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Citation for published version (APA):

Ali, G., Banagaaya, N., Schilders, W. H. A., & Tischendorf, C. (2012). *Index-aware model order reduction for differential-algebraic equations*. (CASA-report; Vol. 1232). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/2012

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 12-32
October 2012

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by

G. Ali, N. Banagaaya, W.H.A. Schilders, C. Tischendorf



Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

RESEARCH ARTICLE

Index-aware Model Order Reduction for Differential-Algebraic Equations

G. Ali^{a,b}, N. Banagaaya^{c*}, W. H. A. Schilders^c, C. Tischendorf^d

^a*Dept. of Mathematics, University of Calabria, Italy (giuseppe.ali@unical.it);*

^b*INFN, Gruppo collegato di Cosenza, Arcavacata di Rende, I-87036 Cosenza, Italy;*

^c*Dept. of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands (n.banagaaya@tue.nl, w.h.a.schilders@tue.nl);*

^d*Mathematical Institute, University of Cologne Weyertal 86-90, 50931 Koeln, Germany (tischendorf@math.uni-koeln.de).*

(Received 00 Month 200x; final version received 00 Month 200x)

We introduce a model order reduction procedure for differential-algebraic equations, which is based on the intrinsic differential equation contained in the starting system and on the remaining algebraic constraints. We implement numerically this procedure and show numerical evidence of its validity.

Keywords: differential algebraic equations; tractability index; model order reduction; modified decomposition of DAEs

AMS Subject Classification: 78M34; 65L80

1. Introduction

Model Order Reduction (MOR) has become a very active area of research within numerical analysis. It aims at quickly capturing the essential features of a problem and its solutions, so as to enable the construction of lower order models that adequately describe processes. Such lower order models can then be used in subsequent simulations, for example in an optimization procedure or in the context of inverse modeling. In many application areas, such simulations can only be carried out if reduced order models are used.

In the history of mathematics we see the desire to approximate a complicated function with a simpler formulation already very early. In the year 1807 Fourier (1768-1830) published the idea to approximate a function with a few trigonometric terms. In linear algebra the first step in the direction of model order reduction came from Lanczos (1893-1974). He looked for a way to reduce a matrix in tridiagonal form [31, 32]. W.E. Arnoldi realized that a smaller matrix could be a good approximation of the original matrix [22]. The ideas of Lanczos and Arnoldi were already based on the fact that a computer was available to do the computations. The question, therefore, was how the process of finding a smaller approximation could be

*Corresponding author. Email: n.banagaaya@tue.nl

automated. The fundamental methods in the area of Model Order Reduction were published in the eighties and nineties of the last century. In 1981 Moore [23] published the method of Truncated Balanced Realization, in 1984 Glover published his famous paper on the Hankel-norm reduction [24]. In 1987 the Proper Orthogonal Decomposition method was proposed by Sirovich [25]. All these methods were developed in the field of systems and control theory. In 1990 the first method related to Krylov subspaces was born, in Asymptotic Waveform Evaluation [26]. However, the focus of this paper was more on finding Padé approximations rather than Krylov spaces. Then, in 1993, Freund and Feldmann proposed Padé Via Lanczos [27] and showed the relation between the Padé approximation and Krylov spaces. In 1995 another fundamental method was published. The authors of [28] introduced PRIMA, a method based on the ideas of Arnoldi, instead of those of Lanczos. This method guarantees that the property of passivity is automatically inherited by the reduced order models.

In more recent years much research has been done in the area of Model Order Reduction. Consequently a large variety of methods is available. Some are tailored to specific applications, others are more general. An example of the latter is SPRIM [29] which attempts to construct reduced order models that mimic the structure of the original problem. This is especially important if the problem on hand contains different types of variables. Another important area of research is MOR for coupled problems, but here no breakthroughs have been obtained to date.

In this paper, we concentrate on another general type of problem. In many application areas, models are described by a system of differential-algebraic equations, in the area of model order reduction referred to as descriptor systems. Such problems can be cast into the form of a state-space system with a singular coefficient matrix multiplying the time derivative. Thus, we have a differential-algebraic equation (DAE).

There are several index concepts which measure how a DAE is far from an ODE. Here we mention the tractability index, which is related to the number of derivatives of the input which enter the solution. In principle, if the matrix pencil of a DAE is regular, it is possible to use conventional MOR techniques to obtain reduced order models, which are generally ODEs. However, as far as their numerical treatment is concerned, the reduced models may be close to higher index models, that is to DAEs. Thus the numerical solution of the reduced models might be computationally expensive, or even not feasible. This problem is very pronounced for system with index higher than 1, but it may occur even if the index of the problem does not exceed 1, as shown by Example 5.5 in Section 5.

To solve such problems in a reliable way, we developed the idea of using the so-called März projectors [15] to split the DAE system into a system of ODE and a system of algebraic equations. Then, conventional MOR techniques can be used to reduce the ODE system. We show that the reduction of the ODE system induced a reduction also of the algebraic equations.

We notice that conventional MOR methods do not work for the algebraic equations, and alternative methods must be used. In [30] a new method for the special case of resistor networks has been described, and one could imagine that similar techniques, based on graph theoretical methods, can be used for the resulting algebraic systems. In this paper, however, we will use a different approach.

In order to describe the ideas in detail, and show the merits of index-aware model order reduction (IMOR), this paper develops the theory and shows applications only for the index-1 case. In a forthcoming paper, we will treat the higher-index cases, for which the ideas are similar, but the analysis is more involved.

The paper is organized as follows. In Section 2 we present the main problem. In

Section 3 we briefly recall the definition of tractability index, and the decomposition of DAEs in differential and algebraic equations, by using März projectors. In particular, we specialize the splitting to index-1 DAEs and introduce an alternative compact splitting. In Section 4 we recall some traditional MOR methods and introduce index-aware MOR (IMOR) methods. For the sake of simplicity we confine our discussion to Krylov-based MOR methods, such as Arnoldi process, and in a final subsection we compare MOR and IMOR methods. In Section 5 we present some numerical examples, divided in small examples and industrial examples. The small examples are used to illustrate the idea of the method, and to show that the splitting of the DAE in differential and algebraic equations is beneficial also for the numerical solution of the system. The industrial examples show the feasibility of the method for real-life applications. The paper is concluded by some final remarks, in Section 6.

2. Statement of the problem

We consider the linear time-invariant control system:

$$\mathbf{E}\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad (1a)$$

$$\mathbf{y} = \mathbf{C}^T\mathbf{x}, \quad (1b)$$

with constant matrices $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{B} \in \mathbb{R}^{n,m}$, $\mathbf{C} \in \mathbb{R}^{n,\ell}$ and vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^\ell$. In (1), n is the state-space dimension, and m and ℓ are the number of inputs and outputs, respectively. The unknown \mathbf{x} depends on the time t , and the prime denotes time derivative. The vector \mathbf{u} , also depending on time, is the input data, while \mathbf{y} is the desired output data and \mathbf{C} is the control output matrix. The system is supplemented with initial data

$$\mathbf{x}(0) = \mathbf{x}_0. \quad (2)$$

We assume that both \mathbf{B} and \mathbf{C} have maximal rank. Usually ℓ and m are required to be (much) smaller than n , in this case we have $\text{rank } \mathbf{B} = m$, $\text{rank } \mathbf{C} = \ell$. In order to have uniqueness of solution, we assume that the matrix pencil (\mathbf{E}, \mathbf{A}) is regular, that is $\det(\mathbf{E}s - \mathbf{A}) \in \mathbb{R}(s) \setminus \{0\}$. If the matrix \mathbf{E} is non-singular the system is an ordinary differential equation (ODE) otherwise it is a differential algebraic equation (DAE). In this paper, we consider the latter case. Thus the initial data \mathbf{x}_0 in (2) must be consistent, since there are constraints on the possible initial conditions that can be imposed on the solutions of equation (1a).

The main goal of MOR is to find a smaller system than (1), which produces an input-output relation similar to the original system. There are many ways to approach this problem. The simplest way makes use of a projection from the solution space \mathbb{R}^n to a subspace \mathcal{V} with sufficiently small dimension $n_r < n$.

If we denote by $\mathbf{V} \in \mathbb{R}^{n,n_r}$ the matrix of an orthonormal basis of \mathcal{V} , that is, $\mathbf{V}^T\mathbf{V} = \mathbf{I}_{n_r}$, then we wish to approximate the solution x by means of the vector $\mathbf{V}\mathbf{x}_r \in \mathcal{V}$, with $\mathbf{x}_r \in \mathbb{R}^{n_r}$. In other words we are discarding the components of \mathbf{x} which lie outside \mathcal{V} , and we are identifying the projection by means of its coordinates with respect to a basis. We replace the original system for $\mathbf{x} \in \mathbb{R}^n$, with output $\mathbf{y} \in \mathbb{R}^\ell$, with the reduced system for $\mathbf{x}_r \in \mathbb{R}^{n_r}$, with output $\mathbf{y}_r \in \mathbb{R}^\ell$,

$$\mathbf{E}_r\mathbf{x}_r' = \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u}, \quad (3a)$$

$$\mathbf{y}_r = \mathbf{C}_r^T\mathbf{x}_r, \quad (3b)$$

with matrices $\mathbf{E}_r = \mathbf{V}^T \mathbf{E} \mathbf{V}$, $\mathbf{A}_r = \mathbf{V}^T \mathbf{A} \mathbf{V} \in \mathbb{R}^{n_r, n_r}$, $\mathbf{B}_r = \mathbf{V}^T \mathbf{B} \in \mathbb{R}^{n_r, m}$, and $\mathbf{C}_r = \mathbf{V}^T \mathbf{C} \in \mathbb{R}^{n_r, \ell}$. In this framework, a MOR method amounts to finding a suitable subspace \mathcal{V} and a projector on it, such that the approximation error $\|\mathbf{y} - \mathbf{y}_r\|$ is small in a suitable norm.

In addition to providing a small approximation error, the main goal of MOR is to find reduced models which preserve some relevant physical properties of the original system (1), such as *stability* and *passivity* (see [19]), or the first moments of the transfer function (see Section 4).

In this paper we present a MOR method which preserves another important property of the original system, that is, the index of the system. There are many index concepts, we consider the tractability index since it can be computed numerically, as discussed in the next section. Roughly speaking, the tractability index gives a measure of how far a DAE is from an ODE, in terms of the derivatives of the input which enters the output. One might expect that this concept can be neglected, since in principle the most common MOR methods based on projection can be applied to a problem of the form (1), provided the matrix pencil (\mathbf{E}, \mathbf{A}) is regular, irrespective of its index. In practice the situation is very different. The reduced system provided by methods like PRIMA is an ODE, but this ODE may generally be close to a DAE, depending on the choice of the matrices \mathbf{B} and \mathbf{C} . Thus, the numerical solution may be very unstable, and sometime it might fail, even for reduced systems originating from index-1 systems, as illustrated in Example 5.5 in Section 5.

3. Index concept and decomposition of DAEs

In this section we review briefly the concept of tractability index of the system (1a). In particular, in the first subsection we present a decoupling of the system which is based on appropriate projectors. In the second subsection we apply the general theory to the decomposition of index-1 systems. Finally, we present a modified decomposition for index-1 systems, which is more practical for applications.

3.1. Tractability index of DAEs

In this subsection we introduce the concept of tractability index of the system (1a). This concept was introduced by März, and this section is based on the material contained in [15, 16].

If the matrix \mathbf{E} is non-singular the system is an ODE, and we say that it has tractability index 0. In the rest of the paper, we assume that the matrix E is singular, so that the system is a DAE. To define the tractability index we introduce a sequence of matrices $E_k, A_k, k \geq 0$. For $k = 0$ we set

$$\mathbf{E}_0 := \mathbf{E}, \quad \mathbf{A}_0 := \mathbf{A}.$$

We introduce the projector \mathbf{Q}_0 onto $\ker \mathbf{E}_0$, and its complementary projector \mathbf{P}_0 characterized by

$$\mathbf{E}_0 \mathbf{Q}_0 = 0, \quad \mathbf{Q}_0^2 = \mathbf{Q}_0, \quad \mathbf{Q}_0 + \mathbf{P}_0 = \mathbf{I}. \quad (4)$$

We then introduce, for $k = 1$, the new matrices

$$\mathbf{E}_1 := \mathbf{E}_0 - \mathbf{A}_0 \mathbf{Q}_0, \quad \mathbf{A}_1 := \mathbf{A}_0 \mathbf{P}_0.$$

By construction, we can rewrite (1a) in the equivalent form:

$$\mathbf{E}_1(\mathbf{P}_0\mathbf{x}' + \mathbf{Q}_0\mathbf{x}) = \mathbf{A}_1\mathbf{x} + \mathbf{B}\mathbf{u}. \quad (5)$$

If the matrix \mathbf{E}_1 is non-singular, the procedure is terminated and we say that the system (1a) has tractability index 1, i.e., it is an index-1 system, otherwise the process continues till we obtain a non-singular matrix.

This procedure can be iterated by induction, as follows. Let $k > 1$. For $j = 0, 1, \dots, k-1$, assume to know the matrices \mathbf{E}_j , \mathbf{A}_j , with \mathbf{E}_j singular, and the projectors \mathbf{Q}_j , $\mathbf{P}_j = \mathbf{I} - \mathbf{Q}_j$, where \mathbf{Q}_j is a projector onto $\ker \mathbf{E}_j$, satisfying the additional condition

$$\mathbf{Q}_j\mathbf{Q}_i = \mathbf{0}, \quad 0 \leq i < j. \quad (6)$$

This condition is crucial to ensure that the relevant products of P 's and Q 's are also projectors. Then, for $j = k$ we define the matrices

$$\mathbf{E}_k := \mathbf{E}_{k-1} - \mathbf{A}_{k-1}\mathbf{Q}_{k-1}, \quad \mathbf{A}_k := \mathbf{A}_{k-1}\mathbf{P}_{k-1},$$

and it is possible to prove that the following equivalent form of equation (1a) holds:

$$\mathbf{E}_k(\mathbf{P}_{k-1} \cdots \mathbf{P}_0\mathbf{x}' + \mathbf{Q}_0\mathbf{x} + \cdots + \mathbf{Q}_{k-1}\mathbf{x}) = \mathbf{A}_k\mathbf{x} + \mathbf{B}\mathbf{u}. \quad (7)$$

If \mathbf{E}_k is invertible and \mathbf{E}_j is singular for $0 \leq j < k$, then system (1a) is said to have tractability index k . It is well-known that a linear DAE with constant coefficients has tractability index k if and only if it has the Kronecker index k (see [20]). In the context of DAEs including over- and underdetermined systems, a proof can also be found in [21]. Note the the projectors that satisfy condition (6) exist in practice and can be computed numerically (cf. [21]).

This procedure leads to a well-defined index concept. In particular, after inverting the matrix \mathbf{E}_k , we can project the resulting system to find an intrinsic differential equation for $\mathbf{x}_P := \mathbf{P}_0\mathbf{P}_1 \dots \mathbf{P}_{k-1}\mathbf{x}$, which we call the ‘‘differential component’’ of \mathbf{x} , and algebraic equations for $\mathbf{x}_{Q,0} := \mathbf{Q}_0\mathbf{x}$, $\mathbf{x}_{Q,1} := \mathbf{P}_0\mathbf{Q}_1\mathbf{x}$, \dots , $\mathbf{x}_{Q,k-1} := \mathbf{P}_0 \cdots \mathbf{P}_{k-2}\mathbf{Q}_{k-1}\mathbf{x}$, which we call the ‘‘algebraic components’’ of \mathbf{x} . The algebraic equations are to be solved iteratively in terms of \mathbf{x}_P , starting from $\mathbf{x}_{Q,k-1}$. In the expression of $\mathbf{x}_{Q,k-2}$ will appear a time derivative of $\mathbf{x}_{Q,k-1}$, and thus of \mathbf{x}_P and \mathbf{u} . At each step of the iteration, a new time derivative will appear. So, in total, the solution of an index- k system will contain $k-1$ derivatives of \mathbf{u} .

3.2. Decomposition of index-1 systems

In this subsection, we concentrate on index-1 systems. Assume system (1a) has tractability index 1, i.e $k = 1$, then equation (7) simplifies to:

$$\mathbf{P}_0\mathbf{x}' + \mathbf{Q}_0\mathbf{x} = \mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{x} + \mathbf{B}\mathbf{u}). \quad (8)$$

Since we have the decomposition of the identity $\mathbf{I} = \mathbf{P}_0 + \mathbf{Q}_0$, and the projectors \mathbf{P}_0 , \mathbf{Q}_0 are mutually orthogonal, then equation (8) is equivalent to the two equations obtained after left-multiplication by \mathbf{P}_0 and \mathbf{Q}_0 :

$$\begin{aligned} \mathbf{P}_0\mathbf{x}' &= \mathbf{P}_0\mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{x} + \mathbf{B}\mathbf{u}), \\ \mathbf{Q}_0\mathbf{x} &= \mathbf{Q}_0\mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{x} + \mathbf{B}\mathbf{u}). \end{aligned}$$

The first equation is an ordinary differential equation for $\mathbf{x}_P := \mathbf{P}_0\mathbf{x}$, the second is an algebraic equation which expresses $\mathbf{x}_Q := \mathbf{Q}_0\mathbf{x}$ in terms of \mathbf{x}_P . We call \mathbf{x}_P the differential component of \mathbf{x} , and \mathbf{x}_Q the algebraic component of \mathbf{x} . The previous equations can be written as:

$$\mathbf{x}'_P = \mathbf{A}_P\mathbf{x}_P + \mathbf{B}_P\mathbf{u}, \quad (9)$$

$$\mathbf{x}_Q = \mathbf{A}_Q\mathbf{x}_P + \mathbf{B}_Q\mathbf{u}, \quad (10)$$

with

$$\mathbf{A}_P := \mathbf{P}_0\mathbf{E}_1^{-1}\mathbf{A}\mathbf{P}_0, \quad \mathbf{B}_P := \mathbf{P}_0\mathbf{E}_1^{-1}\mathbf{B},$$

$$\mathbf{A}_Q := \mathbf{Q}_0\mathbf{E}_1^{-1}\mathbf{A}\mathbf{P}_0, \quad \mathbf{B}_Q := \mathbf{Q}_0\mathbf{E}_1^{-1}\mathbf{B}.$$

This decomposition shows that we can only impose initial data on \mathbf{x}_P . In fact, if we write $\mathbf{x}(0) = \mathbf{x}_P(0) + \mathbf{x}_Q(0)$, we can recover the algebraic part of the initial data by consistency with (10), that is,

$$\mathbf{x}_Q(0) = \mathbf{A}_Q\mathbf{x}_P(0) + \mathbf{B}_Q\mathbf{u}(0). \quad (11)$$

Both equations (9) and (10) are formally defined on the same space \mathbb{R}^n , since we have $\mathbf{x}_P, \mathbf{x}_Q \in \mathbb{R}^n$. Thus, the resulting system is of dimension $2n$. If the tractability index is equal to k , the decoupled system will then be of order $(k+1)n$. However, in all cases, the total rank will always be n . This is one of the limitation of März decomposition. In order to make the computational and model order reduction procedures more efficient, we propose a new way of decoupling index-1 systems using basis column matrices of projector \mathbf{P}_0 and \mathbf{Q}_0 as discussed in the next subsection.

3.3. Modified decomposition of index-1 systems

Let us introduce $n_q = \dim(\ker \mathbf{E})$, $n_p = n - n_q$, and consider a basis $\{\mathbf{p}_1, \dots, \mathbf{p}_{n_p}, \mathbf{q}_1, \dots, \mathbf{q}_{n_q}\}$ in \mathbb{R}^n made of n_q independent vectors $\mathbf{q}_i \in \ker E$ and n_p independent vectors \mathbf{p}_j in the complementary subspace. Then, we can form the matrices $\mathbf{p} := [\mathbf{p}_1 \dots \mathbf{p}_{n_p}] \in \mathbb{R}^{n, n_p}$, $\mathbf{q} := [\mathbf{q}_1 \dots \mathbf{q}_{n_q}] \in \mathbb{R}^{n, n_q}$. We have

$$\mathbf{P}_0\mathbf{p} = \mathbf{p}, \quad \mathbf{P}_0\mathbf{q} = 0, \quad \mathbf{Q}_0\mathbf{p} = 0, \quad \mathbf{Q}_0\mathbf{q} = \mathbf{q}. \quad (12)$$

We can expand \mathbf{x} with respect to the new basis, obtaining

$$\mathbf{x} = \mathbf{p}\boldsymbol{\xi}_p + \mathbf{q}\boldsymbol{\xi}_q, \quad \boldsymbol{\xi}_p \in \mathbb{R}^{n_p}, \quad \boldsymbol{\xi}_q \in \mathbb{R}^{n_q},$$

which implies

$$\mathbf{x}_P = \mathbf{p}\boldsymbol{\xi}_p, \quad \mathbf{x}_Q = \mathbf{q}\boldsymbol{\xi}_q. \quad (13)$$

By construction, $[\mathbf{p} \ \mathbf{q}]$ is invertible, and let $\begin{bmatrix} \mathbf{p}_*^T \\ \mathbf{q}_*^T \end{bmatrix}$ be its inverse. Then we have

$$\mathbf{p}_*^T\mathbf{p} = \mathbf{I}, \quad \mathbf{p}_*^T\mathbf{q} = 0, \quad \mathbf{q}_*^T\mathbf{p} = 0, \quad \mathbf{q}_*^T\mathbf{q} = \mathbf{I}, \quad (14)$$

which gives

$$\boldsymbol{\xi}_p = \mathbf{p}_*^T \mathbf{x}_P = \mathbf{p}_*^T \mathbf{x}, \quad \boldsymbol{\xi}_q = \mathbf{q}_*^T \mathbf{x}_Q = \mathbf{q}_*^T \mathbf{x}.$$

Substituting equation (13) into system (9)-(10), we obtain :

$$\boldsymbol{\xi}'_p = \mathbf{A}_p \boldsymbol{\xi}_p + \mathbf{B}_p \mathbf{u}, \quad (15a)$$

$$\boldsymbol{\xi}_q = \mathbf{A}_q \boldsymbol{\xi}_p + \mathbf{B}_q \mathbf{u}, \quad (15b)$$

with

$$\mathbf{A}_p := \mathbf{p}_*^T \mathbf{A}_P \mathbf{p} \equiv \mathbf{p}_*^T \mathbf{E}_1^{-1} \mathbf{A} \mathbf{p}, \quad \mathbf{B}_p := \mathbf{p}_*^T \mathbf{B}_P \equiv \mathbf{p}_*^T \mathbf{E}_1^{-1} \mathbf{B},$$

$$\mathbf{A}_q := \mathbf{q}_*^T \mathbf{A}_Q \mathbf{p} \equiv \mathbf{q}_*^T \mathbf{E}_1^{-1} \mathbf{A} \mathbf{p}, \quad \mathbf{B}_q := \mathbf{q}_*^T \mathbf{B}_Q \equiv \mathbf{q}_*^T \mathbf{E}_1^{-1} \mathbf{B},$$

Now the decoupled system (15a)–(15b) has the same dimension as the original DAE system.

For comparison with system (1a), we we can rewrite system (15) in the form:

$$\tilde{\mathbf{E}} \boldsymbol{\xi}' = \tilde{\mathbf{A}} \boldsymbol{\xi} + \tilde{\mathbf{B}} \mathbf{u}, \quad (16)$$

where

$$\tilde{\mathbf{E}} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{A}} := \begin{bmatrix} \mathbf{A}_p & \mathbf{0} \\ \mathbf{A}_q & -\mathbf{I} \end{bmatrix} \in \mathbb{R}^{n,n}, \quad \tilde{\mathbf{B}} := \begin{bmatrix} \mathbf{B}_p \\ \mathbf{B}_q \end{bmatrix} \in \mathbb{R}^{n,m},$$

$$\text{and } \boldsymbol{\xi} := \begin{bmatrix} \boldsymbol{\xi}_p \\ \boldsymbol{\xi}_q \end{bmatrix} \in \mathbb{R}^n.$$

This form reveals the interconnection with the original structure of (1a). In fact, if we use in (16) the identities $\boldsymbol{\xi}_p = \mathbf{p}_*^T \mathbf{x}$, $\boldsymbol{\xi}_q = \mathbf{q}_*^T \mathbf{x}$, and the expressions of \mathbf{B}_p , \mathbf{B}_q , we find:

$$\tilde{\mathbf{E}} \begin{bmatrix} \mathbf{p}_*^T \\ \mathbf{q}_*^T \end{bmatrix} \mathbf{x}' = \tilde{\mathbf{A}} \begin{bmatrix} \mathbf{p}_*^T \\ \mathbf{q}_*^T \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{p}_*^T \\ \mathbf{q}_*^T \end{bmatrix} \mathbf{E}_1^{-1} \mathbf{B} \mathbf{u}.$$

Multiplying from the left by $\mathbf{E}_1 [\mathbf{p} \ \mathbf{q}]$, we obtain

$$\mathbf{E}_1 [\mathbf{p} \ \mathbf{q}] \tilde{\mathbf{E}} \begin{bmatrix} \mathbf{p}_*^T \\ \mathbf{q}_*^T \end{bmatrix} \mathbf{x}' = \mathbf{E}_1 [\mathbf{p} \ \mathbf{q}] \tilde{\mathbf{A}} \begin{bmatrix} \mathbf{p}_*^T \\ \mathbf{q}_*^T \end{bmatrix} \mathbf{x} + \mathbf{B} \mathbf{u},$$

which, by comparison with (1a), leads to the identities

$$(\mathbf{E}, \mathbf{A}) = \mathbf{E}_1 \mathbf{W} (\tilde{\mathbf{E}}, \tilde{\mathbf{A}}) \mathbf{W}^{-1}, \quad \mathbf{B} = \mathbf{E}_1 \mathbf{W} \tilde{\mathbf{B}}, \quad (17)$$

where $\mathbf{W} = [\mathbf{p} \ \mathbf{q}]$. This shows that systems (16) and (1a) are equivalent, although the former is easier to solve than the latter, and their solutions are related by

$$\boldsymbol{\xi} = \begin{bmatrix} \mathbf{p}_*^T \\ \mathbf{q}_*^T \end{bmatrix} \mathbf{x} \iff \mathbf{x} = [\mathbf{p} \ \mathbf{q}] \boldsymbol{\xi}$$

Moreover, identity (17) leads to the following theorem.

Theorem 3.1: *The set of the finite eigenvalues of matrix pencil (\mathbf{E}, \mathbf{A}) is equal to the set of eigenvalues of \mathbf{A}_p , i.e. $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{A}_p)$.*

This theorem can easily be proved using the fact that $\det(\lambda\mathbf{E} - \mathbf{A}) = 0$ if and only if $\det(\lambda\tilde{\mathbf{E}} - \tilde{\mathbf{A}}) = 0$, since the two matrix pencils are related by invertible matrices. Then, it is simple to see that $\det(\lambda\tilde{\mathbf{E}} - \tilde{\mathbf{A}}) = (-1)^{n_a} \det(\lambda\mathbf{I} - \mathbf{A}_p)$, thus the thesis.

This theorem implies that the dimension of the differential part of the DAE system is equal to the length of the spectrum of the finite eigenvalues of the matrix pencil (\mathbf{E}, \mathbf{A}) . This will be advantageous in model reduction since the decoupled system also preserves the stability of the DAE system (1). We note that the number of finite eigenvalues of matrix pencil (\mathbf{E}, \mathbf{A}) is equal to the rank of \mathbf{E} for index 1 systems.

Someone may wonder whether the numerical computation of projectors \mathbf{Q}_0 and \mathbf{P}_0 with their respective basis are feasible in the case of large sparse systems arising in actual real-world applications. Actually, the numerical computation of these projectors is feasible and can be done using the sparse LU decomposition- base routine from [17], called LUQ. This routine decomposes a singular sparse matrix \mathbf{E} , into

$$\mathbf{E}_0^T = \mathbf{L}_0 \begin{bmatrix} \mathbf{U}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{R}_0$$

where $\mathbf{L}_0, \mathbf{R}_0 \in \mathbb{R}^{n \times n}$ are nonsingular matrices, $\mathbf{U}_0 \in \mathbb{R}^{r \times r}$ is a nonsingular upper triangular matrix, where r is the rank of \mathbf{E} . The algorithm of this routine is well discussed in [18]. Using this routine as a starting step and using the fact that the nullspace of \mathbf{E}_0 can be computed via its left nullspace of \mathbf{E}_0^T , in [18], a procedure computing projector \mathbf{Q}_0 onto the nullspace of \mathbf{E}_0 is discussed. This same procedure can be used to compute the basis column matrices \mathbf{q} and \mathbf{p} for projectors \mathbf{Q}_0 and \mathbf{P}_0 efficiently. We note that this approach cannot be used on dense matrices, instead we need to use the singular value decomposition (SVD) based methods.

4. Model order reduction

Model order reduction aims at finding a smaller system which preserves certain properties of the original system. In this section we review one of the most commonly used MOR method, that is, the Arnoldi process. Recalling the procedure outlined in section 2, this method amounts to choosing a subspace \mathcal{V} such that the output of the projected system is close to the output of the original system, and this subspace is chosen by requiring that the first moments of the transfer function be preserved. Then we introduce the index-aware MOR method for an index-1 control problem of the form (1), which reaches the same goal by using a modified approach with exploits the simplification offered by the decomposition in differential and algebraic parts, described in the previous section.

4.1. Traditional MOR

MOR techniques based on Krylov subspace methods aim at generating a reduced system which preserves a given number of moments of the transfer function. This is done by using projection methods, as briefly recalled in section 2. There is a large variety of projection methods, in the following we will restrict ourselves to Arnoldi process.

To define the transfer function of the control problem (1), we take its Laplace transform. After simplifying, we can find an explicit expression for the Laplace transform of the output $\mathbf{y}(t)$, denoted by $\mathbf{Y}(s)$, in terms of the Laplace transform of the input $\mathbf{u}(t)$, denoted by $\mathbf{U}(s)$:

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) + \mathbf{G}(s)\mathbf{x}_0. \quad (18)$$

The matrix function $\mathbf{H}(s) := \mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, which relates the transforms of the input and the output, is called transfer function while the function $\mathbf{G}(s) := \mathbf{C}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}$ relates the output to the initial data. We are interested only in the input-output relation, so we can assume $\mathbf{x}_0 = \mathbf{0}$. This assumption cannot be used for systems with higher tractability index.

The relation (18) is valid for any $s \in \mathbb{C}$ if and only if $s_0\mathbf{E} - \mathbf{A}$ is regular. Let us assume that s_0 is such a real number, which exists if the matrix pencil (\mathbf{E}, \mathbf{A}) is regular. Then, for the transfer function $\mathbf{H}(s)$ we have the identity:

$$\mathbf{H}(s) = \mathbf{C}^T(\mathbf{I} + (s - s_0)\mathbf{M})^{-1}\mathbf{R},$$

where $\mathbf{M} := (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}$, $\mathbf{R} := (s_0\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$. From the above identity, we find the formal expansion

$$\mathbf{H}(s) = \sum_{k=0}^{\infty} (-1)^k \mathbf{C}^T \mathbf{M}^k \mathbf{R} (s - s_0)^k =: \sum_{k=0}^{\infty} \mathbf{h}^{(k)} (s - s_0)^k.$$

This expansion defines the moments $\mathbf{h}^{(k)}$, $k = 0, 1, \dots$ of the transfer function $\mathbf{H}(s)$ around $s = s_0$.

We wish to find a reduction procedure which preserves the first r moments of the transfer function. To do so, we consider the order- r Krylov subspace generated by \mathbf{M} and \mathbf{R} , that is, $\mathcal{K}_r(\mathbf{M}, \mathbf{R}) = \text{span}\{\mathbf{R}, \mathbf{M}\mathbf{R}, \dots, \mathbf{M}^{r-1}\mathbf{R}\}$, $r \leq n$, and denote by $\mathbf{V} \in \mathbb{R}^{n,rm}$ the matrix of an orthonormal basis for \mathcal{K}_r , so that $\mathbf{V}^T\mathbf{V} = \mathbf{I} \in \mathbb{R}^{rm,rm}$. Here we are assuming that \mathbf{R} has maximum rank $m < n$. Then we seek an approximate solution of the form $\mathbf{x} = \mathbf{V}\mathbf{x}_r$. Substituting it into (1) leads to the reduced model (3), with $\mathbf{E}_r = \mathbf{V}^T\mathbf{E}\mathbf{V}$, $\mathbf{A}_r = \mathbf{V}^T\mathbf{A}\mathbf{V}$, $\mathbf{B}_r = \mathbf{V}^T\mathbf{B}$, $\mathbf{C}_r = \mathbf{V}^T\mathbf{C}$. The transfer function for this reduced problem is

$$\mathbf{H}_r(s) = \mathbf{C}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r,$$

and it can be proven that its moments around $s = s_0$ coincide with the moments of $\mathbf{H}(s)$ up to order r and $2r$ using PRIMA and SPRIM methods, respectively, to construct the orthonormal matrix \mathbf{V} [28, 29]. Thus the number of matching moments depends on the way orthonormal matrix \mathbf{V} is constructed even though the theory may be the same.

We have to note that, although in principle this procedure can always be used on DAEs with arbitrary index, provided the matrix pencil (E, A) is regular, obtaining a good matching for the moments of the transfer function, nevertheless the resulting reduced models may be difficult to solve or lead to wrong solutions. For this reason we propose a new method in the next subsection.

4.2. Index-aware MOR

We propose a new method for DAEs which we call the index-aware MOR (IMOR). In this method instead of applying model order reduction on system (1) directly,

we apply it on its decoupled system. Then the conventional methods can be used to reduce the differential part and we develop new techniques to reduce the algebraic parts. Since in this paper, we concentrate on index-1 systems, thus using the decoupled system (8) and the control output equation (1b), we obtain :

$$\boldsymbol{\xi}'_p = \mathbf{A}_p \boldsymbol{\xi}_p + \mathbf{B}_p u, \quad (19a)$$

$$\boldsymbol{\xi}_q = \mathbf{A}_q \boldsymbol{\xi}_p + \mathbf{B}_q u, \quad (19b)$$

$$\mathbf{y} = \mathbf{C}_p^T \boldsymbol{\xi}_p + \mathbf{C}_q^T \boldsymbol{\xi}_q, \quad (19c)$$

where $\mathbf{A}_p \in \mathbb{R}^{n_p, n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p, m}$, $\mathbf{A}_q \in \mathbb{R}^{n_q, n_p}$, $\mathbf{B}_q \in \mathbb{R}^{n_q, m}$ have been defined in the previous section, and $\mathbf{C}_p = p^T C \in \mathbb{R}^{n_p, \ell}$, $\mathbf{C}_q = q^T C \in \mathbb{R}^{n_q, \ell}$.

System (19) can be written in the descriptor form:

$$\tilde{\mathbf{E}} \boldsymbol{\xi}' = \tilde{\mathbf{A}} \boldsymbol{\xi} + \tilde{\mathbf{B}} u, \quad (20a)$$

$$\mathbf{y} = \tilde{\mathbf{C}}^T \boldsymbol{\xi}, \quad (20b)$$

where $\tilde{\mathbf{C}} := [\mathbf{C}_p \ \mathbf{C}_q]^T \in \mathbb{R}^{n, \ell}$ and the rest of the matrices are as defined in equation (16). We note that the output solution \mathbf{y} of system (1) and (20) must coincide although their state space \mathbf{x} and $\boldsymbol{\xi}$ may be different. We use the form (20) only for analysis and comparison. Thus to derive the index-aware MOR , we use the form (19) as follows:

We propose an approach to the decoupled control problem (19) which can lead to a natural generalization in the case of higher order systems. We achieve this by first rewriting (19) in two steps, strictly separating the differential and algebraic variables:

$$\boldsymbol{\xi}'_p = \mathbf{A}_p \boldsymbol{\xi}_p + \mathbf{B}_p u, \quad (21a)$$

$$\mathbf{y}_p = \mathbf{C}_p^T \boldsymbol{\xi}_p, \quad (21b)$$

and

$$\boldsymbol{\xi}_q = \mathbf{A}_q \boldsymbol{\xi}_p + \mathbf{B}_q u, \quad (22a)$$

$$\mathbf{y}_q = \mathbf{C}_q^T \boldsymbol{\xi}_q. \quad (22b)$$

The output equation is reconstructed by taking :

$$\mathbf{y} = \mathbf{y}_p + \mathbf{y}_q. \quad (23)$$

Observe that if $\mathbf{C}_q = 0$ the DAE system can be reduced to a differential equation (21) of dimension n_p even before applying the actual reduction procedure. If $\mathbf{C}_q \neq 0$, we proceed as follows: In this approach, the transfer function is decomposed as the sum of the transfer function of the control problem (21) and a modification due to the algebraic component, computed by means of (22).

The order of the control problem (21) can be reduced by means of any conventional MOR methods. For instance we can apply an Arnoldi process, based on the Krylov subspace

$$\mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p) := \text{span} \{ \mathbf{R}_p, \mathbf{M}_p \mathbf{R}_p, \dots, \mathbf{M}_p^{r-1} \mathbf{R}_p \}, \quad r \leq n_p, \quad (24)$$

where $\mathbf{M}_p := (s_0\mathbf{I} - \mathbf{A}_p)^{-1}$ and $\mathbf{R}_p := (s_0\mathbf{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p$. We denote by $\mathbf{V}_p \in \mathbb{R}^{n_p, rm}$ the matrix of an orthonormal basis for $\mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p)$, so that $\mathbf{V}_p^T \mathbf{V}_p = \mathbf{I} \in \mathbb{R}^{rm, rm}$. Then we seek an approximate solution of the form $\boldsymbol{\xi}_p = \mathbf{V}_p \tilde{\boldsymbol{\xi}}_p$, that is, we replace (21) with

$$\tilde{\boldsymbol{\xi}}_p' = \tilde{\mathbf{A}}_p \tilde{\boldsymbol{\xi}}_p + \tilde{\mathbf{B}}_p \mathbf{u}, \quad (25a)$$

$$\tilde{\mathbf{y}}_p = \tilde{\mathbf{C}}_p^T \tilde{\boldsymbol{\xi}}_p, \quad (25b)$$

where $\tilde{\mathbf{A}}_p := \mathbf{V}_p^T \mathbf{A}_p \mathbf{V}_p$, $\tilde{\mathbf{B}}_p := \mathbf{V}_p^T \mathbf{B}_p$, $\tilde{\mathbf{C}}_p := \mathbf{V}_p^T \mathbf{C}_p$.

The reduction procedure for the differential variables induces a reduction procedure for the algebraic variables. To see this, we perform the algebraic step (22) by using the reduced expression for the differential variable,

$$\boldsymbol{\xi}_q^* = \mathbf{A}_q \mathbf{V}_p \tilde{\boldsymbol{\xi}}_p + \mathbf{B}_q \mathbf{u},$$

where $\boldsymbol{\xi}_q^*$ is the approximation of $\boldsymbol{\xi}_q$ induced by the reduction of $\boldsymbol{\xi}_p$. This relation shows that $\boldsymbol{\xi}_q^*$ lives in the subspace

$$\mathcal{V}_q := \text{span}\{\mathbf{B}_q, \mathbf{A}_q \mathbf{V}_p\} = \text{span}\{\mathbf{B}_q\} + \mathbf{A}_q \mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p). \quad (26)$$

We denote by τ the dimension of \mathcal{V}_q , and by $\mathbf{V}_q \in \mathbb{R}^{n_q, \tau}$ the matrix of an orthonormal basis for \mathcal{V}_q , so that $\mathbf{V}_q^T \mathbf{V}_q = \mathbf{I} \in \mathbb{R}^{\tau, \tau}$. Then we can represent the algebraic solution in the form $\boldsymbol{\xi}_q^* = \mathbf{V}_q \tilde{\boldsymbol{\xi}}_q$, that is, we can replace (22) with

$$\tilde{\boldsymbol{\xi}}_q = \tilde{\mathbf{A}}_q \tilde{\boldsymbol{\xi}}_p + \tilde{\mathbf{B}}_q \mathbf{u}, \quad (27a)$$

$$\tilde{\mathbf{y}}_q = \tilde{\mathbf{C}}_q^T \tilde{\boldsymbol{\xi}}_q, \quad (27b)$$

with $\tilde{\mathbf{A}}_q := \mathbf{V}_q^T \mathbf{A}_q \mathbf{V}_p$, $\tilde{\mathbf{B}}_q := \mathbf{V}_q^T \mathbf{B}_q$, $\tilde{\mathbf{C}}_q := \mathbf{V}_q^T \mathbf{C}_q$.

Thus, combining equation (25) and (27) leads to a reduced model given by:

$$\tilde{\boldsymbol{\xi}}_p' = \tilde{\mathbf{A}}_p \tilde{\boldsymbol{\xi}}_p + \tilde{\mathbf{B}}_p \mathbf{u}, \quad (28a)$$

$$\tilde{\boldsymbol{\xi}}_q = \tilde{\mathbf{A}}_q \tilde{\boldsymbol{\xi}}_p + \tilde{\mathbf{B}}_q \mathbf{u}, \quad (28b)$$

$$\tilde{\mathbf{y}} = \tilde{\mathbf{C}}_p^T \tilde{\boldsymbol{\xi}}_p + \tilde{\mathbf{C}}_q^T \tilde{\boldsymbol{\xi}}_q. \quad (28c)$$

The reduced system (28) can also be written in descriptor form:

$$\tilde{\mathbf{E}}_r \tilde{\boldsymbol{\xi}}_r' = \tilde{\mathbf{A}}_r \tilde{\boldsymbol{\xi}}_r + \tilde{\mathbf{B}}_r \mathbf{u}, \quad (29a)$$

$$\mathbf{y}_r = \tilde{\mathbf{C}}_r^T \tilde{\boldsymbol{\xi}}_r, \quad (29b)$$

where

$$\tilde{\mathbf{E}}_r := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{A}}_r := \begin{bmatrix} \tilde{\mathbf{A}}_p & \mathbf{0} \\ \tilde{\mathbf{A}}_q & -\mathbf{I} \end{bmatrix} \in \mathbb{R}^{rm+\tau, rm+\tau}, \quad \tilde{\mathbf{B}}_r := \begin{bmatrix} \tilde{\mathbf{B}}_p \\ \tilde{\mathbf{B}}_q \end{bmatrix} \in \mathbb{R}^{rm+\tau, m},$$

$$\tilde{\mathbf{C}}_r := \begin{bmatrix} \tilde{\mathbf{C}}_p \\ \tilde{\mathbf{C}}_q \end{bmatrix} \in \mathbb{R}^{rm+\tau, \ell}, \quad \text{and} \quad \tilde{\boldsymbol{\xi}}_r := \begin{bmatrix} \tilde{\boldsymbol{\xi}}_p \\ \tilde{\boldsymbol{\xi}}_q \end{bmatrix} \in \mathbb{R}^{rm+\tau}.$$

Hence system (29) is an index-aware MOR (IMOR) model for system (1). This model will always preserve the index of DAE system. We observe that the reduced model (29) can also be obtained by substituting $\xi = \mathbf{V}\tilde{\xi}_r$, where the block diagonal orthonormal matrix $\mathbf{V} = \begin{bmatrix} \mathbf{V}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_q \end{bmatrix}$ into equation (20). Thus the transfer function of the reduced system can be written as:

$$\tilde{\mathbf{H}}(s) = \tilde{\mathbf{C}}_r^T (s\tilde{\mathbf{E}}_r - \tilde{\mathbf{A}}_r)^{-1} \tilde{\mathbf{B}}_r.$$

We note that the IMOR method always preserves the index of the system (1). This method also preserves stability and passivity of the system (1) if and only if the conventional MOR method used to reduce the differential part preserves these properties.

Remark 1: By construction, we have $rm \leq n_p$, and $\tau \leq \min\{(r+1)m, n_q\}$, so the dimension of IMOR model is less than or equal to $n_p + n_q = n$.

4.3. Comparison between traditional and IMOR method

In this subsection we compare the traditional and the index-aware MOR. First, we want to show that the index-aware MOR preserves the desired moments of the original transfer function. To clarify the application of the MOR technique to the decomposed control problem (21)-(22), we compute the transfer function. Of course, by construction, the final transfer function must coincide with the transfer function of the original problem.

Taking the Laplace transform of the two-step formulation (21)-(22), we obtain:

$$s\Xi_p(s) = \mathbf{A}_p\Xi_p(s) + \mathbf{B}_pU(s), \quad (30a)$$

$$\mathbf{Y}_p(s) = \mathbf{C}_p^T\Xi_p(s), \quad (30b)$$

and

$$\Xi_q(s) = \mathbf{A}_q\Xi_q(s) + \mathbf{B}_qU(s), \quad (31a)$$

$$\mathbf{Y}_q(s) = \mathbf{C}_q^T\Xi_q(s), \quad (31b)$$

where $\Xi_p(s)$, $\Xi_q(s)$ are the Laplace transforms of ξ_p , ξ_q , respectively. Also taking the Laplace transform of equation (23) and substituting equations (30) and (31), we obtain:

$$\mathbf{Y}(s) = \mathbf{Y}_p(s) + \mathbf{Y}_q(s),$$

where $\mathbf{Y}_p(s)$, $\mathbf{Y}_q(s)$ are the Laplace transforms of \mathbf{y}_p , \mathbf{y}_q , respectively. The resulting decomposed input-output relations of the differential and algebraic part are:

$$\mathbf{Y}_p(s) = \mathbf{C}_p^T (s\mathbf{I} - \mathbf{A}_p)^{-1} \mathbf{B}_p U(s),$$

$$\mathbf{Y}_q(s) = \mathbf{C}_q^T [\mathbf{A}_q (s\mathbf{I} - \mathbf{A}_p)^{-1} \mathbf{B}_p + \mathbf{B}_q] U(s),$$

The total transfer function is also decompose as

$$\mathbf{H}(s) = \mathbf{H}_p(s) + \mathbf{H}_q(s), \quad (32)$$

where

$$\begin{aligned}\mathbf{H}_p(s) &:= \mathbf{C}_p^T (s\mathbf{I} - \mathbf{A}_p)^{-1} \mathbf{B}_p, \quad \text{and} \\ \mathbf{H}_q(s) &:= \mathbf{C}_q^T [\mathbf{A}_q (s\mathbf{I} - \mathbf{A}_p)^{-1} \mathbf{B}_p + \mathbf{B}_q],\end{aligned}$$

are the transfer functions of the differential component (21) and algebraic component (22), respectively. We note that this transfer function (32) also coincides with the transfer function obtained using system matrices in equation (20).

After performing the reduction of the order of the differential variable, the Arnoldi process preserves the first $2r$ moments of the differential component of transfer function, $\mathbf{H}_p(s)$. The underlying Krylov subspaces depend only on the matrix pencil $(s\mathbf{I} - \mathbf{A}_p)^{-1} \mathbf{B}_p$, which appears also in the proper part of the algebraic component of the transfer function, $\mathbf{H}_q(s)$. Since the improper part of the transfer function is constant, it is also preserved. Thus, the first $2r$ moments of the original transfer function are preserved. This implies the number of matching moments of the IMOR method depend on the method used to reduce the differential part.

It is interesting to comment on the role of the matrices \mathbf{A}_q and \mathbf{B}_q . The first matrix does not affect directly the reduction procedure, it only “transfers” the reduction from the differential to the algebraic component of the solution. Instead, the matrix \mathbf{B}_q is responsible for the improper part of the total transfer function, so it contains the relevant effect of the algebraic variables in the original control problem.

To compare the Krylov subspaces used in the traditional and index-aware MOR procedure, we use (20). We also use the basis column matrices and their respective inverse, and the matrix chain as defined in Section 3.2. The matrices \mathbf{E} , \mathbf{A} , \mathbf{B} , are related to the matrices $\tilde{\mathbf{E}}$, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ by the identity (17). Then, the matrices \mathbf{M} and \mathbf{R} , used to generate the Krylov subspaces of the traditional MOR in Section 4.1, can then be written as follows:

$$\mathbf{M} = \mathbf{W}(s_0 \tilde{\mathbf{E}} - \tilde{\mathbf{A}})^{-1} \tilde{\mathbf{E}} \mathbf{W}^{-1}, \quad \mathbf{R} = \mathbf{W}(s_0 \tilde{\mathbf{E}} - \tilde{\mathbf{A}})^{-1} \tilde{\mathbf{B}}. \quad (33)$$

Recalling the expression of $\tilde{\mathbf{E}}$, $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, and the definition of the matrices \mathbf{M}_p and \mathbf{R}_p , used to generate the Krylov subspaces of the IMOR method in the previous section, Equation (33) simplifies to:

$$\mathbf{M} = \mathbf{W} \begin{bmatrix} \mathbf{M}_p & 0 \\ \mathbf{A}_q \mathbf{M}_p & 0 \end{bmatrix} \mathbf{W}^{-1}, \quad \mathbf{R} = \mathbf{W} \begin{bmatrix} \mathbf{R}_p \\ \mathbf{A}_q \mathbf{R}_p + \mathbf{B}_q \end{bmatrix}.$$

Recalling that $\mathbf{W} = [\mathbf{p} \ \mathbf{q}]$, we obtain:

$$\mathbf{R} = (\mathbf{p} + \mathbf{q} \mathbf{A}_q) \mathbf{R}_p + \mathbf{q} \mathbf{B}_q, \quad \mathbf{M}^i \mathbf{R} = (\mathbf{p} + \mathbf{q} \mathbf{A}_q) \mathbf{M}_p^i \mathbf{R}_p, \quad i = 1, 2, \dots$$

Thus, we have:

$$\begin{aligned}\mathbf{p}_*^T \mathcal{K}_r(\mathbf{M}, \mathbf{R}) &= \mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p), \\ \mathbf{q}_*^T \mathcal{K}_r(\mathbf{M}, \mathbf{R}) &= \text{span} \{ \mathbf{B}_q + \mathbf{A}_q \mathbf{R}_p, \mathbf{A}_q \mathbf{M}_p \mathbf{R}_p, \dots, \mathbf{A}_q \mathbf{M}_p^{r-1} \mathbf{R}_p \}, \\ &\subset \text{span} \{ \mathbf{B}_q \} + \mathbf{A}_q \mathcal{K}_r(\mathbf{M}_p, \mathbf{R}_p) = \mathcal{V}_q.\end{aligned}$$

The above formulas show the relationship between the Krylov subspaces used in the traditional MOR and IMOR methods. We observe that the algebraic reduction

procedure used in the IMOR has no direct counterpart in the traditional MOR procedure. Hence the two MOR methods do not coincide.

5. Numerical aspects and examples

In this section, we present some examples of index-1 systems using the split approach discussed in Section 3.3. We also show the robustness of the proposed IMOR method over the traditional MOR method.

5.1. Examples on decoupling of index-1 systems

In Example (5.1) and (5.2), we illustrate the decoupling of index-1 DAE systems using projectors as discussed in Section 3.3.

Example 5.1 Consider a DAE system with system matrices :

$$\mathbf{E} = \begin{bmatrix} C_1 + C_2 & -C_1 - C_2 & 0 \\ -C_1 - C_2 & C_1 + C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -G_1 - G_2 & G_2 & 0 \\ G_2 & -G_2 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

This DAE model is derived from a circuit network in Figure 1, using modified nodal analysis. In Figure 1, C_i , $i = 1, 2$, is the capacitance of the corresponding capacitor and $G_i = \frac{1}{R_i}$, $i = 1, 2$, is the resistivity of the corresponding resistor with resistance R_i . The derivation of DAE systems from circuits is beyond the scope of this paper

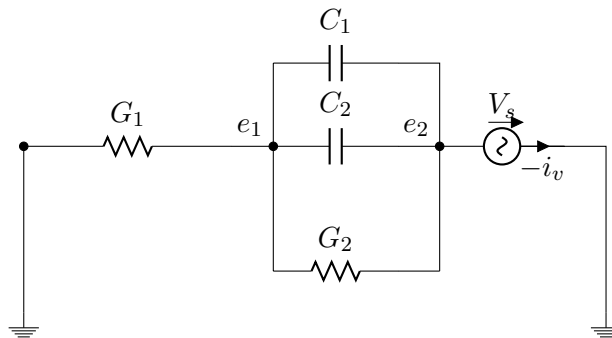


Figure 1. Simple RC circuit network

but we are just interested in the system matrices. We need to find the unknowns $\mathbf{x} = (e_1, e_2, i_v)^T$ given the input function $\mathbf{u} = V_s$. This DAE system is solvable since the polynomial $\det(\lambda\mathbf{E} - \mathbf{A}) = \lambda(C_1 + C_2) + G_1 + G_2$ does not vanish. The system is stable with only one finite eigenvalue given by $\sigma_f(\mathbf{E}, \mathbf{A}) = \{-\frac{G_1 + G_2}{C_1 + C_2}\}$. Thus the differential part of the decoupled system must also have one differential equation and stable.

In order to apply the proposed approach we need to decompose the DAE system into differential and algebraic parts. We set $\mathbf{E}_0 = \mathbf{E}$ and $\mathbf{A}_0 = \mathbf{A}$, then

$$\mathbf{E}_1 = \mathbf{E}_0 - \mathbf{A}_0\mathbf{Q}_0, \quad (34)$$

where \mathbf{Q}_0 is the projector onto $\ker \mathbf{E}_0$ and $\mathbf{P}_0 = \mathbf{I} - \mathbf{Q}_0$. We can choose the

projectors :

$$\mathbf{Q}_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}_0 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using (34) we have,

$$\mathbf{E}_1 = \begin{bmatrix} \frac{2C_1+2C_2+G_1}{2} & \frac{G_1-2C_2-2C_1}{2} & 0 \\ -C_1-C_2 & C_1+C_2 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

We can see that \mathbf{E}_1 is nonsingular and its inverse is given by

$$\mathbf{E}_1^{-1} = \begin{bmatrix} \frac{1}{2(C_1+C_2)} & 0 & \frac{2C_1+2C_2-G_1}{2(C_1+C_2)} \\ -\frac{1}{2(C_1+C_2)} & 0 & \frac{2C_1+2C_2-G_1}{2(C_1+C_2)} \\ -1 & -1 & G_1 \end{bmatrix}.$$

Thus, this is an index-1 system. The bases of projector \mathbf{Q}_0 and \mathbf{P}_0 are given by

$$\mathbf{q} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad (35)$$

and their corresponding inverses are given by

$$\mathbf{q}_*^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{p}_*^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}. \quad (36)$$

The matrices of the decoupled system are

$$\begin{aligned} \mathbf{A}_p &= \mathbf{p}_*^T \mathbf{E}_1^{-1} \mathbf{A} \mathbf{p} = -\frac{G_1+G_2}{C_1+C_2}, & \mathbf{B}_p &= \mathbf{p}_*^T \mathbf{E}_1^{-1} \mathbf{B} = \frac{G_1}{2(C_1+C_2)}, \\ \mathbf{A}_q &= \mathbf{q}_*^T \mathbf{E}_1^{-1} \mathbf{A} \mathbf{p} = \begin{bmatrix} 1 \\ 2G_1 \end{bmatrix}, & \mathbf{B}_q &= \mathbf{q}_*^T \mathbf{E}_1^{-1} \mathbf{B} = \begin{bmatrix} -1 \\ -G_1 \end{bmatrix}. \end{aligned} \quad (37)$$

Thus substituting (37) into (15a)-(15b), we obtain the decoupled system,

$$\xi'_P = -\frac{G_1+G_2}{C_1+C_2} \xi_P + \frac{G_1}{2(C_1+C_2)} \mathbf{u}, \quad (38)$$

$$\xi_Q = \begin{bmatrix} 1 \\ 2G_1 \end{bmatrix} \xi_P - \begin{bmatrix} 1 \\ G_1 \end{bmatrix} \mathbf{u}. \quad (39)$$

We can observe that the decoupled system has one only differential equation and two algebraic equations as expected. We can also see that $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{A}_p)$, thus this example satisfies Theorem 3.1 as expected and the number of differential equations is equal to the number of finite eigenvalues of the matrix pencil (\mathbf{E}, \mathbf{A}) . If we apply initial condition $\xi_P(0) = \mathbf{p}_*^T \mathbf{x}(0)$, where $\mathbf{x}(0)$ is a consistent initial

condition, we obtain the desired solution using the formula :

$$\begin{aligned} \mathbf{x} &= \mathbf{p}\boldsymbol{\xi}_P + \mathbf{q}\boldsymbol{\xi}_Q, \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \boldsymbol{\xi}_P + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\xi}_Q. \end{aligned} \tag{40}$$

Hence the analytic solution of the DAE system can be written as,

$$\mathbf{x} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ i_v \end{bmatrix} = \begin{bmatrix} 2\boldsymbol{\xi}_P - \mathbf{u} \\ -\mathbf{u} \\ 2G_1\boldsymbol{\xi}_P - G_1\mathbf{u} \end{bmatrix}, \tag{41}$$

where

$$\boldsymbol{\xi}_P = e^{-\frac{C_1+C_2}{C_1+C_2}t} \boldsymbol{\xi}_P(0) + \frac{G_1}{2(C_1+C_2)} e^{-\frac{C_1+C_2}{C_1+C_2}t} \int_0^t e^{\frac{C_1+C_2}{C_1+C_2}\tau} \mathbf{u}(\tau) d\tau.$$

Example 5.2 In this example, we consider DAE system with matrices:

$$\mathbf{E} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2k \times 2k}, \quad \mathbf{A} = \mathbf{I} \in \mathbb{R}^{2k \times 2k}, \quad B = \mathbf{1}_{2k} \in \mathbb{R}^{2k}, \tag{42}$$

where $\mathbf{1}$ is a column vector of ones. This system is solvable for all $k \in \mathbb{N}$ since the polynomial $\det(\lambda\mathbf{E} - \mathbf{A}) = (\lambda - 1)^k$ does not vanish. We see that this system is unstable and its matrix pencil (\mathbf{E}, \mathbf{A}) has a finite eigenvalue $\lambda = 1$ with multiplicity k . Thus we expect the differential part of its decoupled system to have at most dimension k and also to be unstable. Introducing $\mathbf{x} = [\mathbf{x}_1 \ \mathbf{x}_2]^T$, the initial condition, $\mathbf{x}(0) = [\mathbf{x}_1(0) \ \mathbf{x}_2(0)]^T$ is consistent if $\mathbf{x}_1(0)$ is chosen arbitrarily, while $\mathbf{x}_2(0)$ has to satisfy $\mathbf{x}_2(0) = -\mathbf{1}_k \mathbf{u}(0)$.

In order to solve this example, we need to first decompose the DAE system into the differential and algebraic equations. Setting $\mathbf{E}_0 = \mathbf{E}$ and $\mathbf{A}_0 = \mathbf{A}$, then

$$\mathbf{E}_1 = \mathbf{E}_0 - \mathbf{A}_0 \mathbf{Q}_0, \tag{43}$$

where \mathbf{Q}_0 is the projector onto $\ker \mathbf{E}_0$ and $\mathbf{P}_0 = \mathbf{I} - \mathbf{Q}_0$. We can choose the projectors :

$$\mathbf{Q}_0 = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{P}_0 = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Using (43) we have,

$$\mathbf{E}_1 = \begin{bmatrix} \mathbf{I} & 2\mathbf{I} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}.$$

We can see that \mathbf{E}_1 is nonsingular and its inverse is given by $\mathbf{E}_1^{-1} = \mathbf{E}_1$. Thus, this system is also an index 1 system. The basis column matrices of projector \mathbf{Q}_0 and \mathbf{P}_0 are given by

$$\mathbf{q} = \begin{bmatrix} -\mathbf{I} \\ \mathbf{I} \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}. \tag{44}$$

and the corresponding inverses are given by

$$\mathbf{q}_*^T = [\mathbf{0} \ \mathbf{I}], \quad \mathbf{p}_*^T = [\mathbf{I} \ \mathbf{I}]. \quad (45)$$

Then, the matrices of the decoupled system are given by:

$$\begin{aligned} \mathbf{A}_p &= \mathbf{p}_*^T \mathbf{E}_1^{-1} \mathbf{A} \mathbf{p} = \mathbf{I}, & \mathbf{A}_q &= \mathbf{q}_*^T \mathbf{E}_1^{-1} \mathbf{A} \mathbf{p} = \mathbf{0}, \\ \mathbf{B}_p &= \mathbf{p}_*^T \mathbf{E}_1^{-1} \mathbf{B} = 2\mathbf{1}_k, & \mathbf{B}_q &= \mathbf{q}_*^T \mathbf{E}_1^{-1} \mathbf{B} = -\mathbf{1}_k. \end{aligned} \quad (46)$$

Thus substituting (46) into (15a)-(15b), we obtain the decoupled system :

$$\xi_P' = \xi_P + 2\mathbf{1}_k \mathbf{u}, \quad (47)$$

$$\xi_Q = -\mathbf{1}_k \mathbf{u}. \quad (48)$$

We can see that $\sigma_f(\mathbf{E}, \mathbf{A}) = \sigma(\mathbf{A}_p)$, thus this example satisfies Theorem 3.1 and the number of differential equations is equal to the number of finite eigenvalues of the matrix pencil (\mathbf{E}, \mathbf{A}) . Regarding (47), the analytic solution of the differential part is given by

$$\xi_P(t) = e^t \xi_P(0) + e^t 2\mathbf{1}_k \int_0^t e^{-\tau} \mathbf{u}(\tau) d\tau.$$

Thus, using the formula $x = p\xi_p + q\xi_q$, we obtain the desired solution of the DAE system (42) given by,

$$\mathbf{x} = \begin{bmatrix} e^t \xi_P(0) + \mathbf{1}_k \mathbf{u} + e^t 2\mathbf{1}_k \int_0^t e^{-\tau} \mathbf{u}(\tau) d\tau \\ -\mathbf{1}_k \mathbf{u} \end{bmatrix}, \quad (49)$$

where $\xi_P(0) = \mathbf{p}_*^T x(0) = \mathbf{x}_1(0) - \mathbf{1}_k \mathbf{u}(0)$.

Hence, we can solve both small and large index 1 systems by first decoupling the DAE system into differential and algebraic equations using projectors.

5.2. MOR examples

In this section, we compare the traditional MOR method with our IMOR method numerically. There exist many traditional MOR techniques but we shall restrict ourselves on the PRIMA method. The PRIMA method uses the matrices $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ of the dynamical system (1) and then obtain reduced model $(\mathbf{E}_r, \mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$ called the PRIMA models, while the IMOR method reduces the matrices $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ indirectly by reducing the decoupled system instead. As a result we obtain reduced model $(\tilde{\mathbf{E}}_r, \tilde{\mathbf{A}}_r, \tilde{\mathbf{B}}_r, \tilde{\mathbf{C}}_r)$, which we call the IMOR model. We have to note that, we used the PRIMA method to reduce the differential part of the decoupled system but also other methods can be used. The comparison between PRIMA models and index aware models will be done using both small and large scale examples below. For comparison purposes we reduce the system to the same dimension.

5.2.1. Small example

Example 5.3 Consider an index-1 Linear time invariant (LTI) dynamical system of dimension 5 with system matrices shown below:

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 \\ \frac{1}{3} & -\frac{5}{6} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B} = [0 \ 0 \ 0 \ 0 \ -1]^T, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T. \quad (50)$$

This DAE system is solvable since the matrix pencil $\lambda\mathbf{E} - \mathbf{A}$ is regular, i.e. $\det(\lambda\mathbf{E} - \mathbf{A}) = \lambda \frac{(5\lambda+3)}{12} \neq 0$ does not vanish identically. This system is stable with finite eigenvalues $\sigma_f(\mathbf{E}, \mathbf{A}) = \{0, -\frac{3}{5}\}$, thus we expect to have a differential part of dimension 2. Its transfer function is given by:

$$\mathbf{H}(s) = \mathbf{C}^T (s\mathbf{E} - \mathbf{A})^{-1} \mathbf{B} = \frac{1}{5s+3} \begin{bmatrix} 2s+3 \\ 3 \end{bmatrix}. \quad (51)$$

Then, we decouple the DAE system into differential and algebraic parts:

$$\boldsymbol{\xi}'_p = \begin{bmatrix} -0.6 & 0 \\ -0.2 & 0 \end{bmatrix} \boldsymbol{\xi}_p + \begin{bmatrix} -0.6 \\ -0.2 \end{bmatrix} \mathbf{u}, \quad (52a)$$

$$\boldsymbol{\xi}_q = \begin{bmatrix} 0 & 0 \\ 0.6 & 0 \\ -0.2 & 0 \end{bmatrix} \boldsymbol{\xi}_p + \begin{bmatrix} -1 \\ -0.4 \\ -0.2 \end{bmatrix} \mathbf{u}, \quad (52b)$$

$$\mathbf{y} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \boldsymbol{\xi}_p + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\xi}_q. \quad (52c)$$

We observe that $\sigma(\mathbf{A}_p) = \sigma_f(\mathbf{E}, \mathbf{A})$ as expected. We can also see that the number of differential equations is $n_p = 2$ and the number of algebraic equations is $n_q = 3$. Thus, the total dimension of the decoupled system is also 5. Using the formula (32) the transfer function of the decoupled system (52) can be decomposed as:

$$\mathbf{H}(s) = \frac{1}{5s+3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \frac{1}{5s+3} \begin{bmatrix} 2s+3 \\ 0 \end{bmatrix}, \quad (53)$$

where $\mathbf{H}_p(s) = \frac{1}{5s+3} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $\mathbf{H}_q(s) = \frac{1}{5s+3} \begin{bmatrix} 2s+3 \\ 0 \end{bmatrix}$ are the transfer functions of the differential and algebraic parts respectively. We can see that that transfer functions (51) and (53) coincide. For comparison system (52) can be written in

descriptor form:

$$\begin{aligned} \tilde{\mathbf{E}} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{A}} = \begin{bmatrix} -0.6 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0.6 & 0 & 0 & -1 & 0 \\ -0.2 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} -0.6 \\ -0.2 \\ -1 \\ -0.4 \\ -0.2 \end{bmatrix} \\ \tilde{\mathbf{C}} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \end{aligned} \quad (54)$$

Then, we need to compare the PRIMA method with the IMOR method. We choose $s_0 = 1$ as the expansion point for both methods. For PRIMA method, we obtained the following reduced model of dimension 3:

$$\begin{aligned} \mathbf{E}_r &= \begin{bmatrix} 0.03 & 0.076596 & 0.012745 \\ 0.076596 & 0.19556 & 0.032541 \\ 0.012745 & 0.032541 & 1.4539 \end{bmatrix}, \quad \mathbf{A}_r = \begin{bmatrix} -0.05 & -0.027534 & -0.12308 \\ 0.27284 & -0.30088 & 0.023584 \\ 0.16101 & -0.11662 & -0.039381 \end{bmatrix}, \\ \mathbf{B}_r &= \begin{bmatrix} 0.1 \\ -0.24531 \\ -0.18533 \end{bmatrix}, \quad \mathbf{C}_r = \begin{bmatrix} 0.5 & 0.3 \\ 0.27534 & 0.76596 \\ 0.24813 & 0.12745 \end{bmatrix}. \end{aligned} \quad (55)$$

Next, we construct a reduced model using the index aware MOR described in section 4.2. If we again choose $s_0 = 1$ as the expansion point and using definition (24), we obtain the orthonormal matrix to reduce the differential part given by:

$$\mathbf{V}_p = \begin{bmatrix} -0.94868 \\ -0.31623 \end{bmatrix}. \quad (56)$$

Then, we use (56) to compute the orthonormal matrix \mathbf{V}_q for the algebraic part by first computing the column matrix \mathcal{V}_q in (26) and then its orthonormal basis which is given by :

$$\mathbf{V}_q = \begin{bmatrix} -0.87758 & 0.37274 \\ -0.45863 & -0.83591 \\ -0.13965 & 0.40288 \end{bmatrix}. \quad (57)$$

Using (56) and (57) we build a block diagonal orthonormal basis matrix given by:

$$\mathbf{V} = \begin{bmatrix} -0.94868 & 0 & 0 \\ -0.31623 & 0 & 0 \\ 0 & -0.87758 & 0.37274 \\ 0 & -0.45863 & -0.83591 \\ 0 & -0.13965 & 0.40288 \end{bmatrix} \quad (58)$$

Applying this column matrix on matrices (54), we obtain the IMOR model of 3

dimension given by:

$$\begin{aligned} \tilde{\mathbf{E}}_r &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{A}}_r = \begin{bmatrix} -0.6 & 0 & 0 \\ 0.23456 & -1 & 0 \\ 0.55225 & 0 & -1 \end{bmatrix}, \quad \tilde{\mathbf{B}}_r = \begin{bmatrix} 0.63246 \\ 1.089 \\ -0.11895 \end{bmatrix}, \\ \tilde{\mathbf{C}}_r &= \begin{bmatrix} 0 & -0.94868 \\ -0.45863 & 0 \\ -0.83591 & 0 \end{bmatrix} \end{aligned} \quad (59)$$

Thus, (55) and (59) are the reduced models of system (50) using the PRIMA and IMOR methods respectively. We observe that the PRIMA model is an ODE while IMOR is a DAE. Thus IMOR preserves the index of the DAE system while PRIMA model does not. We can also observe that the PRIMA model matrices are full while IMOR model matrices are sparse. In Figure 2, we compare the magnitude of the transfer function of the reduced models with the original model. We observe that both transfer functions coincides with that of the original model. In Figure 3, we

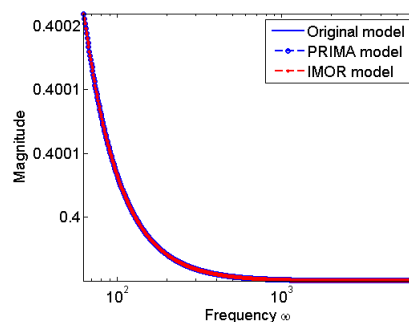


Figure 2. Frequency response

compare the output solutions of the reduced models. We observe that both models lead to good solutions, although IMOR is more accurate since it has a smaller approximation error that of PRIMA model as shown in Figure 4.

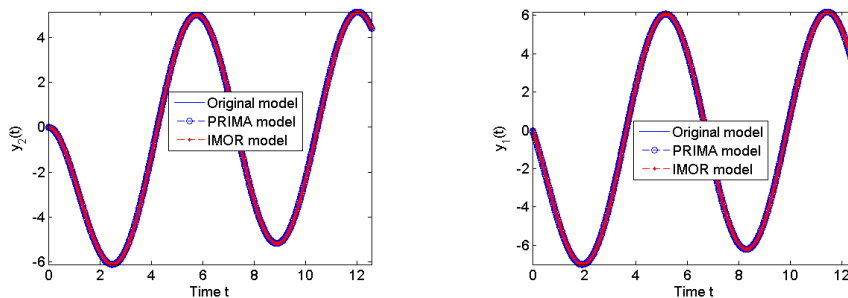


Figure 3. Comparison of the output Solutions, $\mathbf{u}(t) = 10 \sin(t)$, $t \in [0, 2\pi]$

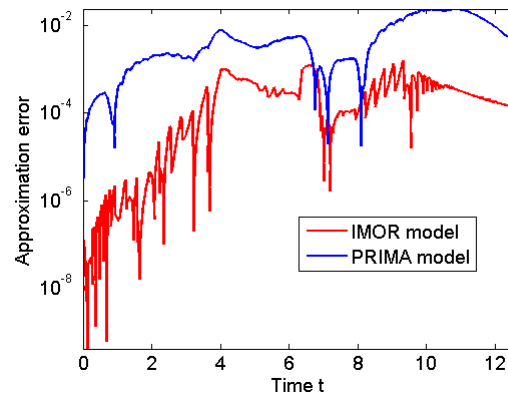
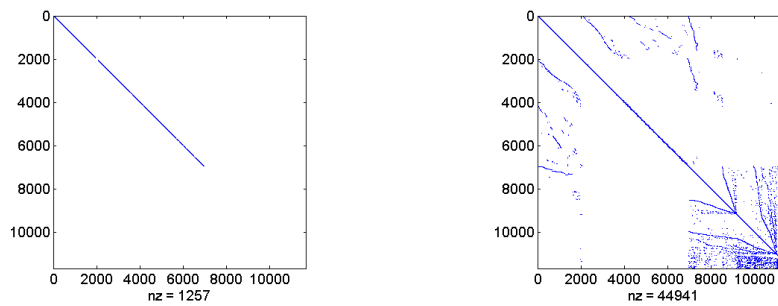


Figure 4. Approximation error

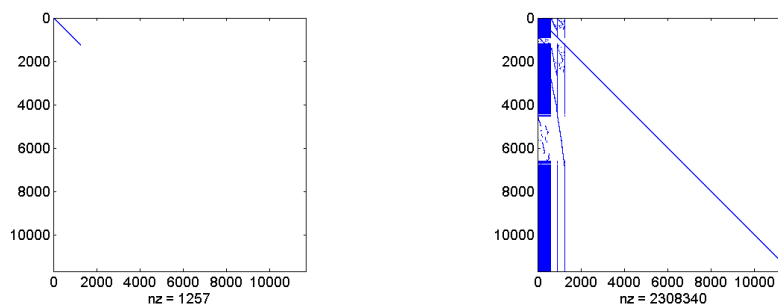
5.2.2. Industrial examples

These benchmark examples below are index 1 systems obtained from Rommes's webpage [14]. These examples are large power system models.

Example 5.4 We consider a power system called nopss, which is a single input-single output (SISO) dynamical system of dimension 11685. The sparsity of its matrix pencil is shown in the Figure 5. We decoupled this DAE system into 1257

Figure 5. Sparsity of matrix pencil (\mathbf{E}, \mathbf{A})

differential equations and 10428 algebraic equations, using the modified decomposition procedure based on projectors. The sparsity of the matrix pencil of the decoupled system is shown in Figure 6. Using $s_0 = 0$ as the expansion point, we

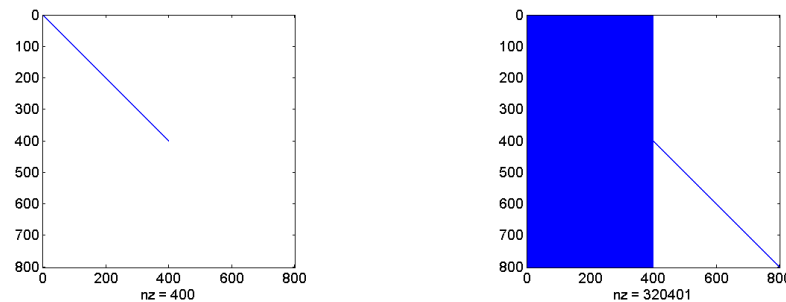
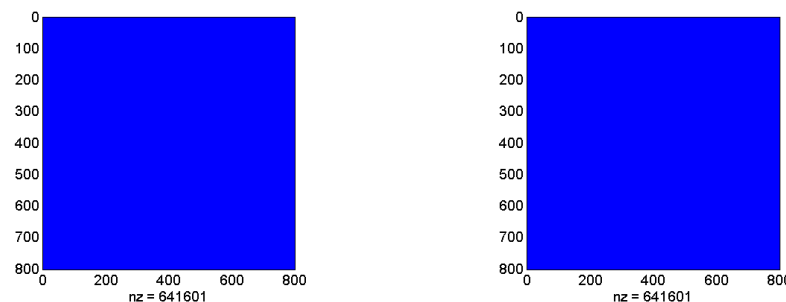
Figure 6. Sparsity of matrix pencil $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$

reduced the DAE system to a total dimension of 801 using the IMOR method. The dimension of the differential and algebraic equations in the reduced model is shown in Table 1. The IMOR model is sparse and preserves the index of the DAE

system as shown in Figure 7. For comparison, we used the PRIMA method and we reduced the DAE system to order 801. The sparsity of matrix pencil of the PRIMA model is shown in Figure 8. We observe that the PRIMA model are very dense and it is an ODE thus does not preserve the index of the DAE system.

Table 1. Dimension of IMOR model

Decoupled system		Reduced Model	
# differential equations	# algebraic equations	# differential equations	# algebraic equations
1257	10428	400	401

Figure 7. Sparsity of matrix pencil $(\tilde{\mathbf{E}}_r, \tilde{\mathbf{A}}_r)$ of IMOR modelFigure 8. Sparsity of matrix pencil $(\mathbf{E}_r, \mathbf{A}_r)$ of PRIMA model

In Figure 9, we compare the magnitude of the transfer functions and their numerical error for the two MOR methods. We can observe that the magnitude of the transfer function coincides with that of the reduced models in both approaches with very small approximation error as shown in Figure 9. In Figure 10, we compare the output solutions of both reduced models with the original model and their respective approximation error. We see that the both methods lead to good solution with small approximation error. In Table 2 we compare the computational cost of solving the original and reduced models. We carried out the experiments using the implicit method implementation in Matlab software to solve the systems at a fixed relative tolerance. We observed that the solver failed to solve the DAE system while it was able to solve the decoupled system in few seconds as shown in Table 2. We can also observe that the IMOR model is cheaper to solve than PRIMA model.

Table 2. Computational time (in seconds)

Relative tolerance	Original Model		Reduced Model	
	DAE system	Decoupled system	PRIMA model	IMOR model
$RelTol = 10^{-6}$	–	26	24	3
$RelTol = 10^{-3}$	–	17	32	1.5

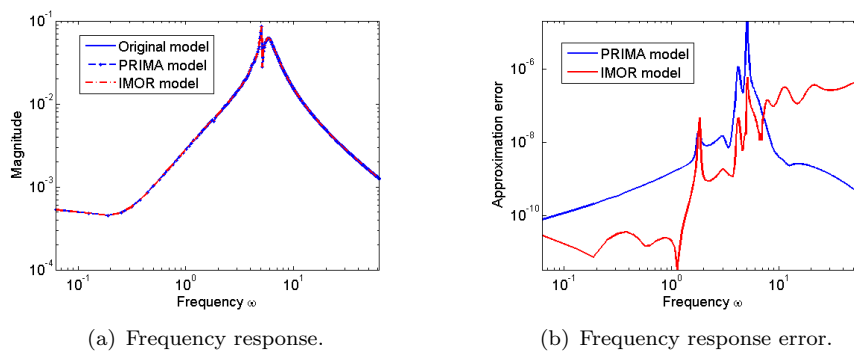
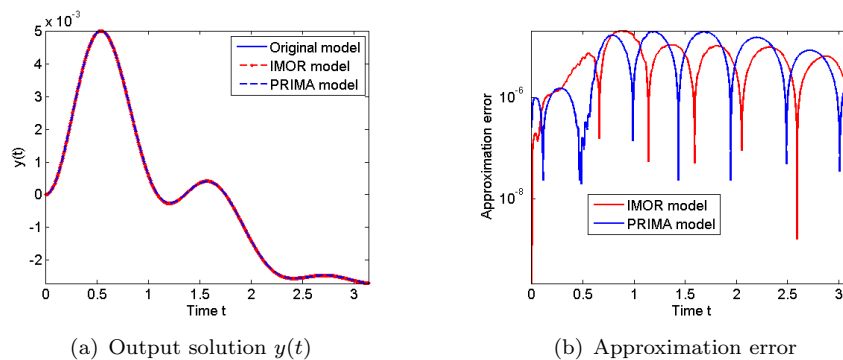
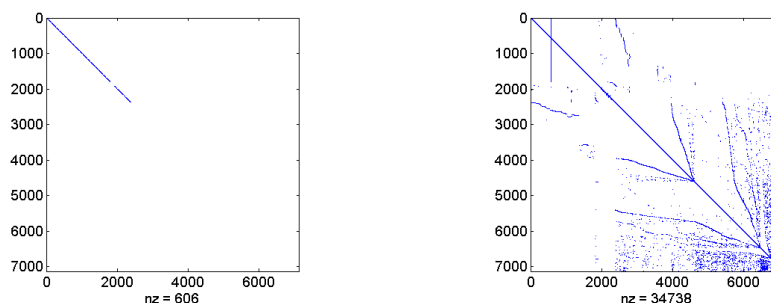


Figure 9. Frequency response and its error

Figure 10. Output Solutions, input function $\mathbf{u}(t) = \sin(t)$, $t \in [0, \pi]$

Example 5.5 We consider another power system called bips98_606, which is a multiple input-multiple output (MIMO) dynamical system with 4 inputs and 4 outputs. This system is of dimension 7135 and the sparsity of its matrix pencil (\mathbf{E}, \mathbf{A}) is shown in the Figure 11. We decoupled this system into 606 and 6529

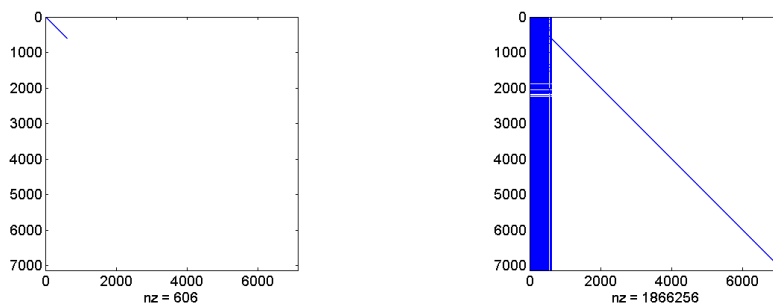
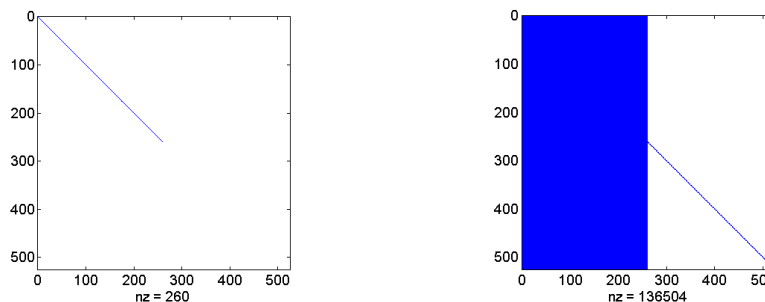
Figure 11. Sparsity of matrix pencil (\mathbf{E}, \mathbf{A})

differential and algebraic equations, respectively. The sparsity of the matrix pencil of the decoupled system in the descriptor form is shown in Figure 12.

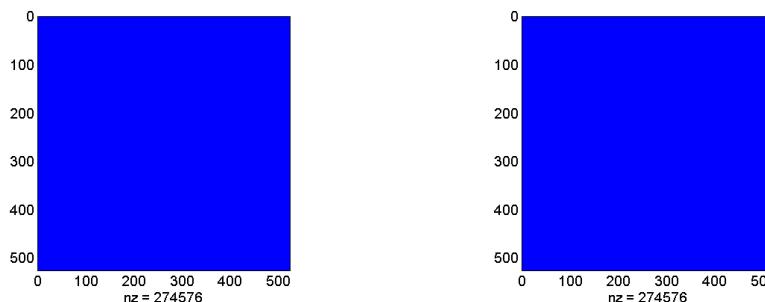
Using the same expansion point as in the previous example, we reduced the DAE system to total dimension 524. The sparsity of the matrix pencil of the IMOR model is shown in Figure 13. The decoupled system was used to 260 and 264 differential and algebraic equations as shown in Table 3. Also for comparison, we reduced the

Table 3. Dimension of IMOR model

Decoupled system		Reduced Model	
# differential equations	# algebraic equations	# differential equations	# algebraic equations
606	6529	260	264

Figure 12. Sparsity of matrix pencil $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$ Figure 13. Sparsity of matrix pencil $(\tilde{\mathbf{E}}_r, \tilde{\mathbf{A}}_r)$ of IMOR model

system to dimension 524 using the PRIMA method. We observed that the PRIMA model is an ODE and it has very dense matrices as shown Figure 14. In Figure 15,

Figure 14. Sparsity of matrix pencil $(\mathbf{E}_r, \mathbf{A}_r)$ of PRIMA model

we compare the magnitude of the transfer functions of reduced models with that of the original model. We observe that IMOR model has more accurate transfer function in the lower frequencies compared to the PRIMA model as shown in Figure 15(b). We solved both reduced models, using $\mathbf{u}(t) = [\sin(t), \sin(t), \sin(t), \sin(t)]^T$ as input function. The PRIMA model fails to give a solution, while the IMOR model leads to good solutions as shown in Figure 16. Figure 17 shows the approximation error of the output solution.

In Table 4, we repeated the same experiments as in the previous example. We observed that also for this example the solver failed to solve the DAE system. We were able to solve the decoupled system in few seconds as shown in the Table 4.

Table 4. Computational time (in seconds)

Relative tolerance	Original Model		Reduced Model	
	DAE system	Decoupled system	PRIMA model	Index-aware model
$RelTol = 10^{-6}$	–	22	–	1.8
$RelTol = 10^{-3}$	–	14	–	1.3

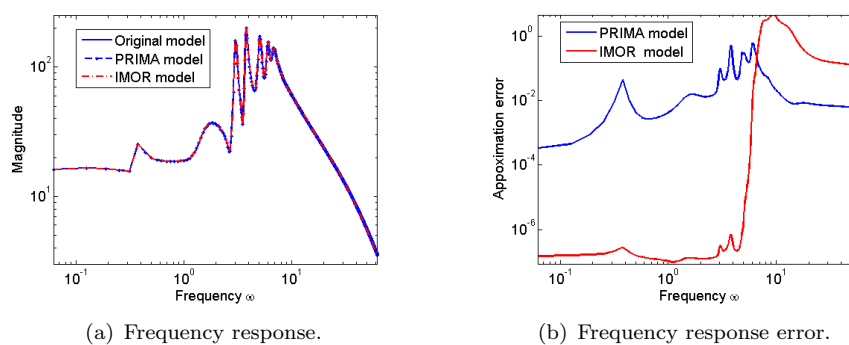


Figure 15. Frequency response and its error.

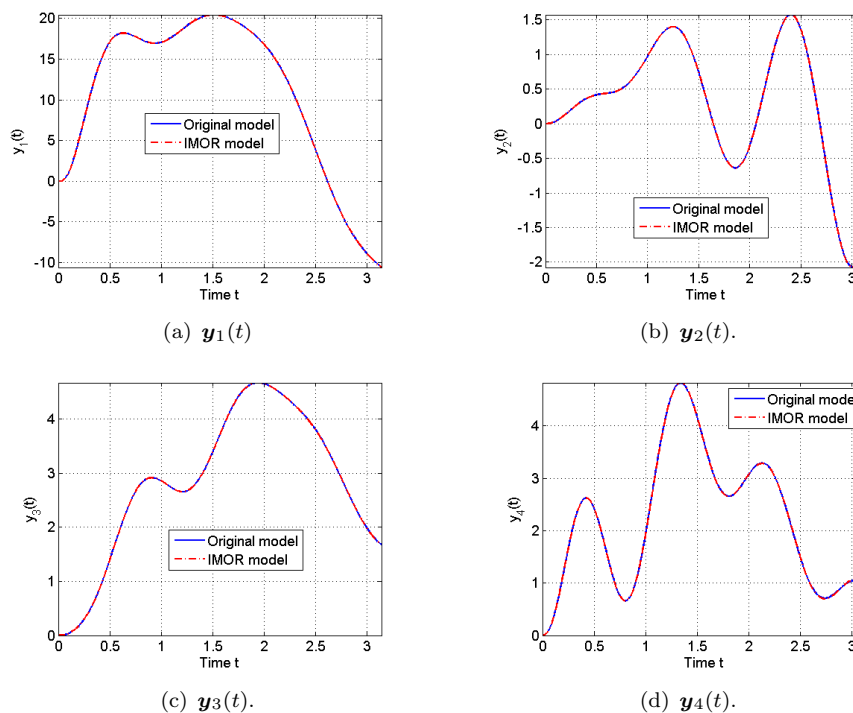


Figure 16. Output solutions of the index-aware model and original model.

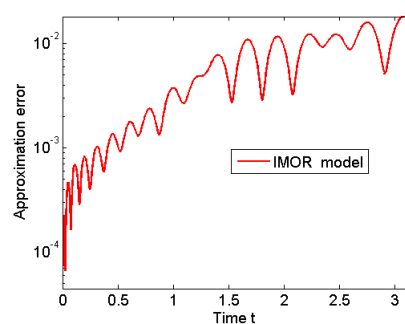


Figure 17. Approximation error

From the above experiments, we can conclude that the IMOR method is a better method than the traditional MOR for reducing DAE systems and can be used on both SISO and MIMO dynamical systems. We have observed that this method leads to sparse and simple reduced models which are easier to solve than the

PRIMA method. By construction, the IMOR method preserves the index of the DAE system. In this paper, we have restricted ourselves to Krylov methods but this method can also be extended to the non-Krylov methods.

6. Conclusion

We have proposed a new MOR method for index-1 DAE, which is based on explicitly splitting into differential and algebraic systems. The method has the attractive property that it preserves the tractability index of the original DAE. Another interesting feature of the method is a reduction of the algebraic variables which is induced by the reduction of the order of the inherent differential system. Moreover, conventional methods, like PRIMA, may lead to reduced models which are difficult to solve numerically, as shown in example 5.5, while reduced models obtained by the IMOR method do not present numerical difficulties.

In real-life problems, our method allows for a more pronounced reduction of the system than with conventional methods, still maintaining very good accuracy of the solution. We also note that the comparison shows a lower frequency response error of our method in the relevant range of frequency, near the frequency used for the moments of the transfer function. Although our method requires the inversion of an order- n matrix, in practical examples, with sparse matrices, this is not an inconvenience. Finally, our method can be extended to systems with higher tractability index. This will be the topic of a forthcoming paper.

Acknowledgements

This work was funded by The Netherlands Organisation for Scientific Research (NWO).

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