Controllability analysis of industrial processes : towards the industrial application

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Controllability Analysis of Industrial Processes.

Towards the industrial application.

Jobert Ludlage

Controllability Analysis of Industrial Processes

Towards the industrial application

PROEFSCHRIFT

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Ter nagedachtenis aan
mijn moeder
en haar onvergetelijke bijdrage.

Voorwoord.

Het is misschien opmerkelijk maar bij het schrijven van dit stukje bekruipt mij een gevoel dat het laatste jaar redelijk zeldzaam was. Ik voel mij dus anders dan anders:

'Ik ben vandaag zo vrolijk, zo vrolijk, zo vrolijk' [1]

Het is voor het eerst dat ik het idee krijg dat *'de moeilijke bevalling'* [Heu91] er bijna opzit. Het afgelopen jaar heb ik mij vele malen afgevraagd wat mij toch bezield kan hebben om het dan toch maar op papier te zetten. Het onderzoek was leuk en ja zelfs het eerste gedeelte van het schrijfwerk was in vele opzichten verhelderend; er onstond een consistent beeld. Het laatste stuk van het tikken kon mij echter duidelijk niet echt bekoren. In deze periode borrelden geregeld vragen op als: *Hoe heb ik mij nu weer tot zoiets kunnen laten verleiden?, Waar doe ik dit nu voor?* en *'Is het dit allemaal wel waard?* Wat ik zo hoor is dit normaal, echt Jan modaal dus. Naarmate de tijd verstrijkt zal het antwoord op deze vragen daarom wel langzaam verschuiven van het zuiver gevoelsmatige maar zeer duidelijke antwoord dat samen met elke vraag opwelde.

Nu kan ik wel klagen, maar het is wonderbaarlijk hoe makkelijk het gezin met de escapade van pa kon om gaan. Het feit dat papa in korte tijd transformeerde van speelmaatje naar workaholic en boekenworm scheen mijn mannen minder van hun stuk te brengen dan papa zelf:

Papa werd namelijk doctor[2]

Ja en dat verklaart natuurlijk alles. Maar mannen we gaan er iets aan doen. Het zal wel wat minder abrupt gaan als de heen transformatie, maar papa is vast van plan om de terug transformatie[3] in te zetten en weer lekker met jullie te gaan spelen. En dan onze mama, die had het ook niet altijd even makkelijk. Haar man had een maitresse. Hij scheen duidelijk meer weg te zijn van zijn PC dan van haar en de kinderen. "Maar chefke het is mij in deze tijd wel duidelijk geworden dat een tweede aspect van mijn vroegere vrijetijdsbesteding nooit meer geheel terug zal keren"[4]. Mijn *chefke* heeft duidelijk haar draai gevonden in de tuin. Ik zal duidelijk hierin de tweede viool gaan spelen, maar chef hij heeft er nog nooit zo netjes bij gelegen. Willy, Theo, Hein en Ben bedankt voor het getoonde begrip en het enorme geduld.

Mijn hockeymaatjes mag ik hier niet vergeten. Hockey betekende even achter mijn PC vandaan. Het was voor mij met name in het laatste jaar enorm ontspannend, ondanks dat het spelletje zelf nogal eens een keer aan mij voorbij ging. Bedankt jongens.

Voor degenen die het nog niet wisten: Ik was een ster in het uitstellen van de laatste fase van het promotieonderzoek. Argumenten te over: Er moest verhuisd worden, verbouwd worden. Natuurlijk waren er ook nog de nieuwe vragen en andere onderwerpen, die eigenlijk iets verder verdiept dienden te worden, want dan ..... Eigenlijk van alles behalve schrijven uiteraard, daar was net geen tijd voor. Kortom ik zag er gewoon tegen op, als een berg zo gezegd. Mijn promotor, baas en hobby maatje T.B. was echter niet van zijn stuk te brengen: Ik zou gaan promoveren. Met wat er lag was het immers zonde om het te laten verzanden. Bovenal het was dan iets meer werk dan de van hem zo bekende *regenachtige zondagmiddag*, maar toch het was al bijna gepiept. Ton, je hebt de wonderbaarlijke eigenschap niet los te laten, maar tegelijkertijd inspirerend te blijven zonder ooit dwingend te worden. Het is dan ook zeker voor een groot gedeelte aan jou te danken dat het er alle schijn van heeft dat je weer eens gelijk gaat krijgen. Ik ben jou maar ook Lianne en je kinderen, die jou toch al zoveel moeten missen, uiterst dankbaar.

Twee mensen die ik uitdrukkelijk persoonlijk wil bedanken voor alles wat zij gedaan hebben zijn mijn ouders. Pa in vele opzichten ben en blijf je mijn grote voorbeeld. Je zult het echter zeker met mij eens zijn dat ik dit proefschrift opdraag aan de nagedachtenis van "ons moeder" en de onvergetelijke bijdrage die zij aan de tot standkoming van dit proefschrift heeft geleverd.

Een van de consequenties van het lange onderzoek is dat het beinvloed wordt door vele mensen. In het bijzonder ben ik Anton Stoorvogel en Siep Weiland erkentelijk voor de bijdrage die zij geleverd hebben aan dit onderzoek. De samenwerking was niet alleen gezellig, maar ook uitermate waardevol. Het proefschrift had er zeker anders uitgezien zonder deze *multidisciplinaire* samenwerking. We vullen elkaar prima aan. Laten we dit vooral voortzetten, bijvoorbeeld onder het motto[5]:

Mathematen en Vonkentrekkers ontwikkelen het samen.

Om niet te langdradig te worden en het risico te lopen iemand over het hoofd te zien wil ik het verder beperken tot een wat onpersoonlijk, maar welgemeend dankwoord:

Hartstikke bedankt.

Jobert.

[1] Herman van Veen: Vrolijk

[2] Citaat Ben Ludlage ter ere van vele gelegenheden geuit.

[3] Wiskundige zullen er opwijzen dat het een wiskundig gegeven is dat de inverse transformatie slechts onder zeer strikte voorwaarden kan bestaan. Heren zij gerust. Hier voldoe ik per definitie aan.

[4] Ja, ja, Anton en Siep, ik weet het wel. Jullie hebben enkele regels later toch gelijk. De wiskunde laat zich niet manipuleren.

[5] Nu we het er toch over hebben: Hoe zit het nu met de verstoringspredictor?

Abstract.

The basic motivation for the work in this thesis was the recognition of the discrepancy between at one hand the availability of very sophisticated controllers and at the other hand the lack in knowledge on how to actual tune these controllers for a given control problem. This discrepancy was found to result in a trial and error approach of the actual design. Certainly for more complex control problems. A more structured approach for the design of the controller is needed. This observation resulted in a closer analysis of the design approach and the information that is needed during the design of an actual problem. It turned out that a missing link during the design is a lack of understanding on the relation between the open loop process behavior and how it restricts the closed behavior[1]. For design it is fundamental to understand how the specific process limits us to achieve a certain required closed loop behavior. In combination with the observation that the model represents the knowledge we have of the process, this resulted in the initial problem statement for this thesis:

> *Develop basic analysis techniques that enable detailed insight in the limitations that stable process behavior puts on the closed loop behavior, without the need of a detailed controller design, based on a given model of the open loop process and the desired behavior of the closed loop.*

After an inventarization of the needs of this type of tool it turns out that input-output controllability is of a much broader use than for control design. Input-output controllability analysis is useful during process design[2], controller design and process operation.

A major difficulty in controller design is the fact that the control design problem is a chicken and egg problem:

[1] The fact that the process behavior itself limits the ability to control a process is not new, e.g. [Zie43, Ros70].

[2] Process design has historically been the major field of application for these analysis techniques.

In order to know the achievable performance we need the controller achieving it. However to design the controller we need to specify the achievable performance.

The goal of input-output controllability analysis can therefore be seen as breaking this loop. The tight relation between controller and closed loop behavior is a fundamental property of closed loop control. From a very strict theoretical point of view the input-output controllability problem seems therefore unsolvable. Moreover input-output controllability is necessarily in itself also a control design approach. The main difference with an ordinary control design technique is that the emphasis is not on obtaining the controller that achieves a certain behavior, but on understanding how the process behavior restricts us in achieving the closed loop requirements.

In order to obtain insight in the input-output controllability of the process we need a factorization of the controller, which results in a direct relation between the process behavior and the closed loop behavior. Morari, e.g. [Mor89] used the IMC scheme as the bridge that relates feedback control to feedforward control. It enables us to translate a feedback control problem to a feedforward control problem. Making use of this relation enables us to interpret the controller as a specific approximate inverse of the process. Hence we obtained an interpretation of the controller that is closely related to the process behavior. Moreover it has a simple linear relation to the requirements posed on the process inputs and outputs. The translation to a feedforward control problem therefore significantly simplifies the relation between open loop and closed loop process behavior.

It turns out that for the input-output analysis additional simplification and standardization is needed to enable us to keep insight in the problem. The equivalence between input-output controllability analysis and controller design makes that there is an inevitable trade-off between the complexity of the approach and the accuracy. What a good compromise is will depend on the purpose of the analysis and the accuracy of the model, i.e. the knowledge we have on the process behavior. In the final design of the controller clearly a more accurate analysis is needed than during the process design phase, where only global insight in the controllability is needed. Moreover in the initial phase of identification, the knowledge we have of the process is still far from accurate and a detailed analysis is not yet possible.

In chapter four we discuss an approach that is based on the basic idea as proposed by Morari [Mor89]. He proposes to split-up the overall problem in essentially two subproblems, using an inner outer factorization. The sensitivity function of the inner transfer matrix represents a measure for the limitation that the non-minimum phase zeros put on the controllability. The principal gains of the outer are used to further analyze the gain behavior of the process. For control design the approach results in a too inaccurate insight[3]. Two major problems were identified. First of all it is difficult to relate the results of the analysis of the subproblems to accurate insight in the limitations the outcome has for the overall problem. The second disadvantage is that the approach is not revealing any detailed information. It is impossible in the approach to obtain detailed insight in the trade-off between the different requirements as posed on the process outputs and inputs. A good understanding of the directional behavior of multivariable systems is therefore a prerequisite. We therefore develop an approach that enables detailed insight in the fundamental mechanisms that govern the algebraic trade-off. In section 4.4 an approach is developed that enables us to manipulate the direction of the influence that non-minimum phase zeros have on the closed loop behavior. This approach clearly reveals the consequences it has for the overall closed loop behavior. In section 4.5 we deal with manipulation of the directionality of the process gains. A basic insight is obtained in the closed loop behavior of ill-conditioned problems and the mechanisms that makes these processes difficult to control. As a result we were able to identify the additional requirements we need the closed loop process to fulfill in order to obtain robust performance. These techniques resulted in a better understanding of the directional behavior of processes. The developed techniques have the disadvantage that they can only be applied to subproblems, whose results are highly dependent. In chapter five we therefore propose to use a different approach that enables us to deal with the overall process behavior. It is assumed that the process outputs are ordered in a descending order of importance, i.e. in accordance with the priority of their corresponding requirement. The input-output controllability analysis then boils down to a sequential procedure were we analyze the closed loop behavior per output. Control of each output is done under the restriction that it does not effect the closed loop behavior of

[3] Morari focused on the use of this technique for process design.

outputs with higher priority. This is equivalent to requiring the closed loop transfer matrices to the output to be lower triangular structure. The approach is clearly more complex than the approach proposed by Morari. It however enables us to analyze the controllability of the overall process based on the requirements. The approach clearly reveals the trade-off's we need to make.

To incorporate nominal internal stability of the closed loop in the above analysis procedure the square down problem was studied. This resulted in new insights in this problem. It is well known that if a dynamic stable controller is used it is always possible to down square a non-square process to a square process, without introducing additional non-minimum phase zeros in the resulting square process. In section 5.4 we however show that not introducing additional non-minimum phase zeros may only be possible at the expense of a high gain of the controller and/or a drastic reduction of the gain of the square process. Introducing one or more non-minimum phase zeros may result in a drastic reduction of this effect. Hence there is a design trade-off between the number of non-minimum phase zeros introduced and the resulting gain of the controller. The above developments are all based on the frequency domain. The most frequently applied multivariable controller in industry is a model predictive controller. This type of controllers is based on a finite time horizon. To asses the controllability for this type of controller, an input-output controllability analysis approach is developed based on the finite time domain. As a result of these developments new insights were obtained on the behavior of the enables a completely new interpretation of the effect that non-minimum phase behavior and the non-squareness of a process has on the controllability of the process. The approach results in a different look at the so-called *waterbed* effect:

> *The waterbed effect is caused by the fact that we can not freely use a certain subspace of the output space of the process for dynamic control.*

For non-square processes the approach clearly reveals the relation between the gain and the number of non-minimum phase zeros introduced during down squaring.

In a last chapter the developed techniques are used to analyze the input-output controllability of a quartz glass process and a polymerization process.

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***Chapter four*:**

Chapter five:

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Glossary of symbols.

$\Re L_\infty^{pxm}$	The class of *pxm* proper rational transfer matrices.
$\Re H_\infty^{pxm}$	The class of stable *pxm* proper rational transfer matrices.
$\Re H_\perp^{pxm}$	The class of antistable *pxm* proper rational transfer matrices, i.e. having all poles outside the unit circle.
$\Re H_\infty^{pxm}(k)$	The class of *pxm* proper rational transfer matrices that have only *k* unstable poles.
$\Re H_\perp^{pxm}(k)$	The class of antistable *pxm* proper rational transfer matrices. have only *k* stable poles.
$\Re H_2$	The subspace of rational functions in H_2.
$\Re L_2$	The subspace of rational functions in L_2.
$C(z)$	Controller in the internal model scheme (figure 2.2.2).
$C_d(z)$	Controller in the feedback path of the control scheme (figure 2.2.2).
$C_{dm}(z)$	Disturbance feedforward controller in a feedback system (figure 2.2.2).
$C_{fb}(z)$	Controller in the unit feedback case (figure 2.2.1).
$C_{sp}(z)$	Setpoint feedforward controller in a feedback system (figure 2.2.2).
$CF_{ij}(M)$	The coupling matrix (section 4.5).
$CF_{ij}(M)$	The coupling factor between the *i*-th and *j*-th singular value of the transfer matrix (section 4.5).
$\mathbb{C}$	The complex numbers.
$\mathbb{C}^p$	The p-dimensional complex numbers.
$cond(X)$	The condition number of the matrix *X*.
$cond_m(X)$	The minimum condition number for the matrix *X*.
D	The unit circle.
$deg(\varepsilon(z))$	The degree of the polynomial $\varepsilon(z)$.

H_2	The subspace of L_2 of functions with an analytic extension outside the unit circle.
H_2^P	subspace of L_2^P of functions with an analytic extension outside the unit circle.
I_{rxr}	The *rxr* identity matrix.
L_2	The space of Fourier transforms of l^2.
L_2^P	The space of Fourier transforms of $(l^2)^P$.
l^2:	A linear inner product space of infinite time signals $x(t)$.
l^2_+:	A linear inner product space of right semi infinite time signals $x(t)$, $t\in[0,\infty)$.
$l^2_+(N)$:	A linear inner product space of right finite time signals $x(t)$, $t\in[0,N]$.
l^2_-:	A linear inner product space of left semi infinite time signals $x(t)$, $t\in(-\infty,0]$.
$l^2_-(N)$:	A linear inner product space of left finite time signals $x(t)$, $t\in[-N,0]$.
$(l^2\ )^P$	A p dimensional inner product space.
$M(z)$	The transfer matrix of a model of the process behavior.
$M_i(z)$	The inner factor after an inner outer factorization.
$M_o(z)$	The outer factor after an inner outer factorization.
m	The number of outputs.
N_{01}, N_{02}	Projector matrices (section 4.4).
$N_O(i, j)$	Finite time observability matrix (equation (3.4.2a) and (3.4.2b)).
N_O	Infinite time observability matrix.
$N_C(i, j)$	Finite time controllability matrix (equation (3.4.3a) and (3.4.3b)).
N_C	Infinite time controllability matrix.
n	The McMillan degree of a transfer matrix..
normrank(.)	The normal rank of a rational matrix.
$P(z)$	Dynamic process behavior.
$P_d(z)$	Dynamic behavior of the measurable disturbance at the output.
P_N	Finite time controllability gramian (equation (3.4.3d)).
P	Infinite time controllability gramian (equation (3.4.3c)).

$P_s(z)$	The (Rosenbrock) system matrix: $P_s(z)=\begin{bmatrix} zI-A & B \\ C & D\end{bmatrix}$.
p	The number of outputs.
Q_N	Finite time observability gramian (equation (3.4.2c)).
Q	Infinite time observability gramian (equation (3.4.2e)).
$S(z)$, $S_o(z)$	The output sensitivity transfer matrix (equation (3.5.5a)).
$S_i(z)$	The input sensitivity transfer matrix (equation (3.5.5e)).
$T(z)$, $T_o(z)$	The output complementary sensitivity transfer matrix (equation (3.5.5b)).
$T_i(z)$	The input complementary sensitivity transfer matrix (equation (3.5.5d)).
T	The infinite lower block Toeplitz matrix (equation (3.4.7b)).
$T(j,i)$	The finite time lower block Toeplitz matrix (equation (3.4.7a)).
X^H	The complex conjugate transpose of the complex matrix X.
$\Gamma(i,j)$	The finite time Block Hankel matrix (equation (3.4.4c)).
Γ	The infinite time Block Hankel matrix.
0_{rxp}	An *rxp* matrix with all entries equal to zero.
$\langle a,b\rangle$	The inproduct (equation (3.2.7c) and (3.2.9b)).
$\Vert\ \Vert_2$	The two norm on a signal (equation (3.2.7a) and (3.2.9c)).
$\Vert\ \Vert$	The vector norm on a signal (equation (3.2.7b)).
$\Vert\ \Vert_\infty$	The infinity norm for a transfer matrix (equation (3.3.6)).
$\Vert\ \Vert_H$	The Hankel norm for a transfer matrix (equation (3.4.5)).
$H^\sim(z)$	The discrete time adjoint of $H(z)$: $H^\sim(z)=H^T(1/z)$.
$H^1(z)$	The inverse of the transfer matrix $H(z)$, i.e. $H(z)H^1(z)=H^1(z)H(z)=I$.
$H^R(z)$	The right inverse of the transfer matrix $H(z)$, i.e. $H(z)H^R(z)=I$.
$H^L(z)$	The left inverse of the transfer matrix $H(z)$, i.e. $H^L(z)H(z)=I$.
$H^\sim(s)$	The continuous time adjoint of $H(s)$: $H^\sim(s)=H^T(-s)$.

CHAPTER ONE

Introduction

1.1 Introduction.

In this chapter we will give the background of the research that resulted in this thesis. The basic motivation for process control is to enable better operation of industrial processes. This statement is fairly general and raises more questions than it gives answers. Process control forms only one aspect that influences the performance of a process and plant. The above statement forces us to position process control as an integral part of the operation of a plant. In this section we will discuss which aspects influence how a plant will be operated in the future. In the first part of this chapter we will discuss what the driving forces are that will change plant operation in the future. This will allow us to identify how plant operation is expected to change in the future. The operation of a plant is not only influenced by what is desired, but also by what is technically possible. We will therefore also take a look at the developments of the different technical aspects related to plant and process operation. It is noted at this point that not only technical, but also organizational aspects influence the performance of the plant. This aspect will however not be explicitly discussed, since it falls outside the scope of the thesis. Next we will discuss the consequences of the observed trends and developments and especially the opportunities offered to obtain a better operation of the plant, i.e. an operation of the plant that is closer to the desired operating conditions. An important aspect in this is specifically the closer cooperation between the different disciplines involved with better process operation. We will then concentrate on the implications this has for control theory and how it is related to the different developments going on in this field.

We will then identify how the topic of this thesis is positioned in the above field and develop a first rough formulation. In chapter two we will discuss the different aspects of this problem in detail and refine the problem formulation.

1.2 Innovation in operation of processes in process industry

In this section we will discuss the trends and developments we observe in how plants and process units are and can be operated. First we will take a closer look at how the market and society influences process industry and process operation in particular (subsection 1.2.1). In subsection 1.2.2 to 1.2.4 we will take a look at the relevant developments that took place in the field of process design, information systems and control theory. In subsection 1.2.5 we will discuss the relation between the different fields, the opportunities this offers for a more efficient operation of processes. In this section use will be made of the terms process plant and process unit. Let us therefore first discuss what we mean with a plant and a process unit, before we continue the discussion. Petrochemical industries and refineries have in general different plants on one site, i.e. one geographical location. A plant, e.g. an ethylene plant, is a production entity consisting of different interconnected processing steps, that produce a (intermediate) product or a certain class of (intermediate) products and can operate more or less independent of other production plants on a site. In most cases if there is a coupling with other plants this is only by incoming products and out going products, except for utilities like electricity, steam, cooling water and fuel. A process unit is again a section of the plant. In many cases it can be identified with a certain steps in the process, e.g. Hydro Cracker, FCC, Crude Unit and so on. A process unit is highly integrated in the plant. Its operation therefore depends to a large extent on other parts of the plant. In other process industries a process is in general a well defined part of the overall production process, e.g. glass furnace, feeder, calciner, rolling mill, ...

1.2.1 Trends in the market and society and their influence on plant operation.

The world market, public opinion and governments have major influence on the way process industry operates. In the last decades these aspects have been the driving forces behind the changes in the way plants are operated. Global competition forces industries to become more competitive and be more alert to changes in the market. This has resulted in a strong need to rationalize production, to cut costs and to realize a certain profitability. We see that production rates, also called throughput, are increased to optimize the use of the production means. Minimization of the time that a process is out of production, due to problems or maintenance becomes more serious. A further consequence is that one tries to minimize inventory. Just In Time (JIT) delivery has become common practice. The market becomes also more dynamic, less predictable and more uncertain, since more competitors enter the market and the market to a large extent may be considered consumer driven. The number of different products to be produced and the required performance of these products increase drastically. As a consequence of these developments we see that production planning becomes more dynamic and that the number of change-overs from one product to another increases. Reduction of change-over times and off-spec production during this time becomes more important. Also the purchase of raw material and intermediate products has become more dynamic. In the refinery for example we see more and more that crude's are (finally) purchased at the last moment. In the past it was possible to schedule crude's, such that the composition of successive crude's only changed gradually. This however becomes more and more difficult and variability in the composition between crude's that are successively processed on a site increases significantly. Production becomes less predictable and therefore has to be more flexible.

1.2.2 The technological life cycle of a plant and its process units.

In the overall life cycle of the process the design step of a process is the most important factor in how process units and the overall plant will perform. In this phase all physical limitations of the process are defined and fixed. This is a trivial, but nevertheless a very important observation, since the consequences of wrong or bad choices made in this stage can have severe and far reaching consequences. New process units ask for large investments, have relatively long development and construction times and therefore have long pay back times. As a consequence the life cycle of process units is long. Most changes to be made to the unit become very expensive after the unit is build. Design of process units evolves only slowly. One heavily relies on the experience build-up with previous units. This is even enforced by the fact that many mechanisms in a process are understood only partially. A major decision factor in the process industry has therefore become experience: How many of these units are in operation or how many of these projects have been successfully completed by the contractor?

The relatively long life cycle of these processes and the changing market make that re-engineering of process units is an important aspect of process design. This upgrading of processes is sometimes called revamping. Energy saving and minimization of pollution and waste products have been important aspects during the last decades. Energy taken out of a unit is again used in an other part of the unit or plant. Certain product streams reenter the process again in front of the unit (up stream), introducing recycle or after the unit (down stream).

An important new aspect in process design and engineering, that has come up in the last decade is the use of computer simulation. The high investments related to a new unit and the high costs that are involved in changing the unit after it is completed make it very appealing to minimize risks in the design phase of the project. Rigorous modeling has always been an important activity in chemical engineering. In the past the usage of rigorous models was restricted to a better understanding of certain aspects and mechanisms of a process. One was unable to use very detailed models of the overall process behavior, due to their complexity. The fast increase in performance of computers has made it possible to actually simulate the overall process behavior, making use of very detailed models of the overall process unit. Hence we see that computer simulation is rapidly developing and is more and more applied in projects. Rigorous simulation packages have become commercially available, e.g. Aspen Plus• for steady state simulation and a product like Speed-Up• for dynamic simulation. Rigorous simulation starts to become an integral part of the design. Most of the models used nowadays are still steady state models. It is however to be expected that dynamic simulation will become more and more important.

The impact rigorous simulation has is not only restricted to support detailed analysis of the process behavior constraints and performance in the design phase. It is probably the primary justification to develop these rigorous models, but if they are available they can have much more impact on the operation of the process unit. The availability of these models enables a much broader use. It makes off-line operator training possible, which enables training of the operator for unexpected events and emergencies. Process monitoring for fault detection and maintenance, trouble shooting, simulation of new operating strategies and so on become possible. In section 1.2.4 and 1.2.5 we will see the potentially central role of these nonlinear models in the field of control and instrumentation. The rigorous model is therefore expected to play a central role in a more optimal and flexible operation of process unit and plant.

1.2.3 Information systems.

Computers and information systems have rapidly developed in the last years. These developments also influenced process industries. Computers have entered in almost all disciplines in industry. Not only has the computer entered the production departments, but also departments like purchasing, sales, planning and administration have become completely dependent on their computer systems. A prerequisite for information systems to function correctly is the availability of correct up to date and to the point information. For production environments real time database and application programs to filter out the desired information are therefore a prerequisite. Major investments are therefore made in this field. More and more all information from any system and to any system is passed through the real time database. Real time databases, like PI• (Oil systems), Setcim• (Aspentech) and INFOplus21• (Aspentech), therefore become more and more the heart of the information and control system on the production floors. It enables to monitor processes on-line. It makes on-line performance monitoring possible. As a consequence it is better possible to identify bottlenecks and to better evaluate the results of improvements and changes made in the process or the way it is operated. The most recent development is the introduction of networks that enable the coupling of these different systems to one big information system. The potential of this development is significant and will have major impact on the whole organization . In principle on-line information is directly available at all levels in the plant. This makes it possible to increase operational flexibility. It decreases reaction time and makes it possible to directly or even anticipatively respond to all kinds of unexpected changes in production and to sudden opportunities offered in the market. We therefore see an increased interest of industry in plant wide information systems.

1.2.4 Control systems and instrumentation in process industry.

Process instrumentation is one of the corner stones in the operation of a process unit. The instrumentation forms the interface between the process and the operator or control system. It has therefore always been a point of attention during the design and redesign of processes. During the years major progress has been made and resulted in nowadays fairly complex and in many cases smart sensor systems. However frequently sensors still form a severe restriction in the operation of a process, despite the progress that is made. Many process variables are still unmeasurable on-line or can only be measured by slow (low sampling frequencies) and very sensitive equipment. Many measurements can only be performed off-line during laboratory tests. Only ones an hour or ones or twice a shift a measurement comes available. In these cases one tries to monitor the behavior of the process by measuring other related variables. In some cases these measurements are used directly. In other cases a calculation is used to determine an estimate of the actually desired measurement. This is called a *soft sensor* or *inferential measurement.*

In the middle of this century processes started to be equipped with SISO control loops. First they were primarily used to control actuators and elementary properties of the process. The operator was in fact performing the actual control of the overall process. In the beginning these controllers were implemented as single analog devices. Today's PID's are implemented in Distributed Computer Systems (*DCS*). or Programmed Logical Control (PCL) systems. This DCS system can be seen as the basic control level of a modern process. On this level the basic functions regarding the operation of the plant are implemented.

In the seventies and eighties the complexity of the processes and the required performance of these processes made that automatic control of the overall process became more and more desirable. The controller therefore has to deal with Multi Input Multi Output (MIMO) processes. The main objective of this type of controller is not to keep certain variables as close as possible to their setpoint, but to push the process to an operating point that enables the process to be operated under the best economic circumstances, without violating the operational limitations and satisfying product specifications. This type of control is frequently called *constraint pushing.* This type of control can only be applied if the controller can cope with constraints. For this type of control PID controllers are not well suited, since it is difficult to deal with the interactive behavior of the process. Initially PID controllers were still used with special routines, i.e. logical decision strategies, to cope with constrained situations. However even for simple problems it resulted in complex and not well understood control schemes. In the refineries we therefore see a new type of MIMO controller appearing. These so called Model Predictive Controller (*MPC*) created the ability to drive processes towards the best economic operating condition. The controllers are currently implemented in a separate computer system. It is however to be expected that these controllers in the future will be integrated in the DCS system, since the computational capabilities of these systems grow rapidly. Examples of commercially available controllers are SMCA (former SETPOINT), DMC (former DMCC), DMCplus (Aspentech), SMOC (Shell) and RMPCT (Honeywell/ Profimatics). These controllers all have more or less the same basic capabilities. The basic control concepts of these controllers are approximately fifteen years old.

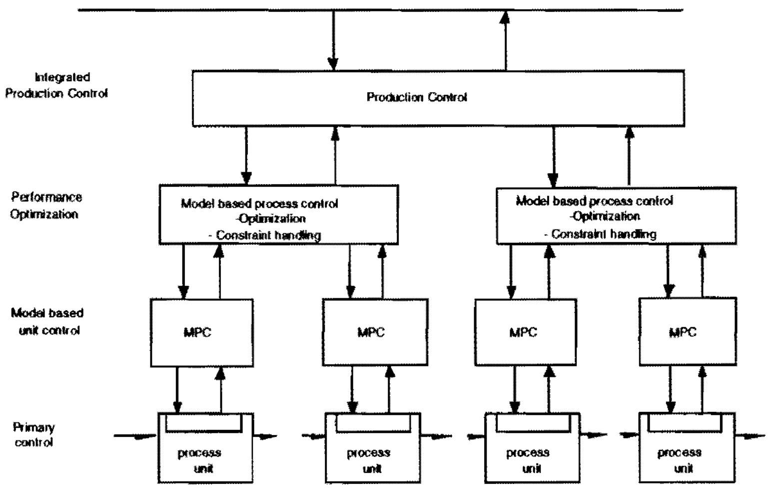

Figure 1.2.1 The control hierarchy of plant control

The next problem to be solved is to determine on-line this best economic operating point. The best economic operating condition will be time dependent, since it depends on a number of different time-dependent conditions, e.g. the state of the plant and that of other units on the plant, disturbances and economic factors. This has resulted in the introduction of *optimizers* on the plant, e.g. DMO (Aspentech) and RT-OPT (Aspentech). In general, optimizers will determine the best economic setpoints for the inputs and outputs of the process on the basis of an economic cost function. These values then serve as targets for the MPC controller. Nonlinear steady state rigorous models of the plant (section 1.2.2) are currently used in these optimizers to calculate the optimal targets.

In fact the primary controllers in the DCS, the MPC controller and the optimizer are all controllers. They however control the plant at different levels and are a member of a control hierarchy (figure 1.2.1). The optimizer controls one to a number of process units. The optimizer is event driven. At the moment that the optimizer detects that the plant is in steady state it will determine the best economic operating conditions for that units. These conditions are passed as targets to the multivariable control level. The MPC controller in turn brings the process unit towards these targets and tries to keep it there. In contrast to the optimizer the MPC runs at a fixed sampling rate (minutes). The SISO controllers in the DCS (sampling time in the range of seconds) receive their setpoints again from the MPC controller.

1.3 Developments in control theory and their relation with process control.

In this section we will discuss the developments in control theory. In the last decade we have seen a rapid development of linear control theory. The aim of this section is not to give a complete overview of all developments in this field. We will specifically concentrate on those developments that have relevance for process control. For the design of MIMO process controllers a model of the process is almost always a prerequisite. In subsection 1.3.1 we will therefore take a closer look at the developments in the field of identification. In the second subsection we will take a closer look at developments for linear controller design.

1.3.1 Identification.

For the application of control on multivariable processes, models of the process behavior have become a prerequisite. For multivariable process control mathematical models are generally obtained by identification, i.e. models obtained on the basis of measured data. In classical identification it was generally assumed that it was possible to find the true model [Eyk74]. Nowadays it is generally accepted that models are only approximations of the real process behavior. The process is therefore never in the model set [Bac87, Ari97]. This is certainly evident for processes, since they are in general distributed non linear systems, while the model is in general lumped linear and finite dimensional [Bac87]. A consequence of this observation is that the classical approaches, based on the estimation of (pseudo) canonical state space models will in many cases not result in satisfactory models. The selection of the correct model structure is doomed to fail, since the process is not in the model set. This observation has been the motivation of the approach proposed in [Bac87] to identify industrial processes, with low order state space models. It also increased the interest in the so called subspace identification techniques, since they enable to directly estimate state space models without selection of a certain canonical form [Lar83, Moo88, Moo89, Ove95, Ver92].

The fact that a model is only an approximation of the real world has two additional consequences that recently received much attention in academia:

1. There is inevitably a discrepancy between the model and the process behavior, i.e. a model error.
2. If the model is only an approximation of reality what is then the best possible approximation for controller design.

The interest in estimating the model errors have of course been enforced with the development of robust control techniques in the eighties. In many approaches one was interested in directly estimating the worst case error [Boo91, Hel91, Jac92]. A second class of approaches is based on probabilistic assumptions, which result in less hard , but possibly more useful model errors [Nin92, Vri94, Zhu90]. The resulting model errors in combination with robust control techniques seem to result in very sluggish control. It is questionable whether this is desired and whether this provides the best achievable result. This observation gives rise to the following observations [Ari97]:

- Conservatism of the model error can be reduced if we can build in prior knowledge in the identification.
- Conservatism can also be reduced if we can make use of structural information still available in the model error.

An interesting and illustrative discussion on this subject can be found in section 2.1 of [Ari97]. From this discussion a significant aspect emerges. From practical experience it appears that designing a controller based on a nominal model with a good robustness margin is well achievable, without the error bounds. Under the nominal operating conditions the model error is only important if we really want to push controlled performance to the limit. For current industrial practice the question whether the controller will operate over a certain operating range is of primary interest. This however means that sufficiently rich data should also be available on these operating conditions. Dedicated tests are in general however very expensive. If for the estimation of realistic error bounds large amounts of rich data are needed we might as well directly estimate a new model for these operating points. It is furthermore questionable if it is at all possible to perform tests under all appearing operating conditions. The inability to obtain accurate bounds on the relevant model error is from an application point of view a severe drawback. A prerequisite is therefore a better understanding of which effects in the process are relevant for control and how their accuracy will limit the closed loop behavior.

Another point of view to deal with the approximate nature of models is to say the following: If the actual process can only be approximated by the model, what is the best approximate model for high performance control design. This question has resulted in a research area frequently indicated as control relevant identification [Gev91, Gev93, Hof95, Li94, Sch92, Zhu90]. In fact the question is to obtain better understanding of what physical behavior is relevant for the closed loop behavior and how it should be taken into account in the model or the model error. In principle the basic underlying problem is again try to obtain a better insight in how the open loop will limit the closed loop behavior.

1.3.2 Multi-input multi-output linear controllers.

Most advanced controller design approaches are based on the minimization of a criterion function. The main idea is to specify the desired behavior of the closed loop system in the weighting functions of the criterion. The most frequently used criterion function is the quadratic criterion. An interesting question is why one wants to choose a quadratic criterion. The answer to this question is simple:

Quadratic criteria have extremely nice and well understood mathematical properties and control theoretical interpretations.

The quadratic criterion is well understood. For linear models the dynamic quadratic criterion is convex and has an analytic solution in the unconstrained case. Many control theoretical results are available, certainly for the infinite horizon problem. As a result a fairly complete theory exists. The approach moreover turns out to give good results in practice. The point to make here is that the criterion is not primarily chosen because it matches best on the control problem at hand. Most requirements have to be approximated into the criterion. Other approaches have been proposed to fit better on the control problem at hand, e.g. [Boy91]. However one still has to approximate many of the requirements in these more complex approaches. In many cases the solutions become infinite dimensional. Moreover the resulting optimization problems are more complex and less well understood. It seems that the choice of the quadratic criterion function is a good choice, both from a theoretical as well as a design point of view. We will therefore restrict to these approaches.

One can discern two main different implementation of controllers based on the minimization of a quadratic criterion:

1. One is based on the off-line minimization of a quadratic criterion function.
2. The other one is based on the on-line minimization of a quadratic criterion function.

Although both techniques are based on a quadratic criterion, we will see that the different way we deal with the optimization problems has a number of consequences that make the approaches in significantly different from a practical point of view.

The first group consists of control techniques like H_2 and H_∞. The development of these controllers started in the sixties. It was observed that PID controllers were not suited for complex dynamic control problems with high performance specifications. The main advantage of the quadratic techniques is that the algorithm determines the best controller given a linear model and an objective function for the specified problem. The optimization is completely based on the model. It was assumed that the model exactly described the process behavior. It was only later that this was recognized as a fundamental problem that frequently resulted in a disappointing behavior of the actually controlled process. In the middle of the seventies the difference between the model and the process behavior, the so-called model error or model uncertainty, became a point of concern. It resulted in a revival of frequency domain for analysis [Doy81] and design [You76, Doy84], since robustness of the controller against model errors is better understood in this domain. This development resulted in 1981 in the formulation by Zames [Zam81] of what is now known as H_∞. The fact that the resulting H_∞-norm is an operator norm made it possible to relate the solution to the so-called small gain theorem to ensure a certain level of robustness against model errors. In the beginning of the eighties one of the major problems was the solution of the criterion. In the mid of the eighties first techniques became available to solve some of the problems [Doy84, Fra87, Kwa86]. It took until the end of the eighties before a general solution became available [Doy89, Sto92]. Remarkable was the fact that the solution boiled down to the solution of two coupled Ricatti equations. The solutions for both H_2 and H_∞ result in finite dimensional closed forms of the controller, i.e. the controller can be described by a finite dimensional transfer matrix or state space model. The order of the controller generically equals that of the total order of the dynamics in the criterion, i.e. the dynamics of the weights and the model.

A clear advantage of both H_2 and H_∞ is that the resulting controller can be implemented as a state space model. As a result the on-line computational requirements are limited. Hence high performance control of fast and stiff systems, i.e. systems with combinations of very fast and slow dynamics, without excessive hardware requirements is possible. The main disadvantage of these controllers is their fixed structure. The controller can not deal with changes in the structure of the process. In the case of saturated actuators ad hoc methods need to be applied [Han87, Cam89, Cam90] to ensure stability of the closed loop system and to constrain the performance degradation.

The Model Predictive Controller (MPC) is different from the above discussed approaches in the sense that it minimizes a quadratic criterion function in real time over a fintie time horizon recursively after each sampling interval [Gar89, Lee96, Qin96]. In refineries and petrochemical industries the model predictive controller has become the standard multivariable controller. This remarkable success is probably partly due to the fact that the basic ideas were presented by people close to this industries [Cut79, Ric78]. An other part of the success of this type of controllers is certainly to the ability of the controller to deal with constraints and more general with a change in the structure of the process. The ability to deal with constraints is a direct consequence of the on-line minimization of the criterion at each sampling instant[1]. This flexibility allows industry to continuously push the process unit to the best economic operating conditions, without violating safety constraints, physical limitations and product qualities, which is generally known as *constraint pushing*. The ability to deal with constraints in real time enables one to cope with changing conditions, like different composition of crudes, which may result in different constraints being restrictive or active. An interesting overview of the different industrially applied MPC techniques can be found in [Qin96].

[1] It is noted here that the inclusion of constraints in the optimization, makes the optimization problem nonlinear. The control law has therefore become nonlinear and hence the controller. Strictly spoken the approach is therfore not linear anymore. Only the model used is linear.

Interesting to note is that for a long time the academic control community was not really interested in Model Predictive Control. Model Predictive Control was mainly based on heuristics. A more fundamental understanding of concepts, like stability and robustness, were lacking. Only in the beginning of the nineties a more general interest in Model Predictive Control revived in academia. It resulted in a rapid increase in a better and more fundamental understanding of the technique. Nominal stability, robustness results and stabilizing controllers, also for the constraint case, have started to emerge [Lee96, Mus93, Raw93]. Recently also first theoretical results emerge on Model Predictive Control for nonlinear processes [May96].

Many of the advantages of Model Predictive Control are directly related to the minimization of the criterion at each sample instant. At the same time the real time optimization is also responsible for the main disadvantage. The minimization of the criterion function at each sample instant is computationally very demanding. In commercial control packages a number of restrictions are put on the complexity of the optimization problem, to reduce the computational load, e.g. the number of future input moves used for optimization over the future horizon are restricted. As a consequence the currently applied Model Predictive Controller is not yet suited for high performance control. To enable high performance control a better understanding is needed in the relation between the degrees of freedom in the optimization and the process behavior.

Interesting to note here is that all the above discussed techniques are based on a criterion in which the designer is supposed to translate the required performance into so-called weight functions. It turns out that this translation of the desired behavior into these weights is not at all a trivial task. This observation seems not to have resulted into an increased interested in academia.

1.4 Consequences of the observed trends for process control.

In this section we will discuss the relation between the different trends we identified in the previous sections.

1.4.1 General trends

From the previous discussion we may conclude that information systems at one hand enable organizations to respond faster to the market. At the other hand the market also enforces companies to alertly respond to changes and opportunities. The required product quality, the reduction of costs and strict legislation will enforce a more strict control. Processes will be operated more and more flexible over a large operational range of the process. It will result in more frequent change-overs from one condition to another and only stay in a certain operating point for a relatively short time.

The way a process can be operated during its life time, is to a large extent fixed during the process design. After the process unit is built the physical behavior is fixed. The limitations on the operation of the unit are therefore fixed in this stage. Control systems are then used to further increase the performance of the process unit. However also the increase in performance that can be obtained by a control system is limited by the physical constraints of the process unit. The opportunities that control offers to further improve the behavior of the process are therefore also fixed in the process design phase. After the unit is designed the limitations become inherent characteristics of the process behavior. This fact together with the increased complexity of processes and the increased significance of control in the operation of process units make control aspects an important aspect of process design. The importance of insight in the control aspects of a design have already been recognized in the beginning of the eighties and have received considerable attention, e.g. [Cao96, Ark86, Mar86, Mor83, Rus87, Sko91 Sko93, Zaf96]. A major problem in the design phase is however a lack of detailed insight in the dynamic process behavior. The introduction of rigorous modeling however offers new opportunities. As dynamic modeling is expected to gain importance it becomes possible to obtain a more detailed insight in the control aspects of a design. It enables an optimal integration of process technological and control theoretical solutions in the design.

One could argue that the control levels in figure 1.2.1 should be replaced by one huge multi-input multi-output control system. This system would even outperform the above hierarchical approach, since it does not introduce an artificially split-up of the control problem in different subproblems. There are in fact two main reasons to split-up control in different levels.

The primary reason is safety and reliability of the process operation. One has to ensure that under all circumstances the process units can be operated safely. The basic safety procedures are therefore situated in the lowest level of the control hierarchy, the DCS. The DCS system is specifically designed to meet these high reliability requirements to ensure maximum safety and minimum damage. Moreover one wants to be able to cope with whatever problems or exceptional situations occur in the plant with a minimum impact on the operation of the overall plant. One therefore wants to be able to take over control of the plant at any level, appropriate for the given situation. A hierarchical architecture fulfills this requirement.

One could of course argue that in principle also these situations can be controlled better by one big overall control system. From a theoretical, practical as well as an implementation point of view the problem is however too complex, e.g. not all problems can be foreseen.. The hierarchy is therefore not expected to change. What we will see instead is an increased complexity of the different subproblems and an increased interaction between the different control levels. The use of a steady state model in the optimization layer assumes that the process is in steady state. The expected increased flexible operation over a larger operating range will make this assumption too restrictive. It can therefore be expected that the dominant slow dynamic process behavior will be incorporated in the rigorous models used for optimization. To ensure consistency with the optimizer the model used in the Model Predictive Controller has to be consistent with the model used by the optimizer. It is however not expected that the next generations Model Predictive Controllers will be fully nonlinear, since this will lead to very complex nonconvex optimization problems. To comply with the high performance requirements over this large range different accurate linear models are needed for the MPC. The controller must be able to on-line switch over from one model to another without loss of performance.

An important aspect in the application of modern controller is the reliability and ability to deal with different situations. This is frequently defined by the fraction of the total operation time of the process that the controller is actually controlling the process. Currently up-times of the controller are required of more than 95%. It is expected that the required up time will be further increased, despite the more strict requirements. As a consequence diagnostic tools are needed that react on those failures and changes in the process behavior that will significantly influence the performance of the controller. An observed change in behavior has to automatically result in an appropriate action of the controller. In principle the controller has to maintain the maximum possible performance, by adapting the control strategy to the new situation. Only in last instance it is allowed to switch off. A prerequisite of on-line adaptation of the control strategy is that we need to know how to adapt the control problem. As a consequence we need insight in the relation the changed behavior and how it affects the closed loop. This discussion bring us to an important and basic question:

Do we actually know how to tune a controller ?

1.4.2 How to tune a multivariable process controller

The question was raised if we know how to tune a controller. At first sight this may seem superfluous and ridiculous. Certainly linear control theory is well developed. Nowadays we have powerful algorithms to determine sophisticated multivariable controllers and tools to analyze its behavior e.g. [And89, Gre95, Kwa72, Mac89, Mor89, Sko96, Zho96]. One could argue that the superfluousness of the question on tuning is confirmed by the fact that in most of these books dealing with these modern approaches no or only minor attention is paid to the selection of the weighting matrices in this literature[1]. In this section we will argue that selection of the weighting matrices of these modern controllers is a major issue of concern. In our opinion the ability to tune the controller in a structured way will determine to a large extend the success of the approach. It is therefore an amazing observation that the subject has not received more attention.

[1] An exceptions must be made for the books of Morari [Mor89] and certainly the book of Skogestad and Postlethwaite [Sko96]. It is however felt by the author that also in these books a principle attempt to tackle the tuning problem is lacking.

In this section we will identify the limitations of currently applied multivariable process control, based on this question. It will enable us to identify areas of research needed for better process control. We will start the discussion with a fictious example.

> A colleague has become seriously ill and you are asked to replace him in a project. In the project a complex process unit has to be controlled. You do not yet have any experience with this type of process. Luckily the project is already in a stage were we can start design. Accurate linear models of the process have already been made. In the project documentation it becomes clear what the control objectives are. So we are almost finished and start up the design package. After definition of the problem structure and loading the model, we are only left with specifying the weights. At this moment the problems start. How to specify the weights? Arrived at this point in the procedure this suddenly is not that trivial anymore. If the problem is complex, i.e. performance requirements are not trivial to accomplish then it might turn out that we are not at all 'almost finished' and run into a lengthy procedure of trial and error.

In the above example we sketched a situation, were a control engineer is assigned to a project in the control design phase. The engineer has no previous experience with the process. Neither did he had time to build-up knowledge and "intuition" on the behavior of the process during the identification. He therefore has to design the controller completely based on the documented requirements and the available model. Any control engineer recognizes this as a difficult situation. The first thing he will probably start with is pick up the phone and start to find out if there is any experience with this type of process within his company. His questions will be focused on how to achieve the performance requirements for the given process. As a consequence he is missing the insight in how the process behavior influences the closed loop behavior and the performance we are able to achieve. This is serious problem if the controller is to be designed under the time pressure in a commercial project.

The problem is however more general than the above example. In general a detailed insight is needed in the relation between the limitations the process behavior puts on the closed loop behavior. If this insight is not available the design of the controller is likely to become a trial and error procedure.

One way out is to tune the controller "robust". This in general means that one tunes the controller such that the closed loop dynamics are slower then the open loop dynamics of the process. In fact this is common practice in industrial projects. This 'quasi steady state' control seems from a practical point of view attractive. It is thought to have the following properties[2]:

1. The models need not be very accurate.
2. The controller will in general be robust.
3. The time needed to design the controller is limited and reasonably predictable.
4. Only limited experience is needed to design the controller.
5. After the client has seen the controller active for a while he is pleased by its performance.

In the above "robust" approach we made implicitly use of two facts:

1. The steady state behavior of the process is easier and better understood than the dynamic problems.
2. As long as we stay well out of the dynamics of the process, most processes are in general less sensitive for drastic performance reductions due to not too severe modeling errors[3]. In SISO control the approach ensures a good phase margin.

In the cases that it works, this seems to solve the tuning problem. If he likes it or not we can always fly in the control expert in case it does not work. Why bother any further? This however strongly depends on how you look at it. We are left with at least one question:

Does the word 'still' in the above discussion mean that it is likely to change in the future and we need to fly in the expert more and more?

From the previous section it is clear that future control problems will become more complex. Future generations of controllers have to deal with:

[2] It is emphasized that the properties are not always observed. There are processes that will never fulfill these properties, based on the above sketched robust approach. (see note 3).
[3] It is emphasized here that the statement is not true in all cases. For some MIMO process the above approach will fail to succeed in a robustly operating control system. For ill-conditioned processes additional requirements have to be fulfilled to obtain a robust performing control system (section 4.5).

- An even larger number of inputs and outputs, that is expected to grow rapidly.
- The more strict requirements will force us 'into the dynamics', i.e. force us to give up on the 'quasi steady state' control[4].
- Increased up-times of the controller.

If one wants to extend the application of multivariable control more widely in process industry one is likely to encounter control problems in which the emphasis is more on variance reduction than it is in most refinery and petrochemical applications, e.g. poly-ethylene reactors. In these cases one has to obtain control significantly faster than the open loop process behavior. In these cases high performance is a prerequisite.

It is therefore expected that this type of control becomes more and more a serious restriction in future applications. Let us therefore investigate what requirements have to be fulfilled to meet this future challenge:

- The models must become more accurate:
  - The applied accuracy of the model must be related to the control problem
  - We must be able to judge also the quality of the dynamic description of the model.
- The controller performance will have to drastically increase:
  - The applied techniques have to become more flexible and tunable on the specific control problem. They have to support high performance control.
  - The tuning of the controllers has to become a structured procedure. It must be able for the control engineer to also tune this controllers for more complex problems, without trial and error.
  - The controller must be maintainable. A prerequisite is that the design decisions made are not dependent on the designer.

[4] Control of stiff systems, i.e. systems that have a combination of fast and slow dynamics, with an MPC controller form a real challenge. In principle a large potential performance increase is possible, since currently only the slow dynamics are considered in the MPC, e.g. a FCC unit. One must however be able to take both the fast and the slow dynamics into account in the controller, to make full use of this potential. For the current generation of controllers this would lead to an unacceptable increase of complexity of the optimization problem. A new generation MPC is needed to deal with the complexity problem.

To meet the above stated requirements for future control projects a lot of applied research has still to be done both in the field of modeling and the field of control. Industrial circumstances and commercial terms must be hard constraints for this research.

1.5 Initial problem statement.

The importance of the dynamic behavior of the process for a save and profitable operation of the plant made Arkun [Ark86] define the term *dynamic operability* as the ability of the plant to maintain satisfactory

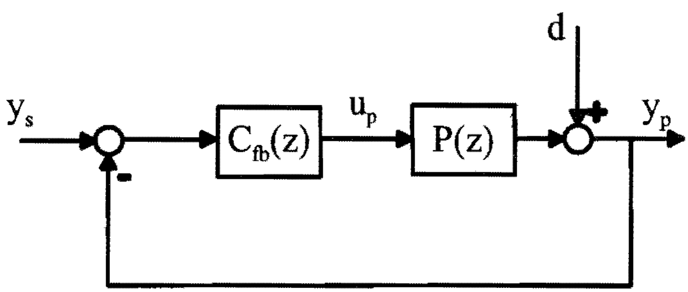

Figure 1.5.1 A unit feedback scheme.

dynamic performance despite uncertainties in the process environment. Uncertainties of the process environment originate from external disturbances, such as changes in raw material product specifications, different market requirements and energy resources. Satisfactory dynamic performance is specified by effective recovery from major disturbances and fast and smooth transition from one set of operating conditions to another. The dynamic operability as defined by Arkun is in fact a closed loop property, that is determined both by the process design and the control design. Arkun notes that the main difficulty with the above definition is the fact that dynamic operability is defined on the closed loop: It is a function of both the process behavior and the controller design, whose effects on the closed loop behavior are difficult to isolate from each other (figure 1.5.1).

This is a serious drawback in the definition. The first effect can only be changed if we change the design of the process unit, which is in general a costly operation. A change in the control design is less drastic and easier to perform. Morari [Mor83] introduced the term *dynamic resilience* to enable a better distinction between these two effects. *Dynamic resilience* of a process can be loosely defined as [Sko91]:

The (best) achievable quality of response which can be obtained for a plant by use of feedback control.

The definition is not very precise, as the authors themselves directly admit. It however clearly states the inevitable relation between the plant behavior and the limitations put on achievable performance for that process. A key idea in the term dynamic resilience is therefore that it is an inherent property of the process and is independent of the selected control structure and parameters. In [Sko91] it is noted that one of course may limit the class of allowed controllers for example to linear controllers. From the definitions it is clear that the main underlying question we want to answer is:

How does the complex physical process behavior limit the closed loop behavior?

Another term frequently used for dynamic resilience is *(input-output) controllability* [Bis93, Cao96, Sko96]. A little more precise is the definition of [Sko96]:

Controllability analysis is the ability to achieve acceptable control performance; that is, to keep the outputs within specified bounds or displacements from their references, in spite of unknown but bounded variations, such as disturbances and plant changes, using available inputs and available measurements.

We again see in this specification that as in the previous definition what is exactly meant with achievable performance is again vague, like 'best' in the initial definition. For SISO systems best possible may in a certain sense still be uniquely definable. For large MIMO systems, where we have a large number of potentially conflicting requirements and restrictions that all have a certain importance, best possible can only be defined based on how we expect the controlled process to perform. The required behavior of the controlled process therefore influences the outcome of an analysis of the controllability of the process.

Arkun identifies a growing need to more rigorous techniques to asses and improve the dynamic operability characteristics of chemical plants [Ark86], page 325:

> *"The practicing design engineer should be able to:*
> *a) correctly quantify the dynamic operability of a process,*
> *b) easily pinpoint the bottlenecks in the plant, and*
> *c) systematically improve the dynamic operability of the process design and the control system design.*

Several other authors [Far93, Mar86, Mor83, Rus87, Sko91] have identified the need for tools that enable us to better understand the restrictions that process behavior can put on the closed loop behavior. It is interesting to note that the need of these type of tools are most directly related to process design. In only a few cases it is pointed out that these tools can also be used to select the control structure [Far92, Sko91, Sko96]. In [Sko96] input-output controllability is used to answer questions like:

1. How well can the plant be controlled?
2. What control structure should be used?
   Questions like which inputs and outputs need to be used for control or what is the best pairing if we use decentralized controllers?
3. How might the process be changed to improve control?
   This in fact refers to the more classical relation between process design and process control.

In the discussions in this section and the previous sections we have seen that detailed knowledge on the process dynamics and the relation it has with the posed control problem is important at different stages in the control design procedure:

1. The identification for controller design.
   We saw that a major problem was to know which parts of the model were essential for control. We asked ourselves what the best model was for controller design, i.e. what should be considered in the model and what part of the process model could be taken into account in the model error. A prerequisite for this operation was a good understanding of the relation between the open loop behavior and how it influences the closed loop behavior.

2. The design of the controller itself and the selection of the weights in the criterion function.
   We concluded that the tuning of a controller was no trivial task and in most cases resulted in either trial and error or in quasi steady state control. Both of these techniques are to fail in the future. We therefore advocated the need for a structured approach. A prerequisite of such a procedure was detailed insight in those aspects that limit the controlled behavior of the process.
3. On-line adaptation of the control strategy, for example if the structure of the process changes, due to a failure of a sensor or saturation of actuators. This in fact may completely change the resilience of the process. Future controllers needed to be able to deal with these situations, with a minimum loss of performance. In order to achive this it may be neccesary to change the control strategy on-line, based on the observed resilience of the process.

Hence we come to a remarkable observation:

> *In all these cases it is necessary to have detailed understanding and knowledge of the dynamic resilience of the process.*

In our opinion controllability analysis is therefore a fundamental issue. It deserves more attention in literature.

The difference in applications of controllability analysis for the different identified fields is mainly determined by the level of detail of the analysis that is desirable, e.g. for controller design a more detailed analysis will be needed then during the evaluation of a resilience of a preliminary design. Note that we considered all the time the resilience of the process. In general however the exact process behavior is unknown. The best knowledge we have of the actual process is the model of the process. Controllability analysis is therefore always performed on the model we have of the process. In the initial phase of the identification the model is still highly inaccurate. It is therefore not useful to do an accurate analysis of the controllability of the process. On the other hand a first analysis gives initial insight in the critical aspects of the process behavior that should be carefully considered and modeled.

In the research here we will restrict ourselves to stable process behavior. Including unstable processes in the consideration will result in additional requirements we need to pose on the closed loop. These requirements will not only put additional limitations on the closed loop behavior we can achieve, but also make the analysis approach more complex. From both a theoretical as well as a more principle point of view this may seem a severe restriction on the applicability of the techniques. From a more pragmatic point of view excluding unstable behavior will not be a real restriction. We can always first design a very robust stabilizing controller for the process and then apply the analysis. One could argue that this is not a very elegant approach. Moreover in industrial practice it is a fact of live. Safety considerations will force us to deal with unstable process behavior with robust (SISO) controllers at the DCS level in the control hierarchy.

Based on the discussion in this section we now come to an initial problem statement for this thesis. We want to develop:

> ***The basic analysis techniques that enable detailed insight into the limitations that stable process behavior puts on the closed loop behavior, without the need of a detailed controller design, based on a given model of the open loop process and the desired behavior of the closed loop.***

In chapter two we will further detail the subject and discuss different aspects of the problem.

1.6 Organization of the thesis.

In this section we will give an overview of the content of the thesis. The thesis exists of five main chapters.

Chapter 2: Analysis of the input-output controllability problem.

In chapter two we will discuss in detail the different aspects related to input-output controllability of a process. We will discuss existing results and the relations input-output controllability analysis has with existing concepts. A closer look will be taken on how controllability analysis can be used profitably in identification and control. We formulate detailed questions we want to solve in this thesis.

Chapter 3: Basic theory.

In this chapter we will define and discuss basic system theoretical concepts that are needed to develop the concepts in the next chapters. The concept of directionality vital for MIMO systems will be developed, both in the frequency domain and the time domain. The Internal Model Control scheme, that will play a central role in the work here, will be introduced in the last section, as a way to make the influence the controller has on the closed loop behavior more transparent. The controller in the IMC scheme can be interpreted as an approximate inverse of the process behavior. As a consequence a more direct relation between the process behavior and the closed loop behavior is obtained.

Chapter 4: A first approach to asses process controllability

In section 4.2 the approach Morari proposed to deal with input-output controllability is discussed. In section 4.3 these techniques are applied on a spraydryer model. The inability to deal with the directionality of MIMO processes will be identified as one of the main short comings of this approach. In the second part of this section we will therefore develop a better insight in the directionality of systems. We will study in more detail the restrictions and opportunities of directionality in a process. In section 4.4 we will study how we can still manipulate the influence of non-minimum phase zeros to a more desirable output direction and what the consequences are of this action. In section 4.5 we will then take a closer look at ill-conditioned process behavior, i.e. processes whose gain is highly directional dependent, and how we still deal with this behavior in a robust way.

Chapter 5: A novel approach to asses process controllability

In this chapter we will take a different approach that enables us to deal with the controllability of the overall process. In section 5.2 and 5.3 the basic idea is discussed. This idea is based on the direct relation between the controller and the approximate inverse of the process behavior. In fact we use a specific approach to build-up the best approximate inverse of the process in a step by step approach, that is directly related to an identified priority in the requirements on the closed loop. The initial approach is based on an analysis of the process as function of frequency. The disadvantage of this approach is that in general stability of the resulting controlled process can not be guaranteed. We therefore extend the frequency domain approach to incorporate stability as a constraint on the analysis (section 5.4 and 5.5).

This approach turns out to give us a better understanding in the so-called square down problem. A classical result in this theory is that it is always possible with a dynamic compensator to obtain for a nonsquare problem a resulting square problem without introducing additional non-minimum phase zeros. We will show however that this might well result in inevitable high gains of the compensator. The only opportunity to reduce the gain of the precompensator is to introduce an additional non-minimum phase zero.

The above approach is completely based on the frequency domain and therefore in principle only suited for analysis of control problems that make use of time invariant linear controllers. Model Predictive Controllers however are in principle time variant controllers. The analysis approach is therefore less suited for this type of controller. In the third part of chapter (section 5.6 and 5.7) we therefore develop a controllability analysis approach based on the finite time domain.

It will turn out that this approach results in completely new insights in the relation between non-minimum phase zeros and non squareness and the resulting closed loop behavior of the process. It results in a good understanding of the relation between the gain of the compensator in down squaring and the introduction of additional non-minimum phase zeros in the resulting square process. An even more appealing result is the simple and intuitive interpretation of the so-called water bed effect for non-minimum phase processes as it occurs in the Bode Sensitivity Integral. (e.g. theorem 5.1 on page 166 and pg. 215 in [Sko96].

Chapter 6: Application of the developed techniques on example processes

In this section we will apply the techniques developed in this thesis on two industrial problems. The frequency domain approach is used to analyze the controllability of a quartz tube glass process. In a second example the time domain approach is applied on a high density polymerization process.

CHAPTER TWO

The Controllability Problem

2.1 Introduction.

In chapter one we discussed the importance of understanding the process behavior and how the process behavior may restrict the ability to control the process. We therefore formulated controllability analysis. The main goal is to obtain insight in the limitations that the process behavior puts on the closed loop behavior, i.e. on the performance of the controlled process. In this chapter we will take a more detailed look at the various aspects of controllability analysis.

The opportunities that control offers are restricted. We will therefore discuss opportunities and limitations of control. In section 2.2 we will take a closer look at how the control problem is specified in the criterion function and what the difficulties are one may encounter. The complexity of the actuall control design is the basis for the need of input-output controllability analysis. In section 2.3 we will review the developments that took place in the field of input-output controllability and related fields of research. In section 2.4 the use of controllability in the subsequent stages of control design is discussed. In section 2.5 we identify the shortcomings of current techniques available and formulate some basic questions we want to further investigate in this thesis.

2.2 *The control problem.*

In chapter one we emphasized the importance of carefully considering the closed loop requirements in the input-output controllability analysis, since control is used to obtain processes with a desired predefined closed loop behavior (figure 2.2.1). We thus have to specify what we mean with desirable closed loop behavior, before we can continue our discussion.

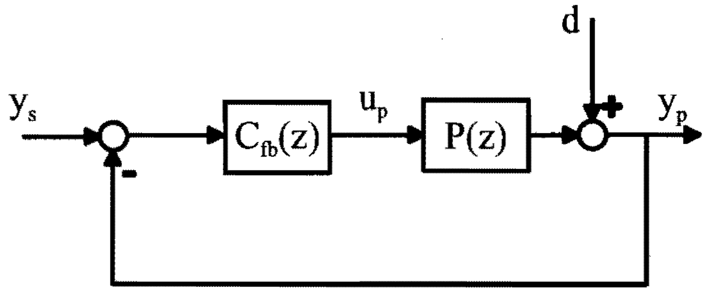

Fig. 2.2.1 The unit feedback scheme.

The specification of the desired behavior for the controlled process seems a simple task, e.g.:

The process outputs should stay at setpoint, i.e. $y_p(t)=y_s(t)$ or stay within a predefined band around the setpoint, e.g. $y_s(t)-\delta_y<y_p(t)< y_s(t)+\delta_y$ for every time instant t.

In many cases it turns out that quantification of requirements is not at all trivial and results in a lot of discussion. Requirements are expressed in terms like *settling time, rise time* and *overshoot* for setpoint behavior, *bandwidth, maximum amplification outside the bandwidth, a minimum attenuation at certain frequencies, minimum variance* and so on. For multivariable processes one is interested specifically in the behavior per output. The relative importance of a requirement will differ per output. A second step is to translate the requirements in the design technique used to design the controller. The actual translation will depend on the opportunities the design technique offers. Here we will focus on techniques that are based on quadratic functions. For example we could specify the above requirement more formally as:

$$\min_{C_{fb}(z)\in S(P)} \left\| W_y (I + PC_{fb})^{-1} W_d \right\|_2 \tag{2.2.1}$$

where $S(P)$ is the class of controllers that result in a stable closed loop behavior.

In the weight W_y the desired behavior at the output and the relative importance of the requirement is reflected. It is emphasized here that the required behavior at the output may have a dynamic nature. In many cases we are not interested in the instantaneous behavior at the output $y(k)$, but more in suppression of the slow variations. This is most easily expressed in the frequency domain and approximated by a frequency dependent function of W_y. For example the purity of a liquid stored in a buffer tank or vessel, is not determined by the momentary composition of the product entering the vessel, but by the impurities of the overall vessel content. This has to be reflected in the weight, for example by choosing the weight equal to a low pass filter whose time constant is related to the average time it takes to fill the vessel. Hence we are not interested in achieving a maximum bandwidth for the disturbance attenuation, but in achieving a maximum attenuation over a fixed frequency range. On the other hand if the product is a glass tube, whose diameter and wall thickness have to fulfill certain quality standards then we need to ensure the instantaneous quality of the tube dimensions, since the quality specs are to be met on any position in the tube. As far as the specification is concerned there is no preference for the attenuation of the disturbances in a certain frequency range.

The weight W_d may be used to reflect additional knowledge we have on the input signals[1]. For example if we have some impression on the spectrum of the disturbance this can be approximated in the weight W_d. For example for the glass tube process we would then incorporate the fact that the disturbances have a certain low pass coloring.

One may extend the above control structure to a two degrees of freedom controller, where we can decouple to a large extend the tracking behavior of the closed loop for setpoint changes from that of the disturbance reduction behavior. This more complex structure may be considered if:

[1] In [Kwa86] it is shown that this weight can also be used to enforce certain other properties of the closed loop. We will not use this here.

- The characteristic behavior of the setpoints significantly differs from the behavior of the disturbance.
- If the required behavior of the closed loop during setpoint changes is different from the behavior for disturbance reduction.

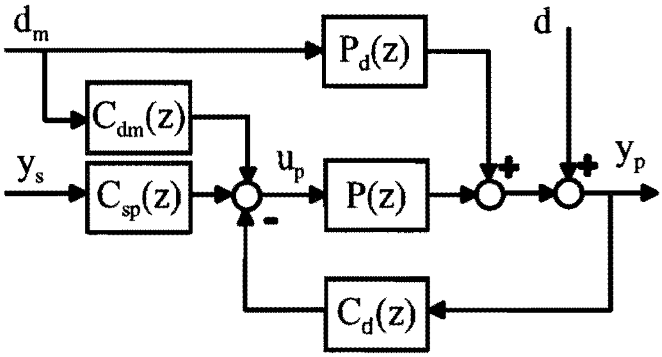

Fig. 2.2.2 The three degree of freedom control scheme. d_m the measurable disturbances and d the unmeasurable disturbances.

In the case that certain disturbances can be measured before they actually affect the behavior at the process outputs one may consider to add an additional feedforward controller in the scheme, i.e. a third degree of freedom[2]. The additional performance obtained with this third feedforward controller will depend on the following conditions:

- Can we make the feedforward bandwidth larger than the feedback bandwidth over a frequency range where the disturbance signal still significantly influences the process outputs.
- The achievable accuracy of the models both from the measured disturbance to the output and the process inputs to the outputs.

A control scheme for the case were we have three degrees of freedom is given in figure 2.2.2. Note however that the addition of these additional controllers does make the design problem more involved. It however does not basically change the design problem. We will therefore restrict ourselves here to a one degree of freedom controller.

[2] This possibility is frequently used in the current generation of Model predictive controllers. The main reason for the frequent application of feedforward disturbance reduction is the fact that the feedback control action of model predictive controllers are in general tuned to be slow.

A second class of controllers is the Model predictive controller. This controller is based on the real time optimization of a criterion function at each sample instant [Qin96, Lee96, Mus93], e.g. [Ric78]:

$$\min_{[u(k)\ \ ,\ u(k+N_c)]} \sum_{j=1}^{N_p} e(k+j)^T Q(j) e(k+j) \tag{2.2.2}$$

with:

$e(k) = y_s(k) - \tilde{y}_p(k)$

- N_c is the number of samples we consider in the optimization. After this (control) horizon it is assumed that the inputs stay constant.
- $\tilde{y}_p(l)$ is the predicted behavior of the process output at sample instant l.
- $y_s(l)$ is the desired behavior at the process output at sample instant l.
- N_p is the number of samples we consider in the prediction of the output. In general $N_p >> N_c$ to try to enforce nominal stability.
- $Q(j)$ The possibly time dependent weighting matrix.

The optimization may be performed subject to constraints. The above optimization is performed at each sample instant. After the optimization only the sample $u(k)$ is actually send to the process.

In model predictive control we have besides the predicted output optimization some additional possibilities to specify the desired output behavior, e.g. constraints and zones[3] [Qin96]. These types of specification add additional options to better specify the actually desired behavior. Many requirements can be translated better into these types of requirement. For example many specifications are dealing with ensuring that the process stays in a save range, i.e. the only requirement we have is that the variable stays below or above a certain value. This requirement is however hard, e.g. temperatures and pressures. Another well known example is the level in vessels. As long as the vessel is far from being full or empty there is in general no need to control the level. Only if the vessel level is close to full or empty control action is required. This requirement fits exactly with the zone specification. The setpoint only roughly approximates what we actually

[3] Zones are nonlinear specifications of the following type: Only if the process variable exceeds a certain predefined range, it will be controlled back into this range as if it is a setpoint. As long as value of the variable is inside the range it is not controlled.

want. Moreover a setpoint always consumes a degree of freedom (section 2.3), whether the variable is well in the zone or not. This can be a clear disadvantage in the case of a process with (significantly) more outputs than inputs. In this case the degree of freedom can be used to achieve some other requirement, if the variable is well in the zone.

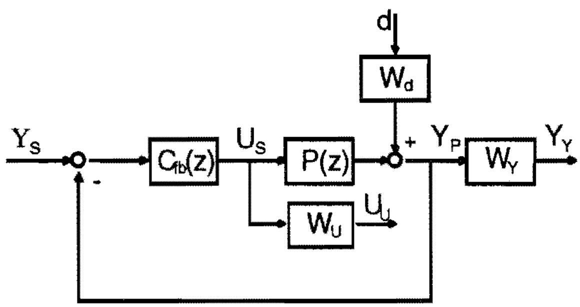

Fig. 2.2.3 An example of a two block criterion.

Until now we only discussed the requirements at the process output. Process inputs also have limitations. Well known are limitations on magnitude and rate of change, related to the physical limitations of the actuators. In refinery industry the process unit is an integral part of a larger plant. As a consequence many of the inputs to the process will not only influence the unit under consideration, but also disturb other units. For example in a distillation process most control inputs not only influence the unit itself but also the up-stream and down stream process. A second example of this interdependency is heat integration. At a number of places heat is extracted from the plant and at other places injected again in the plant. Changing the heat consumption somewhere in the plant will therefore influence other units in the circuit. In refinery industry one is specifically averse from large and fast variations of these type of variables. Inputs are an integral part of the specification of the requirements. The variability of process inputs have to be traded-off carefully against the variations at the process outputs.
In figure 2.2.3 an example of a control problem formulation is given were we are able to trade-off the closed loop behavior of the process at its inputs against the behavior at the output. For example the H_∞ problem formulation of figure 2.2.3 would result in:

$$\min_{C(z)\in S(P)} \left\| \begin{matrix} W_y(I+PC_{fb})^{-1}W_d \\ W_uC_{fb}(I+PC_{fb})^{-1}W_d \end{matrix} \right\|_\infty \tag{2.2.3}$$

An example for the model predictive controller translates to:

$$\min_{[u(k)\ \ldots\ u(k+N_c)]} \sum_{j=1}^{N_p} e(k+j)^T Q(j) e(k+j) + \sum_{j=1}^{N_c} \Delta u(k+j)^T R(j) \Delta u(k+j)$$

subject to: (2.2.4)

$U_{min} \le u(k\_+i) \le U_{max}$ for i=0,...,N_c

with: $\Delta u(l) = u(l) - u(l-1)$

Another aspect of the control loop we need to consider in the design is the robustness of the closed loop behavior for model errors and differences in the behavior at different operating conditions. In criteria like equation (2.2.3) this is accomplished by requiring a certain behavior of the transfer $C(I+PC)^{-1}$, based on the following sufficient condition for robust stability for additive model errors, $\Delta=P-M$ [Doy81, Mac89, Mor89]:

$1/\sigma_1(\Delta)>\sigma_1(C(I+PC)^{-1})$

Also these requirements therefore have to be incorporated in the weighting matrix W_u.

A point of attention in the definition of the weighting matrices is also the resilience of the process, i.e. the physical behavior of the process will limit the achievable performance of the closed loop. Not satisfying the resilience in the weighting matrices may result in a useless controller. It is therefore important that the closed loop behavior as specified in the weighting matrices is in accordance with the resilience of the process.

In section 2.3 we will further discuss the relation between the closed loop behavior and the open loop process behavior. In the discussion a large number of different and conflicting requirements were obtained. In the whole design procedure three basic steps can be discerned:

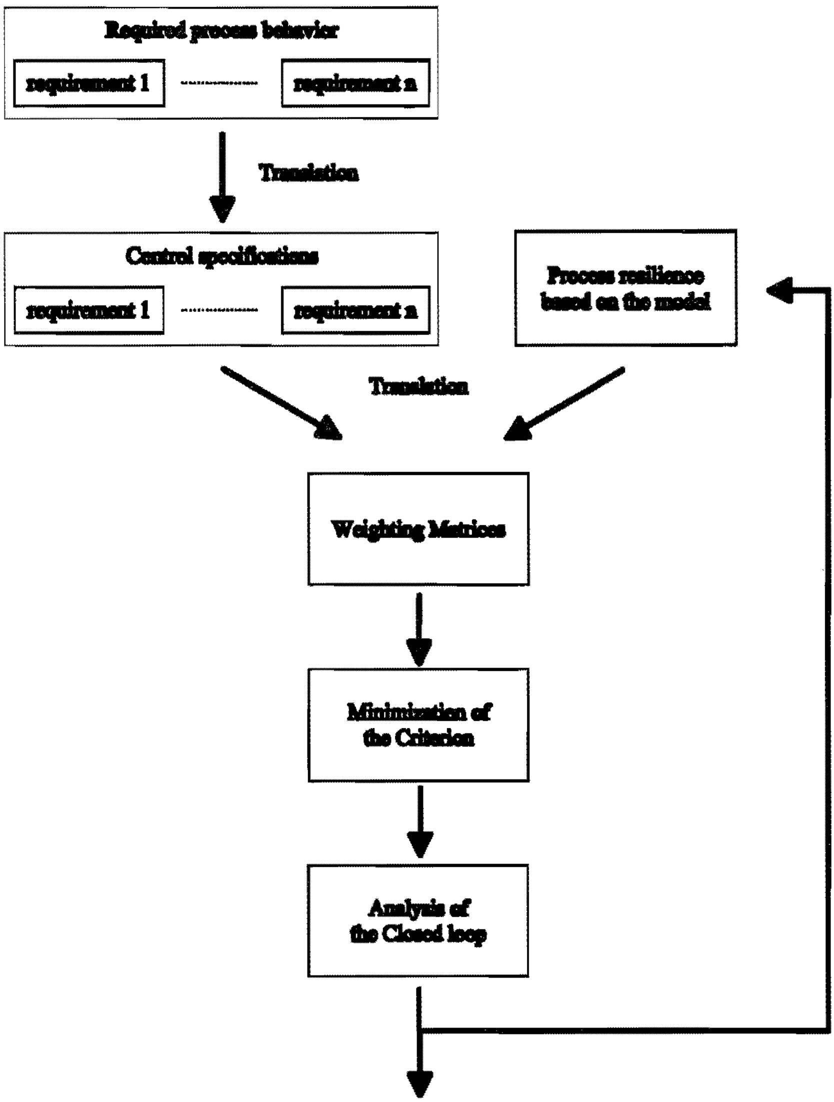

Fig. 2.2.4 The control design procedure.

1. In a first step we tried to quantify each of the requirements, e.g. by specifying rise times settling times, maximum overshoot, the frequency range over which a certain minimum disturbance attenuation is achieved and the maximum amplification of the disturbance outside the bandwidth, actuator restrictions, robustness margins, sensor noise and so on.
2. In a second step we have to approximate all these potentially conflicting requirements in a few weighting matrices. To obtain a sensible problem formulation we also need to ensure that the required behavior is in accordance with the process resilience. The translation of the required behavior into these weights is therefore in general already a compromise and simplification of the actual requirements as defined in step one[4]. In fact a first trade-off between the different requirements, based on the process resilience is actually made in this stage.
3. The actual trade-off of the requirement is made in the minimization of the criterion. It is important to note again that the minimization finds that solution that minimizes the norm of the object function. This can be different from what the designer considers the best trade-off for the actual control problem. The minimization itself is therefore again an approximation of the trade-off we actually want to make.

After analysis of the resulting controller we in general will need to adjust the weighting matrices in step two and perform step three again. The procedure is therefore an iterative procedure (figure 2.2.4). If we want to prevent a trial and error approach then a detailed understanding on how the required closed loop behavior is limited by the open loop process behavior is a prerequisite. The process resilience will be topic of the next section.

[4] One could of course consider to increase the complexity of the criterion function. Increasing the complexity of the optimization criterion has the advantage that in principle one obtains more freedom to specify the requirements. The disadvantage is a more complex trade-off in the optimization that is more difficult to understand and relate to the actual control problem. In most design approaches the complexity is therefore restricted to two block problems like equation (2.2.3) and (2.2.4).

2.3 Process controllability and related fields of research.

In this section we will take a closer look at the restrictions the process behavior poses on the control problem. Ziegler and Nichols already recognized in 1943 [Zie43] that the process behavior and the process equipment form an important limitation in the attainable performance of a closed loop. They state that controller and process form a unit and "credit or discredit for results obtained are attributable to one as much as the other". They observed that for a "miserably designed process" even the finest controller may not deliver the desired performance, despite the fact that they are "able to eke out better performance". They note an important missing characteristic of the process, which they called *"controllability"*, "the ability of a process to achieve and maintain the desired equilibrium value".

Controllability is defined very vague, but in accordance with engineering practice where a plant is called "controllable if it is possible to achieve the specified aims of control, whatever they may be" ([Ros70], pg.161).

In the middle of the sixties Brockett and Mesarovic introduced the term *"reproducibility"* of dynamic system [Bro65] as "the ability of the system to achieve with its output something which is desired of it." Basically the following question is raised, given a desired set of output responses and a set of time dependent inputs signals:

> Is it possible to find for each time function in the set of desired output responses an input signal from the defined set of input signals that generates this time function at the output of the given system.

The aim of the article [Bro65] is to try to formalize this idea.

In the early seventies van der Grinten introduced the term controllability factor ([Gri73], pg. 292 Dutch). He defined this factor as the ratio between the variance of the best possible causal and stable reconstruction of a signal and the variance of the actual signal itself. This factor leads to a simple tool to assess the theoretical limitations of (SISO) closed loop behavior.

In the sixties the term controllability has become equivalent to state controllability as introduced by Kalman. This changed the interpretation of the term controllability completely: A system is state controllable if one is able to bring the system in a finite time interval from an initial state to any other state. The relation the states have with the outputs is not considered. The relation with the old interpretation of the term is then completely lost.

To make a clear distinction with state controllability we will make use of the term input-output controllability. In this section we will discuss the different developments that occurred in the field of input-output controllability or are relevant for this field, based on the different aspects related to input-output controllability.

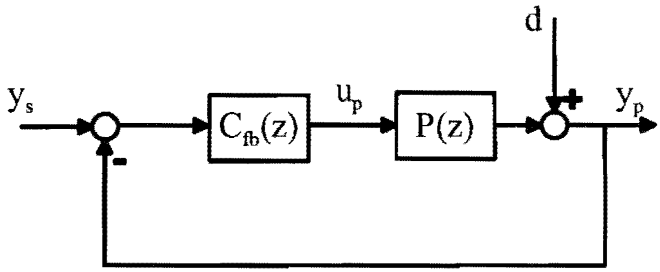

Fig.2.3.1 The unit feedback control scheme.

2.3.1 Functional controllability and degrees of freedom.

In the seventies a main area of research was to find necessary and sufficient conditions for the existence of a solution to the linear time invariant robust servomechanism problem [Dav76, Des80]. In the robust servomechanism problem one is concerned with:

- Asymptotic regulation of disturbances and references signals having a prefixed dynamic (marginally) behavior, i.e. the disturbance and reference are assumed to be combinations of sinusoidal signals with known period time. Amplitude and phase are assumed unknown.
- Robustness for a set of model parameters for a given model.

The question raised is under what conditions it is possible to design a controller that:

- Results in a stable closed loop behavior.
- Asymptotically *rejects*, i.e. blocks, periodic signals with known frequencies and asymptotically track given reference signals, most commonly steps, ramps and parabolic reference signals. The disturbance

$d(k)$ and reference signal $y_s(k)$ are assumed to fulfill the same known differential equation:

$$\sum_{i=0}^{s} \alpha_i x(k-i) = 0, \tag{2.3.1}$$

with $\alpha_0=1$ and $x(k)$ equals $d(k)$ respectively $y_s(k)$

- Ensures robust performance for a predefined set of model errors, i.e. ensure closed loop stability and the asymptotic behavior for all systems in the predefined set[1].

It is shown that [Des80] the above required behavior can indeed be achieved by a controller as long as:

- The rank of the transfer matrix of the system is equal to the number of outputs at those points in the complex plane that equal the roots of equation (2.3.1), i.e. the transfer matrix of the system has no non-minimum phase zeros that equal the roots of equation (2.3.1). In fact we require the system to be functional controllable (see below) at the roots of equation (2.3.1).

This point puts a clear condition on the system behavior. The servo problem is only solvable if the roots of equation (2.3.1) do not coincide with any of the zeros of transfer of the system.

In the servo problem the disturbance is expected to have some finite number of marginally stable modes, with exactly known period times. In a process control problem we are generally confronted with a continuous disturbance spectrum. The rank condition should therefore hold for all frequencies and not only for distinct frequencies.

The rank condition in fact gives us insight in the maximum number of outputs or requirements of the process that can independently be controlled,

[1] It is assumed that only the behavior of the process is uncertain. The polynomial in equation (2.3.1) is assumed to be known exactly. The description of the modeling error is assumed to be parametric based on a state space description of the system, i.e. the actual process is described by $[A+\delta A, B+\delta B, C+\delta C]$, where $[A, B, C]$ describe the nominal model. The uncertainty is described by δA, δB and δC, where it is assumed that the absolute value of each entry of these matrices is bounded by an $\varepsilon>0$ [Dav76, Des80].

i.e. reflects the degrees of freedom the process offers to control the process. The degrees of freedom equal the number of requirements that can be controlled independently. One possibility to obtain insight in the degrees of freedom of the process is to count the number of variables which can be freely assigned in the basic physical equations that make up the problem [Luy96, Pon94]. This approach has the clear advantage that it is based on physical knowledge of the process. A more formal definition of the degrees of freedom in MIMO systems is *functional controllability*[2], as defined by Rosenbrock [Ros70], for square systems. We will use a more general definition for non square systems, as given in [Sko96]:

> *A process P(z) with m inputs and p outputs is functional controllable if the normal rank of the process is equal to p.*

where:

> The *normal rank* of a transfer matrix $P(z)$ is the rank of the matrix almost everywhere in the complex plane, except for a finite number of points, where there is a local drop of rank. This finite number of points exactly coincides with the zeros of the system.

A necessary condition to enable independent control of all outputs is that we need at least as many inputs as we have outputs. This condition is however not sufficient: Each input should contribute differently to the behavior at the outputs, i.e. each input must be independent of the other inputs.

Note moreover that functional controllability itself is also only a necessary condition. It does not consider:

- stability of the closed loop behavior.
- the limitations at the process input.
- the restrictions that robustness put on the nominal closed loop system.

In the next section we will introduce the IMC scheme as a concept that enables us to study the above restrictions on the closed behavior in more detail.

[2] In [Bro65] a same concept was introduced as functional reproducibility (point b of the corollary at pg. 559 [Bro65]).

2.3.2 The Internal Model Control-scheme and input-output controllability analysis.

The first problem we encounter in input-output controllability analysis is the complex relation that the process and the controller have with the closed loop behavior of the process (figure 2.3.1). Morari [Mor83] proposed to replace the unit feedback scheme by the Internal Model Scheme (IMC figure 2.3.2), for the analysis of the input-output controllability of a process. The main advantage of the IMC scheme is its equivalence to a feedforward scheme (figure 2.3.3), in case of a perfect model, i.e. no model error. The feedforward scheme results in a more direct relation between the closed loop behavior and the model. The controller can be seen as an approximate inverse of the process, i.e. the better the controller resembles the inverse of the process, the better the closed loop process output is controlled. This direct relation between the inverse process behavior and the closed loop made Morari [Mor83, Mor87] define the concept of *perfect controller*. It is based on the observation that for a feedforward control scheme the best possible performance at the process outputs is obtained if we choose the controller equal to the inverse of the process[3]. Morari (pg.48, [Mor87]):

> *"Though "perfect control" cannot be achieved, it is of great theoretical and practical interest to determine how closely this ideal can be approached".*

The definition of "perfect" control is completely focused on the output behavior of the closed loop process. In section 2.2 we discussed that this predefined behavior results in requirements and restrictions on both the process inputs and outputs. This trade-off between the behavior at the process inputs and outputs is not directly reflected in the definition of perfect control. We moreover discussed in section 2.2 that the required behavior of a certain output is not always equivalent to requiring a maximum bandwidth. The required quality of a product stored in a tank and the level in a buffer tank were given as examples in section 2.2. In input-output controllability analysis we are therefore not directly interested in how far "perfect control" can be achieved. We are interested in how far we are able to achieve the specified closed loop behavior. What is preventing us from achieving this and what trade-off's are to be made to achieve the

[3] The use of the controllability factor [Gri73] to assess the controllability of a problem is based on this same idea.

closed loop behavior as good as possible. In section 1.5 we therefore explicitly added the required closed loop behavior in the initial problem statement.

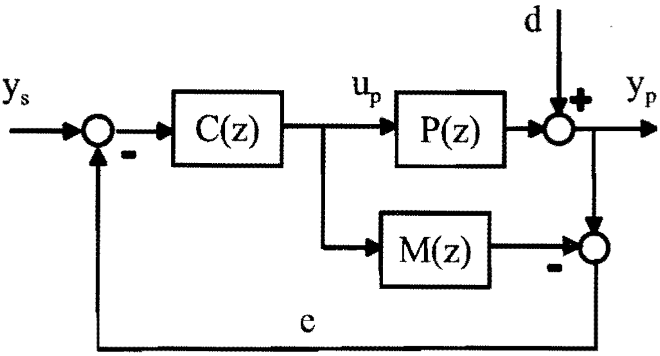

Fig.2.3.2 The internal model control scheme.

Our interest in the concept is the direct relation between the limitations that the process resilience puts on the closed loop behavior and the invertability of the process behavior. In this sense we state that:

Everything that limits a better approximation of the inverse process, potentially limits the input-output controllability of the process:.

If the effect indeed forms a limitation for the input-output controllability and the severity of this limitation is completely determined by the specified closed loop behavior. Hence controller design may be seen as finding an approximate inverse of the process behavior, that results in a closed loop that closest resembles the specified closed loop behavior. The following effects are identified to potentially restrict the input-output controllability [Mor83, Ros70]:

1. Delays, non-minimum phase zeros and unstable poles.
2. The limitations posed on the process input.
3. The robustness against model errors.

An important observation to be made is the crucial role of a model in the analysis. The best knowledge we have of the process is the model. A model therefore always forms the basis for the input-output controllability analysis. Hence the maximum achievable accuracy of the input-output controllability analysis is directly coupled to the accuracy of the model.

In the next subsections we will take a closer look at the relations that these effects have with the closed loop behavior of the process.

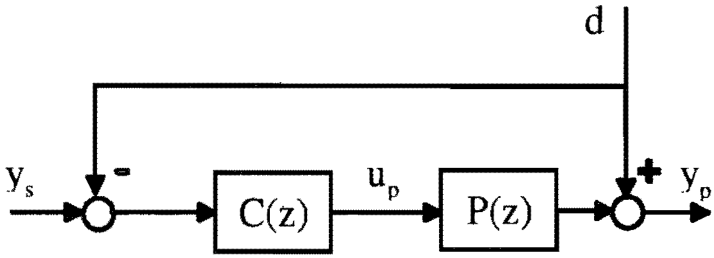

Fig.2.3.3 The feedforward control problem.

2.3.3 Delays, non-minimum phase zeros and unstable behavior.

At the end of the sixties an interest arose in the invertibility of MIMO systems. The importance of the invertibility of a system for decoupling, asymptotic tracking and disturbance reduction was recognized. Applications of invertibility were specifically made to feedforward control problems. In these problems a measurable disturbance is to be reduced at the outputs of the process or equivalently a reference trajectory has to be followed by the outputs of the process (figure 2.3.3). It is therefore directly related to the IMC scheme. It was recognized that the best controller achieving this aim was the inverse process, which results in an instantaneous compensation of the disturbance or tracking of the reference. The problem found was however that it was almost never possible to invert the process behavior.

In the seventies and beginning of the eighties decoupling of the closed loop transfer was an important item of research. A lot of effort was spend to the question of full decoupling, i.e. obtaining a full diagonal system. Questions like under which circumstances is it possible to completely decouple a system were answered, amongst many others [Hau83, Hol85a, Hol85b]. Results included determination of necessary conditions to decouple the non-minimum phase behavior of a closed loop system without introducing additional non-minimum phase zeros and the minimum number of non-minimum phase zeros needed to obtain full decoupling. In the same period we also see an interest in the more fundamental question how non-minimum phase behavior and delays restrict the invertibility of the system.

First results were obtained in the continuous time domain. It was recognized that, except the rank condition as discussed in section 2.3.1, the asymptotic behavior of the system for $s \rightarrow \infty$ was of vital importance for the invertibility of the process [Dav73, Sil69, Sil71, Wol76]. If the singular values of the transfer matrices asymptotically approach zero for $s \rightarrow \infty$ then the transfer can not be inverted by a proper transfer matrix. Techniques were therefore developed to separate the non invertible part of the transfer matrix [Mor71, Sai69, Sil69, Wol76] from the rest. The process model was therefore factored in two parts: One containing the behavior for $s \rightarrow \infty$, i.e. that could not be inverted and one containing the properly invertible behavior. The invertible part of the transfer was then inverted by a controller. Two main streams could be identified in this approach. One was based on the so-called geometric approach. It heavily relies on geometric interpretation of certain subspaces of the state space in realization theory [Kou76, Mor71, Won79]. The other approaches are more directly based on the transfer matrix [Sil69, Sil75, Wol76]. The disadvantage of the first group of techniques and more specifically the last two references is that it is rather abstract and does not have a direct link to the behavior at different outputs. The second group of algorithms has conceptually a direct relation with the outputs of the system. For discrete systems this behavior at infinity of the system is equivalent to the delay structure of the process. In the discussion we will further assume the process model to be a discrete time model.

The Silverman inversion or structure algorithm [Sil69, Sil75] results in a non unique factorization. At that time the consequences of the specific choice of the transformation on the output behavior of the resulting system was not well understood. Only one solution could be given a direct interpretation. The resulting delay structure in this case is a so-called inner or all pass transfer matrix. In fact this can also be shown to be the LQ solution for a non-minimum phase process to a step response [Hol85a]. In this procedure the "invertible" part of the process, that is compensated by the controller in the LQ problem, exactly matches the stable and stably invertible spectral factor of the spectrum expression derived of the transfer matrix of the system. [Doy84].

If steady state decoupling is required then the Wolovich-Falb interactor matrix [Wol76] results in a unique triangular structure. The triangular structure depends only on the ordering of the outputs in the problem. This idea has been used for controller design by [Tsi88]. It has been observed (theorem 4 [Tsi88]) that the resulting triangular structure containing the delays is an optimal controller that minimizes the quadratic error for a step change for each output, under the assumption that the outputs have absolute priority, i.e. first minimize this error for output one then use the freedom left to minimize the quadratic output error criterion for output two and so on. The outputs therefore are ordered in a decreasing importance; output one is the most important output, output two the second important and so on.

In both above approaches it observed that the resulting controller was unstable for non-minimum phase processes. The results therefore have been extended to take also non-minimum phase zeros into account. The inner outer factorizations [Doy84] can also be used to extract the non-minimum phase zeros from the system [Hol85b, Mor87, Sil75, Sko96]. In [Tsi89] a different approach was chosen. They propose to use bilinear transformations to transform the non-minimum phase zeros to infinity and to subsequently use the Wolovich-Falb interactor to extract the singularity at infinity and transform the system back.

For MIMO systems the factorization of the non-minimum phase zero and delay effects is not unique. It was shown that certain interpolation constraints have to be fulfilled, to ensure that the delays and zeros are not inverted [Mor87, Zaf87, Sko96]. These interpolation constraints however do not uniquely fix the non-minimum phase matrix. In MIMO systems one can often move the deteriorating effect of a non-minimum phase zero away from certain outputs to a given set of outputs, which are less important to control. The inner transfer matrix corresponds to the natural direction of the non-minimum phase zeros. No effort is made to push the zero to any direction. It therefore result in a minimum overall interaction and minimum overall control effort. The triangular structures can be seen as an extreme were we pushed the zeros as far as possible to the least important outputs. In principle however infinitely many other solutions exist in between these solutions (figure 2.3.4). It was noted in [Mor87], pg.343, that;

> *"though generically the effect of a zero can be pushed to any arbitrary output, this can cause large interactions and violent moves of the manipulated variables (process inputs) ..... ".*

Not much quantitative results seem to exist on this so-called *algebraic trade-off*, i.e. the relation between pushing the zero to a more desirable output direction and the consequences this has for the closed loop system and the controller.

Another fact not explicitly accounted for in the above approach is the analytic trade-off. For non-minimum phase systems reducing the sensitivity function for one frequency range, will inevitable result in an amplification of the frequency in another frequency band [Boy85, Fre85a, Sko96]. This "waterbed" effect is well known in SISO systems and generalized to MIMO systems, e.g. [Boy85, Fre85a, Fre88c]. These results however only provide insight in the limitations for very simple cases. A general more quantitative insight is not obtained from these complex integral relations. This is probably why for most of the results obtained one standardizes the analytic behavior, by optimizing an integral square error (ISE) or integral absolute error (IAE) criterion for the step response [Hol85a, Tsi88, Tsi89].

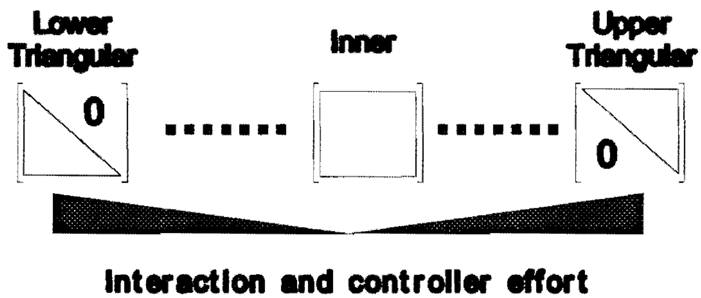

Fig.2.3.4 The N.M.P factorizations for a 2x2 model.

It is a well known fact that unstable poles limit the closed behavior of a process. For unstable processes the IMC scheme is unsuited for implementation, since stability of the closed loop is obtained by pole zero cancellation [Mor87]. For design and analysis the scheme can however still be used [Mor87]. For stability of the closed loop additional requirements have to be fulfilled. The controller has to cancel the unstable poles of the process. It will therefore introduce additional non-minimum phase zeros. Moreover the sensitivity function must cancel the unstable process poles. These requirements put additional interpolation requirements on the closed loop [Sko96] and make the analysis more complex.

2.3.4 Gain behavior of MIMO systems.

In the above discussions on non-minimum phase behavior we completely disregarded the effect of the gain. Although we implicitly obtained a stable inverse of the process in section 2.3.3 it might well result in unacceptably large amplitudes and rates of change at the process inputs. It is from an operational point of view not acceptable, to have too large gains. A second reason to limit the gain of the controller, is the model error. In multivariable systems robust stability and performance of the closed loop against model errors is enforced by limiting the gain [Fra87, Mac89, Mor87, Sko87 Sko96].

Insight in the gain behavior of the system is therefore needed. We can of course add the inputs as additional outputs in the criterion, as is done in most control design techniques. This however does not result in the desired additional insight in the gain behavior of the system. We are therefore back to the problem we discussed on the rank property. If the system was almost singular it is from a practical point of view still rank deficient. Where we draw the line exactly will again depend on the situation. The so-called *principal gains* of a transfer matrix were introduced to cope with this problem [Mac79, Pos81]. These gains are exactly the singular values of the transfer matrix as function of frequency. In the eighties this became the standard tool to analyze the gain behavior of MIMO systems [Mor83, Mor87, Sko91, Sko96]. The singular values are used in many techniques, like LTR, e.g. [Doy82, Fre88a], robust control [Doy84, Doy89, Fra87, Fre88c, Zam81] and robustness analysis [Doy84, Fre89a, Mor87, Sko96]. In fact the singular values of the transfer matrix directly generalize the gain concept of a SISO system.

Some drawbacks of the use of the singular values of the transfer matrix can be identified:

- The singular value decomposition can not be determined as finite dimensional analytic functions.
- The singular value decomposition is scaling dependent.
- The singular values cannot be directly related to the input-output behavior of the system.

These limitations make that the singular values only give a restricted insight in the process behavior. It resulted in several extensions of the concept and other techniques to circumvent these restrictions as far as possible. The fact that the singular values can not be expressed as simple transfer matrices of finite McMillan degree makes that they are not directly suited for most design techniques. The factorization can only be approximated as finite degree transfer functions [Doy84, Kou93]. A further restriction is the complete loss of phase information of the system. As a consequence the analysis can result in a too optimistic view on the input-output controllability, since the non-minimum phase behavior of the system can not be considered. The dependency of the solution on scaling is an unique property of MIMO behavior. The resulting conclusions of the analysis will depend on the applied scaling of the problem. This is of course inherent to the problem as far as it is related to restrictions of the actuators and for example the magnitude of disturbances at the output. It was however also observed that certain properties of the system, specifically related to sensitivity for certain modeling errors were intrinsic system properties. These process properties can never be dependent on the scaling. The choice of a realistic scaling is therefore seen as an important aspect of the analysis [Fre89, Mor87, Sko91, Sko96]. The scaling dependency of the singular value decomposition also resulted in research to obtain scaling independent techniques to asses specifically the robustness behavior of ill-conditioned systems. The minimum condition number and the use of the scaling independent relative gain array (RGA) were proposed [Mor87, Sko87c, Sko96]. The fact that the singular values can not directly be related to the detailed input and output behavior of the process, i.e. be used to assess the directionality of the plant, is a severe drawback. The singular vectors are needed to relate the singular values to a certain output or input. This however complicates the analysis drastically. It resulted in the introduction of additional techniques that enable a more direct relation between the gain and the behavior of the process at the inputs and the outputs of the system. Well known extensions are:

- The introduction of the disturbance direction in the analysis of the gain of the system [Mor87, Sko87b, Sko96]: The ability to reject disturbances of a system does not require all the singular values to be sufficiently large. It only requires the gain to be sufficiently large in the main direction of the disturbance.
- The introduction of μ-analysis [Doy82, Doy84, Fre89a, Pac93] to enable structural information on the model error to be incorporated in the

analysis of the design and to enable the assessment of robust performance.

- The inability to relate the input-output directionality more directly to the singular values made this singular value decomposition less suited to analyze the principle restrictions the process behavior poses on decentralized control schemes [Gro85, Hov92, Mor87, Sko91, Sko96]. Other techniques frequently proposed for this purpose is μ-analysis [Gro86, Mor87, Sko96].

2.3.5 Summarizing the results in input-output controllability analysis.

Let us summarize the results we obtained in this section for as far as it is relevant for input-output controllability analysis. The relation between closed loop behavior and the open loop process behavior is significantly simplified with the introduction of the IMC parameterization. This parameterization results in an affine relation between both the controller and the process behavior, represented by the model and the closed loop behavior. The control design problem is equivalent to finding the best stable approximate inverse of the process behavior. All effects that restrict us in approximating this inverse process by a stable transfer also restrict the input-output controllability of the process. Based on this relatively simple relation a number of techniques have been developed that enable us to obtain insight in how the closed loop behavior is limited by the open loop process behavior. These techniques however focus on one specific aspect that limits the performance of the closed loop, e.g. delays, non-minimum phase zeros or principal gains. It is therefore difficult to estimate the overall effect of these limitations on the closed loop. A second observation of existing tests is that they are highly standardized. Tests for delays and non-minimum phase behavior are limited to optimal solutions of certain criteria, like IAE or ISE. The directionality of the plant can also not accurately be investigated. The singular values are difficult to relate to the input-output behavior. For non-minimum phase effects only triangular and inner structures are used. This makes an analysis of the actual control problem difficult, since it is not possible to relate the limitations the process behavior puts on the specified closed loop behavior.

In the next section we will take a more detailed look at the different uses of controllability analysis.

2.4 The use of controllability analysis.

In chapter one we already discussed the central role of a good understanding on how the process behavior may limit the closed loop behavior of the process. In this section we will take a closer look at how controllability analysis can conceptually be applied in the different stages indicated in figure 2.4.1. The aim of this section is not to result in approaches, but only to indicate the potential advantages of the input-output controllability in the different stages of the design. Most of the research on input-output controllability analysis has historically been devoted to the process design stage [Ark86, Len82, Mar86, Mor83, Rus87]. In the past not much detailed information was available on the dynamic behavior of the process, during the process design stage. In these circumstances it was sufficient to have relative simple tests to scan the dynamic resilience of the process. In the last years detailed rigorous dynamic models become more and more available. It can therefore be expected that in the future more and more detailed dynamic knowledge is already available during the process design. As a result we may expect that more accurate techniques to analyze the process resilience will be needed. As a consequence controllability analysis will more and more resemble the more detailed analysis needed during the control design. In this section we will concentrate on the role of input-output controllability analysis during the controller design.

In figure 2.4.1 we have given an overview of the different stages at which input-output controllability analysis could be used. For an existing process unit we identify three main fields of application:

1. Identification.
2. Control design.
3. On-line process monitoring and adaptation of the control strategy.

In an identification approach on industrial processes a lot of detailed knowledge on the dynamics of the process is needed, before the final test signal can be designed. In general this knowledge is build-up in different steps [Bac87, Bac89, Bac92]:

- Detailed interviews with operators, operational management and different plant engineers.

- Different pretest to obtain insight in the main fast and slow dynamics and disturbance characteristics. Frequently applied tests are:
  - Freerun test to obtain insight in the disturbance behavior
  - Multilevel steps (staircase) test to obtain insight in the non linearity's of the process, gain insight in the slow dynamics and the steady state behavior.
  - A fast Pseudo Random Binary Noise Sequence (PRBNS) to obtain insight in the fast dynamics and delay structure of the process.

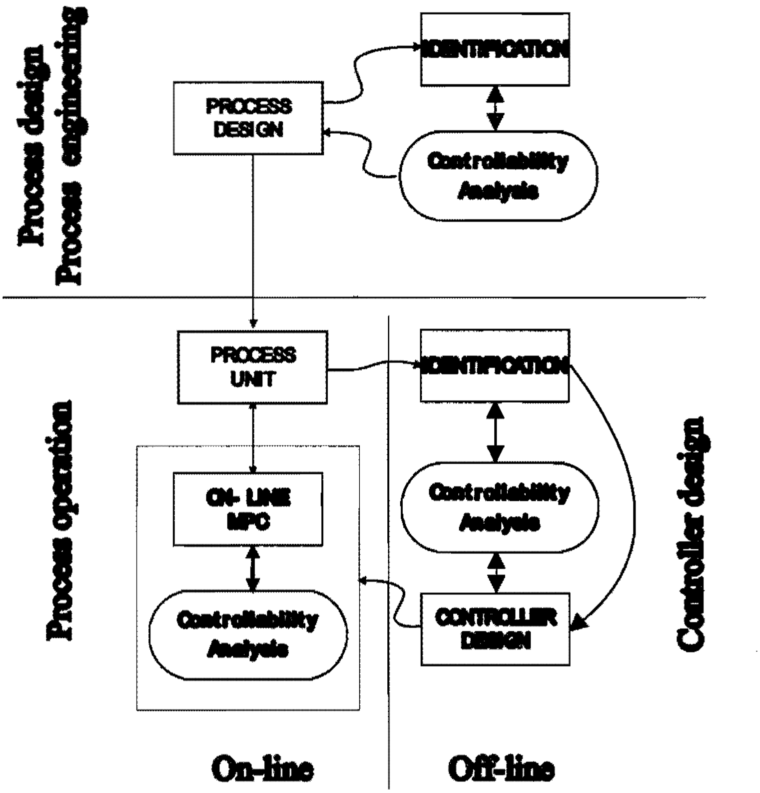

Fig.2.4.1 The central role of controllability analysis.

Based on the knowledge obtained from these tests the final Pseudo Random Noise Sequence is defined. In general it is well possible to make initial models at several occasions during this initial phase of the identification.

These models can then be used to perform initial controllability analysis on the process behavior. The advantage of this approach is that it enables:

- A more detailed insight in expected controlled behavior of the process early in the project, which has both a technical and a commercial significance. It enables for example:
  - A reconsideration of the control strategy already in this early stage. If necessary the control strategy can be adapted, e.g. adjustment of the structure of the control problem, i.e. the selected inputs and outputs.
  - A more accurate estimation of costs and benefits of the project.
- The obtained knowledge and insight in the behavior of the process critical for the control problem at hand, enables a better tuning of the next tests on this problem. e.g.:
  - More accurate knowledge of the expected bandwidth of the closed loop enables us to put more emphasis in the tests around the expected cross-over frequencies.
  - A more detailed knowledge of the disturbance characteristics and the process dynamics enables us to identify the process directions and frequency ranges that are most critical for robustness. In the test most emphasis can then be put on these aspects of the process behavior.

The outcome of the analysis enables us to tune the test signal to the control problem at hand. The less uncertain the preknowledge is the more the test signal can be seen as a set of pseudo random noise sequences filtered by an approximation of the inverse process behavior over a certain bandwidth. This idea is closely related to the approach proposed for ill-conditioned processes [Li96] and to closed loop identification over the frequency range were the controller has high gain [Hof95, Kla95, Sch92].

The level of details expected of the analysis will of course depend on the stage in the identification. It serves no purpose at all to perform a very detailed and accurate analysis if we have an inaccurate model of the process.

In controller design detailed knowledge of how process behavior and uncertainties restrict the closed loop performance of the process will considerably reduce the effort needed to tune the controller. Tuning of the weighting matrices will be less trial and error, but will be based on the process resilience and how it is related to the required closed loop behavior. The tuning part will mainly consist of adapting the weighting matrices to the specific trade-off's made in the optimization of the specific criterion. The amount of detail and accuracy of the input-output controllability analysis will be greatest for this off-line control design.

A last important aspect of controllability analysis is the on-line monitoring. Specifically in Model Predictive Control [Mus93, Qin96, Raw93] we frequently have the situation that certain inputs and outputs are constrained. In many cases these controllers actually push the process in its constraints, due to the fact that they push the process to the most economic operating conditions. It may also happen that certain actuators and sensors fail or that primary controllers are put on hand so that they are no longer available for the MIMO control system as manipulated variables, i.e. inputs. In all these cases the structure of the control problem changes and the process resilience may drastically change. For relative small control problems these ill conditioned situations can still be foreseen and analyzed in advance. For more complex problems the situation becomes too complex to analyze all possibilities. On the other hand the increased up time of the controllers does not allow us to switch off. It is therefore necessary that a supervisory system analyzes the situation and adapts the control strategy at the moment that an ill-conditioned situation occurs. This adaptation should be robust and preferably minimize the performance degradation as far as possible. The resulting controller can however not be expected to be as well designed as in the off-line case. Inevitably the design will be simpler and have a larger robustness margin. As a consequence it may be expected that for on-line adaptation of the strategy the amount of detail will be less than for the off-line design.

2.5 *Discussion and detailed problem statement.*

The problem statement as formulated in section 1.5 has lead us to study input-output controllability in the present chapter. Input-output controllability may help the user to better understand the limitations the model and its uncertainty pose on the control problem.

2.5.1 Discussion on the initial problem statement.

After a critical assessment of controllability analysis one can state that input-output controllability analysis itself is also a control design problem. In the IMC scheme both the design of the controller and the analysis aim at finding a stable approximate inverse of the process behavior. The goal of controllability analysis is however completely different from the goal of controller design. In controller design we are interested in finding the controller which results in a closed loop behavior which fulfills as good as possible the desired behavior. In input-output controllability we are interested in a detailed understanding of the dynamic resilience, i.e. getting a detailed understanding on how the process behavior limits the closed loop behavior and insight the trade-off's we need to make in fulfilling the different requirements. The requirements we pose on the procedure to be applied are therefore different.

In controller design the different trade-off's made are in general too complex to understand in detail. In controllability analysis the problem is therefore simplified and use is made of standardization. Simplification and standardization however results in a less detailed and therefore less accurate answer. In section 2.3 we have seen that in existing techniques delays, non-minimum phase behavior and gain behavior of the process are separately dealt with. The algebraic trade-off and the analytic trade-off for the non-minimum phase behavior are standardized at an inner or triangular structure and a corresponding ISE or IAE criterion. The gain behavior is in general analyzed by the singular values of the invertible part. This standardization enables a relatively simple insight in how each effect separately limits the input-output controllability of the process. In controllability analysis the above sketched procedure has become more or less the standard approach (section 4.2). The disadvantage of the approach is that this standarization results in a too simplified picture of the control problem. Input-output controllability analysis for control needs to be more specific. One needs to be able to relate the behavior of the process directly to the requirements that the closed loop process has to fulfill. It is therefore necessary to directly relate the limitations the process puts on the closed loop to specific outputs and inputs and the specific requirements we have on them. In applying the existing technique on a number of practical control design problems we encountered the following drawbacks (section 4.2 and 4.3):

1. The inner transfer matrix (section 2.3) is not directly related to the requirements posed on the closed loop. Some outputs will be more important than others. In general we would therefore like to turn the influence the zeros have to the least important outputs. This can however only be done if it does not result in an excessive decrease in performance at these least important outputs and does not result in an unrealistic increase of controller gain. The triangular structure in many cases reflects better the relative importance of the required behavior at the closed loop outputs. It exactly enables us to turn the influence of the non-minimum phase zeros as far as possible towards the least important outputs. Control of the important outputs however frequently has an unrealisticallly large impact on the least important outputs. It may furthermore result in an excessive increase of the principal gains of the controller. Both the inner transfer matrix and the triangular transfer matrix are in general not feasible. A more flexible trade-off the different effects is needed.
2. The principal gains of the process are not directly related to inputs and outputs. Hence they are not directly related to the requirements. In principle we would like to use the largest principal gains for the most important outputs. Current techniques do not offer us this possibility.
3. A standard procedure in controllability analysis is to separately deal with non-minimum phase behavior and the gain behavior. In many cases however the combination of the gain and the non-minimum phase behavior restricts the controllability of the process. In these cases the factorization in a part containing the non-minimum phase behavior and a part containing the gain behavior, is highly artificial and will result in inaccurate answers.

The above limitations of the current techniques are specifically present for large systems and when an accurate insight in the process resilience is desired. Other more detailed approaches are therefore needed. These approaches however tend to result in a loss of insight in the process resilience and in its relation to the requirements. Techniques based on optimization of one or more criteria, e.g. [Boy91] are therefore not suited for input-output directionality. On the other hand it is an inevitable fact of live that the best analysis of the controllability is the detailed controller design itself. The main difference between input-output controllability analysis and controller design is the different purpose of the techniques. The analysis does not have to result in the best possible controller, but should clearly reveal the different trade-off's to be made. On the other hand the approach should give us realistic answers directly related to the problem.

We therefore have to make a compromise between accuracy of the result and complexity of the analysis approach. As discussed the compromise will depend on the specific purpose of the controllability analysis. In this thesis we want to better understand the resilience of the process and how it is related to the requirements posed on the closed loop.

> *We will concentrate on developing the basic analysis techniques that provide detailed insight in the resilience of the process and that enable us to assess a more accurate insight in the input-output controllability.*

We will focus on developing more detailed analysis techniques. This will result in a more complex approach. The major question to be answered is therefore:

> *Is it possible to develop a structured procedure that enables us to obtain a detailed and accurate insight in the relation between the process resilience, the requirements and the different trade-off's to be made ?*

A prerequisite for a better analysis approach is a better understanding of the MIMO problem. It is interesting to note that:

1. Small MIMO control problems are considered simpler than problems with a large number of inputs and outputs.
2. Well conditioned problems are perceived to be easier to control and understand than ill-conditioned problems.
3. The additional dimension in the MIMO problem with respect to a SISO system is the directionality.

Reconsidering also the above drawbacks on existing techniques we see that a major deficiency is a lack of insight in the directionality of the non-minimum phase behavior, delays and the principal gain behavior. We therefore try to find an answer in this thesis to the question:

What are the directional restrictions that non-minimum phase behavior, delays and principal gains puts on the input-output controllability of a MIMO system?

In more detail we need to ask ourselves:

1. What freedom is left to change the direction of the restrictions to a more favorable direction, i.e. what are the consequences of changing this direction in terms of:
   - Interaction.
   - Gain of the controller.
2. What makes an ill conditioned process to be perceived as an inherently difficult to control process, i.e.:
   - What mechanisms make the problem difficult to control.
   - What performance can still be achieved for an ill conditioned process.
   - What conditions have to be fulfilled to achieve this performance.

The existing techniques that are available for controllability analysis are all related to the frequency domain. Hence they are less suited for model predictive control that in general results in time varying behavior. We will therefore ask ourselves whether it is possible to develop comparable techniques and understanding for model predictive controllers:

Is it possible to obtain input-output controllability techniques that enable us to obtain understanding on how the process behavior limits control with a receding horizon.

In more detail we need to ask ourselves:

1. How do non-minimum phase zeros and delays manifest themselves over a finite time horizon, i.e.:
   - How does the stability requirement for the controller manifest itself.
   - How does the stability requirement restrict the control problem over the finite horizon.

2. How does the gain behavior of the process manifest itself.
   - How do the directional restrictions manifest themselves.
3. How does directionality appear in the finite time control problem.

2.5.2 Summarizing the problem statement.

Let us summarize the discussion. In this section we further discussed the problem statement we gave in section 1.5:

> We want to develop the basic analysis techniques that enable detailed insight in the limitations that stable process behavior puts on the closed loop behavior, without the need of a detailed controller design, based on a given model of the open loop process and the desired behavior of the closed loop.

This resulted in a number of detailed questions in this section we want to further elaborate. These questions result in the following main points of attention:

1. In the control problem requirements are posed on the closed loop. Many of these requirements are directly related to specific outputs and inputs. We therefore need to understand how the process resilience is related to these specific requirements and how the different requirements can best be traded-off. The problem is that the mechanisms that play a role in the directional behavior of MIMO process behavior are not well understood. A better understanding of the directional behavior of MIMO systems has to be developed.
2. Most existing techniques split-up the analysis problem in different subproblems that are studied independent of each other. This subdivision is in many cases highly artificial, since each subproblem completely disregards the influence of other effects on the overall input-output controllability. It may therefore result in a considerable loss of accuracy. An approach need to be developed that considers the effect all limitations together have on the closed loop behavior.
3. Existing techniques as far as known are based on the frequency domain. Model predictive control, as frequently applied in industry, is based on (constrained) optimization over a finite time horizon. We therefore need techniques and tools that enable us to analyze the process resilience over a finite time horizon.

In chapter three we will introduce some system theoretic concepts and basic tools needed in chapter four and five. In chapter four and five we will then develop the insight and tools needed to find an answer to these questions.

In chapter four we will first shortly summarize the basic existing approach, as proposed by Morari [Mor83] (section 4.2) and show the use and some of the above discussed short comings on an example (section 4.3). In the last part of chapter four we then study the directionality of MIMO systems (section 4.4 and 4.5). In chapter 5 we will introduce an approach that enables us to deal with the overall resilience of the process without subdividing it in separate subproblems. In the first part of this chapter we will develop tools based on the frequency domain. In the second part of the chapter we will deal with the finite time domain input-output controllability analysis.

CHAPTER THREE

Basic Theory

3.1 Introduction.

In this chapter we will discuss some aspects of system and control theory. The concepts introduced and discussed form the basic theory needed for this thesis. In section 3.2 we will take a look at the different representations, i.e. model sets we wil use in this thesis. In section 3.3 we will discuss the frequency domain. In this section the emphasis will be on the directionality of poles, zeros and gains of MIMO systems. Also coprime factorization and inner outer factorizations are introduced in this section. In section 3.4 we will then turn to some aspects of behavior in the time domain, both the finite and infinite time domain. An input output Hankel and a Toeplitz matrices will be introduced. In the last section we will take a closer look at the feedback and introduce the Internal Model Control scheme. The behavior of the IMC scheme is then discussed.

3.2 Multivariable process behavior and its representations.

In this section we will discuss multivariable process behavior. The behavior of a process or system is represented by a model. We will restrict ourselves to those models that will be used in this thesis. The models used are linear time invariant and mostly discrete time (section 3.2.1). In subsection 3.2.2 we will take a look at the representation of signals.

3.2.1 Models of process behavior

Multivariable or *Multi Input Multi Output* (MIMO) models describe the behavior between different inputs and outputs. The models are assumed to be linear and time invariant.

- *Linearity* of a system means that the behavior of the characteristic behavior of the system is not dependent on the amplitude of the input signals, i.e. the *superposition principle* holds:
  1. If $y=Gu$ then $(\alpha y)=G(\alpha u)$,
     with u the input signal, y the output signal and G the relation between the input and output.
  2. If $y_1=Gu_1$ and $y_2=Gu_2$ then $(y_1+y_2)=G(u_1+u_2)$
- *Time invariant* means that the system behavior itself is independent of time. If we put a step on the system in steady state next year it will react in the same way as it will do if we apply the step now.
  If $y(t)=f(u(t))$ then $y(t\text{-}T)=f(u(t\text{-}T))$

Discrete linear time invariant models can be represented in a number of ways. We will shortly review the model sets used in this thesis:

The Finite Impulse Response (FIR) and the Finite Step Response (FSR) model

The impulse response model is based on the observation that in dynamic systems the current output not only depends on the current input, but also on past inputs:

$$y(k) = M_0u(k) + M_1u(k-1) + M_2u(k-2) + M_3u(k-3) + \ldots\ldots\ldots$$

The output signal $y(k)$ is a vector function: $y(k)=\begin{bmatrix} y_1(k) & y_2(k) & . & y_p(k)\end{bmatrix}^T$, with p the number of outputs. As for the outputs also the m inputs are ordered in the same way as a vector, $M_i \in \Re^{pxm}$, a so called *Markov parameter*, is the i-th entry of the FIR model describing the influence the input signal at i samples in the past has on the current output. In principle this model has infinitely many parameters:

$$y(k) = \sum_{i=0}^{\infty} M_i u(k-i) \tag{3.2.1a}$$

For stable systems the influence of the inputs $u(k-i)$ will become negligibly small for i large. In these cases the infinite impulse response model can therefore be approximated arbitrarily close by a Finite Impulse Response (FIR) model:

$$y(t) = \sum_{i=0}^{N_p} M_i u(t-i) \tag{3.2.1b}$$

In the same way as the FIR model we can define the Finite Step Response model. The inputs of the model are however not the absolute values of the input signals but the difference between the current value and the previous value of the input:

$$y(t) = y(t-(N_y+1)) + \sum_{i=0}^{N_p} S_i \Delta u(t-i) \tag{3.2.2}$$

with $\Delta u(t) = u(t) - u(t-1)$.

Note that the step response parameters have the following relation with the FIR parameters:

$$S_i = \sum_{j=0}^{i} M_j \tag{3.2.3}$$

Both the impulse response models and the step response models can only be used to describe stable behavior, since we have to truncate these models after some point. Both model types are therefore not suited to describe unstable process behavior. A further consequence of this observation is that we need a lot of parameters both for the FIR model and the FSR model to describe the behavior if both slow and fast dynamics are present. A fast sampling rate is needed to capture the fast effects in the behavior. As a consequence N_p needs to be large to capture the slow dynamic behavior close to steady state.

The above two model sets are nonparametric model sets, i.e. there is no relation assumed between the successive parameters of the model. These models will in general contain a lot of redundancy, e.g. a first order lowpass filter can be described by two parameters, the gain and the time constant. Parametric models can describe this behavior exactly with a finite number of parameters. Parametric models used in the thesis are the transfer matrix and the state space model.

The Transfer Matrix model:
The transfer matrix $H(z)$ of a system is a direct MIMO generalization of the Single Input Single Output transfer function, as described in undergraduate text books on control, e.g. [Kuo87, Dor80]:

$$y(z) = H(z)u(z) \tag{3.2.4}$$

with:

- $y(z)$ the Z-transform of the output vector $y(k)$
- $u(z)$ the Z-transform of the input vector $u(k)$
- $H(z)$ is an pxm transfer matrix with the i,j-entry equal to[1]:

$$g(i,j)\cdot\frac{\prod_{l=1}^{n_z(i,j)}\left(z-z_l(i,j)\right)}{\prod_{k=1}^{n_p(i,j)}\left(z-z_k(i,j)\right)} = g(i,j)\cdot\frac{z^{n_z(i,j)}+a_1(i,j)z^{n_z(i,j)-1}+\ldots\ldots+a_{n_z}(i,j)}{z^{n_p(i,j)}+b_1(i,j)z^{n_p(i,j)-1}+\ldots\ldots+b_{n_p}(i,j)}$$

The transfer matrix is a strong concept, but is not very practical for explicit use. MIMO process behavior is most frequently described by a state space model.

The State Space model
The State Space (SS) model describes the input output behavior of a system as:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{3.2.5}$$

[1] The fact that we assume that we may write a transfer matrix this way implies that we will deal with finite dimensional systems.

with:
- $A \in \Re^{nxn}, B \in \Re^{nxm}, C \in \Re^{pxn}$ and $D \in \Re^{pxm}$
- p the number of outputs
- m the number of inputs
- n the number of states

A state space model is characterized by the quadruple $[A,B,C,D]$, the so called realization. The vector $x(k) \in \Re^{nx1}$ is an internal variable, the so called state or state vector. It accumulates the history of the system for as far as it is of interest for the future of the system.

The state space model is directly related to:
- The transfer matrix:
  $$H(z) = C(zI - A)^{-1}B + D \tag{3.2.6a}$$
  since the Z-transform of the state space model equals:
  $$zx(z) = Ax(z) + Bu(z)$$
  $$y(z) = Cx(z) + Du(z)$$
- The impulse response model:
  $$\begin{bmatrix} M_0 & M_1 & M_2 & . & M_i & . \end{bmatrix} = \begin{bmatrix} D & CB & CAB & . & CA^{i-1}B & . \end{bmatrix} \tag{3.2.6b}$$
- The step response model:
  $$\begin{bmatrix} S_0 & S_1 & . & . & S_i & . \end{bmatrix} = \begin{bmatrix} D & D+CB & . & . & D+\sum_{j=1}^{i} CA^{j-1}B & . \end{bmatrix} \tag{3.2.6c}$$

A more detailed description of state space models can be found in many standard text books e.g. [Kai80, Kuo80].

The state space model not only results in a compact description of MIMO process behavior. It also has interesting numerical properties. Most calculations related to transfer matrices, impulse or step response models can be performed more efficiently and numerically stable with state space models than with the other models.

3.2.2 Signals and signal spaces

In this section we will introduce signal spaces and define some properties that are related to signals. We will restrict ourselves here to discrete time signals. For a more mathematically rigorous and more complete discussion on signals and their spaces we refer to standard text books, e.g. [Kwa91].

In the thesis use is made of discrete time signals. All discrete time signals are sampled continuous signals. A discrete signal $x(k)$ is in fact nothing else than an ordered sequence of values, that corresponds to the values the continuous signal has at the corresponding sample instances[2]. The samples are taken at equidistant time intervals. The sequence can be of finite or infinite length:

- Finite time signals are signals that only exist over a finite time interval, i.e. from one (finite) time instant to some other (finite) time instant. The sequence is called a *finite time signal.*
- Infinite time intervals can be one sided or two sided. We can distinguish between the following types of *infinite time* sequences:
    1. The signals that always existed in the past but stops to exist after a certain time instant, i.e. systems that "live" from "time instant" $-\infty$ until time instant k. The time axis is therefore unbounded to the left but bounded at the right.
    2. Signals that start at a certain time instant k and will always exist in the further future, i.e. systems that "live" from time instant k until "time instant" ∞ . The time axis is therefore unbounded to the right but bounded to the left.
    3. Signals that live for ever, i.e. signals that both always existed in the past and will always exist in the future. The time axis is therefore unbounded to both sides.

    The first two types of time signals of infinite length are called *left respectively right semi-infinite signals* the last type is called an *infinite time signal* [Kwa91 pg.15].

On the signal spaces, which are assumed to be real, since we only deal with real time dependent signals, we define:

- The *two* norm for a signal $x(t) = \begin{bmatrix} x_1^T(t) & . & . & x_p^T(t) \end{bmatrix}^T$ is defined as:

[2] We will not explicitly deal with sampling effects in this thesis. It is therefore assumed that discrete time signals are exactly representing the continuous behavior over the frequency range of importance.

$$\|x(t)\|_2 = \left(\sum_{j=1}^{p} \sum_{i=-a}^{b} x_j^2(i) \right)^{1/2} \tag{3.2.7a}$$

where a and b are determined by the time axis of the specific signal.

- The (Euler) norm of a vector x is defined as:

$$\|x\| = \sqrt{xx^T} \tag{3.2.7b}$$

- The *inner product* of two signals:

$$\langle x(t), y(t) \rangle = \sum_{j=1}^{p} \sum_{i=-a}^{b} x_j(i) y_j(i) \tag{3.2.7c}$$

with $x(t)$ and $y(t)$ defined as two signals. Note that: $\|x(t)\|_2^2 = \langle x(t), x(t) \rangle$

Concepts like linear dependency, projection and angle between two vectors are based on the inner product. Most classical theoretical models in physics, e.g. classical mechanics and electrical networks, are based on these spaces. The Euclidian norm and the above defined inner product have become the basic metric used in practical Engineering.

To fascilate later discussion we introduce the following linear discrete real signal spaces:

l^2: The linear space of all infinite time signals $x(t)$, with $\|x(t)\|_2^2 < \infty$ and $t \in (-\infty, \infty)$.

l^2_-: The linear space of all right semi infinite time signals $x(t)$, with $\|x(t)\|_2^2 < \infty$ and $t \in [0, \infty)$.

$l^2_-(N)$: The linear space of all right finite time signals $x(t)$, with $\|x(t)\|_2^2 < \infty$ and $t \in [0,N]$.

l^2_+: The linear space of all left semi infinite time signals $x(t)$, with $\|x(t)\|_2^2 < \infty$ and $t \in (-\infty, 0]$.

$l^2_+(N)$: The linear space of all left finite time signals $x(t)$, with $\|x(t)\|_2^2 < \infty$ and $t \in [-N,0]$.

If necessary the dimension of the vector of a time signal will be added as an additional superscript to the space, e.g. $(l^2_{-})^p$.

Note that in the definition of the spaces k has been replaced by zero. This is however without loss of generality, since we can always shift the original time vector with $-k$.

A second space of signals we will frequently use is that of the Fourier transformed signals. The Fourier transform of a signal x is defined as:

$$\hat{x}(z) = \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} x(k) z^{-k} \qquad \text{for all } z \in D \tag{3.2.8}$$

with D the unit circle.

If $x \in (l^2)^p$ then its Fourier transform is in the space L_2^p, which is defined as the class of all functions f from D to $\mathbb{C}^p$, such that[3]

$$\int_D \overline{f^T(z)} f(z) \frac{dz}{z} < \infty \tag{3.2.9a}$$

where:

$$\int_D \overline{f^T(z)} f(z) \frac{dz}{z} = \int_0^{2\pi} \overline{f^T(e^{i\theta})} f(e^{i\theta}) d\theta$$

On this space the following inproduct is defined:

$$\langle f, g \rangle = \int_D \overline{g^T(z)} f(z) \frac{dz}{z} = \int_0^{2\pi} \overline{g^T(e^{i\theta})} f(e^{i\theta}) d\theta \tag{3.2.9b}$$

$$\|f\|_2 = \langle f, f \rangle \tag{3.2.9c}$$

For the relation between the time domain and the frequency domain properties the Parseval theorem is essential:

Let x, $y \in (l^2_{-})^p$ be given.Let $\hat{x}$, $\hat{y}$ denote their Fourier transforms. Then we have:

$$\langle x, y \rangle_{l^2} = \langle \hat{x}, \hat{y} \rangle_{L^2} \tag{3.2.10}$$

and in particular $\|x\|_{l^2} = \|\hat{x}\|_{L^2}$

[3] The other spaces defined in the time domain can all be thought to be extended with zeros outside their domain of definition and as such be a subspace of the space l^2.

If we define for all $x \in (l^2_-)^p$ that $x(k)=0$ for $k<0$ then the Fourier transform $\hat{x}$ can be uniquely extended to an analytic function outside the unit circle. Moreover:

$$\int_D \overline{\hat{x}^T(z)}\hat{x}(z)\frac{dz}{z} = \sup_{\alpha>1} \frac{1}{\alpha}\int_{\alpha D} \overline{\hat{x}^T(z)}\hat{x}(z)\frac{dz}{z}$$

We will denote the space of analytic functions outside the unit circle, which satisfy:

$$\int_D \overline{\hat{x}^T(z)}\hat{x}(z)\frac{dz}{z} < \infty \tag{3.2.11}$$

by H_2. H_2 therefore consists of all Fourier transforms of signals in l^2_-. We will only use L_2^p and H_2^p if we want to explicitly express that we are looking at functions from C to C^p. Otherwise we will simply use L_2 and H_2. If we are only interested in the rational functions in L_2 and H_2, which we denote by $\Re L_2$ and $\Re H_2$.

3.3 The frequency domain and model properties.

In this section a number of properties of linear time invariant systems related to the frequency domain are summarized. We will discuss two main concepts:

- poles and zeros for MIMO systems
- the gain of MIMO systems.

In the last subsection we will take a closer look at some factorizations of transfer matrix.

3.3.1 Poles and zeros of MIMO systems

Poles and zeros of a transfer matrix are an important dynamic concept in control theory. Poles and zeros of Multi Input Multi Output systems can be defined on the basis of the Smith-McMillan form of a transfer matrix. The Smith McMillan form of the transfer matrix is derived from the Smith form of a polynomial matrix. Any rational *pxm* transfer matrix $H(z)$ can be written in the form:

$$H(z)=\frac{1}{d(z)}N(z)$$

with:

- $N(z)$ is a *pxm* polynomial matrix.
- $d(z)$ is the least common denominator of all the elements of $H(z)$

For the *pxm* polynomial matrix $N(z)$ the Smith form is given by ([Mac76]):

$$N(z)=U_1(z)\tilde{S}(z)U_2(z)$$

with:

- $U_1(z)$ and $U_2(z)$ are *pxp* and *mxm unimodular polynomial matrices* respectively, i.e. polynomial matrices with a constant non zero determinant or, equivalently, matrices that have a polynomial inverse.
- $\tilde{S}(z)$ a *pxm* polynomial matrix of the following structure:

  $$\tilde{S}(z)=\begin{bmatrix}\tilde{S}_r(z) & 0\\ 0 & 0\end{bmatrix}$$

  with: $\tilde{S}_r(z)$ a *rxr* diagonal polynomial matrix:

$$S_r(z)=\begin{bmatrix} s_1(z) & 0 & . & 0 \\ 0 & s_2(z) & . & . \\ . & . & . & 0 \\ 0 & . & 0 & s_r(z) \end{bmatrix}$$

with:

- $s_i(z)$ divides $s_{i+1}(z)$.
- r is the normal rank of the transfer matrix as defined in chapter two.
- p the number of outputs of the transfer matrix.
- m the number of inputs of the transfer matrix.

For the transfer matrix $H(z)$ the *Smith-McMillan* form is then defined as ([Kai80] pg.446, [Mac76]):

$$H(z)=U_1(z)S(z)U_2(z) \tag{3.3.1}$$

with:

- $S(z)$ a *pxm* rational matrix of the following structure:

$$S(z)=\frac{1}{d(z)}\tilde{S}(z)=\begin{bmatrix} S_r(z) & 0 \\ 0 & 0 \end{bmatrix}:$$

with: $S_r(z)$ a *rxr* rational diagonal matrix:

$$S_r(z)=\begin{bmatrix} \varepsilon_1(z)/\psi_1(z) & 0 & . & 0 \\ 0 & \varepsilon_2(z)/\psi_2(z) & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \varepsilon_r(z)/\psi_r(z) \end{bmatrix}$$

where:

- $\varepsilon_i(z)$ divides $\varepsilon_{i+1}(z)$.
- $\psi_{i+1}(z)$ divides $\psi_i(z)$.
- $\varepsilon_i(z)$ and $\psi_i(z)$ have no common polynomial divisor.

A polynomial function, say $f(z)$ may always be written as a product of linear factors:

$$f(z)=(z-\lambda_1)^{n_1}(z-\lambda_2)^{n_2}\,........(z-\lambda_s)^{n_s}$$

The elements λ_i are called the roots[1] of the polynomial function. The non-negative integer n_i is called the *multiplicity* of the root λ_i. The *degree* of the polynomial is defined as the sum of the multiplicities:

$$\deg(f(z)) = \sum_{i=1}^{s} n_i$$

The *McMillan degree n* of the transfer matrix is defined as the sum of the degrees of the denominator polynomials:

$$n = \sum_{i=1}^{r} \deg(\psi_i) \tag{3.3.2}$$

The poles and transmission zeros of the transfer matrix $H(z)$ are defined as[2]:

- The *poles* of the transfer matrix are defined as roots of the non zero denominator polynomials ψ_i, $i=1,..,r$.
- The *transmission zeros* of the transfer matrix are defined as roots of the non zero numerator polynomials ε_i, $i=1,..,r$.

If we have v poles (zeros) with a same value then we say the *multiplicity* of the pole (zero) is v. Note however from the Smith-McMillan form of the transfer matrix that we may encounter two types of multiplicity for a pole or zero of a MIMO transfer matrix. The multiplicity may occur only in the polynomial ε_r for a zero and only in the polynomial ψ_1 for a pole. This degenerate case is well known from SISO systems. In this case for a zero at z_i the rank of the matrix $H(z_i)$ decreases with one. For a MIMO transfer matrix we however see from the Smith McMillan form that it also may happen that the zero at z_i may result in a loss of rank of $H(z_i)$ larger than one, say s. In this case the last s polynomials, $\varepsilon_{r-s+1},\ldots,\varepsilon_r$ have as a root the zero at z_i. For example if we have a zero z_i with multiplicity, say two, than we may encounter two situations:

- The rank of the matrix $S_r(z_i)$ in equation (3.3.1) decreases with one. In this case the zero is degenerate, since ε_r must contain two zeros at z_i.

[1] The roots of a polynomial are also called the zeros. We will not use this term here to avoid confusion with the zeros of a transfer matrix.

[2] The pole zero structure at infinity, i.e. $z=\infty$, is not reflected in the Smith McMillan form, since the unimodular matrices destroy the structure at infinity. The infinite structure can be obtained after substitution of $z=\lambda^{-1}$ and evaluation of the Smith McMillan form of $H(\lambda^{-1})$ for $\lambda=0$ ([Kai80] pp. 449-450).

Hence we only have one independent output and input direction given by the corresponding column of $U_1(z_i)$ respectively row of $U_2(z_i)$.
- The rank of the matrix $S_r(z_i)$ in equation (3.3.1) decreases with two. In this case the zero z_i has independent output and input directions, since the unimodular matrices have full rank.

To make a distinction between these different cases possible the so called geometric multiplicity and the algebraic multiplicity were introduced [Mac76]. The *geometric* multiplicity corresponds with the rank deficiency of the transfer matrix $H(z_i)$. The *algebraic* multiplicity equals the total number of zeros at z_i. In the above example the algebraic multiplicity equals two. The geometric multiplicity in the first case equals one and in the second two. If the geometric and algebraic multiplicity are equal for all zeros the system is said to have a *semi-simple (zero) structure*. An equivalent argument results in the same result for poles.

A major difference between SISO systems and a MIMO system is the directionality of poles and zeros of a MIMO system. From the Smith-Mcmillan form of a MIMO transfer matrix we directly see that we can always associate one or more input directions and an equal number of output directions with each zero z_i and each pole p_i. For a zero at z_i the number of independent input directions and output directions is determined by the rank deficiency of the transfer matrix at $H(z_i)$. From the Smith McMillan form it is directly obtained that the zero input direction subspace for z_i, say I_{z_i} ,is equal to the complement of the space spanned by the first r-s columns of $U_2^H(z_i)$, i.e. for all $u_z \in I_{z_i}$ we have $H(z_i)u_z=0$, if the geometric multiplicity of the zero equals s. The output direction subspace for z_i, say O_{z_i} ,is equal to the complement of the subspace spanned by the first r-s columns of $U_1(z_i)$, i.e. for all $y_z \in O_{z_i}$ we have $y_z^H H(z_i)=0$. As a consequence of this directionality we have directions in which the zero or pole is influencing the system behavior and directions in which the zero or pole is totally not influencing the behavior. Based on equivalent arguments we can again define the same spaces for poles.

The above use of the Smith-McMillan form of a transfer matrix is a powerful technique to understand the multivariable concepts related to poles and zeros. For actual calculations and representation of MIMO systems the state space model is preferred above the transfer matrix. The relation between the transfer matrix and the state space model with minimal realization (3.2.6a) is equivalent to:

$$H(z) = \det^{-1}(zI - A)\{C adj(zI - A)B + \det(zI - A)D\}$$

Hence the eigenvalues of the transition matrix A of a minimal realization equal the poles of the transfer matrix[3]:

For a minimal realization $[A, B, C, D]$[4] the zeros of the transfer matrix $H(z)$ equal the generalized eigenvalues of the *(Rosenbrock) system matrix* $P_s(z)$ [Mac76, Ros70]:

$$P_S(z) = \begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix} \tag{3.3.3a}$$

or equivalently equal those values z_i for which $P_s(z_i)$ looses rank

$$rank(P_S(z_i)) < n + normrank(H(z)) \tag{3.3.3b}$$

This relation between the system matrix and the transfer matrix is obtained from the following relation:

$$\begin{bmatrix} (zI-A)^{-1} & 0 \\ C(zI-A)^{-1} & I \end{bmatrix} \begin{bmatrix} zI-A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} I & (zI-A)^{-1}B \\ 0 & C(zI-A)^{-1}B + D \end{bmatrix} = \begin{bmatrix} I & (zI-A)^{-1}B \\ 0 & H(z) \end{bmatrix} \tag{3.3.3c}$$

Note that the most left matrix in equation (3.3.3c) is a nonsingular matrix. The rank of the matrix at the right hand side must therefore equal the rank of the system matrix $P_s(z)$. Note however that the rank of the right handside matrix equals n+*normrank*($H(z)$). Hence the system matrix will loose rank if the transfer matrix $H(z)$ looses rank and the rank deficiency of the system matrix equals the deficiency of the $H(z)$. The input and output directions for a zero follow directly from:

[3] In the case that the realization is a non-minimum realization more zeros and poles will be found by the equations (3.3.3) [Mac76]. These poles and zeros do not influence the input output behavior and are therefore of no interest for this thesis.

[4] In the case that the realization is a non-minimum realization more zeros and poles will be found by the equations (3.3.3) [Mac76]. These poles and zeros do not influence the input output behavior and are therefore of no interest for this thesis.

$$\begin{bmatrix} 0 \\ y(z) \end{bmatrix} = \begin{bmatrix} zI - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} -x(z) \\ u(z) \end{bmatrix}$$

Let us show this for the input direction of the zero. If the system matrix looses rank at a point in the complex plane, say z_i, then an input signal $u(k)$ and an initial state x_0, must exist such that $y(t)=0$ [Mac76][5].

A pole p_j is called asymptotically stable if for $k\to\infty$, the impulse response p_j^k asymptotically approaches zero. If the pole response approaches infinity it is called unstable. From this fact we directly obtain that if and only if:

$|p_j| < 1$, i.e. if p_j is inside the unit circle, it is an *asymptotically stable pole.*

$|p_j| > 1$, i.e. if p_j is outside the unit circle, it is an *unstable pole.*

$|p_j| = 1$, i.e. if p_j is on the unit circle, it is a *marginally stable pole.*

Note that the statement that the behavior of a model is stable is equivalent to stating:

1. for $j\to\infty$, the impulse response A^j approaches zero, since the poles of the transfer are the eigenvalues of the transition matrix A.
2. for $j\to\infty$, the Markov parameters of the impulse response approach zero; $M_j\to 0$, equation (3.2.6b).
3. for $j\to\infty$, the step response parameters converge to a limit, due to equation (3.2.6c).

We define the following classes of transfer matrices with p outputs and m inputs, based on stability considerations:

- $\Re L_\infty^{pxm}$ The class of *pxm* proper rational transfer matrices.
- $\Re H_\infty^{pxm}$ The class of stable *pxm* proper rational transfer matrices.
- $\Re H_\perp^{pxm}$ The class of antistable *pxm* proper rational transfer matrices, i.e. having all poles outside the unit circle.
- $\Re H_\infty^{pxm}(k)$ The class of *pxm* proper rational transfer matrices that have only k unstable poles.
- $\Re H_\perp^{pxm}(k)$ The class of *pxm* proper rational transfer matrices that have only k stable poles.

[5] This is sometimes called blocking of the input signal.

Let us now take a closer look at the behavior of the zeros. It is well known that for a SISO system $H(z)$, with $H(\infty)>0$, the closed loop poles of a feedback system to the process input approach the transmission zeros of $H(z)$, if the gain k is increased, i.e. $k\rightarrow\infty$ [Kuo80, Dor80]:

$$\lim_{k\rightarrow\infty} k(I + kM(z))^{-1} = M^{-1}(z)$$

If a zero of the model is outside the unit circle this results for the above controller in an unstable pole in the closed loop. These zeros are called *non-minimum phase (N.M.P.) zeros.* Zeros that are inside the unit circle are called *minimum phase zeros.* Transfer functions that contain one or more non-minimum phase zeros are systems that have a step response with a so called *inverse response.* For square MIMO systems the same argument can be used to show that the non-minimum phase zeros put equivalent restrictions on the closed loop [Mac76, Kou76]. For nonsquare matrices the situation is more involved. We therefore refer to chapter 5.

Note that:
- For a MIMO system the non-minimum phase zero is only restrictive in the output direction of the non-minimum phase zero and not in the complementary direction.
- For discrete time systems delays are infinite non-minimum phase zeros.

A transfer matrix $X(z)\in \Re L_\infty^{pxm}$ that does contain at least one non-minimum phase zero is called a *non-minimum phase transfer matrix.* In all other cases the transfer is *minimum phase.* Note that for a square transfer matrix $X(z)\in \Re L_\infty^{mxm}$ this is equivalent to $X^{-1}(z)\in \Re H_\infty^{mxm}$. A square stable minimum phase transfer matrix is sometimes called *rational unimodular* or a *unit* [Scha92].

It is emphasized here that a multivariable zero is generically not related to any zero of an entry of the transfer matrix, i.e. a zero of transfer function from input j to output i will in general not be a zero of the MIMO transfer matrix. This is easily understood from the fact that a matrix will in most cases not loose its rank at the moment that one of the entries becomes zero. Although this seems a trivial remark it is frequently not seen or forgotten in practice. An important consequence of this fact is for example that a transfer matrix whose entries all have non-minimum phase zeros, i.e. have an inverse response, can still be minimum phase. However also the opposite can happen. All entries of the transfer matrix have minimum phase zeros, while the transfer matrix itself has some non-minimum phase zeros. In many papers the delay is dealt with separately from N.M.P. zeros. Note however that for discrete time systems a delay is nothing else then a zero at infinity and therefore just a non-minimum phase zero.

The transmission zeros are invariant with respect to feedback operations and non singular input and output transformations [Mac76]. As a consequence the zero structure is fixed at the moment that we have selected the inputs and outputs of the system. The only way to change the zero and/or its input and output directions is to change the input output structure. As a consequence [Mac76]:

> *"Thus the only practical cure for the acute situation associated with a system right-half-plane zero in a feedback loop transmittance is the modification of the system by taking a fresh set of inputs and/or outputs."*

Example 3.3.1

Consider the following transfer function:

$$H(z)=\begin{bmatrix} 0.6/(z-0.4) & 0.5/(z-0.5) \\ 0.5/(z-0.5) & 0.5/(z-0.4) \end{bmatrix}=\begin{bmatrix} 0.6(z-0.5) & 0.5(z-0.4) \\ 0.5(z-0.4) & 0.5(z-0.5) \end{bmatrix}/\left((z-0.5)(z-0.4)\right)$$

From the transfer matrix of the model we obtain that the model has poles at p_1=0.5 and p_2=0.4. The geometric and algebraic multiplicity of both poles is equal to two. We also obtain the delay structure having geometric and algebraic multiplicity equal to two. To further investigate the zero structure of the system we use the Smith-McMillan form of the transfer matrix:

$$\begin{bmatrix} -2 & 4 \\ (z-0.4) & (1-2z) \end{bmatrix}\begin{bmatrix} 50 & 0 \\ 0 & -25(z^2-2z+0.7) \end{bmatrix}/(6z^2-3.6z+1.2)\begin{bmatrix} 1 & 10(1-z)/6 \\ 0 & 1 \end{bmatrix}$$

The roots of the polynomial z^2-2z+0.7 the zeros of the transfer matrix we obtain one non-minimum phase zero at z_1=1.5477, one minimum phase zero at z_2=0.4523.The input and output directions corresponding to the poles and the delays are trivial since they span the whole space. The output direction of the zeros equal for:

- z_1=1.5477: output direction $[0.8858, -0.4640]^T$ and input direction $[0.6742, -0.7386]^T$.

- z_2=0.4523: output direction $[0.9997, 0.0238]^T$ and input direction $[0.6742, -0.7386]^T$.

The non-minimum phase zero will put a restriction on the controllability in the zero direction. Note however that the complementary output direction of the zeros $[0.4640, 0.8858]^T$ and $[0.0238, -0.9997]^T$ are not affected by the zero at 1.5477 respectively 0.4523.

End of example.

3.3.2 The gain behavior of MIMO systems

As for poles and zeros also the gain of a MIMO system is again directional dependent. This is easily seen from the static or steady state gain. Applying a singular value decomposition on the steady state gain of a transfer matrix $H(z)$ results in:

$$H(1) = C(I - A)^{-1}B + D = U\Sigma V^T \tag{3.3.4}$$

with:

- $U \in \Re^{pxp}$ and $U^T U = UU^T = I$.
- $V \in \Re^{mxm}$ and $V^T V = VV^T = I$
- $\Sigma \in \Re^{pxm}$ and:

if $p>m$: $\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}$ and $\Sigma_1 \in \Re^{mxm}$ with diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots\ldots \sigma_m \geq 0$

if $p<m$: $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix}$ and $\Sigma_1 \in \Re^{pxp}$ with diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots\ldots \sigma_p \geq 0$

if $p=m$: $\Sigma = \Sigma_1$ and $\Sigma_1 \in \Re^{pxp}$ with diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots\ldots \sigma_p \geq 0$

If we define the rank of $H(z)$ as r then

- If $r<min(p,m)$ then the last $min(p,m)-r$ diagonal entries of Σ_1 equal zero.
- If $p>r$ then we may write U as: $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$,with:
  - $U_1 \in \Re^{pxr}$ The orthogonal matrix whose first r columns span the *range* of the output space.
  - $U_2 \in \Re^{px(p-r)}$ The orthogonal matrix whose last p-r columns span the *complementary range* of the output space[6].
- If $m>r$ then we may write V as: $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$, with:

[6] The complementary range therefore equals the subspace in the output space that can not be reached by the via the inputs of the system.

- $V_1 \in \Re^{mxr}$ The orthogonal matrix whose first r columns span the *complementary kernel* of the input space.
- $V_2 \in \Re^{mx(m-r)}$ The orthogonal matrix whose last m-r columns span the *kernel or nullspace* of the input space.

The singular value decomposition decomposes the steady state gain in a number of independent steady state gains, i.e. the singular values, with a corresponding input and output direction, the right respectively left singular vectors [Mac79, Mac89]. To use this tool also for a dynamic analysis of a system it has been extended to the frequency domain. The singular value decomposition is then applied as a function of frequency [Mac79, Mac89], like the Bode plot:

$$H(e^{j\phi}) = C\left(e^{j\phi} I - A\right)^{-1} B + D = U(e^{j\phi})\Sigma(e^{j\phi})V(e^{j\phi})^H \qquad (3.3.5a)$$

with the non zero entries of $\Sigma(e^{j\phi})$ again equal to positive real values. The singular values are the so called *principal gains* [Mac79].

The above observed importance of the rank of the transfer matrix formed the basis for the definition of functional controllability in chapter 2. The singular value decomposition now gives a different view point on this problem. As observed in chapter two functional controllability is a minimum requirement for a process to be controllable, i.e. *necessary* but not *sufficient*. It may happen that a transfer matrix has certain directions, which are in principle controllable, but have such a small principal gain, that practically spoken it is not possible to control them, due to limitations at the process inputs.

Example 3.3.2

Assume two constant processes with a transfer matrices:

$$P_1(z) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } P_2(z) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Both processes have a gain per entry equal to one. From the singular value decomposition we however obtain:

$$P_1(z) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

respectively:

$$P_2(z)=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Hence $P_1(z)$ is not invertible, while $P_2(z)$ is perfectly invertible. This major difference in behavior is due to the minus sign in the entry (1,2) of $P_2(z)$. It makes the columns of the transfer matrix orthogonal, while those of $P_1(z)$ are completely dependent. This exactly pinpoints the essential difference between SISO and MIMO control. The additional difficulty of MIMO control is the linear relations between the different columns and rows of the transfer matrix, i.e. the directionality. The singular values reflect these dependencies. From the singular value decomposition of $P_1(z)$ it is directly seen that the output direction $[-1 \;\; 1]^T$ is uncontrollable One degree of freedom is left to control one output direction. This could be output one or output two. For the feedforward control scheme in figure 2.3.3 this would result in the controllers:

$$C(z)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \text{ respectively } C(z)=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

It is therefore possible to control one of the two outputs at the cost of interaction on the other output. The interaction that we obtain at the other output and the additional increase in the gain of the controller is completely determined by the output direction of the largest principal gain. Let us take a look at another constant process Assume that we want to control the first output of a constant rank one $2x2$ process with left singular vector equal to $[a,(1-a^2)^{1/2}]^T$, $|a| \le 1$ corresponding to the nonzero singular value σ_1. Hence the gain of the controller equals $1/(\sigma_1 a)$ and the interaction at output two equals $(1-a^2)^{1/2}/a$. If $|a| << 1$, i.e. σ_1 is coupled primarily to the second output, the interaction and also the gain of the controller become extensively large. On the other hand if the absolute value of a is near one the interaction will be small and the additional gain of the controller equals almost the inverse of the largest singular value.

End of example

The singular values of this decomposition have become a basic tool in MIMO control, since:

1. they give a fast and easy insight in the dynamic behavior of the model.

2. the largest singular value and the smallest singular value enables us to bound from below and above the dynamic gain of the system. Specifically the largest singular value will be frequently used. As we will see later it enables us to introduce robustness measures in MIMO control, analog to the gain margin in SISO control.

Note that the largest singular value of the transfer matrix $H(z)$ can be formalized as:

$$\sigma_1\left(H(e^{j\phi_i})\right) = \sup_{u(e^{j\phi_i})\neq 0} \frac{\left\|y(e^{j\phi_i})\right\|}{\left\|u(e^{j\phi_i})\right\|} = \sup_{u(e^{j\phi_i})\neq 0} \frac{\left\|H(e^{j\phi_i})u(e^{j\phi_i})\right\|}{\left\|u(e^{j\phi_i})\right\|} \qquad (3.3.5b)$$

with $\phi_i \in [0 \quad \pi]$. The so called *infinity norm* on the transfer matrix is defined as the maximum value of this largest singular value:

$$\left\|H(z)\right\|_\infty = \sup_{\|u\|_2\neq 0} \frac{\|y\|_2}{\|u\|_2} = \sup_{\|u\|_2=1} \|y\|_2 = \max_{\phi\in[0 \quad \pi]} \sigma_1\left(H(e^{j\phi})\right) \qquad (3.3.6a)$$

In engineering terms, find the maximum possible gain for the transfer matrix.

The infinity norm is a so called *induced* or *operator norm*. As a consequence of this fact the norm is submultiplicative:

$$\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty \qquad (3.3.6b)$$

Note that based on the Parseval theorem the above norm is also found if we pose the problem as a time domain problem. find $u(t) \in (l_+^2)^m$ such that $\|y(t)\|_2$, with $y(t) \in (l_+^2)^p$ and $x(0)=0$, is maximal:

$$\left\|H(z)\right\|_\infty = \sup_{\substack{u(t)\in(l_+^2)^m \\ \|u(t)\|_2\neq 0}} \frac{\|y(t)\|_2}{\|u(t)\|_2} \qquad (3.3.6c)$$

Remarks:

- In mathematics a distinction is made between the operator and the corresponding matrix. For example the operator is independent of the specific basis chosen for the inputs and outputs. The matrix representation is of course not. We will however not make this formal distinction and use the two concepts as one, which for our purpose will only result in an abuse of notation.

- The input signal and therefore the resulting output signal for which the maximum gain, i.e. the infinity norm, is obtained is a sinusoidal signal. Formally speaking this signal does not belong to l^2_+, since the two norm of the signal is infinite. Distribution theory is needed to extend the space to incorporate also these types of signals as the limit case [Kwa91]. We however disregarded this fact here.
- The singular values are not scaling invariant. A sensible scaling therefore has to be applied to the transfer matrix (see section 3.5).

It is well known that a MIMO system is specifically difficult to control if there is a large difference between the principal gains of the system. In this case the system is called *ill conditioned*. If the principal gains are all in the same range the system is called *spatially round*. An easy indication for the condition of the system is to divide the largest singular value by the smallest singular value, this is the so called *condition number*:

$$cond\left(H(e^{j\phi})\right)=\frac{\sigma_{\max}\left(H(e^{j\phi})\right)}{\sigma_{\min}\left(H(e^{j\phi})\right)} \quad \text{for } \phi\in[0,\pi] \qquad (3.3.7a)$$

The singular values are scaling dependent and therefore the condition number is scaling dependent. As a consequence the conditioning can be bad due to the applied scaling of the transfer. Therefore the scaling independent *minimized condition number* was introduced [Mor89 Sko96]:

$$cond_m(M(e^{j\phi}) = \min_{R,S\in\Re_d} cond\left(SM(e^{j\phi})R\right) \quad \text{for } \phi\in[0,\pi] \qquad (3.3.7b)$$

with:

- $\Re_d$ the set of diagonal matrices with real entries.

The interpretation of the minimum condition number follows directly from equation (3.3.7b). It is the smallest condition number obtainable for the transfer matrix after diagonal scaling. Hence a minimum condition number significantly larger than one indicates that the process is certainly ill-conditioned.

3.3.3 Factorizations of the transfer matrix of MIMO systems.

A model $M(z)\in \Re L_\infty^{pxm}$ can always be factored as a function of two stable transfer matrices $N(z)\in \Re H_\infty^{pxm}$ and $D(z)\in \Re H_\infty^{mxm}$:

$$M(z) = N(z)D^{-1}(z) \tag{3.3.8}$$

This factorization ensures that all non-minimum phase zeros of $M(z)$ must be contained in $N(z)$ and all unstable poles of $M(z)$ must be contained in $D(z)$, since both $N(z)$ and $D(z)$ are stable. A second interesting observation is that the factorization is not unique. Post-multiplying $N(z)$ and $D(z)$ with any $X(z)\in \Re H_\infty^{mxm}$ results in another factorization $N(z)X(z)\in \Re H_\infty^{pxm}$ and $D(z)X(z)\in \Re H_\infty^{mxm}$ of the model. Note that the matrix $X(z)$ may contain non-minimum phase zeros, since we only require stability of the transfer matrix. Hence $N(z)$ may contain additional non-minimum phase zeros compared with $M(z)$ and $D(z)$ may contain additional non-minimum phase zeros compared with the unstable poles of $M(z)$. This is undesirable, since they result in additional restrictions on the invertibility of the transfer matrix. All non-minimum phase zeros that are not related to the unstable poles or non-minimum phase zeros of $M(z)$ have to be non-minimum phase zeros of both $N(z)$ and $D(z)$, since they have to cancel in the product at the right hand side of equation (3.3.8). If the pair $\{N(z), D(z)\}$ does not contain any of these non-minimum phase zeros, it is called a coprime pair. Hence the only common dynamics of $N(z)$ and $D(z)$ are minimum phase zeros and stable poles, i.e.: For $N(z)\in \Re H_\infty^{pxm}$ and $D(z)\in \Re H_\infty^{mxm}$ define:

$$\tilde{N}(z) = N(z)X(z) \text{ and } \tilde{D}(z) = D(z)X(z)$$

If pair $\{N(z), D(z)\}$ is coprime then $\tilde{N}(z) \in \Re H_\infty^{pxm}$ and $\tilde{D}(z) \in \Re H_\infty^{mxm}$ is coprime if and only if $X(z)\in \Re H_\infty^{mxm}$ and $X^{1}(z)\in \Re H_\infty^{mxm}$, i.e. $X(z)$ is a unit.

Let us summarize some facts of coprime factorization of a model [Vid85]:

- The pair of matrices $N(z)\in \Re H_\infty^{pxm}$ and $D(z)\in \Re H_\infty^{mxm}$ is a *right coprime pair* of $M(z)\in \Re H_\infty^{pxm}$ if and only if:

  $$M(z) = N(z)D^{-1}(z) \tag{3.3.9a}$$

  and there exist matrices $X(z)\in \Re H_\infty^{mxp}$ and $Y(z)\in \Re H_\infty^{mxm}$, such that:

  $$X(z)N(z) + Y(z)D(z) = I \tag{3.3.9b}$$

  This equation is called the *right Bezout identity*.

- The pair of matrices $N_l(z)\in\Re H_\infty^{p\times m}$ and $D_l(z)\in\Re H_\infty^{m\times m}$ is a *left coprime pair* of $M(z)\in\Re H_\infty^{p\times m}$ if and only if:

  $$M(z)=D_l^{-1}(z)N_l(z) \tag{3.3.10a}$$

  and there exist matrices $X_l(z)\in\Re H_\infty^{m\times p}$ and $Y_l(z)\in\Re H_\infty^{p\times p}$ such that:

  $$N_l(z)X_l(z)+D_l(z)Y_l(z)=I \tag{3.3.10b}$$

  This equation is called the *left Bezout identity*.
- If $\{N(z), D(z)\}$ and $\{D_l(z), N_l(z)\}$ are a right respectively left coprime factorization of $M(z)\in\Re H_\infty^{p\times m}$ then there exist $X(z)\in\Re H_\infty^{m\times p}$, $Y(z)\in\Re H_\infty^{m\times m}$, $X_l(z)\in\Re H_\infty^{m\times p}$ and $Y_l(z)\in\Re H_\infty^{p\times p}$ such that:

  $$\begin{bmatrix} Y(z) & X(z) \\ -N_l(z) & D_l(z)\end{bmatrix}\begin{bmatrix} D(z) & -X_l(z) \\ N(z) & Y_l(z)\end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix} \tag{3.3.11}$$

  This equation is called the *double Bezout identity*. It can be shown that the factors $X(z)$, $Y(z)$, $X_l(z)$ and $Y_l(z)$ are not unique. However for given $N(z)$, $D(z)$, $N_l(z)$, $D_l(z)$, $X(z)$ and $Y(z)$, the pair $X_l(z)$ and $Y_l(z)$ is uniquely determined ([Sch92] proposition B.2.3).

A specific coprime factorization of a stable transfer matrix, that will play an important role in this work is the (co-) inner outer factorization.

Given a transfer matrix $M(z)\in\Re H_\infty^{p\times m}$, with $p\geq m$ and without zeros on the unit circle a coprime factorization exists, called the *inner outer factorization*:

$$M(z)=N(z)D(z) \tag{3.3.12a}$$

such that:

- $D(z)$ is called the *outer*.
  - $D(z)\in\Re H_\infty^{m\times m}$ and $D^{-1}(z)\in\Re H_\infty^{m\times m}$, i.e. $D(z)$ is a unit.
  - $M^{\sim}(z)M(z)=D^{\sim}(z)D(z)$, where $D^{\sim}(z)=D^{T}(1/z)$, is the so called *adjoint*.
- $N(z)$ is an *inner* transfer matrix also called *lossless* or a (stable) *all-pass* transfer matrix:
  - $N(z)\in\Re H_\infty^{p\times m}$ and all its zeros are non-minimum phase zeros.
  - $N^{\sim}(z)N(z)=I$.

In the case that $p\leq m$ then an equivalent left coprime pair can be defined.

Given a transfer matrix $M(z)\in \Re H_{\infty}^{p\times m}$, with $p\geq m$ and without zeros on the unit circle then there exist a coprime factorization, called the *co-inner outer factorization*:

$$M(z) = D(z)N(z) \tag{3.3.12b}$$

such that:

- $D(z)$ is called the *outer*.
  - $D(z)\in \Re H_{\infty}^{m\times m}$ is an unit.
  - $M(z)M^{\sim}(z)=D(z)D^{\sim}(z)$.
- $N(z)$ is a *co-inner* transfer matrix:
  - $N(z)\in \Re H_{\infty}^{p\times m}$ and all its zeros are non-minimum phase zeros.
  - $N(z)N^{\sim}(z)=I$.

Here we will concentrate further on the inner outer factorization, since the co-inner outer factorization is just the dual. Algorithms to obtain this factorization may be based on determining the stabilizing solution of an Discrete time Algebraic Ricatti Equation (DARE), e.g. [Gu89]. The disadvantage of this approach is however that one is unable to deal with delays directly. Delays first have to be extracted as inner functions, e.g. [Sil69, Sil76], before this approach can be used. In [Ion96] a direct approach is proposed based on the calculation of the stabilizing solution of a Discrete time Algebraic Ricatti system. A last approach is to transform the discrete time to a pseudo continuous time problem using a bilinear transformation [Mey90] and then use the continuous time results [Chu84, Doy84, Gre88]. Here we will not further discuss these algorithms.

From the above properties we directly obtain that the principal gains of the outer factor $D(z)$ equal the principal gains of the model $M(z)$. From the factorization we also obtain that the input directions, i.e. the right principal vectors, of $D(z)$ and $M(z)$ are equal. Moreover all non-minimum phase zeros of $M(z)$ are zeros of $N(z)$ and the only zeros of $N(z)$, since $\{N(z), D(z)\}$ is a right coprime factorization of $M(z)$, $D(z)$ is a unit and $N^{1}(z)$ is antistable if it exists. A consequence of the factorization is that the principal gains of $N(z)$ are equal to one. Therefore only the phase and the directional behavior of the matrix changes as function of the frequency. Hence the factorization is unique, except for a post-multiplication with an orthonormal matrix, i.e.:

If $\{N(z), D(z)\}$ is an inner outer pair of $M(z)$ then $\{N(z)U(z), D(z)U(z)\}$ is an inner outer pair of $M(z)$ if and only if $U(z)=U$ a constant orthonormal matrix, i.e. $U\in\Re^{m\times m}$ and $UU^T=U^TU=I$.

If we therefore choose $U= N^T(1)$ then the inner is decoupled

The poles and zeros of the inner have a very specific relation. In the SISO continuous time case the poles of the inner equal non-minimum phase zeros mirrored around the imaginary axis. The continuous time SISO inner therefore has always the following form:

$$N(z)=\prod_{i=1}^{r}\frac{(a_i-s)}{(s+a_i)}\text{ , with } a_i>0$$

For the discrete time an equivalent relation holds:

$$N(z)=\prod_{i=1}^{r}1/a_i\frac{(a_i-z)}{(z-1/a_i)}=\prod_{i=1}^{r}\frac{(a_i-z)}{(za_i-1)}\text{ , with } |a_i|>1$$

In the MIMO case we essentially have the same situation for square inner transfer matrices. The major difference again is the directionality. In the case of a non square inner function, we may have an inner with less zeros than poles. In this case each zero a_i is again related to a pole at $1/a_i$. The reverse is however not necessarily true: Not each pole p_i needs to be related to a zero at $1/p_i$, as in the square case. In fact there exist nonsquare inners that do not have any zeros. This type of inner is called a *structural inner* or *minimum phase* inner.

Example 3.3.3

Let us continue the example 3.3.1

$$M(z)=\begin{bmatrix}0.6/(z-0.4) & 0.5/(z-0.5)\\ 0.5/(z-0.5) & 0.5/(z-0.4)\end{bmatrix}=\begin{bmatrix}0.6(z-0.5) & 0.5(z-0.4)\\ 0.5(z-0.4) & 0.5(z-0.5)\end{bmatrix}/\left((z-0.5)(z-0.4)\right)$$

Application of the inner outer factorization $M(z)=N(z)D(z)$, results in:

- The inner contains exactly the non-minimum phase zeros of the model:

$$N(z)=\begin{bmatrix}0.7385 & -0.6742\\ 0.6742 & 0.7385\end{bmatrix}\begin{bmatrix}1/z & 0\\ 0 & \dfrac{1-0.6461z}{z(z-0.6461)}\end{bmatrix}\begin{bmatrix}0.7385 & 0.6742\\ -0.6742 & 0.7385\end{bmatrix}$$

Observe that non of the entries of the step response gives any sign of an inverse response (figure3.3.1). In fact the inverse response is completely compensated by the influence of the complementary direction. Applying

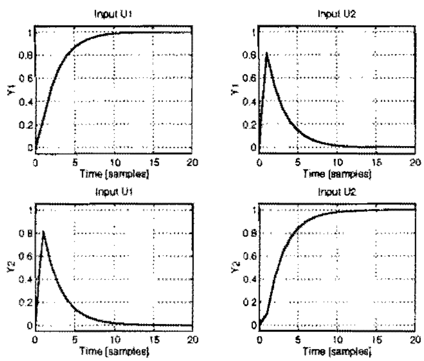

Figure 3.3.1 Step response of the inner. 1st column response of the system to a step at input 1. 2nd column response of the system to a step at input 2.

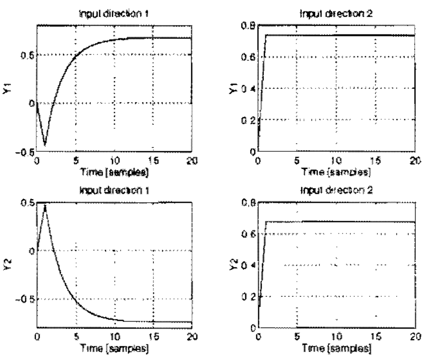

Figure 3.3.2 Step response of the inner. 1st column is the step response in the direction [*0.6742,0.7385*]. 2nd column is thestep response in the direction [*0.7385,–0.6742*]. Notice the NMP behavior.

however a step in the input direction $[-0.6742, 0.7385]^T$ will reveal after one delay the complete inverse response at output 1 and 2 (figure 3.3.2 first column). Applying a step in the input direction $[0.7385, 0.6742]^T$ will reveal only one sample delay at the outputs (figure 3.3.2, second column). The last input direction clearly shows much faster responses than the first one, which is a clear advantage for control. Note that each separate column of the transfer matrix is a structural inner.

end of example.

In the next section we will discuss some further properties of the inner function.

3.4 The time domain and model properties.

In this subsection a number of properties of stable, linear, time invariant systems will be described that are relevant for understanding the thesis.

Assume we have a model $H(z)$ with realization $[A,B,C,D]$. The state space model is not a unique representation of the input-output behavior of the process. Infinitely many state space models represent the same input-output behavior. Any nonsingular matrix $T \in \Re^{n \times n}$ can be used to obtain a new realization of the same input-output behavior:

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}\tilde{x}(k) + \tilde{B}u(k) \\ y(k) &= \tilde{C}\tilde{x}(k) + Du(k) \end{aligned} \tag{3.4.1}$$

with: $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$ and $\tilde{x}(k) = Tx(k)$

The transformation of the state space model is known as a *similarity transformation*. The similarity transformations are frequently used to obtain state space representations with specific properties [Kai80]. A realization is called *observable* if all states can be observed in a finite time at the output. If the complete state space can be reached from the inputs in a finite time then the realization is called *controllable*.

A realization that is both controllable and observable is called a *minimal* realization. It is always possible to reduce a non minimal realization to a minimal realization by using a similarity transformation [Kai80]. In this thesis we will assume the realizations to be minimal unless explicitly stated otherwise.

In general the observability matrix is defined as:

$$N_O = \left[C^T \quad \left(CA^1\right)^T \quad . \quad . \quad \left(CA^{N-1}\right)^T \quad \left(CA^N\right)^T \right]^T$$

In chapter five, where the finite time domain approach is developed, we will need a more general definition of the finite observability matrix. The finite observability matrix $N_0(j,i)$ is therefore defined more general as[1]:

$$N_O(j,i) = \left[\left(CA^j\right)^T \ \left(CA^{j+1}\right)^T \ . \ . \ \left(CA^{i-1}\right)^T \ \left(CA^i\right)^T \right]^T \tag{3.4.2a}$$

with $j \leq i$

We will more encounter the case that $j \geq i$, i.e.[2]:

$$N_O(j,i) = \left[\left(CA^j\right)^T \ \left(CA^{j-1}\right)^T \ . \ . \ \left(CA^{i+1}\right)^T \ \left(CA^i\right)^T \right]^T \tag{3.4.2b}$$

with $j \geq i$

We have chosen this nonstandard definition for the observability matrix to facilate the subsequent discussions. Note that $N_0(0,\infty)$ corresponds to the standard definition of the observability matrix. We will use the short hand notation N_0 instead of $N_0(0,\infty)$. The state space of dimension n is observable if the rank of the observability matrix N_0 is equal to n. It can be shown that for the observability test the first n times p rows of N_0 are sufficient, i.e. $N_0(0,n-1)$, with n the dimension of the state space. All successive rows are dependent on the rows of $N_0(0,n-1)$ by the *Cayley-Hamilton* theorem [Gan74, Cha92].

The finite time observability gramian Q_N, as:

$$\sum_{i=0}^{N} \left(A^T\right)^i C^T C A^i = N_O^T(0,N) N_O(0,N) = N_O^T(N,0) N_O(N,0) \tag{3.4.2c}$$

It is easy to show the relation between Q_N and Q_{N+1}, based on the above relation:

$$A^T Q_N A + C^T C = Q_{N+1} \tag{3.4.2d}$$

For a stable system we see that $i \to \infty$ results in $A^i \to 0$. As a consequence we

[1] An example is given by:

$$N_O(3,8) = \left[\left(CA^3\right)^T \ \left(CA^4\right)^T \ . \ . \ \left(CA^7\right)^T \ \left(CA^8\right)^T \right]^T$$

[2] An example is given by:

$$N_O(8,3) = \left[\left(CA^8\right)^T \ \left(CA^7\right)^T \ . \ . \ \left(CA^4\right)^T \ \left(CA^3\right)^T \right]^T$$

obtain that for $N \to \infty$ Q_N converses to a constant matrix, say Q, the so called *observability gramian* which fulfills:

$$A^T QA + C^T C = Q \tag{3.4.2e}$$

For a stable realization we obtain that it is observable if the matrix Q is positive definite.

The controllability matrix is defined conform the definition of the observability matrix:

$$N_C(j,i) = \begin{bmatrix} A^j B & A^{j+1} B & . & . & A^{i-1} B & A^i B \end{bmatrix} \text{ with } j \le i \tag{3.4.3a}$$

$$N_C(j,i) = \begin{bmatrix} A^j B & A^{j-1} B & . & . & A^{i+1} B & A^i B \end{bmatrix} \text{ with } j \ge i \tag{3.4.3b}$$

We will again use N_c instead of the full notation $N_c(0,-\infty)$. Based on dual arguments we can show that the state space is controllable if the matrix N_c is of rank n. A second test for the controllability of a stable realization is again based on the existence of a positive definite solution of the Lyapunov equation P, the so-called *controllability gramian*:

$$APA^T + BB^T = P \tag{3.4.3c}$$

If the matrix P is positive definite then the state space model is completely controllable. The finite time controllability gramian P_N can again be defined as:

$$\sum_{i=0}^{N} A^i BB^T \left(A^T\right)^i = N_C(0,N)N_C^T(0,N) = N_C(N,0)N_C^T(N,0) \tag{3.4.3d}$$

For the finite time controllability gramian we obtain again:

$$AP_N A^T + BB^T = P_{N+1} \tag{3.4.3e}$$

Define the following realizations:

- A realization is a *balanced realization* if the controllability and controllability gramians are equal: $P=Q$.
- If Q equals the identity matrix it is called an *output balanced realization.*
- If P equals the identity the realization is called *input balanced.*

For a stable system with a minimal realization it is always possible to find a similarity transform such that the realization is balanced [Glo84, Heu89].

The controllability matrix determines the relation between the state at the current time instant and the past behavior at the inputs of the system

$x(0) = N_C u(t)$, with $u(t) \in (l_-^2)^m$. N_c therefore compresses the past of the system, as far as it is relevant for the future behavior in the states of the model. The observability matrix on the other hand relates this compressed past to the future behavior at the outputs: $y(t) = N_O x(0)$, with $y(t) \in (l_+^2)^p$. The matrix $\Gamma = N_o N_c$ relates the past behavior of the inputs to the future behavior of the outputs. This matrix Γ, that transforms the past of the system to the future, is called the (block) *Hankel matrix*. The Hankel matrix plays a crucial role in modern control theory. Analog to the convention for the observability and the controllability matrix we define the Hankel matrix as:

$$\Gamma(k,l,j,i) = N_O(k,l)N_C(j,i) \tag{3.4.4a}$$

where we again introduce some shorthand notations:

- $\Gamma = \Gamma(0,\infty,0,-\infty) = N_O N_C$ (3.4.4b)
- $\Gamma(l,i) = \Gamma(0,l,0,i) = N_O(0,l)N_C(0,i)$ (3.4.4c)

The block Hankel matrix and the impulse response are related as:

$$\Gamma = \begin{bmatrix} CB & CAB & CA^2B & . & . \\ CAB & CA^2B & CA^3B & & . \\ CA^2B & CA^3B & CA^3B & & . \\ . & & & & . \\ . & . & . & . & . \end{bmatrix} = \begin{bmatrix} M_1 & M_2 & M_3 & . & . \\ M_2 & M_3 & M_4 & & . \\ M_3 & M_4 & M_5 & & . \\ . & & & & . \\ . & . & . & . & . \end{bmatrix} \tag{3.4.4d}$$

Note the special structure of the matrix: the block entry $(j+1,i)$, i.e. $M_{j+1,i}$, always equals the block entry $(j,i+1)$ and therefore the entry $(j-1,i+2)$ and so on. In fact the matrix fulfills the following property: All block entries whose sum of the block column number and the block row number is equal have a same block entry. This is characteristic for this type of matrix and is generally known as the *Block Hankel structure*. The singular values of the above infinite Hankel matrix are called the *Hankel singular values* of the system. The corresponding singular vectors are called the *Schmidt* vectors.

We can ask ourselves for a stable system, which past input signal pattern has most influence on the future outputs of the system in a two norm sense, i.e. find $u(t) \in (l_-^2)^m$ such that $\|y(t)\|_2$, with $y(t) \in (l_+^2)^p$, is maximal and m and p the number of inputs respectively outputs of a stable realization $[A,B,C,D]$. The solution is the so-called *Hankel norm* of the system $H(z)$:

$$\|H(z)\|_H = \sup_{u(t)\in(l_-^2)^m} \frac{\|\Gamma u(t)\|_2}{\|u(t)\|_2} \tag{3.4.5}$$

and equals the largest singular value of the infinite Hankel matrix, i.e. the largest Hankel singular value. The Hankel norm equals the maximum amplification of past input signals to future output signals. The right Schmidt vector of the Hankel matrix corresponding to the Hankel norm therefore exactly reveals the input sequence that results in this largest influence of the past on the future. The corresponding left Schmidt vector shows how this influence is seen in the future outputs of the system. Observe that for a minimal realization the rank of the matrix Γ will equal n since the rank of both N_o and N_c is equal to n. As a consequence only a n-dimensional subspace of the input space will contribute to future behavior of the system, i.e. all input signals that are orthogonal to the right Schmidt vectors will not influence the future outputs of the system. Along the same line of reasoning we see that the influence of the past on the future is only seen in an n-dimensional subspace of the future outputs, spanned by the columns of N_o

The Hankel singular values can be determined from the observability gramian Q (equation (3.4.2e) and the controllability gramian P (equation (3.4.3c) . The i-th Hankel singular value is given by the square root of the i-th eigenvalue of QP: $\lambda_i^{1/2}(QP)$. To see this, observe that the Hankel singular values are the square root of the eigenvalues of $\Gamma^T\Gamma$. We thus obtain using equation (3.4.4a) and (3.4.2c) that: $\Gamma^T\Gamma = N_C^T N_O^T N_O N_C = N_C^T Q N_C$. Define $X = \begin{bmatrix} N_C^T & N_{C\perp}^T \end{bmatrix}$, where $N_{c\perp}$ spans the nullspace of Γ and use the fact that $\lambda_i(A) = \lambda_i(X^{-1}AX)$ results in:

$$X^{-1}\Gamma^T\Gamma X = \begin{bmatrix} N_C^T & N_{C\perp}^T \end{bmatrix}^{-1} N_C^T N_O^T N_O N_C \begin{bmatrix} N_C^T & N_{C\perp}^T \end{bmatrix} = \begin{bmatrix} QP & 0 \\ 0 & 0 \end{bmatrix}$$

The nonzero singular values of Γ therefore equal the square root of the eigenvalues of QP.

The observability gramian Q thus equals the energy transfer from the states, to the future of the system under the assumption that u(i)=0 for i≥0:

$$\sum_{i=0}^{\infty} y^T(i)y(i) = x^T(0)N_O^T N_O x(0) = x^T(0)Qx(0)$$

The interpretation of the controllability gramian P is somewhat more involved. P inverse equals the minimum energy needed at the input of the system in the past to bring the system at t=0 to a certain state, i.e. for any $x \in \Re^n$ we obtain:

$$\min_{u(t)\in l_-} \|u(t)\|_2 = x^T P^{-1} x$$

subject to:

$$x(0) = x$$

$$x(k+1) = Ax(k) + Bu(k)$$

The Hankel matrix plays a crucial role in all kind of model reductions schemes [Heu91, Enn84]. Model reduction techniques enable us to simplify or better approximate a high order model with a low order model, e.g. to come from a step response model to a low order state space model. These techniques fall outside the scope of this thesis and are not discussed here.

If we assume that the maximum singular value of the Hankel matrix has multiplicity one then the maximum gain from past inputs to future outputs the input $u(t)$ for the Hankel matrix must be exactly in the direction of the corresponding Schmidt vector. As a second consequence we also see that the output sequence corresponding to this gain is unique. Taking a closer look at the direction of the input and output vector we see two effects that influence the direction of this vector:

1. The distribution of the signal as a function of time, i.e. the input sequence must be a very specific sequence as function of time.
2. It must have a specific distribution over the different inputs of a multi input system.

For a MIMO system the dynamic behavior is also determined by the distribution of the time dependent sequence over the different inputs respectively outputs. This is the *directionality* of a MIMO system.

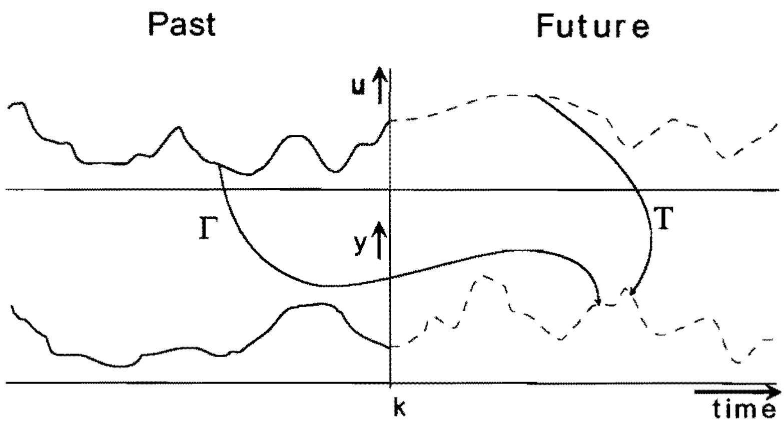

Fig.3.4.1 Graphical interpretation of the Hankel and Toeplitz matrix.

We introduced the Hankel operator of $H(z)$ as the influence that the past inputs of the system have on the future outputs of the system: $\Gamma: u(t) \in (l_-^2)^m \rightarrow y(t) \in (l_+^2)^p$ (figure 3.2.1). Another operator of interest is how future inputs will influence the future outputs. This results in the so-called Toeplitz operator T: $u(t) \in (l_+^2)^m \rightarrow y(t) \in (l_+^2)^p$ (figure 3.4.1). If we assume the system to be in steady state we obtain for the matrix T of the above operator:

$$T = \begin{bmatrix} M_0 & 0 & . & . & . \\ M_1 & M_0 & 0 & & . \\ M_2 & M_1 & M_0 & 0 & . \\ M_3 & M_2 & M_1 & M_0 & . \\ . & . & . & . & . \end{bmatrix} = \begin{bmatrix} D & . & . & . & . \\ CB & D & 0 & & . \\ CAB & CB & D & 0 & . \\ CA^2B & CAB & CB & D & . \\ . & . & . & . & . \end{bmatrix} \quad (3.4.6)$$

where T is a lower block triangular matrix with an infinite number of rows and columns. Note the specific structure of the matrix again. This structure is known as the block Toeplitz structure. In contrast to the Hankel operator the rank of this block Toeplitz matrix is infinite. The infinite rank is easily understood from the fact that each block columns starts one block row lower. Each next block column will therefore increase the order. Like we did for the Hankel matrix we will introduce the following notation for the Toeplitz matrix. The finite Toeplitz matrix, which relates the behavior from the input at time instant *j* to time instant *i* to the output at time instant *k* to *l* is notated as $T(k,l,j,i)$, where we again introduce some shorthand notations:

- $\Gamma = \Gamma(0,\infty,0,\infty)$ (3.4.7a)
- $\Gamma(l,i) = \Gamma(0,l,0,i)$ (3.4.7b)

From the Toeplitz and the Hankel operators as we introduced them above we obtain the following general expression for the future behavior of the output:

$$y(t) = \Gamma u_p + \mathrm{T} u_f \tag{3.4.8a}$$

with:

- $u_p \in (1^2_-)^m$ the input signal in the past
- $u_f \in (1^2_+)^m$ the input signal in the future

The finite variant of this operator is well known in model predictive control:

$$y_f = \Gamma(N_{f_y}, N_p)u_p + \mathrm{T}(N_{f_y}, N_{f_u})u_f \tag{3.4.8b}$$

with:

- $u_p \in (1^2_-(N_p))^m$ the input signal taken into account over the past time horizon $[-N_p,0]$.
- $u_f \in (1^2_+(N_{f_u}))^m$ the input signal over the finite future time horizon $[0, N_{f_u}]$.
- $y_f \in (1^2_+(N_{f_y}))^p$ the predicted output signal in the finite future $[0, N_{f_y}]$.

Note that the future horizon over which we assume the inputs to move, N_{f_u}, the so-called control horizon, can be different from the horizon over which we predict the output, N_{f_y}, the so-called prediction horizon.

Moreover besides neglecting inputs far in the past, i.e. before sample $-N_p$ in the above predicted fuutre output, we also see that the influence the inputs have on the 'far' future, i.e. after the prediction horizon N_{f_y}, is not considered in the above model. If we do take this 'far' future int account we obtain:

$$y_f = \Gamma(\infty, N_p)u_p + \mathrm{T}(\infty, N_{f_u})u_f \tag{3.4.8c}$$

A last point we want to discuss for a better understanding of the Toeplitz matrix is the fact that it can be written as a composition of smaller Toeplitz matrices and Hankel matrices. It is for example easy to check that:

$$T(0,2N,0,2N)=\begin{bmatrix} T(0,N,0N) & 0 \\ \Gamma(0,N,N,0) & T(0,N,0N) \end{bmatrix} \tag{3.4.8d}$$

In this section we introduced a number of concepts related to the representation of process behavior with linear time invariant models. Some basic time domain properties of these systems have been introduced.

3.5 *Feedback control and the IMC scheme.*

In control theory there are two basic principles; *feedforward* and *feedback* control. The main difference between feedforward and feedback control is the assumption we make on the knowledge we have on the process and the disturbances acting on the process. In feedforward control (figure 3.5.1)

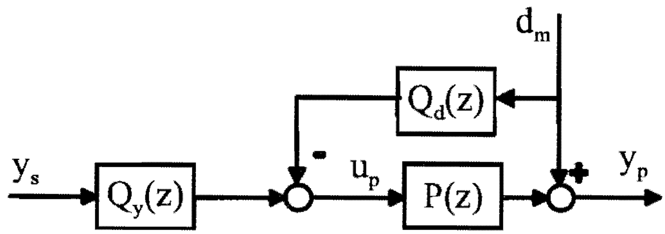

Fig.3.5.1 The Feedforward Control scheme.

exact knowledge of the process and the disturbances acting on the output are assumed. For feedback the knowledge on the process need not be perfect and the disturbance need not be known. If a disturbance $d_m(z)$ can be measured before it actually appears on the process output then a feedforward controller can already take action before the output is affected by the disturbance:

$$y_p(z) = \left(P_d(z) - P(z)Q_d(z)\right)d_m(z) + P(z)Q_y(z)y_s(z) \qquad (3.5.1)$$

From this formula we directly obtain that the problem is affine in the controller.

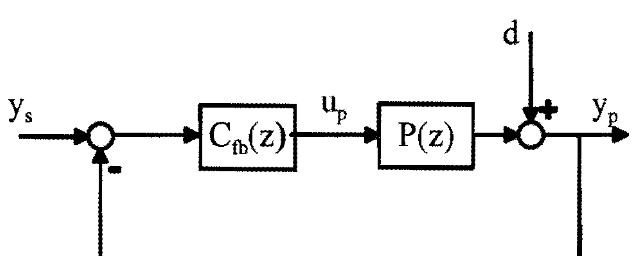

Fig.3.5.2 The Feedback Control scheme.

If $P(z) \in \Re H_\infty$ then for $Q_d(z) \in \Re H_\infty$ and $Q_y(z) \in \Re H_\infty$ the closed loop transfers are stable. The design problem in (3.5.1) is therefore relatively simple. It can however only be applied on a stable or stabilized process, since pole zero cancellation is the only possibility we have to stabilize the process. In fact the problem (3.5.1) can be written as two separate problems. Each of which is a specific case of the model matching problem [Doy84, Fra87]:

$$T_{cl}(z) = T_1(z) - T_2(z)Q(z)T_3(z) \qquad (3.5.2)$$

The main advantage of the feedforward scheme is its main disadvantage: There is no feedback term. Process models are always approximations and not all influences acting on process outputs are measurable. The feedforward scheme can not deal with this uncertainty. The basic idea of feedback control is to feed the output signal back to the controller (figure 3.5.2). In this way the controller is able to correct the behavior *after* it is observed at the outputs of the process:

$$e(z) = \left(I + P(z)C_{fb}(z)\right)^{-1}\left(y_s(z) - d(z)\right) \qquad (3.5.3)$$

Advantages of feedback control are that knowledge on the disturbance is not necessary and the process behavior may be unstable and uncertain to a certain extend. Feedback is at the same time the main disadvantage of the concept. It may result in unstable behavior. Over the years stability of the feedback loop has been a topic of great concern in control engineering. Nowadays we are able to factor all controllers that stabilize a given process [Vid85]. This factorization of all stabilizing controllers reduces the design problem again to the model matching problem (3.5.2) [Fra87, Sch92]. An interesting control concept that tries to combine the advantages of both feedback and feedforward is the Internal Model Control (IMC) scheme (figure 3.5.3). It is based on the observation that feedback is only needed to enable the controller to correct for the unknown behavior. The signal $e(z)$ that is fed back to the controller input is the difference between the process output $y_p(z)$ and the model output $y_m(z)$, i.e. the by the model unexplained part of the process behavior. The IMC structure has been introduced by Zames [Zam79, Zam80] as the so called *model reference transformation* and became also known as *Q factorization*, due to his use of the symbol Q for the transformed controller. The attention this controller structure received in the chemical process control community is due to Brosilow [Bro79] and Garcia and Morari [Gar82, Gar85]. The later ones introduced the term Internal Model Control scheme. One of the reasons for its popularity is the bridge this scheme forms between different control strategies, like Smith predictors, inferential control, Model Predictive Control strategies and fixed control structures obtained from H_2 and H_∞. A second reason is surely the direct relation the scheme has with feedforward control, which makes it easy to understand.

3.5.1 The Internal Model Control scheme.

In this section we will develop the internal model scheme based on its relation to the unit feedback scheme. The concept of internal stability was introduced to guarantee the overall stability of the closed loop:

> A closed loop system is *internally stable* if a bounded signal injected anywhere in the closed loop generate bounded signals at any other point in the loop.

The feedback system in figure 3.5.4 is internally stable if and only if the following transfer matrix is stable [Fra87],:

$$\begin{bmatrix} u_p(z) \\ y_p(z) \end{bmatrix} = \begin{bmatrix} \left(I + C_{fb}(z)M(z)\right)^{-1} & C_{fb}(z)\left(I + C_{fb}(z)M(z)\right)^{-1} \\ M(z)\left(I + C_{fb}(z)M(z)\right)^{-1} & \left(I + M(z)C_{fb}(z)\right)^{-1} \end{bmatrix} \begin{bmatrix} d_i(z) \\ d(z) \end{bmatrix} \tag{3.5.4a}$$

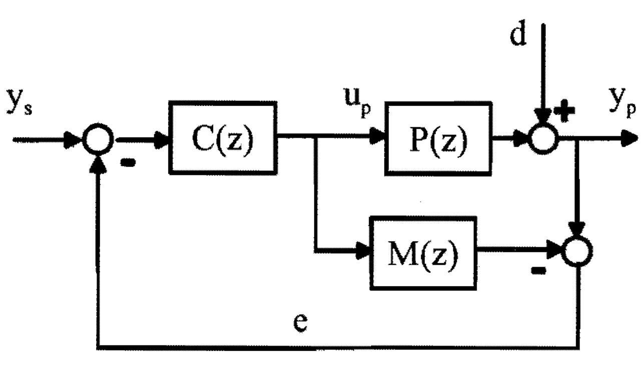

Fig.3.5.3 The IMC scheme.

Equation (3.5.4) can now be used to factor all controllers that stabilize the above closed loop. To achieve this use is made of a coprime factorization of the model $M(z)$. Define $(N(z), D(z))$ and $(N_l(z), D_l(z))$ to be a right and left coprime factorization of $M(z)$:

$$M(z) = N(z)D^{-1}(z) = D_l^{-1}(z)N_l(z) \tag{3.5.4b}$$

Define the corresponding Bezout factors ($X(z)$, $Y(z)$) respectively ($X_l(z)$ $Y_l(z)$) that result in a double coprime factorization:

$$\begin{bmatrix} X(z) & -Y(z) \\ -N_l(z) & D_l(z) \end{bmatrix} \begin{bmatrix} D(z) & Y_l(z) \\ N(z) & X_l(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3.5.4c}$$

All stabilizing controllers $C_{fb}(z)$ ([Fra87] section 4.4 theorem 1) are then given by the factorization:

$$C_{fb}(z) = \left(Y_l(z) - D(z)C(z)\right)\left(X_l(z) - N(z)C(z)\right)^{-1} \tag{3.5.4d}$$

or equivalently:

$$C_{fb}(z) = \left(X(z) - C(z)N_l(z)\right)^{-1}\left(Y(z) - C(z)D_l(z)\right) \tag{3.5.4e}$$

where $C(z)\in\Re H^\infty$, i.e. any stable transfer matrix. If $M(z)\in\Re H^\infty$ then a possible right and left coprime factorization is given by $N(z) = N_l(z) = M(z)$ and $D(z) = D_l(z) = I$ with Bezout factors $Y(z) = Y_l(z) = 0$ and $X(z) = X_l(z) = I$. This results in the control scheme as drawn in figure 3.5.5, which is directly seen to be equivalent to the IMC scheme of figure 3.5.3. In a more formal way we thus derived the relation the IMC scheme has with the more conventional unit feedback controller.

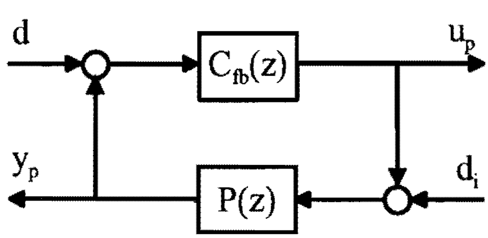

Fig.3.5.4 Internal stability and unit feedback.

If we assume the model to be exactly equal to the process, $P(z)=M(z)$, i.e. no model error, we see that $e(z)$ exactly equals the disturbance $d(z)$. The scheme can then be redrawn as a feedforward problem (figure 3.5.6). In this so-called nominal case the *(output) sensitivity* and the *(output) complementary sensitivity* equal[1]:

$$S_o(z) = I - M(z)C(z) \tag{3.5.5a}$$

[1] In the text the subscript o is skipped for notational convenience, if no confusion occurs.

$$T_O(z) = M(z)C(z) \tag{3.5.5b}$$

For the process input and the output we obtain respectively:

$$\begin{aligned} y_p(z) &= S_O(z)d(z) + T_O(z)y_s(z) \\ u_p(z) &= C(z)(y_s(z) - d(z)) \end{aligned} \tag{3.5.5c}$$

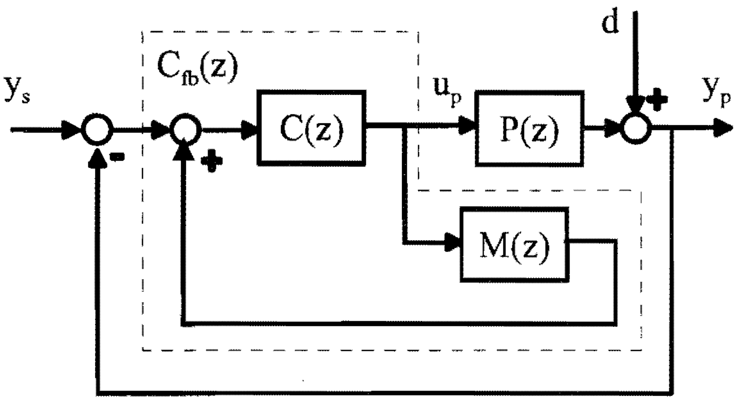

Fig.3.5.5 The redrawn IMC scheme is equivalent to the unit feedback scheme.

For this case the IMC controller $C(z)$ is affine in the above closed loop transfer matrices. In fact all closed loop transfer functions are affine in $C(z)$, e.g. the transfer of disturbance at the process input to the controller output, the process input and the process output are given by respectively:

$$T_I(z) = C(z)M(z) \tag{3.5.5d}$$

$$S_I(z) = I - C(z)M(z) \tag{3.5.5e}$$

$$M(z)S_I(z) = S_O(z)M(z) \tag{3.5.5f}$$

where $T_I(z)$ and $S_I(z)$ are called respectively the *input complementary sensitivity* and the *input sensitivity*. A requirement for most controllers is that the outputs of the controlled process asymptotically converge to the value of the setpoint, after a step change in the setpoint or disturbance. From the final value theorem we obtain that for step changes this is fulfilled if and only if:

$$C(1) = M^{-1}(1)$$

This is called *integral control* or equivalently the controlled process has type one behavior. The requirement to asymptotically converge to a ramp function, so called type two behavior, a parabolic function, so called type three behavior and higher order functions is in general of less importance for the classical process control problem[2]. In [Mor89] (theorem 7.5-1, pg.153) the necessary and sufficient conditions for a system type m system are given. Note that perfect control (chapter 2) is obtained if $C(z)$ equals $M^{-1}(z)$. An important consequence of this observation is:

All effects that restrict the causal and stable invertability of the model are exactly the effects that potentially limit the performance of the controlled process.

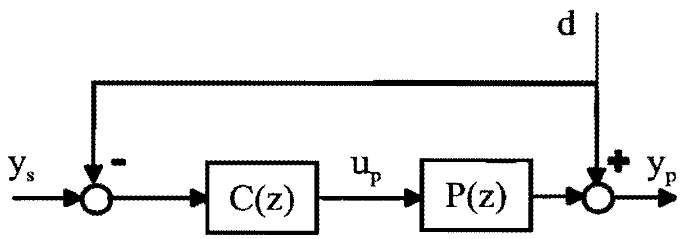

Fig.3.5.6 Equivalent Feedforward scheme.

This relation between closed loop performance and invertability of the process will be used extensively in the next chapters. Until here we restricted ourselves to the nominal case. The process behavior was completely described by the model behavior. This will however never be the case in real life. In the next subsection we therefore take a closer look at robustness.

From the relation between the IMC scheme and all stabilizing controllers we obtain that the IMC scheme is not a stabilizing control scheme anymore for unstable systems, since $(N(z), D(z))=(M(z), I)$ is not a coprime factorization anymore, since the factors are not stable. The reason for this is directly obtained from the feedforward nature of the control scheme. The input output behavior of the system can only be stabilized by pole zero

[2] One might argue that these asymptotic properties are of importance for transition control, i.e. controlling the process from one operating point to another operating condition. Current developments in the market make transition control indeed important for industry. These transitions however have to be made in a certain predefined minimum time and in general with a minimum production of off-spec products. Asymptotic requirements are therefore insufficient. It is better described by a tracking problem, where the controller has to follow a predefined path from one operating condition to another.

cancellation of the unstable system poles with zeros of the controller. It is therefore not possible to implement a controller in the IMC scheme for an unstable process. For analysis purposes the situation is different. Morari has shown, [Mor89, theorem 5.1-1], that if additional interpolation constraints, i.e. $S_I(z)M(z) \in \Re H^\infty$, $M(z)C(z) \in \Re H^\infty$ and $C(z)M(z) \in \Re H^\infty$, with $C(z) \in \Re H^\infty$ then the IMC scheme is internally stable (See also [Bha86]). The internal stability is however based on pole zero cancellation. The actual controller should therefore always be implemented making use of an observer based controller. It is well known that oscillatory behavior can not be robustly controlled by pole zero cancellation [Tsa90, Sef90, Smi90]. As a consequence of the observed pole zero cancellation in the IMC scheme it must be emphasized it is not advisable to use the IMC scheme for implementation in these cases.

3.5.2 Robustness analysis in the IMC-scheme

In this paragraph we will concentrate on robustness analysis in the IMC-scheme. As already stated there will always be a difference between the process and the model behavior. Moreover it is even possible to deliberately introduce the model error to obtain for example stability [Tel93a]. This idea is not considered here. In the IMC scheme the model error is most naturally expressed as an additive error $P(z) = M(z) + \Delta(z)$. Other model errors are directly related to this additive model error, e.g. multiplicative input uncertainty $\Delta_i(z)$ is defined as $\Delta(z) = M(z)\Delta_i(z)$ and the multiplicative output error is defined by $\Delta(z) = \Delta_o(z)M(z)$. Unless explicitly stated otherwise we will deal with an additive uncertainty.

In the case that the process $P(z)$ and the model $M(z)$ are not equal we obtain, after some algebraic manipulations, for the transfer matrices of the IMC scheme:

- Transfer from output disturbance and setpoint to process input $T_c(z)$:
  $T_c = C(I - T_\Delta)$ (3.5.6a)
- Transfer from setpoint to process output, i.e. the output complementary sensitivity, $T_{out}(z)$:
  $T_{out} = T_o + S_o \cdot T_\Delta$ (3.5.6b)
- Transfer from output disturbance to process output, i.e. the output sensitivity, $S_{out}(z)$:

$$S_{out} = S_o - S_o \cdot T_\Delta \tag{3.5.6c}$$

where:

- $T_o(z)$ and $S_o(z)$ are the nominal output complementary sensitivity, respectively output sensitivity.
- $T_\Delta(z)$ is a closed loop transfer matrix resulting from the model error $\Delta(z)$:

$$T_\Delta = \Delta C(I + \Delta C)^{-1} \tag{3.5.6d}$$

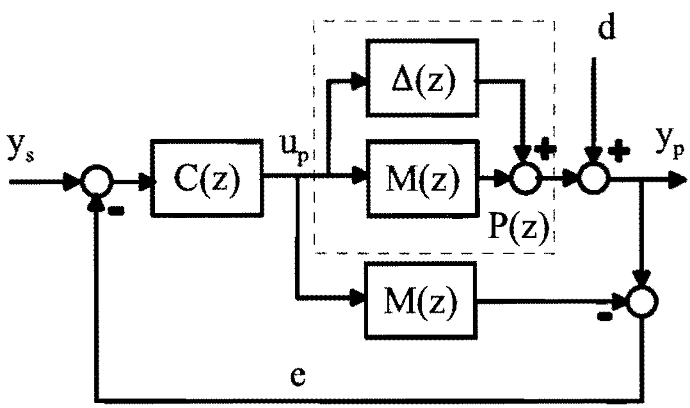

Figure 3.5.7 The IMC-scheme with an additive model error.

From the above closed loop transfer matrices we directly obtain that the stability of the control scheme is completely determined by the inverse of $(I+\Delta C)$. Assume that process and model are both stable, i.e. $\Delta(z) \in \Re H^\infty$. Assume furthermore that the nominal closed loop behavior is stable. Closed loop stability is then guaranteed if and only if there is no perturbation in the set of possible model errors that makes $det(I+(M(e^{j\phi})+\Delta(e^{j\phi}))C(e^{j\phi}))=0$, i.e.:

$$\left\{\forall \phi \in [-\pi, \pi] \mid \ \det\left(I + \Delta(e^{j\phi})C(e^{j\phi})\right) \neq 0\right\} \tag{3.5.7a}$$

If there would be a perturbation in the set that makes this determinant equal to zero for any frequency then this would change the number of encyclements in the Nyquist diagram and according to the Nyquist criterion make the closed loop unstable. Hence if the largest eigenvalue of ΔC is smaller than 1 the system will stay stable. A sufficient, but not necessary requirement is then:

$$\left\{\forall \phi \in [-\pi, \pi] \mid \ \rho\left(\Delta(e^{j\phi})C(e^{j\phi})\right) < 1\right\} \tag{3.5.7b}$$

where $\rho(\,.\,)$ is the *spectral radius*, the absolute value of the largest eigenvalue. Using the facts that the spectral radius is bounded above by the largest singular value, i.e. $\rho(X)\leq\sigma_1(X)$ and the fact that $\sigma_1(XY)\leq\sigma_1(X)\sigma_1(Y)$, results in the frequently used robustness criterion for additive model errors:

$$\left\{\forall\phi\in[-\pi,\pi]\;\middle|\; l_a^{-1}(e^{j\phi})>\sigma_1\left(C(e^{j\phi})\right)\right\} \tag{3.5.7c}$$

where l_a bounds the uncertainty from above:

$$\left\{\forall\phi\in[-\pi,\pi]\;\middle|\; \sigma_1\left(\Delta(e^{j\phi})\right)<l_a(e^{j\phi})\right\}.$$

The above criterion (3.5.7c) can directly be deduced from the small gain theorem [Zam66].

A possibility we have to reduce conservatism further is making use of additional information on the modeling error structure. For example if we can bound the uncertainty in each entry of the transfer matrix separate, i.e. $\left|\Delta^{ij}\right|<l_a(i,j)$, with Δ^{ij} the (i,j) entry of the matrix Δ or know that the uncertainty is due to the actuators at the input of the process, i.e. $\left|\Delta_I^{ij}\right|=0$ if $i\neq j$ and $\left|\Delta_I^{ii}\right|<l_I(j,j)$. In these cases the above singular value test is conservative. For this purpose Doyle [Doy82, Doy84, Doy87, Mac89, Mor89, Pac93, Sko96] introduced μ-*analysis*.

Assume given an uncertainty $\Delta(z)$ of the following block diagonal structure, skipping the dependency on z:

$$\Delta(z)=\begin{bmatrix}\Delta_{11} & 0 & . & 0\\ 0 & \Delta_{22} & . & .\\ . & . & . & 0\\ 0 & . & 0 & \Delta_{nn}\end{bmatrix} \tag{3.5.8a}$$

with Δ_{jj} is a linear time invariant stable full block of arbitrary dimensions with $\sigma_1(\Delta_{jj})\leq 1$. We will denote the set of all uncertainty matrices fulfilling equation (2.5.8a) as $\boldsymbol{B}\Delta_1$. It is always possible with scaling to ensure that the maximum uncertainty is smaller or equal to one: If $\sigma_1(\Delta_{jj})\leq l_j$ then $\sigma_1(\Delta_{jj}/l_j)\leq 1$.

From the Nyquist criterion we have seen that the above loop will be destabilized if a perturbation in the set results in $\det\left(I + \Delta(e^{j\phi})T(e^{j\phi})\right) = 0$ for a certain ϕ. In this equation the transfer matrix $T(z)$ represents the interconnection between the nominal stable closed loop system and the uncertainty. Now define the *structured singular value* of the transfer $T(z)$ as:

$$\mu(T(e^{j\phi})) = \begin{cases} 0 \text{ if } \det\left(I + \Delta(e^{j\phi})T(e^{j\phi})\right) \neq 0 \text{ for any } \Delta \text{ in the set } \boldsymbol{B}\Delta_1 \\ \left\{ \min\limits_{\Delta \in \boldsymbol{B}\Delta_1} \sigma_1(\Delta) \colon \det\left(I + \Delta(e^{j\phi})T(e^{j\phi})\right) = 0 \right\}^{-1} \end{cases} \tag{3.5.8b}$$

The so called μ-*norm* of the transfer $T(z)$ is defined as:

$$\|T\|_\mu = \max_{\phi \in [-\pi, \pi]} \mu(T(e^{j\phi})) \tag{3.5.8c}$$

Based on the above norm Doyle [Doy82] defined the following robust stability criterion. A feedback system remains stable for all $\Delta(z) \in \boldsymbol{B}\Delta_1$ if and only if:

$$\|T\|_\mu < 1 \tag{3.5.8d}$$

One of the major problems with μ-analysis is the determination of function μ. A number of approaches have been proposed to approximate μ. Most of these techniques are based on bounding it from above [Doy82, Mac89, Sko96]:

$$\mu(T(e^{j\phi})) \leq \min_{D \in \boldsymbol{Dz}} \sigma_1\left(D(e^{j\phi})T(e^{j\phi})D(e^{j\phi})^{-1}\right) \tag{3.5.8e}$$

with $\boldsymbol{Dz}$ the set of block diagonal matrices, whose block structure corresponds to the structure of the uncertainty. Each block diagonal entry is of the form $d_i I$, with I the identity matrix of appropriate size. As for the singular values the above upper bound has to be calculated frequency point after frequency point. The above μ-analysis may seem a complex approach. A complex notation is needed and a lot of technicalities are hampering the theory and the solution of the problem. The basic concept is simple however. In many cases we have knowledge on the structure of the modeling error. For example if simple rigorous models are determined one is in general capable to indicate which parameters are uncertain and where simplifications in the model have been made, e.g. [Doy87, Smi87]. This results in a highly structured uncertainty matrix. New identification approaches enable us to bound the model error made:

- The uncertainty of each entry of the transfer, i.e. a structured additive error [Zhu93, Hak94]
- Bound certain parameters [Fal93]
- Determine and bound the directionality in the error [Ari97].

In all these cases we therefore have some structure in the model error. Applying the singular value test would completely disregard the block diagonal structure of the error, which may lead to arbitrary conservative errors. Using μ can therefore significantly reduce the conservatism in the error. During the years a lot of extensions, applications and generalizations of the μ-approach have been proposed by several authors. Here we are only interested in the basic concept.

Robust stability is a necessary, but not a sufficient condition for robust performance. If $\|\Delta T_C\|_\infty < 1$ then we obtain from (3.5.6):

$$y_p = T_o y_s + S_o d + S_o \left(\sum_{i=1}^{\infty} (\Delta C)^i \right) (y_s - d) \tag{3.5.9a}$$

where we used the fact that for $\|\Delta T_C\|_\infty < 1$:

$$T_\Delta = \Delta C(I + \Delta C)^{-1} = \sum_{i=1}^{\infty} (-1)^{i-1} (\Delta C)^i = \Delta C(\sum_{i=0}^{r} (-\Delta C)^i + (-\Delta C)^{r+1} (I + \Delta C)^{-1}) \tag{3.5.9b}$$

Note the role of the nominal sensitivity. For a sensible design robust performance problems are therefore most likely to occur in the frequency range around the cross-over frequency, i.e. frequencies for which $S_0(z)$ approaches one from below and $C(z)$ still has a significant gain. Robust performance requirements urge us to obtain accurate models around the cross-over frequency. A further consequence is that the accuracy of the model for the frequency range much smaller than the closed loop bandwidth is of less importance as long as stability is guaranteed.

If $\{\forall \phi \in [-\pi, \pi] : \sigma_1(C(e^{j\phi})) \ll 1 / \sigma_1(\Delta(e^{j\phi}))\}$ then $T_\Delta \approx \Delta C$. We then obtain at the process output:

$$y_p \approx T_o y_s + S_o d + S_o \Delta C (y_s - d) \tag{3.5.10}$$

In fact the equations (3.5.10) is the mathematical justification of an intuitive approach frequently used in practice:

Enforce robustness of the loop by requiring the amplification of the external signals towards the process inputs to be limited.

This intuitive approach will in general lead to a too conservative design of the controller, i.e. a very sluggish control, specifically for MIMO systems. It is an easy, simple and save way to ensure robustness of the control system, that is well applicable to many situations. In many cases however a significant increase of performance is still possible.

If a robust performance specification is stated as an infinity norm it can be rewritten in a robust stability requirement [Doy84, Mac89]. Let us give an example. Assume robust performance of the sensitivity function for multiplicative input uncertainty is to be analyzed. i.e. $\Delta(z) = M(z)\Delta_I(z)$. Substitution of this expression in (3.5.6c) results for the actual sensitivity function in:

$$S_{out}(z) = S_o(z) - S_o M(z)\Delta_I(z)(I + C(z)M(z)\Delta_I(z))^{-1}C(z) \qquad (3.5.11a)$$

The following equivalent problem formulation is obtained if we now want $S_{out}(z)$ to have better performance then a certain minimum bound $l_p(z)$ for a given uncertainty set: $\sigma_1(\Delta_I) < 1$:

$$\left\|S / l_p\right\|_\infty < 1 \text{ for all } \Delta_I \text{ with } \sigma_1(\Delta_I) < 1 \qquad (3.5.11b)$$

(3.5.11b) can be seen as an artificial robust stability requirement. It is equivalent to:

$$\det\left(I + \Delta_p S / l_p\right) \neq 0$$

$$\text{for any } \Delta_p \text{ with } \sigma_1(\Delta_p) < 1 \text{ and any } \Delta_I \text{ with } \sigma_1(\Delta_I) < 1 \qquad (3.5.11c)$$

The following μ-analysis problem is obtained, after some algebraic manipulations:

$$\Delta(z) = \begin{bmatrix} \Delta_p(z) & 0 \\ 0 & \Delta_I(z) \end{bmatrix} \text{ and } T(z) = \begin{bmatrix} S_o(z) & S_o(z)M(z) \\ C(z) & C(z)M(z) \end{bmatrix} \qquad (3.5.12)$$

As a consequence we see that robust performance problems can be reformulated into a structured robust stability problems. The criterion in equation (2.4.12) will play an important role in section 4.5 were we will analyze the behavior of a class of systems whose closed loop performance is sensitive for differences between the process behavior and the model behavior.

3.5.3 Scaling of control problems.

As is readily known, scaling influences the outcome of the design of a controller. For MIMO systems scaling not only influences the magnitudes of the gain, but also influences the directional properties of the process to be controlled. To clearly state the effect scaling has on the problem let us take a close look at a simple example.

Example 3.5.1

Assume the following process model to be given:

$$P(z) = \frac{1}{5\sqrt{2}}\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{(z-0.6)}{4(z-0.9)} \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}\right)$$

From the model we directly obtain that the singular values equal 1 and $\left|\frac{\left(e^{-i\phi}-0.6\right)}{4\left(e^{-i\phi}-0.9\right)}\right|$ for $0 \leq \phi \leq \pi$, with the left and right largest singular vector equal to $1/\sqrt{2}[1, 1]^T$ respectively $1/5[4, 3]^T$. The second left and right singular vectors also have a constant direction $1/\sqrt{2}[-1 , 1]^T$ and $1/5[-3 , 4]^T$. The zero and pole have the same direction as the second singular value.

Apply as a scaling; $[1 , 0.1]^T$ for the outputs and $[1 , 10]^T$ for the inputs. After scaling the zero output and input direction of the scaled model are $[0.0995 , 0.9950]^T$ respectively $[-0.0748 , 0.9972]$ and for the pole output and input direction $[0.9912 , -0.1322]^T$ respectively $[-0.9950, 0.0995]$. The directionality have therefore completely changed. Moreover we see that the directions of pole and zero are not the same anymore. Also the singular values of the original model and the scaled version have completely changed (figure 3.5.8) The scaling transformed the well conditioned model in an ill-conditioned scaled model.(figure 3.5.9) .

End of example

Scaling may therefore drastically change the behavior with respect to the original behavior. On the other hand scaling can of course never change the actual control problem, since the physical behavior of the plant can never be changed by scaling the model. However:

Scaling will change the appearance of the process behavior during the design of the controller.

Decisions made will be based on this scaled model, since we design the controller on the scaled model. A good scaling is one that fascilates the design procedure and contributes in this way to a more transparent design procedure. Scaling should be based on the physical reality. For most control problems this will boil down to scaling the variations on the inputs and outputs to one and the same range.

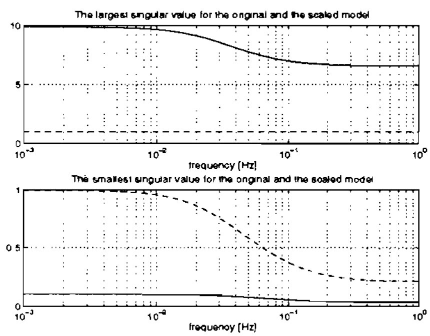

Fig. 3.5.8 The first and second singular values of the unscaled model (dashed) and after scaling (solid).

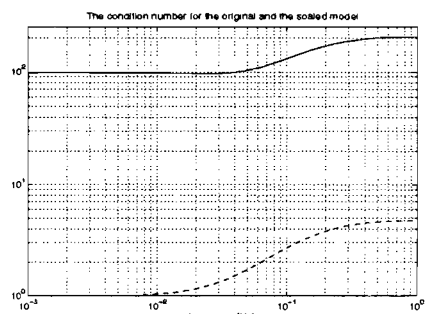

Fig. 3.5.9 The condition number of the unscaled model (dashed) and after scaling (solid).

The inputs should reflect the range over which the actuators are expected to vary or are allowed to vary. If constraints of the actuators form no limitation one might scale the inputs by the expected variations. If the actuators are limited by constraints on magnitude and/or rate of change and these limitations are likely to be violated then a better approach is to take these limitations as the scaling factors for that input, i.e. multiply the inputs with a corresponding low pass filter. Scaling of the output may be done in accordance with the average variations the outputs are allowed to vary. If outputs are constrained and these constraints are likely to be violated the constraining value is a better scaling factor. The result of the scaling will be that all inputs and outputs have an equal range they will vary or are allowed to vary, while the problem still reflects the physical reality.

Let us return for a moment to the example. It might at first seem strange to scale the model as we did in the example, i.e. scale it such that the control problem becomes ill conditioned. If however the disturbance at output one is only a tenth of that of output 2, it is a fact we have to live with. The controller has to compensate a ten times larger variation on output two. A same remark of course holds for the inputs. If we do not perform the scaling at the input we need to consider that input one can be used over a range of ten, while input one can only be used over a range one.

As a consequence scaling can well fascilate the design. It can however not change the physical reality. It should therefore always be based on the physical problem. If reality is denied during the design, it can well result in a disappointing behavior of the closed loop.

CHAPTER FOUR

A First Approach to Asses Input-Output Controllability

4.1 Introduction

In chapter one and two we introduced input-output controllability analysis and discussed the relevance of these technique for process design and control design. In 1983 Morari published a first paper in a series of papers [Mor83, Hol85a, Hol85b, Mor87, Sko87a] on the relation between the closed loop process behavior and the designed process. The main goal of his work was to provide insight in the implications that process design decisions have on the input-output controllability of the process, i.e. on the achievable closed loop behavior, during the process design phase. In section 4.2 we will summarize this approach. In section 4.3 the approach is used to analyze the behavior of a spraydryer model. The main purpose of this analysis is to obtain insight in the short comings of this approach for the input-output controllability analysis for controller design purposes.

One of the major shortcomings of the approach that are identified in the first two sections is a lack of understanding of directionality. For controller design understanding of the directionality is of major importance, since requirements and restrictions are in most cases directly related to process inputs and outputs. Input-output controllability techniques for the purpose of controller design must result in insight in how the directionality of the process is related to the required closed loop behavior. As was already observed in chapter three directionality is the major difference between SISO and MIMO systems. It is therefore a striking observation that the knowledge we have on the directional behavior of MIMO systems is still very restricted. In section 4.4 and 4.5 directionality and their effects on the closed loop behavior are therefore investigated.

In section 4.4 the focus will be on the directionality of non-minimum phase behavior. An approach will be developed that enables us to obtain more direct insight in the relation between the direction of a non-minimum phase zero and the effect the zero has on the behavior of the closed loop. In section 4.2 and 4.3 it will become clear that it is possible to turn the direction in which the influence of a non-minimum phase zero exhibits itself in the closed loop to a more desirable direction. In section 4.4 a good insight is obtained in the relation between the closed loop output direction that is not influenced by the zero, the consequence this has for the direction that is effected by the zero and for the gain behavior of the controller and the output direction of the non-minimum phase zero.

In section 4.5 we will further investigate the directionality of the gain and how it will limit the desired closed loop behavior. We will focus on ill-conditioned processes, since they are known to be inherently difficult to control. In section 4.5 we will reveal the reason for this behavior and will state conditions that are to be fulfilled and limitations it poses on the closed loop to robustly control these processes.

In section 4.6 we will then summarize the results obtained in this chapter. The results are discussed in relation to the characteristics we desire for an input-output analysis approach for controller design.

4.2 Input-output controllability analysis; A basic approach.

In this section we will summarize the basic idea proposed by Morari [Mor83, Mor89] to analyze the input-output controllability of a process. In chapter two we concluded that input-output controllability analysis can be seen as a control design problem itself. The goal of the analysis is to obtain insight, while the goal of control design was to obtain a well behaved closed loop process. The requirements posed on a controllability analysis technique therefore differ from control design. A balance was needed between the complexity of the approach and the accuracy of the analysis. In this trade-off the aim of the analysis was identified to be a key factor. Morari focused specifically on the design phase of process units. In this phase the available process knowledge is still very limited and uncertain. Morari required a controllability analysis tools to be [Mor83]:

- Simple: The analysis should be possible on rough inaccurate process knowledge, with a limited effort.
- unambiguous: The analysis only should depend on the process behavior and not depend on the controller structure or the control design engineer.
- transparent: The analysis should be easy understandable and result in clear answers.

Most controllability approaches, also for controller design [Mor89, Sko96], are based on this basic framework. We will therefore take this approach as a starting point for our discussions.

In section 2.3 we already saw that Morari proposes to make use of the Internal Model Scheme, which makes the problem essentially feedforward. As a consequence the controller can be seen as stable approximator of the inverse process behavior. Hence input-output controllability analysis is equivalent to understanding the effects that restrict the stable inversion of the process and how this limits the closed loop behavior. From section 2.3 and 3.5 we directly obtain that non-minimum phase effects can not be inverted by a stable function.

A second restriction on the invertibility of the process behavior are the restricted magnitude and rate of change allowed at the input signal [Mor83].

These input restrictions are generally posed in the time domain. In the analysis it is however neither possible nor useful to try to ensure that these bounds are always fulfilled. A more realistic approach is to limit the maximum gain of the controller as a function of frequency. A further advantage of this approach is that it enables an unified treatment of restrictions on the input signals and robustness aspects of the closed loop [Sko87a].

Morari proposes to deal with non-minimum phase and gain effects separately, in order to further simplify the input-output controllability analysis of the process behavior. The separation of these effects can be accomplished by the application of the inner outer factorization on the process model. The inner outer factorization of the model $M(z)\in\Re H_\infty^{pxm}$ is defined as (section 3.3):

$$M(z) = M_i(z)M_o(z) \tag{4.2.1}$$

where:

- $M_i(z)\in\Re H_\infty^{pxp}$ inner transfer matrix and $M_i(1)=I$, i.e. represents the non-minimum phase behavior of the process.
- $M_o(z)\in\Re H_\infty^{pxm}$ and outer, i.e. represents the gain behavior of the process.

This basic procedure is summarized in figure 4.2.1. In chapter two we already discussed different existing techniques that enable us to analyze the behavior of the outer and inner system. In this section we only shortly summarize the main topics relevant to the above analysis approach.

At first sight it may seem attractive to use the Bode plot to analyze the gain behavior of a MIMO processes. For a process with p outputs and m inputs pxm magnitude plots are obtained. Moreover from example 3.3.2 we have seen that an essential part of the information is contained in the phase plots corresponding to the system. The overall behavior of the model is therefore contained in the combination of the phase and the magnitude plots.

Certainly for larger systems an analysis of these plots will not provide the desired insight. The singular values or principal gains of the transfer matrix are a better alternative to obtain insight in the multivariable behavior of the transfer matrix $M(z)\in\Re H_\infty^{pxm}$ [Mor83]. Observe that:

- The restrictions the process inputs put on the input-output controllability are directly related to the gain of the controller $C(z)$ (figure 3.5.6 and equation 3.5.6c) and are therefore directly related to the inverse of the singular values of the transfer matrix of the model.

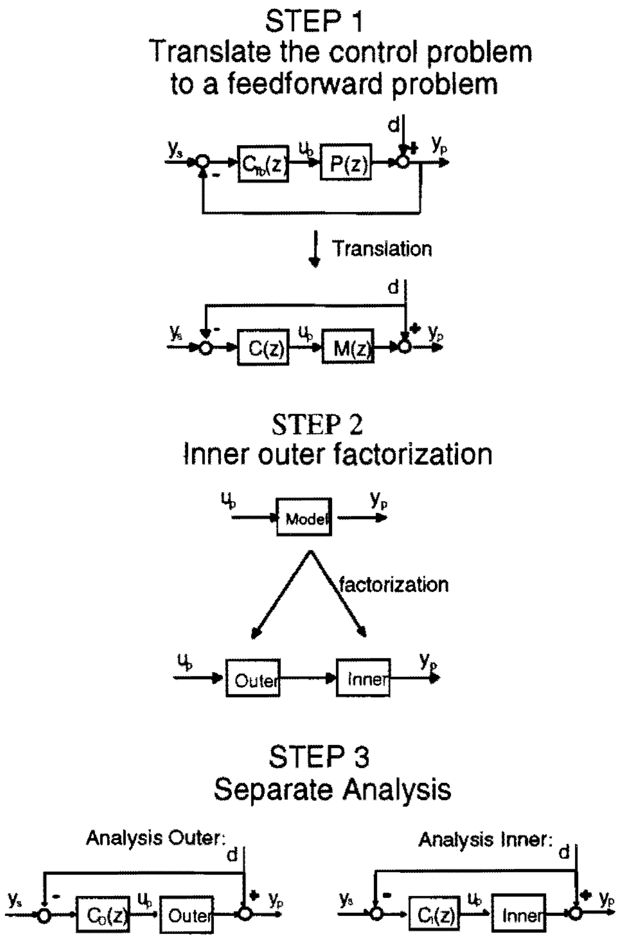

Fig.4.2.1 The basic input-output controllability approach.

- Robustness for model uncertainty is also directly related to the gain of the controller (figure 3.5.7 and equation 3.5.7c) and therefore again to the singular values of the transfer matrix of the model.

The principal gains are therefore used to analyze the gain behavior of the outer transfer matrix $M_0(z)$.

Non-minimum phase behavior limits closed loop performance, because it can not be canceled by a stable controller. In chapter two we discussed that there are essentially two different trade-off's to be made. The analytic trade-off, also known as the water bed effect, and the directional or algebraic trade-off [Fre88]. The main problem is the complexity related to the algebraic as well as the analytic trade-off [Boy85, Fre85a, Fre88, Mor87, Mor89, Sko96]. This is probably the reason to use the inner transfer matrix $M_i(z)$ in equation (4.2.1) as the basic transfer matrix to analyze the effect finite non-minimum phase zeros and delays have on the dynamic resilience of the process. In this case the controller in figure 4.2.1 for the inner transfer matrix equals the identity, i.e. $C_i(z)=I$. The sensitivity function of the steady state decoupled inner transfer matrix is then used to quantify the effect the non-minimum phase behavior has on the closed loop bandwidth:

$$S_i(z) = I - M_i(z)\text{, with } M_i(1)=I \qquad (4.2.2)$$

The singular values of $S_i(z)$ reveal the limitations the non-minimum phase behavior puts on the bandwidth of the closed loop system (Best case equals the smallest singular value of $S_i(z)$ and worst behavior the largest singular value of $S_i(z)$). Note that the statically decoupled inner transfer matrix results in the following properties for the sensitivity:

- The system possesses integral action, i.e. $S_i(1)=0$
- The maximum gain is limited to two, i.e. $\|S_i\|_\infty \le 2$

The procedure for the inner transfer matrix completely standardizes the analysis for the non-minimum phase behavior of the process[1] and is therefore completely independent of the design engineer.

The approach highly simplifies the analysis due to the simplification and standardization applied in the approach. The above properties on simplicity,

[1] The inner transfer matrix is an optimal two norm solution for a step input, with a random input direction $\eta \in \Re^{m \times 1}$ (See also section 4.4):

$$\min_{C(z)\in \Re H_\infty} \left\| (I - M_i(z)C(z))/(z-1) \right\|_2 = \left\| (I - M_i(z))/(z-1) \right\|_2$$

unambiguousness and transparency, as defined by Morari, are therefore certainly fulfilled. Remember however that the approach was meant for comparison and analysis of process designs. For this purpose no accurate analysis is directly desired. For control design it is expected that the approach is too simple. The approach does not enable relating the process behavior to limitations of the behavior at certain outputs of the closed loop. This is expected to reveal itself as a major drawbacks for control design purposes. In the next section we will apply the above approach on a spraydryer model, to further analyze the usefulness of the approach for controller design.

4.3 *Application on a spray-dryer process.*

In this section we will demonstrate the use of the input-output controllability analysis approach, as discussed in section 4.2, for a spray-dryer. Spray drying is a frequently-used process throughout industry. This process is applied in a large variety of production processes, specifically in chemical and food industry, e.g. milk, fruit and vegetable extracts, detergents, salts, polymers and so on. The dryer in this section is used for milk powder production. Control of the spray-dryer operation should be very tight, for both economic operation and quality ensurance. From a control point of view the process is interesting, due to the strong interaction and a large difference in time constants. In subsection 4.3.1 the process is discussed. An accurate rigorous simulation model of the dryer [Wij92] is used as process in this section. In section 4.3.2 the analysis approach discussed in section 4.2 is applied to the dryer process. In section 4.3.3 we will discuss the results in relation to the requirements an analysis approach for control

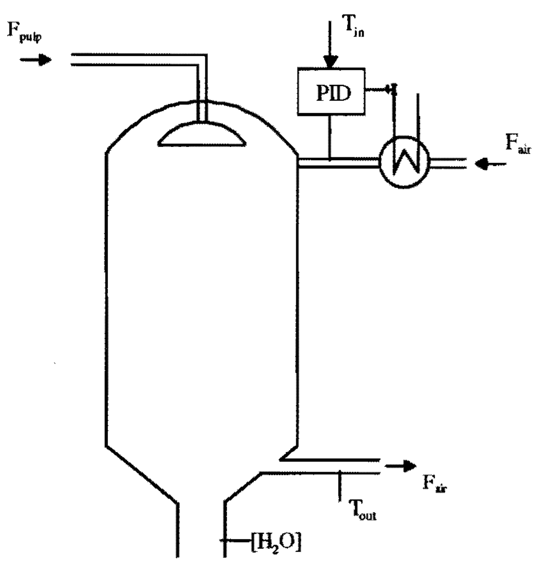

Fig. 4.3.1 A schematic overview of a spray-dryer.

design should fulfill.

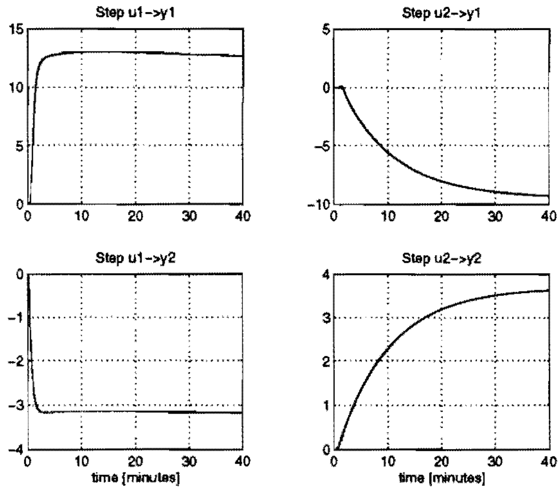

Fig.4.3.2 The step response of the linear model of the spray-dryer under nominal conditions.

4.3.1. The spray-dryer process

Spray drying is a widely spread process throughout industry. Spray drying is used to remove liquids, generally water, from solutions and suspensions. Spray drying is in particular used to dry heat sensitive products. In this example the dryer is used in milk powder production. The temperature for

	Name	Nominal conditions	Physical unit	Number of extracted delays
Output 1	H_2O-concentration	0.094	kgH_2O/kgSolid	5 samples
Output 2	Air outlet temperature	68.2	°C	2 samples
Input 1	Pulp feed-rate	1	liter/sec	0 samples
Input 2	Air inlet temperature	200	°C	4 samples

Table 4.3.1 The nominal conditions during the test and the a-priori extracted delays in samples (sampling time 5 seconds).

this process is critical as certain proteins denature at temperatures above approximately 70°C. The spray-dryer unit usually consists of a cylindrical shaped chamber with a conical bottom (figure 4.3.1). The feed , i.e. the suspension to be dried, is sprayed from the top of the chamber through a rotary or pressurized nozzle atomizer. Heated air enters from the top, such that an efficient and rapid mixing with the suspension is ensured. The air entering the tower is heated by a locally controlled heat exchanger. The local controller is used to control the temperature of the air entering the tower at the top. The dried powder leaves the dryer at the bottom separate from the exhaust air. Important process variables are the moisture content and the temperature of the moisture at the outlet of the dryer. For milk a maximum for the water concentration left in the powder is imposed for sanitary reasons. The moisture content of the dried powder is not to exceed 3%. The temperature of the solid is not to exceed 70°C, to prevent denaturation of proteins. The highest powder temperature is reached at the bottom of the tower as the evaporation rate has been drastically reduced. Here the outlet air and the powder will have approximately the same temperature. Controlled variables for the spray-dryer are therefore the moisture concentration of the powder and the outlet air temperature. The moisture is measured by an infrared absorption transducer and is measured 15 seconds after leaving the tower. The outlet air temperature is measured 10 seconds after leaving the tower. Major disturbances for the dryer are the humidity of the inlet air and the water content of the feed, deposition of powder on the wall of the dryer, wear and pollution of the nozzle. Heat losses may influence operation, depending on the situation. Manipulated variables, i.e. process inputs, are the feedrate of the suspension and the local controller setpoint of the inlet air of the spray-dryer. The identification approach, as developed in [Bac87], was applied to obtain an accurate linear model from the spray-dryer around its nominal operating point [Wij92]. The nominal conditions and the identified transport delays at the input and output of the process are given in table 4.3.1 [Wij92]. The step responses of the scaled model are shown in figure 4.3.2.

4.3.2 Analysis on a Spray-dryer model

The variations of the output and range of the input, table 4.3.2, are used to scale the model used for the controllability analysis. The idea is to scale all signals such that their magnitude in the frequency domain is smaller than or equal to one. Hence the restriction on the process input is not to exceed one. The step response of the scaled model is given in figure 4.3.2.

The minimum condition number is frequently used as an indication of the directional dependency of the gain of the process (section 3.3). A minimum condition number significantly larger than one indicates that the process is ill-conditioned. For the spray-dryer we obtained for the static minimum condition number 9.39. Looking at the step response we indeed see a strong

	Name	variations	Physical unit
Output 1	H_2O-concentration	0.01	$kgH_2O/kgSolid$
Output 2	Air outlet temperature	1.4	°C
Input 1	Pulp feed-rate (maximum change allowed)	.06	liter/sec
Input 2	Air inlet temperature (maximum change allowed)	17	°C

Table 4.3.2 The estimated average variation of the output around the nominal conditions and the maximum range of the inputs.

coupling between the two outputs. Remember that this is in accordance with the discussion on the process. The control problem is therefore ill-conditioned. Hence we may expect it to be sensitive for relative small model errors. From the step response we furthermore observe that the air inlet temperature is a much slower input then the feedrate, due to the heat exchanger dynamics. Let us further analyze the model, to obtain a better insight in the dynamic behavior of the process. An inner outer factorization is applied to the process transfer to enable a separate analysis of the non-minimum phase behavior and the gain behavior of the process conform the approach discussed in section 4.2:

$$M(z) = M_i(z)M_o(z)$$

Five non-minimum phase zeros;[-1.36, 1.68±6.76*i*, -0.71±0.86*i*] are present in the model, apart from the input output delays. The effect of the non-minimum phase behavior is analyzed based on the sensitivity of the static decoupled inner transfer matrix$M_i(z)$ (equation (4.2.2)):

$$S(z) = I - M_i(z)$$

The largest principal gain of the sensitivity function becomes one approximately at f_{I1}=0.23 per minute (figure 4.3.3). For the smallest principal gain we obtain approximately f_{I2}=0.36 per minute. It is interesting to note that the principal gains in figure 4.3.3 do not differ significantly, while output one has 15 seconds more delay than output two (table 4.3.1).

Hence, the other non-minimum phase effects must be coupled more to output two than to output one. This is verified by rewritting the inner transfer $M_i(z)$ as the product of two inner transfers:

$$M_i(z) = \begin{bmatrix} z^{-5} & 0 \\ 0 & z^{-2} \end{bmatrix} M_{i2}(z)$$

with:

- $M_{i2}(z)$ a statically decoupled inner transfer matrix containing the other non-minimum phase effects.

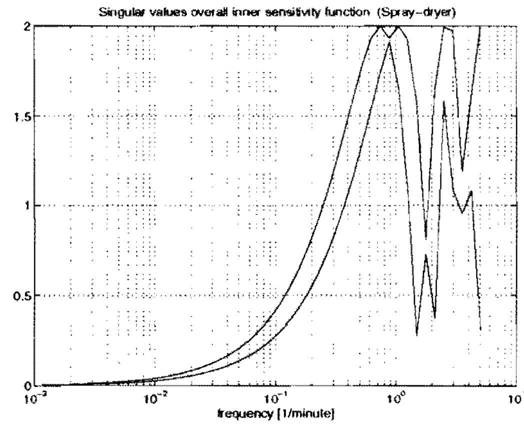

Fig.4.3.3 The principal gains of the sensitivity transfer matrix of the statically decoupled inner.

Figure 4.3.4 verifies that the delays of output one are more restrictive than the delays of output two. In the overall inner transfer matrix $M_i(z)$ this effect is compensated by the fact that the other non-minimum phase zeros are coupled more to output two (figure 4.3.5). It is important to note that the output direction of the influence that the non-minimum phase effects of $M_{i2}(z)$ have on the closed loop may be manipulated to a more favorable direction, e.g. move the influence of the non-minimum phase effects more to output one[1]. The current approach does however not offer this opportunity. Moreover there is no good insight in the consequences of manipulation of these directions.

[1] The direction of the influence the output delays have on the closed loop process outputs can of course not be manipulated, since they are directly coupled to the outputs.

From the step response of the process we already observed the ill-conditionedness of the process and the fact that the air inlet temperature reacts much slower than the feedrate. To further investigate this let us take a look at the principal gains and the condition number of the transfer matrix. Let us

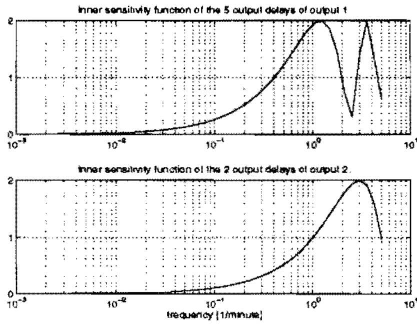

Fig.4.3.4 Bode magnitude plot of the diagonal entries of the sensitivity transfer matrix $I\!-\!M_{ii}(z)$.

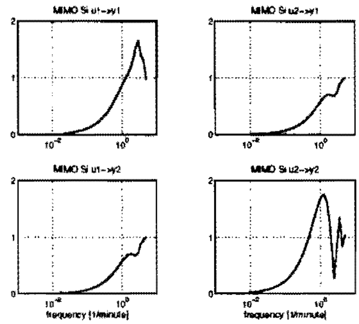

Fig.4.3.5 The Bode magnitude plot of the sensitivity transfer matrix $I\!-\!M_{ij}(z)$.

assume that the outputs are at least statically still independently controllable, despite the bad condition. If performance on both outputs is desired over a frequency range in which the small singular value of the process already starts to roll-off then the large gain of any controller in the IMC-scheme achieving this performance has to start increasing if the small process singular value starts to drop-off, since the controller should still approximate the inverse process in this frequency range to obtain the desired performan-ce. At a certain moment we may expect that both robustness and limitations of the inputs will become a bottle neck. Central question is how this restriction translates to the performance at the outputs of the closed loop process. The IMC controller is frequently represented as the inverse of the minimum phase part multiplied by a transfer matrix $F(z)\in\Re H_\infty^{2x2}$ used for the tuning of the controller:

$$C(z) = M_o^{-1}(z)F(z)$$

For the closed loop complementary behavior of the process this directly results in:

$$T_o(z) = M(z)C(z) = M(z)M_o^{-1}(z)F(z) = M_i(z)F(z)$$

For a minimum phase process the transfer matrix $F(z)$ equals the closed loop complementary sensitivity function and is therefore directly related with the process outputs. The relation with the process inputs is more complex. It is more difficult to choose $F(z)$ such that the restrictions at the input are fulfilled. A choice frequently made for the structure of the transfer matrix $F(z)$ is a diagonal transfer matrix, with diagonal entries equal to a lowpass filter with steady state gain one. This structure is an excellent choice only if the left principal vectors of the transfer matrix of the process almost equal the identity or the principal gains of the model are close together. For the spray-dryer the dynamics are concentrated at the input of the process. The use of

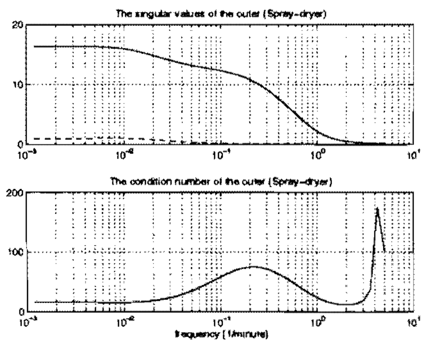

Fig.4.3.6 Principal gains and condition number of the transfer matrix.

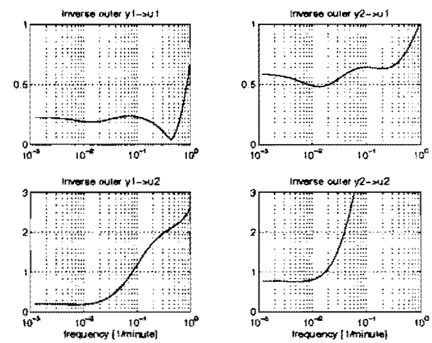

Fig.4.3.7 The Bode magnitude plot of the inverse of the outer transfer matrix of the spray-dryer.

the diagonal structure for $F(z)$ will therefore result in a small closed loop bandwidth of the whole process. This is directly seen from the Bode magnitude plot of the inverse outer transfer matrix (figure 4.3.7). The diagonal structure of the transfer matrix $F(z)$ forces us to choose the cut-off frequencies of both filters based on the small model principal gain, to reduce the gain of the second output of the controller. Hence we make no use of the possibility the largest principal gain offers to control part of the process over a larger bandwidth. If more performance is desired then another structure for the transfer matrix $F(z)$[2]. A better structure for $F(z)$ is one that directly influences the principal gains, e.g.:

$$F(\omega) = U(\omega)F_d(\omega)U^H(\omega)$$

[2] In section 4.5 we will moreover see that a certain minimum difference in bandwidth between the small and large principal gain directions is a prerequisite for robust control of an ill-conditioned process.

with U the left singular matrix of the singular value decomposition of the outer transfer matrix:

$$M_o(\omega) = \mathrm{U}(\omega)\Sigma(\omega)\mathrm{V}^{\mathrm{H}}(\omega)$$

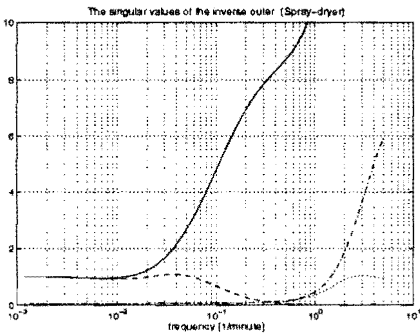

Fig.4.3.8 The principal gain of the inverse outer and controller based on the second approach.

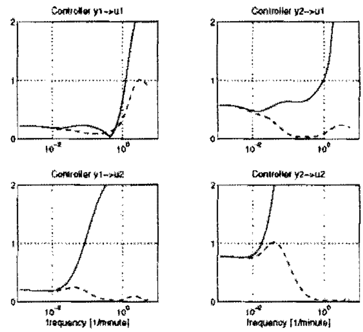

Fig.4.3.9 Bode magnitude plot of the inverse outer and the controller based on the second approach.

which results for the controller and the closed loop behavior of the outer transfer matrix in:

$$C(\omega) = \mathrm{V}(\omega)\Sigma^{-1}(w)\mathrm{F_d}(\omega)\mathrm{U}^{\mathrm{H}}(\omega) \text{ resp. } T_{O_O}(\omega) = \mathrm{U}(\omega)F_d(\omega)\mathrm{U}^{\mathrm{H}}(\omega)$$

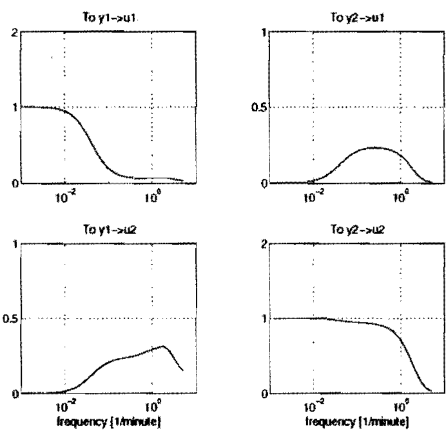

Fig.4.3.10 Bode magnitude plot complementary sensitivity of the overall process $M(z)$.

The resulting behavior of this approach is shown in figure 4.3.8 to figure 4.3.10. From figure 4.3.8 and 4.3.9 we obtain that the largest principal gain of the controller is restricted by the amplitude of the entry (2,2) of the transfer matrix of the controller. The smallest principal gain of the controller is specifically limited by the amplitude of the entry (1,1) of the transfer matrix of the controller. From figure 4.3.10 we observe that the fast dynamics of the closed loop transfer matrix are almost completely coupled to output two. This behavior is however not a consequence of the requirements we posed on the closed loop behavior of the process. It is completely determined by the analysis approach. From the step response of the open loop system we see that the fast process dynamics are coupled significantly to both output one and two. It must therefore also be possible for example to control output one fast instead of output two. If we have information on the direction of the dominant fast disturbances then one also may consider to try to turn the fast dynamics towards this output direction. We however do not have the tools yet to change the direction of the gain, such that the obtained closed loop behavior is more in line with the desired closed loop behavior.

4.3.3 Conclusions and consequences.

A restriction encountered twice in the above analysis is the inability to manipulate the directionality of the process dynamics, with the current analysis approach. First we encountered this problem for the non-minimum phase behavior and secondly for the gain behavior of the outer transfer matrix. The outcome of the analysis resulted in a overall closed loop response that was fast for output two and slow for output one. This particular directional behavior is a consequence of the analysis approach and is not expected to be due to the physical restrictions or process behavior. We however do not yet have the tools to further investigate this.
In the current approach we can not manipulate the closed loop output direction of the largest principal gain or the closed loop direction of the non-minimum phase effects. Therefore no full use is made of the opportunities a MIMO process offers to manipulate the behavior. In fact not much is known on the mechanisms that play a role in rotating the direction of the largest principal gain to a more favorable direction.

A last point of serious concern is the ill-conditionedness of the process. In the above analysis we found that the process was extremely ill-conditioned, which is generally known to make the closed loop sensitive for modeling errors. Consequently accurate models will be needed, e.g. [Li96]. It is however also expected that the conditioning of the problem puts additional restrictions and requirements on the design of the closed loop. What additional requirements and how they restrict the performance is not at all clear. In literature much attention has been paid to this type of processes. Most results in this field are focused on mathematical techniques, e.g. μ-analysis and μ-synthesis, e.g. [Doy82, Doy84, Mac89, Sko96]. The disadvantage of these mathematical techniques is however that they do not result in good understanding and insight in the basic mechanism that make these processes so difficult to control.

The above analysis with the existing input-output controllability tools make clear that we are not able to deal properly with the directionality of MIMO processes. Moreover the basic mechanisms that are involved in manipulating the directions of both principal gains and non-minimum phase zeros seem not to be well understood. The ability to manipulate and understand the directionality of a MIMO process is a prerequisite for input-output controllability analysis for controller design, since requirements and restrictions posed on the closed loop are in general posed per input and output.

In the next two sections we will therefore further analyze the directional behavior of MIMO processes. In section 4.4 we will analyze the opportunities and restrictions of manipulating the direction of non-minimum phase zeros. In section 4.5 we will focus on the directional behavior for ill-conditioned processes.

4.4 A generalized factorization approach.

In the previous sections we have indicated the important role the inner matrix plays on the analysis of the effect the non-minimum phase behavior has on the closed loop. The inner solution can be shown to be the optimal two norm or ISE (integral square error) solution for a random step change in the disturbance or reference. Hence if we want to minimize the integral square error of an inner transfer matrix $M_i(z)\in\Re H_\infty^{p\times m}$, with $M_i(1)=I$, for a step change with a random input direction then an optimal feedforward controller achieving this minimum square error is given by the identity, i.e.:

$$\min_{C(z)\in\Re H_\infty^{m\times p}} \|y\|_2$$

given:

$$u(z) = \eta/_{(z-1)} \text{ with any } \eta\in\Re^{m\times 1} \text{ not equal to the zero vector.}$$

which is equivalent to:

$$\min_{C(z)\in\Re H_\infty^{m\times p}} \|(I - M_i(z)C(z))/(z-1)\|_2 = \|(I - M_i(z))/(z-1)\|_2$$

This property is however not the reason for the use of the inner transfer matrix in the analysis approach of section 4.2. The use of the inner to assess the restriction the non-minimum phase behavior puts on the controllability of the process simplifies the controllability analysis significantly. The user does not have to make any design choices, i.e. does not have to make any design trade-off. The trade-off between bandwidth over which the disturbance is attenuated against the amplification outside this frequency range (the waterbed effect or analytical trade-off) and the direction of the influence the non-minimum phase zeros have on the controlled process are fixed. On the other hand this makes the approach rigid. The outcome of the analysis is not sufficiently detailed for MIMO controller design. The opportunity to change the directionality of the influence the non-minimum phase effect has on the closed loop output of the process is not considered. This last opportunity is relevant for control design, since it enables us to better design the closed loop behavior in accordance with the requirements. For example if output one is more important than output two the inner is not the ultimate answer. In fact we would like to turn away the influence of the zero from the most important outputs to the less important outputs. It is therefore necessary to have a better insight in how the directionality of the closed loop behavior can be influenced and the consequences this has for other closed loop relations. In chapter two we already discussed the approach taken by Tsiligiannis and Svoronos [Tsi88, Tsi89] to choose a triangular structure instead of the inner solution. The approach is based on the Wolovich & Falb interactor matrix [Wol76] and can be shown to results again in an optimal ISE solution[1] for the case that the outputs have an absolute priority, i.e. output one is the most important output, then output two and so on (section 2.3). This idea to prioritize outputs directly coincides with the required behavior in the above example. In many cases the prioritized outputs will reflect the required behavior more closely than the inner matrix approach. The approach therefore seemed attractive. However applying the interactor approach on a number of examples revealed that the solutions obtained are far from realistic. The influence that outputs with a higher priority have on the less important outputs may be extremely large and the consequences for the controller gain extremely severe. In many cases the triangular structure is therefore a too severe requirement.

In this section a better understanding is developed of the directional restriction that non-minimum phase behavior puts on the closed loop design. An alternative factorization of the process in a biproper unit and a stable non-minimum phase system will be introduced. In section 4.4.1 we will develop the approach for delays. In subsection 4.4.2 the approach is generalized to enable us to deal also with finite non-minimum phase zeros. In section 4.4.3 an example is discussed.

4.4.1 Delays

In this section we will study the directional behavior of delays in more detail. The approach we will follow here is closely related to the Silverman structure algorithm [Sil69, Sil71, Sil75]. The main difference between the Silverman structure algorithm and the approach presented here is the explicit relations the degrees of freedom in the factorization have with the output behavior of the delay in the non-minimum phase factor. Let us first state the basic factorization theorem that will be used to ultimately factor the process in two parts; one being biproper and one representing the behavior at

[1] They show that in an analog way also an optimal IAE (integral absolute error) solution can be obtained with the same directional interpretation [Tsi89].

infinity. After stating of this theorem we will further discuss the use for understanding the directional restrictions delays put on the MIMO control problem.

Theorem 4.4.1

Given a state space model $[A, B, C, D]$ of $M(z)\in \Re H_\infty^{pxm}$ and apply any nonsingular constant matrix $U_1 \in \Re^{pxp}$ such that:

$$U_1 D = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} D = \begin{bmatrix} D_{11} \\ 0 \end{bmatrix} \tag{4.4.1a}$$

with

- $D_{11} \in \Re^{(p-r)xp}$ and of full row rank.

Define the realization $M_1(z)\in \Re H_\infty^{pxm}$ as:

$$\left[A_1, B_1, C_1, D_1\right] = \left[A, B, V^{-1}\begin{bmatrix} \tilde{C}_{11} \\ \tilde{C}_{21} A \end{bmatrix}, V^{-1}\begin{bmatrix} D_{11} \\ \tilde{C}_{21} B \end{bmatrix}\right]$$

with:

- $\begin{bmatrix} \tilde{C}_{11} \\ \tilde{C}_{21} \end{bmatrix} = \begin{bmatrix} U_{11} C \\ U_{21} C \end{bmatrix}$
- $V \in \Re^{pxp}$ any non singular matrix.

then we obtain the factorization:

$$M(z) = N_1(z) M_1(z) \tag{4.4.1b}$$

with $N_1(z)\in \Re H_\infty^{pxp}$:

$$N_1(z) = U^{-1}\begin{bmatrix} I_{(p-r)x(p-r)} & 0 \\ 0 & 1/zI_{rxr} \end{bmatrix} V \tag{4.4.1c}$$

Proof: [Sil69]

In the above theorem we have thus rewritten the transfer as the product of two transfer matrices $N_1(z)$, containing the delay as only dynamics and $M_1(z)$ containing exactly p–m delays less than $M(z)$. One could say we extracted one level of delays from the transfer matrix $M(z)$. The above theorem can be applied repetitively on the successive transfer matrices $M_i(z)$ until its direct feed through matrix has full row rank. The maximum number of steps needed to extract the delays equals the McMillan degree of $M(z)$, based on the theorem of Caley and Hamilton [Gan74]. A sharper bound can be defined on the number of iteration [Sil69].

If the model does not contain any finite non-minimum phase zeros the above approach results in a coprime factorization of the transfer matrix of the model $M(z) \in \Re H_\infty^{p\times m}$, with $p \le m$:
with:

$$M(z) = N(z) M_0(z) \tag{4.4.2}$$

- $N(z) \in \Re H_\infty^{p\times p}$ containing the non-minimum phase behavior, called the *non-minimum phase factor*[1]
- $M_0(z) \in \Re H_\infty^{p\times m}$ with $M_0^{-R}(z) \in \Re H_\infty^{p\times m}$, called the *minimum phase factor*[2].

The above approach in fact extracts the delays from the transfer matrix "level after level". It can moreover be shown that the above procedure to extract the delays can be formulated as a state feedback problem with a non singular input transformation [Sil71].

In theorem 4.4.1 we see that a lot of freedom is still left in the factorization:
1. The matrix U is not uniquely determined.
2. The matrix V is completely free to choose.

We will use this freedom to parameterize the freedom left in the directional behavior of $N(z)$. In the approach of Morari the non-minimum phase behavior is analyzed separately from the analysis of the behavior of the principal gains. This is equivalent to assuming that the minimum phase factor is completely invertible. From a control point of view we therefore require the following steady state condition on the transfer matrix $N(z)$, to ensure integral control:

$$N(1) = I \tag{4.4.3}$$

Apply the QR decomposition[3] [Go89] of the direct feed through matrix D of $M(z)$ as:

$$D = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \tag{4.4.4a}$$

The orthogonal matrix Q_2 spans the kernel or nullspace of D. The requirement (4.4.1a) implies that the matrix U_{21} is spanned by the rows of Q_2^T. This is known as the interpolation requirement. The matrix U_{11} is free to choose as long as U is a non singular matrix. Mathematically expressed as:

[1] The name for the transfer matrix is introduced to fascilate the discussion.
[2] The name for the transfer matrix is introduced to fascilate the discussion.
[3] equivalently a singular value decomposition may be used.

$$U^{-1} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & \Phi_{22} \end{bmatrix} \tag{4.4.4b}$$

From the steady state requirement in equation (4.4.3) and the transfer matrix $N(z)$ in equation (4.4.1c) we directly obtain that $V=U$. We moreover obtain that we can choose $\Phi_{11}=I_{(p-r)x(p-r)}$ and $\Phi_{22}=I_{rxr}$. The actual degree of freedom in the directionality is therefore given by Φ_{12}:

$$N_1(z) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} I & \Phi \\ 0 & I \end{bmatrix} \begin{bmatrix} I_{(p-r)x(p-r)} & 0 \\ 0 & 1/zI_{rxr} \end{bmatrix} \begin{bmatrix} I & -\Phi \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} \tag{4.4.5a}$$

where we skipped the subscript for notational convenience. Note that equation 4.4.5a is equivalent to:

$$N_1(z) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} I_{(p-r)x(p-r)} & (1/z-1)\Phi \\ 0 & 1/zI_{rxr} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} \tag{4.4.5b}$$

The choice of Φ therefore determines the algebraic trade-off. Note that choosing $\Phi=0$ results in the inner solution used in the previous section. The analytic trade-off is fixed by the choice of the pole at zero. In principle however the approach can directly be generalized to any choice of the pole. This will however change the bandwidth of the sensitivity function $S(z)=I-N(z)$. Note that if we assume the model $M(z)\in \Re H_\infty^{pxp}$ to be inner, then we obtain from the factorization (4.4.1b) of the inner:

$$M_o^{-1}(z) = I - Q_1 \Phi Q_2^T \left(1 - z^{-1}\right) \tag{4.4.5c}$$

This expression directly reveals that Φ equals the additional effort (gain) that is needed to change the inner behavior into the more favorable directional behavior of $N(z)$ for the closed loop.

The following example is used to clarify the above extraction idea.

Example 4.4.1: The 2x2 case

Assume given the inner $M(z)\in \Re H_\infty^{pxm}$, with normal rank two, which contains one delay with output direction $d_d=[-x\ 1]^T$. The question is to find first order, 2x2 transfer matrices $N(z)$ which yield the factorization (4.4.2), such that (4.4.3) is fulfilled. From zero direction and the fact that the transfer matrix $M(z)$ is inner we obtain using (4.4.5), that:

$$M(z)=\begin{bmatrix}1 & -x\\ x & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}\begin{bmatrix}1 & x\\ -x & 1\end{bmatrix}/\left(1+x^2\right)$$

where:

$$Q=[Q_1 \quad Q_2]=\begin{bmatrix}1 & -x\\ x & 1\end{bmatrix}/\left(1+x^2\right)^{1/2}$$

$N(z)$ is directly obtained form equation (4.4.5b):

$$N(z)=\begin{bmatrix}1 & -x\\ x & 1\end{bmatrix}\begin{bmatrix}1 & (z^{-1}-1)\Phi\\ 0 & z^{-1}\end{bmatrix}\begin{bmatrix}1 & x\\ -x & 1\end{bmatrix}/\left(1+x^2\right)$$

The minimum phase factor of the coprime factorization (4.4.2) therefore equals:

$$M_O(z)=N^{-1}(z)M(z)=\begin{bmatrix}1 & -x\\ x & 1\end{bmatrix}\begin{bmatrix}1 & (1-z^{-1})\Phi\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & x\\ -x & 1\end{bmatrix}$$

Interesting to note is that:

- $\Phi=-x$ results in the lower triangular matrix
- $\Phi=1/x$ results in the upper triangular matrix.

These two cases in fact directly result in the solution found by the approach in [Tsi88].

End example

From equation (4.4.5b) we obtain that the transfer matrix $N(z)$ can also be written as:

$$N(z)=N_{01}+N_{02}z^{-1} \tag{4.4.6}$$

with $N_{01}+N_{02}=I$:

- $N_{01}=Q_1Q_1^T+Q_1\Phi Q_2^T$
- $N_{02}=Q_2Q_2^T-Q_1\Phi Q_2^T$

An interesting interpretation of equations (4.4.6) follows from the fact the matrices N_{01} and N_{02} are complementary oblique projectors. For an oblique projection N on the subspace W_1 along the subspace $W_2^{\perp}$ we have the following matrix expression [Ove95]:

$$N=U_1\left(V_1^TU_1\right)^{-1}V_1^T \tag{4.4.7}$$

where:

- W_1 is spanned by the columns of U_1
- W_2 is spanned by the columns of V_1

In the next lemma we will show that the matrices N_{01} and N_{02} are indeed complementary projectors.

Lemma 4.4.1
An alternative formula for the projection matrices N_{01} and N_{02} of equation (4.4.6) for the realization given in theorem 4.4.1 is defined by:

$$N_{01} = Q_1\left(M_1^T Q_1\right)^{-1} M_1^T \tag{4.4.8a}$$

$$N_{02} = M_2\left(Q_2^T M_2\right)^{-1} Q_2^T \tag{4.4.8b}$$

where:
- Q is determined by the QR decomposition of D conform equation (4.4.1a).
- M_2 is a rank r orthogonal pxr matrix which is free to choose
- M_1 is an orthogonal matrix that spans the orthogonal complement of M_2

The relation between Φ and the orthogonal matrix M=[M_1 , M_2] is given by:

$$\Phi = -Q_1^T M_2\left(Q_2^T M_2\right)^{-1} = \left(M_1^T Q_1\right)^{-1} M_1^T Q_2 \tag{4.4.8c}$$

Proof:: See appendix 4.A.

Hence:
- N_{01} is the projection on the subspace spanned by Q_1 along the subspace spanned by M_2.
- N_{02} is the projection on the subspace spanned by M_2 along the subspace spanned by Q_1.

Let us take a closer look at the interpretation of this lemma
The image of N_{01} exactly equals the image of D, since both are spanned by the columns of Q_1. From equation (4.4.8a) we moreover obtain directly that in the output direction M_1 the desired behavior is instantaneously achieved the desired. This can only be achieved at the cost of an undesired behavior of N(z) in the direction M_2, since N_{01} is rank deficient. The behavior in the direction M_2 is given by:

$$M_2 M_2^T N_{01} = M_2\left(M_2^T Q_1\right)\left(M_1^T Q_1\right)^{-1} M_1^T \tag{4.4.8c}$$

From equation (4.4.6) and equation (4.4.8b) we see that only one sample moment later the complementary projector $N_{02}=I-N_{01}$ corrects the undesired behavior in the direction M_2. Let us try to further clarify this in a simple example.

Example 4.4.2

Let us continue example 4.4.1 Alternatively we may write $N(z)$ conform equation (4.4.6) and take a closer look at the projection interpretation of N_{01} and N_{02} in relation to $N(z)$. The cosine of the angle, say θ, between the subspaces spanned by Q_1 and M_1 is equal to $Q_1^T M_1$. Hence $Q_2^T M_1$ equals minus the sine of this angle θ, which is also easily verified from figure

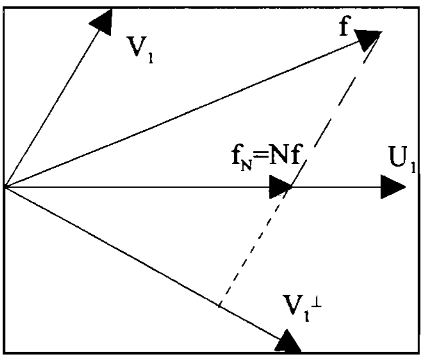

Fig.4.4.1 Projection on W_1 along $W_2^{\perp}$.

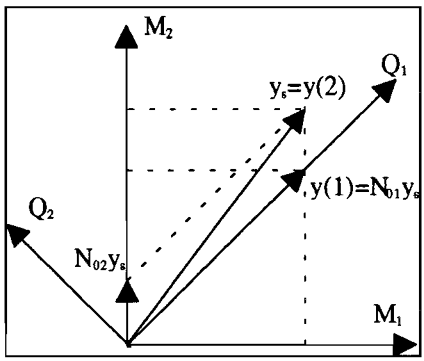

Fig.4.4.2 The input direction $M_1=[1\ 0]^T$ for x=1.

4.4.1. Substitution in equation (4.4.8c) therefore yields $\Phi=tan(\theta)$. We thus obtain for $N(z)$:

$$N_{01} + N_{02}z^{-1} = \cos^{-1}(\theta)\left(\begin{bmatrix}1\\x\end{bmatrix}M_1^T + M_2 z^{-1}\begin{bmatrix}-x & 1\end{bmatrix}\right)\Big/\left(1+x^2\right)^{1/2}$$

If the influence of the delay is turned to output two then $M_2=[0,\ 1]^T$ and $\cos^{-1}(\theta)=(1+x^2)^{1/2}$ and consequently:

$$N(z) = N_{01} + N_{02}z^{-1} = \begin{bmatrix}1 & 0\\x & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\-x & 1\end{bmatrix}z^{-1}$$

The first output is therefore directly at setpoint. However at the cost of an interaction x on output 2. This interaction is replaced by the desired behavior at the second output after one sample. The additional control action needed to achieve this is given by equation (4.4.5c):

$$M_o^{-1}(z) = I + (1 - z^{-1})\begin{bmatrix} 1 \\ x \end{bmatrix} x \begin{bmatrix} -x & 1 \end{bmatrix} \Big/ \left(1 + x^2\right)^{1/2}$$

See figure 4.4.2. The cost in the sense of the interaction for one sample and the additional control action needed is directly coupled to the magnitude of x, to the coupling of the zero to output two. If the zero is

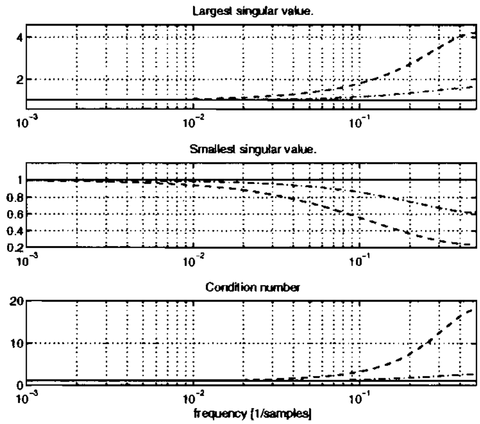

Fig.4.4.3 Principal gains and condition number of $N(z)$, with x=2 and Φ equal to: 0 (solid), 1/2 (dash dot) and 2 (dashed).

strongly coupled to output 2 then x will be small. So if $\theta \rightarrow \pi/2$ then $\tan(\theta) \rightarrow \infty$. Or equivalently: The more orthogonal M_1 is to Q_1 the larger the interaction and the control action needed. The principal gains of the controller as a function of β and the frequency $z = e^{j\varphi}$, with $\Phi \geq 0$ equal:

$$\sigma_1 = \sqrt{1 + \beta^2(1 - \cos(\varphi)) + \beta\sqrt{\left(2 + \beta^2(1 - \cos(\varphi))\right)(1 - \cos(\varphi))}}$$

$$\sigma_2 = \sqrt{1 + \beta^2(1 - \cos(\varphi)) - \beta\sqrt{\left(2 + \beta^2(1 - \cos(\varphi))\right)(1 - \cos(\varphi))}}$$

In figure 4.4.3 the principal gains are plotted as a function of frequency φ, for different values Φ.

End of example.

If we make use of the concept of canonical angles [Ove95] then the results obtained in example 4.4.2 can directly be extended to the case of a p dimensional output space. We will however not further pursue this idea here, since it does not result in any additional insight in the algebraic trade-off. As already noted the above applied approach is closely related to the Silverman structure algorithm. As a consequence all results [Sil69. Sil71, Sil75] can directly be used here.

4.4.2 Factorization of finite non-minimum phase zeros.

The above developed tools make it possible to separate the delays from the other dynamics. If the transfer matrix $M(z)$ contains non-minimum phase zeros then also these zeros should be extracted from the transfer matrix to enable us to analyze the complete non-minimum phase behavior. We would like to generalize the approach in section 4.4.1 such that it also be used for non-minimum phase zeros: Apply per non-minimum phase zero an analog approach as discussed in section 4.4.1 for the delays that results in the factorization:

$$M(z) = N_1(z) M_1(z) \tag{4.4.9}$$

such that $N_1(z)$ contains the non-minimum phase zero behavior and $M_1(z)$ does not contain the zero anymore. The generalization is accomplished using the idea described in [Tsi89], who developed a technique to accommodate for finite non-minimum phase zeros in the interactor matrix. They propose to transform each non-minimum phase zero to infinity, using a bilinear transformation. In the new domain the zero therefore equals a delay. Hence the approach developed in section 4.4.1 can then be used to perform the factorization. After extraction the two transfer matrices obtained are transformed back to the original domain, using the inverse transformation. Two different bilinear transformations are proposed for this purpose. One that results for $N_1(z)$ in the optimal IAE solution for a step change:

$$z = \frac{z_i q}{q + z_i - 1} \text{ with inverse transform } q = \frac{(1 - z_i) z}{z - z_i} \tag{4.4.10a}$$

The other yields for $N_1(z)$ an ISE optimal solution for a step change:

$$z = \frac{1+z_i p}{p+z_i} \text{ with inverse transform } p = \frac{1-z_i z}{z-z_i} \tag{4.4.10b}$$

We will use the second bilinear transformation in this section. In principle however the other transformation could also be used. Application of this idea on the results of section 4.4.1 results in the following theorem.

Theorem 4.4.2

Given a transfer matrix $M(z) \in \Re H_\infty^{pxm}$, $m \geq p$, which has a full rank direct feed through term and a real non-minimum phase zero at z_i, whose geometric multiplicity equals the algebraic multiplicity. The transfer matrix can be factored as:

$$M(z) = N_1(z) M_1(z) \tag{4.4.11a}$$

where:

- $N_1(z) \in \Re H_\infty^{pxp}$ the non-minimum phase factor containing zeros at z_i and $N(1)=I$.

$$N_1(z) = N_{01} + N_{02}\left(\frac{z-z_i}{1-zz_i}\right) \tag{4.4.11b}$$

- $M_1(z) \in \Re H_\infty^{pxm}$ a transfer matrix that for which the non-minimum phase zero z_i is replaced by $1/z_i$.

with N_{01} and N_{02} determined by:

$$N_{01} = Q_1 Q_1^T + Q_1 \Phi Q_2^T \tag{4.4.11c}$$

$$N_{02} = Q_2 Q_2^T - Q_1 \Phi Q_2^T \tag{4.4.11d}$$

with:

- $\Phi \in \Re^{rx(p-r)}$ matrix
- Q_1 Q_2 and r, the geometric multiplicity, are determined by the QR decomposition of the matrix $D+C(z_iI-A)^{-1}B$:

$$D + C(z_i I - A)^{-1} B = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} \text{ and } R \text{ has full row rank } p\text{-}r.$$

Proof: See appendix 4.A

From the theorem we directly observe that N_{01} and N_{02} are projector matrices and alternative formulation of the degrees of freedom is therefore again as in lemma 4.4.2. If we choose Φ equal to the zero matrix, we have the inner matrix again. As was the case for the delays we obtain that the change in gain behavior, due to turning the influence of the zero to another output direction, with respect to the inner, is given in the next equation:

$$X_0(z) = I - Q_1 \Phi Q_2^T (z - z_i)/(1 - z_i z) \tag{4.4.12}$$

The algorithm has to be applied again recursively:

- For each non-minimum phase zero at a different location in the complex plane.
- If the geometric multiplicity is smaller than the algebraic multiplicity for the zero.

In general the second argument will be more an academic problem than a practical one, since finite non-minimum phase zeros will almost always have multiplicity one.

Theorem 4.4.2 can also be used to extract complex conjugated non-minimum phase zeros. We could successively extract the zero z_i and its complex conjugated zero $\bar{z}_i$. The additional restriction is however that the product of the two corresponding complex transfer matrices results in a second order real transfer matrix $N_1(z) \in \Re H_\infty^{p \times p}$:

$$N_{1a}(z) N_{1b}(z) = \left(N_{01} + N_{02}\left(\frac{z - z_i}{1 - z z_i} \right) \right) \left(N_{11} + N_{12}\left(\frac{z - \bar{z}_i}{1 - z \bar{z}_i} \right) \right) \tag{4.4.13a}$$

We were not able to find explicit elegant expressions for the freedom left in the projector for the case of a complex conjugated pair of zeros.

4.4.3 Application of the technique

Assume $M(z) \in \Re H_\infty^{3x3}$ to be a statically decoupled inner transfer matrix containing four non-minimum phase zeros. Two zeros have the same location in the complex plane, but are independent, i.e. the geometric and the algebraic multiplicity is two. The output direction and location in the complex plane of the zeros is given in table 4.4.1. We assume the model

	zero location	transposed output direction
zero 1	1.5	[-0.6608 , 0.7474 , -0.0686]
zero 2a	5	[-0.5636 , 0.8196 , -0.1029]
zero 2b	5	[0.3216 , 0.1029 , -0.9413]
zero 3	-1.2	[-0.4081 , 0.6721 , -0.6179]

Table 4.4.1 Zero location and output direction of the zeros.

outputs to be ordered in a descending relative importance, based on the closed loop specifications for this process. Models obtained by identification will rarely have multiple finite zeros. This does not hold for delays, although delays are extracted from the data before identification as far as

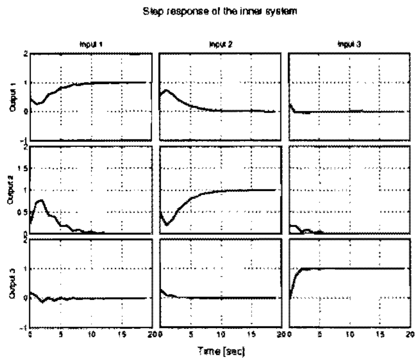

Fig.4.4.4 The step response of the inner.

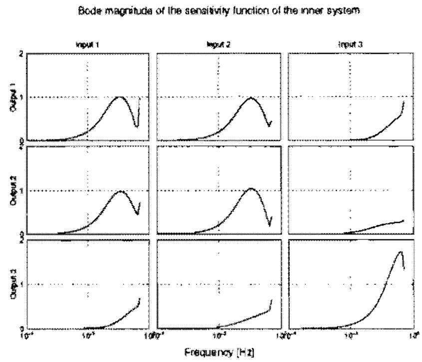

Fig.4.4.5 The Bode magnitude plot of the sensitivity function of the inner.

possible [Bac87], not all delays can be removed due to a lack of degrees of freedom to do so. The step response of the model is shown in figure 4.4.4 and the Bode magnitude plot for the sensitivity function of the inner transfer matrix (equation 4.2.2) is given in figure 4.4.5. For this model the zero at $z=1.5$ will put the worst restriction on the input-output controllability, secondly the zeros located at $z=5$ and then the zero at $z=-1.2$. This last zero does not limit the bandwidth of the closed loop system seriously. The most restrictive ones are the zeros at $z=1.5$ and $z=5$. The zero at $z=-1.2$ has however a different impact on the system behavior. The corresponding pole of the inner transfer matrix, $p_3=1/z_3=-1/1.2$, introduces a fast alternating lightly damped behavior at the output of the model. In the last part of this subsection we will further discuss this behavior.

Assume we want to find a controller that tries to turn the influence the zeros have on the closed loop to a more favorable direction, i.e. find $C(z) \in \Re H_\infty^{3x3}$ such that: $T(z)=M(z)C(z)$ has a more desirable behavior. We will assume that the outputs are ordered in descending order of importance of the corresponding requirement, i.e. output one the most important, output two the second important. and output three the least important output. Hence the most important is to minimize the influence of the zeros on output one and then on output two.

The main problem with the technique developed in this section is that generically we can only deal with one zero at a time. Hence we need to deal with the zeros one after the other. The following steps are applied to successively turn the zeros at 1.5, 5 and -1.5 to a more desirable direction:

1. Turn the zero at 1.5 to a more desirable direction, since this has the severest impact on the closed loop behavior.
2. Turn the zeros at 5 to a more desirable direction in the second step, since its influence is less restrictive than the zero at 1.5.
3. Turn the zero of.-1.2 to a more desirable direction.

Hence the procedure in which the zeros are dealt with is determined by the severity of the limitation that a zero puts on the closed loop.

Let us make a start with the first step, i.e. turn the influence of the zero located at 1.5 to a more desirable direction. We factor the model in two statically decoupled inner transfer matrices:

$$M(z) = M_1(z)N_2(z)$$

with:

- $M_1(z)$ the inner transfer matrix containing as only zero the zero at 1.5
- $N_2(z)$ the inner transfer matrix containing two independent zeros at 5 and one zero at -1.2.

In order to turn the influence of the zero at 1.5 to a more favorable direction we can now apply the theory of section 4.4.2 on the inner transfer matrix $M_1(z)$. The best solution is to turn the influence of the zero completely to the

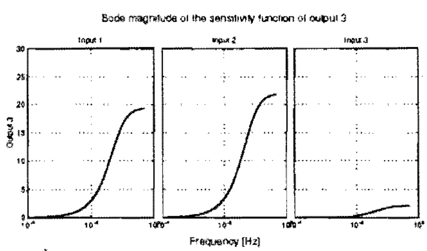

Fig.4.4.6 The Bode magnitude plot of the sensitivity function of output 3, when the influence of zero is turned to output 3.

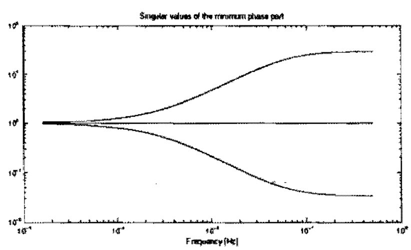

Fig.4.4.7 The principal gains of the triangular system obtained after turning the influence of the zero at z=1.5 to output 3.

third output, based on the descending order of importance of the outputs. As a direct consequence of the zero being coupled to all outputs (table 4.4.1) this is indeed possible. Hence M_1 and M_2 in lemma 4.4.2 equal

$M_2^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $M_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. From table 4.4.1 we see however that

the zero is only weakly coupled to output three. It is therefore to be expected that the proposed triangular structure will have a large influence on the performance of output 3. The Bode magnitude plot of the sensitivity function of output 3 (figure 4.4.6) and the principal gains of the triangular system (figure 4.4.7) confirm this. Hence, the effort needed to turn the zero in this direction will seriously effect the minimum phase behavior, which is obtained after substitution of equation (4.4.8) into equation (4.4.12) and noting that $\Phi = -Q_1^T M_2 \left(Q_2^T M_2\right)^{-1} = \begin{bmatrix} 9.63 & -10.90 \end{bmatrix}^T$ and $\|\Phi\|_2 = 14.54$.

The large value for Φ implies that the minimum phase behavior of the system is drastically changed. Hence it is not realistic to assume that the above factorization is feasible from a practical point of view. We therefore propose to remove the influence of the zero from the first output. We may also decouple the influence of the zero from the third output, since the zero is almost decoupled from this output. In the case the influence of the zero is removed from output one and three we obtain:

$M_2^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $M_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\Phi = \begin{bmatrix} -0.88 & -1.00 \end{bmatrix}^T$ and $\|\Phi\|_2 = 1.33$, which

is acceptable. The Bode magnitude plot of the sensitivity function, obtained after turning the influence of the non-minimum phase zero to output two (solid line) is given in figure 4.4.8. In the case that we turn the influence of the zero away from output one alone we are still left with one degree of freedom. This degree of freedom can for example be used to minimize the influence that turning the influence of the zero has on the other outputs[1]. This solution is also plotted in figure 4.4.8 (dashed line). Output one of the sensitivity matrix is not plotted in figure 4.4.8, since it equals zero in both cases. We will assume that for the extraction of the first zero we have chosen the solution that removes the influence of the zero at $z=1.5$ from the first and the third output. We obtain for the overall problem:

[1] This approach minimizes the norm and not the influence on specific output. It may happen that the interaction for a specific output is not decreased, but increased.

$$M(z) = T_1(z)X_1(z)N_2(z)$$

with:

- $T_1(z)$ the transfer whose zero is located at 1.5 and does not influence output one and three.
- $X_1(z)$ the minimum phase behavior due to changing the direction of the influence of the zero from output one and output three, i.e.: $M_1(z)=T_1(z)X_1(z)$

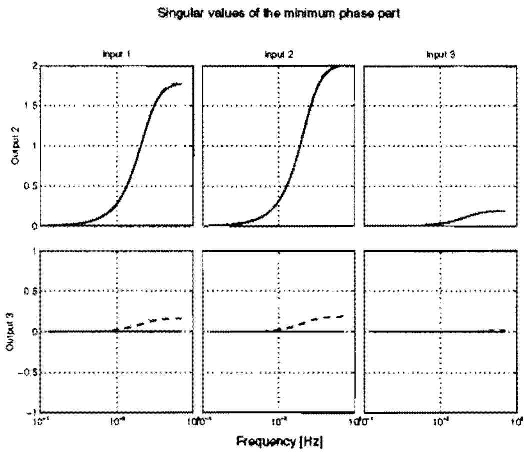

Fig.4.4.8 Bode magnitude plot of output 2 of the sensitivity matrix after tuning the influence of the zero to output 2 (solid) & to output 2 and 3 (dashed).

In the second step we need to turn the direction of the zeros at z=5 to a more desirable direction. From table 4.4.1 we see that the coupling of the zeros to output three is relatively strong, while the influence on output one and two is of more or less the same magnitude. We choose to decouple the influence of the zero from output one, since output one has preference above output two and we only one direction that is not hampered by the zeros at z=5.

In order to obtain a problem equivalent to the extraction of the initial zero we need to apply the following initial two steps:

a) In order to extract the zeros located at z=5 as an inner we apply theorem 4.4.2 on $X_1(z)N_2(z)$, with Φ=0:

$$X_1(z)N_2(z) = M_2(z)N_3(z) \tag{4.4.14a}$$

with:

- $M_2(z)$ the inner transfer matrix containing the two zeros located at 5
- $N_3(z)$ the non-inner transfer matrix containing a zero at -1.2.

b) We need to translate the desired direction of the influence zero at the output of $M(z)$, say m_2, to a direction at the output of $M_2(z)$, say m_{2T}. From the relation:

$$M(z) = T_1(z)M_2(z)N_3(z)$$

we thus obtain that:

$$m_{2T} = T_1^{-1}(5)m_2 \tag{4.4.14b}$$

We can now use theorem 4.4.2 on the transformed problem in the equations (4.4.14) to find the solution for:

$$M_2(z) = T_2(z)X_2(z)$$

As a result we obtain for $M(z)$:

$$M(z) = T_1(z)T_2(z)X_2(z)N_3(z) = T_{12}(z)X_2(z)N_3(z)$$

The Bode magnitude plot of the sensitivity function of $T_{12}(z)$ is given in figure 4.4.9.

For the last step we in principle have to follow the same procedure as for the

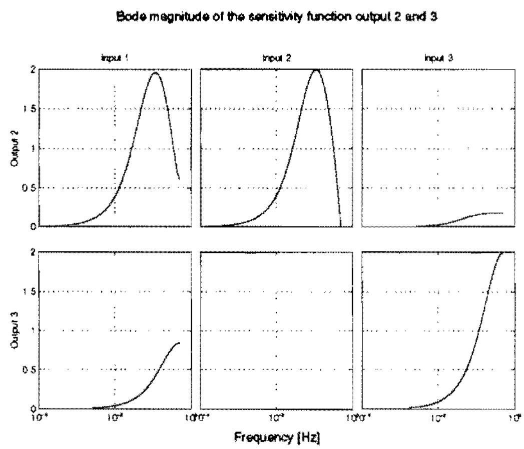

Fig.4.4.9 The sensitivity $I\text{-}T_{12}(z)$ after turning the influence of the zero to output 2 and output 3.

zeros at z=5. In order to extract the zero located at $z=-1.2$ as an inner we apply theorem 4.4.2 on $X_2(z)N_3(z)$, with Φ=0:

$$X_2(z)N_3(z) = M_3(z)N_4(z) \tag{4.4.14c}$$

with:

- $M_3(z)$ the inner transfer matrix containing the zero located at -1.2
- $N_4(z)$ the minimum phase transfer matrix.

The effect a zero at -1.2 has on the closed loop behavior of the process is very limited. We therefore choose the inner transfer $M_3(z)$ as the solution for the zero at $z=-1.2$. Hence we obtain for the controller $C(z)$:

$$C(z) = N_4^{-1}(z) \tag{4.4.15a}$$

and for closed loop complementary sensitivity transfer matrix $T(z)$:

$$T(z) = T_1(z)T_2(z)M_3(z) = T_{12}(z)M_3(z) \tag{4.4.15b}$$

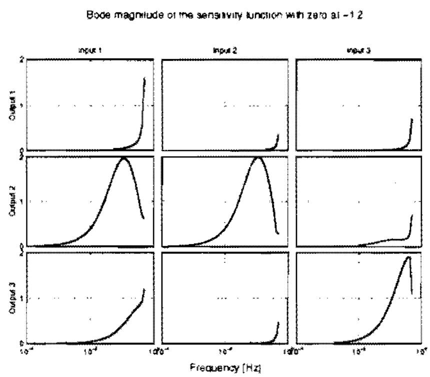

Fig.4.4.10The sensitivity function $I-T(z)$ with negative pole at -1/1.2.

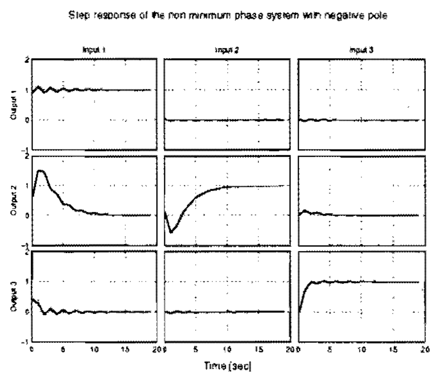

Fig.4.4.11 Step response of $T(z)$. Note the negative poles, -1/1.2,behavior.

The Bode magnitude plot of the $I-T(z)$ is given in figure 4.4.10. An interesting effect is seen from the step response of $T(z)$ (figure 4.4.11). The outputs of $T(z)$ has a clear undesirable alternating behavior, due to the negative pole at $-1/1.2$. The alternating behavior can be removed from the output if the pole located at $-1/1.2$ is replaced by a non negative pole. For example by a pole at $z=0$. Replacing the pole can be done completely in line with the developed theory. The inner transfer matrix $M_3(z)$ can be written conform equation (4.4.11b) as:

$$M_3(z) = N_{01} + N_{02}\left(\frac{z+1.2}{1+1.2z}\right)$$

The inner can then be rewritten as:

$$M_3(z)X_3^{-1}(z) = M_3(z)\left(N_{01} + N_{02}\left(\frac{1+1.2z}{2.2z}\right)\right) = \left(N_{01} + N_{02}\left(\frac{z+1.2}{2.2z}\right)\right)$$

We then obtain that the overall controller in this case equals the inverse of $X_3(z)N_4(z)$. From the step response of the resulting transfer matrix $T(z)=T_{12}(z)T_3(z)$ it is observed that the alternating behavior is completely

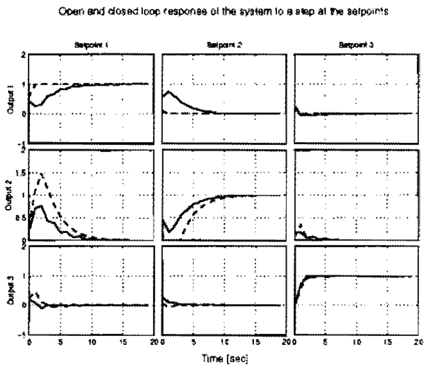

Fig.4.4.12 Step response $M(z)$ (solid) and step response of $T(z)=T_{12}(z)T_3(z)$ (dashed).

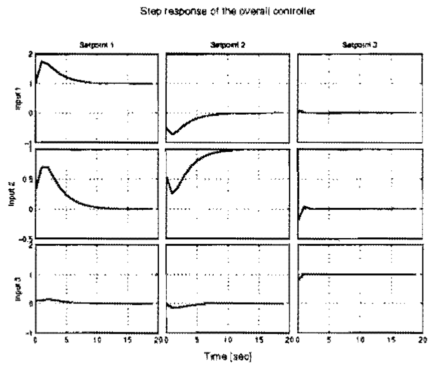

Fig.4.4.13 Step response of the overall controller $C(z)=(X_3(z)N_4(z))^{-1}$.

removed from the process outputs (figure 4.4.12).

Remark:

Identification frequently results in models containing negative poles and zeros. In general the negative poles and zeros do not represent true process behavior. This undesired (model) characteristics merely results from noise, disturbance and test signal limitations. One therefore needs to consider to remove this behavior from the model.

4.4.4 Summary and discussion on the results obtained

In section 4.3 we proposed to use the inner function to analyze the non-minimum phase behavior of a model. The main disadvantage of the approach was the inability to change the directional behavior of the zeros. To fascilate the analysis, it would be appealing to fix the input output structure of the non-minimum phase transfer matrix. An, at first sight, interesting approach, was to analyze the behavior of the non-minimum phase behavior based on a triangular transfer matrix. The advantage of the triangular structure is that it can directly be related to the prioritized requirements, if we order the outputs according to these priorities. The idea was proposed in [Tsi88, Tsi89], to deal with non-minimum phase behavior during controller design. In many cases applying this idea results in an unrealistic behavior: As a result of the triangularization the behavior of the outputs with lowest priority became unrealistically large and results in a drastic increase of the controller gain. The reason for this fact is that some of the non-minimum phase zeros are not significantly coupled to the outputs with lowest priority. In these cases the interaction on these outputs and the gain of the controller will be unrealistically large. In the analysis we therefore have to consider the output directions of the zero.

In this section an approach was developed to obtain this insight in the directionality of non-minimum phase behavior. The freedom to manipulate the direction in which the zeros influence the output behavior were factored per zero by a matrix Φ or by the matrix M_1 spanning the subspace of the output space not effected by the zero. Turning the influence of the zero from its natural direction towards M_2, the complementary subspace of M_1, resulted in an increased gain in the opposite direction, i.e. M_2 and an increased gain of the controller. The severity of this effect was completely determined by the principle angles between the subspace M_1 and the complementary zero direction. The more the spaces are aligned the less severe these effects are.

One of the main problems in the developed approach is that the extraction is applied per zero, one after the other, which makes it difficult to oversee what the consequences are of extracting one zero on the other zeros still to be extracted. This surely limits the accuracy of the results, if we have many zeros and outputs. The approach tends to become iterative and potentially trial and error, if the number of zeros increases and the influence the zeros have on the resilience of the model is more or less in the same range. In these cases a more sophisticated approach is desired.

Turning the direction in which the zeros influence the closed loop, also influences the minimum phase behavior of the model. The partition of the behavior in a non-minimum phase and a minimum phase factor therefore becomes even more artificial and potentially troublesome. Certainly if both effects limit the performance in the same frequency range. Hence the approach will tend to an iterative analysis of the non-minimum phase effect and the minimum phase effect. Although in principle straightforward the approach can easily lead to a trial and error approach and accurate insight in the relation between the limiting effects of the process behavior and the closed loop requirements tend to get lost. At least for complicated problems with many zeros and complex minimum phase behavior.

In the next section we will take a closer look at the directionality of minimum phase or gain behavior. As for the zeros we will deal with turning the gain behavior of the process to a more desirable direction.

4.5 *Ill-conditioned control problems.*

In section 4.2 no attention was paid to the directionality of the gain. We already concluded that the directionality was of crucial importance to understand the relation between requirements and dynamic resilience. In this section we will therefore take a closer look at the gain behavior and especially the directionality of the gain. It is generally known that an ill-conditioned process is very difficult to control. The directionality of the gain behavior is very pronounced in ill-conditioned processes. We will therefore use this type of process as a carrier for this section. The results obtained however have a more general value than only for ill-conditioned processes. In subsection 4.5.1 we will introduce the problem. To obtain a better understanding of the problem we will analyze ill-conditioning and the consequences for control in more detail in subsection 4.5.2. In section 4.5.3 we will deal with a simple example of a distillation column as frequently used in literature on robust control of ill-conditioned processes. In section 4.5.4 we will summarize and discuss the results obtained. We will furthermore discuss a procedure to deal with ill-conditioned control problems.

4.5.1 An introduction to the problem

It is known that ill-conditioned processes are difficult to control. The gain of an ill-conditioned process highly depends on the direction. In some directions the gain is large and in others the gain is small, i.e. there is a large difference in magnitude between the largest and smallest principal gains. It has been observed by several authors that:

- The closed loop behavior of these processes can be very sensitive for certain type of model errors.
- The closed loop disturbance reduction behavior can differ significantly depending on the points in the loop the disturbance enters.

The most primitive solution to circumvent these effects is to reduce the number of requirements or to restrict performance to the frequency range, where the process is spatially round [Fre85b]. These solutions are potentially very restrictive. The performance obtained may be significantly less than the potential performance [Fre85b]. Another approach applied is to restrict the bandwidth over which we control each principal gain to the range were its magnitude is still sufficiently large. Inversion of the principal gain is stopped at the moment that the principal gain becomes too small. In fact Kouvakis [Kou93] proposed to solve MIMO control problems for generalized predictive control in this way. This will of course result in better performance than the first approach. A disadvantage of this approach is that the requirements are not considered. It may still be possible to turn the influence of the largest principal gains in a more desirable direction.

Moreover an important class of control problems is excluded. All problems that are ill-conditioned over the frequency range relevant for control can not be dealt with. A well known example of such a process is the distillation column operating at high purity. For a binary high purity distillation column it results in control only in the direction that makes one quality better and the other worse. Hence the controller will not react to a disturbance that decreases or increases the quality of both products. This is of course unacceptable. More sophisticated approaches and a detailed insight are needed.

Amongst others Freudenberg and coworkers [Fre87, Fre88, Fre89a, Fre89b], Doyle and coworkers [Doy82, Doy84], Morari and Skogestadt [Mor89, Sko86] did a lot of research to these type of processes. Much of this research is focused on μ, as introduced by Doyle [Doy82]. The main advantage of using μ is that it allows us to deal with structure in the problem. The importance of explicitly accounting for structure in the control design of ill-conditioned problems can be understood as follows: It has been observed that closed loop behavior of ill conditioned processes may react completely different to disturbances and model uncertainties entering at different places in the loop, e.g. good nominal disturbance attenuation of output disturbances and a reasonable margin for additive uncertainty do not guarantee a sufficient behavior for disturbances entering at the process inputs and for uncertainty modeled at the process inputs. It is therefore proposed to take sources entering at different spots in the closed loop into account in the criterion function. An example of such a criterion is given in equation.(3.5.12). In this case uncertainty at the process input and disturbances entering at the process output are considered, which is reflected in the choice of the structure of $\Delta(z)$ in equation (3.5.12). However most robust design techniques can not cope with this explicit structure imposed by the problem. Significant conservatism is therefore potentially introduced. The attention μ theory receives in robust control is due to its ability to explicitly make use of structure. A major drawback of μ theory is that it is a mathematical optimization that does not result in fundamental insight in the mechanisms. The understanding of these mechanisms is however a prerequisite. Not only for input-output controllability analysis , but also for controller design itself. Not understanding the fundamental relations will result in a trial and error approach in the design.

This fact has been recognized by Freudenberg, Looze and coworkers. They have made a start to come to a more fundamental understanding of the ill conditioned open loop behavior in relation to the resulting restrictions on the closed loop [Fre87, Fre88, Fre89a, Fre89b]. In this section we will extend their results to obtain a better understanding of the problem. Let us make a start with analyzing the situation.

A frequently proposed measure to judge the conditioning of the process is the condition number of the transfer matrix as a function of frequency (equation (3.3.7a)), since it is a measure for the directionality of the gain. However a process whose transfer matrix is ill-conditioned is not always difficult to control:

- A process, which is known to have a diagonal structure, e.g.:

  $$M(z)=\begin{bmatrix} g_1(z) & 0 \\ 0 & g_2(z) \end{bmatrix}$$

  is not more difficult to control than the two independent SISO processes on the diagonal, independent of the value of the condition number. To make it a process that is difficult to control also the structure is of importance. For ill-conditioned processes also the directionality of the principal gains needs to be uncertain, as will come out of the discussion in this section.
- The principal gains and therefore the condition number are known to be scaling dependent. The condition number of the process is dependent on the physical units used (section 3.5). The units used can however never change the physical behavior. It can only be the physical behavior that makes the process inherently difficult to control.

A large condition number is therefore only necessary, but not sufficient for the problem to be difficult to control. To circumvent the above problems a frequently proposed indicator is the so called minimum condition number [Mor89, Sko96] and the Relative Gain Array [Mor89, Sko96].

If this minimum condition number or the RGA is large then the process is known to be difficult to control [Mor89, Sko96]. What if the minimum condition number is small but the condition number of the scaled model is large. The scaling applied may result in an ill-conditioned control problem. However a prerequisite for a sensible problem formulation is that scaling is in accordance with physical reality, as discussed in section 3.5. In case this scaling is not in accordance with physical reality, we may expect the minimum condition number to give a too optimistic view. At first sight this statement seems to contradict existing results with respect to the minimum condition number. It will turn out in this section that we need to make a distinction between robust stability and robust performance. Robust stability is scaling independent. The minimum condition number is therefore a good indicator for potential stability problems, due to small model errors. However robust performance results are indeed scaling dependent. In order to obtain sensible results a physical relevant scaling is needed.

The Relative Gain Array is also proposed as a measure for the conditioning of the problem. The main advantage is that the relative gain array is scaling independent. It is closely related to the minimum condition number. From the above discussion on scaling it follows that the RGA is no good measure for robust performance.

It is emphasized that in most cases really ill-conditioned processes, i.e. having a condition number larger than 10 to 15, are uncontrollable. At least from an industrial point of view. This can already be seen from the following consideration: To achieve the desired behavior in the low gain direction of the process relatively large changes of the actuator are desired, i.e. large amplitude range. On the other hand for the high gain direction very small and accurate changes of the actuators are needed, i.e. high resolution [1].

[1] In the case that the conditioning is directly coupled to inputs, i.e. one input has a large gain and one a small gain, the actuators will not form a principle restriction. An example of such a situation is found in the tube glass process in chapter 6.

In most cases actuators currently in use in industry do not meet these two conflicting requirements. The conditioning of the problems we actually have to deal with in the controller design are therefore always limited.

4.5.2 Analysis of the problem

In this section we will make an attempt to better understand the mechanisms that make ill-conditioned problems that difficult to control. We will start the discussion with discussing the existing results. We will then further develop the insight in the basic mechanisms, based on these results.

A frequently used criterion for control design is a two block criterion that contains the output sensitivity and the controller outputs, e.g.:

$$\sum_{k=0}^{\infty}\left\{(y(k)-\hat{y}(k))^T Q(y(k)-\hat{y}(k))+u(k)Ru(k)\right\} \tag{4.5.1a}$$

or in the frequency domain:

$$\left\|\begin{bmatrix} W_1(I-M(z)C(z)) \\ W_2C(z)\end{bmatrix}\right\|_\alpha \quad \text{with } \alpha \text{ equals 2 or } \infty \tag{4.5.1b}$$

It has been observed that ill-conditioned processes designed with the above type of criteria are likely to be sensitive for disturbances entering the closed loop at the process inputs and input uncertainty [Fre88c]. The behavior is possibly not as we might expect from the transfer matrices of the criterion function (4.5.1). This can be understood from the following reasoning. Based on the criterion (4.5.1b) we may expect that the output sensitivity $S_o(z)$ and the controller transfer $C(z)$ are well shaped and have the desired properties. A well shaped transfer matrix $C(z)$ and $S_o(z)$ does not necessarily guarantee that also $T_i(z)=C(z)M(z)$ and consequently $S_i(z)$ and $So(z)M(z)$ have the desired behavior. For a design based on criteria like (4.5.1b) it is well possible that the right principle vectors belonging to large principal gains of $C(z)$ are not aligned with the corresponding left principle vectors of the small principal gains of $M(z)$. A well known bound on the maximum gain of $T_i(z)$ is:

$$\sigma_1(T_i)\le\sigma_1(M^{-1})\sigma_1(T_o)\sigma_1(M)=cond(M)\sigma_1(T_o) \tag{4.5.2}$$

In principle it is a conservative bound. As we will see below it is in fact the worst case situation. Note that a similar upper bound can be based on the controller, since $T_i(z)=C(z)T_o(z)C^{1}(z)$:

$$\sigma_1(T_i) \leq cond(C)\sigma_1(T_o) \tag{4.5.3}$$

A sufficient condition to obtain a well shaped $T_i(z)$ is therefore to design a controller $C(z)$ with a condition number close to one. This requirement is a very severe one, since it means that we do not control the process anymore in the direction of the small principal gains. Fortunately it is not a necessary condition.

Assume the transfer model of $M(z) \in \Re H_\infty^{p\times m}$ can be decomposed by a singular value decomposition in two subsystem one with high gain and one with low gain [Fre85b][2]:

$$M = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix} \tag{4.5.4}$$

with:

- $\Sigma_1 \in \Re^{r\times r}$ and $\Sigma_2 \in \Re^{(s-r)\times(s-r)}$, $s=\min(p,m)$ and $\sigma_1(\Sigma_2) << \sigma_r(\Sigma_1)$.

Rewriting (4.5.4) as a dyadic decomposition result in:

$$M = U_1\Sigma_1 V_1^T + U_2\Sigma_2 V_2^T \tag{4.5.5}$$

Substitution of equation (4.5.5) in:

$$T_i(z) = M^{-1}(z)T_o(z)M(z)$$

results in[3]:

$$T_i = V_1\Sigma_1^{-1}U_1^H T_o U_1\Sigma_1 V_1^H + V_1\Sigma_1^{-1}U_1^H T_o U_2\Sigma_2 V_2^H + \\ V_2\Sigma_2^{-1}U_2^H T_o U_1\Sigma_1 V_1^H + V_2\Sigma_2^{-1}U_2^H T_o U_2\Sigma_2 V_2^H \tag{4.5.6}$$

The third term of the decomposition, $\Sigma_2^{-1}U_2^H T_o U_1\Sigma_1$, is of primary interest. For an ill-conditioned process this term may be very large, unless T_o is designed such that is not too large. After substitution of M in T_o this results in: $\Sigma_2^{-1}U_2^H T_o U_1\Sigma_1 = V_2^H C U_1\Sigma_1$. This is the main cause of the potential disappointing behavior of T_i. We have to ensure in the design that $U_2^H T_o U_1$ is sufficiently small for the frequency range over which the condition number is large, to circumvent problems due to small modeling errors at the process inputs. Based on the relation $T_o(z) = M(z)T_i(z)M^{-1}(z)$ analog restrictions are obtained: $\Sigma_1 V_1^H T_i V_2\Sigma_2^{-1} = \Sigma_1 V_1^H C U_2$.

[2] For notations convenience the dependency of the frequency is not explicitly stated.
[3] For notations convenience the dependency of the frequency is not explicitly stated.

From the above analysis it is clear that in the criterion both the closed loop transfer matrices related to behavior at the process inputs as well as at the process output and the controller itself should be taken into account. Freudenberg pointed out that the above two requirements on $V_2^H C U_1 \Sigma_1$ and $\Sigma_1 V_1^H C U_2$ may be insufficient to guarantee robust stability if simultaneous divisive output and input uncertainty occur in the loop. An additional requirement has to be posed to ensure robust stability for this case.

This robust stability problem for simultaneous divisive output and input uncertainty can be stated as a μ problem, conform equation (3.5.8) for the transfer matrix M_T [Fre89a]:

$$M_T = \begin{bmatrix} S_o & S_o M \\ -C & -T_i \end{bmatrix} \tag{4.5.7}$$

Freudenberg was able to deduce an interesting upper bound on $\mu(M_T)$:

$$\alpha - \beta \leq \mu(M_T) \leq \alpha + \beta \tag{4.5.8}$$

with:

- $\alpha = \left(\sigma_1(S_o M)\sigma_1(C)\right)^{1/2}$
- $\beta = \max\left(\sigma_1(S_o), \sigma_1(T_i)\right)$

From these bounds we directly obtain that, besides for the previously discussed two requirements which are reflected in β, we also have to account for the principal gains of the off diagonal blocks, i.e. limit α. Note that considering this term causes the extreme sensitivity for the simultaneous occurrence of the errors, despite of a fair robustness margin for each error separate.

The criterion matrix (4.5.7) has a second interpretation that is much more appealing for control design (section 3.5). The criterion enables us to deal with robust performance of the system for relative input uncertainty, conform equation (3.5.11a):

$$S_{out}(z) = S_o(z) - S_o M(z)\Delta_I(z)(I + C(z)M(z)\Delta_I(z))^{-1}C(z) \tag{4.5.9}$$

Translation of the above requirements to the robust performance case results in the following four requirements that have to be fulfilled for an ill conditioned model:

1. The first requirement is to keep the principal gains of the controller restricted such that the input requirements and robust stability criteria for additive model errors are fulfilled.
2. For nominal performance we need to give S_o the desired behavior.
3. Limit the principal gains of T_i to ensure robust stability against relative uncertainty at the process inputs. Crucial for this requirement is the behavior of $U_2^H T_o U_1$.
4. In order to ensure robust performance also a requirement is posed on the relation between the behavior of $S_o M$ and C, which is reflected in α.

The first two requirements are also posed on a well conditioned control problem. The last two requirements are specific for ill-conditioned problems. Let us further investigate these last two requirements to obtain an understanding of the underlying mechanisms.

The IMC controller C can be parameterized as:

$$C = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2^{-1} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \qquad (4.5.10a)$$

with Q_1 and Q_2 the design parameters. Substitution of C in $U_2 T_o U_1$ yields:

$$U_2^H T_o U_1 = Q_2 U_1 \qquad (4.5.10b)$$

Note that only if the last row of Q_2 is completely aligned with the first column of U_1 we obtain the upper bound (4.5.2). The input direction of Q_2 therefore needs to be perpendicular to U_1. No alignment between Q_2 and U_1 is obtained if we factor Q as:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix} \cdot \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} \qquad (4.5.10c)$$

From (4.5.10b) we also see that equation (4.5.2) is too restrictive. For the controller we thus obtain as a parameterization:

$$C = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2^{-1} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix} \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} \qquad (4.5.11)$$

In order to minimize μ, we may try to minimize the upper bound (4.5.8). We therefore have to ensure that if $\sigma_1((I-MC)M)$ is large then $\sigma_1(C)$ should be small and visa versa. This is however a severe restriction if the model is ill-conditioned. If performance is desired in all directions, then $\sigma_1(C)\sigma_1(M)$ is necessarily large. Hence $\sigma_1(I\text{-}MC)$ needs to be small. It therefore requires a significant attenuation of $(I\text{-}MC)$ in the input direction aligned with the model output direction U_1. Let us investigate this behavior in more detail. Note that equation (4.5.9) is equivalent to:

$$S_{out} = S_o\left(I - M\,\Delta_I\, C(I + M\,\Delta_I\, C)^{-1}\right) = S_o(I + M\,\Delta_I\, C)^{-1} \qquad (4.5.12)$$

The amount of alignment between $\sigma_1((I\text{-}MC)M)$ and $\sigma_1(C)$ is completely determined by the input uncertainty Δ_I. Central question is to what extend Δ_I is able to align the large principal gain of C with that of M. Using equations (4.5.4) and (4.5.11) results in:

$$M\Delta_I C = U\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}\begin{bmatrix}V_1^H\\ V_2^H\end{bmatrix}\Delta_I\begin{bmatrix}V_1 & V_2\end{bmatrix}\begin{bmatrix}\Sigma_I^{-1} & 0\\ 0 & \Sigma_2^{-1}\end{bmatrix}QU^H \qquad (4.5.13a)$$

From (4.5.13a) we directly see that, with $\sigma_1(\Delta_I)\leq\delta$, maximum alignment is obtained if:

$$V_1^H\Delta_I V_2 = \delta I \qquad (4.5.13b)$$

In this case we obtain for $(I+M\Delta_I C)^{-1}$:

$$\begin{bmatrix}(I+\delta Q_{11})^{-1} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}I & -\delta Q_{12}\\ 0 & I\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & (I+\delta Q_{22})^{-1}\end{bmatrix} \qquad (4.5.13c)$$

Neither robust stability nor performance of the closed loop for this specific input uncertainty is affected by the conditioning of the process. From (4.5.13a) we directly obtain that in the case of:

1. a full *mxm* uncertainty block, there always is a matrix that fulfills this property. A worst case Δ_I, is given by:

$$\Delta_I^{worst} = \begin{bmatrix}V_1 & V_2\end{bmatrix}\begin{bmatrix}0 & \delta I\\ \delta I & 0\end{bmatrix}\begin{bmatrix}V_1^H\\ V_2^H\end{bmatrix} \qquad (4.5.13d)$$

2. a diagonal *mxm* matrix with independent entries, the situation depends on the direction of the principle vectors. This is clearly seen from the two input case.

We will take a closer look at at static two input example.

Example 4.5.1

In the case of a static transfer with two inputs and a diagonal input uncertainty then $V\Delta_I V^T$ can be written as:

$$\begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix} \Delta_I \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

We now have to maximize $V_1\Delta_I V_2^T$, which results, with $|\delta_i| \leq |\delta|$ for $i=1,2$ in:

$$\max_{\delta_1,\delta_2} \left| V_1^H \Delta_I V_2 \right| = |\delta \sin(2\varphi)|$$

which is attained for $\delta_1=\delta$ and $\delta_2=-\delta$ and $\delta_1=-\delta$ and $\delta_2=\delta$. The maximum error therefore depends on the angle φ and is obtained for: $V_1^T=\pm[\ 1, -1\ ]/\sqrt{2}$. Hence the more the inputs are coupled to a specific principal gain the less influence the uncertainty will have on the robust performance.

For $\delta_1=\delta$ and $\delta_2=-\delta$ or $\delta_1=-\delta$ and $\delta_2=\delta$, the transfer matrix $T_\Delta=M\Delta_I C(I+M\Delta_I C)^{-1}$ may be approximated by, since $\sigma_2/\sigma_1<<1$:

$$\delta U \begin{bmatrix} X_1^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \cos(2\varphi)Q_{11} & \sin(2\varphi)\sigma_1\sigma_2^{-1}Q_{22} \\ 0 & \cos(2\varphi)Q_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X_2^{-1} \end{bmatrix} U^H$$

with:

$$X_1 = I + \delta\cos(2\varphi)Q_{11} \text{ and } X_2 = I + \delta\cos(2\varphi)Q_{22}$$

From this equation we directly obtain that:

1. The robust stability requirement in the above equation is completely determined by $(I+\delta\cos(2\varphi)Q_{11})^{-1}$ and $(I+\delta\cos(2\varphi)Q_{22})^{-1}$. Robust stability is therefore not influenced by the conditioning of the model. Moreover the closed loop is most sensitive for input uncertainty if each diagonal entry is coupled to one input, i.e. $\varphi=(\pi\pm k\pi)/2$, with k any integer.
2. If $V=I$, i.e. $\varphi=0$, then no additional interaction will occur. The more the influence of each uncertainty block is spread over the principal gains the more interaction can be expected.
3. If the closed loop is sensitive for changes in performance then it is insensitive for stability problems due to the input uncertainty and visa versa.

4. The only opportunity we have to reduce the influence of σ_1/σ_2 on T_Δ is to reduce Q_{22}, i.e. reduce performance of the closed loop in the direction of the second principal gain. The influence T_Δ has on the output, conform equations (3.5.6), can be reduced by increasing the performance of the closed loop in the direction of the first principal gain, i.e. decreasing the magnitude of $I-Q_{11}$.

End of example.

Independent of the number of inputs of the problem the worst case diagonal modeling error is always given by $|\delta_i| = |\delta|$ for $\forall i \in \{1,2,...,m\}$. For an m-input problem the maximum interaction between the j-th and i-th principal gains, due to diagonal input uncertainty, is given by:

$$\max_{\delta_1,..,\delta_m} \left|V_j^H \Delta_I V_i\right| = |\delta| \sum_{k=1}^{m} \left|V_{kj}\right|\left|V_{ki}\right| \leq |\delta|$$

where equality will hold if and only if $|V_{kj}| = |V_{ki}|$ for $\forall k \in \{1,..,m\}$.The absolute value of each entry will then equal either 0 or $1/\sqrt{n_d}$ where n_d equals the number of non zero entries. Introduce the *(principal gain) coupling factor*:

$$CF_{ji}(M) = \sum_{k=1}^{m} \left|V_{kj}\right|\left|V_{ki}\right| \left(\sigma_j / \sigma_i\right) \qquad (4.5.14)$$

The coupling factor indicates the worst case coupling that may occur between the i-th and j-th principal gain, due to diagonal input uncertainty. For a transfer matrix M we thus obtain rxr coupling factors, with r the normal rank of M. The coupling factors can be represented in a rxr matrix, the *(principal gain) coupling matrix*:

$$C(M) = \begin{bmatrix} CF_{11}(M) & . & CF_{1i}(M) & . & CF_{1r}(M) \\ . & & . & & . \\ CF_{i1}(M) & . & CF_{ii}(M) & & . \\ . & & . & . & . \\ CF_{r1}(M) & . & . & . & CF_{rr}(M) \end{bmatrix} \qquad (4.5.15)$$

The matrix $C(M)$ is a rxr frequency dependent function with real entries that enables a fast insight in potential robust performance problems, due to diagonal uncertainty. The coupling matrix has the following structure, easily verified using (4.5.14) and the properties of the principal gain decompostition:

1. $0 \le CF_{ij}(M) \le \sigma_i/\sigma_j$
2. $CF_{ii}(M)=1$
3. $CF_{ij}(M) \ge CF_{ji}(M)$, for $j>i$
4. If equality in 3 holds then $CF_{kl}(M)=CF_{lk}(M)$, for $k,l=\{i,...,j\}$, since $\sigma_i=\sigma_{i+1}=..=\sigma_j$

Hence, to asses potential robust performance problems only the upper triangular matrix is important. To fascilate later discussions we will introduce the following definitions:

1. If $CF_{ij}(M)>>1$ then the i-th principal gain is said to be *strongly coupled* with the j-th principal gain.
2. If $CF_{ij}(M)<<1$ then the i-th principal gain is said to be *weakly coupled* with the j-th principal gain.

Note that strongly coupled principal gain pairs can only occur in the upper right part of the coupling matrix. Moreover a principal gain pair that is strongly coupled is also accompanied by a weakly coupled pair. The reverse is not true.

The above discussion was completely focused on diagonal uncertainty. If the uncertainty is not diagonal anymore, but full block, then the degradation of performance is not restricted by structural restrictions of the input and we always obtain *full coupling*:

$$CF_{ij}(M)=\sigma_i/\sigma_j$$

In this case strongly coupled principal gain pairs are also weakly coupled, since $CF_{ij}(M)=(CF_{ji}(M))^{-1}$.

After we identified the potentially troublesome parts in the behavior, the question arises what we can do to design a controller that prevents this effect to happen. We assume the transfer matrix can be decomposed as in equation (4.5.4), with $\sigma_1(\Sigma_2)<<\sigma_r(\Sigma_1)$. We furthermore assume that the diagonal entries of Σ_i, $i=\{1,2\}$, are all in the same order of magnitude, to fascilate the discussion. We obtain as an approximation for S_{out}, with the controller parameterization (4.5.11), full coupling between the principal gains and $\sigma_1(\Sigma_2)<<\sigma_r(\Sigma_1)$:

$$S_{out} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} I - Q_{11} & \delta(I - Q_{11})\Sigma_1\Sigma_2^{-1}Q_{22} - Q_{12} \\ 0 & I - Q_{22} \end{bmatrix} \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} \quad (4.5.16)$$

From the derivation of this equation (Appendix 4.B) we obtain that the more ill conditioned the process is the more accurate the approximation is. The sensitivity of the performance for input uncertainty is completely determined by:

$$(I-Q_{11})\Sigma_1\Sigma_2^{-1}Q_{22}. \quad (4.5.17)$$

An important observation for control design is therefore that a certain gap between the bandwidth over which the two principal gains are controlled is necessary. Only after Q_{22} has become sufficiently small we may reduce performance in the direction of the large principal gain. The minimum attenuation needed over this frequency range is directly related to the ratio of the corresponding principal gains σ_i/σ_j. A useful rule of thumb and a sufficient condition directly follow from this observation:

$$(I-Q_{11})Q_{22} \leq \alpha_d\, cond^{1}(M)/\delta \quad (4.5.18)$$

with:

- α_d the maximum allowed performance reduction for a given δ.

A frequently applied remedy against poor performance in practice is to reduce the bandwidth of the closed loop. A consequence of the above is that for an ill-conditioned process reducing the overall bandwidth of the closed loop, by no way is a guarantee for more robust control.

A remaining interesting question is: Is possible to turn the direction of S_o to a direction that is more in line with the requirements posed on the closed loop? In fact the question is of more general interest and independent of the conditioning of the problem. We will assume again that the process can be decomposed, conform equation (4.5.5), one with high gain over a large bandwidth and one whose gain can only be inverted over a smaller frequency range.

The output direction U_1 can therefore be controlled over a larger frequency range than U_2. Interesting to know is how we can still use these large principal gains to fulfill the requirements as good as possible. During the discussion we assume that the first outputs are ordered in an ascending order of importance, i.e. the first outputs are the most important ones. The question then becomes: Are we able to obtain a complementary sensitivity T_o with the following structure:

$$T_o = \begin{bmatrix} X & 0 \\ * & * \end{bmatrix}$$

where X should be as close as possible to the identity and * indicates that no performance is desired. Factoring the controller conform (4.5.7) and assuming for the moment that Q_{22} is already almost zero, i.e. the small principal gains are not controlled anymore results for T_o in:

$$T_o \approx \begin{bmatrix} U_{11}Q_{11}U_{11}^H + U_{11}Q_{12}U_{12}^H & U_{11}Q_{11}U_{21}^H + U_{11}Q_{12}U_{22}^H \\ U_{21}Q_{11}U_{11}^H + U_{21}Q_{12}U_{12}^H & U_{21}Q_{11}U_{21}^H + U_{21}Q_{12}U_{22}^H \end{bmatrix}$$

Choosing $Q_{11}=f_1 I$ and $Q_{12}=f_1 U_{11}^{-1}U_{12}$, yields:

$$T_o \approx \begin{bmatrix} f_1 I & 0 \\ f_1 U_{21}U_{11}^{-1} & 0 \end{bmatrix} \tag{4.5.19}$$

where f_1 is a scalar low pass filter with steady state gain one. In the above analysis we assumed that the Q_{22} is neglectably small. Of special interest is of course the range where Q_{22} starts to deviate from one, but is still not neglectable. In this range the approximation does not hold. Define $Q_{22}=f_2 I$, with f_2 a scalar lowpass filter with steady state gain one, i.e. it determines the bandwidth over which we invert the small principal gains. We therefore have to compensate Q_{12} for the fact that Q_{22} is not zero. This results in the following controller:

$$C = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma_I^{-I} & 0 \\ 0 & \Sigma_2^{-I} \end{bmatrix} \begin{bmatrix} f_1 I & (1-f_2)U_{11}^{-1}U_{12}f_1 \\ 0 & f_2 I \end{bmatrix} \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} \tag{4.5.20a}$$

Output complementary sensitivity:

$$T_o = \begin{bmatrix} f_1 I & 0 \\ (1-f_2)U_{21}U_{11}^{-1}f_1 & f_2 I \end{bmatrix} \tag{4.5.20b}$$

Input complementary sensitivity:

$$T_I = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} f_1 I & (1-f_2)\Sigma_I^{-I}U_{11}^{-1}U_{12}\Sigma_2 f_1 \\ 0 & f_2 I \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix} \tag{4.5.20c}$$

It is therefore indeed possible to make use of the degrees of freedom that the directions of the principal gains offer to turn the influence of the gains in the same way as for non-minimum phase effects, both the severity of the influence control of the first set of outputs has on the second set of outputs and the additional gain needed in the controller are completely determined by the directionality of the principal gains. If these directions are only weakly coupled to the first outputs the consequences will be large. In this case we obtain an ill-conditioned controller.

In the next section we will deal with a simple model of a high purity binary distillation column, as is frequently used in literature on robust control.

4.5.3 Analysis of a high purity distillation column model.

In this section we will take a closer look at an example process that is extremely ill-conditioned. This process, a high purity distillation process, is frequently used as an example in robust control literature, e.g. [Mor89, Sko88a, Sko88b, Sko90]. A clear description of the process can be found in the book of Morari [Mor89]. We will take the most simple first order model of this distillation column as used in literature [Mor89 Sko89a], since we are primarily interested in discussing the results we found in subsection 4.5.2 and not in actually controlling a column:

$$M(s)=\begin{bmatrix}0.878 & -0.864\\ 1.082 & -1.096\end{bmatrix}/\,(\tau s+1)\text{, with } \tau\text{=75 minutes}$$

The model is only a very crude model. In reality the column dynamics are much more involved. Moreover columns operating at high purity are highly nonlinear, even after linearizing the outputs with a logarithmic function. The inputs of the model are the reflux and reboil heat and the outputs are the linearized top and bottom compositions, i.e. the impurity of the top and bottom [Mor89]. The singular value decomposition of the steady state gain is:

$$M(0)=U\Sigma V^{T}=\begin{bmatrix}0.878 & -0.864\\ 1.082 & -1.096\end{bmatrix}$$

$$=\begin{bmatrix}0.625 & 0.781\\ 0.781 & -0.625\end{bmatrix}\begin{bmatrix}1.972 & 0\\ 0 & 0.0139\end{bmatrix}\begin{bmatrix}0.707 & 0.708\\ -0.708 & 0.707\end{bmatrix}^{T}$$

As can be seen from the singular value decomposition of the steady state gain the process is ill-conditioned (condition number equals 141.7), which is in this case certainly not due to scaling (minimum condition number 138.3). From the fact that the dynamics are scalar (figure 4.5.1a) we directly obtain that the process is conditioned this way over all frequencies. Moreover from the V matrix we see that with respect to robust performance we have the worst possible situation for an ill conditioned process: Each input has an approximately equal influence on both principal gains. It can therefore be expected that the performance of the closed loop process is extremely sensitive for input uncertainty. An interesting model therefore to use as example for this section.

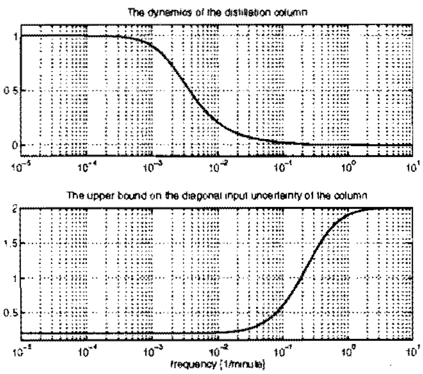

Fig.4.5.1 a:The scalar dynamics of the column b: Bound on the input uncertainty.

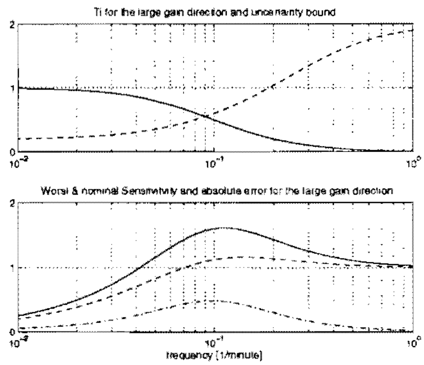

Fig.4.5.2 a: T_o in the direction of the largest. principal gain (solid), uncertainty (dashed). b: S_{out} (solid), S_o (dashed), S_oT_Δ (dash dot) worst case direction & $\delta_1=\delta_2=1$.

A bound on the diagonal input uncertainty, which represents the actuator uncertainty, is given by:

$$W_I(s) = 0.5\frac{2s+1}{0.5s+1}$$

The input uncertainty is therefore 20% for low frequencies increasing to one at approximately one minute and reaching two in the high frequency range (figure 4.5.1b). The uncertainty can be used to model unknown dynamics, specifically those related to the actuators ,e.g. actuator nonlinearities, finite resolution of the actuator and some transport delay. The process model was discretized with a sampling time of 3 seconds. In this analysis attention will be focused on aspects that are causing problems specific for ill-conditioned processes. No limitations on the magnitude of signals at the process inputs are considered, to enable a comparison with μ results described in the afore mentioned papers, e.g. [Mor89].

From the previous subsection we know that we have to control the large principal gain more tight, i.e. over a substantially larger bandwidth than the small principal gain. Robust stability and performance considerations for the case the diagonal input uncertainty equals $\delta_1=\delta$ and $\delta_2=\delta$ will limit the bandwidth over which we can control the large principal gain. This will essentially determine the behavior of Q_{11}=¦$_1$[4], i.e.:

> Based on the input uncertainty $W_I(z)$, the response of the complementary sensitivity in the direction U_1, Q_{11} has to be determined such that still good and robust performance and stability properties are attained for $\delta_1=\delta_2=\delta$.

The exact choice of the pole locations, order and type of the lowpass filter f_1 is of course to a certain extend arbitrary. We however have to fulfill the above requirement of good nominal and robust performance. We choose a second order lowpass filter with a double pole with a cut-off frequency of 0.1 1/minute (figure 4.5.2).

The choice of Q_{11} also fixes the upper bound on the performance we can attain for the small principal gain, i.e. on Q_{22}, since we need a certain gap between the largest and smallest principal gain to ensure robust performance in case the input uncertainty equals $\delta_1=\delta$ and $\delta_2=-\delta$. To ensure robust performance we have to fulfill (4.5.18). So if $Q_{22}\approx I$ we need an attenuation of the sensitivity in the direction of the large gain of α_d times the condition number. Since the condition number is very large, we needed a significant gap between the cut-off frequencies of Q_{22} and Q_{11}, with $\alpha_d=1$ a factor 10^2 as can be seen from figure 4.5.3a. We therefore adapted the lowpass filter f_1 to have an increase of 40dB/decade for I-Q_{11} over the frequency range from 0.01/minute to the cut-off frequency (figure 4.5.3a). The choice of 0.01/minute is based on the trade-off we have to make between the increased gain in Q_{11} around the cut-off frequency , due to the zero, which results in decreased robust performance for uncertainty of the form $\delta_1=\delta_2=\delta$ and the reduction of the gap between the two cut-off frequencies, i.e. increasing the overall nominal performance and robust performance of the system (figure 4.5.3b). The ultimately obtained lowpass filter f_1 has a double pole at 0.9838 and a zero at 0.9984. Based on the new f_1 we designed f_2.

[4] It can be any principal gain that forms the restriction. For this example it is the first one.

The trade-off for this filter, $Q_{22}=I_2$, is between fulfilling equation (4.5.18), i.e. to ensure robust performance in the face of uncertainty of the form $\delta_1=\delta$ and $\delta_2=-\delta$ and maximum nominal performance for the direction U_2. For f_2 a second order filter with complex conjugated poles at 0.9965±0.0004i was

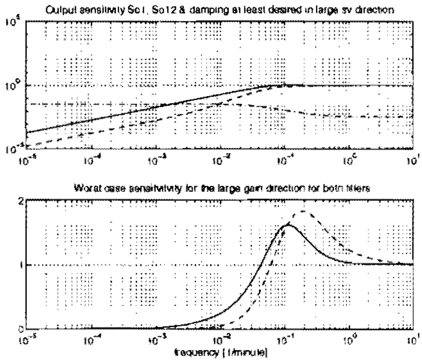

Fig.4.5.3 a: Sensitivity functions for the 2 filters f_1 (solid) and (dashed) and the function $W_1\mathrm{cond}(M)^{-1}$ (dash dot) b: Worst case response for the 2 filters f_1 for model error $\delta_1=\delta_2=1$.

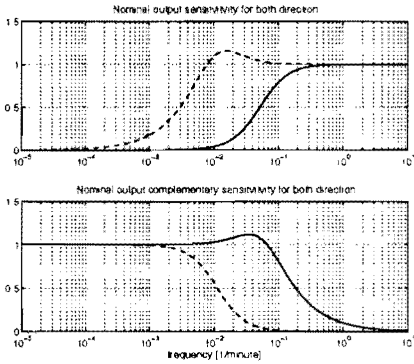

Fig.4.5.4 a: $(I-Q_{11})$ (solid) and $(I-Q_{22})$ (dashed). b: Q_{11} (solid) and Q_{22} (dashed).

chosen. The principal gains of the resulting sensitivity and complementary sensitivity functions are given in figure 4.5.4. The resulting errors T_Δ for both extremes for the diagonal input uncertainties $\delta_1=\delta_2=\delta$ and $\delta_1=-\delta_2=\delta$ are given in figure 4.5.5a. The principal gains of S_o for both errors and the nominal values are given in figure 4.5.5.b. To confirm the approach we used μ-analysis to check the result. To do this we used the performance weight for the sensitivity, as defined in the above papers:

$$W_p(s) = 0.5\frac{s+0.1}{s}$$

$\mu(M_{T_w})$ then becomes, conform equation (3.5.8), with:

- The uncertainty structure $\Delta(z)\in \boldsymbol{B}\Delta$:

$$\Delta(z) = \begin{bmatrix} \Delta_{2x2} & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{bmatrix}$$

- M_{T_w} the weighted criterion of M_T as defined in equation (4.5.7):

$$M_{Tw} = \begin{bmatrix} W_p(s) & 0 \\ 0 & I \end{bmatrix} M_T \begin{bmatrix} I & 0 \\ 0 & W_I(s)I \end{bmatrix}$$

The resulting μ-bound is plotted in figure 4.5.6a. From this plot we see that the μ bound is above one for the lower frequencies, indicating that the robust performance criterion is not fulfilled (μ=1.6). This is a result of the fact that the nominal performance in the direction of the small principal gain of the model, $I-Q_{22}$, is violating the performance specification. Some more fine tuning of the filters is needed to fulfill the specification. This however is not the aim of an analysis technique as developed here. For this further optimization μ-synthesis is well suited. The algorithm is better capable of

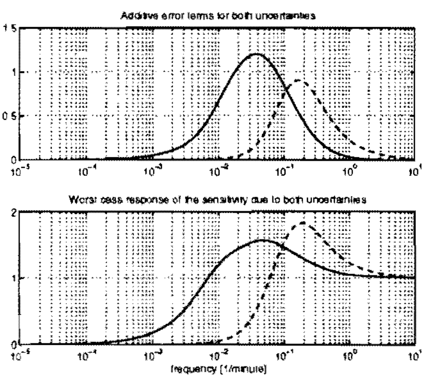

Fig.4.5.5 a: $\sigma_1(T_\Delta)$ due to $\delta_1=-\delta_2=1$ (solid) and $\delta_1=\delta_2=1$ (dashed). b: $\sigma_1(S_{out})$ for both errors.

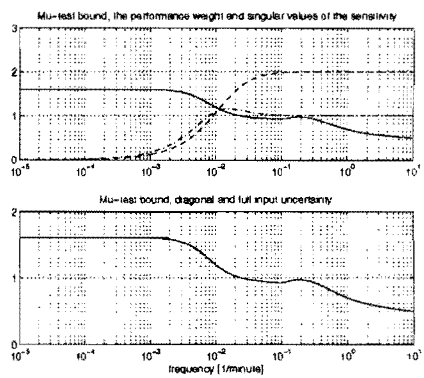

Fig.4.5.6 a: μ (solid), $(I-Q_{22})$ (dashed) & bound on performance (dash dot). b: μ for full block input uncertainty.

doing this optimization than any user. The analysis technique should not be used to design the controller, but to understand the optimization problem. The analysis performs enables us just this. It results in the insight needed to adjust the weights of the criterion if the required performance is not met. In figure 4.5.6b we plotted the μ bound for both the case that the input uncertainty has a diagonal structure and the case it has a full 2x2 block structure. We directly see that both bounds are equal. This confirms the conclusion in section 4.5.2, that if the inputs have approximately equal influence on both principal gains there will be no difference between the full block uncertainty and the diagonal uncertainty, both uncertainty descriptions have the worst case uncertainty in their set. In the example the left and right principle vectors were constant matrices over the whole frequency range. In general however they will be functions of frequency. In this case we have to perform the singular value decomposition as a function of the frequency, which is known not to be possible analytically, with finite rational matrices. In this case we have to perform the analysis at a number of selected frequency points. In contrast to the above simple example the resulting controller can not be directly constructed anymore from the analysis.

4.5.4 A procedure for ill-conditioned problems and a discussion on the obtained results

Let us summarize what we obtained in the previous subsections on control of an ill-conditioned control problem. Ill-conditioning is known to make the control problem difficult. A detailed analysis of how the conditioning of the process restricts the input-output controllability was developed.

From this analysis it became clear that robust stability is not affected by the conditioning of the process. One of the consequences of this observation is that it is scaling independent. Robust performance however is dependent on the condition of the problem. This effect could be explained from the coupling that potentially may occur between the large principal gains of the process and the inverse of the small principal gains of the process in the controller. In the case of a full input uncertainty block it turned out that full coupling was always possible, resulting in potentially extreme sensitivity for input uncertainty. The severity of the potential reduction in performance for diagonal input uncertainty is determined by how the inputs are coupled to the principal gains of the process. Maximum reduction in performance due to diagonal input modeling errors is obtained, if two strongly coupled principle vectors have the same non-zero entries and if the angle they make with the elementary vectors for the non-zero entries has an absolute value equal to $\pi/4$. Almost no additional reduction, due to diagonal input uncertainty is obtained, if two principal gains are weakly coupling. Minimum reduction of performance will occur if both vectors have no non zero entries in common. In the case of strong coupling we have to minimize the magnitude of the term in (4.5.17) to obtain robust performance, else a drastic reduction of performance will occur. The absolute influence of the degradation at the output can be reduced, by increasing the attenuation of S_o in the direction of the corresponding large principal gain. As a direct severe consequence of this observation we obtain that any controller or controller structure that is not able to achieve this will result in poor performance, independent whether it is tuned for low performance or high performance.

In exactly the same way as for the non-minimum phase effects it is possible to turn the influence of the large principal gain to a more favorable position. The amount of additional interaction on the other outputs and the additional effort that is needed from the controller are completely determined by the coupling of the outputs still to be controlled to these large principal gains.

A robust closed loop behavior that satisfies the restrictions posed and results in high performance, has to fulfill the following properties with respect to the process principal gains:

A) For all processes the design has to fulfill, with respect to the gain, the following properties:
1. The maximum required amplitudes of the process inputs have to stay within the limits
2. Uncertainty restrictions have to be satisfied, e.g. $\sigma_1(T_i\Delta_i)<<1$ and/or $\sigma_1(C\Delta_p)<<1$.
3. The output sensitivity may be increased, within the limitations stated in point one and two.
4. The large principal gains can be turned to a more desirable direction within the limitations given by point one and two. Turning the direction will result in additional amplification in the complementary direction and additional gain of the controller, conform the equations (4.5.20).

B) For frequency ranges were the model is ill-conditioned we moreover have to fulfill:
1. $V_2^H C U_1 \Sigma_1$ are sufficiently small with respect to the condition number of the model, which is ensured by the parameterization of the controller conform equation (4.5.11)
2. For strongly coupled ill conditioned subsystems *i*, *j*, with $j>i$, $\sigma_1\left((I-Q_{ii})Q_{jj}\right)<\left(\Sigma_i\Sigma_j^{-1}\right)$ should be fulfilled for robust performance.

Based on these properties we propose the following input-output controllability analysis should be applied for an ill-conditioned process:

1. *Properly scale the model.*
2. *Apply a singular value decomposition of the transfer matrix.*
3. *Determine the potential coupling between the principal gains with a large difference in magnitude and select the strongly coupled pairs.*
4. *Estimate the bandwidth that is attainable per principal gain, based on the robust stability criterion for diagonal input uncertainty and the limitations of the magnitude of the input signals of the process.*
5. *Based on these basic bandwidths we have to further analyze the behavior of So and ensure that in the input direction corresponding to strongly coupled principal gains sufficient attenuation of the term (4.5.17) is obtained to obtain the desired level of robustness.*
6. *In a last step we may use the possibility to turn the influence of the large principal gains towards more desirable directions.*

A major disadvantage of the developed analysis approach is that we are not able to deal with nominal stability. Even in the case of a square minimum phase process nominal stability is not ensured.

In the case of a non-minimum phase system we have to first factor out the non-minimum phase behavior and separately deal with both effects. Even if this is perfectly possible for both effects, then this will still result in an iteration between the analysis of the non-minimum phase behavior and the minimum phase behavior, which may result in loss of insight.

In case of a non square process the stability becomes even more an issue, since we implicitly or explicitly have to square down the process. It is known that this is always possible, without introducing additional non-minimum phase zeros, using an dynamic compensator. It seems however not be widely known that this inevitably results in an increased gain of a stable controller. The above analysis technique can not deal with the stability issue and results in an antistable controller (see section 5.4). The analysis will therefore not account for the additional effort needed to obtain a stable controller.

4.6 Conclusions and summary.

In this section we started to look at techniques that enabled us to obtain insight in the input-output controllability of a process. We were interested in understanding and quantifying the basic mechanisms and trade-off's that limit input-output controllability of the process. We started with the basic approach proposed by Morari [Mor83] for the analysis and comparison of the input-output controllability of different process designs. The approach can be summarized in three main steps (figure 4.2.1):

1. Translate the feedback problem into a feedforward problem, based on the internal model scheme.
2. Separate the non-minimum phase and the minimum phase behavior, using an inner outer factorization.
3. Analyze the restrictions that the non-minimum phase and the minimum phase system pose for each factor separately:
   - The principal gains of the sensitivity matrix of the inner are used to judge the limitations the non-minimum phase behavior poses on the frequency range over which the set of outputs can be controlled.
   - The principal gains of the outer are used to estimate the frequency range over which the controller can invert the process behavior.

The above approach is a very simple and easy to use procedure, which enables a first insight in the input-output controllability of the process. The estimates we obtain from this analysis are however too inaccurate for a detailed analysis as needed for controller design. This became clear during the discussion on the application of the approach on a spraydryer. A number of drawbacks on this approach have been identified. The major drawback we identified is that the outcome of the analysis was not directly related to the behavior of the closed loop process on specific inputs and outputs. In fact the whole directional behavior of the process, is not covered in the analysis. This is a fundamental deficiency in the approach, since most closed loop specifications are in general posed per input and output or set of inputs and outputs. It is therefore important to obtain insight in the relation between the principal gains and the inputs and outputs, i.e. directional behavior of the MIMO system. The central question is: If the process is not completely input output controllable anymore:

- What opportunities are still left to use the controllable part of the process to satisfy specific requirements, i.e. turn the controllable part of the process to a prespecified subspace in the output space.
- What are the consequences of turning the controllable behavior to a specific direction, for the behavior of the other outputs, i.e. the complement of the above output subspace.
- What are the consequences for the principal gains of the controller.

In section 4.4 and 4.5 this problem was studied for non-minimum phase behavior and for minimum phase behavior respectively. A good insight was obtained in the fundamental mechanisms that enable us to turn the influence of certain process behavior to a more desirable (closed loop) output direction. An analysis method is developed that shows that the more aligned the output direction is with the output direction of the complementary zero output direction or the large gain, the less influence it will have on the performance of the complementary outputs and the less additional effort is needed of the controller.

In section 4.5 we moreover studied the input-output controllability of ill-conditioned processes. These processes are experienced as potentially difficult to control, due to the extreme directional dependency of the gain. The exact conditions under which these processes become difficult to control have been established. A good understanding of the effects that make these processes difficult to control was developed. We moreover have determined the conditions the controller has to fulfill to ensure robust performance of these systems. A disadvantage of the tool is that nominal stability of the controller is not considered. Moreover the results are obtained from the application of singular value decomposition of the transfer matrix at distinct frequency points. No closed form, like a state space realization of these results exists.

The, in this chapter, developed techniques give us a good insight in directional effects of multi input multi output behavior. They enable us to understand how we can use the directionality to realize our control objectives and they show what the restrictions and consequences are for the controlled process. If the process is minimum phase or the non-minimum phase behavior is restricted to a few non-minimum phase zeros the approach will result in a good and accurate insight in the input-output controllability related to the requirements. The actual use of these techniques to judge the overall controllability of a general process is however limited. If the input-output controllability of a process is limited both by the gain and the non-minimum phase behavior then it becomes difficult to accuratly judge the controllability.

The main reason for the limited value in this case, is the fact that we actually have split the problem in small subproblems that all influence each other. We not only split-up the behavior of the process in a minimum phase and a non-minimum phase part, but also deal essentially with each non-minimum phase zero one after the other. A way to look at the overall procedure based on the developed tools is as follows:

After extraction of the delays coupled to specific outputs, take the most dominating non-minimum phase zero and extract it from the transfer matrix: $M(z) = N_1(z)M_1(z)$. The extraction $N_1(z)$ can be applied such that the influence the zero has on the output is turned to the most favorable direction. This extraction however also influences the behavior of $M_1(z)$, i.e. it changes the directionality of the other zeros and the behavior of the principal gains. After the extraction of the first zero we may extract the second zero from $M_1(z)$. Note that the directionality of the zero is also influenced by the extraction we applied to extract the first zero. After extracting all non-minimum phase zeros, say n:

$$M(z) = \left(\prod_{i=1}^{n} N_i(z) \right) M_n(z)$$

We start to analyze the behavior of the principal gains of $M_n(z)$. The behavior of these principal gains is however not only determined by the behavior of the principal gains of the process, but also by the applied factorization of the zeros.

In the above procedure we in fact extract each zero one after the other without taking in consideration the effect it has on the rest of the system. However optimality of each subsequent step does in general not guarantee overall optimality of the final solution. In no way we can therefore guarantee that the end result is in any way optimal or even close to what is actually achievable with the process in relation to the requirements.

In the next chapter we will therefore propose and develop a different approach that enables us to deal with all restrictions at the same time.

Appendix4.A Proofs of section 4.4.

Lemma 4.A.1

Assume $M(z) \in \Re H_\infty^{pxm}$ with realization $[A, B, C, D]$ is factored conform equation (4.4.1b) then the realization of $M_1(z) \in \Re H_\infty^{pxm}$ is given by:

$$\left[A, B, N_{01}C + N_{02}CA, D + N_{02}CB\right] \tag{4.A.1a}$$

Proof of Lemma 4.A.1

The series expansion of $M(z)$, i.e. infinite impulse response model, equals:

$$M(z) = D + CBz^{-1} + CABz^{-2} + CA^2Bz^{-3} + \ldots\ldots \tag{4.A.1b}$$

For $M_1(z)=N^1(z)M(z)$ we then obtain after using the properties of N_{01} and N_{02}:

$$(D + N_{02}CB) + (N_{01}C + N_{02}CA)Bz^{-1} + (N_{01}C + N_{02}CA)ABz^{-2} + .. \tag{4.A.1c}$$

which has as a realization (4.4.7).

Proof of Lemma 4.4.1:

Follows directly from the properties of projectors and theorem 4.4.1.

Proof of Theorem 4.4.2:

As explained in the text the idea is to transform the finite zero z_i to infinity. This is done with the following transformation:

$$z = \frac{1 + z_i p}{p + z_i} \tag{4.A.2a}$$

Applying this transformation to the state space realization of $M(z)$, i.e.:

$$M(z) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.A.2b}$$

results in following realization in the p-domain $M(p)$ [Saf87]:

$$M(p) = \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix} = \begin{bmatrix} (z_i A - I)(z_i I - A)^{-1} & (z_i^2 - 1)(z_i I - A)^{-1} B \\ C(z_i I - A)^{-1} & C(z_i I - A)^{-1} B + D \end{bmatrix} \tag{4.A.2c}$$

From (4.A.2c) we see that the rank deficiency of D_p indeed equals the deficiency for the zero of z_i. Now applying theorem 4.4.1 on $M(p)$ results in:

$$L_0(z) = N_{01} + N_{02}\, p^{-1} \tag{4.A.2d}$$

with:

$$N_{01} = V_1 V_1^T + V_1 \Phi V_2^T$$

$$N_{02} = V_2 V_2^T - V_1 \Phi V_2^T$$

and

- Φ any real $rx(p\text{-}r)$ matrix
- V_1 V_2 and r are determined by the QR decomposition of $D+C(z_iI-A)^{-1}B$:

$$D + C(z_i I - A)^{-1} B = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \cdot \begin{bmatrix} R \\ 0 \end{bmatrix}$$

$$rank(D + C(z_i I - A)^{-1} B) = p - r$$

Substitution of $p = \dfrac{1 - z_i z}{z - z_i}$ in (4.A.2d) results in formula (4.4.11b).

Appendix4.B Deduction of equation (4.5.16).

Assume M and C factored conform equation (4.5.4) respectively (4.5.11). We then obtain for $M\Delta_I C$:

$$M\Delta_I C = \delta U \begin{bmatrix} 0 & \Sigma_1 \Sigma_2^{-1} Q_{22} \\ \Sigma_2 \Sigma_1^{-1} Q_{11} & \Sigma_2 \Sigma_1^{-1} Q_{12} \end{bmatrix} U^H$$

consequently:

$$I + M\Delta_I C = U \begin{bmatrix} I & \delta\Sigma_1 \Sigma_2^{-1} Q_{22} \\ \delta\Sigma_2 \Sigma_1^{-1} Q_{11} & I + \delta\Sigma_2 \Sigma_1^{-1} Q_{12} \end{bmatrix} U^H$$

which may be approximated by, since $\Sigma_2\Sigma_1^{-1} << I$:

$$I + M\Delta_I C \approx U \begin{bmatrix} I & \delta\Sigma_1 \Sigma_2^{-1} Q_{22} \\ 0 & I \end{bmatrix} U^H$$

We thus obtain for the inverse that it approximates:

$$(I + M\Delta_I C)^{-1} \approx U \begin{bmatrix} I & -\delta\Sigma_1 \Sigma_2^{-1} Q_{22} \\ 0 & I \end{bmatrix} U^H$$

After substitution of this approximate relation in equation (4.5.12) we obtain equation (4.5.16). Note that T_o is directly obtained from the fact that $T_o = I - S_o$.

CHAPTER FIVE

A Novel Approach to Asses Process Controllability

5.1 Introduction.

Closed loop performance at the output of the process is always to be attained under a number of restrictions posed on the behavior of the loop. We have seen that control of one output or output direction can lead to a drastic reduction in the controllability of other outputs and may have a severe influence on other properties desired of the closed loop, depending on the specific open loop process behavior and the specific requirements we have on the closed loop behavior. In these cases a detailed knowledge of the relation between the required closed loop behavior and the open loop process behavior will lead to a well defined trade-off between the conflicting requirements, to a reformulation of the specifications or even a change in the structure of the modeled process, e.g. add additional inputs or remove outputs. In chapter four we developed tools that enabled us to get an understanding of the directionality of non-minimum phase zeros and principal gains of multi input multi output systems. The developed tools were however not generally applicable. In cases were we have a large number of dynamic effects all limiting the closed behavior in the same frequency range no accurate insight could be obtained, despite the fact that we were well able to accurately deal with each subproblem separately. The separation in subproblems is highly artificial. Dealing with subproblems was needed to simplify the analysis. For each of these subproblems we had a direct relation between the restricting effect itself and the requirements. We are however not interested in how each limitation itself restricts the design, but how the total of restrictions limits the overall closed loop behavior. In chapter two we already concluded that input-output controllability analysis in itself is a control design problem. The goals of controllability analysis and control design are however different. In the analysis we are interested in obtaining insight in the fundamental restrictions the process poses on the closed loop behavior and the different trade-off's, that result from these restrictions. In control design we are primarily interested in designing a controller that results in a well behaved closed loop behavior that confirms with the closed loop objectives. In most control design techniques the insight in the fundamental trade-off's is completely lost, due to the complexity of the design technique itself. A way to circumvent the loss in insight in the problem is to simplify and standardize the problem. In chapter four we have chosen to achieve this goal by the subdivision of the overall MIMO problem in simpler subproblems. In this chapter we will take a different approach, to circumvent the limitations of the approach of chapter four.

The approach to be developed here is based on the idea of prioritizing the requirements. For analysis purposes it is well possible to replace the relative importance of requirements for the closed loop by an absolute priority. In this way we are able to obtain an approach that is more accurate than the approach used in chapter four, without loozing insight in the fundamental trade-off's we need to make during the design.

Assume given a model $M(z) \in \Re H_\infty^{p \times m}$. As a first step we order the outputs in accordance with the relative importance of the required closed loop behavior of this output with respect to the other requirements: The first output is the most important requirement, the second output corresponds to the second important requirement and so on.

As we will see in section 5.2 the approach in fact boils down to chosing a lower triangular structure for the output complementary sensitivity transfer matrix of the process, i.e. $T_0(z)$ in equation (3.5.5b):

> The lower triangular structure can be interpreted as first controlling output one as good as possible, without considering the consequences on the other outputs. After output one is controlled we want to control output two as good as possible without effecting the closed loop behavior of output one and so on.

Remark:

> It is emphasized here that we do not advocate the use of absolute priorities for the design of the controller. The absolute priority is used for the analysis to obtain insight in the different trade-off's. Based on the insight obtained it may well be possible that during the analysis we need to change the order of the requirements or that we have to take a certain linear combination of outputs as an output during the analysis. In section 5.2 the above described idea will be formalized and further discussed.

In section 5.3 we will develop a non-parametric approach. We first develop the approach for steady state. In applying the same approach at discrete frequency points, the approach can directly be extended to also perform a dynamic analysis. A main disadvantage of the approach of 5.3 is that nominal stability of the closed loop system is not considered, i.e. the resulting controller may be unstable. The restrictions that non-minimum phase effects and non square problems pose on the control problem can not be analyzed. Hence the analysis may result in too optimistic results.

In section 5.4 we will therefore include stability in the analysis. Nominal stability will result in an additional restriction on the input-output controllability of the process. This section also results in new insights in the down squaring problem. From a practical point of view it is not always possible to square down a problem, without introducing additional non-minimum phase zeros. At first sight this seems to be contradicting a well known fact, that it is always possible to square down a problem with a dynamic compensator, without introducing additional non-minimum phase zeros in the closed loop complementary sensitivity matrix. It will be shown that down squaring without introducing additional non-minimum phase zeros may result in inevitably large principal gains for the controller. In section 5.5 the results of section 5.4 will incorporated in the input-output controllability analysis.

Model Predictive Controllers (MPC) are the only MIMO controllers widely used in industry. The model predictive controller at each sample moment calculates an optimal input sequence over a finite time horizon. This however makes MPC a finite time domain approach. A finite time analysis approach is desired, that enables us to analyze the controllability problems for this type of controllers. In section 5.6 we investigate the basic tools that are needed to obtain insight in how controllability problems manifest in the finite time domain. These results turn out to be very elegant. They relate concepts as non-minimum phase zeros and closed loop stability to the infinite time horizon behavior. The technique for example clearly reveals how non-minimum phase effects behave over a finite time horizon. The relation between the finite and infinite time domain results in a new interpretation of the Bode sensitivity integral. In section 5.7. we will take a look at how the concepts developed in section 5.6 can be incorporated in a finite time input-output controllability analysis approach for MIMO. It turns out that this time domain approach is much simpler and in many respects more elegant than the frequency domain approach.

5.2 The new controllability approach: The basic approach.

In this section we will introduce the basis of a new approach for controllability. The resulting procedure can be seen as a step by step controller design. Each of these steps is essentially a SISO or simpler MIMO design. The specific sequence of steps made in the procedure are based on a prioritization of the performance requirements. The requirements are first ranked in a descending order of importance. Each of the performance requirements is linked with an output of the process. In each step of the procedure we analyze the controllability of an output. Starting with the first output, i.e. the most important requirement, then the second one and so on. During the analysis of each output we not only have to fulfill the restrictions at the process inputs and to achieve a certain level of robustness for modeling uncertainty, but we also need to ensure that no (serious) deterioration of performance occurs at the outputs that have been analyzed in previous steps, i.e. have a higher priority. The analysis approach for a model with p outputs can be written as the following iterative loop:

FOR i=1, .., p

> *Determine what restricts us from fulfilling the required closed loop behavior for output i, under the condition that control of output i does not change the closed loop behavior of the outputs 1 to i−1.*

Next output

In chapter three the IMC scheme was introduced (figure 3.5.3). If we assume no model error , i.e. the process and model are equal we found this scheme to be equivalent to the feedforward control scheme in figure 5.2.1. The controller in this concept is therefore closely related to the behavior of the model. All effects that prevent the construction of a stable inverse of the model over the desired frequency range, directly restricts the input-output controllability of the process. In this section we again make use of this fact. For $M(z)\in \Re H_\infty^{p\times m}$ the controlled setpoint behavior to the model inputs and outputs is given by:

$$\begin{bmatrix} y_p(z) \\ u_p(z) \end{bmatrix} = T_{ff}(z)\big(y_s(z)-d(z)\big) \qquad (5.2.1a)$$

with:

- $T_{ff}(z) \in \Re H_\infty^{(p+m)xp}$:

$$T_{ff}(z) = \begin{bmatrix} T_o(z) \\ C(z) \end{bmatrix} = \begin{bmatrix} M(z) \\ I \end{bmatrix} C(z) \qquad (5.2.1b)$$

The potential trade-off amongst the outputs and inputs are directly reflected in equation (5.2.1): To achieve performance over a certain frequency range at the process outputs, we need the controller $C(z) \in \Re H_\infty^{mxp}$ to approximately invert the process over its entire relevant frequency range. On the other hand this may result in a too large principal gain of the controller or a too high gain at the process input, which again restricts us in doing so.

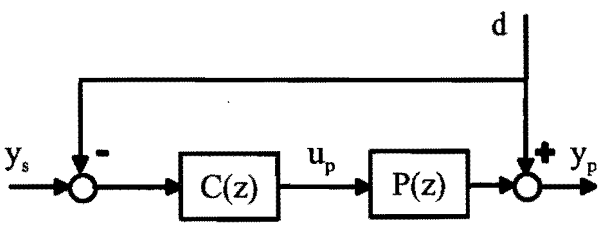

Fig.5.2.1. Feedforward control scheme If $P(z)$ equals $M(z)$ then this scheme is equivalent to the IMC scheme.

Observe that if we define $C(z)=D(z)Q_n(z)$, with $D(z) \in \Re H_\infty^{mxm}$, i.e. stable, we obtain the following equivalent control problem. Design $Q_n(z)$ such that:

$$T_{ff}(z) = \begin{bmatrix} T_o(z) \\ C(z) \end{bmatrix} = \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} Q_n(z) \qquad (5.2.1c)$$

has the desired behavior. For a stable closed loop behavior both $N(z)$ and $D(z)$ need to be stable. Moreover these transfers are right coprime factors of the model $M(z)$. In fact any choice of a stable and square $Q_n(z) \in \Re H_\infty^{mxm}$ will result in new coprime factors of the process model $M(z)$:

$$\begin{bmatrix} N(z) \\ D(z) \end{bmatrix} Q_n(z) = \begin{bmatrix} N_e(z) \\ D_e(z) \end{bmatrix} = M_e(z) \qquad (5.2.1d)$$

with

- $N_e(z) \in \Re H_\infty^{pxm}$ and $D_e(z) \in \Re H_\infty^{mxm}$ a set of right coprime factors of the model $M(z)$.
- $M_e(z) \in \Re H_\infty^{(p+m)xm}$ the extend model used for design

Remark:
1. It is well known that there exists a coprime factorization that makes the extended model $M_e(z)$ inner [McF90, Sch92]. The factorization is obtained after applying the inner-outer factorization of $\begin{bmatrix} M(z) \\ I \end{bmatrix}$.
   Choosing the normalized right coprime factorization therefore results in an inner process representation $M_{nrc}(z)$..
2. For a fat process, i.e. $p \leq m$, there is freedom left at the outputs of the process. In this case it is useful to choose . $M_e(z) \in \Re H_\infty^{(p+m)\times m}$ equal to:
$$M_e(z) = \begin{bmatrix} N_1(z) & 0 \\ D_1(z) & D_2(z) \end{bmatrix} \tag{5.2.1e}$$
   with:
   - $N_1(z) \in \Re H_\infty^{p\times p}$, $D_1(z) \in \Re H_\infty^{m\times p}$.
   - $D_2(z) \in \Re H_\infty^{m\times(m-p)}$ is an annihilator of the model $M(z)$, i.e. projects in the nullspace of $M(z)$.

   The annihilator can be used to achieve additional requirements, e.g. minimize the amplitude of certain controller output signals or add additional outputs. In this sense we therefore will never deal with fat processes.
3. Note the close relation that the factorization (5.2.1) has with the behavioral framework, as introduced by Willems [Wil89, Wei91].

To keep the discussion simple we will not explicitly use the factorization (5.2.1) any further and concentrate on the process outputs. In order to better understand the trade-off between the different outputs we factor the model $M(z) \in \Re H_\infty^{p\times m}$ per output.

Lemma 5.2.1
Any model $M(z) \in \Re H_\infty^{p\times m}$ can be factored as:
$$M(z) = \begin{bmatrix} m_1(z) \\ \cdot \\ \cdot \\ m_p(z) \end{bmatrix} = M_I(z) M_g(z) E(z) \tag{5.2.2}$$
with:

- $m_i(z) \in \Re H_\infty^{1\mathrm{x}m}$ The MISO transfer matrix from all inputs to the i-th model output.
- $M_I(z) \in \Re H_\infty^{p\mathrm{x}p}$ and diagonal:

$$M_I(z) = \begin{bmatrix} n_1(z) & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & 0 \\ 0 & . & 0 & n_p(z) \end{bmatrix} \tag{5.2.3a}$$

$n_i(z) \in \Re H_\infty^{1\mathrm{x}1}$ An inner transfer function.

- $M_g(z) \in \Re H_\infty^{p\mathrm{x}p}$ and diagonal:

$$M_g(z) = \begin{bmatrix} g_1(z) & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & 0 \\ 0 & . & 0 & g_p(z) \end{bmatrix} \tag{5.2.3b}$$

$g_i(z) \in \Re H_\infty^{1\mathrm{x}1}$ A minimum phase transfer function.

- $E(z) \in \Re H_\infty^{p\mathrm{x}m}$:

$$E(z) = \begin{bmatrix} e_1(z) \\ . \\ . \\ e_p(z) \end{bmatrix} \tag{5.2.3c}$$

  - $e_i(z) \in \Re H_\infty^{1\mathrm{x}m}$ A structural co-inner transfer matrix..

Proof: See appendix 5.A.

Remark:

The factorization of each output model $m_i(s) \in \Re H_\infty^{1\mathrm{x}m}$:

$m_i(s) = n_i(s) g_i(s) e_i(s)$

will be called the *inner-outer structural co-inner factorization*.

This factorization can directly be generalized for any transfer matrix $M(s) \in \Re H_\infty^{1\mathrm{x}m}$, $m>p$.

Let us give an interpretation for the different factors of equation (5.2.2), before we continue. Per definition (see section 3.3) the inner and co-inner transfer matrices have their principal gain equal to one. Hence the principal gain for each model $m_i(z)\in\Re H_\infty^{1\times m}$ is equal to the principal gain of the minimum phase transfer function $g_i(z)\in\Re H_\infty^{1\times 1}$, i.e. for $\phi\in[0,\pi]$ we obtain (equation (3.3.5b)):

$$\left|g_i(e^{j\phi})\right| = \max_{u(e^{j\phi})\neq 0}\frac{\|m_i u\|}{\|u\|} \tag{5.2.4}$$

The structural co-inner transfer matrix $e_i(z)\in\Re H_\infty^{1\times m}$ has per definition no non-minimum phase zeros (section 3.3). Hence the inner transfer function $n_i(z)\in\Re H_\infty^{1\times 1}$ contains all non-minimum phase zeros of the model $m_i(z)$. Therefore $e_i(z)$ represent the input direction of the $m_i(z)$. Moreover the complete multivariable behavior of the model $M(z)$ is determined by the transfer matrix $E(z)\in\Re H_\infty^{p\times m}$.

Factor the controller $C(z)\in\Re H_\infty^{m\times p}$ in accordance with lemma 5.2.1 per column:

$$C(z)=\begin{bmatrix}C_1(z) & C_2(z) & . & C_p(z)\end{bmatrix} \tag{5.2.5a}$$

Hence, the absolute priorities at the closed loop outputs, we need the column $C_i(z)$ to be in the kernel or nullspace of the model spanned by the rows 1 to $i-1$ of the model:

$$\begin{bmatrix}m_1(z)\\ .\\ m_{i-1}(z)\end{bmatrix}C_i(z)=0 \tag{5.2.5b}$$

The analysis of the input-output controllability has to be performed under this restriction, i.e.: Find $C_i(z)\in\Re H_\infty^{m\times 1}$ fulfilling equation (5.2.5b), such that: $m_i(z)C_i(z)$ has the desired behavior. A further consequence of equation (5.2.5b) is that we have the following equality:

$$\begin{aligned}&\begin{bmatrix}m_1(z)\\ .\\ m_{i-1}(z)\end{bmatrix}\begin{bmatrix}C_1(z) & . & C_{i-1}(z)\end{bmatrix}\\ &=\begin{bmatrix}m_1(z)\\ .\\ m_{i-1}(z)\end{bmatrix}\left\{\begin{bmatrix}C_1(z) & . & C_{i-1}(z)\end{bmatrix}+C_i(z)\begin{bmatrix}A_1(z) & . & A_{i-1}(z)\end{bmatrix}\right\}\end{aligned} \tag{5.2.5c}$$

with $A_i(z)$ any transfer function that can be used to minimize the influence that control of output 1 to i–1 has on output I. Let us give a simple static example to clarify the above procedure.

Example 5.2.1

Assume we have a process model M with only two outputs and at least two inputs and a static transfer matrix:

$$M = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

Factoring the model transfer conform equation (5.2.2) yields:

$$M = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} e_1(z) \\ e_2(z) \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

From linear algebra we obtain that all controllers that fulfill the above stated prioritized strategy are given by:

$$\mathrm{C} = \left\{ \begin{bmatrix} \mathrm{e}_1^{\mathrm{T}} & 0 \end{bmatrix} + \mathrm{E}_1^{\perp\mathrm{T}} \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \right\} \begin{bmatrix} \mathrm{g}_1^{-1} & 0 \\ 0 & \mathrm{g}_2^{-1} \end{bmatrix} \tag{5.2.6a}$$

where:

- $E_1^{\perp} \in \Re^{m \times 1}$ is an orthogonal vector that spans the kernel or nullspace of m_1 or equivalently spans the complementary space of e_1, i.e. $E_1^{\perp} e_1^{\mathrm{T}}=0$ and $E_1^{\perp} E_1^{\perp\mathrm{T}}=I$, i.e. $E_1^{\perp}=[\ 1, -2\ ]/\sqrt{5}$.
- $\alpha_i \in \Re$, i=1,2 are the design parameters.

Since $E_1^{\perp}$ spans the kernel of the first output, i.e. $m_1 E_1^{\perp\mathrm{T}}=0$, α_1 can be used to minimize the influence that control of the first output has on the second output. Preferred is of course that control of the first output does not effect the second output, i.e.:

$$\alpha_1 = -\left(e_2 E_1^{\perp T}\right)^{-R} \left(e_2 e_1^T\right) = 3$$

Since the first output has the highest priority we do not want control of output 2 to interfere with the behavior of output 1. To ensure this, also the nullspace of e_1 must be used for the control of the second output.

$$\mathrm{MC}_2 = \mathrm{M}\left\{\mathrm{E}_1^{\perp\mathrm{T}} \alpha_2\right\} \mathrm{g}_2^{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ which yields } \alpha_2 = \left(e_2 \cdot E_1^{\perp T}\right)^{-R} = \sqrt{10}$$

The additional effort needed to control output 2 independent of output 1 is completely determined by the angle between the vectors e_1 and e_2, say

ϕ. The more these vectors are aligned, the smaller $e_2E_1^{\perp T}=sin\phi=1/\sqrt{10}$ and the larger $e_2e_1^T=cos\phi$.

The term $sin\phi$ exactly determines the additional cost we have to pay for controlling output 2 under the restriction that C_2 lies in the nullspace or kernel of e_1, i.e. with the restriction that we do not reduce the performance at output one. The controller becomes more and more dominated by the output direction $E_1^{\perp T}$, if ϕ decreases. For small angles the condition number of the controller increases drastically. The sinus of the angle therefore determines directly the restrictions the MIMO behavior of the process puts on the overall controllability of the model.

In the next section we will make use of the following relation between the controller and the LQ decomposition of E. The LQ decomposition, i.e. the dual QR decomposition [Go89], equals:

$$E = RQ = \begin{bmatrix} 1 & 0 \\ \left(e_2 e_1^T\right) & \left(e_2 E_1^{\perp T}\right) \end{bmatrix} \begin{bmatrix} e_1 \\ E_1^{\perp} \end{bmatrix} \tag{5.2.6b}$$

and E^{-1}:

$$E^{-1} = Q^T R^{-1} = \begin{bmatrix} e_1^T & E_1^{\perp T} \end{bmatrix} \left[\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3/\sqrt{10} & 1 \end{bmatrix} \right] \tag{5.2.6c}$$

For the gain the approach can be directly extended to the non-static case by applying the same factorization for selected discrete frequencies. In this way it is possible to obtain, detailed insight in the controllability of the process as a function of frequency.

End of example

In section 5.3 we will further discuss the direct extension of the approach developed in the example to the general dynamic case. The main disadvantage of this extension is that we can not guarantee stability of the controller unless the model is square and minimum phase. In section 5.5 we will include nominal stability in the above described approach. In section 5.7 we will extend the above procedure to the finite time domain.

5.3 *A procedure to obtain a structured approach of controllability analysis.*

In section 5.2 we developed a new conceptual approach to perform an input-output controllability analysis of a model. In example 5.2.1 we already looked at a 2x2 case. Note that if we perform the analysis at different distinct frequencies then the approach can directly be extended to also cover the dynamic behavior of the model. In fact this is the basic idea of this section. We will assume the plant to be minimum phase, since the approach developed in this section is not able to deal with non-minimum phase behavior[1]. Hence we may choose $M_I(z)=I$ in equation (5.2.2).

5.3.1 An initial approach

In order to simplify notation we will develop the procedure for the steady state case. The whole procedure is equivalent for each arbritrary selected other frequency point. Based on the close relation between how we construct the inverse of the model and the LQ-decomposition on $E(1)$, as observed in example 5.2.1, we define the LQ factorization of E[2]:

$$E(1)=RQ=\begin{bmatrix} R_{11} & 0 & . & 0 \\ R_{21} & R_{22} & . & . \\ . & . & . & 0 \\ R_{p1} & R_{p2} & . & R_{pp} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ . \\ q_p \end{bmatrix} \tag{5.3.1}$$

with: $Q\in\Re^{p\times m}$ with $QQ^T=1$.

The factorization exactly reflects the priorities we have in the requirements. The whole multivariable behavior of the model is determined by R and is directly related to the output priorities. To understand the multivariable nature of the model and the restrictions it puts on control we need to closer study the matrix R. First of all $R_{11}=1$, since $e_1e_1^T=1$. From the example we obtain that q_i is the direction we can best use for control of output i. It is the vector closest to the direction e_i^T that is orthogonal to the input directions of

[1] Remark: In the case we analyze the dynamics of a non-square process, stability of the controller is not guaranteed by the approach. The conclusions resulting from this approach may therefore be too optimistic. We come back to the stability issue in section 5.4.

[2] We will skip the dependency of the frequency from here on, since steady state is assumed.

the outputs with higher priorities, i.e. e_1 to e_{i-1}. For output i we thus obtain that R_{ij} for j=1 to i–1, are the dependencies of the i-th output on the outputs with higher priority. R_{ii} indicates the magnitude of that part of e_i that is independent of the outputs with higher priority. From $\|e_i\|_2 = 1$, we obtain:

$$\sum_{j=1}^{i} R_{ji}^2 = 1 .$$

The inverse of R_{ii} is to be interpreted as the additional price we have to pay to control output i independent of the outputs 1 to i–1. So R_{ii} directly reflects the consequences of the prioritizing. Based on the relevance of this interpretation of the diagonal terms of R we factor it separately:

$$R = R_P R_{IA} = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & R_{22} & . & . \\ . & . & . & 0 \\ 0 & . & . & R_{pp} \end{bmatrix} \begin{bmatrix} 1 & 0 & . & 0 \\ R_{22}^{-1}R_{21} & 1 & . & . \\ . & . & . & 0 \\ R_{pp}^{-1}R_{p1} & R_{pp}^{-1}R_{p2} & . & 1 \end{bmatrix} \tag{5.3.2}$$

The matrix R in fact summarizes the whole multivariable nature of the control problem and reflects the essential trade-off to be made. R_P reflects the effects that the priorities have on the controllability of the outputs. R_{IA} directly displays the interdependencies of the different priorities for the multivariable model behavior. We will therefore call R_{IA} the *prioritized control interaction matrix* or just the *prioritized interaction matrix*.

We thus obtain for the factorization of the model at steady state:

$$M = M_g \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & R_{22} & . & . \\ . & . & . & 0 \\ 0 & . & . & R_{pp} \end{bmatrix} \begin{bmatrix} 1 & 0 & . & 0 \\ R_{22}^{-1}R_{21} & 1 & . & . \\ . & . & . & 0 \\ R_{pp}^{-1}R_{p1} & R_{pp}^{-1}R_{p2} & . & 1 \end{bmatrix} Q \tag{5.3.3}$$

Let us turn to the inverse of the prioritized control interaction matrix to obtain a better insight in how the controller is actually related to both the different requirements and the model behavior. Denote the inverse of M_g, R_P and R_{IA}, as respectively C_g, C_P and C_{IA} results for equation (5.3.3) in:

$$C = Q^T C_{IA} C_P C_g \tag{5.3.4}$$

with: $C_g = M_g^{-1}$, $C_P = R_P^{-1}$, $C_{IA} = R_{IA}^{-1}$

The relation that C_{IA} has with the control problem can be represented in two different ways, an additive representation and a multiplicative representation. We will state these representations in the form of two lemmas.

Lemma 5.3.1 The additive formulation
Assume given a process model $M \in \Re^{pxm}$ factored as in equation (5.3.3). The controller C that achieves perfect control, i.e. equals the inverse of M, can be factored as:

$$C = Q^T C_{IA} C_P C_g \qquad (5.3.5a)$$

with:

- $Q \in \Re^{pxm}$ is defined in equation (5.3.1)
- $C_g \in \Re^{pxp}$ is defined in equation (5.3.4)
- $C_P \in \Re^{pxp}$ is defined in equation (5.3.4)
- $C_{IA} \in \Re^{pxp}$ a lower triangular matrix:

$$C_{IA} = C_P \begin{bmatrix} 1 & 0 & . & . & 0 \\ \chi_{21} & \chi_2 & . & & . \\ \chi_{31} & \chi_{32} & \chi_3 & & . \\ . & & . & . & 0 \\ \chi_{p1} & . & . & \chi_{p(p-1)} & \chi_p \end{bmatrix} \qquad (5.3.5b)$$

with:

- $\chi_i = e_i q_i^T = R_{ii}$
- $\chi_{(i+1)i} = e_{i+1} q_i^T = R_{(i+1)i}$ and $\chi_{ji} = e_j P(j-1, i+1) q_i^T$ for $j > i+1$ and $P(j-1, i+1)$ is a product of oblique projections:
  $P(s,t) = (I - q_t^T (e_t q_t^T)^{-1} e_t) P(s, t-1) =$
  $$\prod_{k=s}^{t} (I - q_k^T (e_k q_k^T)^{-1} e_k) = \prod_{k=s}^{t} P(k)$$

Proof: See appendix 5.B

The interpretation of the matrix C_{IA} is that it removes the influence that control of the setpoints of the high priority outputs has on the lower priority outputs, i.e. removal of the lower triangular part of the output complementary sensitivity T_O. C_P on the left hand side of equation (5.3.5a) can be seen as the additional cost we have to pay, for removing the interaction of the first $i-1$ columns of C on the i-th row of E, without influencing the first $i-1$ rows of E.

A multiplicative version of Lemma 5.3.1 can be obtained from the proof of lemma 5.3.1:

Lemma 5.3.2 The multiplicative formulation
Assume given a process model $M\in\Re^{p\times m}$ factored as in equation (5.3.1). The controller C that achieves perfect control, i.e. equals the inverse of M, can be factored as:

$$C = \tilde{C}_{IA} C_P C_g \tag{5.3.6a}$$

where:

- $Q\in\Re^{p\times m}$ is defined in equation (5.3.1)
- $C_g\in\Re^{p\times p}$ is defined in equation (5.3.4)
- $C_p\in\Re^{p\times p}$ is defined in equation (5.3.4)

The matrix $\tilde{C}_{IA}$ is a lower triangular matrix that can be represented as:

$$\tilde{C}_{IA} = \begin{bmatrix} \tilde{C}_{IA_1} & . & \tilde{C}_{IA_i} & . & \tilde{C}_{IA_p} \end{bmatrix} \tag{5.3.6b}$$

where:

$$\tilde{C}_{IA_i} = (\prod_{k=i+1}^{p} P(k)) q_i^T \tag{5.3.6c}$$

with $P(k)$ again the oblique projection matrix:

$$P(k) = (I - q_k^T (e_k q_k^T)^{-1} e_k) \tag{5.3.6d}$$

Proof: See appendix 5.B

The above lemma can be interpreted per column of the controller as a step by step, reduction of the influence that control of output i with q_i^T has on the outputs with lower priority. The formulation with the oblique projection clearly shows how it is achieved. To remove the influence of the control of output i on output $i+1$ the vector q_i^T is projected in the space spanned by the orthogonal complement of e_{i+1}, i.e. in $e_{i+1}^{\perp}$. This is seen from:

$$P(k) = (I - q_k^T (e_k q_k^T)^{-1} e_k) = e_k^{\perp T} (q_k^{\perp} e_k^{\perp T})^{-1} q_k^{\perp} \tag{5.3.6e}$$

Per definition $q_{i+1}^{\perp} q_i^T = 1$, i.e. the vector q_i is in kernel of q_{i+1}, which means that the projection does not change the length of the resulting vector in the direction q_i. Moreover the increase in magnitude of the resulting vector after the projection is determined by $\left\|(e_{i+1}^{\perp} q_i^T)\right\|_2$ or equivalently the more orthogonal q_i is to e_{i+1} the smaller the increase in magnitude. As a next step of the procedure for the resulting vector the same procedure is performed for e_{i+2}. A direct consequence of lemma 5.3.2 is the following recursive representation of the matrix $\tilde{C}_{IA}$:

Corollary 5.3.1

The matrix $\tilde{C}_{IA}$ in lemma 5.3.2 can equivalently be represented in a recursive way. If we define the matrix Θ_1 as:

$$\Theta_1 = q_1^T \tag{5.3.7a}$$

and Θ_i for i=2 to p as:

$$\Theta_i = \left[P(i)\Theta_{i-1} \quad q_i^T\right] \tag{5.3.7b}$$

with $P(i)$ as defined in equation (5.3.6c) we obtain that Θ_p equals $\tilde{C}_{IA}$.
Proof: The corollary is a direct consequence of lemma 5.3.2.

An interesting interpretation of the above approach is that it builds the controller completely based on the priority scheme: First output one is controlled. Output two is then controlled under the restriction that the performance on output one is not effected and so on. In each step of the construction of the inverse we thus see what the costs of the previous actions are for the controllability of the other outputs.

Until here we assumed that the number of inputs and outputs of the model were equal. Note that in the case that we have more outputs than inputs, i.e. $p>m$, we know in advance that the last p-m outputs are uncontrollable. It is therefore useless to take these last requirements into consideration in the problem. A better approach is to reduce the number of outputs in advance. This can be done by just skipping the least important or the most dependent combinations, before starting the procedure. Another possibility is to take linear combinations of outputs as one "output" during the analysis. In the case that we have more inputs than outputs, $p<m$, we could use the additional freedom for example to minimize the maximum gain per input.

Let us summarize the above input-output controllability analysis for static models. In the above described procedure a clear insight can be obtained in the relation between the specifications and the model behavior. It enables the designer to trade-off exactly those requirements that are difficult or not achievable with the given model behavior. This insight may also lead to an adaptation of the number of requirements to be fulfilled or to an adjustment of the number of inputs. The factors in the decomposition of equation (5.3.3) have a clear relation to the control problem:

1. The matrix Q gives the different orthogonal directions we have to use to control each output independent of the outputs with higher priority.
2. The matrix R_p directly gives the additional costs, i.e. the reduction in gain, we pay for each output to be controlled without influence on the outputs with higher priorities.
3. The lower triangular matrix R_{IA} contains essentially all information on the multivariable model behavior. The inverse of this matrix (C_{IA}) has a nice interpretation: each entry j of a column is in fact the cost we pay to remove the interaction of the controller on output j under the assumption that the outputs with higher priority have already been dealt with.
4. The interpretation of M_g is the maximum gain per output. The larger this gain the easier it is to control the output.

Remark:

The above developed approach can of course also be applied per input instead of per output. It is in fact dual. This results in a better insight in the relation to the inputs. It however makes the interpretation of the relation to the outputs and therefore the required behavior more difficult.

5.3.2 The approach and the restrictions posed on the controller.

The frequency domain is not really suited to handle real constraint type of requirements, as posed on the inputs. We therefore propose to deal with this type of requirement in a less strict way. The idea is to guarantee that the restrictions are fulfilled 'on average'. The frequency domain is well suited to deal with this more loose formulation of the restriction i.e. the gain to each of the model inputs should be restricted. We therefore scale the model inputs with a constant or a lowpass filter (section 3.5). This enables us to approximate both the speed (rate of change) restriction and the magnitude requirement on an input by one position constraint on the scaled model, i.e. $\|u_i(j\omega)\|_2 \leq 1$. Depending on the specific situation we translate this requirement to a restriction on the maximum gain per controller output or per SISO transfer of the controller. The gains can be easily monitored during the successive steps of the analysis. In case a relative large gain occurs during the analysis, the step by step approach enables us to directly trace back the cause of this gain and to reconsider the relevant design trade-off.

In process control we need to trade-off the behavior at the controlled process outputs against the behavior at the controlled process inputs. To better deal with the magnitude restrictions put on the different process inputs we have to relate the controlled behavior at the controlled process output to that of the behavior at the controlled process inputs. As discussed in section3.5 the controller in the IMC scheme the controller can be seen as a particular approximation of the inverse process model. This concept of approximate inverse of the process model makes the relation between the closed loop process inputs and outputs more transparent. We therefore want to make explicite use of the inverse model in the factorization of the controller:

$$C = Q^T C_{IA} C_P C_g K \tag{5.3.8a}$$

with:

- $Q \in \Re^{pxm}$ is defined in equation (5.3.1).
- $C_g \in \Re^{pxp}$ is defined in equation (5.3.4).
- $C_P \in \Re^{pxp}$ is defined in equation (5.3.4).
- $C_{IA} \in \Re^{pxp}$ is defined in equation (5.3.5).
- $K \in \Re^{pxp}$ a lower triangular transfer matrix containing the degrees of freedom.

The output performance is directly related to the tuning transfer matrix K:

$$T_o = MC = K = \begin{bmatrix} k_1 & 0 & . & 0 \\ k_{21} & k_2 & . & . \\ . & & . & 0 \\ k_{p1} & . & k_{p(p-1)} & k_p \end{bmatrix} \tag{5.3.8b}$$

The interpretation of this factorization is very attractive and intuitive: We replace the model behavior M, by inverting it, and replace it by the more desired behavior K. The price we have to pay however is a complex relation between the tuning parameters and the model inputs. The relation between the tuning parameters and the model inputs is given by $C=Q^TX$, where the entries of X are defined as:

- $x_{ji} = 0$, for $j<i$
- $x_{ii} = R_{ii}^{-1} g_i^{-1} k_i$ (5.3.8c)
- $x_{ji} = R_{jj}^{-1} \sum_{l=i}^{j} \chi_{jl} R_{ll}^{-1} g_l^{-1} k_{li}$, for $j>i$ and $k_{jj}=k_j$
- R_{ij} defined in equation (5.3.1).

From (5.3.8c) we see that the relation between k_{ji} and the controller outputs is more complicated. An apparent reaction to this observation is to choose a different parameterization of the tuning parameters, e.g.:

$$C = Q^T \overline{C}_{IA} C_P C_g \widetilde{K} \tag{5.3.9a}$$

with:

- Q, C_p and C_g are defined as above.
- $\widetilde{K} \in \Re^{p \times p}$ and diagonal, with the i-th diagonal entry equal to k_i.
- $\overline{C}_{IA}$ is defined as:

$$\overline{C}_{IA} = \begin{bmatrix} 1 & 0 & . & 0 \\ \chi_2^{-1} \chi_{21} \widetilde{k}_{21} & 1 & . & . \\ . & & . & 0 \\ \chi_p^{-1} \chi_{p1} \widetilde{k}_{p1} & . & \chi_p^{-1} \chi_{p(p-1)} \widetilde{k}_{p(p-1)} & 1 \end{bmatrix} \tag{5.3.9b}$$

with χ_i and χ_{ji} are as defined in (5.2.7)

The parameterization to the process inputs is less complex than the relation in (5.3.8c). However the relation to the model output is now as complex as the relation to the inputs in the previous parameterization:

$$T_o = MC = \begin{bmatrix} k_1 & 0 & . & 0 \\ t_{21}^o & k_2 & . & . \\ . & & . & 0 \\ t_{p1}^o & . & t_{p(p-1)}^o & k_p \end{bmatrix} \tag{5.3.9c}$$

with:

- $t^o_{ji} = X_{ji}\left(\sum_{l=i}^{j} R_{jl} R_{ll}^{-1} \chi_{jl} \tilde{k}_{li}\right) R_{ii}^{-1} k_i$, $X_{ji} = g_j / g_i$, $\chi_{jj} = R_{jj}$ and $\tilde{k}_{ii} = 1$

(5.3.9.d)

It is clear that there is a trade-off between complexity of the closed loop representation at the model inputs against that at the model outputs. This trade-off seems a basic property of MIMO systems and solvable by a "smartly chosen" parameterization. In the next section a step by step procedure is proposed that enables us to keep insight in the behavior at both model input and output (section 5.3.3).

Another restriction, we have to carefully consider during the design, is robustness of a controller. Robustness especially becomes important if we design high performance controllers, i.e. controllers that try to achieve maximum performance for a given model. These controllers are known to be sensitive for differences between the actual process behavior and the assumed behavior, i.e. the model behavior. Our aim in this section is not directly to come-up with a new or better robustness criterion, but to obtain a better understanding where robustness problems can be expected for a given model. If the model is a good reflection of the process behavior then a robustness analysis can completely be based on this model. In section 4.5 it was found that the sensitivity of the closed loop behavior for modeling errors is determined by the combination of the open loop model behavior and the requirements posed on the closed loop. An important consequence of detailed insight in the relation between requirements and model behavior at one side and the sensitivity for model uncertainty on the other side is that robustness of the controller can be assessed without detailed knowledge of the uncertainty. Based on the model behavior we are already able to accurately trade-off the potential robustness problems against the desired (nominal) closed loop behavior. A detailed knowledge of the model error will only reduce conservatism in the sense that the same trade-off's can be made more accurate and tight.

5.3.3 An integral controllability analysis for a two by two model.

In this section we will take a closer look at the simple case that we only have two outputs and inputs available. We will assume that the two outputs are ordered in descending order of importance. Factor the model conform equation (5.3.3) as:

$$M = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \cos(\phi_{12}) & \sin(\phi_{12}) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{5.3.10a}$$

with ϕ_{12} the angle between the first and second row of the model. For the controller we make use of the parameterization conform equation (5.3.9a):

$$C = \begin{bmatrix} q_1^H & q_2^H \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sin(\phi_{12})^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\cos(\phi_{12})k_{IA} & 1 \end{bmatrix} \begin{bmatrix} g_1^{-1} & 0 \\ 0 & g_2^{-1} \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \tag{5.3.10b}$$

For the controller to fulfill the robustness requirements (section 4.5), it has to fulfill a number of restrictions on the gain. Loosely formulated:

1. The gain of the controller should be restricted, i.e. its principal gains should be limited to fulfill the restrictions at the model input and some level of robustness against additive model uncertainty.
2. The large gain of the controller should in no way be coupled to the large gain of the model to ensure nominal performance and a certain level of robustness against relative output and input uncertainty. To fulfill this requirement both the closed loop transfers matrices of T_0 and T_I should have largest principal gains not much larger than one.
3. To reduce the influence of input uncertainty on the performance of the closed loop we also had to ensure that the attenuation of the sensitivity function in the input direction that is aligned to the output direction of the largest principal gain of the transfer of the model is sufficiently large with respect to the magnitude of the inverse of the small principal gain.

The triangular decomposition used in this section makes a direct application of the results of section 4.5 not possible. Let us analyze this further.

The first requirement can be analyzed using equation (5.3.10b) The second requirement by analyzing both the output and input complementary sensitivities:

$$T_O = MC = \begin{bmatrix} k_1 & 0 \\ X_g \cos(\phi_{12})(1 - k_{IA})k_1 & k_2 \end{bmatrix} \tag{5.3.11a}$$

respectively:

$$T_I = CM = \begin{bmatrix} q_1^T & q_2^T \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ (k_2 - k_{IA}k_1)\tan(\phi_{12})^{-1} & k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{5.3.11b}$$

with $X_g = g_2/g_1$.

From equation (5.3.11b) we directly observe that:

- An ill conditioned problem is not more sensitive for robust stability with respect to input uncertainty than a well-conditioned problem. Making use of the invariance of the eigenvalues for a similarity transformation results in the sufficient stability criterion: $|k_i| < \sigma_1(\Delta_I)$ for i=1,2.
- Robust stability is independent of the output gains of the model. This is in fact the result we already found; the determinant and the eigenvalues are invariant for a similarity transformation. This is easily understood from the fact that stability is not related to the magnitude of external signals.

To further fascilate the discussions we assume that full coupling of the row vectors in Q (section 4.5) is possible, i.e. the set of possible modeling errors always contains at least one member Δ^W such that $Q_1\Delta_I^W Q_2^H=\delta I$. $M\Delta_I C$ in equation (4.5.12) equals for the worst case input uncertainty Δ_I^W:

$$M\Delta_I^W C = \delta\begin{bmatrix} -\tan(\phi_{12})^{-1}k_{IA}k_1 & X_g^{-1}\sin(\phi_{12})^{-1}k_2 \\ X_g\left(\sin(\phi_{12}) - \cos(\phi_{12})\tan(\phi_{12})^{-1}k_{IA}\right)k_1 & \sin(\phi_{12})^{-1}k_2 \end{bmatrix}$$

(5.3.11c)

To analyze the behavior we distinct between three different types of ill-conditioning:

1. The principal gains of R_{21} are close to one and the entries of M_g are in the same order of magnitude, i.e. $X_g \approx 1$ and $sin(\phi_{12})<<1$.
   - For the frequency band where this occurs we have to limit the gain of the controller from a certain freqeuncy on by reducing the influence of $sin(\phi_{12})^{-1}$ on the controller outputs (equation (5.3.10b)). Therefore both k_{IA} and k_2 have to role-off from one to zero. Note from (5.3.10b) that if g_1 is sufficiently large it is still possible in principle to choose k_{IA} equal or close to one. If we however keep k_{IA} close to one this will result in a significant magnitude of $T_I(2,1)$, i.e. in the order of magnitude of $tan(\phi_{12})^{-1}$. To keep this interaction limited we need to reduce k_{IA} also, i.e. $k_{IA}=k_2/k1$. Hence, we have to tolerate interaction on $T_O(2,1)$ (equation (5.3.11a)) for the frequency range in which the first output is still controllable. The influence is limited since it is bounded

above by X_g, which is approximately equal to one. The transfer $M\Delta_I^W C$ can be approximated by:

$$M\Delta_I^W C = \delta\begin{bmatrix}1\\1\end{bmatrix}\sin(\phi_{12})^{-1}k_2\begin{bmatrix}-k_1 & 1\end{bmatrix}$$

For $S_O T_\Delta$ we then obtain:

$$S_O M\Delta_I^W C = \delta\begin{bmatrix}1\\(1-k_2)\end{bmatrix}\sin(\phi_{12})^{-1}(1-k_1)k_2\begin{bmatrix}-k_1 & 1\end{bmatrix}$$

A necessary condition for robust performance therefore is that $\delta(1-k_1)k_2 << sin(\phi_{12})$ or $k_1 \approx 1$ We then obtain for $S_O\ T_\Delta$:

$$S_O T_\Delta = \delta\begin{bmatrix}1\\(1-k_2)\end{bmatrix}\sin(\phi_{12})^{-1}(1-k_1)k_2\begin{bmatrix}-1 & 1\end{bmatrix}$$

2. There is a significant difference in magnitude between the entries of g_1 and g_2, i.e. $X_g<<1$ or $X_g>>1$ and $cos(\phi_{12})$ significantly smaller than one.
   - Assume first that $X_g<<1$ In the case the ill-conditioning is caused by $g_2<<g_1$,we obtain from (5.3.10b) that k_2 has to role-off from one to zero to keep the controller gain limited. There is no need to reduce k_{IA}. From (5.3.10a) we see that it is even preferable to keep k_{IA} equal to one. This will result in an interaction term $T_I(2,1)$ unequal to zero (5.3.10b), which is however completely independent of g_2 and essentially determined by $tan(\phi_{12})^{-1}$. Equation (5.3.10c) can now be approximated by the upper triangular matrix:

$$M\Delta_I^W C = \delta\ \sin(\phi_{12})^{-1}\begin{bmatrix}-\cos(\phi_{12})k_{IA}k_1 & X_g^{-1}k_2\\ 0 & k_2\end{bmatrix}$$

   which will be dominated by the upper right entry. S_O is well approximated by a diagonal transfer with the i-th diagonal entry equal to $(1\text{-}k_i)$, i=1,2, since $X_g<<1$. For $S_O T_\Delta$ we then obtain:

$$\delta\begin{bmatrix}(1-k_1)/\alpha_1 & 0\\ 0 & (1-k_2)\end{bmatrix}\begin{bmatrix}\cos(\phi_{12})k_{IA}k_1 & X_g^{-1}\sin(\phi_{12})k_2\\ 0 & k_2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/\alpha_2\end{bmatrix}$$

   with: $\alpha_1 = (\sin(\phi_{12}) - \cos(\phi_{12})k_{IA}k_1\delta)$ and $\alpha_2 = (\sin(\phi_{12}) + k_2\delta)$. For the uncertainty error it is therefore necessary that $(1\text{-}k_I)$ is sufficiently small in relation to $X_g^{-1}k_2$. Again we have to make the bandwidth of the first output larger to ensure robust performance.

Note that for the error term it is not necessary that k_{IA} reduces in combination with k_2.

- In the case that X_g>>1 k_1 has to role of from one to zero to keep the gain of the controller limited. In this case it is important to keep k_{IA} equal to the identity for T_0. For T_1 it is preferred to choose k_{IA} equal to k_2/k_1. However the choice of k_{IA} equal to one only has limited consequences if $cos(\phi_{12})$ is not close to one. On the other hand not making k_{IA} equal to one results in an excessive increase of $T_0(2,1)$. k_{IA} therefore has to be chosen equal to one as long as $X_g k_1$ is sufficiently large. We now see that entry (1,2) is insignificant for the error term equation (5.3.11c) and can be approximated by zero. Applying the same procedure as in the previous case results in the requirement for robust performance that (1-k_2) is sufficiently small in relation to essentially $X_g k_1$. If this situation occurs one should consider to change over to an upper triangular structure. The consequences of changing over to this structure are limited for output one, since X_g^{-1} is small.

3. The magnitude of $cos(\phi_{12})$ is close to one and there is a significant difference in magnitude between the entries of g_1 and g_2
   - If both X_g and $sin(\phi_{12})$ are very small, the situation is in principle the same as in the first case. We have to role off k_{IA} and k_2 as discussed.
   - A last possibility is of course the case that the magnitudes of g_1. and the principal gains of R_{22} are small. In this case essentially no performance is left anymore. In this case the factorization is not usable. Note that this case coincides with the case in section 4.5 that the large principal gain is essentially coupled to the second output.

The insight obtained in the case of the 2x2 model can directly be extended to the case of more outputs. The applied technique is however not directly generalized to the case that we have more outputs and/or inputs. The analysis becomes far more complex in this general case. A more tight procedure is needed therefore.

5.3.4 A general procedure for the new controllability analysis approach

In this section we will present a procedure that enables us to build up a detailed insight in the controllability of the model. The procedure is however no direct extension of the previous section. As we already observed in section 5.3.1 the complexity of the analysis increases rapidly with the number of outputs and inputs of the model. Key to simplification of the complex relations is a step by step procedure, dealing with one output after another. In the discussion we will assume that an initial analysis is applied. Hence, we assume that each output can be controlled over a certain frequency range. Moreover we assume that the factorization in equation (5.3.3) will not give rise to situations that are uncontrollable, due to the fact that the chosen parameterization does not at all fit to the natural model behavior. We will furthermore assume here that the situation where X_{ji} is too large, with $j>i$, does not occur. In such cases, we strongly recommend to change-over to an upper triangular structure for these conflicting outputs, as discussed[3]. This however does not basically change the approach.

As a first step we investigate the frequency range over which we can control each prioritized output, without influencing the outputs with higher priority.

$$C_1 = Q^T C_p C_g K_1 \tag{5.3.12a}$$

with:

- $K_1 \in \Re^{p\times p}$ a diagonal matrix with diagonal entries, k_i, a lowpass filter with steady state gain equal to one.

The initial complementary output sensitivity therefore equals:

$$T_{O1} = M_g R_p R_{IA} C_p C_g K_1 =$$

$$\begin{bmatrix} k_1 & 0 & . & 0 \\ g_2 R_{21} g_1^{-1} k_1 & k_2 & . & . \\ . & . & . & 0 \\ g_p R_{p1} g_1^{-1} k_1 & g_p R_{p2} R_{22}^{-1} g_2^{-1} k_2 & . & k_p \end{bmatrix} \tag{5.3.12b}$$

[3] This change is equivalent to choosing the new columns:

$\bar{q}_1^H = (I - e_j^H e_j) e_1^H / \left\| (I - e_j^H e_j) e_1^H \right\|$ & $\bar{q}_j^H = (I - e_1^H e_1)^{-1} e_j^H / \left\| (I - e_1^H e_1)^{-1} e_j^H \right\|$.

These columns can then be used to determine the new interaction terms for the first and j-th column being respectively; $E\bar{q}_1^H = \begin{bmatrix} \bar{R}_{11}^H & . & \bar{R}_{(j-1)1}^H & 0 & \bar{R}_{(j+1)1}^H & . & \bar{R}_{p1}^H \end{bmatrix}^H$ and

$E\bar{q}_j^H = \begin{bmatrix} \bar{R}_{1j}^H & 0 & . & 0 & \bar{R}_{jj}^H & . & \bar{R}_{pj}^H \end{bmatrix}^H$

In the choice of the bandwidth of each lowpass filter, k_i, we already have to consider the fact that we still need to compensate the influence that the outputs with higher priority have on the lower priority outputs, i.e. compensation of the lower triangular structure T_{01}.

In section 4.2 we used the principal gains of the model to asses the input-output controllability of the model. These principal gains gave us an indication of the bandwidth over which we may expect the model to be controllable. A good indication for the choice of the bandwidth of the lowpass filters are therefore the principal gains. The bandwidth of each filter k_i ,say $B(k_i)$, must correspond to the frequency range, say $B(\sigma_j)$, over which we consider the principal gain σ_j of the model to be controllable. Note that output i is not necessarily coupled to the i-th principal gain. We however need to have the following relation:

For $i=1,..,p$ we have at least $p+1-i$ lowpass filters, say k ,for which $B(k)\leq B(\sigma_i)$ and equality holds for at most as many filters as we have principal gains equal to σ_i.

Let us turn to the minimization of the lower triangular part of the complementary sensitivity matrix T_{01}. As a first step for output two we want to minimize the influence control of output one has on this output. According to corollary 5.3.1 we can perfectly accomplish this by premultiplying by the projection matrix in (5.3.6d):

$$P(2)q_1^H g_1^{-1} k_1 = (I - q_2^T R_{22}^{-1} e_2) q_1^H g_1^{-1} k_1 \tag{5.3.16a}$$

In general equation 5.3.16a will not be realizable, due to a too large magnitude of the inverse of R_{22}. We therefore have to role-off the compensation. Define:

$$P(2,k_{21}) = (I - q_2^T R_{22}^{-1} k_{21} e_2) \tag{5.3.16b}$$

The resulting controller after the second step then equals:

$$C_2 = \begin{bmatrix} \Theta_2 & q_2^T & . & q_p^T \end{bmatrix} C_p C_g K_1 \tag{5.3.16c}$$

with:

$$\Theta_2 = P(2,k_{21}) q_1^T \tag{5.3.16d}$$

Note that in most cases k_{21} will be equal to or close to k_2 (section 5.3.2).

The analysis of output three, which is similar to the one for output two results in:

$$C_3 = \begin{bmatrix} \Theta_3 & q_3^T & \cdot & q_p^T \end{bmatrix} C_p C_g K_1 \tag{5.3.17a}$$

with:

$$\Theta_3 = \begin{bmatrix} P(3,k_{31})P(2,k_{21})q_1^T & P(3,k_{32})q_2^T \end{bmatrix} \tag{5.3.17b}$$

Note that we may choose different filters for the projection $P(3,k)$ for the first and second column of the controller. The above procedure is repeated until we reached the analysis of the last output.

5.3.5 Concluding remarks

In this section we developed a procedure that enables us to asses the controllability in a very structured and step by step approach. In each step we can estimate the consequences of that step on the controllability of the other outputs and reconsider the design decisions made in previous steps. If problems occur, at a certain step then the approach enables an easy trace back where the main conflicts are with the higher priorities and what design parameters need to be reconsidered. The price we have to pay is that the approach has become more complex in relation to the approach in chapter three.

Let us summarize the basic properties of the approach we developed in this section:

1. It is assumed that the outputs of the process can be ordered in decreasing order of importance.
2. Apply the factorization of lemma 5.2.1 (equation (5.2.2)) per discrete frequency point, with $M_i(z)=I$. The factors M_g and E have the following interpretation:
   - M_g contains the largest possible gain that can be achieved for each output per discrete frequency point. It therefore gives insight in the maximum achievable frequency range over which the output may be controlled. A large difference in magnitude $g_i >> g_j$ indicates potential robust performance problems at output i.
   - E contains the multivariable relations between the different outputs. It determines the dependencies between the different outputs
3. Apply the LQ-decomposition (equation (5.3.1)) on E. The factors Q and R have the following interpretation:

- R is lower triangular that expresses the actual dependencies between the different outputs, with $\sum_{j=1}^{i} R_{ji}^2 = 1$. The dependencies are expressed in accordance with the assumed priorities at the output:
  - R_{ii} the diagonal term express how independent we can still control output i if we are not allowed to affect the outputs with higher priority, i.e. output one to $i-1$ and do not have to consider the outputs with lower priority, i.e. $i+1$ to p. The smaller R_{ii} the more difficult it is to control. It results in high gains of the controller and potential robustness problems.
  - R_{ji} with $j<i$, expresses the dependency that output i has with the direction that is used to control output j. The closer R_{ji} is to one the more difficult it is to control output i indenpedent of output j.
- Q contains the corresponding unitary input directions.

4. Use the additive or multiplicative formulation for the controller and analyse the controller per output. Starting from output one to the last output determine the tuning parameters sequentially.

A major draw back of the described approach is that we did not at all consider nominal stability. Even in the case that we have a minimum phase square model, nominal stability is not proven. Note that in the above procedure the design of the controller in each step is essentially a square down problem. If we take the square down problem into consideration in each step then also nominal stability can be taken into account. In the next section we will therefore take a closer look at this square down problem.

5.4 *The tools needed to deal with stability; Down squaring of non-square processes.*

In the previous sections we dealt with an approach that enabled us to obtain better insight in the controllability of a process. The main drawback of this approach was that we could not deal with the stability of the resulting controller. In the case of a square non-minimum phase model we had to perform again a inner outer factorization on the model. We could then apply the approach developed on the outer part of the process model. Even for this square minimum phase part no stability results were given in the previous section. The directional effects of these non-minimum phase zeros had to be dealt with separately. Again the interaction between minimum phase and non-minimum phase behavior is difficult to investigate properly, due to the separation. In this section we will introduce and develop the basic tools that enable us to deal explicitly with stability. The actual extension of the approach in section 5.3, to include nominal stability, will be undertaken in the next section. In this section use is made of the continuous time representations. In continuous time the results are easier to derive and result in more elegant expressions. As is readily known the results obtained in the continuous time can be used directly for the discrete time domain using the bilinear transformation. Note from section 5.3 that the problem we have to solve is essentially a classical square down problem:

Given a structural co-inner transfer matrix as defined in equation (5.2.3c), find a stable approximate inverse.

In this section we follow a new approach to deal with the square down problem. (Other approaches can be found in for instance [Col89, Le92, Mac76, Sab88]). As a result of this alternative approach we will obtain a result, that is of basic importance for understanding the square down problem. We will see that there is a trade-off between the gain of the square down controller and the number of additional non-minimum phase zeros we introduce with down squaring. The approach is based on [Sto94].

Down squaring can be loosely defined as designing a controller, or even better compensator that reduces a non square model, i.e. a model with more inputs than outputs or visa versa, to a square model, i.e. a model with an equal number of inputs and outputs. The compensator is only used to obtain a square model. These compensators are used for example in combination with PID controllers or to design controllers using Nyquist type of techniques [Mac89]. The simplest and therefore very attractive down squaring is of course to select a subset of inputs or outputs that make the model square. This is in fact the way most square modeles are obtained. In a number of cases this will however lead to a poorly defined problem. It may for example result in an ill-conditioned problem, introduction of nasty non-minimum phase behavior or a relatively large uncertainty in measurements or a reduction of attainable bandwidth, due to slow sensors and actuators. In these cases we would like to be able to take more inputs or outputs into account in the controller. Applying this controller results in a better controllable model. We will restrict ourselves, without loss of generality, to the case of more inputs than outputs. The dual case can be found in [Sto94].

In first instance it was proposed to use a constant precompensator to achieve the down squaring. It was shown that this constant matrix could be too restrictive. It was proven that in some cases it was impossible to obtain a constant square down matrix that did not introduce additional non-minimum phase zeros [Dav83, Sab88]. Several authors proved that with a dynamic precompensator it was always possible to square down a model without introducing additional non-minimum phase. zeros, e.g. [Le92, Sab88]. The problem seemed solved. As far as we know it is first observed in [Sto94] that the above down square operation, using a stable precompensator, may lead to a drastic reduction of the gain of the squared down model or equivalently can only be achieved with a very high gain of the compensator. This high gain will of course significantly reduce the controllability of the model. We will see that it can be better to introduce additional non-minimum phase zeros which drastically reduce the necessary gain of the precompensator. A direct consequence of this observation is that we need tools to trade-off additional non-minimum phase effects against gain behavior in the design of the precompensator. In subsection 5.4.1 we will develop tools that help us make this trade-off.

First of all we will factor the transfer matrix $M(s)\in \Re H_\infty^{p\times m}$, with $m>p$, as the cascade of two transfer matrices a square $G_0(s)$ and a non-square structural co-inner transfer matrix $G_s(s)$.

Lemma 5.4.1
For a stable transfer matrix $M(s) \in \Re H_\infty^{pxm}$, with $m>p$ and realization $[A,B,C,D]$, there exists a factorization

$$M(s)=G_0(s)G_s(s) \tag{5.4.1}$$

where:

- $G_S(s) \in \Re H_\infty^{pxm}$ is a structural co-inner transfer matrix.
- $G_O(s) \in \Re H_\infty^{pxp}$

Proof: See appendix 5.C

The factorization in lemma 5.4.1 enables us to factor $M(s)$ in two transfer matrices; $G_s(s)$ that is related to the non-squareness of the model and a square transfer matrix $G_o(s)$. $G_s(s)$ only contains the directional effects at the input of the model. It contains no non-minimum phase zeros and all the principal gains equal one. The principal gains of $G_o(s)$ equal the principal gains of $M(s)$ and the non-minimum phase zeros of $G_o(s)$ equal those of $M(s)$.

The factorization enables us to take a closer look at how the non squareness of the model restricts the controllability of the model. In subsection 5.4.1 we will develop the theory that enables us to obtain this insight. In subsection 5.4.2 we will take a look at how we can use the technique to analyze the behavior of a structural co-inner. In subsection 5.4.3 we will summarize this section and discuss the results obtained.

5.4.1 Down squaring of structural inners.

In this section we will restrict ourselves to the structural co-inner $G_s(s)$. The structural or minimum phase co-inner and inner concept was introduced by [Tsa90, Yeh90]. A structural or minimum phase co-inner is a non-square co-inner, $m>p$, that contains no non-minimum phase zeros and therefore has a stable inverse. The fact that a structural inner does not contain non-minimum phase zeros means that it does not posses zeros at all [Tsa90]. In [Yeh90] a stable inverse was found for a structural inner. We will give the dual version here for the co-inner, together with a stable right annihilator.

Theorem 5.4.1
Let $G(s) \in \Re H_\infty^{pxm}$ be a co-inner transfer matrix, i.e. $m>p$, with realization $[A,B,C,D]$, controllability gramian P and observability gramian Q. If $G(s)$ is

structural co-inner then (I-QP) is positive definite and there exists a stable right inverse and an inner right annihilator, .i.e. there exist a $G_R(s)$, and a $G_A(s)$ respectively such that:

$$G(s)\cdot\begin{bmatrix}G_R(s) & G_A(s)\end{bmatrix}=\begin{bmatrix}I & 0\end{bmatrix} \tag{5.4.2a}$$

A particular $\begin{bmatrix}G_R(s) & G_A(s)\end{bmatrix}$ is given by the following realization:

$$\left[A,\ \ (I-PQ)^{-1}\begin{bmatrix}BD^T & BD_\perp^T\end{bmatrix},\ \ -(B^TQ+D^TC),\ \ \begin{bmatrix}D^T & D_\perp^T\end{bmatrix}\right] \tag{5.4.2b}$$

where $D_\perp$ is any $mx(m$-$p)$ matrix that fulfills: $D_\perp D_\perp^T=I$ and $DD_\perp^T=0$
Proof: See appendix 5.C

The above stable inverse (5.4.2b) is certainly not unique. Any sum of a stable function multiplied with the right annihilator with the above right inverse is again a stable right inverse of the structural co-inner As a consequence we obtain a characterization of all stable inverses of $G(s)$:

$$G_{RI}(s)=\begin{bmatrix}G_R(s) & G_A(s)\end{bmatrix}\begin{bmatrix}I\\ \mathrm{A}(s)\end{bmatrix} \tag{5.4.3}$$

where $A(s)$ is any arbitrary stable matrix of appropriate dimensions, i.e. $A(s)\in\Re H_\infty^{(m-p)xp}$. Theorem 5.4.1 states that any structural co-inner always has a stable right inverse. This is equivalent to the statement that we can always square down a model without introducing additional non-minimum phase zeros. The properties of this right inverse are however not discussed. In a number of cases we found that the infinity norm of the above stable inverse, equation (5.4.2b), was very large. This raised the question if the high gain is due to the specific choice of the specific inverse or is more fundamental, i.e. due to the behavior of the co-inner. To further investigate the high gain behavior note that an inverse of $G(s)$ is given by $G^{\sim}(s)$. This inverse is all-pass, i.e. has all principal gains equal to one, but is anti stable. The high gain therefore seems more fundamental. To see this let us factor the stable inverse as a function of this unstable inverse:

$$G_{RI}(s)=\begin{bmatrix}G^{\sim}(s) & G_A(s)\end{bmatrix}\begin{bmatrix}I\\ \overline{\mathrm{A}}(s)\end{bmatrix} \tag{5.4.4a}$$

where $\overline{A}(s)$ is any matrix of appropriate dimensions that results in a stable $G_{RI}(s)$. Since $G(s)G^{\sim}(s)=I$, $G_A(s)$ is only used to stabilize $G_{RI}(s)$. In the proof of lemma 5.4.6 we will see that our choice of $\overline{A}(s)$ for the right inverse of

(5.4.4a) is in fact the H_2 optimal solution. After some algebraic manipulation (see appendix 5.C) we obtain that the specific $\bar{A}(s)$ that results in the stable right inverse of theorem 5.4.1 has realization:

$$\left[-A^T,\ \ C^T,\ \ -C_{\perp}P(I-QP)^{-1},\ \ 0\right] \tag{5.4.4b}$$

The first matrix on the right side of equation (5.4.4a) is all pass. The gain behavior is therefore completely determined by the behavior of $\left[I \quad \bar{A}^T(s)\right]^T$. Note that the function $\bar{A}(s)$ is completely unstable. The infinity norm of $\bar{A}(s)$ is larger or equal to the largest Hankel singular value, which equals the square root of the largest eigenvalue of $(I\text{-}PQ)^{-1}PQ$. The infinity norm of the right inverse in theorem 5.4.1 will therefore be large if $(I\text{-}PQ)$ is close to zero. That the high gain is indeed independent of the specific choice of the stable inverse of $G(s)$ is stated in the next lemma:

Lemma 5.4.2

Let $G(s)\in\Re H_\infty^{\,p\times m}$, $m>p$, be a structural co-inner transfer matrix with realization $[A,B,C,D]$, controllability gramian P and observability gramian Q. A lower bound on the infinity norm of any stable right inverse of $G(s)$ is given by the square root of the largest eigenvalue of the inverse of $(I\text{-}QP)$. In other words for all stable right inverses of $G(s)$, $G_{RI}(s)$, we have:

$$\left\|G_{RI}(s)\right\|_\infty \geq \lambda_1((I-PQ)^{-1/2}) \tag{5.4.5}$$

with:

- $\lambda_1(.)$ the largest eigenvalue.
- $G_{RI}(s)$ is factored conform equation (5.4.3).

Moreover there exists a right inverse for which equality (5.4.5) holds.
Proof: See appendix 5.C

From the lemma we see that the closer the largest Hankel singular value of $G(s)$ is to one the more effort is needed to invert the structural inner transfer matrix. In [Yeh90, Sto94] it was found that for an inner transfer matrix there is a direct relation between the Hankel singular values and the number of non-minimum phase zeros. We will give the result for co-inner transfer matrices, which is the dual of the lemma stated in [Sto94].

Lemma 5.4.3
Given $G(s) \in \Re H_\infty^{p \times m}$ with realization $[A,B,C,D]$, controllability gramian P and observability gramian Q. If $G(s)$ is an inner or co-inner transfer matrix then the number of Hankel singular values, given by $\lambda_i^{1/2}(PQ)$, that are equal to one is equal to the number of zeros of $G(s)$, counting multiplicity.

A direct consequence of the lemma is that a square inner transfer matrix can never be minimum phase. After all in that case all the Hankel singular values are equal to one [Glo84]. The concept of structural inner and structural co-inner is therefore directly related to non square transfer matrices. In [Sto94, Yeh90] it was shown that any non square inner can be factored as a square inner containing all non-minimum phase zeros and a structural part, containing no zeros at all. Here we state the co-inner variant.

Lemma 5.4.4
Let $G(s) \in \Re H_\infty^{p \times m}$ be a co-inner transfer matrix then it is always possible to factor $G(s)$ as:

$$G(s) = G_{sq}(s) G_{mp}(s)$$

where $G_{sq}(s) \in \Re H_\infty^{p \times p}$ is square inner and $G_{mp}(s) \in \Re H_\infty^{p \times m}$ is structural co-inner.

Assume $G(s)$ has a minimal balanced realization $[A,B,C,D]$ then there exists a diagonal matrix Q with:

$$QA + A^T Q + C^T C = 0$$

$$AQ + QA^T + BB^T = 0$$

where $Q \in \Re^{n \times n}$ equals:

$$Q = \begin{bmatrix} I & 0 \\ 0 & Q_2 \end{bmatrix} \text{ and } Q_2 < I$$

If we decompose A, B, C conform Q we obtain for the realization of $G_{sq}(s)$:

$$\begin{bmatrix} A_{11}, & C_1, & -C_1^T, & I \end{bmatrix}$$

and for $G_{mp}(s)$:

$$\begin{bmatrix} A_{22} & B_2 & C_2 & D \end{bmatrix}$$

Proof: See appendix 5.C

If a Hankel singular value equals one then the co-inner transfer matrix contains a non-minimum phase zero and is therefore not invertible by a stable transfer matrix. If the largest Hankel singular value is close to one it is natural to expect that $G(s)$ loses rank approximately somewhere in the right half plane, i.e. $G(s)$ contains an "almost zero". The concept of *almost zeros* was introduced in [Kar83, Kar84]. They defined almost zeros as the local minima of the smallest principal gain of the transfer. In this concept it is possible to understand the high gain of the controller. A stable function can only compensate a very small value of the co-inner in the right half plane, with a very high gain at the imaginary axis, since it is not allowed to place a pole in the right half plane, i.e.:

Define $\nu = \min_{s \in C^+} \sigma_{\min}\left(G(s)\right)$ then any stable right inverse has to fulfill

$\left\| G_{RI}(s) \right\|_\infty \geq \nu^{-1}$.

We were not able to mathematically relate the gain behavior of the stable right inverse to the almost zeros. We however observed in examples, that if a singular value of $(I\text{-}PQ)$ was close to zero we could always find a ν close to zero. This indicates that introducing additional non-minimum phase zeros in the square model obtained from the down squaring operation, could reduce the gain of the squaring down compensator significantly. This fact is also confirmed by the following interesting property of the complementary inner of $G_A(s)$. The realization of a complementary inner of the right annihilator in theorem 5.4.1, say $G_A^{\perp}(s)$, is given by:

$$\left[A,\ \ Q^{-1}C^T,\ \ -(B^TQ+D^TC),\ \ D^T\right] \tag{5.4.6a}$$

Post multiplying $G(s)$ with $G_A^{\perp}(s)$ results after some algebraic manipulation in the square inner function $G_{AI}(s)$ with realization:

$$\left[A,\ \ Q^{-1}C^T,\ \ -C,\ \ I\right] \tag{5.4.6b}$$

whose non-minimum phase zeros equal the eigenvalues of $-A^T$. The down squaring operation therefore results in a square inner transfer matrix, whose non-minimum phase zeros z_i are directly related to the poles of the co-inner. As a result we see that it is possible to down square a model with a compensator that is an inner transfer matrix, if we allow the introduction of n non-minimum phase zeros, equal to the eigenvalues of $-A^T$. The introduction of these non-minimum phase zeros restricts the frequency range over which we can invert $G(s)$. The severity of the restrictions is determined by the location of the zeros and therefore the poles of $G(s)$. The result of this corollary and the relation with almost zeros seems to indicate that there is a relation between the introduction of non-minimum phase behavior in $G(s)G_{RI}(s)$ and the gain of $G_{RI}(s)$. In case there is such a relation we are able to trade-off the gain of the compensator against the restrictions the non-minimum phase behavior poses on the controlled loop. We can try to restrict $G_{AI}(s)$ to those non-minimum phase zeros that are far outside the frequency range for control or are directly related to an almost zero. In this case we may expect the principal gain of the precompensator to be reduced.

A first approach to trade-off the gain of the compensator against the additionally introduced non-minimum phase zeros in the resulting square transfer matrix after down squaring is based on the poles of $G(s)$. In fact one chooses the zero locations equal to the inverse of certain poles of the structural co-inner $G(s)$. This approach is established in the next lemma.

Lemma 5.4.5
Let $G(s)\in\Re H_\infty^{p\times m}$, $m>p$, be a structural co-inner transfer matrix with realization $[A,B,C,D]$, controllability gramian P and observability gramian Q. Assume that the realization of $G(s)$ is such that A is in a real Schur form and $Q=I$, which is possible without loss of generality. We want to introduce k non-minimum phase zeros, located at the inverse of the eigenvalues of A_{11}, i.e. $1/\lambda(A_{11})$. If we factor the realization of $G(s)$ accordingly:

$$A=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix},\ B=\begin{bmatrix} B_1 \\ B_2\end{bmatrix},\ C=\begin{bmatrix} C_1 & C_2\end{bmatrix} \text{ and } P=\begin{bmatrix} P_{11} & P_{21}^T \\ P_{21} & P_{22}\end{bmatrix}$$

then we obtain for the stable right precompensator of $G(s)$, say $G_{RI}^k(s)$:

$$G_I(s)=G(s)G_{RI}^k(s) \tag{5.4.7a}$$

with:

- $G_I(s)\in\Re H_\infty^{p\times p}$ a square inner with realization:

$$\begin{bmatrix} A_{11}, & C_1^T, & -C_1, & I\end{bmatrix} \tag{5.4.7b}$$

- $G_{RI}^k(s)\in\Re H_\infty^{m\times p}$ a stable right transer $G_{RI}^k(s)=\begin{bmatrix} G_R^k(s) & G_A(s)\end{bmatrix}\begin{bmatrix} I \\ \mathrm{A}(s)\end{bmatrix}$

  where $A(s)$ is any stable transfer and $\begin{bmatrix} G_R^k(s) & G_A(s)\end{bmatrix}$ has a realization:

$$\left[A,\ \ (I-P)^{-1}\left[\begin{bmatrix}C_1^T\\0\end{bmatrix}+BD^T \quad BD_\perp^T\right],\ \ -(B^TQ+D^TC),\ \ \begin{bmatrix}D^T & D_\perp^T\end{bmatrix}\right] \tag{5.4.7c}$$

with $D_\perp$ as in theorem 5.4.1.

The infinity norm of any $G_{RI}^k(s)$ is limited by:

$$\left\|G_{RI}^k(s)\right\|_\infty \geq \lambda_1^{1/2}\left(I_n+\begin{bmatrix}0 & I_{n-k}\end{bmatrix}P(I-P)^{-1}\begin{bmatrix}0\\I_{n-k}\end{bmatrix}\right) \tag{5.4.7d}$$

with:
- $\lambda_1(.)$ the largest eigenvalue.
- I_s a sxs identity matrix.

Proof: See appendix 5.C

Remark:
1. Note the close relation between (5.4.7c) and equation (5.4.2b) and (5.4.6). Equation (5.4.7c) is equal to (5.4.2b), for $G_1(s)=I$,. Moreover we obtain that $G_R^k(s)$ in (5.4.7c) equals (5.4.6b), by using the co-inner property $PC^T=-BD^T$, if $G_1(s)$ equals (5.4.6b).
2. It is easy to check that the above factorization is equivalent to applying an output injection $H=\begin{bmatrix}C_1\\0\end{bmatrix}$

Let us take a closer look at equation (5.4.7d) to obtain a better understanding of the relation between the non-minimum phase zeros and the minimum gain of the precompensator. Apply a singular value decomposition on P of lemma 5.4.5:

$$P=\begin{bmatrix}P_{11} & P_{21}^T\\P_{21} & P_{22}\end{bmatrix}=U^T\Sigma U \text{ and } U=\begin{bmatrix}U_1 & U_2\end{bmatrix}=\begin{bmatrix}U_{11} & U_{12}\\U_{21} & U_{22}\end{bmatrix}$$

As a consequence of $Q=I$ we see that Hankel singular values of $G(s)$ equal the square root of the diagonal entries of Σ. We then obtain:

$$I+\begin{bmatrix}0 & I_{n-k}\end{bmatrix}P(I-P)^{-1}\begin{bmatrix}0\\I_{n-k}\end{bmatrix}=I+U_2^T\Sigma(I-\Sigma)^{-1}U_2 \tag{5.4.7e}$$

The reduction of the gain is therefore determined by the orthonormal matrix U_2. The minimum norm is attained for $U_2=[0\ I]^T$, since the entries of Σ are in descending order. We can however not influence, either U or Σ. Only in the case of almost zeros a significant reduction is obtained if we allow the corresponding non-minimum phase zeros to be extracted. From the proof of Lemma 5.4.5, it is obtained that U is exactly the transformation needed to bring the original A matrix of $G(s)$ in a real Schur form, if the original realization is output balanced. Assume we have k almost zeros. The first k singular values of $P(I-P)^{-1}$ will therefore be extremely large. From the proof of lemma 5.4.4 we see that in this case A_{21} is very small and hence, U_{21} will be small. Extracting the influence of the almost zeros, by introducing non-minimum phase behavior in $G_1(s)$, therefore significantly reduces the norm. In all other cases the influence of the zeros is less understood. Note however that the introduction of additional zeros in these cases will have a limited effect on the gain of the compensator, since the corresponding entry of $P(I-P)^{-1}$ is not very large. It is therefore crucial for a restricted magnitude of the precompensator to remove the almost zeros from the transfer matrix.

In the above approach we selected the location of the non-minimum phase zeros in the squared down transfer matrix and used lemma 5.4.5 to determine the gain of the precompensator. Another approach is to require the principal gain of the precompensator to be smaller or equal to a certain predefined maximum gain. The location of the zeros then follows from this requirement. This problem turns out to be a specific H_∞ control problem. To establish the relation with a H_∞ problem formulation we proceed as follows: The relation between the non-minimum phase zeros and the gain of the compensator can be formulated as finding the minimum gain of $G_{RI}(s)$ if we allow $G_{AI}(s)=G(s)G_{RI}(s)$ to contain a prescribed number of non-minimum phase zeros. This problem is equivalent to allowing the precompensator to have a prescribed number of unstable poles, since $G(s)G_{RI}(s)=G_{AI}(s)$ is equivalent to $G(s)G_{RI}(s)G_{AI}(s)^{-1}=I$. The solution is given by the next lemma:

Lemma 5.4.6

Let $G(s)\in\Re H_\infty^{p\times m}$, $m>p$, be a structural co-inner transfer matrix with realization $[A,B,C,D]$, controllability gramian P and observability gramian Q. We then obtain that for $G_{RI}(s)$ a right inverse of $G(s)$, with at most $l-1$ unstable poles the minimum achievable infinity norm is restricted by:

$$\underset{A(s)\in\Re H_\infty^{(m-p)xp}}{\text{MIN}} \quad \|G_{RI}(s)\|_\infty \geq \lambda_i((I-PQ)^{-1/2}) \tag{5.4.8}$$

with:

- $\lambda_i(.)$ the i-th largest eigenvalue.
- $G_{RI}(s)$ is factored conform equation (5.4.3).

Moreover there exists a controller that attains the lower bound.
Proof: See appendix 5.C

The lemma thus states that the minimum infinity norm attainable by any right inverse of $G(s)$, say $G_{RI}(s)$, with i unstable poles is determined by the square root of the i-th largest eigenvalue of $(I\text{-}QP)^{-1}$. To obtain the stable compensator we have to extract the unstable poles from G_{RI}(s). To extract these unstable poles of $G_{RI}(s)$ we perform a coprime factorization of $G_{RI}(s)$:

$$G_{RI}(s) = G_{RI}^S(s)G_I^{-1}(s)$$

where $G_I(s)$ is a square inner and $G_{RI}^S(s)$ stable. Post multiplying the structural co-inner $G(s)$ with $G_{RI}^S(s)$ then results in the inner function $G_I(s)$. $G_I(s)$ therefore exactly contains the non-minimum phase zeros introduced to reduce the gain of the square down transfer matrix. The right coprime factorization with inner $G_I(s)$ is a standard problem, see e.g. [Doy84]. For completeness we state the factorization in the next lemma.

Lemma 5.4.7
Given a transfer matrix $G(s)\in \Re H_\infty^{pxm}$, with $m>p$ and realization $[A,B,C,D]$. There exists a right coprime factorization, $G_{RI}(s) = G_{RI}^S(s)G_I^{-1}(s)$ such that $G_I(s) \in \Re H_\infty^{pxp}$ is a square inner containing the k unstable poles. The realization of $\begin{bmatrix} G_I(s) \\ G_{RI}^S(s) \end{bmatrix}$ is given by:

$$\left[A+BF, \quad BU, \quad \begin{bmatrix} F \\ C+DF \end{bmatrix}, \quad \begin{bmatrix} U \\ DU \end{bmatrix}\right] \tag{5.4.9a}$$

with:

- $U\in \Re^{pxp}$ and $UU^T=I$.
- $F=B^TX^{-1}$ and X the solution of the Ricatti equation: $A^TX+XA+XBB^TX=0$

If $G_{RI}(s) \in \Re H_{\infty}^{m \times p}(k)$ is a right inverse of the structural co-inner transfer $G(s)$ we obtain that:

$$G_I(s) = G(s)G_{RI}^S(s) \tag{5.4.9b}$$

Proof: See appendix 5.C

Remark:
1. In principle we may choose U freely. However choosing U equal to the transpose of $(F(A+BF)^{-1}B+I)$ results in an inner with steady state equal to the identity .
2. It is interesting to see the close resemblance between (5.4.7e) and (5.4.8). The two equations are equal to each other if U_2 equals $[0\ I_{n-k}]$. It looks like the algorithm tries finds a solution for the compensator such that the zeros, i.e. the unstable poles to be extracted are related to the first k singular values of $P(I-P)^{-1}$ via a U_2 equal to $[0\ I_{n-k}]$. A better understanding of this relation might contribute to a better understanding of the non square H_∞ problem. Moreover this understanding will probably make the factorization of lemma 5.4.8 not necessary. Further research into this topic is certainly desired.

The possibly unstable right inverses $G_{RI}(s)$ that achieve a certain infinity norm can be obtained using essentially the Hankel Norm approximation theory of Glover [Glo84, Glo89]. A precompensator for the so called suboptimal solution, i.e. $\gamma^2 \neq \lambda_i((I-QP)^{-1})$, with γ the maximum allowed infinity norm of the precompensator, was obtained in [Sto94]. Here we will state an alternative solution, which enables us to obtain all precompensators that have an infinity norm smaller than or equal to γ.

Lemma 5.4.8

Let $G(s) \in \Re H_\infty^{p \times m}$, $m>p$, be a structural co-inner transfer matrix with realization $[A,B,C,D]$, controllability Gramian P and observability gramian Q. All right inverses of $G(s)$ which fulfill the following inequality:

$$\lambda_i((I-QP)^{-1}) > \|G_{RI}(s)\|_\infty^2 > \lambda_{i+1}((I-QP)^{-1})$$

have at least i unstable poles and are given by:

$$G_{RI}(s) = \Im_l(J,K) \tag{5.4.11a}$$

J has the realization :

$$\left[A+\overline{Z}^{-1}PC^TC,\ \ \overline{Z}^{-1}P\begin{bmatrix}C^T & C_\perp^T\end{bmatrix},\ \begin{bmatrix}-\gamma^2B^TQ-(\gamma^2-1)D^TC\\ C\end{bmatrix},\ \begin{bmatrix}D^T & D_\perp^T\\ I & 0\end{bmatrix}\right] \tag{5.4.11b}$$

with:

$$\overline{Z}=(I-\gamma^2(I-QP)) \tag{5.4.11c}$$

and K is any stable transfer matrix that fulfills:

$$K\in\Re H_\infty^{(m-p)xp}\qquad \|K\|_\infty\le\sqrt{(\gamma^2-1)}$$

Proof: See appendix 5.C

Remark:

From lemma 5.4.8 we see that the additional pole that becomes unstable if we reduce γ below $\lambda_i((I\text{-}PQ)^{-1/2})$ is entering the right half plane from infinity.

The freedom left in the solution of theorem 5.4.2 and lemma 5.4.8 can in principle be used to influence the location of the unstable pole and the magnitude of the transfer matrix of $G_{RI}(s)$ as a function of the frequency. The direct relation this freedom has to the location of unstable poles and the shape of $G_{RI}(s)$, is not well understood. It is expected that the freedom can at one hand be used to further reduce the gain of the controller over a certain frequency range, while keeping the maximum gain in the frequency domain below the desired value gamma. The freedom can also be used to push the zeros we introduce as far away as possible from the imaginary axis into the right half plane. This will however mean that the zeros move further away from their optimal location and the gain of the precompensator will increase. A sensible use of the freedom left in the H_∞ solution is however in general a problem. Analytic approaches are hampered by the constraint on the infinity norm . A frequently proposed solution is to use the minimum entropy solution or central solution (put all degrees of freedom equal to zero), which is known to bound the H_2 norm of the above solution. The disadvantage of this approach is however that it does not have a direct relation to the shape nor to the location of the unstable poles of $G_{RI}(s)$.

Remark:

In [Sto93] it is shown that the minimum entropy solution is not invariant for the bilinear transformation. After bilinear transformation the minimum entropy solution in the continuous domain will therefore not equal the discrete minimum phase solution. The continuous domain solution that results in the discrete time minimum entropy solution is obtained by choosing a specific weight in the entropy function [Sto93].

A possible way to use the degrees of freedom in a more favorable way is making use of the relation the entropy has with the H_2 problem as observed in the remark. We could use the weight in the entropy equation to put more emphasis on the low frequency behavior of the compensator. A second possibility is to formulate the problem as an optimization problem. This in many cases leads to complex nonlinear optimization problems. An interesting possibility may be offered by the developed theory on LMI's. This technique offers the possibility to reformulate some problems into convex optimization problems [Boy94, Gah96]. We will not deal with this problem further here. A good choice, or better understanding of the influence of the degrees of freedom on the behavior of the precompensator and the resulting inner is left for future research.

In order to reduce the maximum gain of the precompensator we followed two approaches. In lemma 5.4.8, we reduced the maximum gain below a predefined value. If necessary we have to allow the introduction of non-minimum phase zeros. The location of the zeros is completely determined by the algorithm. In lemma 5.4.5 we showed that the introduction of non-minimum phase zeros equal to minus the conjugated poles of the structural inner reduces the gain of the precompensator. We can however ask ourselves what the influence on the gain of the precompensator will be if we allow the introduction of a number of non-minimum phase zeros with a predefined direction and location in the complex plane. Assume we predefine an inner transfer matrix $G_I(s)$ and want to find a stable minimum H_∞ precompensator $G_{RI}(s)$, such that $G_I(s) = G(s)G_{RI}(s)$.

This approach will be used in section 5.5 to remove the influence of control of the high priority outputs on the low priority outputs. The idea to solve the above problem is simple. The set of precompensators fulfilling $G_I(s) = G(s)G_{RI}(s)$ is given by:

$$G_{RI}(s) = \begin{bmatrix} G_R(s) & G_A(s) \end{bmatrix} \begin{bmatrix} G_I(s) \\ \mathrm{A}(s) \end{bmatrix} \tag{5.4.12a}$$

where $G_R(s)$ and $G_A(s)$ are defined in theorem 5.4.1. The compensator thst minimizes the maximum gain is thus found by solving:

$$\underset{\mathrm{A}(s)\in\Re\mathrm{H}_\infty^{(m-p)xp}}{\mathrm{MIN}} \quad \|G_{RI}(s)\|_\infty \tag{5.4.12b}$$

This idea is used to obtain the following result.

Lemma 5.4.9

Given a *pxp* inner transfer matrix $G_I(s)$ with realization $[A_I,B_I,C_I,D_I]$ and a structural co-inner transfer matrix $G(s)$ with realization $[A,B,C,D]$. All precompensators $G_{RI}(s)$ with at most i–1 unstable poles, with $G_I(s) = G(s)G_{RI}(s)$, fulfilling:

$$\|G_{RI}(s)\|_\infty \geq \lambda_i^{1/2}\left((I-PQ)^{-1}P(Q - X_S P_I X_S^T)\right) + 1 \tag{5.4.12c}$$

with:

- P,Q The controllability respectively observability gramian of $G(s)$.
- P_I The controllability gramian of $G_I(s)$.
- X_S The solution of the following Sylvester equation:

$$X_S A_I + A^T X_S + C^T C_I = 0 \tag{5.4.12d}$$

Proof: See appendix 5.C

Remark:

1. A characterization of the controllers achieving (5.4.12c) is obtained from a reasoning analog to the one used in the proof of lemma 5.4.8.
2. Lemma 5.4.9 is closely related to lemma 5.4.5: For example the minimum realization of $G_R(s)G_I(s)$ in the proof of lemma 5.4.9 (appendix 5.C) equals the realization in equation (5.C.4a) of the proof of lemma 5.4.5 for $P_I = I$ and $X_S = \begin{bmatrix} -I \\ 0 \end{bmatrix}$. A closer study of lemma 5.4.5, lemma 5.4.9 and lemma 5.4.8 may enable us to trade-off the location of the non-minimum phase zeros, introduced after the down squaring operation, against the H_∞-norm of the controller. This is left for future research.

5.4.2 An example of the analysis of a strutural inner

In this subsection we will analyze the behavior of the structural inners corresponding to one output of the discrete time model of a scaled tube glass

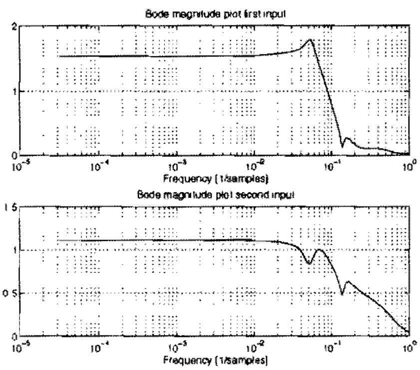

Fig.5.4.1 Bode magnitude plot of the model of section 5.4.2.

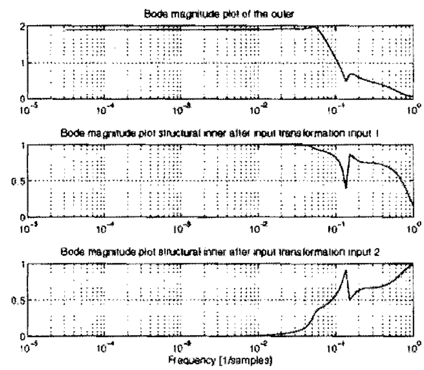

Fig.5.4.2 Bode magnitude plot of the structural inner transfer after the input transformation.

process without delays. The tube glass process will be further discussed in chapter 6. Here we will see the model just as a *1x2* model. The analysis here is completely done in the discrete time domain. In the calculations a bilinear transformation is used to get the results as they were deduced in the previous subsection. The Bode magnitude plots of the model are given in figure 5.4.1. Both inputs have more or less equal influence on the output until the 0.03 1/sample, the second input starts to drop-off. The first input starts to drop off, rapidly, after some increase at 0.06/samples. The equal influence on the output for both inputs at lower frequencies, is due to the scaling we applied for this example. To analyze the restrictions stable down squaring puts on the controllability we first factor the model conform lemma 5.4.1:

$$G(z) = G_o(z)G_s(z)$$

In this section we will mainly be interested in the behavior of the structural co-inner $G_s(z)$. The behavior of the inputs is reflected in the structural inner. A change of the bases of the inputs, based on the steady state behavior at the input of the $G_s(z)$ is performed:

$$G(z) = G_s(z)U^T \qquad (5.4.13)$$

where U is an orthonormal 2x2 matrix: $U = \begin{bmatrix} G_s(1) \\ U_2 \end{bmatrix}$ and U_2 spans the complement of $G_s(1)$. The advantage of this change in coordinates to the steady state is that we can see the dynamic behavior evolve as a function of frequency. This transfor-mation not only makes it easier to interpret, since steady state is in general best understood and known, it also enables us to get an insight in how the natural behavior changes as function of frequency. It enables a trade-off of complexity against performance. The Bode magnitude plot of the outer and of the structural inner in this new basis is given in figure (5.4.2). From this plot we see that after static decoupling the model to this output behaves essentially as a SISO system unto a frequency of approximately 0.04 1/sample. After this frequency the output behavior starts to become more involved, i.e. more MISO. The behavior changes gradually to a behavior opposite to the steady state. In the frequency range between 0.1/sample and 0.2/sample we even see a very short frequency peak., in the opposite direction. From the Bode magnitude plot of the outer (figure 5.4.2) we see that this change of the behavior coincides with the sharp role-off of input one (figure 5.4.1). It is also this range in which we may expect to encounter problems inverting the behavior of the co-inner, with a stable function. In section 5.4.1 we developed the tools to obtain a more detailed insight in this inversion problem. It is readily known that the inversion can always be accomplished without introducing an additional zero. Using lemma 5.4.6 we however see that the cost of a stable inversion is a very high gain of the compensator, i.e. a minimum H_∞ norm equal to 28058, which is of course unrealistic. From table 5.4.1 we see that allowing the introduction of one non-minimum phase zero drastically decreases the minimum of the maximum gain. From this table we moreover see that introducing a second non-minimum phase zero bring the maximum gain down to an acceptable value. In the table also the location of the zeros for which the minimum H_∞ norm is achieved are given.

Remark:

The zeros do not correspond with the minimum achievable norm, but correspond to a H_∞-norm of at most 1.0001 higher than the minimum, as indicated in table 5.4.1, since we used a suboptimal algorithm, which is unable to solve the problem for the actual minimum.

The algorithm will locate the allowed number of non-minimum phase zeros such that the influence the co-inner minimum outside the unit circle has on

Number of zeros introduced by down squaring and their location.

	1	2	3	4	5	6	7	8	9	10	11
	8.3	8.6	8.6	8.6	8.6	8.6	8.6	8.6	8.6	8.6	8.6
		-1.2	-1.2	-1.2	-1.2	-1.2	-1.2	-1.2	-1.2	-1.2	-1.2
l			-4.2	15	-6.6	215	-31.7	-15.2	-10.6	-6.5	-6.0
o				1.3	1.0+0.4i	1.0+0.4i	0.9+0.4i	0.9+0.4i	0.9+0.4i	0.9+0.4i	0.9+0.4i
c					1.0-0.4i	1.0-0.4i	0.9-0.4i	0.9-0.4i	0.9-0.4i	0.9-0.4i	0.9-0.4i
a						1.7	1.37	1.2+0.2i	1.2+0.2i	1.0+0.2i	1.0+0.2i
t							1.09	1.2-0.2i	1.2-0.2i	1.0-0.2i	1.0-0.2i
i								-1.3	-1.4	-2.0	-2.0
o									3.5	1.3+0.5i	1.3+0.5i
n										1.3-0.5i	1.3-0.5i
											1.12

$\min\limits_{A(s)}\|G_{RI}(s)\|_\infty$ for the different number of zeros introduced by the down squaring

0	1	2	3	4	5	6	7	8	9	10	11
28058	74	3.5	1.18	1.07	1.04	1.01	1.00	1.00	1.00	1.00	1.00

Table 5.4.1 Location of the additional non-minimum phase zeros introduced by down squaring and the minimum ∞-norm for the different compensators.

the gain of the stable compensator is minimized. In this sense the location is optimal. The gain of the compensator equals one for the inner compensator. This compensator introduces eleven zeros in the resulting square model. Almost zeros, i.e. local minimum corresponding to $\lambda(PQ)$ close to one, will strongly attract the zeros to be introduced by the square down operation. The first zero introduced will be directly placed close to the smallest minimum, since there is a large gap between the first and second Hankel singular value. This is directly seen in table 5.4.1. The first zero introduced does not significantly move if more zeros are introduced. It is dominated completely by the minimum, somewhere around 8.5. The same effect occurs for the second zero that remains in the neighborhood of -1.2. If some Hankel singular values are close together, the zeros will be placed such that they mask the local minimum with respect to the unit circle as well as possible. Hence, it may happen that for a certain number of zeros it is an advantage to locate a zero close to the unit circle, while there is no local minimum in the vicinity of the zero. Introducing an additional zero may then drastically change the location of the zeros in the complex plane. The severity of the restriction the non-minimum phase zeros poses on the controllability of the model is of course determined by the number of zeros and the location in the complex plane. It is therefore not impossible that the compensator introducing less zeros is more restrictive then a compensator introducing more zeros. This effect has however not been observed yet. As in chapter three the restrictions that the additional zeros pose on the controllability can be visualized by the Bode magnitude plot of the sensitivity function. In figure 5.4.3 the Bode magnitude plot of the sensitivity of the inner transfer matrices obtained with the compensators that introduce two, three and four non-minimum phase zeros is plotted. The Bode magnitude plot of the three compensators are plotted in figure 5.4.4. The introduction of the fourth and more zeros results in no significant change anymore in the gain behavior of the compensator. The influence these zeros have on the controllability of the closed loop is however severe. It results in a drastic decrease of the closed loop bandwidth. Extracting these zeros seems not useful. From the Bode magnitude plot of the compensators we see for the lower frequencies that the gain for the second input is, in contrast to the first input, different for each compensator.

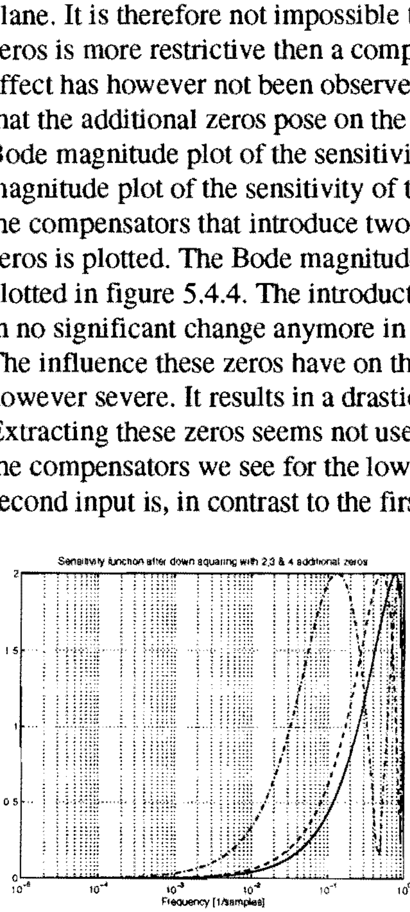

Fig.5.4.3 Bode magnitude plot of the sensitivity of the inner transfer after down squaring: 2 zeros (solid), 3 zeros (dashed) & 4 zeros (dash dot).

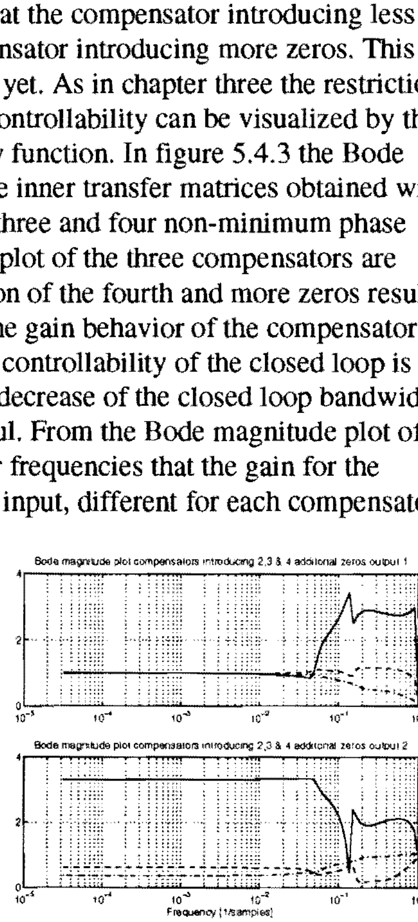

Fig.5.4.4 Bode magnitude plot of the pre-compensators introducing 2, 3 & 4 additional non-minimum phase zeros.

This is due to the fact that this input for very low frequencies exactly coincides with the direction of the right annihilator, i.e. the direction used to stabilize the compensator. In fact for the whole frequency range the gain in the direction of the complementary annihilator equals one by construction. The constant gain in the direction of the complementary annihilator, input 2 for very low frequencies, is of course due to the near optimal solution of the H_∞ solution. As for non-minimum phase zeros there will be an analytic trade-off for this gain. It is expected that it is possible to make the gain of the second output small without a drastic increase of the infinity norm for low frequencies.

Remark:

> In principle we could also try to use the freedom left in the H_∞ solution to accomplish this. As discussed in the previous subsection this is not an analytically solvable problem and will result in an optimization problem.

5.4.3 Summarizing the results and conclusions

A dynamic precompensator can always down square a model, without introducing additional non-minimum phase zeros in the down squared model. It may however result in a high gain of the precompensator. The minimum infinity norm of the compensator is directly related to the largest Hankel singular values of this structural co-inner transfer matrix. We claimed that if the Hankel singular value is close to one this indicates the existence of a so called almost zero. The location of the almost zeros corresponding to the square of the Hankel singular values very close to one can be approximated by minus the related poles of the A-matrix. To ensure a reasonable gain behavior of the controller we have to introduce as many non-minimum phase zeros as we have singular values that are close too one. This knowledge enables us to trade-off the gain of the compensator against the introduction of non-minimum phase zeros. The influence the non-minimum phase zero will have on the controllability of the model is determined by the location in the complex plane and the direction of the zeros. No complete insight was obtained in the relation between gamma and the influence of the zeros on the controllability of the plant. It is important to remember that the zero locations are not fixed. There is a clear trade-off between the maximum gain of the controller the shape of the Bode magnitude plot and the location of the zero.

In the next section we will take a closer at how we can use the here developed theory to consider nominal stability in the controllability analysis.

5.5 *The approach to asses process controllability with stability.*

In the previous section we dealt with an approach that enabled us to consider nominal stability during the design of a square down compensator for a non square process. In this section we will make use of the techniques developed in section 5.4 to include nominal stability in the controllability procedure developed in section 5.3. The approach developed in section 5.3 is not able to deal with stability of the controller. Hence the results obtained from the analysis are too optimistic in the case of a non-minimum phase process and /or non-square processes. In this section we will embed nominal stability in the procedure. Nominal stability introduces an additional trade-off between performance on the different process outputs and the gain of the controller. Essentially the approach here is a direct extension of section 5.3 with the addition of nominal stability. In contrast to section 5.3 we will however make use of an additive formulation of the inverse (Lemma 5.3.1). The reason for this change of approach is of a technical nature:

- In general the LQ factorization of the process will result in an unstable L and/or Q matrix.
- A second complication is that applying the factorization and the multiplicative formulation tend to results in a drastic increase of the order of the resulting state space models of L and Q.

We therefore return to the more basic approach to construct the approximate inverse as chosen to determine the inverse in the first part of section 5.2. This change in approach does not essentially change the procedure of section 5.3. As in the previous section the complete development of the technology will be in continuous time domain. The bilinear transformation is used to transform the discrete time problem to continuous time domain.

In this section we will make use of the *inner-outer structural co-inner factorization*, as introduced in section 5.2[1]: A biproper stable process model $M(s)\in \Re H_\infty^{p\times m}$, $m>p$, can always be written as:

$$M(s) = N(s)G(s)S(s) \tag{5.5.1}$$

[1] The factorization was introduced in section 5.2 in the discrete domain. The continuous time is however directly related. The continuous domain is only used to keep a direct relation with the results from the previous section.

with:
- $N(s) \in \Re H_\infty^{p\times p}$ an inner transfer matrix.
- $G(s) \in \Re H_\infty^{p\times p}$ a biproper minimum phase transfer matrix.
- $S(s) \in \Re H_\infty^{p\times m}$ the structural co-inner transfer matrix.

Remark:

The requirement that the transfer matrix is biproper is not a restriction. Infinity zeros have to be extracted before solving the above factorization. A continuous time version of the theory developed in section 4.4 can be used for this purpose. In this thesis we are primarily interested in discrete time systems. We therefore apply the factorization on systems that are obtained after application of a bilinear transformation of a discrete time system. Hence, the point -1 in the discrete time domain will be transformed to infinity. However poles and zeros close to -1 will in general not occur in the models. Their behavior is not part of the process behavior, but generally due to sampling of the data, i.e. quantization and aliasing.

We assume again that the required closed loop behavior is defined at the model outputs in a decreasing order of importance, conform section 5.2. The model of the process $M(s) \in \Re H_\infty^{p\times m}$ is factored again per output as, conform equation (5.2.2):

$$M(s) = \begin{bmatrix} m_1(s) \\ m_2(s) \\ \cdot \\ m_p(s) \end{bmatrix} \tag{5.5.2}$$

The controller $C(s) \in \Re H_\infty^{m\times p}$ is factored per column conform equation (5.2.5a) and has to fulfill equation (5.2.5b), since we assume again absolute priorities. Hence we obtain that each column $C_i(s) \in \Re H_\infty^{m\times 1}$, $I=1,.., p$, has to fulfill:

$$C_i(s) = \begin{bmatrix} \overline{C}_i(s) & S_{1,i}^A(s) \end{bmatrix} \begin{bmatrix} I \\ A_j(s) \end{bmatrix} \tag{5.5.3}$$

with:
- $\overline{C}_i(s) \in \Re H_\infty^{m\times 1}$ and $S_{1,i}^A(s) \in \Re H_\infty^{m\times(m-i)}$ fulfill respectively:

$$\begin{bmatrix} m_1(s) \\ . \\ m_p(s) \end{bmatrix} \overline{C}_i(s) = \begin{bmatrix} 0 \\ 0 \\ t_i(s) \end{bmatrix} \text{ and } \begin{bmatrix} m_1(s) \\ . \\ m_i(s) \end{bmatrix} S_{1,i}^{A}(s) = 0$$

where $t_i(s) \in \Re H_\infty^{1x1}$,i.e. entry (i,i) of the output complementary sensitivity matrix $T(s)$ and $A_i(s) \in \Re H_\infty^{(m-i)x1}$ is the right annihilator of the first i rows of the model and is used to minimize the influence of this controller column on the model outputs $i+1,..,p$, i.e. design of the entries (j,i), $j>i$, of $T(s)$[2]:

$$\begin{bmatrix} t_{i+1,i}(s) \\ . \\ t_{p,i}(s) \end{bmatrix} = \begin{bmatrix} m_{i+1}(s) \\ . \\ m_p(s) \end{bmatrix} C_i(s)$$

The input-output controllability analysis is performed again per output of the model and in accordance with the priorities, conform section 5.2. Assume that we finished the analysis for the first $i-1$ outputs of the model and start with the analysis of the i-th output. As a first step we need to ensure that control of the i-th output does not influence the outputs 1 to $i-1$. Hence $C_i(s)$ needs to be spanned by the columns of the right annihilator $S_{1,i-1}^{A}(s)$, equation (5.5.3). We therefore define a reduced system for the i-th output and apply the inner-outer structural inner factorization, equation (5.5.1), on this reduced system:

$$m_i(s) S_{1,i-1}^{A}(s) = n_i(s) g_i(s) S_i(s) \tag{5.5.4a}$$

with:

- $n_i(s) \in \Re H_\infty^{1x1}$ the inner transfer function of the reduced system containing the non-minimum phase behavior coupled to the output.
- $g_i(s) \in \Re H_\infty^{1x1}$ the outer transfer function of the reduced system.
- $S_i(s) \in \Re H_\infty^{1x(m-i+1)}$ the structural co-inner transfer of the reduced system.
- $S_{1,i-1}^{A}(s) \in \Re H_\infty^{mx(m-i+1)}$ an inner transfer matrix that is a right annihilator of $S_j(s)$, for $j=1,.., i-1$, which equals the identity for $i=1$[3].

[2] A second possibility is to use $A_i(s)$ to minimize the principal gain of the controller or the gain to certain outputs of the controller.

[3] The existence of such a right annihilator follows directly from theorem 5.4.1.

We can now use the theory developed in section 5.4, e.g. lemma 5.4.8, to determine a precompensator $S_i^R(s) \in \Re H_\infty^{mx(m-i+1)}$, such that:

$$S_i(s) S_i^R(s) = I_i(s) \tag{5.5.4b}$$

with:

- $I_i(s) \in \Re H_\infty^{1x1}$ the inner transfer function that contains the additional non-minimum phase zeros introduced by the down square operation.

In equation (5.5.4b) we trade-off the frequency range over which we can invert the reduced system against the principal gain of the precompensator $S_i^R(s)$. After design of the precompensator a SISO control problem is left:

$$m_{ii}(s) = m_i(s) S_{1,i-1}^A(s) S_i^R(s) = I_i(s) N_i(s) O_i(s) \tag{5.5.4c}$$

The analysis of this problem will result in the SISO controller $C_{ii}(s)$. Hence for $\overline{C}_i(s)$ in equation (5.5.3c) we obtain:

$$\overline{C}_i(s) = S_{1,i-1}^A(s) S_i^R(s) C_{ii}(s) \tag{5.5.4d}$$

As a next step in the procedure the influence that the controller columns 1,.., $i-1$ have on output i has to be reduced. For $j=1,..,i-1$ we have to find a $(m-i+1)x1$ transfer matrix $A_i^j(s)$ such that:

$$m_i(s) C_j^i(s) = m_i(s)\left\{C_{1,j}^{i-1}(s) + S_{1,i-1}^A(s) A_i^j(s)\right\} \tag{5.5.5a}$$

has a more desired behavior and the precompensator $C_j^i(s) \in \Re H_\infty^{mx1}$. Note that the superscript i of the controller indicates that the influence of the controller on the outputs $j+1$ to i is minimized, i.e.:

$$C_j^i(s) = \overline{C}_j(s) + \sum_{k=j}^{i-1} S_{1,k}^A(s) A_k^j(s) \tag{5.5.5b}$$

A sufficient condition for stability of the precompensator $C_j^i(s)$ is of course to require $A_i^j(s) \in \Re H_\infty^{(m-i)x1}$. In the case that $m_i(s) C_j^{i-1}(s)$ does not have any additional non-minimum phase zeros compared to $m_i(s)$ then the best choice for the direction of $A_i^j(s)$ is of course equal to $S_{i+1}^R(s)$. This is the stable precompensator with an acceptable gain that is the best projector in the complementary nullspace of $S_{i+1}(s)$, under the restriction that it projects only in the nullspace of $m_1(s)$ to $m_i(s)$. If $m_i(s) C_j^{i-1}(s)$ contains additional non-minimum phase zeros above those of $m_i(s)$ it may be worthwhile to apply lemma 5.4.9 to obtain a better solution. Let us take a closer look at this case.

To simplify the discussion we assume both $C_j^{i-1}(s)$ and $m_i(s)$ to be minimum phase transfer matrices[4]. Apply the following inner-outer factorization:

$$m_i(s)C_j^{i-1}(s) = I_{i,j}(s)O_{i,j}(s) \tag{5.5.6a}$$

with:

- $I_{ij}(s) \in \Re H_\infty^{1\times 1}$ The scalar inner transfer function
- $O_{ij}(s) \in \Re H_\infty^{1\times 1}$ The scalar outer transfer function

We then obtain for the minimization of the interaction after substitution of (5.5.6a) in equation (5.5.5a):

$$m_i(s)C_j^i(s) = \left(I_{i+1,j}(s) + X(s)A_j^i(s)\right)O_{i,j}(s) \tag{5.5.6b}$$

with: $X(s) = m_i(s)S_{1,i-1}^A(s)O_{i,j}^{-1}(s)$

Note that $I_{i+1,j}(s)+X(s)A_j^i(s)$ is again a square down problem. The difference with the problem in equation (5.5.4b) is the fact that $I_{i+1,j}(s)$ must not equal to identity, but has to equal an inner transfer. We may then use the fact that $m_i(s)S_{1,i-1}^A(s)A_i^j(s)$ must contain the non-minimum phase zeros of $I_{ij}(s)$, which may result in a reduction of the gain of $A_j^i(s)$ (using lemma 5.4.9)[5]. To further clarify the above procedure let us take a closer look at the tube glass example used in section 5.4.2.

Example 5.5.1

In this example we continue the example of section 5.4.2 and add the second output to the scaled model. In the analysis we assume that the output discussed in the section 5.4.2 is the most important output. The process outputs are therefore ordered according to the priority they have. For the process we apply the inner-outer structural co-inner factorization

[4] This is in fact without loss of generality and only done to simplify further discussion. We can always first apply an inner-outer factorization on both transfer matrices. For equation (5.5.6b) we then obtain:

$$m_i(s)C_j^i(s) = I_m(s)\left(I_{i+1,i}(s) + X(s)\tilde{A}_i^{i+1}(s)\right)O_{i+1,i}(s)I_C(s)$$

with $I_m(s)$ the inner factor of $m_i(s)$, $I_C(s)$ the inner factor of $C_j^{i-1}(s)$ and $A_j^i(s) = \tilde{A}_i^i(s)O_{i,j}(s)I_C(s)$. This however results again in equation (5.5.6c).

[5] The problem is different for the last output, since equation (5.5.6b) is no square down problem. Hence no additional gain is obtained by the above formulation.

per output (equation 5.5.2) and construct $E(z)$ (equation 5.5.3). In figure 5.5.1 both outer transfers and the L-term of the LQ factorization of $E(z)$ are given. The inner factors of both outputs equal one, since no non-minimum phase behavior is coupled to the outputs. From figure 5.5.1a we observe that the X_g term as defined in section 5.3.2, i.e. $X_g=g_2/g_1$, is close to one. From the R_{22} term in figure 5.5.1b we directly obtain that both outputs are almost independent of eachother in the frequency range of importance. As we have seen in section 5.4 the annihilator of output 1 is also used to stabilize the controller for column one. Based on this observation we may conclude that it must be possible to obtain a well conditioned control problem, under the assumption that stability of the controller can be guaranteed, without excessive use of the annihilator.

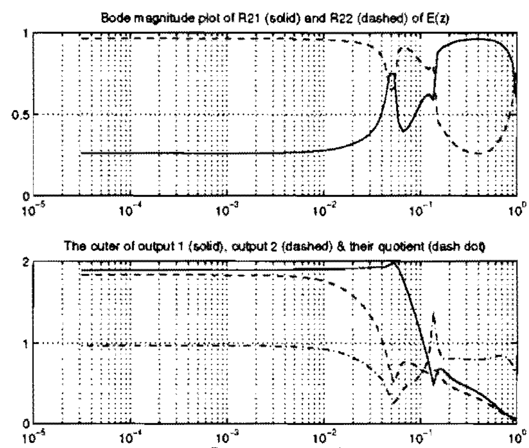

Fig.5.5.1 a) Bode magnitude plot of entry (2,1) (solid) & (2,2) (dashed) of the transfer matrices R of the LQ factorization of E. b) Outer of output 1 (solid), output 2 (dashed) & their quotient $O2/O1$ (dash dot).

Stabilizing the first column of the controller to square down the co-inner of the first output is already extensively discussed in section 5.4.2. From this section we obtain that the compensators that introduce two and three non-minimum phase zeros, say $C_{12}(z)$ respectively $C_{13}(z)$, are poten-tial candidates for the stable compensator for this control problem. This conclusion is based on both the gain behavior of the transfers of the setpoint to the controller outputs and the restrictions that the non-minimum phase zeros pose on the bandwidth over which we can control output one. We have to analyze the influence of both compensators on the second output to decide which of the two compensators is best suited. The influence that both compensators have on the second output are given in figure 5.5.2a. We see that $C_{13}(z)$ has significantly less influence on output two than $C_{12}(z)$. This can easily be understood from the fact

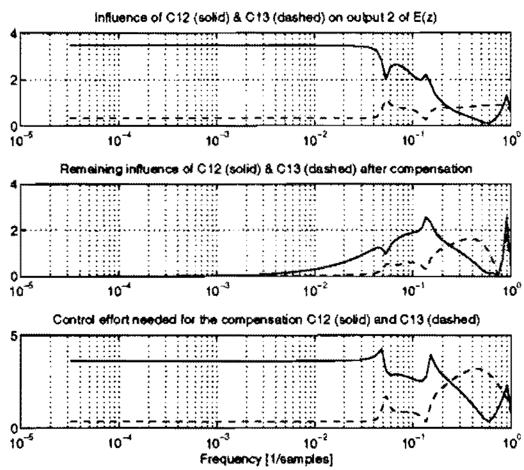

Fig.5.5.2 a) Bode magnitude plot of the influence of $C_{12}(z)$ and $C_{13}(z)$ on output 2 of $E(z)$. b) Output 2 of $EC_{12}(z)$ & $EC_{13}(z)$ after compensation c) Principal gain of the control effort needed to remove this influence. (C_{12} (solid) & C_{13} (dashed)).

that R_{12} is very small and R_{22} almost every where close to one. From figure 5.4.6b we observe that the gain in the direction of the annihilator of $C_{12}(z)$ is significantly larger than the gain of $C_{13}(z)$ in this direction. The influence on the second output will therefore also be significantly larger. To select the best compensator we have to know how well we can remove the influence from output two. To judge the ability to remove the influence from output two, we choose $A_1^2(s)$ in equation (5.5.5) as:

$$A_1^2(z) = -\left(S_2(z)S_1^A(z)\right)_O^{-1}\left(S_2(z)S_1^R(z)\right) \tag{5.5.7a}$$

with:

- $\left(S_2(z)S_1^A(z)\right)_O$ equals the outer factor of $S_2(z)S_1^A(z)$

- $S_1^R(z)$ is equal to $C_{12}(z)$ respectively $C_{13}(z)$

This effort needed to accomplish this compensation, i.e. $A_1^2(s)$, is plotted for both controllers $C_{12}(z)$ and $C_{13}(z)$ in figure 5.5.2c. The remaining influence of the compensator on output two, i.e.

$$\left(I-\left(S_2(z)S_1^A(z)\right)_I\right)\left(S_2(z)S_1^R(z)\right) \tag{5.5.7b}$$

with: $\left(S_2(z)S_1^A(z)\right)_I$ equals the inner factor of $S_2(z)S_1^A(z)$.

is plotted for both $C_{12}(z)$ and $C_{13}(z)$ in figure 5.5.2b. As a stable compensator for output one we decide to choose $C_{13}(z)$, based on the above results and the restricted loss of performance on output one.

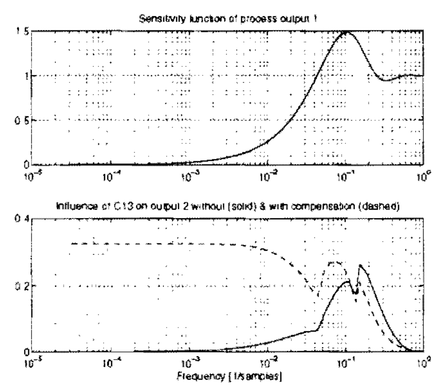

Fig.5.5.3 a) Bode Magnitude plot of entry (1,1) of the sensitivity transfer. b) Bode magnitude plot of entry (2,1) of the sensitivity transfer with (solid) & without compensation (dashed).

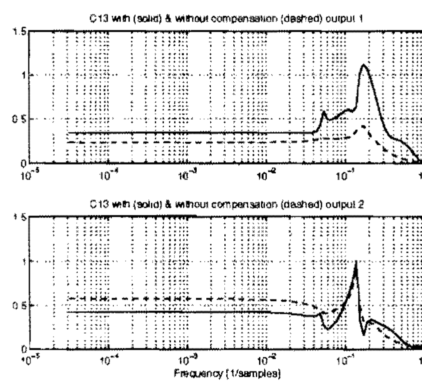

Fig.5.5.4 Bode magnitude plot of the first column of the controller output 1 (plot a) and output 2 (plot b), with and (solid) without compensation of the influence on process output 2 (dashed).

As a next step in the procedure we design the controller for output one. In principle we can choose for a number of approaches from a very simple low pass filter times the inverse $g_1(z)$ to very sophisticated techniques like H_∞ and so on. Here for the analysis we will choose for the simple approach and tune a second order low pass filter with two real poles. The selection of the poles is chosen such that the we maximize the bandwidth of the controlled output, given the restrictions at the controller outputs and the interaction on output two. This resulted in a behavior at the process outputs as given in figure 5.5.3. Both the case with no compensation of the influence of control of output one and the above discussed maximum reduction case are given for the second output. From

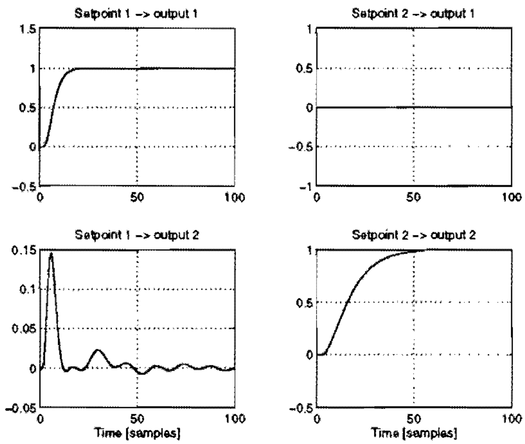

Fig.5.5.5 The step response of the closed loop.

figure 5.5.4b we see that the performance is restricted by the magnitude of the controller output between 0.1/sample and 0.2/sample. It is noted here that if an amplification of the disturbance outside the bandwidth, with a factor two is too large also the non-minimum phase zeros introduced by the down square operation can be a bottle neck. Figure 5.5.4 also reveals that if a maximum compensation of the influence on output two is applied, the restrictions on the controller outputs are only slightly exceeded at controller output one for a frequency close to 0.2/sample. There is no additional trade-off between the setpoint behavior at output one and output two and the gain behavior at the controller output, since the violation is not significant and the restrictions are not hard constraints as discussed in chapter three. $A_1^2(s)$ is therefore chosen equal to the expression in equation (5.5.7a). We now only need to design the controller for the setpoint of output two. The design is performed basically in the same way as the design for output one. The step responses of the overall system are given in figure 5.5.5.
End of example.

In this section we sketched an approach that also incorporates stability of the controller in the analysis. The procedure is not the only one possible.

Certainly at a more detailed level variations are possible. The fact that we gradually build up the knowledge and understanding of the process behavior and the relation it has to the requirements is essential in the above approach. It results in a good understanding of the different trade-off's we have to make and where the closed loop behavior is sensitive for differences between model and process behavior and changes in the requirement. The level of detail of the approach presented here makes that it is advisable to first apply an initial analysis, e.g. first apply some of the basic analysis as discussed in chapter three. As a second step apply a simple not too detailed analysis based on the approach as developed in section 5.3. The level of detail and the corresponding complexity of the approach moreover implies that the procedure should only be applied to this level of accuracy if the model and requirements are accurate. As a further consequence of the level of detail it is not always completely possible to foresee what the exact consequences of a certain choice on the lower priorities are. The procedure however enables the user to easily trace back the cause, if a problem occurs at a certain step in the procedure. It are these basic properties that are essential for a good controllability analysis procedure. All techniques developed so far are based on the frequency domain. In section 5.6 we take a closer look at the finite time domain.

5.6 Controllability analysis in the finite time domain.

The controllability analysis techniques we developed in chapter three and the preceding sections of this chapter are completely based on a frequency domain analysis. As a consequence of this observation, one could state that the value of the analysis for time domain design techniques is limited. From a strict theoretical point of view these techniques can only be used for controller design techniques that are based on an unconstrained quadratic criterion over an infinite time horizon, e.g. LQ-type of techniques, based on the Parseval theorem. On the other hand we discussed in chapter one, that an important class of MIMO controllers are based on optimization of a finite time horizon criterion under constraints. Model Predictive Controllers are the only MIMO controllers that are frequently used in industry and have become a more or less standard tool in petrochemical and chemical industry. The applicability of the developed techniques seems therefore seriously limited. Remember however that the major goal of the development was to obtain insight in how the process behavior restricts control of the process in accordance with required behavior. These restrictions are thus determined by the physics and the requirements we pose on the unit and independent of the techniques used to obtain this insight. The principle problem is therefore a translation problem, in the sense that we have to translate the restrictions as they manifest themselves in the frequency domain to the finite time domain. Important is to obtain insight in how the restrictions manifest themselves in the finite time domain and how they limit controllability. In this section we will develop this insight from an engineering point of view. In section 5.6.1 we will first formulate the control problem for the finite time horizon. Based on this formulation we will then discuss the basic idea to follow to develop the controllability analysis approach. The approach makes use of the concepts we developed in preceding sections. In section 5.6.2 we first discuss how non-minimum phase and structural effects manifest themselves in the finite time domain and how they restrict the controllability in this domain. In section 5.6.3 we will then develop the overall approach. In the next subsection the example of section 5.5 will then be used to further clarify the approach. In the last subsection, section 5.6.5, we will discuss the results we obtained.

Remark:

It is emphasized that in this section the concepts of inner, structural co-inner and outer are used as they were defined in the previous chapters, i.e. on the infinite horizon, since we want to relate infinite time properties to finite time behavior[1].

5.6.1 The basic idea

In this section we will start to develop a time domain approach that enables us to deal with the finite time horizon control problem. The behavior of the model at the outputs from time instant k to $k+N_y$, say $Y(k,k+N_y)$, can be determined from:

$$Y(k,k+N_y)=\begin{bmatrix} y(k) \\ y(k+1) \\ . \\ . \\ y(k+N_y) \end{bmatrix} = O(N_y)x(k)+T(N_y,N_u)U(k,k+N_u) \quad (5.6.1)$$

with $O(N_y)$ and $T(N_y,N_u)$ the finite observability matrix respectively the input-output Toeplitz matrix:

$$O(N_y)=\begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{N_y} \end{bmatrix} \text{ respectively } T(N_y,N_y)=\begin{bmatrix} D & 0 & . & . & 0 \\ CB & D & 0 & & . \\ . & . & . & . & . \\ . & & . & D & 0 \\ CA^{N_y-1}B & . & . & CB & D \end{bmatrix},$$

where we assumed for notational convenience we assume that $N_u=N_y$.

From chapter two we have seen that Model Predictive Control strategies are all based on this relation. The idea was to find at time instant k an input sequence $U(k,k+N_u)$ that minimized the difference between a desired future output behavior and the actual output behavior in some sense. The number of samples over which we predict the output, N_y, is called the *prediction horizon* and the number of samples over which we determine the best input sequence, N_u, is called the *control horizon* (chapter two). In chapter two we discussed the quadratic criterion:

[1] It is possible to define systems that have these same properties over a finite time horizon.

$$\min_{U(k,k+N_p)} \left\| \Delta Y^T W_y \Delta Y + U^T(k,k+N_p) W_u U(k,k+N_p) \right\|_2 \qquad (5.6.2)$$

where:

- $\Delta Y = Y_d(k,k+N_y) - Y(k,k+N_y) =$
$$Y_d(N_y) - O(N_y)x(k) - T(N_y,N_u)U(k,k+N_u)$$
- $Y_d(N_y)$, W_y, W_u, N_y and N_u are tuning variables, whose use and tuning will depend on the specific approach chosen.

In general the criterion is solved under certain restrictions. Most of these restrictions were posed in the form of constraints on the solution of the criterion, e.g. constraints on the magnitude and rate of change of the control signals. After the solution was obtained only $u(k)$ was actually send to the process. The optimization was then again repeated at the next time instant. One of the problems encountered in the above approach was to ensure nominal stability of the controller, i.e. how to deal with non-minimum phase behavior. In case all constraints have a minimum phase behavior, the problem was solved by introducing a final penalty term in the criterion. One such an approach was given by Rawlings et al. [Raw92, Mus93], who introduced a quadratic penalty on the final states, i.e. $x^T(k+N_u)Ox(k+N_u)$, where O is the observability gramian.

If we assume that the number of inputs equals the number of outputs, $N_y=N_u$, W_y equals identity and W_u equals zero, then the best possible behavior at the model output is given by:

$$U(k,k+N_u) = \left(T(N_y,N_y)\right)^{-1}\left(Y_d(N_y) - O(N_y)x(k)\right) \qquad (5.6.3)$$

As was the case for the infinite time approach developed in the previous sections, this will in general not be possible, due to the above restrictions, like stability and constraints, posed on the above problem. However from the above equation we directly observe the crucial role of the Toeplitz matrix $T(N_y, N_u)$, which represents the response of the model to the input $U(k,k+N_u)$ over the finite horizon. To obtain insight in the controllability of the model we have to analyze the (approximate) invertability of the matrix $T(N_y, N_u)$. The central question for controllability analysis is therefore how to determine the approximate inverse that at one hand fulfills the restrictions and at the other hand approximates the desired output behavior best.

As was the case for the infinite time domain, we may expect that non-minimum phase zeros and a too small gain of the model will force us to choose an approximate inverse or generalized inverse of the Toeplitz matrix $T(N_y, N_u)$ To study the behavior of the Toeplitz matrix we will make use of the singular value decomposition and the LQ-decomposition. The singular value decomposition of $T(N_y, N_u)$ is:

$$T = U_T \Sigma_T V_T^T \tag{5.6.4}$$

where per definition:

- $U_T \in \Re^{pN_y x pN_y}$ is an orthogonal matrix, i.e. $U_T U_T^T = U_T^T U_T = I$
- $\Sigma_T \in \Re^{pN_y x mN_u}$ is a diagonal matrix with positive semidefinite entries, ordered in descending order, i.e. $\sigma_i \geq \sigma_{i+1} \geq 0$.
- $V_T \in \Re^{mN_u x mN_u}$ is an orthogonal matrix, i.e. $V_T V_T^T = V_T^T V_T = I$

The definition of the singular value decomposition is repeated here to help understand the following interpretation of the above matrices. The diagonal entries of the matrix Σ_T can be seen as the principal gains of the behavior, like in the frequency domain, i.e. the maximum gains, but of orthogonal signal patterns, in a space of time and direction. U_T represents the different corresponding signal pattrens at the outputs and the matrix V_T decomposes the input space in a set of orthogonal directions. Interesting is that the input respectively output space spanned by respectively U_T and V_T combine the directional behavior and the behavior in time.

In the next section we first take a closer look at how non-minimum phase behavior and non square behavior translates to the finite time domain. To establish this relation we will again make use of the inner and structural co-inner concepts, as defined in chapter three on the infinite time horizon.

5.6.2 Non-minimum phase behavior and inners in the finite time domain.

In this section we will take a closer look at the behavior of the input-output Toeplitz matrix of inner and co-inner systems. In the previous sections we have seen that the inner and co-inner systems play a crucial role in the understanding of nominal stability of the closed loop behavior. We will therefore use these concepts again in the approach here. The study of the Toeplitz matrices of these systems will result in a clear insight in how non-minimum phase behavior and non squareness manifest itself in the finite time domain and in how these effects restrict the process controllability. The concepts developed here will not only result in a good understanding of the restrictions on the finite time domain, but also result in a much better general understanding of these effects and their relation to the controllability of a process. It will result in a better relation between the infinite time domains and the finite time behavior. In fact this insight leads to a new definition of non-minimum phase behavior. This new insight will also result in a much better understanding of analytical trade-off's that occur for systems defined on a finite time horizon in controlling non-minimum phase zeros. It gives a direct and clear insight and understanding of the complex Bode sensitivity integral relation, i.e. the trade-off between bandwidth of the system and amplification outside the bandwidth. It couples more direct the almost zero concept, as introduced in section 5.4, with that of actual zeros.

As a start of this section let us take a closer look at the behavior of the square *NxN* Toeplitz matrix of a scalar inner function with only one zero and a realization $[A,B,C,D]$. Simulations with these types of systems resulted in a number of highly interesting observations. These observations are here stated in the form of a corollary, that will be a direct consequence of the theorem 5.6.1 to be stated after the initial discussion.

Corollary 5.6.1

Assume given a SISO inner transfer function with only one zero and a minimum realization $[A,B,C,D]$. The input-output Toeplitz matrix of this system is given by:

$$T(N,N)=\begin{bmatrix} D & 0 & . & . & 0 \\ CB & D & 0 & & . \\ . & . & . & . & . \\ . & & . & D & 0 \\ CA^{N-1}B & . & . & CB & D \end{bmatrix} \tag{5.6.5a}$$

The singular value decomposition of this Toeplitz matrix results in the following partitioning:

$$T(N,N)=\begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \mathrm{I}_{N-1} & 0 \\ 0 & \varepsilon \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{5.6.5b}$$

with:

- $U_2 \in \Re^{Nx1}$ and equals a scaled exponential function of the:

$$U_2 = \begin{bmatrix}1 & A & A^2 & . & . & A^N\end{bmatrix}^T \Big/ \left(\sum_{i=0}^{N} A^{2i}\right)^{1/2} \tag{5.6.5c}$$

- $U_1 \in \Re^{Nx(N-1)}$ spans the orthogonal complement of U_2.
- $I_{N-1} \in \Re^{(N-1)x(N-1)}$ is an $(N\text{-}1)x(N\text{-}1)$ identity matrix.
- $\varepsilon > 0$ is a small scalar value close to zero:

$$\varepsilon = \left|DA^N\right| \tag{5.6.5d}$$

- $V_2 \in \Re^{Nx1}$ and equals a scaled exponential function of the non-minimum phase zero of the system, i.e. A^{-1}:

$$V_2 = \begin{bmatrix}1 & A^{-1} & A^{-2} & . & . & A^{-N}\end{bmatrix}^T \Big/ \left(\sum_{i=0}^{N} A^{-2i}\right)^{1/2} \tag{5.6.5e}$$

- $V_1 \in \Re^{Nx(N-1)}$ spans the orthogonal complement of V_2.

Proof: See appendix 5.D.

Interesting to note is that the singular vectors corresponding to the singular value ε are exponential functions, corresponding to the value of the pole and the non-minimum phase zero, for the left, respectively right singular vector. The scalar ε .is a function of $|A|^N$. The actual distance of ε from zero therefore depends on the value of the pole, i.e. A and the number of samples N. ε is therefore a decreasing exponential function of the number of samples N, whose convergence rate is determined by the value of the pole. As a consequence we obtain that for increasing N the process gain in the input direction V_2 decreases to zero and it becomes more and more impossible to generate the signal pattern U_2 at the output of the process with sufficient amplitude. As we have seen the signal pattern V_2 is generated by the zero of the system. For $N=\infty$, ε equals zero. As a consequence V_2 equals the truncated zero input direction. The above singular value decomposition therefore seems to represent the finite time behavior of the non-minimum phase zero.

Example 5.6.1

Let us show the above effects for the Toeplitz matrix using an example of an inner with one pole. The behavior of the smallest singular value of the input-output Toeplitz matrix $T(N,N)$ as function of the dimension, i.e. N for poles at $\begin{bmatrix}0.9 & 0.99 & 0.999\end{bmatrix}$ is given in figure 5.6.1. It is easy verified that the smallest singular value of the Toeplitz matrix of a

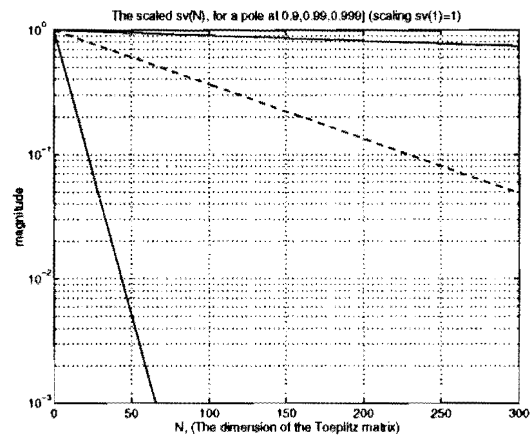

Fig.5.6.1 The smallest singular value of the Toeplitz matrix for poles at 0.9, 0.99 and 0.999, as function of the size of T.

negative pole has exactly the same behavior as the positive pole. The difference in behavior between a positive and negative pole is reflected in a different behavior of the singular vectors corresponding to the smallest singular value. The singular vector of a negative pole is an alternating exponential function. This is illustrated in figure 5.6.2 for poles at $\begin{bmatrix}-0.9 & 0 & 0.9\end{bmatrix}$ and a Toeplitz matrix with N=300. Note that the left singular vectors correspond to the stable pole and the right singular vectors to the non-minimum phase zero.

End of example

Assume we have more outputs than inputs. In this case it was observed that like for the square inner (N-n) singular values equaled one again. The n other singular values where smaller than one. Their behavior as function of the number of samples in the Toeplitz matrix was however different. As for the square inner they seemed to be a non-decreasing function of the number of samples, N. They however did not asymptotically converge to zero, but to

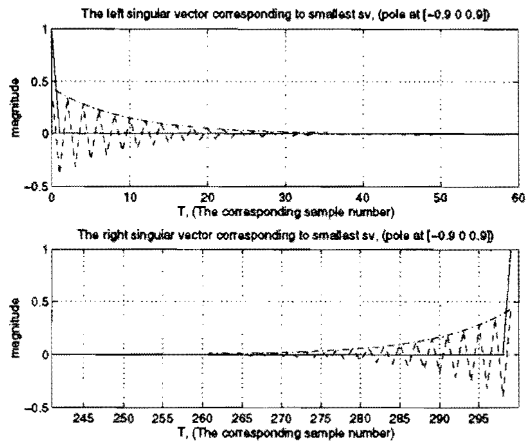

Fig.5.6.2 Behavior of the last left & right singular vector of T(300,300) for poles at -0.9, 0 & 0.9.

constant values. As it turned out these values equal the square root of the eigenvalues of I-PQ. Let us give an example to show the effects observed.

Example 5.6.2

As an example we will take a closer look at the behavior of the finite Toeplitz matrix of the structural co-inner of output two of the scaled delayless glass tube model as also used in section 5.4.2 and example 5.5.1. As was the case for the square inner it was observed that all singular values of $T(N,N)$ were equal to one, except for the last n singular values, with n the dimension of the state vector for a minimal realization. In contrast to the inner these last eleven singular values do not converge to zero as a function of N, but to a constant value (figure 5.6.3). It was found that the singular values of the Toeplitz matrix asymptotically converge, as function of N, to the eigenvalues of $(I\text{-}PQ)^{1/2}$, with Q and P the observability respectively controllability gramian of the realization of the structural co-inner (figure 5.6.4). Remember from section 5.4 that the i-th singular value exactly correspond to the inverse of the minimum infinity norm of any precompensator that results in a square inner with exactly (i-1) non-minimum phase zeros after down squaring. The corresponding right and left singular vectors are given in figure 5.6.5 respectively figure 5.6.6. From these figures we observe that both the corresponding left and the right singular vectors are decreasing functions.

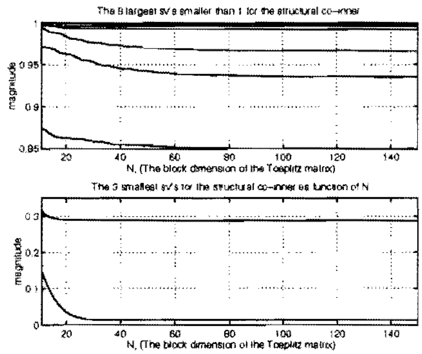

Fig.5.6.3 The behavior of the 11 smallest singular values of T(300,300) as function of N. a) 8 largest singular values. b) 3 smallest singular values (the smallest equals 3.5 10^{-5}).

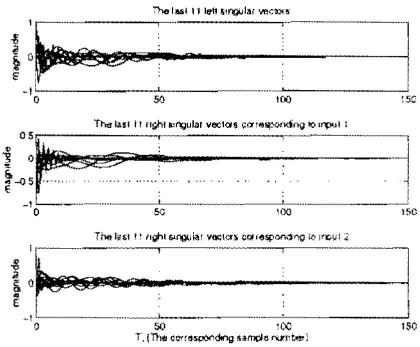

Fig.5.6.4 The left and the right singular vector related to the smallest singular value of T(300,300) as function of N. plot a) left singular vector. plot b) right singular vector input 1. plot c) input 2.

As a consequence the Toeplitz matrix has an inverse with finite singular values and the singular vectors corresponding to the singular values larger than one are again asymptotically decreasing functions. The inverse therefore is a stable function. This is an interesting observation, since the inverse of the realization is an antistable function as seen in section 5.4.

End of example

An interesting question is whether we can better understand the above observed remarkable behavior of the singular values of the Toeplitz matrix for inner systems, i.e. $p \geq m$. To do this remember that an inner transfer is nondissipative. Hence, all energy entering the system therefore has to leave the system "sooner or later". More formally expressed:

$$\left\{\forall u(t) \in l_+^2 \mid \|y(t)\|_2 = \|u(t)\|_2\right\} \tag{5.6.6a}$$

Assume given a *pxm* inner with $p \geq m$ and a minimum realization $[A,B,C,D]$. A direct consequence of the nondissipative is that for any input signal and any N, we must have:

$$\sum_{i=0}^{N} u^T(i)u(i) = \sum_{i=0}^{N} y^T(i)y(i) + \sum_{i=N+1}^{\infty} y^T(i)y(i) \tag{5.6.6b}$$

If we want to find the part of the system that is responsible for the smaller singular values we need to search for those input signals that transfer minimum energy to the output from time instant $i=0$ to time instant $i=N$, which means minimizing the first term on the right hand side of the above equation. Minimizing this term is equivalent to maximizing the second term. This last maximization is however equivalent to the maximization of the energy transfer from the past to the future, i.e. finding the Hankel singular values of the Hankel matrix for a finite length input signal. We must find those input signals that result in a maximum amplification by this Hankel operator. It is this subspace of the input space that spans the subspace of the right singular vectors corresponding to the small singular values of the Toeplitz matrix. If we obtained this subset of input signals, we can study how they are transformed via the Toeplitz matrix to the output of the system. Based on this idea we were able to obtain an expression of the behavior of the singular values and right singular vectors of the finite square input-output Toeplitz matrix of arbitrary length in the next theorem.

Theorem 5.6.1

Given a *pxm* inner transfer with minimum realization $[A,B,C,D]$, n the dimension of the A matrix and Q the observability gramian. For any length $N>n$, we obtain for the singular value decomposition of the input-output Toeplitz matrix $T(N,N)$, constructed as in equation (5.6.5a)

$$T(N,N) = U_T \Sigma_T V_T^T = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix} \begin{bmatrix} I_{mN-n} & 0 \\ 0 & E \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{5.6.7a}$$

with:

- $V_2 \in \Re^{mNxn}$ with $V_2^T V_2 = I$ are the right singular vectors corresponding to the n singular values smaller than one:

$$V_2^T = W_T^T \left(N_c(N,0) N_c^T(N,0) \right)^{-1/2} N_c(N,0) \tag{5.6.7b}$$

where the matrix $N_c(N,0)$ equals the finite controllability matrix as defined in section 2.3.

- $V_1 \in \Re^{mNx(mN-n)}$ with and $V_1^T V_1 = I$ spans the orthogonal complement of V_2, i.e. the subspace of the right singular vector corresponding to singular values equal to one.
- $I_{mN-n} \in \Re^{(mN_y-n)x(mN_y-n)}$ is an identity matrix of the dimension *(mN-n)*
- The matrices Z_T, W_T and E are determined by the singular value decomposition:

$$\left(I - Q(N_c(N,0)N_c^T(N,0))\right)^{1/2} = Z_T E W_T^T \qquad (5.6.7c)$$

with Z_T, W_T the matrices containing the left and right singular vector and E a diagonal matrix containing the singular values on the diagonal.

- $U_1 \in \Re^{pNx(mN-n)}$ with $U_1^T U_1 = I$ and spanning the left singular vectors corresponding to the singular values equal to one.
- $U_2 \in \Re^{pNxn}$ with $U_2^T U_2 = I$ are the left singular vectors corresponding to the n singular values smaller than one. U_2 is spanned by the columns of the matrix:

$$-\begin{bmatrix} C \\ CA \\ . \\ . \\ CA^N \end{bmatrix} P(A^T)^{N+1} + \begin{bmatrix} (CPA^T + DB^T)(A^T)^N \\ (CPA^T + DB^T)(A^T)^{N-1} \\ . \\ . \\ (CPA^T + DB^T) \end{bmatrix} \qquad (5.6.7d)$$

- $U_3 \in \Re^{pNx(p-m)N}$ with $U_3^T U_3 = I$ and spanning the left singular vectors corresponding to the complementary range of $T(N,N)$.

Proof: See appendix 5.D

Remarks:

1. Note that in the case of a square transfer matrix, i.e. $m=p$, the expression in equation (5.6.7d) simplifies to minus the observability matrix: $-N_o(0,N) = -\begin{bmatrix} C^T & (CA)^T & . & (CA^N)^T \end{bmatrix}^T$, due to the fact that for a square inner we have $DB^T = -CPA^T$.
2. It is a known fact that the right singular vector matrix of the Hankel matrix are spanned by the columns of the observability matrix. As a

consequence of this observation we see that the sign of the output signal for a square inner after the control horizon is opposite to that of the output over the control horizon. If we extend the horizon with one sample we need to increase the amplitude of the input signal with respect to the previous sample since we need to compensate for the effects of the previous inputs on the output in the last sample. However the amount of energy accumulated in the system will also increase and result in a larger effect on the output after the horizon. This implies again that the magnitude of the control signal has to be larger for each successive sample. The magnitude of the control signal will therefore increase with increasing horizon. This explains why these n_z singular values asymptotically approach zero as a function of the length of the control horizon. Note that this effect does not occur in the case of a non-square inner. In this case we see that the longer the horizon becomes the more the last column in equation (5.6.7e) will dominate the behavior of the output at the end of the control horizon. At a certain moment the sign will become positive and equal to that after the control horizon. The behavior at the end of the control horizon will then equal the sign of the output signal after the control horizon. Increasing the control horizon will not result in unstable behavior of the square inner. Hence, the singular values will converge asymptotically to a nonzero value.

3. The change in sign at the end of the control horizon which occurs in the Toeplitz matrix of a square inner may well be a property that characterizes the behavior of non-minimum phase zeros. In this case it might be a good characterization of non-minimum phase behavior in a general Toeplitz matrix. It would enable in this case an easy check for non-minimum phase behavior in a general finite Toeplitz matrix. Further research is needed to check if this observation is correct.
4. We have chosen to study the inner, $p \geq m$, because we than can directly apply the above reasoning to extend the theory. The co-inner, i.e. the case with more inputs than outputs, is the dual of the above problem.

The Toeplitz matrix enables us to analyze the behavior of non-minimum phase zeros and almost zeros in the finite time domain. Over a finite horizon it is theoretically possible to invert the complete Toeplitz matrix. The costs in terms of energy of the generation of these output signal patterns are however high. First of all the amount of energy needed to generate the signal pattern with sufficient energy over the finite horizon will be high and exponentially increasing as a function of the location of non-minimum phase zeros in the complex plane. The energy not coming out the system over the finite horizon is however accumulated in the process and will therefore come out of the system after this horizon. The effect control of these output directions over the finite horizon has on the behavior of the system after this time horizon is therefore also exponentially increasing with the time horizon over which it is controlled. Over time horizons that are long with respect to the time constant related to the non-minimum phase zeros, it will therefore be impossible to generate these signal patterns with significant energy. Both the energy needed to generate it and the consequences after the time horizon will be too large to be realistic. From a practical point of view control of the output over longer time horizons is therefore restricted to the space U_1. We will discuss this further in section 5.6.3. An important consequence of the discussion is that the above factorization results in a very nice and unique characterization of a non-minimum phase zero in the finite time domain. Moreover this characterization is consistent with the infinite time behavior, since for $N \to \infty$ the well known infinite Toeplitz matrix of the system is obtained.

In lemma 5.4.6 we found that for a co-inner process the gain of the precompensator introducing $(i-1)$ non-minimum phase zeros in the square downed process is always larger than or equal to the i-th eigenvalue of $(I - QP)^{-1/2}$. The singular values of the Toeplitz matrix $T(N,N)$, with $N \to \infty$, that are unequal to one, have exactly the same singular values. This characterization clearly reveals the behavior of almost zeros and the close relation they have to actual non-minimum phase zeros. It also directly reveals why introducing a non-minimum phase zero after down squaring may reduce the minimum achievable infinity norm of the precompensator. In this case we ensure that the controller is not using a certain subspace of the input space of the Toeplitz matrix. The more this direction coincides with the direction of the smallest singular value of the Toeplitz matrix the more effect it will have on the minimum achievable maximum gain of the controller, i.e. on its largest singular value, since the controller equals the pseudo inverse of the Toeplitz of the model, were we did not invert the model in the direction of the zero.

Remark:

A direct consequence of this relation is that the left singular vector corresponding to the smallest singular value is always generated by exactly one non-minimum phase zero. Equivalently the output direction of the minimum singular value of the Toeplitz matrix is always spanned by the inverse of one exactly one non-minimum phase zero. The left singular vectors of the two smallest singular values are always spanned by exactly two of these zeros and so on.

From the above theorem we thus obtained a connection between the frequency domain definition of non-minimum phase zeros and their behavior in the time domain. It not only gives an infinite time characterization, but also a clear characterization over a finite horizon, where the infinite time is just the limiting situation of the finite time characterization. It is even worthwhile to consider this characterization as the definition of non-minimum phase behavior. It directly relates the restrictions that non-minimum phase zeros pose on the controllability to certain subspaces that can not be controlled. A basis for this subspace is spanned by the exponential behavior of the zeros. As a consequence of not being able to control these directions, a subspace of the output is not reachable anymore. Note that characterizing the limitation in this way is much clearer than the frequency domain characterization, were the Bode sensitivity integral relation was needed.

A topic of interest not yet explicitly discussed is how directional behavior of the MIMO system appears in the input-output Toeplitz matrix. In this part we will focus attention to the output space, i.e. the left singular vectors of the Toeplitz matrix. For the right singular vectors, i.e. the input space exactly the same procedure can be followed. The problem and the way of solving it is to a large extend equivalent to the discussion in section 3.5.

We obtained from the above discussion that it was impossible to invert the input-output Toeplitz matrix of the model in the direction of the non-minimum phase zeros over a longer time horizon. First of all the energy needed to invert the Toeplitz matrix, exponentially increases with the horizon N_u in the zero direction. The amplitudes of the input signals needed will grow unacceptably. Secondly we have seen that the energy accumulated in the system will also grow exponentially as function N_u. This energy will more and more dominate the behavior of the model after the time horizon and rapidly become unacceptable. As a consequence it will in general not be possible to control the process in these directions. As in section 3.5 we can ask ourselves in how far we are able to turn the influence of these zeros away from the most important outputs and what the consequences are.

To fascilate the analysis of the directional behavior we in general make use of an ordering in the input-output Toeplitz matrix equivalent to the prioritized strategy we introduced in the previous section. Hence we obtain for a system with p outputs and m inputs, the following Toeplitz matrix $T(N_y, N_y)$:

$$\begin{bmatrix} T_{1,1}(N_y,N_y) & T_{1,2}(N_y,N_y) & . & . & T_{1,m}(N_y,N_y) \\ T_{2,1}(N_y,N_y) & T_{2,2}(N_y,N_y) & & & . \\ . & & . & & . \\ . & & & . & T_{p-1,m}(N_y,N_y) \\ T_{p,1}(N_y,N_y) & . & . & T_{p,m-1}(N_y,N_y) & T_{p,m}(N_y,N_y) \end{bmatrix} \tag{5.6.8}$$

with $T_{ij}(N_y,N_u)$ the input-output Toeplitz matrix from input j to output i.

Take the control and prediction horizon equal to N. The idea is to find a controller that turns the influence of the zeros to a more favorable direction, without changing the structure of the zeros or equivalently:

$$T_T(N,N) = T_I(N,N)T_C(N,N) \tag{5.6.9}$$

with:

- $T_I(N,N)$ the input-output Toeplitz matrix of the inner.
- $T_T(N,N)$ the input-output Toeplitz matrix of the system after turning the influence of the zeros.
- $T_C(N,N)$ the input-output Toeplitz matrix of the controller that turns the influence of the zeros.

It is of course essential that we do not change the zero structure. The Toeplitz matrix $T_T(N,N)$ should therefore have the same zeros structure as $T_I(N,N)$, i.e. have the same number of singular values and singular left vectors that converge to zero for an increasing number of samples. To enable us to obtain insight in the possibilities and consequences of turning the influence of zeros in the output space basically the same approach is followed as in section 3.5. We first take a closer look at the case that we only have two outputs, to obtain a good understanding of what can be achieved and what the consequences are of these actions. From the two output case it is straightforward to extend the results to more outputs.

Theorem 5.6.2

Given a square two input two output inner, with n non-minimum phase zeros. If the time horizon N of the block square Toeplitz matrix $T_I(N,N)$ is larger than or equal to the number of non-minimum phase zeros n, i.e. $N \geq n$, then there exist a singular value decomposition of this Toeplitz matrix $T_I(N,N)$ of the form:

$$\begin{bmatrix} Z_{11} & 0 & Z_{12}\Sigma_c & -Z_{12}\Sigma_n \\ 0 & Z_{21} & Z_{22}\Sigma_n & Z_{22}\Sigma_c \end{bmatrix} \begin{bmatrix} I_{N-n} & 0 & 0 & 0 \\ 0 & I_{N-n} & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & V_X^T E V_X \end{bmatrix} \begin{bmatrix} \tilde{V}_1^T \\ \tilde{V}_3^T \\ \tilde{V}_2^T \\ \tilde{V}_4^T \end{bmatrix} \tag{5.6.10a}$$

with:

- $Z_{ij} \in \Re^{(N-n)xn_j}$ an orthogonal matrix, i.e. $Z_{ij}^T Z_{ij} = I$
- $\Sigma_n \in \Re^{nxn}$ a diagonal matrix with diagonal entries σ_i in descending order and $\sigma_i \geq 0$.
- $\Sigma_c = (I - \Sigma_n^2)^{1/2}$

A parameterization of a set of controller matrices that do not use the input space related to the small singular values in E and change the influence the zeros have on the different outputs is given by:

$$T_C(N,N) = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_3 & \tilde{V}_2 & \tilde{V}_4 \end{bmatrix} \begin{bmatrix} I_{N-n} & 0 & 0 & 0 \\ 0 & I_{N-n} & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_{11}^T & 0 \\ 0 & Z_{21}^T \\ Z_{12}^T & 0 \\ 0 & Z_{22}^T \end{bmatrix} \tag{5.6.10b}$$

with:

- $\begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} I & \Lambda \end{bmatrix} \begin{bmatrix} (I - \Sigma_n^2)^{1/2} & \Sigma_n \\ -\Sigma_n & (I - \Sigma_n^2)^{1/2} \end{bmatrix}$
- $\Lambda \in \Re^{nxn}$ the free parameter.

The resulting transfer $T_T(N,N)=T_I(N,N)T_C(N,N)$ is given by:

$$T_T(l,l)=\begin{bmatrix} Z_{11}Z_{11}^T & 0 \\ 0 & Z_{21}Z_{21}^T \end{bmatrix}+\begin{bmatrix} Z_{12} & 0 \\ 0 & Z_{22} \end{bmatrix}\begin{bmatrix} (I-\Sigma_n^2)^{1/2} \\ \Sigma_n \end{bmatrix}\begin{bmatrix} X_1 & X_2 \end{bmatrix}\begin{bmatrix} Z_{12}^T & 0 \\ 0 & Z_{22}^T \end{bmatrix} \tag{5.6.10c}$$

Proof: See appendix 5.D.

Let us first illustrate the theorem with three examples, before making a start with the discussion:

1. The inner transfer matrix is of course obtained if we choose $\Lambda=0$ or equivalently
   $$\begin{bmatrix} X_1 & X_2 \end{bmatrix}=\begin{bmatrix} (I-\Sigma_n^2)^{1/2} & \Sigma_n \end{bmatrix}$$
2. If we choose $\Lambda=(I-\Sigma_n^2)^{-1/2}\Sigma_n$,i.e. $\begin{bmatrix} X_1 & X_2 \end{bmatrix}=\begin{bmatrix} (I-\Sigma_n^2)^{-1/2} & 0 \end{bmatrix}$ the influence of the zero is completely removed from the first output. At the cost of an increased interaction on the second output.
3. If we choose $\Lambda=(I-\Sigma_n^2)^{1/2}\Sigma_n^{-1}$ or equivalently $\begin{bmatrix} X_1 & X_2 \end{bmatrix}=\begin{bmatrix} 0 & \Sigma_n^{-1} \end{bmatrix}$ the influence of the zero is completely removed from the second output. At the cost of an increased interaction on the first output.

Remarks:

1. In theorem 5.6.2 we made the assumption that the number of samples N is larger than the number of zeros, i.e. $N>n$ and n the dimension of E. This is in fact a technical assumption needed to use lemma 5.C.1 in the proof of the theorem. This assumption can be removed. In general however it is not a restriction, since we are interested in the behavior for longer horizons.
2. In theorem 5.6.2 we excluded the singular directions V_4 corresponding to the singular values smaller than one from the control actions. We argued that this ensured nominal stability. The essence of the theorem 5.6.2 is not that we should exclude these directions from the control action, but that we should not try to invert the process behavior over the prediction horizon. As we will discuss later really excluding these directions from the control actions, i.e. choosing the gain zero, also implies that the states of the system equal zero at time $N+1$. This is undesirable from a control point of view. Another choice of the gain

in these directions will therefore be suggested. In this subsection we will however assume the gain in this direction to equal zero.
3. An important difference with section 3.5 is that we need not extract the zeros separately. All zeros can directly be dealt with in one step.
4. Note that for the right singular vector matrix an equal structure can be found as in theorem 5.4.2 for the left singular vector matrix. This theorem is in fact the dual of theorem 5.4.2. It will however in general not be possible to obtain a singular value decomposition in which both the left and the right singular vector have the structure of theorem 5.4.2.

Let us take a closer look at results found in the theorem. Especially the left singular vector space obtained from the decomposition in the theorem is of interest. From this matrix we directly obtain that per output a subspace Z_{i1} of the output space is related to the singular values equal to one that is only related to this output i. For this subspace, i.e. these signal patterns, no trade-off between performance at different outputs is necessary. The complementary subspace of the above subspace, related to the singular values equal to one, is of more importance. It determines the trade-off between the different outputs and is closely related to the subspace related to the non-minimum phase behavior of the process. This subspace is related to the squaring down problem: Assume we are only interested in output one. Equation (5.6.10a) reduces to $T_{011}(N,N)$:

$$\begin{bmatrix} Z_{11} & 0 & Z_{12}(I-\Sigma_n^2)^{1/2} & -Z_{12}\Sigma_n \end{bmatrix} \begin{bmatrix} I_{l-n} & 0 & 0 & 0 \\ 0 & I_{l-n} & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & V_X^T E V_X \end{bmatrix} \begin{bmatrix} \tilde{V}_1^T \\ \tilde{V}_3^T \\ \tilde{V}_2^T \\ \tilde{V}_4^T \end{bmatrix} \qquad (5.6.11a)$$

Using the orthogonal transformation $\begin{bmatrix} (I-\Sigma_n^2)^{1/2} & \Sigma_n \\ -\Sigma_n & (I-\Sigma_n^2)^{1/2} \end{bmatrix}$ on the last two columns in the left singular vector matrix results in the singular value decomposition $T_{101}(N,N)$ equals:

$$\begin{bmatrix} Z_{11} & Z_{12} \end{bmatrix} \begin{bmatrix} I_{l-n} & 0 & 0 \\ 0 & \left(I_n + \Sigma_n V_X^T (EE^T - I_n) V_X \Sigma_n\right)^{1/2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_1^T \\ \tilde{V}_{N_1^{\perp}}^T \\ \tilde{V}_{N_1}^T \end{bmatrix} \qquad (5.6.11b)$$

with:

- $\tilde{V}_{N_1^\perp}^T = \left(I_n + \Sigma_n V_X^T (EE^T - I_n) V_X \Sigma_n\right)^{-1/2} \begin{bmatrix}(I-\Sigma_n^2)^{1/2} & -\Sigma_n V_X^T E V_X\end{bmatrix} \begin{bmatrix}\tilde{V}_2^T \\ \tilde{V}_4^T\end{bmatrix}$

- $\tilde{V}_{N_1}$ is equal to a orthogonal matrix whose columns span the orthogonal complement of $\begin{bmatrix}\tilde{V}_1 & \tilde{V}_{N_1^\perp}\end{bmatrix}$

A second point of interest is the relation that Σ_n must have with Q_1, the observability gramian for output one. From theorem 5.6.1 we know that the left singular vectors corresponding to the singular values less then one are spanned by the columns of the observability matrix, which equals using the convention as in equation (5.6.8):

$$\begin{bmatrix} -Z_{12}\Sigma_n \tilde{V}_X^T \\ Z_{22}(I-\Sigma_n^2)^{1/2}\tilde{V}_X^T \end{bmatrix} = \begin{bmatrix} N_{o1}(0,N) \\ N_{o2}(0,N) \end{bmatrix} \left(N_{o1}^T(0,N) N_{o1}(0,N) + N_{o2}^T(0,N) N_{o2}(0,N)\right)^{-1/2} \tag{5.6.12a}$$

with $N_{oi}(0,N)$ the observability matrix for the i-th output:

$$N_{oi}(0,N) = \begin{bmatrix} C_i^T & . & . & \left(C_i A^l\right)^T \end{bmatrix}^T$$

As a consequence we obtain that Z_{1i} is spanned by the columns of $N_{oi}(0,N)$ and the diagonal entries of Σ_n equal the singular values of

$$\left(\left(N_{o1}^T(0,N) N_{o1}(0,N)\right)\left(N_{o1}^T(0,N) N_{o1}(0,N) + N_{o2}^T(0,N) N_{o2}(0,N)\right)^{-1}\right)^{1/2},$$

i.e.:

$$\lambda_i\left(\left(N_{o1}^T(0,N) N_{o1}(0,N)\right)\left(N_{o1}^T(0,N) N_{o1}(0,N) + N_{o2}^T(0,l) N_{o2}(0,N)\right)^{-1}\right)^{1/2} \tag{5.6.12b}$$

Remark:

1. From the discussion for a non-square inner we moreover obtained that for $N \to \infty$ the singular values unequal to one asymptotically approach the square root of the eigenvalues of $(I - Q_1 P)^{1/2}$. From equation (5.6.11b) we see that the singular values smaller than one approach

$(I-\Sigma_n^2)^{1/2}$, since $EE^T \to I$ for $N \to \infty$. Moreover
$N_{o1}^T(0,N)N_{o1}(0,N) \to Q_1$ and
$N_{o1}^T(0,N)N_{o1}(0,N) + N_{o2}^T(0,N)N_{o2}(0,N) \to Q$ for $N \to \infty$. Together with the fact that $P=Q^{-1}$ for a square inner we indeed obtain for $N \to \infty$ that the eigenvalues of $\left(I_n + \Sigma_n V_X^T(EE^T - I_n)V_X\Sigma_n\right)$ approach the eigenvalues of $(I-Q_1P)$.

2. The whole procedure can be extended to the case general case of p outputs, without fundamental differences with respect to the two output case (see appendix C).
3. The whole discussion here has been focused on the left singular vectors. This procedure can also be applied to the right singular vectors. This case is in fact dual to the above, i.e. apply the above theory on a system with a minimum realization $[A^T, C^T, B^T, D^T]$.

Example 5.6.3

Let us take a look at a *2x2* square inner, obtained from the *2x2* scaled tube process in example 5.5.1. Application of the decomposition in theorem 5.6.2 on the Toeplitz of the inner. with N chosen sufficiently large with respect to the dynamics, results in four singular values smaller

NMP coupling with:	**state 1**	**state 2**	**state 3**	**state 4**
output 1 (Σ_n)	1.0000	0.9995	0.5353	0.0097
output 2 $((I-\Sigma_n^2)^{1/2})$	0.0001	0.0322	0.8447	1.0000

Table 5.6.1 Coupling of the non-minimum phase effects to output one (first row) and output two (second row).

than one. Consequently we have four non-minimum phase zeros. The coupling of the zeros to output one and output two, Σ_n respectively $(I-\Sigma_n^2)^{1/2}$ of the zero patterns are given in the first and second column of table 5.6.1. From this table we obtain that the effect of two non-minimum phase zeros is mainly coupled to output one and the other two are more coupled to output two. The complete removal of the influence of the non-minimum phase behavior from output one, can only be achieved at the cost of a large control effort and a large influence on the second output. The same holds if we try to completely cancel the non-minimum phase behavior from output two (table 5.6.2) It is therefore not realistic to try to completely remove the influence from one of the outputs. A compromise has to be made. If we assume again, as in the previous examples that output one is more important than output two. A realistic choice then seems to be to only turn two of the non-minimum phase effects away from output one and leave the other two effects unaffected. We then obtain for Λ:

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & (I-\Sigma_2)^{-1/2}\Sigma_2 \end{bmatrix}$$

with Σ_2 a diagonal matrix whose diagonal entries equal the smallest two singular values of Σ_n. In figure 5.6.5 the step responses of the controlled process are given. As a result we obtain from the step response that there is no complete decoupling anymore, since there is a small influence of control of output two on output one. The influence of control of output one on output two is however also restricted.

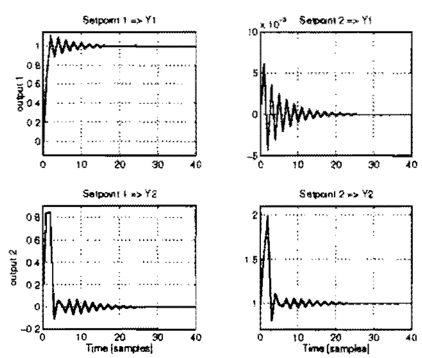

Fig.5.6.5 Step response of the controlled inner with 4 non-minimum phase zeros.

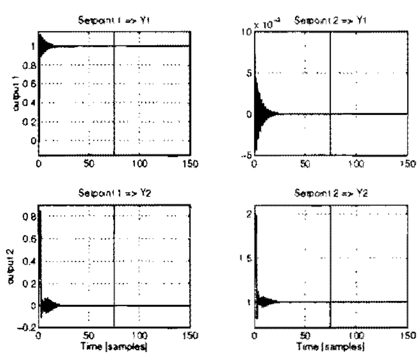

Fig.5.6.6 Step response of the controlled inner with constraint that steady state is reached at T=75 samples.

From theorem 5.6.1 we obtain that the right singular vectors related to the singular values smaller than one of the inner span the same subspace as the rows of the finite controllability matrix. Completely excluding this subspace from the controller consequently results in a zero state vector at time N in the future. This of course enforces stability of the controller. It however does not result in a desirable behavior in the far future, i.e. after time N. If we assume that the situation is unchanged after time N, we would like the controlled process to be in the steady state corresponding to the desired output behavior. This is however in general not achieved with a steady state value of the states equal to zero. More desirable is to bring the states to the steady state corresponding with the output. We therefore propose to make the states at $N+1$ equal to the steady state values corresponding with the desired output value. In this case applying the corresponding steady state input after time N will result in the steady state behavior (figure 5.6.6). As we will discuss in section 5.6.3 this also guarantees stability.

End of example.

	state 1	state 2	state 3	state 4
Σ_n^{-1}	8982	31.037	1.1839	1.0000
$(I-\Sigma_n^2)^{1/2}\Sigma_n^{-1}$	8982	31.021	0.6337	0.0097

Table 5.6.2 Additional control effort needed to turn the nmp effects away from output two (row one) & the effect this has on output one (second row).

In the above discussion we assumed the inner to be square, i.e. have an equal number of inputs and outputs. In the case that we have more outputs than inputs, $p>m$, the structure of the left singular vector matrix U is more complex and looses some of its nice properties. The additional complexity of the U matrix is due to the existence of a complementary range. The range of the Toeplitz operator is therefore only a subspace of the overall output space $l^2_+(N)^p$. This complementary range is spanned in equation (5.6.7a) by the columns of U_3. For controller design this means that we are not able anymore to reach the complete output space. As a consequence it is not possible anymore to freely assign the output behavior of each output. A coupling between the outputs occurs. Assume we partition the inner system $M(z)\in \Re H_\infty^{p\times m}$, with $p>m$, at the output in two parts:

$$M(z)=\begin{bmatrix} M_1(z) \\ M_2(z) \end{bmatrix} \tag{5.6.13}$$

with: $M_1(z)\in \Re H_\infty^{p_1\times m}$, $M_2(z)\in \Re H_\infty^{p_2\times m}$ and $p_2=p-p_1$. For the Toeplitz matrix of the system we obtain:

$$T(N,N)=U_T\Sigma_T V_T^T=\begin{bmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{bmatrix}\begin{bmatrix}I_{mN-n} & 0\\ 0 & E\end{bmatrix}\begin{bmatrix}V_1^T\\ V_2^T\end{bmatrix} \tag{5.6.14a}$$

with:

- $U_{k1}\in\Re^{p_kNx(mN-n)}$ and $U_{k2}\in\Re^{p_kNxn}$ $U_{k3}\in\Re^{p_kNx(p-m)N}$, with k=1,2.

Now define:

$$\begin{bmatrix}U_{p1}\\ U_{p2}\end{bmatrix}=\begin{bmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{bmatrix} \tag{5.6.14b}$$

The first block row U_{p1} corresponds to the first p_1 outputs and the second block row U_{p2} to the other p_2 outputs. From the first part of this subsection we know that if the first p_1 outputs contain n_z additional non-minimum phase behavior then the infinite dimensional ($N=\infty$) Toeplitz matrix for the first p_1 outputs must have a corresponding number of singular values equal to zero. In equation (5.6.14b) there must therefore necessarily be a dependency in the submatrix U_{p1}. If we choose the number of samples of the horizon (N) sufficiently large in conjuction with the dynamic behavior of the non-minimum phase zeros then n_z of the columns of U_{p1} must be almost dependent. This dependency amongst the columns of U_{p1} has to increase asymptotically with an increasing number of samples (N) of the horizon. Assuming n_z non-minimum phase zeros result for the singular value decomposition of U_{p1} in:

$$U_{p1}=\begin{bmatrix}Z_{11} & Z_{12}\end{bmatrix}\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}\begin{bmatrix}W_1^T\\ W_2^T\end{bmatrix} \tag{5.6.14c}$$

with:

- $Z_{11}\in\Re^{pNx(mN-n_z)}$ and $Z_{12}\in\Re^{pNxn_z}$ with $[Z_{11}\ Z_{12}]$ an orthonormal matrix.
- $\Sigma_2\in\Re^{n_zxn_z}$ a diagonal matrix asymptotically approaching zero if $N\to\infty$.
- $\Sigma_1\in\Re^{(pN-n_z)x(mN-n_z)}$ is a diagonal matrix.
- $W_1\in\Re^{mNx(mN-n_z)}$ and $W_2\in\Re^{mNxn_z}$

From equation (5.6.14c) and the fact that the columns of U are orthonormal we obtain:

$$\begin{bmatrix}U_{p2}W_1 & U_{p2}W_2\end{bmatrix}=\begin{bmatrix}Z_{21}(I-\Sigma_1^2)^{1/2} & Z_{22}(I-\Sigma_2^2)^{1/2}\end{bmatrix} \tag{5.6.14d}$$

and since Z_{11} is perpendicular to Z_{12} we must have Z_{21} perpendicular to Z_{22}. The left singular matrix U in equation (5.6.7a) therefore equals to:

$$U\begin{bmatrix} W_1 & W_2 \end{bmatrix} = \begin{bmatrix} Z_{11}\Sigma_1 & Z_{12}\Sigma_2 \\ Z_{21}(I-\Sigma_1^2)^{1/2} & Z_{22}(I-\Sigma_2^2)^{1/2} \end{bmatrix} \tag{5.6.15a}$$

In the square case we found that the trade-off's between the first block of outputs and the second block of outputs was restricted to a n_z dimensional subspace of the space spanned by the first block column in (5.6.15a), i.e.:

$$\Sigma_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & (I-\Sigma_2^2)^{1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.6.15b}$$

As a result we obtain that the nice structure of the U matrix in equation (5.6.10a). For a square inner the trade-off's we needed to between the two output blocks are therefore restricted to a n_z dimensional subspace. In the non square case this nice property is completely lost, all outputs are completely coupled.

Remark:

The right singular value matrix is again a square matrix. For this matrix we can again find a decomposition that satisfies (5.6.15b). This fact can be obtained in the same way as we have proven the structure of the left singular vector matrix in theorem 5.4.2. As for this case the results can directly be extended to more than two inputs or blocks of inputs.

5.6.3 Extension of the finite time domain results for co-inner and general systems.

In this subsection we will use the results obtained in the previous subsection to further extend the insight in the relation between the open loop and the closed loop process behavior. We will focus on how the restrictions the process poses on the closed loop behavior, manifest themselves in the finite time domain approach. In chapter three we have seen that non squareness, the non-minimum phase behavior and the gain behavior of the process restrict the ability to manipulate the closed loop behavior of this process. In this section we will see that the first two effects translate to the finite time domain into a same principle restriction of the control problem, the existence of practically spoken uncontrollable subspaces. The output space that can be fruitfully used for control is not of full rank. The controllability problem in the finite time domain can essentially be seen as:

Finding that subspace of the output space of the Toeplitz operator that can best be used to meet the required closed loop behavior.

The observation that these two effects translate to essentially the same restriction enables us to better understand the closed loop behavior of a process and the different trade-off's that occur. Something that was difficult in the frequency domain. Let us first summarize the results obtained from a controller design point of view.

In section 5.6.2 an approach was developed to understand how non-minimum phase effects manifest themselves in finite time problems. We have seen that the Toeplitz matrix has all its singular values equal to one except for the n_z smallest singular values for a square inner with n_z non-minimum phase zeros. These singular values became arbitrarily small for longer control horizons, i.e. $\sqrt{I - QP_N} \rightarrow 0$ for $N \rightarrow \infty$. In fact the effect on the output is more and more postponed until after the control horizon, i.e. $QP_N \rightarrow I$ for $N \rightarrow \infty$. This behavior could be explained from the fact that in the corresponding subspace of the output space the sign of the output changes. At each next sample time more energy is needed therefore to compensate the undesired behavior of previous inputs, since it is opposite to the desired behavior. However the undesired behavior at the next time sample is increased. Inverting this subspace will therefore result in an unstable behavior for a receding horizon controller. As a consequence however only a subspace of the output space can be manipulated. This is exactly the restriction that non-minimum phase behavior puts on the controllability of the process.

Let us turn to the effect that non squareness of a process has on the controllability. It was shown that it is possible to obtain a square process without introduction of additional non-minimum phase zeros, by the use of a dynamic precompensator. In section 5.4 we however showed that this may result in an arbitrary high gain of the precompensator. It can result in very large amplitudes at the process inputs and a non robust design, which is from a control point of view undesirable. In section 5.4 it was established that the introduction of additional non-minimum phase zeros may be needed to limit the gain of the compensator. Hankel theory was used to proof this. The relation between the gain of the precompensator and the additionally introduced non-minimum phase zeros in the resulting process became not clear. The finite time domain approach for non square inner (section 5.6.2) resulted in a complete new insight in the square down problem. We found that a subspace of the output space of the Toeplitz matrix was coupled to singular values smaller than one. If we still want to use the corresponding subspace then the gain of the compensator needs to be the inverse of the corresponding singular values smaller than one. So the smaller a corresponding singular value the larger the gain of the compensator in this direction needs to be. If the process singular values become too small this is not possible anymore. At one hand the gain of the controller becomes too large and at the other hand almost all this energy will be transferred to the outputs after the control horizon. As a consequence the subspace corresponding to too small singular values can not be used anymore for control. This is exactly the restriction that non squareness of the process may pose on the control problem. Note that excluding a certain subspace from inversion in fact has the same consequence for the control problem as a non-minimum phase zero; a certain output subspace is not freely manipula-table. It therefore manifests itself in the control problem in a same way as a non-minimum phase zero. As a result a better insight was obtained between the relation between the gain of the precompensator and the additionally introduced non-minimum phase zeros of the resulting square process. The better the left singular vectors corresponding to the non one singular values of the *pxp* inner of the non-minimum phase zeros match the left singular vectors corresponding to the smallest singular values of the co-inner, the more the gain of the precompensator will be reduced, since we do not control it in the zero direction for stability reasons.

In the previous discussion we claimed that the decomposition of the Toeplitz matrix of a structural co-inner is just the dual of the structural inner. This does however not hold for the relation that the right singular vectors corresponding to the singular values not equal to one have with the controllability gramian. In the case of a co-inner the space spanned by these singular vectors does not coincide anymore with the subspace spanned by the controllability matrix N_c. Let us show the above in an example.

Example 5.6.4

Let us take a look again at the structural co-inner obtained from the first output of the scaled tube glass production process of section 5.4.2. Apply a singular value decomposition on the Toeplitz matrix of the structural

V_2	90	90	90	90	90	90	90	89	87	84	81
V_3	0	0	0	0	0	0	0	1	3	6	9

Table 5.6.3 The canonical angles in degrees between the space spanned by the controllability matrix and the subspaces V_2 and V_3.

co-inner. The singular values smaller than one are plotted in figure 5.6.7. To obtain the relation between the right singular vector matrix V_2 of the Toeplitz and the controllability matrix $N_C(N,0)$ we determined the canonical angles between these subspaces and the canonical angles between $N_C(N,0)$ and the nullspace spanned by V_3 (table 5.6.3). From this table we see that the subspace spanned by $N_C(N,0)^T$ is indeed spanned by the columns of V_2 and V_3 and orthogonal to the subspace spanned by the columns of V_1.

End of example.

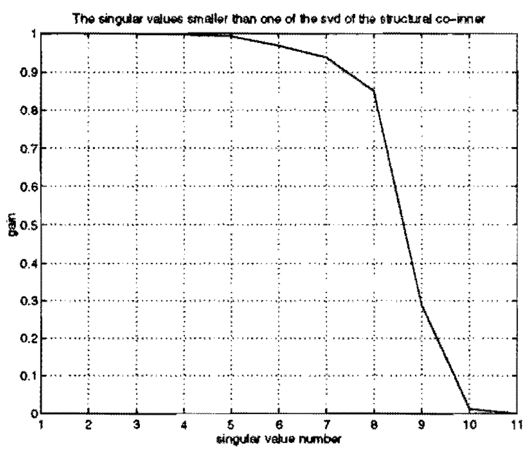

Fig.5.6.7 The smallest 11 singular values of the Toeplitz matrix of the system in example 5.6.4.

The importance of the above observation, using the singular value decomposition of the structural co-inner is the fact that we can not completely control the state at $N+1$ anymore. As a consequence we can not guarantee stability anymore. To deal with stability we propose to use a different factorization of the Toeplitz of the structural co-inner. Instead of using $T(N,N)$ use the matrix $T(2N,N)$ to also take the behavior after the control horizon into account.

Theorem 5.6.3

For a *pxm* structural co-inner, with minimal realization $[A, B, C, D]$ and $p \geq m$ with Toeplitz matrix $T(2N,N)$ has a factorization of the form:

$$T(2N,N) = \begin{bmatrix} U_{11} & U_{12} & U_{13}\left(I - \Sigma_{ns}^2\right)^{1/2} \\ 0 & 0 & U_{23}\Sigma_{ns} \end{bmatrix} \begin{bmatrix} I_{pN-n} & 0 & 0 & 0 \\ 0 & \Sigma_1 & 0 & 0 \\ 0 & 0 & U_{ns}\Sigma_2 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \\ V_3^T \\ V_4^T \end{bmatrix} \tag{5.6.16}$$

with:

- $U_{11} \in \Re^{pNx(pN-n)}$, $U_{12} \in \Re^{pNxn}$ and $U_{i3} \in \Re^{pNxn}$ with $U_{ij}^T U_{ij} = I$ and i=1,2, j=1,2.
- $V_1 \in \Re^{(pN-n)xmN}$, $V_2 \in \Re^{nxmN}$, $V_3 \in \Re^{nxmN}$ and $V_4 \in \Re^{(pN-n)xmN}$, with $V_i^T V_i = I$ and i=1,..,4.
- $U_{ns} \in \Re^{nxn}$ with $U_{ns} U_{ns}^T = U_{ns}^T U_{ns} = I$.
- $\Sigma_{ns} \in \Re^{nxn}$ a diagonal matrix with positive semidefinite entries smaller or equal to one.
- $\Sigma_1 \in \Re^{nxn}$ a diagonal matrix with positive semidefinite entries smaller or equal to one.
- $\Sigma_2 \in \Re^{nxn}$ a diagonal matrix with positive semidefinite entries smaller or equal to one.

The factorization has the following properties:

1. The columns of V_3 span the same space as the rows of the controllability matrix $N_c(N,0)$.
2. The matrices V_i, with i=1, .., 4 are mutually orthogonal.
3. The matrix U_{11} is orthogonal to U_{12}.
4. The matrices U_{12} and U_{13} are **not** mutually orthogonal.

Proof: See appendix 5.C.

In the above theorem we factored the Toeplitz essentially in two subsystems one that is related to the states at time instant N+1 and a part that is independent of these states and does not influence the behavior of the outputs after time instant N. The advantage of this factorization is the direct relation we obtain again between the behavior until sample instant N, which we will call the *near future*, and the behavior after time instant N, which we will call the *far future*. The idea is as follows: If we require the system inner at time instant $N+1$ to be in steady state then the dynamics are in rest and stability is guaranteed. In Model Predictive Control the control objective is in general to control step changes with an end error equal to zero. We will therefore assume that the steady state we want to reach is for a step change. If the system is in steady state then after time instant N the behavior is completely described by the steady state solution. For a state space model a steady state solution fulfills the following condition:

$$\begin{bmatrix} X_{ss} \\ Y_{ss} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{ss} \\ U_{ss} \end{bmatrix} \tag{5.6.17a}$$

If we want the system to be in steady state after time instant N then we obtain for the system output after this time instant that it must satisfy:

$$\begin{bmatrix} Y_{ss} \\ \cdot \\ \cdot \\ Y_{ss} \end{bmatrix} = U_{22}\Sigma_{ns}\tilde{X}_{ss} + T(N,N)\begin{bmatrix} U_{ss} \\ \cdot \\ \cdot \\ U_{ss} \end{bmatrix} \tag{5.6.17b}$$

Note that last term on the left side is exactly the finite step response of the system from 0 to N. If we introduce the step response model equation (2.3.2), we obtain for the steady state solution of the state:

$$\tilde{X}_{ss} = \Sigma_{ns}^{-1}U_{22}^{T}\left(\begin{bmatrix} I \\ I \\ \cdot \\ I \end{bmatrix} Y_{ss} - \begin{bmatrix} S_0 \\ S_1 \\ \cdot \\ S_{N-1} \end{bmatrix} U_{ss}\right) \tag{5.6.17c}$$

where we assumed that all diagonal entries of Σ_{ns} are larger than zero. Interesting to note is that a trade-off between the output behavior of the far future against that of the near future is possible. For each state we can trade-off the effect on the near future against that of the far future by looking at the corresponding entries of Σ_{ns}.

A last point of interest is the fact that n singular values of the part that is independent of the states at time $N+1$ are smaller than one. Some of these values can be so small that they are actually not invertible and seem to give the restriction that the non squareness of the system poses on the controllability of the system. Let us clarify this with an example.

Example 5.6.5

On the Toeplitz matrix of the system defined in example 5.6.4 we apply the factorization of equation (5.6.16). The last eleven singular values of the part of the decomposition that are independent of the state at $N+1$, i.e. the entries of Σ_1, are plotted in figure 5.6.8. Comparing these singular values with those of figure 5.6.7 indicates that they are the same. A more accurate study shows that there are small differences. In table 5.6.4 the inverse of both sets of singular values are given.

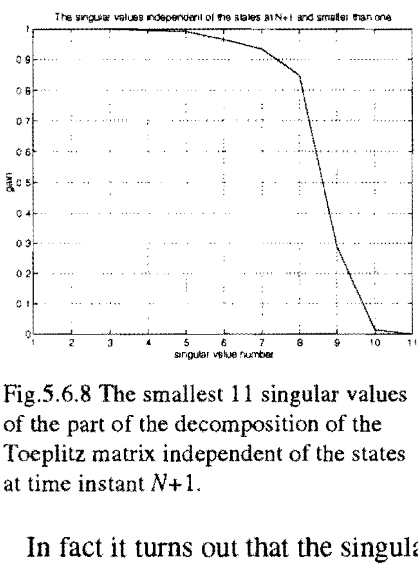

Fig.5.6.8 The smallest 11 singular values of the part of the decomposition of the Toeplitz matrix independent of the states at time instant $N+1$.

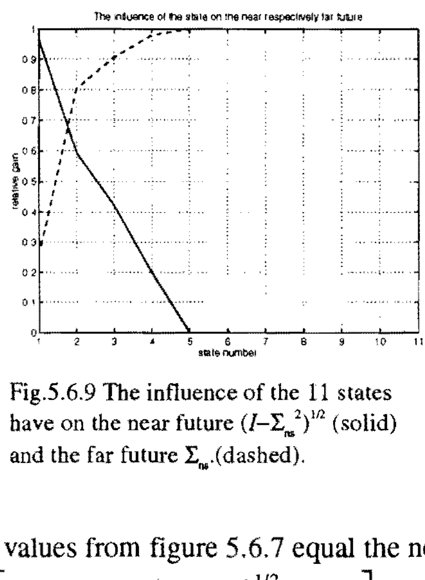

Fig.5.6.9 The influence of the 11 states have on the near future $(I-\Sigma_{ns}^{2})^{1/2}$ (solid) and the far future Σ_{ns}.(dashed).

In fact it turns out that the singular values from figure 5.6.7 equal the non zero singular values of the matrix $\left[U_{12}\Sigma_1 \quad U_{13}\left(I-\Sigma_{ns}^2\right)^{1/2}U_{ns}\Sigma_2\right]$.

State	1	2	3	4	5	6	7	8	9	10	11
Σ_2	**1.00**	**1.00**	**1.00**	**1.00**	**1.01**	**1.03**	**1.07**	**1.18**	**3.47**	**73.64**	**28050**
Σ_1	**1.00**	**1.00**	**1.00**	**1.00**	**1.01**	**1.03**	**1.07**	**1.18**	**3.48**	**73.68**	**28225**

Table 5.6.4 The singular values as plotted in figure 5.6.7 (first row) and figure 5.6.8 (2nd row).

The actual restriction a co-inner system with a fixed steady state behavior poses on the problem is therefore determined by the singular values of Σ_1 that are not invertible any more. For this system we see that this holds for the last two or three singular values. In figure 5.6.9 the dependencies of the near future and the far future on the states, $(I-\Sigma_{ns}^{2})^{1/2}$ respectively Σ_{ns}, are given. Interesting to note is that indeed a trade-off between the near and far future is possible. The idea is that states that only have a weak coupling with the far future (small entries of Σ_{ns}), but whose output directions, i.e. related column of U_{13}, have sufficient coupling with the columns of U_{12} that correspond to singular values of Σ_1 can be used to increase performance in the near future without severe deterioration of the far future. We will make use of this fact to control a square system.
End of example.

Remark:
This factorization has a clear relation with the square inner. In case that $m=p$ we obtain $\Sigma_1=0$ and the factorization reduces to the one of theorem 5.6.1. after applying the singular value decomposition on $(I-\Sigma_{ns}^{2})^{1/2}U_{ns}\Sigma_2$.

For a square general system that is not inner we will make use of essentially the same ideas as in theorem 5.6.3. We separate the Toeplitz matrix again in a part that has no influence on the far future and a part that influences the far future. For a *pxm* system, with $m \geq p$ and minimal realization $[A, B, C, D]$ the Toeplitz matrix $T(2N,N)$ has a factorization of the form:

$$T(2N,N)=\begin{bmatrix} U_{11} & U_{12} & U_{13}\left(I-\Sigma_{ns}^{2}\right)^{1/2} \\ 0 & 0 & U_{23}\Sigma_{ns} \end{bmatrix}\begin{bmatrix} \Sigma_{11} & 0 & 0 & 0 \\ 0 & \Sigma_{12} & 0 & 0 \\ 0 & 0 & U_{ns}\Sigma_2 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \\ V_3^T \\ V_4^T \end{bmatrix} \tag{5.6.18}$$

where $U_{12}\in\Re^{pNxn}$, $V_2\in\Re^{nxmN}$, $V_4\in\Re^{((m-p)N-n)xmN}$ and $\Sigma_{12}\in\Re^{nxn}$ are not present if the process is square.

The main difference with the factorization of an inner is that the first block of singular values will not equal the identity anymore. Furthermore the matrices $U_{11}\in\Re^{pNx(pN-n)}$, $U_{13}\in\Re^{pNxn}$ are in general not orthogonal. This last fact discerns the above factorization of a direct singular value decomposition on the Toeplitz matrix. Direct application of the singular value decomposition therefore does not have the direct relation between the states. The factorization in equation (5.6.18) enables to directly consider stability in the analysis (equation 5.6.17). Again it is possible to make a trade-off between the states at $N+1$ and the behavior of the output in the near future. Note however that in this case we loose the fact that the controller is in steady state at time instant $N+1$. As a consequence we are not able anymore to formally guarantee stability of the controller over the receding horizon. We will see however that for a square system this trade-off is more important, since U_{11} does not span the whole output space. Remember however from the square inner how the behavior of a non-minimum phase effect manifests itself. The singular values of the NxN block Toeplitz matrix corresponding to the states decreased asymptotically towards zero as a function of N. At the other hand we saw in the proof of theorem 5.6.1 that the influence on the far future increased to one with increasing N. As a consequence we also need to see this effect in the Toeplitz of a non inner model. This last fact is easily understood from the following reasoning: It is always possible to factor a process in an inner and an outer part. Since for the inner part a certain subspace of the near future will become unreachable with increasing N this should also hold for the total system. At the other hand for this direction all energy accumulated in the inner has to come out again after the control horizon. The accumulated energy however also has to come out of the overall system. For a sufficiently long horizon we therefore may expect that non-minimum phase effects are related to those states that have there main influence on the far future. As long as we choose only those states that have a small influence on the far future for further optimization of the near future, we may expect the system to stay stable. A more formal proof is desired.

Assume the process to be square, i.e. $p=m$ to simplify the discussion. The fact that the large diagonal matrix $\Sigma_{11} \in \Re^{(pN-n)x(pN-n)}$ is unequal to identity makes the analysis more complex. An additional trade-off is needed based on the entries of Σ_{11}. The additional problem is that it is not possible to discern hard between a controllable and a uncontrollable number of singular values, since the entries of Σ_{11} will in general smoothly decrease in value. An approach is therefore needed that enables us to obtain insight in the behavior of this subsystem. We may proceed in several ways. One is to compare the two norm approximation of the open loop behavior against the two norm approximation of the perfect controller for the subsystem that is independent of the state at time instant N+1. Let us clarify what we mean. The best controller for the subsystem $U_{11}\Sigma_{11}V_1^{\mathrm{T}}$ in two norm equals the pseudo inverse, i.e. $V_1\Sigma_{11}^{-1}U_{11}^{\mathrm{T}}$. We now define the *time gains* for a given desired output step response, say y_{ref}, as:

$$g_1 = \Sigma_{11}^{-1}U_{11}^T y_{ref} \tag{5.6.19a}$$

The idea behind the use of the time gain is to compare a chosen reference with the time gains found for the open loop step response and the time gain of the two-norm solution of the perfect controller. The output signal generated by this system then equals:

$$y_1 = U_{11}U_{11}^T y_{ref} \tag{5.6.19b}$$

Let us give an example, before we further discuss this approach.

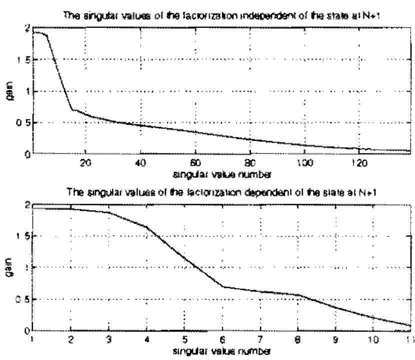

Fig.5.6.10 The singular values Σ_1 (first plot) and Σ_2 of the factorization in equation (5.6.18) of the system.

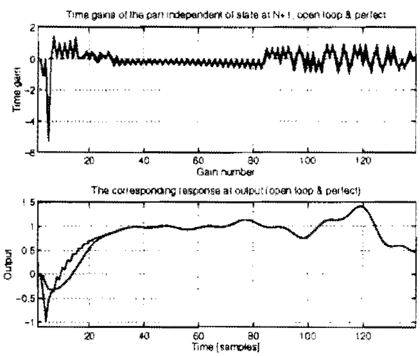

Fig.5.6.11 The time gain and corresponding output of the subsystem, independent of the states at time N+1.

Example 5.6.6a

Given a 1x1 system with an eleventh-order minimal realization $[A,B,C,D]$ that contains four non-minimum phase zeros located in the complex plane at 1.12, 8.59, -5.96, -2.06 and -1.19. For the Toeplitz matrix, with N=150, we applied the factorization of equation (5.6.18). The singular values of Σ_1 and Σ_2 are given in figure 5.6.10. Note that the entries of Σ_1 change smoothly and it is not possible to discern a controllable and uncontrollable subspace. In figure 5.6.11 we plotted the behavior for g_1 and y_1 for a reference signal y_{ref} equal to the open loop step response respectively the step itself, i.e. $y_{ref} = [1 \quad . \quad . \quad 1]^T$.

End of example.

At first sight the output response of this system might seem disappointing. Note however that we did not consider the subsystem related to the states at time $N+1$. Implicitly we have therefore chosen these states equal to zero. In contrast to a fat system this has severe consequences for the performance of the controlled system. We therefore have to include the state dependent part in the analysis. Let us first consider the case were we want all states to be in steady state using equation (5.6.17c):

$$g_1 = \Sigma_{11}^{-1} U_{11}^T \left(y_{ref} - U_{12} (I - \Sigma_s^2)^{1/2} \tilde{X}_{ss} \right) \qquad (5.6.20a)$$

The output signal generated by this system then equals:

$$y_1 = U_{11} U_{11}^T \left(y_{ref} - U_{12} (I - \Sigma_s^2)^{1/2} \tilde{X}_{ss} \right) + U_{12} (I - \Sigma_s^2)^{1/2} \tilde{X}_{ss} \qquad (5.6.20b)$$

Let us continue the example with this idea.

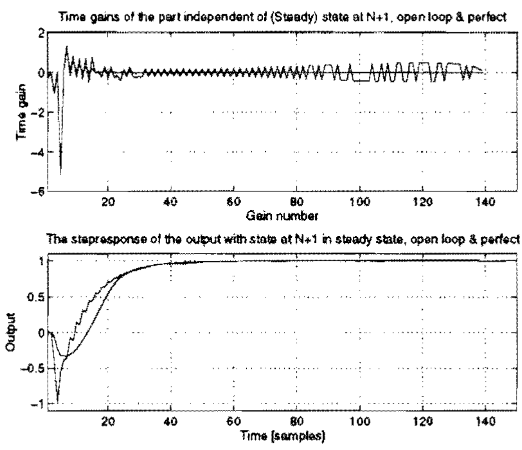

Fig.5.6.12 The time gain and output response of the system with the states at time $N+1$ fixed at the steady state value.

Example 5.6.6b

We now apply equation (5.6.20) to the factorization of the Toeplitz of example 5.6.6a for the same two references. The resulting time gains,

equation (5.6.20a) and the output response, equation (5.6.20b) are given in figure 5.6.12. Note that indeed the states where needed to obtain a good response at the output of the system.
End of example.

An interesting observation is that the entries of Σ_s determine the relative importance that a state has on the far future and $(I-\Sigma_s^2)^{1/2}$ the influence on the near future. Note that if an entry of Σ_s is small then the influence of the corresponding state on the far future is small. The influence on the near future however is significant. In this case we could decide to use this state to further optimize the near future output at the cost of a restricted influence on the behavior in the far future. In this case we split the state vector in two parts $\tilde{X}_s = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix}$, where we will use the upper part of the state vector to further optimize the near future. The lower part will be used to minimize the dynamic behavior of the output in the far future. To simplify the discussion we first split up the matrices U_{12}, U_{22} and Σ_s according to the split-up of the state vector, i.e. for i=1,2

$U_{i2} = \begin{bmatrix} U_{i21} & U_{i22} \end{bmatrix}$ and $\Sigma_s = \begin{bmatrix} \Sigma_{s1} & 0 \\ 0 & \Sigma_{s2} \end{bmatrix}$. We then obtain for $\tilde{X}_2$, since $U_{222}^T U_{221}=0$, conform equation (5.6.17c):

$$\tilde{X}_2 = \Sigma_{s2}^{-1} U_{222}^T \left(\begin{bmatrix} I \\ I \\ \cdot \\ I \end{bmatrix} Y_{ss} - \begin{bmatrix} S_0 \\ S_1 \\ \cdot \\ S_{N-1} \end{bmatrix} U_{ss} \right) \tag{5.6.21a}$$

We then obtain for the near future output:

$$\begin{bmatrix} g_1 \\ \tilde{X}_1 \end{bmatrix} = \left(\begin{bmatrix} U_{11} & U_{121} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & (I-\Sigma_{s1}^2)^{1/2} \end{bmatrix} \right)^{-R} \left(y_{ref} - U_{122}(I-\Sigma_{s2}^2)^{1/2} \tilde{X}_2 \right) \tag{5.6.21b}$$

The output signal generated by this system in the near future then equals:

$$y_1 = U_{11} g_1 + U_{121}(I-\Sigma_{s1}^2)^{1/2} \tilde{X}_{s1} + U_{122}(I-\Sigma_{s2}^2)^{1/2} \tilde{X}_2 \tag{5.6.21c}$$

The output in the far future will equal:

$$y_2 = U_{221} \Sigma_{s1} \tilde{X}_{s1} + U_{222} \Sigma_{s2} \tilde{X}_2 \tag{5.6.21d}$$

Let us take a look at this for the example 5.6.6.

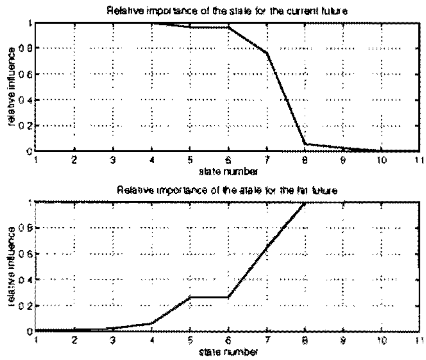

Fig.5.6.13 Relative importance of the state for the near future $(I-\Sigma_s^2)^{1/2}$ (1st plot) and the relative importance for the far future Σ_s (2nd plot).

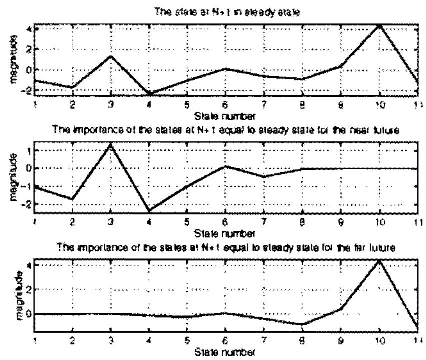

Fig.5.6.14 Steady state values of the states (1st plot), their influence on near future (2nd plot) and the far future (3rd plot).

Example 5.6.6c

For the factorization of the Toeplitz matrix of the system in example 5.6.6a Σ_s and $(I-\Sigma_s^2)^{1/2}$ are given in figure 5.6.13. From this plot we see that the first six states mainly influence the near future output behavior. The last four states have their influence mainly in the far future. This is verified by figure 5.6.14, were we plotted the influence the steady state of the open loop system as used in example 5.6.6b has on the near and far future. To ensure a good behavior of the far future we will use only the first 6 states to further optimize the near future behavior. It turns out however that the additional freedom does not result in increased performance. It seems that the optimal solution is already reached in case we fixed the states at steady state. We compared the solution found with this approach with the step response the inner obtained from the non-minimum phase zeros of the system, since this response is known to be the optimal stable 2-norm solution for a step input. As could be expected the two responses coincide completely. As a consequence we indeed also ensured stability by fixing the last five states.

End of example.

In the above discussion we already indicated that the time gain is only one possible way to analyze the behavior of the part of the system that is independent of the states. The idea of the time gain is completely based on the two norm for a step change. This means that according to this approach the performance obtained in the above example is the best possible in two norm sense for a step change. This two norm optimality need however not to coincide with what we actually want of the controlled system. Let us return to our example to see this

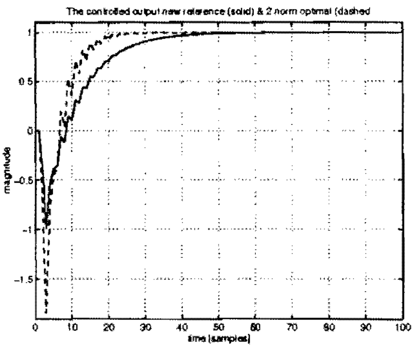

Fig.5.6.15 The two norm optimal step response (solid) and the one obtained for the modified criterion (dashed).

Example 5.6.6d

We are interested in a system that takes less time to come into steady state. We do not mind in an increased inverse response when the system can still be made faster. This can for example be achieved by taking as a reference signal y_{ref} the step response of reference system whose dynamics equal the inner system based on the non-minimum phase zeros of the process except that the slowest pole 1/1.12 is replaced by a faster pole at for example 0.8 and the steady state gain of the system is corrected again to equal one. Using this reference in the above proposed approach results in a much faster response (figure 5.6.15).

End of example.

Another example of a possible approach that depends less on the specific properties of the criterion and that is more restricted by the process behavior itself is to use a LQ decomposition of the $U_{11}\Sigma_1$ or of $[U_{11}\Sigma_1 \; U_{121} (I-\Sigma_{s1}^2)^{1/2}]$. L is of the form:

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

with:

- $L_1 \in \Re^{txt}$ is a full matrix and $L_2 \in \Re^{(pN-t)x(pN-t)}$ is an upper triangular matrix.

where t is equal to n in the first case or equal to n_2, i.e. the dimension of Σ_2, in the second case. The idea is that it is most important to make the output at time N equal to the reference, second important is to make time N–1 equal to the reference and so on. After j steps one obtains for the manipulated L matrix:

$$\tilde{L}^j = \begin{bmatrix} L_{11} & L_{12}L_{32}^{-1} \\ L_{21} & L_{22}L_{32}^{-1} \\ 0 & I_j \end{bmatrix} \text{ with } L_2 = \begin{bmatrix} L_{21} & L_{22} \\ 0 & L_{32} \end{bmatrix}$$

At each step we thus see the consequences of the previous time instants to the near future. The large number of steps in the procedure make it rather complex. Of course many more approaches are possible. More research and experience is needed to find the best suited approach.

5.6.4 Conclusions and discussion

This section has resulted in techniques that provide us with a very good and new insight in the relation between the behavior of the model and the possible closed loop behavior. As far as known it has resulted in a completely new characterization of system behavior in the finite time domain. Although the techniques were developed for the finite time domain they in fact deepened insight in general and independent of the domain. The relation between the characterization of the behavior of systems over a finite and an infinite time horizon has always been a problem. The here developed insight makes this relation directly apparent. The finite time Toeplitz matrix of the square inner turned out to have a very nice structure that clearly revealed the effect a non-minimum phase zero has on the process behavior and the restriction it poses on control. For non-minimum phase behavior we are not allowed to use certain subspaces over a long period of time. At least if stable behavior is desired over a longer time horizon. For a restricted period of time it is possible to use these subspaces. However, at a price of undesired behavior after the initial time period. The magnitude of this behavior grows exponentially with the time horizon as a function of the location of the zeros in the complex plain. In case of receding horizon control it will therefore not be possible to use these specific subspaces for control, since we need to guarantee stability of the controller. As a result non-minimum phase zeros can now be defined independent of the time horizon. Moreover the difficult and not well understood behavior of non-minimum phase zeros in the closed loop as it appears in the frequency domain in the Bode sensitivity integral, see for example [Boy85], becomes crystal clear. It just boils down to the inability to invert the model in the output direction of the zeros over longer time horizons, with finite amplitude. As a consequence, part of the output space is not free to manipulate. To obtain nominal stability we therefore proposed to fix this subspace to be in steady state at the end of the control horizon. A prerequisite therefore is that we have a control horizon of sufficient length. The possibilities left to manipulate the non-minimum phase behavior is to try to turn the influence of these effects to a more desirable direction. From theorem 5.6.2 we obtained that for the square inner the actual trade-off is in fact restricted to a n-dimensional subspace. For a structural inner, $p \geq m$, we have seen that turning the direction of the non-minimum phase effect to a less important output is more complex, since it is not restricted any more to the n-dimensional subspace. This is due to the existence of a complementary range.

If we use the term direction here we in fact mean the direction of the Toeplitz matrix. As a consequence we may try to turn the effect not only to a certain output direction of the model but also in time. This trade-off as function in time translates itself to the Bode sensitivity integral relation, i.e. the analytic trade-off in the frequency domain. It is felt that the approach presented here can help to develop a better understanding of this analytic trade-off. Further research is however needed to develop this understanding.

In section 5.4 we established the relation between the gain of the square down controller and the number of additional non-minimum phase zeros we introduce in the down squaring. The approach was based on Hankel theory. The function these non-minimum phase zeros fulfilled and the relation the gain had to the position of the zero in the complex plane were only partly understood. The here developed theory increased this insight considerably. For a certain subspace the singular values become so small that they are not invertible anymore. This subspace can therefore not be freely used for control. Excluding a subspace has however the same effect on the near future as the introduction of additional non-minimum phase zeros. The more the direction of these zeros coincides with the direction of the smallest singular values the more effect it will have on the gain of the controller, i.e. on the magnitude of the control signals.

For structural co-inners and more general systems the problem was that the direct relation between the finite time controllability matrix and a certain subspace of the left singular vectors of the Toeplitz matrix was lost. We therefore proposed to adapt the factorization. For the structural co-inner we suggested a new factorization of the Toeplitz matrix in theorem 5.6.3. The main idea was to split the factorization of the Toeplitz in two main parts. One that was related to the state at time $N+1$, i.e. to the behavior in the far future, i.e. the behavior of the process after the control horizon and one that was independent of the behavior of the output behavior in the far future. Both subsystems however influence the near future, i.e. the output behavior over the control horizon. In this way we again have the possibility to ensure stability of the control actions. It was found that for the part of the Toeplitz matrix that was independent of the far future there are $pN-n$ singular values equal to one and exactly n singular values that were smaller than one. These n singular values smaller than one are unique for a fat system. These singular values potentially form the restriction the non square system poses on the control problem. If these singular values become too small, they will not be invertible anymore.

For a non inner system we proposed the same factorization as we introduced for the co-inner, since also for a non-inner system the direct relation between the states at time $N+1$ and the singular value decomposition is lost. The additional problem for a non inner system is that the nice structure of singular values of the part of the Toeplitz that is independent of the state at time $N+1$ is lost (equation 5.6.18). We discussed two approaches that enable the user to judge the controllability of this part of the system. For the first approach we introduced the time gains that enabled us to compare the behavior of an output reference signal against that of the open loop behavior and the two norm optimal solution. As discussed the time gain approach heavily depends on properties of the two norm criterion. A second approach was therefore proposed, based on the LQ-decomposition. This approach did not suffer from a dependency on a criterion at the cost of more complexity. We did not claim that one of these solutions is in any sense the final solution to this problem. To come to a well balance approach that provides a good insight in the gain properties of the system more research and certainly more experience is needed.

The developments started in this section form only an initial start. Many items are not yet completely known. An open issue still is the stability of the approach. Specifically if we want to use part of the states for optimization of the near future, this is still an open issue. As discussed it is currently based on intuition. One way out to ensure stability is not to assume the control signal to be constant but to be generated by an infinite time controller, for example a state feedback. The basic idea here is that the control system must be stable if it is possible to obtain steady state conditions at the output after the control horizon. This seems a very appealing idea that can be further extended. More research is however needed.

It is interesting to obtain a factorization of the Toeplitz matrix that enables us to unify the two factorizations we proposed here. In fact it is expected that it is possible to obtain one generally applicable factorization, that clearly displays the different trade-off's. (A partial relation has already been established in the proof of theorem 5.6.3). Another topic of great interest is robustness. A more formal insight in robust stability and performance issues in the finite time domain still is a difficult item. The currently developed frame work might well be the frame work to study these effects. Full understanding of the consequences the developed approach has on our understanding of systems and signal theory is just started and still has to be extended further.

The results obtained in this section have more significance and potential than controllability analysis and Model Predictive control alone. The theory on the finite Toeplitz operators of inner systems, their extensions to more general systems and their relation to dynamic systems and signals potentially results in a complete new framework for control theory with consequences on the way we look at and deal with dynamic systems and signals in the future. It for example enables us to unify the more signal oriented approaches, like the behavioral framework as originally defined by Willems [Wil89], subspace techniques [Lar83, Moo88, Moo89, Ove95, Ver92], polynomial basis, like Laguerre, Kreitz [Nur87, Wal89, Heu91] and others [Hue91, Hof94] and wavelets with the more classical techniques based on system behavior itself. It is also expected to have an impact on the understanding of the relation between identification and controller design.

In the next section we will use the here developed techniques to develop a finite time controllability analysis approach for potentially large MIMO systems.

5.7 A finite time controllability analysis approach for MIMO systems.

In the first part of this section we will develop an input-output controllability analysis based on the finite time horizon. In the last part of the section we will discuss the application of the approach on a scaled glass tube process.

5.7.1 Analysis approach for the finite time domain

In this section we will develop a general applicable approach to analyze the controllability of a process based on the finite time horizon. The approach is based on the priority scheme and sequential analysis approach as proposed in section 5.2. The outputs are ordered again in a decreasing order of importance; output one most important and so on. In many control problems integral control of the process is an important property of the closed loop system. In model predictive control the control problem is frequently splitted in two subproblems:

1. The steady state control problem. Analysis of the dynamic problem.
2. The dynamic control problem, i.e. bring the process in a predefined way to the steady state solution.

Examples of techniques that explicitly use this approach are e.g. DMC™, DMCplus™ and the approach of Rawlings [Mus93]. In [Mus93] the dynamic problem is reformulated to a problem where we need to control the process from an initial offset back to zero. This way of splitting the problem in two subproblems and reformulating the dynamic problem to an initial condition problem can also be used for the analysis. Moreover it simplifies the approach. We propose to use the same approach for the input-output controllability analysis to be developed and therefore split-up the analysis in two steps:

1. Analysis of the steady state problem.
2. Analysis of the dynamic control problem.

The steady state analysis can be performed with the techniques developed in section 5.3. We will in this section therefore concentrate on an approach to analyze the input-output controllability of the dynamic control problem, i.e. controlling the system from an initial offset back to zero.

As for the frequency domain approach discussed in section 5.5, we first factor the process per output in an inner outer and structural co-inner part. The factorization for the i-th output is applied on that part of the model of the output that is independent of outputs with a higher priority, output 1 to $i-1$. To determine the independent part of the model we determine a stable transfer matrix of the kernel operator[1] of the model for output 1 to $i-1$, say $S_{1,i-1}^{A}(z)$ (section 5.5):

$$\begin{bmatrix} m_1(z) \\ \cdot \\ \cdot \\ m_{i-1}(z) \end{bmatrix} S_{1,i-1}^{A}(z) = 0 \tag{5.7.1}$$

A realization of this matrix based on a state space model of output 1 to $i-1$ is given by theorem 5.4.1. The part of the model related to output i that is independent of the part of the model related to the outputs 1 to $i-1$ is given by:

$$m_i(z) S_{1,i-1}^{A}(z) \tag{5.7.2}$$

As a second step in the procedure we start to analyze the controllability of the outputs recursively, based on the priorities: First output one than output 2 and so on.

Let us take a closer look at the procedure for output i: Apply the factorization equation (5.6.18) on the Toeplitz matrix of the part of the model of output i that is independent of the outputs with a higher priority. Based on the techniques developed in the previous section we can then analyze the possibilities this system offers for control. From the analysis we obtain an expected control signal for control of this output, say u_i. Define the *output interaction vector*, say y_{IAi}, as the Toeplitz matrix of the model from output $i+1$ to p, say $T_{i+1,p}(N,N)$ post multiplied by u_i:

$$y_{IAi} = T_{i+1,p}(N,N) u_i$$

In the last step of the procedure we adjust the control signal to minimize the influence that control of the outputs with higher priority have on the lower priority outputs. This reduction can again be achieved according to the

[1] It is emphasized that the concepts of inner outer structural inner and right annihilator that are used in this section are defined on the infinite time horizon.

approach developed in section 5.6 by taking the reference signal equal to the output interaction vector.

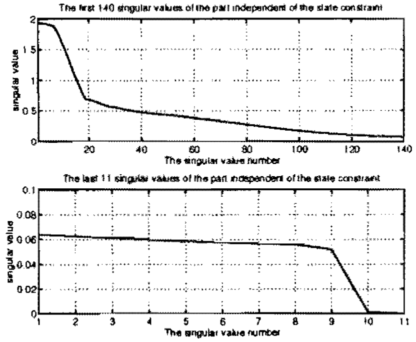

Fig.5.7.1 Singular values of the Toeplitz matrix. a) the first 140 singular values (Σ_{11}). b) the last 11 singular values (Σ_{12}).

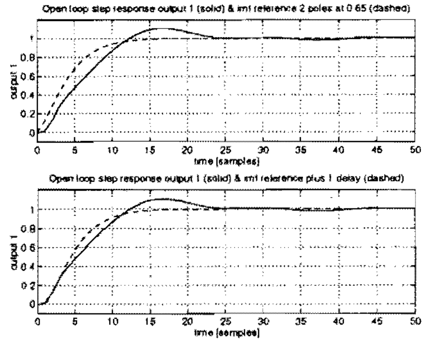

Fig.5.7.2 a) The open loop step response output 1 & the IMF reference trajectory (2 poles at 0.65). b) as a) but reference with one additional delay.

5.7.2 The approach applied on the scaled tube glass process

In this section we will make use of the delayless and scaled model of the tube glass production process, as used in example 5.5.1. The emphasis is again on the techniques developed and not on the analysis of this particular process. We will again assume the ordering of the outputs as in example 5.5.1, i.e. output one the most important output and order the model accordingly:

$$M(z) = \begin{bmatrix} m_1(z) \\ m_2(z) \end{bmatrix}$$

As a first step in the procedure we determine the right annihilator of the first output, i.e. the controller whose image equals the nullspace of output one. Based on the right annihilator we determine the realization of the part of output two that is independent of the behavior of the first model, say $m_{2\perp 1}(z)$.

Let us start with the analysis of output one. The singular values Σ_{11} and Σ_{12} of the part of output one that is independent of the state at $N+1$ are given in figure 5.7.1. We see that the singular values of Σ_{11} smoothly decease. For Σ_{12} we see a drop in its values; the last two singular values become very small. For control we will therefore not use the last two singular values of Σ_{12}. To obtain fast insight in the controllability of the process output we used the following approach to obtain an output reference. For this first output an inner-outer structural co-inner factorization was performed. We constructed the square $1x1$ system from the inner and outer. From section 5.5. and 5.4

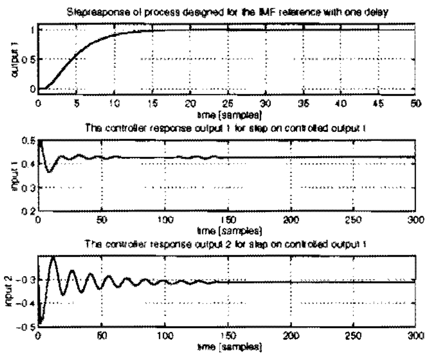

Fig.5.7.3 Closed loop step response of output 1 & control signals (last 2 plots).

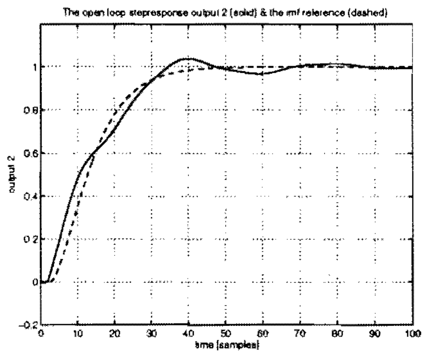

Fig.5.7.4 Open loop step response of part of output 2 that is independent of output 1 (solid). The selected IMF reference for output 2 (dashed).

we know this system represents the best possible square system for this output. Implicit Model Following [Bac87] was used to determine a good controller for this square model. A second order reference with two equal poles was chosen. The resulting response is used as an initial reference trajectory in the procedure and compared with the open loop step response of output one to a step on the inputs whose magnitudes equal the pseudo inverse of the static gain of the first process output, i.e. $m_1^{-R}(1)$ (figure 5.7.2, upper plot). Based on the comparison of the two responses one extra delay was added to the reference signal to obtain a better fit in the initial part of the response (figure 5.7.2, lower plot). Comparing the closed loop response with the open loop behavior shows that the we did not actually increase the speed of the response, but essentially removed the oscillating behavior of the open loop response. To ensure the stability of the receding horizon we fixed the state at the end of the horizon at its steady state value. The resulting behavior of the process output and at the process inputs is given in figure 5.7.3. Note that indeed at the end of the control horizon (N=150) the system is in steady state if we extend the input signal with the steady state value. In the last two subfigures we see the control signals that are needed to generate this output signal. Note specifically over the control horizon of input one that from sample 90 to 150 a small oscillation appears in the signal. This oscillation is expected to be due to the steady state constraint at the end of the horizon.

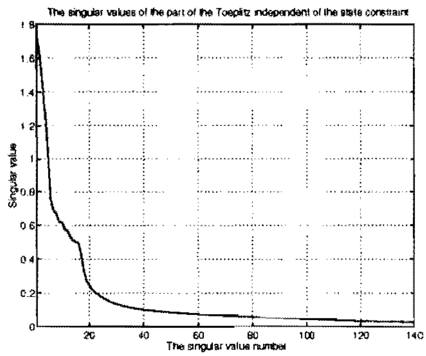

Fig.5.7.5 The singular values Σ_{11} of the Toeplitz matrix of the part of the model of output 2 that is independent of the model of output 1.

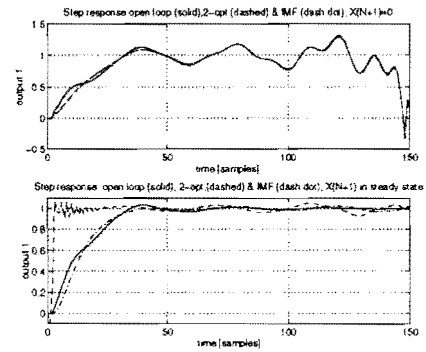

Fig.5.7.6 Response of the 2nd output to a step change: open loop (solid), 2 norm optimal (dashed) & IMF (dash dot). a) states at time N+1 equal zero. b) states at time N+1 equal steady state value.

Let us start to analyze output two. As was the case for output one we again use the IMF design approach to quickly obtain a first reference signal. A good reference signal was determined (two poles at 0.85) and is given together with the open loop step response in figure 5.7.4. Figure 5.7.5 shows the values of Σ_{11}. Remember that for output two the subsystem corresponding to the singular values Σ_{12} does not exist, since the system is made square using the right annihilator of output one. To obtain insight in to Σ_{11}, corresponding subspace the two norm approximation for the reference signal in equation (5.6.21) made equal to the open loop step response, the two norm optimal step response and the IMF reference trajectory under the assumption that the state at time N equals the zero resp. the steady state condition are given in figure 5.7.6. From these two figures we see first of all that the value of the states at time N are of essential importance for the behavior of the output over the control horizon. Moreover the fact that we fix all states to equal the steady state value at time N restricts performance over the control horizon. A better trade-off between the control horizon and the far future is therefore needed. To do this we need to study the relative importance of the different states for the near and the far future (see section 5.6). From this relation (figure 5.7.7) we obtain that the first seven states could be used for a further optimization of the near future. The behavior at output two and the control signals applied at the process inputs are given in figure 5.7.8 together with the IMF solution for the infinite time horizon control problem. Note the small differences between the two responses. The

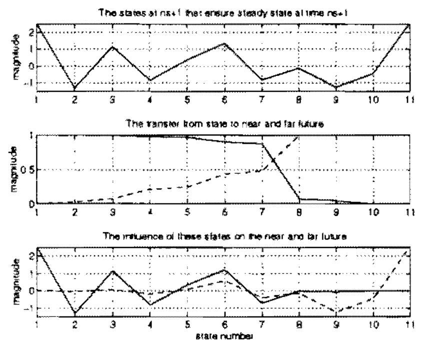

Fig.5.7.7 a) Steady state value of the 11 states. b) Influence of the state on the near & far future. c) Influence of the steady state of the states on the near & far future.

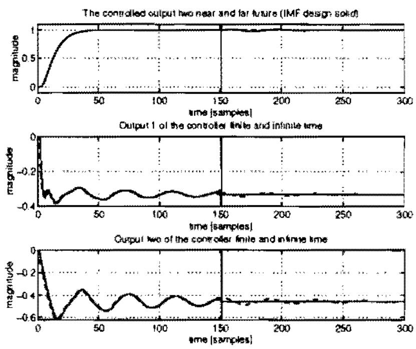

Fig.5.7.8 Step response of output 2 & the process inputs for the finite time solution (dashed) & infinite time solution (solid).

differences are due to our choice of the finite control horizon N=150, which is clearly seen from the control signals to be too short with respect to the infinite time solution. The last step is now to reduce the influence of the control actions for output one on output two. We essentially follow again the same approach as in the previous section. Only the reference signal for this problem equals the interaction on output two times the reference signal found during the analysis of output two. The open loop and the compensated behavior of output two for a unit step on the setpoint of output one and the extra control action needed to achieve this is given in figure 5.7.9. The overall behavior at the outputs for a unit step at setpoint one and two is given in figure 5.7.10 and the corresponding control signals in figure 5.7.11.

In the above example we showed the use of the analysis approach we developed in this section. From this example we clearly see the trade-off that exists between the near and the far future. This is especially the case for square processes. Remember that the steady state was only used to ensure stability of the control actions for the receding horizon controller. Note however from the above that requiring steady state behavior in itself sometimes is restrictive and not strictly necessary. From this observation a new idea on how to deal with the states emerged. The idea will be further discussed in the next section.

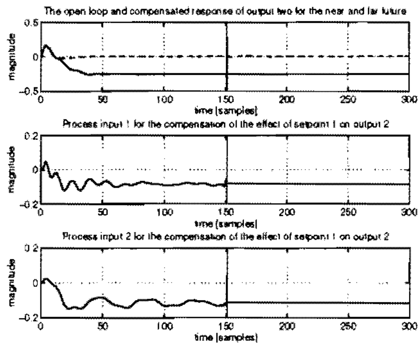

Fig.5.7.9 Response of output 2 to a step change of setpoint of output 1 before (solid) & after compensation (dashed). last 2 plots control signals needed for compensation.

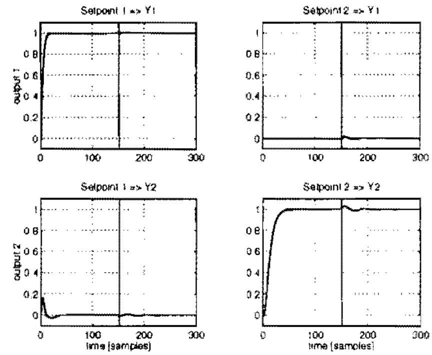

Fig.5.7.10 Step response model of the resulting system at the process outputs.

It must have been noticed by the reader that the approach as we developed it for the finite time is more elegant than the frequency domain approach. It is much easier to understand and interpret the different trade-off's that are

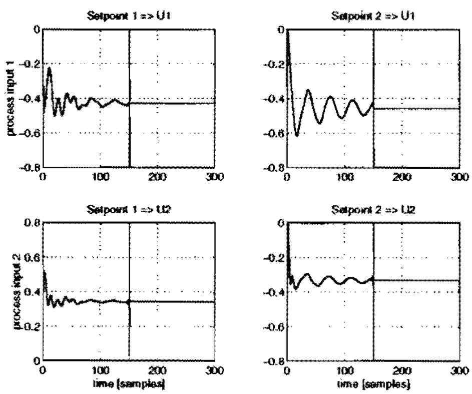

Fig.5.7.11 Step response model of the resulting system at the process inputs.

made than is possible in the frequency domain. The time domain approach results in a better understanding on how different effects, i.e. non-minimum phase zeros, non-squareness and gain, limit the performance at the outputs. Hence, it seems worthwhile to apply the finite time analysis with a sufficiently long control horizon and use the results in the actual control design.

5.7.3 Summarizing the analysis approach

For a stable model $M(z)\in \Re H_{\infty}^{p\times m}$ the procedure is summarized as follows: Starting with output one, the most important output, apply the following procedure per output, until all degrees of freedom are used:

1. Apply an inner outer structural co-inner factorization on the model of output i, $m_i(z)$: First determine a transfer for the kernel of the first $i-1$ outputs, say $S_{1,i-1}^{A}(z)$ (section 5.5). For output one , take $S_{1,0}^{A}(z) = I$.

   The independent part of he model of output i is given by $m_i(z)S_{1,i-1}^{A}(z)$.
2. Analyze the behavior of $m_i(z)S_{1,i-1}^{A}(z)$ (section 5.6) and determine the interaction it causes on the outputs with lower priority. The procedure to follow per output is as follows: Analyze the values of Σ_{12}, if present, and determine the invertible part of this subspace. The controllability of the output can now be analyzed in two steps:
   - first fix all states at the end of the control horizon at steady state and determine the response at the output for this system.
   - Include part of the states in the optimization for the near future, if this does not result in a satisfactory behavior. We therefore have to select part of the states that are best suited for this purpose based on Σ_z (equation 5.6.18).
3. As a last step we minimize the interaction that control of higher priority outputs has on the lower priority outputs. If we use the output interaction vector as reference then the procedure to follow is essentially the same as in step two. In this step we however have to pay attention to the dependency between the different columns of the controller and monitor the robustness of the controller in this way.

5.8 Summary.

In this chapter we proposed a strategy for the analysis of the input-output controllability analysis that enabled us to analyze the resilience of the model in a more direct relation to the requirements posed in the control problem. In chapter two we already concluded that also input-output controllability analysis is in principle a control design. The emphasis of the analysis should however be on understanding the relation between process resilience and requirements posed in the control problem. The approach therefore has to be transparent and unambiguous. In chapter four it was proposed to split the problem into subproblems. The major drawback of the approach was the inability to what the resulting input-output controllability of the overall process behavior will be on the basis of the insight obtained in the controllability of the subproblems. Moreover it turned out that in the existing approach it was difficult to analyze the behavior directly in relation with the actually required closed loop behavior of the process.

In chapter five we therefore proposed to use a completely different approach to the input-output controllability approach. The approach is based on the assumption that:

- the requirements are coupled to outputs of the process.
- the outputs, i.e. the requirements, can be ordered in a decreasing order of importance.

This results in the following basic iterative approach for a model with p outputs:

FOR i=1, .., p

Determine what restricts us from fulfilling the required closed loop behavior for output i , under the condition that control of output i does not change the closed loop behavior of the higher priority outputs 1 to i–1.

Next output,

Hence the approach boils down to finding for each output an approximate inverse that results in a lower triangular output complementary sensitivity transfer matrix, due to the assumed absolute priority that a requirement at output i has above the requirements posed on the outputs $i+1$ to p. The procedure successively analyzes all outputs starting with output one in this way.

The procedure developed in section 5.3 is a completely non analytic approach, based on the LQ-decomposition, i.e. the dual of the QR decomposition. As for the principal gains the LQ-decomposition is determined for a number of frequencies. The drawback of the approach is that stability of the controller and hence nominal stability of the closed loop, can only be guaranteed for a minimum phase square system. For non-minimum phase and or non square systems stability of the controller is not ensured anymore. As it turns out the approach may result in these cases in too optimistic a view on the controllability of the process.

An aproach was therefore developed that enabled us to incorporate nominal stability of the controller in the approach. This approach ,based on the concept of structural co-inner transfer matrices, results in a better insight in the squaring down problem. It is well known that down squaring of a process with a dynamic controller can always be achieved without introducing non-minimum phase zeros in the resulting square process. However down squaring of a process without introducing additional non-minimum phase zeros may result in unacceptably large gains of the compensator and/or a drastic difference in the magnitude of the principal gains of the square and the non square process. It turned out that introducing additional non-minimum phase zeros could drastically reduce the effect down squaring has on the principle gains. It was found that for a structural co-inner there is a direct relation between the number of non-minimum phase zeros we introduce and the minimum achievable infinity norm of the controllers that result in a square inner process after down squaring:

> *Hence there is a direct trade-off between the magnitude of the principal gains of the process after down squaring and the principle gains of the controller and the number additional non-minimum phase zeros introduced.*

The input-output controllability analysis techniques discussed until now were completely based on the frequency domain. However model predictive controller is by far the most applied MIMO control strategy in process industry. It was therefore inportant to develop an approach that enabled us to analyze the controllability of the process for this type of controller. An approach was therefore developed that enabled us to obtain insight in the controllability of the process over a finite horizon. It turns out that inner and co-inner systems, as they are defined on the infinite time horizon, also have a very specific behavior over the finite horizon. This enables a completely new interpretation of the effects that non-minimum phase behavior and the non squareness of a process have on the controllability of the process. This new interpretation is completely consistent with the infinite time interpretation. It turns out that for a non-minimum phase system a certain subspace in the output space exists that can not be freely used for control. It is shown that if this subspace is used to control the output over the control horizon this has severe consequences for the process behavior after the control horizon. The approach therefore results in completely different look at the so called *waterbed* effect:

> The waterbed effect is a consequence of the fact that we can not freely use a certain subspace of the output space for dynamic control of the process.

Analog considerations can be used to better understand the additional restrictions that non squareness of the process poses on the control problem. It moreover clearly reveals the relation between the gain and the number of additional non-minimum phase zeros introduced during down-squaring.

5.A Appendix A Proofs of section 5.2

Proof of Lemma 5.2.1

We may write the model $M(z)\in \Re H_\infty^{pxm}$ as:

$$M(z)=\begin{bmatrix} m_1(z) \\ \cdot \\ \cdot \\ m_p(z) \end{bmatrix} \tag{5.A.1}$$

- Applying per output a co-inner outer factorization, equation (3.3.12), which results for $i=1,..,p$ in:
  $m_i(z) = g_i(z)S_i(z)$
- We then apply again an inner outer factorization on the co-inner factor $S_i(z)$, which results for $i=1,..,p$ in:
  $S_i(z) = n_i(z)e_i(z)$

Now $n_i(z)\in \Re H_\infty^{1x1}$ is inner and contains all non-minimum phase zeros of the co-inner $S_i(z)$, conform section 3.3. Hence since $S_i(z)\in \Re H_\infty^{1xm}$ is co-inner and $n_i(z)\in \Re H_\infty^{1x1}$ contains all non-minimum phase zeros we have that the outer $n_i(z)\in \Re H_\infty^{1xm}$ is a structural co-inner.

Remark: See also lemma 5.4.3 for the continues version.

5.B Appendix B Proofs of section 5.3

Proof of Lemma 5.3.1

The representation is in fact a direct consequence of the fact that $C_{IA}C_p$ is the inverse of R and $R=EQ^T$:

$$EQ^T = \begin{bmatrix} e_1 \cdot q_1^T & 0 & . & 0 \\ . & e_2 \cdot q_2^T & . & . \\ . & . & . & 0 \\ e_p \cdot q_1^T & . & e_p \cdot q_{p-1}^T & e_p \cdot q_p^T \end{bmatrix} \tag{5.B.1}$$

From a mathematical point of view the proof is trivial. It however results in a good insight of the different dependencies and how they are related to the controller. We will perform a constructive proof of the i-th column of C_{IA}:

$$Eq_i^T \chi_i^{-1} = \begin{bmatrix} 0 \\ . \\ 0 \\ 1 \\ * \\ . \\ * \end{bmatrix} \begin{matrix} \\ \\ \leftarrow row\ \mathrm{i-1} \\ \leftarrow row\ i \\ \leftarrow \mathrm{row\ i+1} \\ \\ \\ \end{matrix} \tag{5.B.2}$$

To remove the interaction on row $i+1$, i.e. $e_{i+1}q_i^T\chi_i^{-1}$, without effecting the first i rows of the matrix, we have to use the complement of the space spanned by the first i rows of E. This complementary space is spanned exactly by the last columns of Q^T, i.e. q_{i+1}^T, q_{i+2}^T,.... Moreover the vector q_{i+1} is the vector in this subspace that is closest to the direction e_{i+1}. As a direct consequence we like to use q_{i+1}^T to compensate the interaction, i.e. $e_{i+1}(q_i^T\chi_i^{-1} + q_{i+1}^T\alpha_{i+1,i}) = 0$. which results for the parameter $\alpha_{i+1,i}$ in:

$$\alpha_{i+1,i} = -(e_{i+1}q_{i+1}^T)^{-1}e_{i+1}q_i^T\chi_i^{-1} = -\chi_{i+1}^{-1}e_{i+1}q_i^T\chi_i^{-1} \tag{5.B.3a}$$

or equivalently for the overall vector in:

$$(q_i^T - q_{i+1}^T\chi_{j+1}^{-1}e_{i+1}q_i^T)\chi_i^{-1} = \begin{bmatrix} q_i^T & q_{i+1}^T \end{bmatrix}\begin{bmatrix} \chi_i^{-1} & 0 \\ 0 & \chi_{j+1}^{-1} \end{bmatrix}\begin{bmatrix} \chi_i \\ -(e_{i+1} \cdot q_i^T) \end{bmatrix}\chi_i^{-1} \tag{5.B.3b}$$

In the next step we want to cancel the influence of the above vector on e_{i+2} in the same way as before: $e_{i+2}((q_i^T - q_{i+1}^T\chi_{j+1}^{-1}e_{i+1}q_i^T)\chi_i^{-1} + q_{i+2}^T\alpha_{i+2,i}) = 0$, results in:

$$\alpha_{i+2,i} = -(e_{i+2}q_{i+2}^T)^{-1}e_{i+2}(q_i^T - q_{i+1}^T\chi_{j+1}^{-1}e_{i+1}q_i^T)\chi_i^{-1} \quad (5.B.4a)$$

which is equivalent to:

$$\alpha_{i+2,i} = -\chi_{j+2}^{-1}e_{i+2}P(i+2)q_i^T\chi_i^{-1} \quad (5.B.4b)$$

where P(i+2) equals the oblique projection:

$$P(i+2) = (I - q_{i+1}^T\chi_{j+1}^{-1}e_{i+1}) = (I - q_{i+1}^T(e_{i+1}q_{i+1}^T)^{-1}e_{i+1}) \quad (5.B.4c)$$

It is therefore the projection along the direction e_{j+1} onto the orthogonal complement of q_{i+1}. The new i-th column of $Q^TC_{IA}C_p$ then equals:

$$\begin{bmatrix} q_i^T & q_{i+1}^T & q_{i+2}^T \end{bmatrix}\begin{bmatrix} \chi_i^{-1} & 0 & 0 \\ 0 & \chi_{i+1}^{-1} & 0 \\ 0 & 0 & \chi_{i+2}^{-1} \end{bmatrix}\begin{bmatrix} \chi_i \\ -(e_{i+1}\cdot q_i^T) \\ -e_{i+2}\cdot P(i+2)\cdot q_i^T \end{bmatrix}\chi_i^{-1} \quad (5.B.4d)$$

Induction is to be used to complete the proof.

Proof of Lemma 5.3.2

The proof follows trivially from the left side of equation (5.B.3b)

$$(q_i^T - q_{i+1}^T\chi_{j+1}^{-1}e_{i+1}q_i^T)\chi_i^{-1} = (I - q_{i+1}^T(e_{i+1}q_{i+1}^T)^{-1}e_{i+1})q_i^T\chi_i^{-1} \quad (5.B.5a)$$

which is equivalent to:

$$(q_i^T - q_{i+1}^T\chi_{j+1}^{-1}e_{i+1}q_i^T)\chi_i^{-1} = P(i+1)q_i^T\chi_i^{-1} \quad (5.B.5b)$$

where $P(i+1)$ is the oblique projection onto the orthogonal complement of e_{i+1}^T along q_{i+1}^T. The fact that the projection is along q_{i+1}^T means that the length of the vector in the direction spanned by the orthogonal complement of q_{i+1}^T is not changed. Note that q_i^T falls completely in this complement. The resulting vector therefore lies in the orthogonal complement of e_{i+1}^T and has a unchanged length in the direction q_i^T. To remove the interaction between the vector in (5.B.5b) and the next row of E, i.e. e_{i+2}, we have again to construct an oblique projection that does not change the length of the vector in the original direction but transforms it to the space orthogonal to e_{i+2}. This projection is given by equation (5.B.4c). We therefore obtain as new vector for the i-th column of $Q^TC_{IA}C_p$:

$$(I - q_{i+1}^T\chi_{j+1}^{-1}e_{i+1})P(i+1)q_i^T\chi_i^{-1} = P(i+2)P(i+1)q_i^T\chi_i^{-1} \quad (4.B.5c)$$

which can be continued until $i+(p-i)$.

5.C *Appendix 5.C Proofs of section 5.4.*

Proof of lemma 5.4.1

The factorization $M(s)=G_0(s)G_s(s)$ with $M(s)\in\Re H^{p\times m}$ $m>p$, can be obtained in two steps: First perform the outer/co-inner factorization [Doy84]:

$$M(s)=N^{-1}(s)N_I(s)$$

with:

- $N(s)\in\Re H^{p\times p}$ is a square stable minimum phase transfer matrix whose singular values of the transfer matrix equal that of $M(s)$. A realization of $N(s)$ is given by: $\left[A+HC,\quad H,\quad UR^{-1/2}C,\quad UR^{-1/2}\right]$ (5.C.1a)
- $N_I(s)\in\Re H^{p\times m}$ is a co-inner, i.e. $N_I(s)^{\sim}N_I(s)=I$. A realization of $N_I(s)$ is given by: $\left[A+HC,\quad B+HD,\quad UR^{-1/2}C,\quad UR^{-1/2}D\right]$ (5.C.1b)

withh:

- $U\in\Re^{p\times p}$ and $UU^T=I$
- $R=D^TD$
- $H=-(-XC^T+BD^T)R^{-1}$ and X fulfills the Algebraic Ricatti Equation:

$$(A-BD^TR^{-1}C)X+X(A-BD^TR^{-1}C)^T+XC^TR^{-1}CX+B(I-D^TR^{-1}D)B^T=0$$

In the second step we have to factor out possible non-minimum phase zeros from $N_I(s)$, using Lemma 5.4.3. Multiplying the outer with the transfer containing these non-minimum phase zeros then results in the factorization.

Proof of theorem 5.4.1

For $G(s)\in\Re H^{p\times m}$ structural co-inner, i.e. $m>p$, with realization $[A,B,C,D]$, controllability grammian P and observability grammian Q we obtain as a realization of $G(s)[G_R(s)\ G_A(s)]$:

$$[\begin{bmatrix} A & -(BB^TQ+BD^TC) \\ 0 & A \end{bmatrix},\begin{bmatrix} BD^T & BD_\perp^T \\ (I-QP)^{-1}BD^T & (I-QP)^{-1}BD_\perp^T \end{bmatrix},$$

$$\begin{bmatrix} C & -(DB^TQ+C) \end{bmatrix},\begin{bmatrix} I & 0 \end{bmatrix}]$$

Applying the state transformation $T_s=\begin{bmatrix} I & -(I-QP) \\ 0 & I \end{bmatrix}$ (5.C.2) on the above realization results in:

$$[\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ (I-QP)^{-1}BD^T & (I-QP)^{-1}BD_\perp^T \end{bmatrix}, [C \quad 0], [I \quad 0]]$$

Use the following square inner $\begin{bmatrix} G^T(s) & G_\perp^T(s) \end{bmatrix}^T$ properties (lemma 2.10 [Glo89]) to proof the inner property for $G_A(s)$:

1. $AP + PA^T + BB^T = 0$
2. $QA + A^TQ + C^TC = 0$
3. $Q_\perp = P^{-1} - Q = P^{-1}(I - PQ)$
4. $D_\perp B^T + C_\perp P = 0$
5. $DB^T + CP = 0$
6. $DD^T = I$ and $D_\perp D_\perp^T = I$

The controllability gramian P_A of $G_A(s)$ is determined by:

$$AP_A + P_A A^T + (I - PQ)^{-1} BD_\perp^T D_\perp B^T (I - PQ)^{-T} = 0$$

Substitution of inner property 3 and 4 in this equation results in $P_A=(I-PQ)^{-1}P$. For the observability gramian Q_A of $G_A(s)$ is determined by:

$$A^TQ_A + Q_A A + (QB + C^TD)(B^TQ + D^TC) = 0$$

Substitute inner property 6, 1, 2 and 5 in this equation results in $Q_A=Q(I-PQ)$. Now lemma 2.10 of [Glo89] can be used to verify that $G_A(s)$ is inner.

Proof of equation (5.4.4)

This follows directly from making $\begin{bmatrix} G^\sim(s) & G_A(s) \end{bmatrix} \begin{bmatrix} I \\ A(s) \end{bmatrix}$ minimal by applying the similarity transform $T = \begin{bmatrix} I & 0 & -I \\ 0 & I & -P(I-QP)^{-1} \\ 0 & 0 & I \end{bmatrix}$ and using the co-inner properties and the fact that $P(I-QP)^{-1}$ is the inverse of the solution of the Lyapunov equation:

$$A^TQ_\perp + Q_\perp A + C_\perp^T C_\perp = 0$$

Proof of lemma 5.4.2

This is a direct consequence of lemma 5.4.6.

Proof of lemma 5.4.3
See [Sto94]

Proof of lemma 5.4.4
Let $G(s)$ be a co-inner transfer matrix, with a balanced realization with gramians equal to $P=Q=\begin{bmatrix} I & 0 \\ 0 & Q_2 \end{bmatrix}$ and $Q_2 < \mathrm{I}$. Partition the realization conform the gramians:

$$\left[\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D\right]$$

From the co-inner properties we obtain:

$$B_1 B_1^T = C_1^T C_1 \tag{5.C.3a}$$

$$A_{12} Q_2 + A_{21}^T = -B_1 B_2^T \tag{5.C.3b}$$

$$A_{12} + A_{21}^T Q_2 = -C_1^T C_2 \tag{5.C.3c}$$

$$C_1 = -D B_1^T \tag{5.C.3d}$$

$$C_2 = -D B_2^T Q_2^{-1} \tag{5.C.3e}$$

Post multiplying (5.C.3c) by Q_2 and subtracting it from (5.C.3d) and then using (5.C.3d) and (5.C.3e) results in:

$$A_{21}^T (I - Q_2^2) = -B_1 (I - D^T D) B_2^T \tag{5.C.3f}$$

Now using (5.C.3a) and (5.C.3d) results in $B_1 (I - D^T D) B_1^T = 0$ and hence $B_1 (I - D^T D) = 0$ which results in $A_{21} = 0$ and $A_{12} = -C_1^T C_2$. From $B_1 (I - D^T D) = 0$ we obtain $B_1 = -C_1^T D$ and for the realization of $G(s)$:

$$\left[\begin{bmatrix} A_{11} & -C_1^T C_2 \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} -C_1^T D \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D\right]$$

which is equal to $G_{sq}(s) G_{mp}(s)$

Proof of equation (5.4.6)
The complementary inner of the right annihilator follows directly from the definition of the complementary inner. The realization of $G(s)G_A^{\perp}(s)$ directly follows from applying the state similarity transform T as in (4.B.2) on the non-minimal realization obtained $G(s)G_A^{\perp}(s)$.

Proof of lemma 5.4.5

We first show the existence of the realization of $G(s)$ in lemma 5.4.5. Assume a minimum realization $\left[\tilde{A},\tilde{B},\tilde{C},\tilde{D}\right]$ for $G(s)$, with controllability and observability graimians equal to respectively $\tilde{P}$ and $\tilde{Q}$. As a first step apply the simularity transform $T_1 = \tilde{Q}^{T/2}$, where $\tilde{Q}^{T/2}$ is the Cholesky factor of $\tilde{Q}$. Apply on the resulting realization the simularity transformation $T_2 = V^T$, where V is determined by the singular value decomposition of the observability matrix of the new realization, say $\overline{P}$, i.e. $\overline{P} = V\Sigma V^T$. The realization is now output balanced with a diagonal P. Now apply a real Schur decomposition on the obtained state transition matrix A and sorting in the upper triangular block the poles related to the non-minimum phase zeros to be introduced in the square down problem results in the realization in lemma 5.4.5. The real Schur decomposition and the ordering of the eigenvalues of the state transition matrix resulting from the previous state transformations, say $\hat{A}$, is equivalent to an orthonormal transformation U, i.e. $A = U\hat{A}U^T$. The controllability matrix then equals $P = U\Sigma U^T$ and the observability matrix is not changed by the last two simularity transforms. The factorization equation (5.4.7a) has to fulfill equation (5.4.7b). Write $G(s)$ as: $G(s) = G_I(s)\overline{G}(s)$, with $G_I(s)$ given in equation (5.4.8a). $\overline{G}(s)$ is all-pass, but not necessarily stable. $\overline{G}^T(-\bar{s}) = G^T(-\bar{s})G_I^T(-\bar{s})$ is an inverse of $\overline{G}(s)$. Apply the simularity transformation $T = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ on the realization obtained for $G^T(-\bar{s})G_I^T(-\bar{s})$, yields the following realization for the inverse of $\overline{G}(s)$, i.e. for $\overline{G}^T(-\bar{s})$:

$$\left[\begin{bmatrix} A_{11} & 0 \\ 0 & -A_{22}^T \end{bmatrix} \begin{bmatrix} -C_1^T \\ -C_2^T \end{bmatrix} \left[\left(B_1^T + D^T C_1\right) \quad B_2^T\right] \quad D^T\right] \tag{5.C.4a}$$

From the above realization we see that the stable poles of the realization of $\overline{G}^T(-\bar{s})$ are related to the additional non-minimum phase zeros that are introduced by the down squaring operation. The transfer equals the sum of a stable, say $\overline{G}_S^T(-\overline{s})$, with realization $\left[A_{11} \quad -C_1^T \quad \left(B_1^T + D^T C_1\right) \quad D^T\right]$ and an anti-stable part $\overline{G}_U^T(-\overline{s})$, with realization $\left[-A_{22}^T \quad -C_2^T \quad B_2^T \quad 0\right]$. Stabilization of $\overline{G}_U^T(-\overline{s})$ can be obtained by using a right annihilator of $G(s)$. We will make use of a completely unstable annihilator ,ie. the inverse of the complementary co-inner of $G(s)$. This inverse $G_\perp^T(-\overline{s})$ has realization $\left[-A^T \quad -C_\perp^T \quad B^T \quad D_\perp^T\right]$. We thus have to find a matrix $H(s)$ such that $\left[\overline{G}_U^T(-\overline{s}) \quad G_\perp^T(-\overline{s})\right]\begin{bmatrix} I \\ H(s)\end{bmatrix}$ is stable. A minimum realization for $\left[\overline{G}_U^T(-\overline{s}) \quad G_\perp^T(-\overline{s})\right]$ is obtained after applying the similarity transform

$$T = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}:$$

$$\left[\begin{bmatrix} -A_{11}^T & 0 \\ -A_{21}^T & -A_{22}^T \end{bmatrix} \begin{bmatrix} 0 & -C_{\perp_1}^T \\ -C_2^T & -C_{\perp_2}^T \end{bmatrix} \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} \begin{bmatrix} 0 & D^T \end{bmatrix}\right] \tag{5.C.4b}$$

For this realization we need to find a state feedback $\begin{bmatrix} 0 \\ F \end{bmatrix}$ and input tranformation $\begin{bmatrix} I \\ X \end{bmatrix}$ such that the above equation is stable. A stabilizing solution for this problem is $F = C_\perp Q_\perp^{-1}$ and $X = 0$, where $Q_\perp$ is the observability graimian of the complementary co-inner $G_\perp(s)$. Since it is the complementary inner we obtain from the fact that $\left[G^T(s) \quad G_\perp^T(s)\right]^T$ is a square inner transfer matrix:

$$Q_\perp = P^{-1} - Q = P^{-1}(I - P) \tag{5.C.4c}$$

This will turn out to be the H_2 optimal solution to the problem. Noting that $Q_\perp A Q_\perp^{-1} = -\left(A^T + C_\perp^T C_\perp Q_\perp^{-1}\right)$ and applying the state simularity transformation $T = Q_\perp$ results after adding $\overline{G}_S^T(-\overline{s})$ in the stable right inverse $G_{RI}^k(s)$ has as a non minimal realization:

$$\left[\begin{bmatrix} A_{11} & 0 \\ 0 & A \end{bmatrix} \quad \begin{bmatrix} -C_1^T \\ -Q_\perp^{-1}\begin{bmatrix} 0 \\ C_2^T \end{bmatrix} \end{bmatrix} \quad \left[\left(B_1^T + D^T C_1\right) \quad \left(B^T Q_\perp^T + D^T C_\perp\right)\right] \quad D^T \right]$$

Using the following properties of the square inner $\left[G^T(s) \quad G_\perp^T(s)\right]^T$:

1. $Q_\perp = P^{-1} - Q = P^{-1}(I - P)$
2. $D_\perp B^T + C_\perp P = 0$
3. $DB^T + CP = 0$
4. $D^T D + D_\perp^T D = I$

results in the following equivalence: $B^T Q_\perp^T + D^T C_\perp = -\left(B^T + D^T C\right)$. If use this relation and apply the simularity transform $T_1 = \begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}$ on the above realization of $G_{RI}^k(s)$, we obtain as a realization for $G_{RI}^k(s)$ after removing the unobservable part with $T_2 = \begin{bmatrix} -I & 0 & 0 \\ 0 & 0 & -I \end{bmatrix}$:

$$\left[A \quad X \quad -\left(B^T + D^T C\right) \quad D^T\right] \tag{5.C.4.d}$$

This results in, after using $Q_\perp^{-1} = P(I - P)^{-1} = -I + (I - P)^{-1}$:

$$X = (I - P)^{-1}\left(\begin{bmatrix} C_1^T \\ 0 \end{bmatrix} + BD^T\right) \tag{5.C.4e}$$

The stable annihilator is equivalent to the one used in theorem (5.4.1). To obtain the bound on the infinity norm we use the fact that the norm is invariant for multiplication of the criterion with an allpass function, i.e.:

$\left\|G_{RI}^k(s)\right\|_\infty = \left\|\begin{bmatrix} G_A^{\perp^T}(-\bar{s}) \\ G_A^T(-\bar{s}) \end{bmatrix} G_{RI}^k(s)\right\|_\infty$, where the realization of $G_A(s)$ and $G_A^\perp(s)$ are given in equation (5.4.2b) and (5.4.6a). For the upper part of the norm at the right hand side of the above equation we obtain $G_A^{\perp^T}(-\bar{s})G_{RI}^k(s)$ is a square inner, with realization $\left[-A_{22}^T \quad -C_2^T \quad C_2^T \quad I\right]$.

Use $-\left(B^T + D^T C\right)\left(B^T + D^T C\right) = (I-P)A + A^T(I-P)$ and apply the state simularity transformation $T = \begin{bmatrix} I & -(I-P) \\ 0 & I \end{bmatrix}$ on the realization of $G_A^T(-\bar{s})G_{RI}^k(s)$ results in: $\left[-A_{22}^T \quad -C_2^T \quad -\left(C_\perp (I-P)^{-1} P\begin{bmatrix} 0 \\ I \end{bmatrix}\right) \quad 0\right]$, which is completely unstable and therefore equals the H_2 optimal solution of the problem. The controllability gramian of the above realization equals identity. After using the fact that $-C_\perp C_\perp^T = (P^{-1} - I)A + A^T(P^{-1} - I)$ we obtain for the observability gramian: $\begin{bmatrix} 0 & I \end{bmatrix}(I-P)^{-1} P\begin{bmatrix} 0 \\ I \end{bmatrix}$. Using $P = U\Sigma U^T$ we obtain that: $\begin{bmatrix} 0 & I \end{bmatrix}(I-P)^{-1} P\begin{bmatrix} 0 \\ I \end{bmatrix} = U_2^T \Sigma (I-\Sigma)^{-1} U_2$.

From the Hankel approximation theory we obtain that the infinity norm must be larger than or equal to $\lambda_1^{1/2}\left(I + \begin{bmatrix} 0 & I \end{bmatrix}(I-P)^{-1} P\begin{bmatrix} 0 \\ I \end{bmatrix}\right)$.

Proof of lemma 5.4.6

To find the right inverse with minimum infinity norm we minimize this norm over all right inverses of $G(s)$ with at most k unstable poles. Using formula (5.4.2) we thus obtain:

$$\underset{A(s)\in \Re H_\infty^{(m-p)xp}(k)}{MIN} \left\| \begin{bmatrix} G_R(s) & G_A(s) \end{bmatrix} \cdot \begin{bmatrix} I \\ \mathrm{A}(s) \end{bmatrix} \right\|_\infty < \gamma \tag{5.C.5a}$$

Use the invariance of the infinity norm for multiplication with an allpass:

$$\underset{A(s)\in \Re H_\infty^{(m-p)xp}(k)}{MIN} \left\| \begin{bmatrix} (G_A^\perp(s))^\sim G_R(s) \\ (G_A(s))^\sim G_R(s) \end{bmatrix} - \begin{bmatrix} 0 \\ \mathrm{A}(s) \end{bmatrix} \right\|_\infty < \gamma \tag{5.C.5b}$$

A minimal realization of $\begin{bmatrix} (G_A^\perp(s))^\sim G_R(s) \\ (G_A(s))^\sim G_R(s) \end{bmatrix}$ equals

$\left[-A^T, C^T, \begin{bmatrix} CQ^{-1} \\ -C_\perp P(I-QP)^{-1} \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix}\right]$ This realization is obtained after applying $T_s = \begin{bmatrix} I & -(I-QP)Q \\ 0 & I \end{bmatrix}$ on the non minimal realization obtained from the multiplication. From this minimal realization we see that the resulting system is anti stable. Moreover $G_A^{\perp^T}(-\bar{s})G_R(s)$ is allpass. We thus obtain that the minimization criterion in (5.C.4) is equivalent to:

$$\underset{A(s)\in\Re H_\infty^{(m-p)xp}(k)}{MIN} \quad \left\| G_A^T(-\bar{s})G_R(s) - A(s) \right\|_\infty \leq \sqrt{(\gamma^2 - 1)} \qquad (5.C.5c)$$

The controllability gramian and observability gramian of the antistable realization of $G_A^T(-\bar{s})G_R(s)$ equal $P_S = -Q$ and $Q_S = -(I-PQ)^{-1}P$. For the Hankel singular values of the antistable realization we then obtain:

$$\lambda_i^{1/2}(Q_S P_S) = \lambda_i^{1/2}((I-PQ)^{-1}PQ)$$

Since the realization of $G_A^T(-\bar{s})G_R(s)$ is antistable the minimization of equation (5.C.5c) equals a Hankel approximation problem [Glo89,Glo84]. Assume that the Hankel singular values are ordered in descending order. For the minimum infinity norm of (5.C.5c) we obtain from [Glo89] that the minimum infinity norm γ will fulfill $\gamma^2 - 1 \geq \lambda_{k+1}(PQ(I-PQ)^{-1})$ if we introduce k unstable poles, which is equivalent to:

$$\gamma^2 \geq \lambda_{k+1}((I-PQ)^{-1}) \qquad (5.C.6a)$$

which proofs the lemma 5.4.2 and lemma 5.4.6

Proof of lemma 5.4.7

This can be found in for example [Doy84].

Proof of lemma 5.4.8

A solution of the problem is called suboptimal if in equation (5.C.6a) γ is such that the inequality is strict. If equality holds the solution is called optimal. From the Hankel theory we are able to give a generalized realization of the controller A(s) ([Glo89] (section 3)) for the suboptimal solution to equation (5.C.5c):

$$A(s) = -\Im_l(\bar{J}(s), K(s)) \qquad (5.C.6b)$$

where:

- A realization of $\bar{J}(s)$ is given by:

$$\left[\overline{Z},\ \ \overline{Z}A+PC^TC,\ \ P\begin{bmatrix}C^T & C_\perp^T\end{bmatrix},\ \begin{bmatrix}-C_\perp PQ(I-PQ)^{-1}\\ C\end{bmatrix},\ \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\right] \tag{5.C.6c}$$

with: $\overline{Z}=(I-\gamma^2(I-PQ))$

- $K(s)$ is any transfer that fulfills:

$$K(s)\in\Re H_\infty^{(m-p)xp}\ \text{ and }\ \|K(s)\|_\infty<\sqrt{(\gamma^2-1)} \tag{5.C.6d}$$

The controller $G_{RI}(s)$ is given by:

$$G_{RI}(s)=\begin{bmatrix}G_R(s) & G_A(s)\end{bmatrix}\begin{bmatrix}I\\ \Im_l(\overline{J}(s),K(s))\end{bmatrix} \tag{5.C.6e}$$

which is equivalent to $G_{RI}(s)=\Im_l(J(s),K(s))$ (5.C.6f)

with $J(s)$ equal to:

$$J(s)=\begin{bmatrix}G_R(s) & G_A(s) & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}I & 0\\ \overline{J}_{11} & \overline{J}_{12}\\ \overline{J}_{21} & \overline{J}_{22}\end{bmatrix} \tag{5.C.6g}$$

In the suboptimal case the non-minimal realization of $J(s)$, say $[A_{nm}, B_{nm}, C_{nm}, D_{nm}]$, equals:

$$A_{nm}=\begin{bmatrix}A & (I-PQ)^{-1}BD_\perp^TC_\perp PQ(I-QP)^{-1}\\ 0 & A+\overline{Z}^{-1}PC^TC\end{bmatrix},$$

$$B_{nm}=\begin{bmatrix}(I-PQ)^{-1}BD^T & (I-PQ)^{-1}BD_\perp^T\\ PC^T & PC_\perp^T\end{bmatrix},$$

$$C_{nm}=\begin{bmatrix}-(B^TQ+D^TC) & -D_\perp^TC_\perp PQ(I-QP)^{-1}\\ 0 & C\end{bmatrix},\ D_{nm}=\begin{bmatrix}D^T & D_\perp^T\\ I & 0\end{bmatrix}$$

Applying the similarity transform $T=\begin{bmatrix}I & (I-PQ)^{-1}\overline{Z}\\ 0 & I\end{bmatrix}$ on this realization results in the suboptimal solution for J,

$$\left[A+\overline{Z}^{-1}PC^TC\ \ \overline{Z}^{-1}P\begin{bmatrix}C^T & C_\perp^T\end{bmatrix}\ \begin{bmatrix}-\gamma^2B^TQ-(\gamma^2-1)D^TC\\ C\end{bmatrix}\ \begin{bmatrix}D^T & D_\perp^T\\ I & 0\end{bmatrix}\right] \tag{5.C.7a}$$

with $\overline{Z} = (I - \gamma^2(I - PQ))$ (5.C.7b)

and $K \in \Re H_\infty^{(m-p)xp}$ $\quad \|K\|_\infty \leq \sqrt{(\gamma^2 - 1)}$ (5.C.7c)

In the case that we choose γ equal to $\lambda_i^{1/2}((I - QP)^{-1})$ we obtain a so called optimal solution [Glo89]. The matrix Z is singular and the above state space solution doesnot hold anymore. It is however outside the scope of this work.

Proof of lemma 5.4.9

To minimize the gain of the compensator we have to find one compensator that results in the desired square inner process $G_I(s)$. The right annihilator can then be used to minimize the H_∞-norm of the compensator analog to the proof of lemma 5.4.6. Note that a compensator that results in the desired process is given by the right inverse found in theorem 5.4.1 post multiplied with the inner $G_I(s)$. The set of compensators that fulfills $G_I(s) = G(s)G_{RI}(s)$ is then given by:

$$G_{RI}(s) = \begin{bmatrix} G_R(s) & G_A(s) \end{bmatrix} \begin{bmatrix} G_I(s) \\ \mathrm{A}(s) \end{bmatrix} \tag{5.4.12a}$$

where $G_R(s)$ and $G_A(s)$ are defined in theorem 5.4.1. The compensator which minimizes the maximum gain is thus found by solving:

$$\underset{\mathrm{A}(s)\in\Re\mathrm{H}_\infty^{(m-p)xp}}{\mathrm{MIN}} \quad \|G_{RI}(s)\|_\infty \tag{5.4.12b}$$

To find the norm we proceed along the same lines as in the proof of lemma 5.4.6 and make use of the invariance of the H_∞-norm for allpass functions. Using the same allpass as in lemma 5.4.6 results for (5.4.12b) in:

$$\underset{A(s)\in\Re H_\infty^{(m-p)xp}(k)}{MIN} \quad \left\| \begin{bmatrix} (G_A^\perp(s))^\sim G_R(s) \\ (G_A(s))^\sim G_R(s) \end{bmatrix} G_I(s) - \begin{bmatrix} 0 \\ \mathrm{A}(s) \end{bmatrix} \right\|_\infty < \gamma \tag{5.C.8a}$$

Note again that $(G_A^\perp(s))^\sim G_R(s)G_I(s)$ is allpas results in the equivalent criterion:

$$\underset{A(s)\in\Re H_\infty^{(m-p)xp}(k)}{MIN} \quad \left\| G_A^T(-\bar{s})G_R(s)G_I(s) - \mathrm{A}(s) \right\|_\infty \leq \sqrt{(\gamma^2 - 1)} \tag{5.C.8b}$$

In contrast to lemma 5.4.6 $(G_A(s))^\sim G_R(s)G_I(s)$ also contains the stable poles of $G_I(s)$. These poles have to be removed first. To do this we apply an additive decomposition on $(G_A(s))^\sim G_R(s)G_I(s)$:

$$(G_A(s))^{\sim} G_R(s)G_I(s) = X_S(s) + X_{AS}(s) \tag{5.C.8c}$$

with $X_S(s)$ the stable transfer matrix and $X_{AS}(s)$ the anti stable part. The stable part can directly be compensated by $A(s)$, i.e. $A(s)=-X_S(s)+A_S(s)$. The H_∞ problem to be solved then equals:

$$\underset{A_S(s)\in\Re H_\infty^{(m-p)xp}(k)}{MIN} \quad \|X_{AS}(s) - A_S(s)\|_\infty \le \sqrt{(\gamma^2 - 1)} \tag{5.C.8d}$$

which can be dealt with again by applying the standard theory as developed by Glover [Glo89, Glo84]. The realization for $(G_A(s))^{\sim} G_R(s)G_I(s)$ equals:

$$\left[\begin{bmatrix} -A^T & C^T C_I \\ 0 & A_I \end{bmatrix}, \begin{bmatrix} C^T D_I \\ B_I \end{bmatrix}, \begin{bmatrix} -C_\perp P(I-QP)^{-1} & 0 \end{bmatrix}, 0\right] \tag{5.C.8e}$$

After using some inner properties of the realization of $G_I(s)$ it is easy to see that the controllability grammian equals: $Q_T = \begin{bmatrix} -Q & 0 \\ 0 & P_I \end{bmatrix}$, with P_I the controllability gramian of the realization of $G_I(s)$. We now have to find a simularity transform $T_S = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ that makes the (1,2) block of the transition matrix zero, i.e.: $XA_I + A^T X + C^T C_I = 0$ (5.4.12d)

which is a Silvester equation, that has an unique solution since the spectra of A_I and $-A^T$ are disjunct. We then obtain for the realization:

$$\left[\begin{bmatrix} -A^T & 0 \\ 0 & A_I \end{bmatrix}, \begin{bmatrix} C^T D_I + XB_I \\ B_I \end{bmatrix}, \begin{bmatrix} -C_\perp P(I-QP)^{-1} & C_\perp P(I-QP)^{-1} X \end{bmatrix}, 0\right] \tag{5.C.8f}$$

The realization of the antistable $X_{AS}(s)$ then equals:

$$\left[-A^T, C^T D_I + XB_I, -C_\perp P(I-QP)^{-1}, 0\right] \tag{5.C.9}$$

Now noting that the (1,1) block of $T_S Q_T T_S^T$ equals the controllability gramian of the antistable realization results in $(Q - X_S P_I X_S^T)$. The observability gramian equals that of the antistable realization in lemma 5.4.6 and results in the bound on the infinity norm of lemma 5.4.9:

$$\|G_{RI}(s)\|_\infty \ge \lambda_i^{1/2}\left((I-PQ)^{-1} P(Q - X_S P_I X_S^T)\right) + 1 \tag{5.4.12c}$$

5.D *Appendix 5.D Proofs of section 5.6.*

Proof of Corollary 5.6.1
This result is a direct consequence of theorem 5.6.1 and lemma 5.6.1

Proof of Theorem 5.6.1
An inner transfer is non dissipative. As a consequence we have that for any input signal all energy entering the system therefore has to leave the system "sooner or later". More formally expressed:

$$\left\{\forall u(t)\in l_+^2 \;\middle|\; \|y(t)\|_2=\|u(t)\|_2\right\} \tag{5.6.6a}$$

Assume we have a *pxm* inner with *p*≥*m* and a minimum realization $[A,B,C,D]$ with *n* the dimension of the state space. A consequence of the nondissipativeness is that for any input signal $u(t)\in l_+^2(N)$ and any *N*, we have:

$$\sum_{i=0}^{N} u^T(i)u(i)=\sum_{i=0}^{N} y^T(i)y(i)+\sum_{i=N+1}^{\infty} y^T(i)y(i) \tag{5.6.6b}$$

Minimizing the first term on the right hand side of the above equation is equivalent to maximizing the second term on the right hand side. This last maximization is equivalent to maximizing the energy transfer from the past to the future, i.e. finding *n* nonzero singular values of the Hankel matrix Γ:

$$\Gamma(N,\infty,N,0)=\begin{bmatrix} CA^NB & CA^{N-1}B & . & . & CB \\ CA^{N+1}B & CA^NB & . & . & CAB \\ . & . & & & . \\ . & . & & & . \end{bmatrix}=N_0N_C(N,0) \tag{5.D.1}$$

This can be accomplished if we see the above problem as a special case of theorem 5.6 of [Wei91] pg.64. The matrix that maps the past inputs $u(t)$ to the state at time instant 0, say f_- exactly equals the finite controllability gramian $N_C(N,0)$. Note moreover that the pseudo inverse of N_0 is exactly the map that maps the past future behavior of the system back to the state at time instant 0. Defining this mapping as f_+, results in $f_+=N_0^T(N_0N_0^T)^{-1}=N_0^TQ^{-1}$. It is directly verified that $\Gamma(N+1,\infty,0,N)=f_+^T(f_+f_+^T)^{-1}f_-$. Based on this observation we directly obtain from theorem 5.6 [Wei91], that the Hankel singular values are equal to the square root of the eigenvalues of $P_N Q$, with $P_N = N_C(N,0)N_C^T(N,0)$. We will take a slightly different route to directly obtain the left singular vectors. First note that the eigenvalues of $P_N Q$ equal the singular values of $P_N^{T/2} Q P_N^{1/2}$ Using the short hand notation Γ for the above Hankel matrix, results in $\Gamma^T\Gamma = f_-^T Q f_-$. Assume x to be an right singular vector of $P_N^{T/2} Q P_N^{1/2}$ with σ_i^2 the corresponding singular value i.e. $P_N^{T/2} Q P_N^{1/2} x = \sigma_i^2 x$, due the fact that matrix is symmetric. Define $u_- = f_-^T P_N^{-T/2} x$ and observe that $\Gamma^T \Gamma f_-^T P_N^{-T/2} x$ equals:

$$\left(f_-^T Q f_-\right) f_-^T P_N^{-T/2} x = f_-^T Q P_N^{1/2} x = f_-^T P_N^{-T/2} P_N^{T/2} Q P_N x = \sigma_i^2 f_-^T x$$

As a consequence u_- must a singular vector of $\Gamma^T\Gamma$ and σ_i^2 the corresponding singular value. Conversely assume u_- to be a singular vector of $\Gamma^T\Gamma$ and σ_i^2 the corresponding eigenvalue, i.e. $\Gamma^T\Gamma u_- = \sigma_i^2 u_-$, then $P_N^{-1/2} f_- \Gamma^T \Gamma u_- = P_N^{T/2} Q x_i = P_N^{T/2} Q x_i = \sigma_i^2 P_N^{-1/2} f_-^T u_- = \sigma_i^2 P_N^{-1/2} x_i$. As a consequence $P_N^{-1/2} x$ must be an eigenvector of $P_N^{T/2} Q P_N^{1/2}$ and σ_i^2 the corresponding eigenvalue and therefore σ_i a singular value. The left singular vectors of the Hankel matrix are therefore given by

$$V_2 = N_c^T(j,0) P_N^{-1/2} W_T \tag{5.6.7c}$$

with W_T the matrix with right singular vectors of $P_N^{T/2} Q P_N^{1/2}$.

From equation (5.6.6b) and the fact that this holds for any input signal we obtain that the singular values of $T(N,N)$ smaller than one equal the square root of the eigenvalues of $(I-QP_N)$ and that V_2 equals the corresponding right singular vectors. The singular values of the Toeplitz matrix that are smaller than one are spanned by the columns of $N_C^T(N,0)$. To determine the left singular vectors of the Toeplitz matrix we restrict ourselves to inputs in this subspace:

$u_- = N_c^T(N,0) P_N^{-1/2} w$, with $w \in \Re^{nx1}$. For y_- we then obtain:

$$y_- = T(N,N) N_c^T(N,0)\left(N_c(N,0) N_c^T(N,0)\right)^{-1/2} w \tag{5.D.2a}$$

Let us first take a closer look at the square inner: For a discrete square inner the following holds: $DB^T = -CPA^T$ and $BB^T = P - APA^T$. Substitution of these relations in equation (5.D.2a) yields:

$$y_- = -\begin{bmatrix} C \\ CA \\ . \\ . \\ CA^N \end{bmatrix} P(A^T)^{N+1} P_N^{-1/2} w \tag{5.D.2b}$$

with: $P_N = N_c(N,0) N_c^T(N,0)$. Hence the output directions of the Toeplitz matrix corresponding to singular values of this matrix that are unequal to one are spanned by:

$$N_o(0,N)\left(N_o^T(0,N) N_o(0,N)\right)^{-1/2} Z_T \tag{5.D.2c}$$

with Z_T an orthonormal matrix still to be determined. The range for which the singular values of the input output Toeplitz matrix are one equals of course the orthogonal complement of $N_o(0,j)$ and has of course a dimension equal $p(N+1)-n$, with p the number of outputs and n the number of states, i.e. the number of poles. For the non square inner p>m the expression $DB^T = -CPA^T$ does not hold anymore, since it only holds for square matrices. In the case of a non square transfer matrix we obtain instead (chapter two): $DB^T = -\left(C\left(P + P_\perp\right)A^T + D_\perp B_\perp{}^T\right) = -\left(CQ^{-1}A^T + D_\perp B_\perp{}^T\right)$.

In this case we obtain for equation (5.D.2a):

$$y_- = -\left(\begin{bmatrix} C \\ CA \\ . \\ . \\ CA^N \end{bmatrix} P(A^T)^{N+1} - \begin{bmatrix} (CPA^T + DB^T)(A^T)^N \\ (CPA^T + DB^T)(A^T)^{N-1} \\ . \\ . \\ (CPA^T + DB^T) \end{bmatrix}\right) P_N^{-1/2} w \tag{5.D.2d}$$

which can be shown to equal:

$$y_- = \begin{bmatrix} -DB^T(A^T)^N \\ (CBB^TA^T - DB^T)(A^T)^{N-1} \\ (C(\sum_{l=0}^{1} A^l B(BA^l)^T)A^T - DB^T)(A^T)^{N-2} \\ . \\ C(\sum_{l=0}^{N-1} A^l B(BA^l)^T)A^T - DB^T \end{bmatrix} P_N^{-1/2} w \tag{5.D.2e}$$

Proof of theorem 5.6.2

Define the input output Toeplitz two by two inner system conform equation (5.6.10), with E equal to the zero matrix. The input output Toeplitz matrix of the inner then fulfills the equations (5.6.7) of theorem 5.6.1. The Toeplitz matrix of the resulting system with the influence of the zeros turned to a certain direction can be written as:

$$T_T(l,l) = T_I(l,l)T_C(l,l) \qquad (5.6.9)$$

with $T_C(l,l)$, the Toeplitz matrix of the controller that achieves the desired rotation of the influence of the zeros on the output of the system without altering the zero structure. The controller can only use the input directions corresponding to singular values equal to one to obtain performance at the output: $T_C(l,l) = V_1 W_C$ (5.D.3a)

V_1 is defined in theorem 5.6.1 and W_C is the free parameter that can be used to turn the influence of the zeros to more desired directions. As a result we obtain for T_T :

$$T_T(l,l) = U_1 W_C \qquad (5.D.3b)$$

From the partitioning in equation (5.6.8) we obtain that the first l rows of U_1, say U_{11}, are related to the first output and the second l rows, say U_{21}, are related to the second output. The assumption that the number of samples is larger than the number of zeros, i.e. $l>n$ and n the dimension of E, enables us to apply lemma 5.D.1 on U_1 in equation (5.6.). From lemma 5.D.1:

$$U_1 = \begin{bmatrix} Z_{11} & 0 & Z_{12} & 0 \\ 0 & Z_{21} & 0 & Z_{22} \end{bmatrix} \begin{bmatrix} I_{l-n} & 0 & 0 \\ 0 & I_{l-n} & 0 \\ 0 & 0 & \Sigma_n \\ 0 & 0 & (I-\Sigma_n^2)^{1/2} \end{bmatrix} \begin{bmatrix} W_1^T \\ W_3^T \\ W_2^T \end{bmatrix} \qquad (5.D.4)$$

We thus see that the trade-off of the influence of the zero between the different outputs, is determined by the part that is related to the input subspace W_2. Substitution of equation (5.D.5) of lemma 5.D.1 in the singular value decomposition of $T_I(l,l)$ results in:

$$\begin{bmatrix} Z_{11} & 0 & Z_{12}\Sigma_n & -Z_{12}\Sigma_C \\ 0 & Z_{21} & Z_{22}\Sigma_C & Z_{22}\Sigma_n \end{bmatrix} \begin{bmatrix} I_{l-n} & 0 & 0 & 0 \\ 0 & I_{l-n} & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & V_X^T E V_X \end{bmatrix} \begin{bmatrix} \tilde{V}_1^T \\ \tilde{V}_3^T \\ \tilde{V}_2^T \\ \tilde{V}_4^T \end{bmatrix}$$

with $\Sigma_C = (I-\Sigma_n^2)^{1/2}$, $\tilde{V}_i = V_1 W_i$ for i=1,2,3 and $\tilde{V}_4 = V_2 V_X$. As a consequence the trade-off between the different outputs is completely determined by the subspace of the input space spanned by the columns of $\tilde{V}_2$. We thus obtain for a $T_C(l,l)$:

$$T_C(l,l) = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_3 & \tilde{V}_2 & \tilde{V}_4 \end{bmatrix} \begin{bmatrix} I_{l-n} & 0 & 0 & 0 \\ 0 & I_{l-n} & 0 & 0 \\ 0 & 0 & X_1 & X_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_{11}^T & 0 \\ 0 & Z_{21}^T \\ Z_{12}^T & 0 \\ 0 & Z_{22}^T \end{bmatrix}$$

For $T_T(l,l)$ we obtain:

$$\begin{bmatrix} Z_{11}Z_{11}^T & 0 \\ 0 & Z_{21}Z_{21}^T \end{bmatrix} + \begin{bmatrix} Z_{12} & 0 \\ 0 & Z_{22} \end{bmatrix} \begin{bmatrix} \Sigma_n \\ (I-\Sigma_n^2)^{1/2} \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} Z_{12}^T & 0 \\ 0 & Z_{22}^T \end{bmatrix}$$

The first matrix on the right hand side of the above equation is a block diagonal matrix with as diagonal blocks two orthogonal projectors. Of interest for the rotation of the zero influence is of course the last term:

$$T_{T2}(l,l) = \begin{bmatrix} Z_{12} & 0 \\ 0 & Z_{22} \end{bmatrix} \begin{bmatrix} \Sigma_n \\ (I-\Sigma_n^2)^{1/2} \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} Z_{12}^T & 0 \\ 0 & Z_{22}^T \end{bmatrix}$$

The above equation can directly be reparameterized as:

$$\begin{bmatrix} Z_{12} & 0 \\ 0 & Z_{22} \end{bmatrix} \begin{bmatrix} \Sigma_n & -\Sigma_C \\ \Sigma_C & \Sigma_n \end{bmatrix} \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_n & \Sigma_C \\ -\Sigma_C & \Sigma_n \end{bmatrix} \begin{bmatrix} Z_{21}^T & 0 \\ 0 & Z_{22}^T \end{bmatrix}$$

with: $\begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \end{bmatrix} \begin{bmatrix} \Sigma_n & (I-\Sigma_n^2)^{1/2} \\ -(I-\Sigma_n^2)^{1/2} & \Sigma_n \end{bmatrix}$

This operation should result in a gain one in the desired output direction. This is therefore again an oblique projection as we used it also in section 3.5: We therefore obtain:

$$X_T = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} I & \Lambda \end{bmatrix} \begin{bmatrix} \Sigma_n & (I-\Sigma_n^2)^{1/2} \\ -(I-\Sigma_n^2)^{1/2} & \Sigma_n \end{bmatrix}$$

with $\Lambda \in \Re^{nxn}$ the free parameter.

Lemma 5.D.1

Given a matrix $U \in \Re^{(2l)x(2l)}$ with $UU^T = U^TU = I$. Partition U as

$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, with $U_{11} \in \Re^{lx(2l-n)}$ and $l>n$ and apply a singular value decomposition on U_{11}:

$$U_{11} = Z\Sigma_{11}W^T = \begin{bmatrix} Z_{11} & Z_{12} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & \Sigma_{n_2} & 0 \end{bmatrix} \begin{bmatrix} W_1^T \\ W_2^T \\ W_3^T \end{bmatrix} \tag{5.D.5a}$$

with:

- $I_{n_1} \in \Re^{n_1 x n_1}$ an identity matrix
- $\Sigma_{n_2} \in \Re^{n_2 x n_2}$ a diagonal matrix whose diagonal entries fulfill $\sigma_1 \geq \cdots \geq \sigma_{n_2 - 1} \geq \sigma_{n_2} \geq 0$
- $W_i \in \Re^{(2l-n)xn_i}$ and $W_i^T W_i = I$ and $Z_{1i} \in \Re^{lxn_i}$ and $Z_{1i}^T Z_{1i} = I$
- $n_1 = l - n$, $n_2 = n$ and $n_3 = n_1$

The following relation between the submatrices then exists:

$$U = \begin{bmatrix} Z_{11} & Z_{12} & 0 & 0 \\ 0 & 0 & Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & \Sigma_{n_2} & 0 & -\Sigma_C \\ 0 & 0 & I_{n_1} & 0 \\ 0 & \Sigma_C & 0 & \Sigma_{n_2} \end{bmatrix} \begin{bmatrix} W_1^T & 0 \\ W_2^T & 0 \\ W_3^T & 0 \\ 0 & V_X^T \end{bmatrix} \tag{5.D.5b}$$

with:

- $V_X \in \Re^{nxn}$, with $V_x^T V_x = I$ and $V_x = -U_{12}^T Z_{12}(I - \Sigma_{n_2}^2)^{-1/2}$
- $Z_{2i} \in \Re^{lxn_i}$, with $Z_{1i}^T Z_{1i} = I$ and determined by:

$$\begin{bmatrix} Z_{21} & Z_{22} \end{bmatrix} = U_{21} \begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ (I - \Sigma_{n_2}^2)^{-1/2} & 0 \\ 0 & I_{n_1} \end{bmatrix}$$

Proof of Lemma 5.D.1:

Given the orthonormal matrix, which is partitioned as

$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, with $U_{11} \in \Re^{lx(2l-n)}$, $U_{12} \in \Re^{lxn}$, $U_{21} \in \Re^{lx(2l-n)}$, $U_{22} \in \Re^{lxn}$ and $l>n$. The orthonormality, i.e.

$U_{11}U_{11}^T+U_{12}U_{12}^T=I$ and the dimensions of the partitioning of the matrix result in the following rank conditions:

1. $l-n \leq rank(U_{11}) \leq l$
2. $0 \leq rank(U_{12}) \leq n$

As a direct consequence U_{12} has to fulfill:

$$U_{12} = \pm Z_{12}\left(I - \Sigma_{n_2}^2\right)^{1/2} V_X^T \tag{5.D.6a}$$

Let us choose the minus sign as solution. For the singular value decomposition of U_{11} we therefore must fulfill equation (5.D.5a). From $U_{12}^T U_{12}+U_{22}^T U_{22}=I$ we then obtain:

$$U_{22} = Z_{22}\Sigma_{n_2} V_X^T \tag{5.D.6b}$$

and therefore from $U_{21}U_{21}^T+U_{22}U_{22}^T=I$ and $U_{11}^T U_{11}+U_{21}^T U_{21}=I$ we get:

$$U_{22} = \begin{bmatrix} Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 & I_{n_1} \\ 0 & \Sigma_{n_2} & 0 \end{bmatrix} \begin{bmatrix} W_1^T \\ W_2^T \\ W_3^T \end{bmatrix} \tag{5.D.6c}$$

which results in equation (5.D.5b).

Extension of the result to the case that we have three outputs.

Lemma 5.D.2

Given a matrix $U \in \Re^{(3l)x(3l)}$ with $UU^T=U^TU=I$. Partition U as:

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{23} \end{bmatrix} \tag{5.D.7a}$$

with $U_{ji} \in \Re^{lxn_i}$ where $j \in \{1,2,3\}$, $i \in \{1,2\}$, $n_1=l-n$, $n_2=n$ and $l>n$. For U exists a factorization $U = ZSV^T$ (5.D.7b)

with:

- $$Z = \begin{bmatrix} Z_{11} & 0 & 0 & Z_{12} & 0 & 0 \\ 0 & Z_{21} & 0 & 0 & Z_{22} & 0 \\ 0 & 0 & Z_{11} & 0 & 0 & Z_{32} \end{bmatrix} \tag{5.D.7c}$$

CHAPTER SIX

Application of the Developed Techniques on Example Processes.

- $$S=\begin{bmatrix} I_{3n_1} & 0 & 0 & 0 \\ 0 & \Sigma_1 & 0 & (I-\Sigma_1^2)^{1/2} \\ 0 & -\left((I-\Sigma_1^2)(I-\Sigma_2^2)\right)^{1/2} & \Sigma_2 & \Sigma_1(I-\Sigma_2^2)^{1/2} \\ 0 & -\Sigma_2(I-\Sigma_1^2)^{1/2} & -(I-\Sigma_1^2)^{1/2} & \Sigma_1\Sigma_2 \end{bmatrix} \tag{5.D.7d}$$
- $$V=\begin{bmatrix} W_1 & W_2 & W_3 & W_4 & W_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_X \end{bmatrix} \tag{5.D.7e}$$

with:

- $\Sigma_1 \in \Re^{nxn}$ a diagonal matrix whose diagonal entries fulfill $1\geq\sigma_1\geq ..\geq\sigma_n\geq 0$
- $\Sigma_2 \in \Re^{nxn}$ a diagonal matrix whose diagonal entries fulfill $\sigma_1 \geq \cdots \geq \sigma_{n_2-1} \geq \sigma_{n_2} \geq 0$
- $Z_{ji} \in \Re^{lxn_i}$,with $Z_{ji}^T Z_{ji} = I$, where $j\in\{1,2,3\}$, $i\in\{1,2\}$, $n_1=l-n$, $n_2=n$
- $W_j \in \Re^{(3l-n)x(l-n)}$ with $j\in\{1,2,3\}$ and $W_j^T W_j = I$.
- $W_j \in \Re^{(3l-n)xn}$ with $j\in\{4,5\}$ and $W_j^T W_j = I$
- $V_X \in \Re^{nxn}$,with $V_x^T V_x = I$

Proof of Lemma 5.D.2:

The proof of the lemma is essentially the same as that of lemma 5.D.1. The orthonormality, i.e. $U_{i1}U_{i1}^T+U_{i2}U_{i2}^T=I$ and the dimensions of the partitioning of the matrix result in the following rank conditions for $i\in\{1,2,3\}$:

1. $3l-n \leq rank(U_{i1}) \leq 3l$
2. $0\leq rank(U_{i2}) \leq n$ (5.D.8a)
3. $rank(U_{i1})+rank(U_{i2})=l$

As a direct consequence U_{11} and U_{12} have to fulfill:

$$U_{11}=\begin{bmatrix} Z_{11} & Z_{12} \end{bmatrix}\begin{bmatrix} I_{l-n} & 0 \\ 0 & \Sigma_1 \end{bmatrix}\begin{bmatrix} W_1^T \\ W_4^T \end{bmatrix} \tag{5.D.8b}$$

$$U_{12}=\pm Z_{12}\left(I-\Sigma_1^2\right)^{1/2} V_X^T \tag{5.D.8c}$$

Let us choose the plus sign as solution. From the fact that $U_{12}^TU_{12}+U_{22}^TU_{22}+U_{32}^TU_{32}=I$ we obtain:

$$U_{22} = \pm Z_{22}\Sigma_1\left(I-\Sigma_2^2\right)^{1/2}V_X^T \tag{5.D.9a}$$
$$U_{32} = \pm Z_{32}\Sigma_1\Sigma_2V_X^T \tag{5.D.9b}$$

Let us choose the plus sign for both solutions. As a consequence of the rank condition (5.D.8a) we obtain:

$$U_{21} = \begin{bmatrix}\tilde{Z}_{21} & \tilde{Z}_{22}\end{bmatrix}\begin{bmatrix} I_{l-n} & 0 \\ 0 & \left(I-\Sigma_1^2(I-\Sigma_2^2)\right)^{1/2}\end{bmatrix}\begin{bmatrix}W_2^T \\ \tilde{W}_{21}^T\end{bmatrix} \tag{5.D.9c}$$

$$U_{31} = \begin{bmatrix}\tilde{Z}_{31} & \tilde{Z}_{32}\end{bmatrix}\begin{bmatrix} I_{l-n} & 0 \\ 0 & \left(I-\Sigma_1^2\Sigma_2^2\right)^{1/2}\end{bmatrix}\begin{bmatrix}W_3^T \\ \tilde{W}_{31}^T\end{bmatrix} \tag{5.D.9d}$$

From the fact that $U_{11}^TU_{11}+U_{21}^TU_{21}+U_{31}^TU_{31}=I$ that $W_1 \perp W_2 \perp W_3$ and as further consequence:

$$W_4\left(I-\Sigma_1^2\right)W_4^T + \tilde{W}_{21}\left(I-\Sigma_1^2(I-\Sigma_2^2)\right)\tilde{W}_{21}^T + \tilde{W}_{31}\left(I-\Sigma_1^2\Sigma_2^2\right)\tilde{W}_{31}^T = W_4W_4^T + W_5W_5^T$$

or equivalently:

$$\tilde{W}_{21}\left(I-\Sigma_1^2(I-\Sigma_2^2)\right)\tilde{W}_{21}^T + \tilde{W}_{31}\left(I-\Sigma_1^2\Sigma_2^2\right)\tilde{W}_{31}^T = W_4\Sigma_1^2W_4^T + W_5W_5^T \tag{5.D.9e}$$

Together with the fact that $U_{11}U_{21}^T+U_{12}U_{22}^T=I$, $U_{11}U_{31}^T+U_{12}U_{32}^T=0$ and $U_{21}U_{31}^T+U_{22}U_{32}^T=0$ this results in the solution (5.D.7).

Generalization to the p-output case

From lemma 4.C1 and Lemma 5.D.2 it is easy to see that in case of p outputs we obtain the following form for the left singular vector matrix:

$$U = ZSY^T = \begin{bmatrix}Z_1 & Z_2\end{bmatrix}\begin{bmatrix} I_{pn_1} & 0 & 0 \\ 0 & R_1 & R_2\end{bmatrix}\begin{bmatrix}W_1^T & 0 \\ W_2^T & 0 \\ 0 & V^T\end{bmatrix} \tag{5.D.10}$$

with:

- $Z = \begin{bmatrix}Z_1 & Z_2\end{bmatrix} \in \Re^{plx(p(l-n)+pn)}$ and $ZZ^T = Z^TZ = I$, where:

  $Z_1 \in \Re^{plxp(l-n)}$ is a block diagonal matrix with j-th diagonal block entry $Z_{1j} \in \Re^{lx(l-n)}$ and $Z_{1j}^TZ_{1j} = I$. $Z_2 \in \Re^{plxpn}$ is a block diagonal matrix with j-th diagonal block entry $Z_{2j} \in \Re^{lxn}$ and $Z_{2j}^TZ_{2j} = I$
- $W = \begin{bmatrix}W_1 & W_2\end{bmatrix} \in \Re^{(pl-n)x(p(l-n)+(p-1)n)}$ with $WW^T = W^TW = I$.

- $V \in \Re^{nxn}$ with $VV^T = V^T V = I$.
- $R = \begin{bmatrix} R_1 & R_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \in \Re^{((p-1)n+n)x((p-1)n+n)}$ with $RR^T = R^T R = I$

  and $R_{11} \in \Re^{(p-1)nx(p-1)n}$ is a triangular matrix.

The matrix R in fact determines the trade-off's we need to make if we want to manipulate the behavior of the Toeplitz of a *pxp* inner transfer matrix with n non-minimum phase zeros.

Proof of Theorem 5.6.3

From the Toeplitz $T(2N,N)$ matrix take first the part that is related to the far future: This part of the system represents the influence of the past on the future and is therefore equivalent to the Hankel matrix (section 2.,3):

$$T(2N,N) = \begin{bmatrix} T(N,N) \\ H(0N,N,0) \end{bmatrix}$$

As a first step we apply a singular value decomposition on this Hankel matrix $H = U_H \Sigma_H V_H^T$ and reorder the factorization such that the n non zero singular values are not eqaul to the firsst upper left diagonal terms, but equal to last n values on the diagonal entries of Σ_H. Postmultiply the overall Toeplitz $T(2N,N)$ with the reordered V_H results in:

$$T_T(2N,N) = T(2N,N)\begin{bmatrix} V_{H1} & V_{H2} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} T_{T1} \\ 0 \end{bmatrix} & T_{T2} \end{bmatrix}$$

with:

- $T_{T1} \in \Re^{pNxmN-n}$ and $T_{T2} \in \Re^{2pNxn}$
- $V_{H1} \in \Re^{mNxmN-n}$ and $V_{H2} \in \Re^{mNxn}$

The above factorization directly follows from the ordering we choose. As a consequence T_{T2} contains the part of the system that influences the far future of the system. Applyiung a singular value decomposition on T_{T2} results in:

$$T_{T2} = \begin{bmatrix} U_{T21} \\ U_{T22} \end{bmatrix} \Sigma_2 V_{T2}^T$$

As a next step apply a singular value on U_{T22} $U_{T2} = U_{23} \Sigma_{ns} U_{ss}$. The fact that U_{T22} and U_{T21} form a orthonormal matrix results in

$U_{T21} = U_{13}\left(I - \Sigma_{ns}^2\right)^{1/2} U_{ss}$.

Now apply a singular value decomposition on the part that is independent of the far future:

$$T_{T1} = U_{T1}\begin{bmatrix}\Sigma & 0\end{bmatrix}\begin{bmatrix}V_{T11}^T \\ V_{T12}^T\end{bmatrix}$$

with:

- $U_{T1} \in \Re^{pNxpN}$ and
- $V_{T11} \in \Re^{mNxpN}$ and $V_{T12} \in \Re^{mNx(m-p)N-n}$

The nullspace of the system is now given by $V_4 = V_{H1}V_{T12}$. What stays to proof is that the structure of Σ equals:

$$\Sigma = \begin{bmatrix} I_{pN-n} & 0 \\ 0 & \Sigma_1 \end{bmatrix}$$

with $\Sigma_1 \in \Re^{nxn}$ diagonal with entries smaller or equal one.
This is however directly obtained from theorem 5.6.2. To see this first extend the co-inner with its complement $[A,B,C_\perp,D_\perp]$ to a square *mxm* inner and apply the factorization of theorem 5.6.2. From this factorization then removing the part corresponding to the complementary system results in equation (5.6.11a). Hence, we obtain that Σ_1 equals $(I-\Sigma_n^2)^{1/2}$ in equation (5.6.11a).

Note that in fact the whole part that is independent of the far future in both factorizations is equal to each other. Moreover the parts that are dependent on the state at time $N+1$ should also be equivalent.

6.1 Introduction.

In this chapter we will show how the analysis techniques developed in the previous chapters may be applied to analyze the controllability of a process. In section 6.2 the dynamic behavior of the quartz glass tube process will be analyzed, making use of the frequency domain tools developed in the first five sections of chapter five. In section 6.3 use is made of the finite time domain techniques to analyze the dynamic resilience of a high density polyethylene reactor.

6.2 *Application of input-output controllability analysis on a glass tube process.*

In this section we will deal with drawing of quartz glass tubes. This process has become a standard process in control literature [Bac87, Fal94, Hak94, Ove95, Zhu90]. In this section we will analyze the behavior of the process for three different operating conditions, corresponding to three fairly different products, called product A, product B and product C. The main differences between the three products are characterized by differences in the diameter and the wall thickness of the tube. The first two products have the same wall thickness, but the second one has a significantly larger diameter. The last product has the largest diameter and a thicker wall. For each product different models were available from a joint development project of companies and universities. The models of the tube glass process, were obtained by three different modeling approaches [Fal94, Hak94, Ove95]. A short introduction on the tube glass production process is given in section 6.2.1. In section 6.2.2 an input-output controllability analysis is performed for product A to obtain a better understanding of the relation between the open and the closed loop behavior for this product. In section 6.2.3 we will study the behavior of the process for the two other products. Central in this section will be the question whether it is possible to control the process for

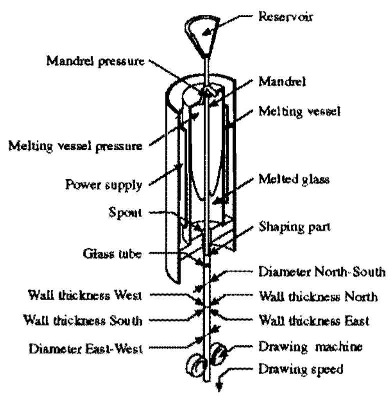

Fig. 6.2.1 A schematic overview of the glass tube process.

these three products with only one controller. In section 6.2.4 we will summarize and further discuss the results.

6.2.1 The glass tube process

A schematic outline of the most important parts of the tube glass process are given in figure 6.2.1. The raw material, high purity SiO_2 (sand), enters from above the furnace vessel of the process. In an electrically heated vessel the sand is heated to melted glass. In the bottom of the furnace there is a hole, called the spout, in which a hollow pipe, the mandrel, is accurately positioned. Along this mandrill the glass is drawn out of the furnace. The shaping part of the process is in fact a distributed parameter system [Bac87]. The geometry of the tube is determined in an area were the glass is still sufficiently weak. The shaping of the tube is however assumed to happen in a point just below the mandrill, since the models used for identification are lumped parameter models. Two important parameters that determine the geometry of the tube are the wall thickness (W) and the diameter (D). In the production process, 4 wall thicknesses (North, South, East and West) and 2 diameters (North/South and East/West) are measured in-line. Two process variables are used to directly affect the shape of the tube with the controller , the mandrel pressure (MP), i.e. the pressure of an inert gas that is fed into the mandrel at the top and the drawing speed (DS) with which the glass is drawn out of the furnace [Bac87]. The sensors have been positioned at a certain distance of the furnace, due to the extreme high temperature at the spout. The steady state behavior of this two by two system is easily understood. Increasing the mandrill pressure will result in a larger pressure difference between the in and outside of the tube, which result in "blowing-up" the tube. Therefore a smaller wall thickness and a larger diameter result. Increasing the drawing speed will result in an increased length of tube produced per unit of time. This results in a reduction of diameter and wall thickness, since the amount of material drawn from the furnace per unit of time is kept constant. Hence, the following steady state relation is obtained:

	drawing speed	Mandrill pressure
diameter	–	+
Wall thickness	–	–

Table 6.2.1 The sign of the steady state relation between the inputs and outputs of the tube glass process.

The process is a time variant non-linear distributed system [Bac87]. The models applied for control are linear time invariant lumped models of a finite order, estimated in a certain operating region. Hence the models only approximate the actual process behavior. The dynamic behavior depends on the specifications of the particular tube produced, as it turned out from the identification of the process for the three different tube sizes, product A, product B and product C. The diameter is most critical for production. Tight control of the diameter is therefore seen as most important.

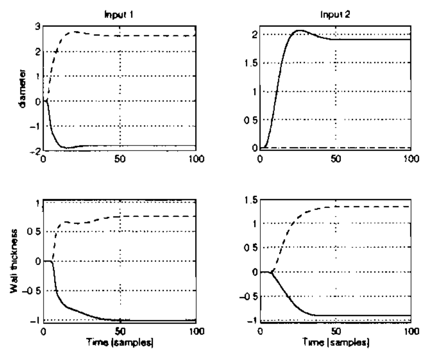

Fig.6.2.2. Stepresponse of the open loop model (solid) and the trianguralized model (dashed).

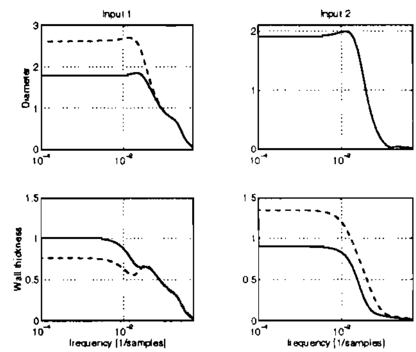

Fig.6.2.3. Bode magnitude plot of the open loop model (solid) and the trianguralized model (dashed).

6.2.2 Analysis of the glass tube production process for product A[1].

The step responses of the diameter and wall thickness to a unit step on the drawing speed and the mandrill pressure are given in figure 6.2.2., solid line. This plot verifies the steady state behavior identified in Table 6.2.1. The delays mainly are due to the distance between the sensors and the shaping point of the tube. The Bode magnitude plot is given in figure 6.2.3., solid line. From these two plots we see that the dynamics of both outputs to the diameter are approximately equally fast. The dynamics to the wall thickness contain both slow and fast dynamics. The drawing speed to wall thickness contains both dynamics and the pressure to wall thickness only the slow dynamics. An interesting question is whether we can use the fast dynamics to control both outputs. A simple reflection of this idea in a

[1] The process data used in this section is scaled. A scaling is however applied that preserves those characteristics that are relevant for control of the process.

control strategy, just to clarify what we mean, is to use fast dynamics of the drawing speed to control the wall thickness and to use the pressure for fast control of the diameter. This idea can of course only work if the fast dynamics enable independent control of both outputs. From the principal gains we directly obtain that this is not the case (figure 6.2.4a): The small principal gain does not contain a significant fast component. From the singular directions (figure 6.2.5) we see that both inputs equally contribute to both principal gains for the low frequency range. For the high frequency range the drawing speed dominates the behavior of the largest principal gain and the pressure that of the smallest gain. For the output direction we see that for the low frequency range the largest principal gain is coupled to the first output and the smallest one to the wall thickness. For the high frequency range the second singular value also influences the second output. To decrease the condition number of the process (figure 6.2.4b, solid line), one could argue that the output scaling should be changed. This however violates physical reality; disturbances on the diameter are larger than those on the wall thickness. The chosen scaling is therefore the right one.

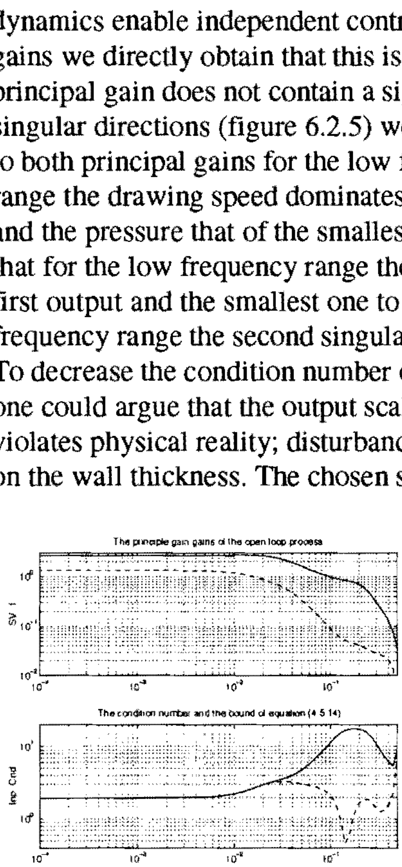

Fig.6.2.4. a) Principal gains of the open loop model b) condition number (solid) and the bound (4.5.14) (dashed).

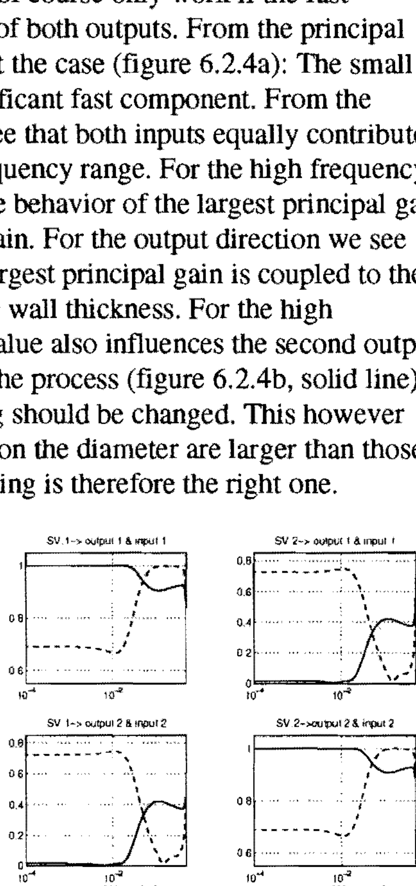

Fig.6.2.5. Principal input and output directions Output direction (solid). Input direction (dashed).

It has already been noticed that control of the diameter is experienced as more important than control of the wall. The above analysis indicates that we can indeed use the largest principal gain to control the diameter. Note however that this will result in some additional interaction on the wall thickness for higher frequencies, since we need to slightly turn the influence of the largest singular value towards output one (conform equation 4.5.19).

For further analysis the approach of section 5.5 is followed, with diameter having absolute priority. The process is factored per output conform equation (5.5.2). The first step determines the stable compensator of the diameter, conform the approach of section 5.4. We choose to introduce two

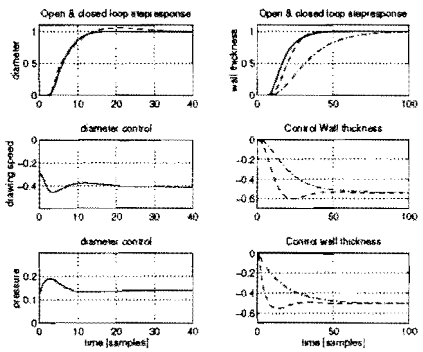

Fig.6.2.6. The step response of the sub-controllers for the setpoint of the diameter (left) and wall thickness (right). Solid response (scaled) triangular system.

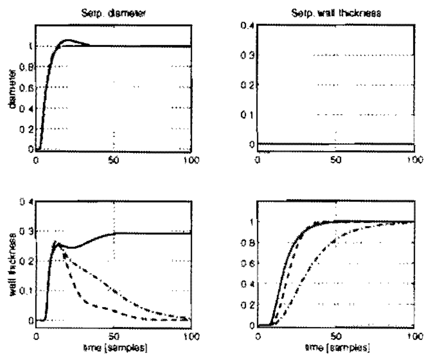

Fig.6.2.7. Step response of the overall nominal model: design 1 (dashed) design 2 (dash dot). Solid is the scaled triangular system.

additional non-minimum phase zeros after down squaring of output one, say $S_1^R(z)$. This choice is directly clear from the minimum infinity norms for the

N.M.P. zeros	**0**	**1**	**2**	**3**
$\|S_1^R(z)\|_\infty$	8997	36.60	1.137	1.081
location zero 1	–	-376	61.18	59.88
location zero 2	–	–	-4.642	-4.722
location zero 3	–	–	–	1.036

Table 6.2.2 The minimum infinity norms for the precompensators introducing respectively 0, 1, 2 and 3 N.M.P. zeros after down squaring and their location.

different precompensators and the location of the additional non-minimum phase zeros introduced by the down squaring operation (table 6.2.2). The inner stable right annihilator of the first output ($S_{1,1}^A(z)$) is then used to control output two. The step response and the Bode magnitude plot of the resulting triangular process are the dashed lines in figure 6.2.2 respectively figure 6.2.3. Note that both plots are in accordance with the expectations:

- The diameter has a fast response.
- The part of the wall thickness that depends on the control of the diameter contains both a fast and slow component.
- The part of the wall thickness that is independent of the diameter only contains a slow component.

We have reduced the control problem to the design of two SISO controllers[2]. For the SISO design we make use of an enhanced version of the implicit model following algorithm, as used for this process by Backx [Bac87]. The approach is attractive for controllability analysis, due to its simplicity. The only tuning variable is the time constant of the first order reference trajectory. It turned out however that for more complex dynamics the approach is too restrictive. The following main drawbacks were identified:

- The inability to consider the behavior at the process inputs in the design
- The restricted ability to shape the process dynamics with the first order reference.

The basic approach is therefore modified in two ways[3]:

1. The degree of the reference function is not restricted to first order systems, but may be chosen arbitrarily.
2. In the criterion function not only the outputs are considered, but also the influence on the rate of change of the process inputs, drawing speed and pressure.

The above approach resulted in the desired balance, between the complexity of the approach and the accuracy of the results obtained. It is emphasized that any other design approach could be used for this problem. Important however is to obtain insight in the controllability of the system. The more freedom a design approach offers the better we can approach the limits of the problem. It however also tends to make the design less clear and more dependent on the user, as discussed in chapter two.

[2] In principle we have three subproblems. We may also see the reduction of the interaction as a separate control problem. We will use the controller for the setpoint of the wall also for the reduction of the interaction (see the discussion below). Note the improvement we may obtain with this third design is always a marginal one, since we are only left with one direction we can still fully use for higher frequencies.

[3] The design approach is strictly spoken not a SISO approach, but a MISO approach, with the proposed extensions

The resulting controller for the diameter, say $C_{1,1}(z)$ and for the independent part of the wall thickness, say $C_{2,2}(z)$ are given in figure 5.2.6 (dashed line). The resulting overall controller then equals:

$$C_1(z) = \begin{bmatrix} S_1^R(z) & S_1^A(z) \end{bmatrix} \begin{bmatrix} C_{1,1}(z) & 0 \\ -C_{2,2}(z)m_2(z)S_1^R(z)C_{11}(z) & C_{2,2}(z) \end{bmatrix} \qquad (6.2.1)$$

where the $m_2(z)$ equals the transfer of the model to the wall thickness. The controller was designed to obtain maximum performance for both outputs. The resulting closed loop step response is given in figure 6.2.7 (dashed lines) and the output sensitivity figure 6.2.8. Simulation of this controller, using the different models for product A as process, shows that the diameter controller is sensitive for relatively small high frequency differences in the

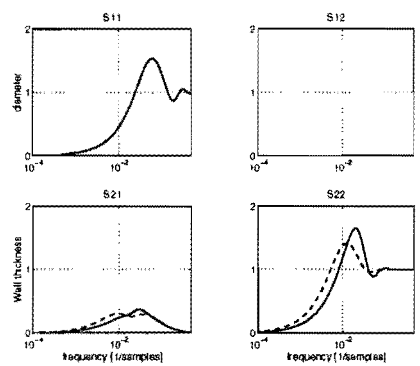

Fig.6.2.8. Output sensitivity function of the closed loop nominal model. Design 1 (solid) and design 2 (dashed).

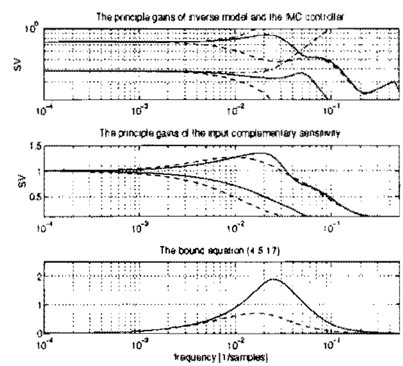

Fig.6.2.9. Robustness measures design 1 (solid) & 2 (dashed): Principal gains of the controller, the input complementary sensitivity, the bound (equation 4.5.17).

model, e.g. (figure 6.2.10). Control actions for the wall thickness clearly influences the behavior of the diameter. This behavior could not be expected from the principal gains of the controller (figure 6.2.9a dashed line) or the input complementary sensitivity (figure 6.2.9b dashed line). Plotting the bound of equation (4.5.17) (figure 6.2.9c solid line) reveals however a sensitivity for errors that couple the largest principal gain of the model and the controller in the frequency range between 0.01 and 0.1/sample. In a second design the closed loop bandwidth of the wall thickness is further reduced, which results in a reduction of the bound of equation (4.5.17), (figure 6.2.9c). The corresponding behavior of the nominal closed loop are given by the dash dotted line in the figures 6.2.6 to 6.2.9. The simulations show the increased performance of the closed loop system (figure 6.2.10).

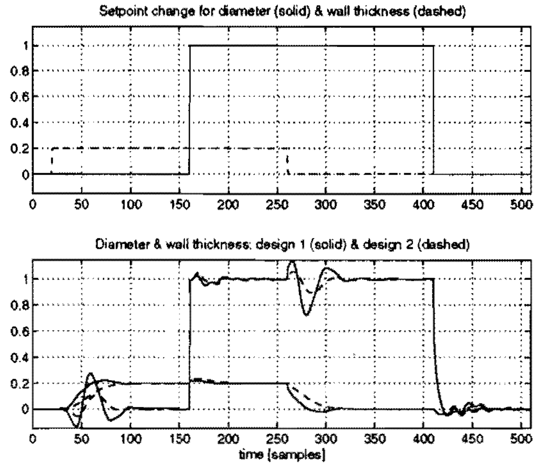

Fig.6.2.10. Simulation with different model as a process. design 1 (solid) & design 2 (dashed).

6.2.3 The behavior of the process for the two other products.

It was experienced by production that especially for larger diameters, the performance of existing controllers was significantly less than for product A. The question arises if it is possible to control the process with one linear controller and what the consequences are of this decision. Models were also identified for two other products with larger diameters, called product B and product C. The diameter of product B is significantly larger than the diameter of product A. The diameter of the product C is again larger than the one of product B. For product C also the wall thickness was larger. The models of the different products are indicated by a subscript that corresponds to the product, $M_i(z)$, $i=A,B,C$. From the stepresponses (figure 6.2.11) it is seen that the dynamics and gain indeed differ significantly for the three products. Both the gain and the dynamics exhibit a nonlinear behavior as function of the product dimensions. The transport delays increase with a larger dimension of the tube. Furthermore an increase in wall thickness is observed. Most of the increase in the delay is explained by the fact that the furnace operates at a constant glass production rate. The larger diameter of product B and the larger diameter and wallthickness of product

C therefore mean a reduced drawing speed and hence a larger transport delay. In the IMC scheme these changes in the delay can be compensated in the model, since the drawing speed can be measured on-line. Moreover the restrictions the increased delay time put on the bandwidth of the closed loop can be enforced by lowpass filtering of the feedback signal $e(z)$ (figure 3.5.3), whose time constants depend on the delay time. The gain of the drawing speed increases for the larger diameters and wallthickness, while

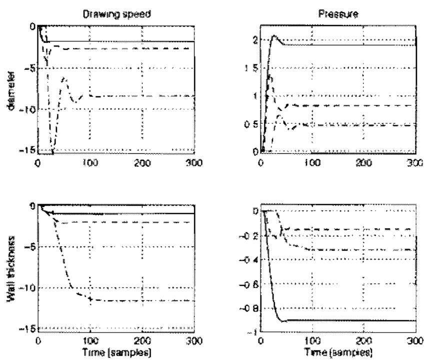

Fig.6.2.11. Step response of the model for product A (solid), the model for product B (dashed) and product C (dash dot).

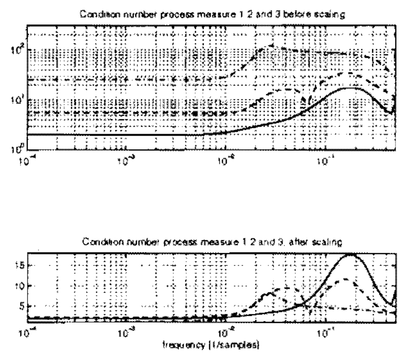

Fig.6.2.12. a) condition number without extra scaling. b) condition number after input scaling of the model for product A (solid), B (dashed) and C (dash dot).

the influence of the pressure reduces. The process becomes more and more ill-conditioned as is observed from the condition number (figure 6.2.12a). In section 3.6. we discussed that the actuators form a principle restriction on controlling ill-conditioned processes, since current industrial actuators are incapable to cover a very large amplitude range and at the same time have a very large absolute accuracy, i.e. be able to very precisely achieve a certain value independent of its magnitude. For the tube process we however have a special situation. It can already be seen from the step response that a significant part of the conditioning problem is directly related to changes in the behavior of each separate input. In this case the ill-conditioning can be seen as a scaling problem. If for a certain product large amplitudes are needed, we do not need a large absolute accuracy at the same time. Hence if the pressure system and the drawing machine can cope with the changing requirements then the problem stays well controllable (figure 6.2.12b)[4]. The Bode magnitude plots of the rescaled process are given in figure 6.2.13. The input scaling is obtained by an optimization of the steady state condition number:

$$\min_{\gamma_1 \in \Re} cond\left(M_i(1)\begin{bmatrix} 1 & 0 \\ 0 & \gamma_1 \end{bmatrix} \right) \quad i= B,C \tag{6.2.2a}$$

In a second step a least squares fit was applied to fit the steady state gain of the models of product B and product C on that of product A:

$$\min_{\gamma_2 \in \Re} \left\| vec(M_A(1)) - vec(M_i(1))\gamma_2 \right\| \quad i= B,C \tag{6.2.2b}$$

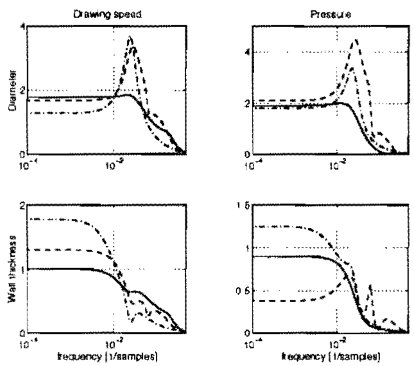

Fig.6.2.13. Bode magnitude plot of the scaled models for product A (solid), B (dashed) and C (dash dot).

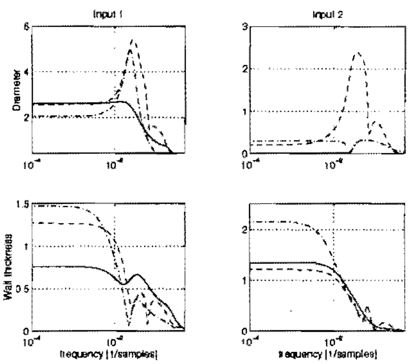

Fig.6.2.14. Bode magnitude plot of $M_i(z)[S_1^R(z), S_{1,1}^A(z)]$, using the scaled models. product A (solid), B (dashed) and C (dash dot).

If we postmultiply the models with the trianguralizing controller $[S_1^R(z), S_{1,1}^A(z)]$ of product A we see that the directional behavior of the models of product B and product C for low frequencies do not drastically differ from the behavior for product A (figure 6.2.14). For higher frequencies the directionality of the system changes. The major difference between the

[4] This requirement may of course have drastic consequences for the actuators itself. For small dimensions of the tube geometry actuators have to follow small steps accurately. For large dimensions the actuators have to be able to follow large steps in the input signals. However these requirements do not occur at the same time, since they depend on the tube dimensions. It is therefore possible to adapt the actuators to the different products, e.g. change the type of control valve in the pressure system.

models relevant for control is the oscillating behavior that occurs for the models with larger diameter. It is not at all trivial how to make the controller robust against the occurrence of this oscillation, unless we ensure that the oscillating behavior is not excited. This however results in a significant reduction of the frequency range over which the model can be controlled. More conservative control of at least the diameter for product A is the consequence. Let us design a controller based on the first product that is also robustly controlling the other two models, based on the above analysis, to verify the validity of the above conclusions. The aim is therefore not to obtain the best possible controller for this case, but only to verify the validity of the conclusions. It is assumed that the model is adapted for the

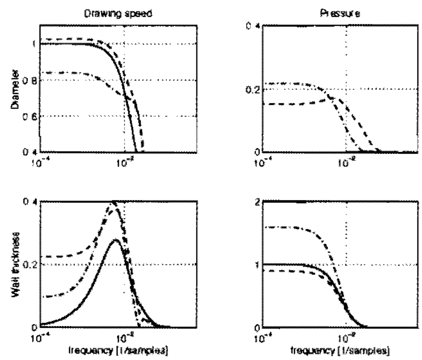

Fig.6.2.15. Bode magitude plot of the feed forward controlled and scaled models for product A (solid), B (dashed) and C (dash dot).

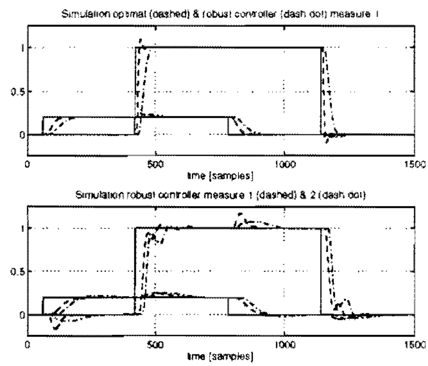

Fig.6.2.16. Setpoint behavior of the closed loop models. a) Product A: with original controller (dashed), & robust controller (dash dot). b) Robust controller: product B (dashed) and C (dash dot).

different actuator gains and the transport delays. The structure for the controller is as in equation (6.2.1). The main difference is that in this case the dynamics of the controllers $C_{1,1}(z)$ and $C_{2,2}(z)$ are replaced by third order lowpass filters. The bandwidth of $C_{1,1}(z)$ and $C_{2,2}(z)$ are taken such that the oscillation in the diameter of $M_i(z)[S_1^R(z)\ ,\ S_{1,1}^A(z)]$), $i=A,B,C$, are sufficiently attenuated. The Bode magnitude plot of the resulting feedforward controlled models, $M_i(z)C(z)$, $i=A,B,C$ are given in figure (6.2.15). From this plot we see that the gain of the diagonal transfers of the resulting model for product C clearly differs from the one for product A. We therefore apply an adaptation of the output scaling for this model. The bandwidth of the controller was reduced such that it turned out that the lowpass filter was not needed for the feedback signal $e(z)$ (figure 3.5.3), whose time constants depend on the delays. A simulation of the closed loop behavior of this controller with all three models for step changes on the diameter and wall thickness setpoints is given in figure 6.2.16. The resulting internal model scheme used for this simulation with the adaptation mechanisms is given in figure 6.2.17.

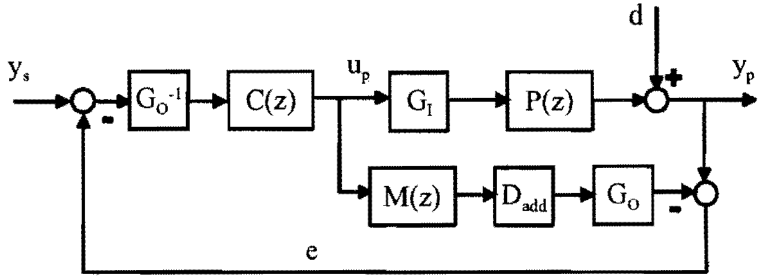

Fig.6.2.17. The internal model control scheme used for the simulations of the controlled behavior of the three productsocks G_I, G_O are diagonal consttant blocks and D_{add} is the diagonal output delay block. These three blocks depend on the product dimensions.

6.2.4 Conclusions and discussion.

From the above analysis we have seen that the process can indeed be controlled over these three models with one fixed controller in combination with an adaptation scheme for the actuator and sensor gains and the time delays[5]. The main bottleneck is the oscillating behavior of the diameter that occurs in the models of product B and product C. This oscillating behavior forces us to reduce the bandwidth of the controller, since we need to supress it. The price we have to pay as a consequence is therefore a reduction in performance, especially for the diameter.

More detailed approaches to deal with the input-output controllability analysis problem are of course possible, e.g.: As a starting point for the controllability analysis not the model of a certain operating condition is

[5] In the analysis we have seen that for robust control of the different products with one controller it is a prerequisite to have an adaptation mechanism for the gain and the transport delays. A pre requisite for application of the idea is therefore know-ledge on the relation between the product dimensions and the gain and the delays.

chosen, but an average model of the behavior at different operating points is considered, with or without an additional uncertainty description. The main question is how to obtain a good nominal model. In the above analysis the oscillating behavior of the diameter was the main bottleneck in achieving performance. The controller bandwidth was chosen such that the oscillation was sufficiently surpressed. It is however not at all clear if this is indeed the only possible way. A more sophisticated approach is to formulate the oscillation as a μ problem, where the uncertainty description is related to the location in the complex plane of the poles responsible for the oscillating behavior. Whether this results in a significant improvement of performance is not clear however. The main problem is in fact a lack of insight in the process mechanisms that play a prime role in robustly controlling oscillating behavior.

In the analysis of the tube glass process we concentrated on the setpoint behavior of the process. The disturbance behavior may result in a different outcome of the analysis. In this case we need to pay more attention to the analytic trade-off. The transport delays and other non-minimum phase effects result in an amplification of the disturbances outside the closed loop bandwidth. The disturbance however still has a contribution outside this bandwidth, which can only be tolerated to a certain extend, since the diameter and wall thickness of the tube should stay within specification anywhere on the tube. The analytic trade-off for disturbance reduction can therefore be different from that of the step analysis.

6.3 The finite time approach applied on a high density polyethylene process.

In this section we will discuss a high density polyethylene polymerization process. The process is regularly used type of process in the polymers industry. Polyethylene has become one of the most important plastics, based on the production volume (36 million ton/year (1991) [Pos93]). The operation conditions for the fluidized bed process are less difficult (20-30bar, 85-95°C) than the alternative free radical based high pressure tubular polymerization reactor (2000bar, 300°C). A major advantage of the fluidized bed reactor for polyethylene production is that the solid polymer product is obtained directly from the reactor, without further seperation of the product from other reactants and solids. The process is highly exothermic and capable to produce a wide range of polyethylene grades covering a broad range of densities and molucar weight distributions. The discussions in this section are based on a rigorous model of the reactor [Maz96, Pos92]. In section 6.3.1 we will first discuss the process [Pos93]. In section 6.3.2 we will analyze the behavior.

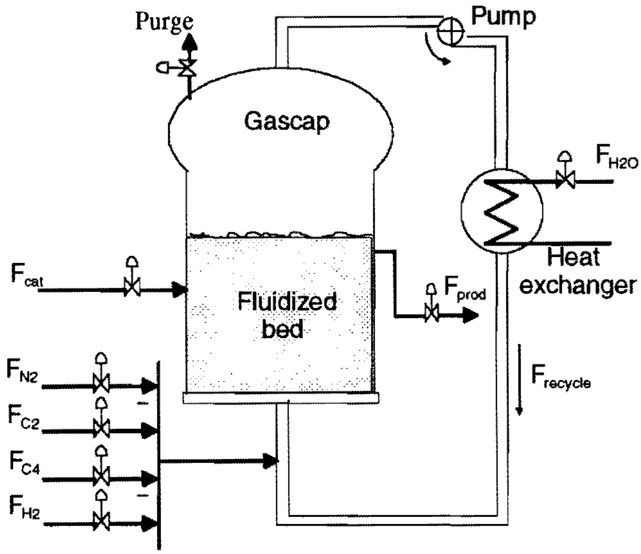

Fig.6.3.1 Schematic overview of a high density gasphase polyethylene reactor.

6.3.1 The fluidized bed polymerization process and the control strategy.

A schematic outline of the high density polyethylene process is given in figure 6.3.1. Monomer (ethylene or shortly C_2), co-monomer (butene or shortly C_4[1]) and hydrogen (H_2) feed streams are fed to a circulating gas stream that feeds the fluidized bed. Catalyst and cocatalyst are injected in the reactor at different points along the reactor wall[2]. The fluidized bed consists of an emulsion phase and a bubble phase. In the emulsion phase, the monomers polymerize on contact with the catalyst forming growing polymer particles. Through the emulsion phase goes a stream of gas bubbles with a high speed (the bubble phase). The gas bubbles supply the emulsion phase with monomer, comonomer and hydrogen. The reaction rate of the monomer is larger than the reaction rate of the comonomer and hydrogen. Therefore the flow of C_2 is much larger than that of C_4 and H_2. The bubbles stream is also used to remove the reaction heat of the exothermic reaction. On top of the fluidized bed is a so called gascap that is used to separate the solid particals from the gas flow. This gas flow enters the recycle loop were it is cooled in a counter current heat exchanger. Water is used as cooling medium. After leaving the heat exchanger the gas is mixed with the fresh input flows and enters the reactor bed again.

The process is operated in an unstable temperature working point. The unstable behavior is caused by the exothermic reaction and the (non-linear) dependency of the reaction rate on the temperature. The exothermic character of the reaction dominates both the process design and process operation. The process unit is designed and operated such that a maximum of heat can be removed. The balance between heat production and heat removal is achieved by:

- The composition of the gas flow through the bed. For this purpose the gas stream also contains the inert gas nitrogen (N_2). The nitrogen is used to remove heat and reduce conversion.
- The high velocity of the gas flow through the reactor causes a low single pass conversion (6% [Pos93]).

A PID controller that manipulates the flow of cooling water to the heat exchanger is used to control the bed teperature and therefore stabilize the process. The temperature setpoint is to be changed with a very small rate of change to prevent a local increase of the temperature in the bed, the so-called hot spots, that may result in a runaway of the reactor. Hence,

[1] As co-monomer propene and pentene are also used, to obtain different product properties

[2] The cocatalyst is used to activate the actual catalyst.

temperature setpoint manipulations are of limited value for dynamic control of the reactor, during production.

The density and the melt index determine the quality of the product. Measuring these properties is not feasible in-line. No reliable sensors exist. The analysis is a very elaborate laboratorium process. The analysis is performed only a few times per shift. The sampling rate is too slow for control. This problem can be solved in two ways:

- Use of a software program that calculates the reactor conditions needed to obtain the right polymer properties after each measurement of the melt index and the density. The calculations are based on the steady state relation between the density and melt index and the reactor conditions. These reactor conditions are then used as setpoints for the controller. Hence producing the right polymer quality reduces to controlling the reactor conditions.
- Inferential measurements. A model is used for on-line estimation of the current melt index and density, based on the measured process conditions. The measured melt index and density are used to update the model. The inferential models used may reach from simple steady state correlation based function upto sophisticated nonlinear rigorous models or neural networks.

For our purpose it is sufficient to assume that an accurate inferential measurement of density and melt index can be realized. We assume it to be available.

The quality of the polyethylene grade is mainly determined by the following reactor conditions:

1. The ratio of the partial pressure of comonomer and monomer.
2. The ratio of the partial pressure of hydrogen and monomer.
3. The type of catalyst and comonomer used.

The second important requirement is the production rate. The production rate is restricted by the cooling capacity of the reactor. The nominal operating conditions are chosen such that approximately 80% of the total cooling capacity is used. The last 20% is used for control of the reactor to ensure safe operation under all conditions. The nominal production rate is mainly determined by the following conditions:

1. The bed temperature.
2. The catalyst feedflow.
3. The concentration of reactants in the gas flow.
4. The pressure in the reactor

5. The level in the reactor.

For control of the process use can be made of the following manipulated variables:

1. Monomer (ethylene), comonomer (butene) and hydrogen flows.
2. The catalyst feed.
3. The bed temperature.
4. The level.
5. The nitrogen flow and the purge.

Nitrogen is an inert gas. If additional nitrogen is introduced in the reactor it will not leave the gas stream anymore unless the purge is used to blow off part of the gas stream[3]. The disadvantage of using the purge is of course that not only nitrogen is removed, but also the valuable reactants. Nitrogen and purge are therefore expensive control variables that are preferably not used. Hence purge and nitrogen are not used to dynamically control the production quality, during production of a certain grade. These variables are only used for emergency actions and to speed-up some of the grade changes.

For the control strategy we may discern two different situations:

1. Control of the process during production of a specific grade. During production we are interested in producing polymer within specification at a maximum production rate, limited by the cooling capacity.
2. Control during a grade change, i.e changing over from production of one grade to production of another grade. In this case we are interested in minimizing the time needed to change-over and to minimize the off spec product produced during the grade change.

The difference in requirements and conditions between grade-changes and production asks for a different control strategy. In the subsequent discussion we will limit ourselves to control of the process during production.

As we discussed above nitrogen and purge are not used as manipulated variables to dynamically control the process during production of a specific grade. We can discern two control levels. The PID-controllers and the MPC controller. The reason for splitting the control problem in two parts is as

[3] The only way that nitrogen leaves the gas stream is by leakage together with the final product leaving the reactor. The amount that leaves the system in this way is however so small that it can be neglected for control purposes.

follows: The PID controller to control the temperature is left in the control scheme as a primary stabilizing control loop. The main reason to leave the temperature controller in the loop is safety. A second PID controller is used to control the level[4]. A third PID controller is used to control total pressure. The reason for applying this control loop can be understood as follows: The ratios of the partial pressure of butene over the partial pressure of ethylene and the partial pressure of hydrogen over ethylene determine to a high extend the properties of the polymer, i.e. the melt index and the density. Analysis of the gas composition takes approximately five minutes. Hence these conditions are known each five minutes with a delay time of five minutes, which is rather slow. On the other hand the overall pressure, which equals the sum of the partial pressures, is available at a very high sampling rate and can therefore be controlled fast. The pressure controller can therefore be used to reduce fast disturbances. The basic idea is to keep the overall pressure constant without changing the partial pressure ratios. Therefore the pressure controller manipulates a total flow consisting of the ethylene, butene and the hydrogen flow. The flows for C_4 and H_2 are determined by the flow of ethylene and the factors R_4 respectively R_2:

$$F_{tot} = F_{C_2} + F_{C_4} + F_{H_2} = (1 + R_4 + R_2)F_{C_2}$$

The factors R_4 and R_2 account for the difference in reactivity of the different reactants. As a consequence we may expect the pressure to be corrected such

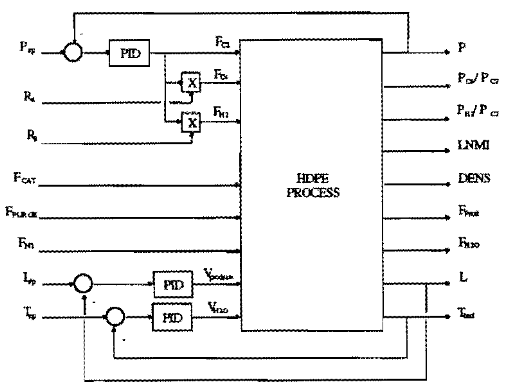

Fig.6.3.2 A schematic overview of the process with primary controllers.

[4] The level could also be included in the MPC. The reason for leaving the level controller as

that the overall pressure does not change the partial pressure ratios. The second advantage of this approach is that we can use R_4 and R_2 as the manipulated variables for the multivariable controller. In this way we may expect the relation between these flow ratios and the ratios of the partial pressures and therefore the melt-index and the density to be more linear than between the flow and the melt index and density. A last advantage of splitting the control level in two parts is reliability. The MPC is in general implemented on a host system. The PID controllers are implemented in the DCS. If for some reason the multivariable controller is not available or the host system or network fails the operator can still control the process using the setpoints of the PID controllers. The second level of control is the model predictive controller running on a sampling time of six minutes (the sampling time of the partial pressure measurement). The melt-index is linearized by taking the natural logarithm of the melt-index. In figure 6.3.2 a schematic overview is given of the model predictive controller.

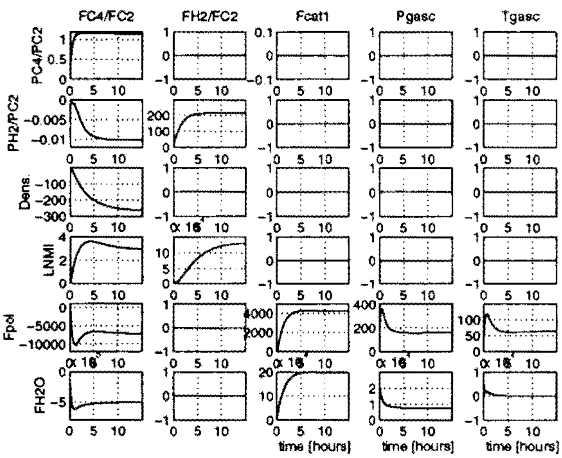

Fig. 6.3.3 The step response of the process after implementation of the primary controllers (physical units).

6.3.2 The behavior of the reactor during production and the proposed control strategy.

The step response of the high density polyethylene process with primary controllers on temperature in the gascap, pressure in the gascap and level is

it is, is that it does not interfere with the other control objectives during normal production.

given in figure 6.3.3. The purge and the N_2 flow are not incorporated, since they are not used during normal production of a certain grade unless strictly necessary. From the step response we directly obtain that the response of the model has a block triangular structure:

- The pressure setpoint, temperature setpoint and the catalyst flow only influence the production rate and the water flow for the cooler, together with the F_{C4}/F_{C2}. The flow ratio F_{H2}/F_{C2} is too small to have any noticable influence on the production rate.
- The flow ratios F_{C4}/F_{C2} and F_{H2}/F_{C2} are the only inputs that influence the product quality and the partial pressure ratios P_{C4}/P_{C2} and P_{H2}/P_{C2} [5].

Let us first take a closer look at the production rate. The response of catalyst flow is slow, since the catalyst needs to be activated first. To prevent the occurance of hot spots and possible run aways, the temperature setpoint can only be manipulated with a limited rate of change. The only independent output that can have a fast response to the production rate is the pressure setpoint. The main interest of production control is to maximize the production rate. Hence, we are interested in the low frequency behavior of the production rate. The structure of the process behavior and the specific requirements make that we can see the control problem for quality and production rate as two separate problems:

- The flow ratios F_{C4}/F_{C2} and F_{H2}/F_{C2} are used to control the product quality, i.e. density and melt index.
- The catalyst flow, pressure setpoint and temperature setpoint are used to control the production rate.

This approach simplifies the control problem significantly.

The production control problem is a very simple problem. Production will be limited by the cooling capacity of the reactor. The control problem is therefore to slowly drive the process towards maximum production subject to eighty to ninety percent of the cooling capacity[6]. Let us turn to control of the product quality. The transfer of the flow ratios F_{C4}/F_{C2} and F_{H2}/F_{C2} to the product quality, density and the melt index have again a triangular structure. Note that the partial pressure ratios, P_{C4}/P_{C2} and P_{H2}/P_{C2} and the product quality are completely dependent. Note however also that the models

[5] In the actual process there is some dynamic influence of pressure and temperature on these outputs. The influence is very fast and therefore neglecteded in the model.

[6] The spare capacity is left for safety reasons.

coupled to the partial pressure ratios, P_{C4}/P_{C2} and P_{H2}/P_{C2}, are faster than the models to the quality parameters, density and melt index. The frequency range over which we can reduce disturbances on the partial pressure ratios is larger than for the product quality. We therefore propose to use a cascade controller for the control of the product quality (figure 6.3.4):

- The inner loop controls the partial pressure ratios, P_{C4}/P_{C2} and P_{H2}/P_{C2} (Y_{P1} in figure 6.3.4).
- The outer loop controls density and melt index (Y_{P2} in figure 6.3.4).

In appendix 6.A it is shown that the cascade control strategy boils down to an IMC structure as in figure 6.3.4. Hence the controllability analysis of the inner loop control problem is independent of that of the outer loop control problem. In the next subsections we will discuss the analysis of these controllers.

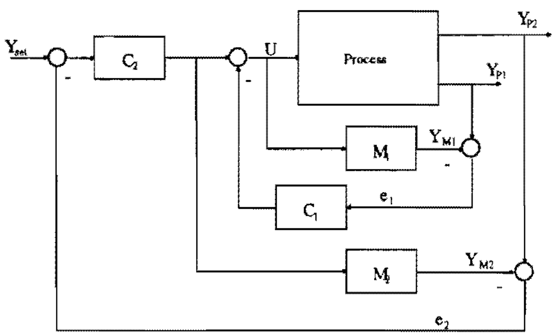

Fig. 6.3.4 The MPC control strategy for product quality control.

6.3.3 Quality control.

As a first step the steady state controller for the transfer from the flow ratios F_{C4}/F_{C2} and F_{H2}/F_{C2} (U in figure 6.3.4) to the product quality, density and natural logarithm of the melt index is determined. The step response of the steady state controlled process is given in figure 6.3.5. The approach we follow to design the controller is essentially the approach developed in section 5.7. The difference between the approach here and the one followed in section 5.7 is that we deal with the two outputs together. The reason for the difference with the approach of section 5.7 is that no ordering can be applied, since both density and melt index are equally important. Based on the step response of the steady state controlled process to the density and the natural logarithm of the melt index we have chosen:

- The control horizon to be 150 samples.
- The IMF reference trajectory for the density and the natural logarithm of the melt index. The step response of the reference and of the diagonal model entries are shown in figure 6.3.5.

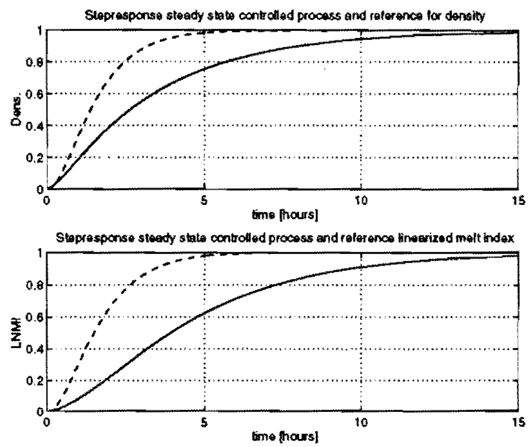

Fig.6.3.5. Step response of steady state controlled process (diagional entries) & the reference trajectories for quality control.

As in section 5.7 we apply the factorization (5.6.18) of the Toeplitz matrix *T*(2*N*,*N*) of the model from the flow ratios to the density and the natural logarithm of the melt index[7]:

$$T=\begin{bmatrix} U_{11} & U_{12}(I-\Sigma_s^2)^{1/2} \\ 0 & U_{22}\Sigma_s \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & U_{ss}\Sigma_2 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{6.3.1}$$

The states can fully be used for control of the far future (equation 5.6.21b):

$$\tilde{X}_{ss}=\Sigma_s^{-1}U_{22}^T\left(\begin{bmatrix} I \\ . \\ I \end{bmatrix}-\begin{bmatrix} S_0 \\ . \\ S_{N-1} \end{bmatrix}\right) \tag{6.3.2}$$

No trade-off between far and near future, as in example 5.6.6c, is needed. The resulting singular values Σ_1 are given in figure 6.3.6a[8]. The correspon-

[7] Note that since the process is square the kernel space V_4 is empty in this case.

[8] The singular value decomposition of the part of the Toeplitz matrix that is independent of the states at future time instant *N* (see section 5.7).

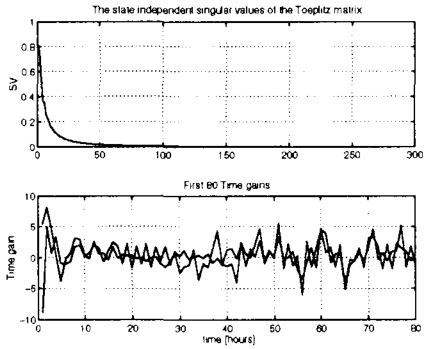

Fig.6.3.6. Singular values Σ_1 of the model, corresponding time gains: density (solid), natural logarithm melt-index (dashed).

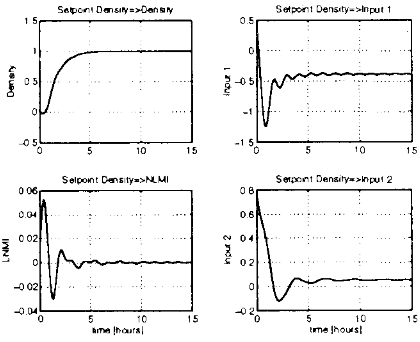

Fig.6.3.7. The response of the controlled process to a step on the density setpoint. process outputs (1st col) & inputs (2nd col).

ding time gains for the the IMF reference trajectory to density, say $y_{ref}(1)$, and the natural logarithm of the melt index , say $y_{ref}(2)$, equation (5.6.19a), i.e. $g_1(i)= \Sigma_1^{-1}U_{11}^T\left(y_{ref}(i)-U_{12}(I-\Sigma_s^2)^{1/2}\tilde{X}_{ss}\right)$ for i=1,2, are given in figure 6.3.6b. Based on figure 6.3.6 only the first 35 respectively 25 singular values of Σ_{11} were considered in the control input signal, say u_C, for respectively the density and the natural logarithm of the melt index:

$$u_C = V_1\left[\tilde{g}_1(1) \quad \tilde{g}_1(2)\right]+V_2\Sigma_2^{-1}U_{ss}^T\tilde{X}_{ss} \tag{6.3.3}$$

with: $\tilde{g}_1(i)=\begin{bmatrix} I_i & 0 \\ 0 & 0\end{bmatrix} g_1(i)$ for i=1,2

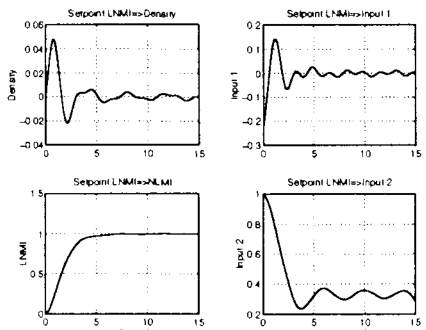

Fig.6.3.8. The response of the controlled process to a step on the natural logarithm of the melt index setpoint.

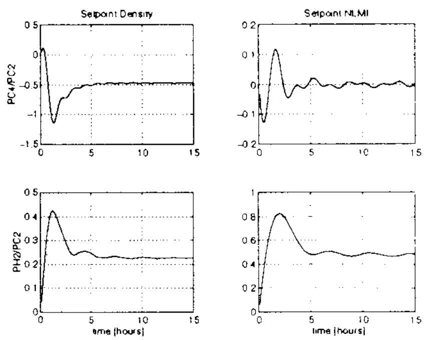

Fig.6.3.9. Response of the partial pressures to a step on the setpoint of natural logarithm of melt index and the density.

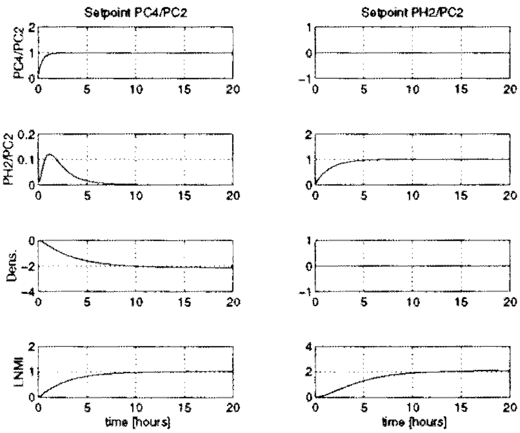

Fig.6.3.10. Partial pressure ratio control. Step response of steady state controlled process.

where: I_1 is an 35x35 identity matrix and I_2 is an 25x25 identity matrix. The step response of the controlled model for the density and the corresponding response at the process inputs, the flow ratios F_{C4}/F_{C2} and F_{H2}/F_{C2} are given in figure 6.3.7. The corresponding responses for the natural logarithm of the melt index is given in figure 6.3.8. In figure 6.3.9 the step responses for the partial pressure ratios P_{C4}/P_{C2} and P_{H2}/P_{C2} are given. In subsection 6.3.4 we will take a closer look at the inner loop controller for the partial pressures.

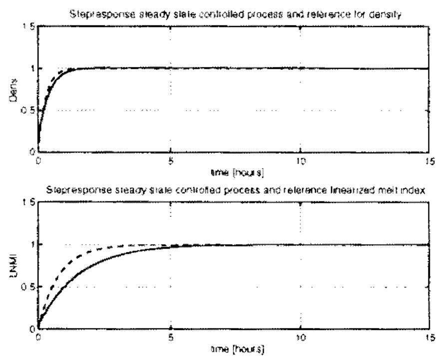

Fig.6.3.11. Step response of diagonal model entries to the partial pressure ratios (solid) & reference trajectoriess (dashed).

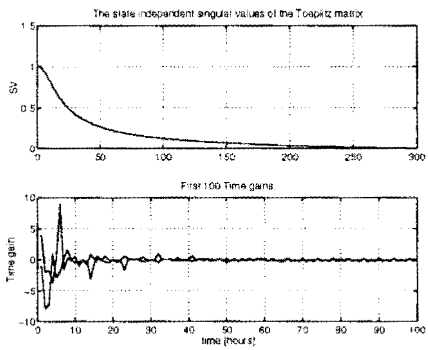

Fig.6.3.12.The singular values Σ_1 of the model & the corresponding time gains: P_{C4}/P_{C2} (solid) and P_{H2}/P_{C2} (dashed).

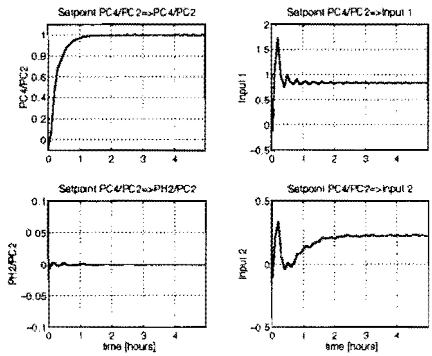

Fig.6.3.13. Response of the controlled process to a step on setpoint of P_{C4}/P_{C2}. process outputs (first column) & inputs (second column).

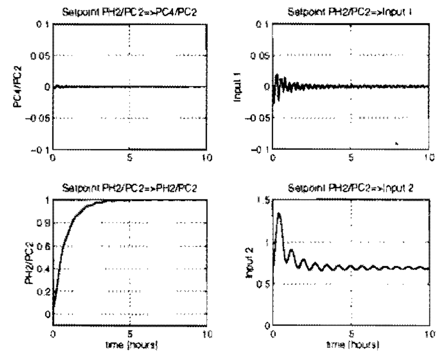

Fig.6.3.14. Response of the controlled process to a step on setpoint of P_{H2}/P_{C2}. process outputs (first column) & inputs (second column).

6.3.4 Partial pressure ratio control.

For control of the partial pressure ratios the same approach is followed as for the density and melt index in section 6.3.3. The steady state controlled process from the flow ratios F_{C4}/F_{C2} and F_{H2}/F_{C2} to the partial pressure ratios P_{C4}/P_{C2} and P_{H2}/P_{C2} are given in figure 6.3.10. The control horizon for this problem is chosen equal to fifty. The reference trajectories are again chosen on the basis of the step response of the diagonal entries (figure 6.3.11). The singular values Σ_i and the resulting time gains are given in figure 6.3.12. The resulting responses at the partial pressure ratios P_{C4}/P_{C2} and P_{H2}/P_{C2} and the corresponding responses at the process inputs, F_{C4}/F_{C2} and F_{H2}/F_{C2} are given in figure 6.3.13 respectively 6.3.14.

CHAPTER SEVEN

Conclusions and Recommendations

Appendix 6.A Cascade control with an Internal Model Controller.

We will derive in this appendix an equivalent IMC scheme for the unit feeed back cascade cascade controller of figure 6.A.1.

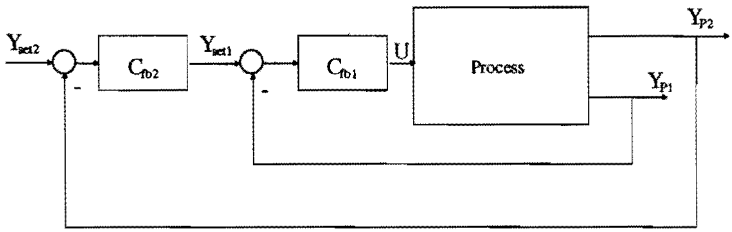

Fig. 6.A.1 The unit feed back cascade controller.

Under the condtion that the model equals the process the IMC scheme of the inner loop can be written as a feed forward scheme (figure 3.5.6). The IMC control scheme for the inner loop of a cascade controller can be written as:

$$y_{p1} = d - M_1 C_1 d + M_1 C_1 y_{set1}$$

For the process M_2 we thus obtain:

$$y_{p2} = M_2 C_1 y_{set1}$$

The IMC control problem for the outer loop thus becomes:

$$y_{p2} = d_2 - (M_2 C_1)\tilde{C}_2 d_2 + (M_2 C_1) y_{set2}$$

Now if we define $C_2 = C_1 \tilde{C}_2$ we obtain for the outer loop problem:

$$y_{p2} = d_2 - M_2 C_2 d_2 + M_2 C_2 y_{set2}$$

which is independent of the inner loop.

7.1 Conclusions.

For controller design it is fundamental to understand how the process behavior limits us to achieve a certain closed loop behavior. This resulted in the initial problem statement for this thesis:

> *Develop basic analysis techniques that enable detailed insight in the limitations that stable process behavior puts on the closed loop behavior, without the need of a detailed controller design, based on a given model of the open loop process and the desired behavior of the closed loop.*

This insight is not only useful for control design itself, but also for process design, identification and on-line monitoring of the closed loop process.

A major problem with existing techniques is the inability to accurately deal with the directional behavior of MIMO systems:

> *What are the directional restrictions that non-minimum phase behavior, delays and principal gains put on the input-output controllability of a MIMO system?*

Newly developed techniques enable us to understand the relation between the directional behavior of the open loop process and of the closed loop process, both for the non-minimum phase behavior (section 4.4) as for the gain of the process (section 4.5). To obtain a well controlled robust closed loop behavior for ill-conditioned processes a new tuning procedure is proposed.

A clear disadvantage of the techniques in chapter four is that the insight obtained is not directly related to the overall control problem. The tools only enables us to manipulate the direction of one effect at a time:

- The gain behavior of the process is analyzed separate from the non-minimum phase zeros.
- Non-minimum phase zeros are analyzed one zero after the other.

The mutual dependency of the different steps on each other is not considered in the approach.

To asses the overall controllability a completely different and new approach is developed in chapter five:

Arranging the process outputs in a descending order of importance of the corresponding requirement and requiring the complementary sensitivity at the process output to be lower triangular enables the assessment of the input output controllability in a structured sequential approach.

To assure closed loop stability the down squaring problem is studied and results in a completely new insight in this problem (section 5.4):

The existence of a trade-off between the number of non-minimum phase zeros introduced in the square process and the gain of the down squaring controller.

In section 2.5 we asked whether it is possible to develop a technique that enables us to better asses the input-output controllability of a process for model predictive control:

Is it possible to obtain techniques that enable us to obtain understanding on how the process behavior limits control with a receding horizon.

Analysis of the behavior of inner and co-inner systems over the finite time horizon results in completely new insight in non-minimum phase systems and non-square systems (section 5.6):

- For stable control of a system with n non-minimum phase zeros a n dimensional subspace of $l^2_-(N)$ can not be freely assigned.
- For a structural inner with McMillan degree n there exists a n dimensional subspace of $l^2_-(N)$ whose invertability depends on the non-one singular values of the corresponding Toeplitz matrix.

The new characterization of these effects:

- clearly reveals the effect they have on the behavior of the closed loop.
- is completely consistent with the frequency domain characterization.
- results in a better understanding of the well known waterbed effect for non-minimum phase systems and enables an analytic trade-off.

A procedure was proposed to deal with the input output controllability analysis of a process over the finite time horizon, based on the obtained insight (section 5.7).

7.2 Recommendations for future research.

In this section we will discuss some open issues and further developments.

7.2.1 General remarks concerning the input output controllability approach.

It is assumed that the process is stable. This assumption is for the input-output controllability analysis of industrial MIMO processes not a very severe restriction, since current industrial practice is that primary controllers, (PID) are generally used to stabilize the process behavior. However unstable poles are restricting the input-output controllability of the process. A better understanding of how these poles limit the input-output controllability of a process is needed. The IMC frame work can also be used for the input-output controllability analysis of unstable processes [Mor89]. Complicating factor is the need to fulfill interpolation constraints on different transfer matrices [Mor89], which makes the analysis complex.

One would like to be able to bring in available knowledge on the disturbances in the analysis of the input output controllability:

$$(I + MC)M_d = M_d + MCM_d$$

which may be rewritten as:

$$(I + MC)M_d = M_d + M\tilde{C}$$

with M_d minimum phase. The disturbance information significantly increases the complexity of the analysis, since M_d is in general a full matrix.

In section 5.4 we discussed that introducing non-minimum phase zeros in the square process after down squaring may significantly decrease the infinity norm of the controller. There is however no direct relation known between the location and output direction of these zeros and the gain of the controller. Two different approaches are proposed:

1. A number of zeros is selected such that an acceptable controller gain is obtained. After the controller is determined the location and direction of the additional zeros can be determined (lemma 5.4.8)
2. Define the square inner that results after down squaring and determined the reduction in the gain of the controller (lemma 5.4.9).

A more direct relation between zero location, zero direction and controller gain may be obtained by:
1. Exploring the close relation between lemma 5.4.9 and lemma 5.4.5.
2. Exploring the relation the singular vectors, related to the small singular values of the finite time domain representation of the inner (theorem 5.6.1), must have with the behavior of the introduced non-minimum phase zeros.

7.2.2 Finite time domain characterization.

The finite time domain characterization is as far as known completely new. Opportunities and consequences this development may have can however not yet be overseen. It is felt however that the new characterization of non-minum phase behavior and non-squareness of the process may well have significant impact on our understanding of multivariable controlled process behavior. Examples of potential future results directly related to control are:
- A topic for further research directly related to controllability analysis is the stability of the receding horizon problem. In chapter five we enforced stability of the closed loop by requiring the problem to be in steady state or to be almost in steady state at the end of the horizon. This requirement is however too strict, since we only need to ensure that the process can still be controlled after the end of the horizon. Hence the requirement is actually that there must exist a control signal after the end of the future horizon that fulfills all requirements. One possible way is to extend the control signal after the horizon with the state feedback solution of the infinite time horizon LQ problem. Further study is needed.
- Extending the results to cover unstable process behavior in the approach.
- Incorporating robust performance and robust stability in the approach.
- There seem to be close relations to orthogonal basis functions [Heu91, Ari97, Hak94]. One of the advantages of using orthonormal basis functions for identification is the fact that one can incorporate pre-knowledge on the process dynamics. The more accurate this knowledge the less parameters one has to estimate and hence the more accurate the model [Hof94]. A problem in applying these techniques is that only the infinite time basis is orthonormal. This may cause problems for short data sequences. The techniques developed here may be useful to find a basis that is orthonormal on the finite time horizon.

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Samenvatting.

Industriele toepassing van multivariabele regelsystemen hebben in de laatste twee decennia een grote vlucht genomen. Heden ten dage vormen regelproblemen bestaande uit twintig tot dertig uitgangen en tien tot vijftien ingangen geen uitzondering. Ook de aan de regeling gestelde eisen, ten aanzien van nauwkeurigheid, betrouwbaarheid en flexabiliteit zijn gedurende deze tijd sterk toegenomen. Een gevolg van deze ontwikkelingen is dat zowel het ontwerp als de afregeling van deze regelaars erg complex is geworden. Een beter en meer gedetailleerd inzicht in de regelbaarheid van het proces is noodzakelijk.

Centraal staat de vraag in hoeverre het gedrag van het proces, het zogenaamde open lus gedrag, beperkingen legt op het geregelde gedrag, het zogenaamde gesloten lus gedrag. Hoewel inzicht in de regelbaarheid van het proces essentieel is voor het ontwerp is hierover slechts weinig bekend. Het bovenstaande probleem resulteert in een vicieuze cirkel: Om het gesloten lus gedrag te kennen heeft men de corresponderende regelaar nodig. Om deze regelaar te onwerpen heeft men echter het te verkrijgen gesloten lus gedrag nodig. Om deze cirkel te doorbreken is gebruik gemaakt van het 'internal model controller scheme'. Het grote voordeel van deze formulering is dat het ontwerp van de regelaar equivalent is aan het vinden van een stabiele benaderende inverse van het proces gedrag. Hierdoor is, gebruikmakende van een model van het proces, een directe en inzichtelijke relatie verkregen tussen het procesgedrag (het model) en het gewenste gesloten lus gedrag. Bestaande technieken op dit gebied zijn in het algemeen beperkt tot hoe een specifiek aspect van het procesgedrag de regelbaarheid in het algemeen beinvloedt. Een gedetailleerd inzicht hoe dit doorwerkt op de verschillende gestelde eisen, welke eisen strijdig zijn en hoe zij bijgesteld dienen te worden, geven deze technieken in het algemeen niet.

In het eerste gedeelte van het onderzoek is getracht om deze bestaande technieken verder te ontwikkelen om meer gedetailleerd inzicht te verkrijgen in de regelbaarheid van een proces en de relatie die dit heeft tot de gestelde eisen. De nieuw ontwikkelde inzichten leiden tot een goed inzicht in de mechamismen die een rol spelen in het beinvloeden van de richting waarin zowel niet minimum fase nulpunten als ook de singuliere waarden van het open lus systeem, zich manifesteren in de gesloten lus overdrachtsmatrix. Verder resulteren de ontwikkelde technieken in een inzicht in het specifieke gedrag van slecht geconditioneerde systemen. Een belangrijk nadeel van de ontwikkelde technieken blijft echter dat zij slechts inzicht verschaffen over de beperking die deelaspecten van het procesgedrag hebben op de regelbaarheid. Een gedetailleerd inzicht in regelbaarheid van het totale proces wordt hierdoor in sterke mate bemoeilijkt.

In het tweede gedeelte van het onderzoek is daarom gekozen voor een geheel nieuw concept gebaseerd op de gestelde eisen aan de regeling. Het is mogelijk gebleken op basis van dit concept te komen tot een veel gedetailleerder inzicht in de regelbaarheid van het regelprobleem: De regelbaarheid van het proces gerelateerd aan de gestelde eisen. De regelbaarheid kan worden onderzocht gebaseerd op zowel het frequentie domein als op eindige tijdbasis. De frequentie domein techniek is specifiek bedoeld voor toepassing met tijdinvariante regelaars met een vaste structuur. De ontwikkelingen leiden tot nieuwe inzichten in het vierkant maken van niet vierkante processen. De eindige tijd techniek is specifiek bedoeld voor toepassing met model predictieve regelingen. Het is met name de eindige tijdsdomein formulering die heeft geleid tot geheel nieuwe fundamentele inzichten in de relatie tussen procesgedrag en regelbaarheid. De toepasbaarheid van beide methodieken wordt geillustreerd aan een industriële applicatie.

Curriculum vitae.

December 24, 1957	Born in Oegstgeest, the Netherlands.
September 1970 - June 1976	Secondary school: HAVO, Cobbenhagen College, Tilburg, the Netherlands.
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September 1980 - December 1981	Communications officer in the Dutch Army.
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May-1987 - September 1989	CACSD Engineer; Nederlandse Philips bedrijven b.v., HTG Glas, Eindhoven, the Netherlands.
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Stellingen

behorende bij het proefschrift

Controllability Analysis of Industrial Processes
Towards the industrial application

van

Jobert Ludlage

Eindhoven, 11 November 1997

1) Vooringenomenheid ten opzichte van ideeën van derden die niet stroken met het eigen beeld vormen een struikelblok voor een vrucht-bare relatie en goede samenwerking. Het zich bewust zijn van dit feit kan de vooruitgang in met name het eigen vakgebied bevorderen.

2) Voor een goede regeling is het van belang zowel langzame als snelle procesdynamica in rekening te brengen (zie dit proefschrift). Men is echter van nature geneigd met name de trage dynamische effecten onvoldoende in ogenschouw te nemen. *'Wie dan leeft wie dan zorgt'*, *'Dat zien we dan wel weer'*, *'niet verder zien dan de neus lang is'* zijn veel gebezigde uitdrukkingen in onze taal die hiernaar verwijzen. Men mag verwachten van beleidsmakers en bestuurders binnen de overheid en bedrijfsleven dat juist zij deze trage aspecten voldoende onderken-nen en er adeqaat op reageren. Het in de laatste decennia gevoerde landbouwbeleid, de opbouw en afbraak van de sociale zekerheid doet echter vermoeden dat ook de visus van sommige van deze lieden een zeer beperkte reikwijdte heeft. Vanuit een eindige tijd benadering lijkt de conclusie dat de gekozen predictie horizon te kort is om enige vorm van robuust gedrag te kunnen garanderen voor de hand te liggen. (Zie hoodstuk 5 van dit proefschrift.)

3) Regelbaarheidsanalyse is slechts een gereedschap. Het voorkomt niet dat men het boerenverstand moet gebruiken om te komen tot een solide ontwerp. (Zie dit proefschrift.)

4) Een explosieve groei van zowel kwaliteit als kwantiteit van de regel-technische toepassingen in de procesindustrie is slechts dan te ver-wachten wanneer het niet lineaire gedrag van processen structureel wordt meegenomen (Zie hoofdstuk 1 van dit proefschrift). Een veelheid aan praktische en theoretische problemen gaan echter gepaard met het expliciet meenemen van het niet-linear gedrag in de probleem formule-ring. Dit grotendeels braakliggende terrein vormt dan ook de ultieme uitdaging voor het regeltechnisch onderzoek in het komend decennium.

5) Wiskundige modellen en technieken roepen bij velen een beeld op van onomstotelijkheid en almachtigheid. De wiskunde is echter meestal te beperkt om realistische problemen te beschrijven en op te lossen. De gevonden oplossing hangt dan ook veelal meer af van de gekozen

aannamen en benaderingen dan de gekozen methodiek. (Zie dit proefschrift).

6) Fuzzy logic is een in vele disciplines van de regeltechniek ondergewaardeerde techniek. Deze techniek is bij uitstek geschikt voor het combineren van diverse vormen van zowel kwantitatieve alsook kwalitatieve kennis van het procesgedrag.

7) In wetenschappelijke kringen is het goed gebruik geworden te veronderstellen dat men minimaal gepromoveerd dient te zijn om een kans te maken op een vruchtbare en geslaagde academische carriere. Zeker voor de toegepast technische wetenschappen zijn relevantere criteria denkbaar. Dit zeker gezien de huidige ontwikkelingen, waarin kwaliteit van het onderwijs, toepasbaarheid van onderzoek en derde geldstroom steeds centraler staan.

8) Goede sociale wetgeving, arbeidsomstandigheden wetgeving en strenge milieuwetgeving worden veelal genoemd als oorzaak voor een verslechterende concurrentiepositie op de wereldmarkt. Gezien het grote belang dat de samenleving hecht aan deze wetgeving kan men stellen dat juist het hier straffeloos en onbelast importeren uit landen met een op dit gebied gebrekkige wetgeving strijdig is met het vrije markt beginsel. Geimporteerde goederen uit deze landen dienen dan ook extra belast te worden.

9) De Nederlandse regering belijdt een groot belang te hechten aan één veralgemeniseerde Europese markt. Het werkelijke belang dat de Nederlandse regering aan deze markt hecht valt echter direct af te meten aan het gevoerde nationale belastingbeleid met betrekking tot benzine.

10) Er is geen regelbaarheidsanalyse voor nodig om te kunnen conclu-deren dat het financieel belonen van een onderwijsinstelling per gediplomeerde '*schoolverlater*', op termijn moet leiden tot een verlaging van het nivo van de opleiding.

11) Het gebruikmaken van de mogelijkheden tot inspraak bij een wijziging van een bestemmingsplan leidt noch tot een groter vertrouwen in het democratische gehalte van politiek en overheid noch tot meer betrokkenheid bij de politiek.

12) Het is algemeen bekend dat het goede voorbeeld goed doet volgen. Het wordt dan ook wel gezien als een goede manier voor een manager om zijn ondergeschikten aan te zetten tot het leveren van extra prestaties. Men mag dit misschien als een noodzakelijke voorwaarde beschouwen. Het is echter zeker geen voldoende voorwaarde om mensen tot een boven normale prestatie te bewegen. Op een langere termijn kunnen dergelijke prestaties slechts verwacht worden, wanneer een manager zijn mensen kan inspireren en binden. Dit is echter alleen mogelijk als er sprake is van wederzijds vertrouwen en respect voor elkaars persoonlijke sterktes, zwaktes, ideeën en kunde.

13) Het gelijkheidsprincipe, de legitimiteit en anonimiteit die regelgeving per definitie aan de uitvoerder ervan verschaft draagt er zorg voor dat op het moment van totstandkomen de mens reeds ondergeschikt is aan de regel.

14) Het Nederlands parlementair systeem wordt in hoge mate gekenmerkt door de aanwezigheid van vele kleine partijen van allerlei richting. De kleine politieke partijen zijn in meerdere opzichten te vergelijken met de katalysator in een polymerisatie reactor (Zie hoofdstuk 6.3 van dit proefschrift.)

15) In het huidige rationele wereldbeeld tracht men emoties volledig ondergeschikt te maken aan de ratio. Het grote belang van emotionele factoren in het nemen van beslissingen is echter een publiek geheim. Juist het belang dat emoties spelen bij alle beslissingen is een van de grootste taboe van onze tijd.