179.

4. Het bewijs dat Mathews en Walker met behulp van een variatie-principe geven van de volledigheid van het stelsel eigenfuncties van de Sturm-Liouville operator, is onvolledig.

   J.Mathews en R.L.Walker, "Mathematical Methods of Physics", Benjamin, New York, 1964, blz. 320 v.v..

5. In de basisopleiding van natuurkundestudenten aan de T.H.E. krijgen asymptotische methoden te weinig aandacht.

6. De verzameling van strafwetten die voldoen aan de definitie gegeven in de katholieke moraaltheologie, is leeg.

   H.Jone, "Katholieke Moraaltheologie", Romen, Roermond, 1953, blz.34.

Eindhoven, 27 september 1974 Th.M.M.Verheggen