Performance analysis of manufacturing networks : surplus-based control

DOI:
10.6100/IR738942

Document status and date:
Published: 01/01/2012

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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PERFORMANCE ANALYSIS OF MANUFACTURING NETWORKS: SURPLUS-BASED CONTROL

Performance analysis of manufacturing networks: surplus-based control

Konstantin Starkov

The research leading to these results has received partial funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. INFSO-ICT-223844.

This work has been carried out under the auspices of the Engineering Mechanics research school.

A catalogue record is available from the Eindhoven University of Technology Library ISBN: 978-90-386-3266-7

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Cover design by Konstantin Starkov and Marina Khlytina.

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van de
rector magnificus, prof.dr.ir. C.J. van Duijn, voor een
commissie aangewezen door het College voor
Promoties in het openbaar te verdedigen
op donderdag 15 november 2012 om 16.00 uur

door

Konstantin Starkov

geboren te Moskou, Rusland

Dit proefschrift is goedgekeurd door de promotoren:

prof.dr.ir. J.E. Rooda
en
prof.dr.ir. I.J.B.F. Adan

Copromotor:
dr.ir. A.Y. Pogromsky

Acknowledgements

This dissertation would not have been possible without the excellent research environment provided by my promoters Koos Rooda and Ivo Adan. I would like to sincerely thank them for the supervision and support that they have provided in my research and education related activities. Particularly I would like to thank Koos Rooda for teaching me structure and patience, and Ivo Adan for supporting my creativity and for his very detailed and constructive feedback on the results of my research. I feel lucky to have Alexander Pogromsky as my co-promotor. During these past 4 years he was my mentor, my coach and my friend always open for a short advise or for a long educative discussion. Among a lot of other valuable things, he taught me how to develop my research and teaching skills. During this last 4 years I was also lucky to be able to collaborate with Boris Andrieysky, Alexey Matveev and Varvara Feoktistova. I thank them for their contribution to Chapters 2 and 7 of this thesis. Also I would like to acknowledge that this research was made possible thanks to the European Community's Seventh Framework Programme and the C4C project. A lot of creative ideas that helped me to improve my research were born during the quarterly C4C project meetings.

I am very thankful with my doctorate committee members Alexey Matveev, Henk Zijm, and Geert-Jan van Houtum for their suggestions and constructive remarks, which enabled me to improve the quality of my dissertation.

On a more personal note I would like to thank Mieke Lousberg for her interest and care, for helping me with all the administrative and also out of office matters that I encountered during my first 3 years at TU/e. My gratitude also extends to Nicol Palmer for her friendliness and administrative help.

I would like to thank all my colleges from System Engineering and Manufacturing Network groups for being open, friendly, and for creating a pleasant working atmosphere. Particularly my gratitude for creating the excellent working atmosphere extends to my (former) roommates and colleagues at the university, i.e., Damian, Jasen, Ricky, Casper, Rolf, Dirk, Allan, Frans, Henk, Dennis, Albert, Bert, Pascal, Asia, Erjen, and Michel. Also I am grateful with my former colleagues Qin Li and Mihaly Petreczky, for their friendship, openness and goodwill. Special gratitude goes to my former colleague and my friend Evgeniy Ivanov for lifting up my spirits during the difficult times, as well as his wife Marina for her hospitality and professional help with the cover design of this thesis.

Finishing a PhD project would, of course, not be possible without a proper recreational activities and "goede conversaties". For this I would like to thank Fausto and Pily, Minerva and Hans, Mike P., Hans and Idoia, Dion, Jop and Elena, and the singing quire “Brabant Coro Band”of Parochie Heilig Hart van Jezus.

On even more personal note, I would like to thank my parents Konstantin Starkov and Valentina Starkova for believing in me, always encouraging me for a continuous improvement and hard work, and for being there for me. Finally I would like to emphasize my never ending gratitude to my wife Maria for all her love, understanding, care and patience during the past 10 years of our relationship and for recently giving me the best gift in the world, i.e. my son Andrei.

Summary

Performance analysis of manufacturing networks: surplus-based control

In the modern market, keeping high competition in brands and varieties in type of products is the way for survival of manufacturing industries. Therefore production control methods with capabilities of quick responses to rapid changes in the demand and efficient distribution of the raw material throughout the network are of importance among leading manufacturers. Nowadays, the production control problem has been widely studied and a lot of valuable approaches including queuing theory, Petri nets, dynamic programming, linear programming, hybrid systems were proposed and some of them are implemented. Though up to this moment many methods have been developed, the factory performance remains a challenging problem for further research. Motivated by this problem we study the performance of several manufacturing networks operated by surplus-based control. In the surplus-based control, decisions are made based on the demand tracking error, which is the difference between the cumulative demand and the cumulative output of the network. The studied networks are a single machine, a manufacturing line, a multi-product manufacturing line, a re-entrant machine and a re-entrant line. The performance analysis is based on the performance factors such as demand tracking errors and inventory levels. Specifically, given the presence of unknown but bounded production speed perturbations as well as demand rate fluctuations, we investigate how close the cumulative production output of a manufacturing network follows its cumulative production demand under a surplus-based control policy.

The research is subdivided into theoretical analysis, simulation-based analysis and experimental analysis parts.

Theoretical analysis
By means of analytical tools, the relation between the production demand tracking accuracy and the inventory levels of the networks is investigated. In order to find this relation, classical tools from control theory are used. Models of production flow processes are formulated by means of difference as well as differential equations. In order to analyze their performance, optimal control theory and Lyapunov

theory approaches are exploited.

Simulation based analysis

By means of simulation tools, the theoretical results on performance are evaluated by time-based simulation models. Thus, all theoretical results are illustrated and confirmed by computer simulation. Also two comparative studies are conducted. The first comparative study is realized in order to test the theoretical results on more accurate models, which are event-based. The results are shown to be in agreement with the theory. The second comparative study is on time-based models, where the behavior of a line, a single re-entrant machine and a re-entrant line is tested under three commonly used surplus-based production policies. The performance of each network is evaluated and the results are presented.

Experimental analysis

An experimental prototype is invented, designed and developed for education and research purposes. The prototype is a hardware tool that serves as a liquid-based emulator of manufacturing network processes. In its core, the liquid-based emulator consists of several electrical pumps and liquid reservoirs. The electrical pumps emulate manufacturing machine behavior, while the liquid reservoirs serve as the intermediate product storages, also called buffers. In the platform, pumps and tanks can be interconnected in a flexible manner. In that way the prototype permits an easy and intuitive way of studying manufacturing control techniques and performance of several network topologies. A detailed system description is provided. Several network configurations and experimentations are presented and discussed.

Samenvatting

Prestatie analyse van fabricage netwerken: surplus-gebaseerd regelen

In de huidige markt overleven fabricage industrieën door een grote competitie in merken en variëteiten in product types. Hierdoor zijn methodes voor het regelen van productie (production control) met vlugge responsmogelijkheden voor snelle veranderingen in de vraag en efficiënte verdeling van grondstoffen door het netwerk erg belangrijk voor fabrikanten. Het probleem voor regelen van productie is uitgebreid bestudeerd en veel waardevolle aanpakken zoals wachtrij-theorie, Petri nets, dynamisch programmeren, lineair programmeren en hybride systemen zijn voorgesteld en sommige van deze zijn geimplementeerd. Tot op dit moment zijn er vele methodes ontwikkeld, maar de prestaties in een fabriek blijft een uitdagend probleem voor verder onderzoek. Gemotiveerd door dit probleem bestuderen wij de prestatie van verscheidene fabricage netwerken bestuurd met een surplus-gebaseerde regeling. In een surplus-gebaseerde regeling worden beslissingen gemaakt op basis van de productie volgfout (production tracking error), het verschil tussen de cumulatieve vraag en cumulatieve productie van het netwerk. De bestudeerde netwerken zijn een enkele machine, een fabricage lijn, een fabricage lijn met meerdere producten, een herintredende (re-entrant) machine en herintredende lijn van machines. De prestatie analyse is gebaseerd op prestatie factoren zoals productie volgfouten en bufferwaardes. We onderzoeken hoe dicht de cumulatieve productie van een fabricage netwerk de cumulatieve vraag volgt onder een surplus gebaseerd beleid, gegeven de aanwezigheid van onbekende maar begrensde perturbaties in productiesnelheden en fluctuaties in de vraag.

Het onderzoek is onderverdeeld in theoretische analyse, analyse gebaseerd op simulaties en experimentele analyse.

Theoretische analyse
Door middel van analytische hulpmiddelen is de relatie tussen de nauwkeurighed van het volgen van de vraag en de bufferwaardes van het netwerk onderzocht. Om deze relatie te vinden zijn klassieke methodes van de regeltechniek theorie gebruikt. Modellen van productie processen zijn geformuleerd door differentiaalvergelijkingen alsmede verschilvergelijkingen. Om de prestaties te analyseren is de theory van optimaal regelen (optimal control) en de Lyapunov theory gebruikt.

Analyse gebaseerd op simulaties
Door middel van simulatie hulpmiddel zijn de theoretische resultaten van de prestatie beoordeeld met tijd gebaseerde simulatie modellen. Dus alle theoretische resultaten zijn geïllustreerd en bevestigd door middel van computer simulaties. Ook zijn twee vergelijkbare onderzoeken gedaan. Het vergelijkbare onderzoek is gerealiseerd om de theoretische resultaten van meer gedetailleerde gebeurtenis-gebaseerde (event-based) modellen te testen. Het is aangetoond dat het resultaat in overeenstemming is met de theorie. Het tweede vergelijkende onderzoek was over tijd-gebaseerde modellen, waar het gedrag van een lijn, een enkele herintredende machine en een herintredende lijn is getest onder drie veel gebruikte surplus-gebaseerde strategies. De prestatie van elk netwerk is geëvalueerd en de resultaten zijn gepresenteerd.

Experimentele analyse
Een experimenteel prototype is bedacht, welke is ontworpen en vervaardigd voor educative- en onderzoeksdoelen. Het prototype is een hardware hulpmiddel dat dient als een vloeistof-gebaseerde emulator van processen in fabricage netwerken. Hoofdzakelijk bestaat deze emulator uit verschillende elektrische pompen en water reservoirs. De elektrische pompen imiteren gedrag van machines en de water reservoirs dienen als tussentijdse opslagplaatsen voor producten, ook wel buffers genoemd. In het platform kunnen pompen en tanks op een flexibele manier met elkaar verbonden worden. Op deze manier staat het prototype een makkelijke en intuïtieve manier van het studeren van fabricage regeltechnieken en prestatie van verschillende netwerk typologieën toe. Een gedetailleerde systeembeschrijving is voorzien. Verschillende netwerk configuraties en experimenten zijn gepresenteerd en bediscussieerd.

(Van enkele termen is de Engelse vertaling toegevoegd, omdat de Engelstalige termen volledig ingeburgerd zijn in het nederlandse vakjargon.)

Contents

1

Introduction

1.1 Overview of manufacturing networks

The concept of manufacturing network has several interpretations, that are all related, but may vary depending on a level of abstraction that is used. Within a factory a manufacturing network can be viewed at four different levels of abstraction (see Rooda and J.Vervoort (2007) and references therein). At low level a manufacturing network consists of machines and buffers, interconnected in any possible or desired configuration. A machine can be understood as hardware (equipment or tool) used in a factory to produce a certain product (intermediate or final) from raw materials and a buffer as a storage for raw materials or produced products. A group of machines that are typically scheduled as one entity as it is located in the same area in a factory, where each machine within a group performs a similar task, is called a manufacturing cell. A higher level of abstraction is at area level, which is a location within a factory where manufacturing cells are situated. Finally, a system composed of manufacturing areas and individual machines or cells can be seen as a factory or a plant. More globally, a network composed of raw material suppliers, factories that manufacture products out of the raw material, distribution centers and customers can be seen as a manufacturing network at a supply chain level.

From Chryssolouris (2006) a manufacturing process consists of one or more physical mechanisms that transform the material's shape and/or form and/or properties. The objective of this material transformation into goods is focused on satisfaction of human needs.

Manufacturing processes can be classified as *continuous processes* and *discrete part processes*. This classification also applies to industries, where these processes occur (see Pinedo (2009)).

In continuous manufacturing industries, such as paper mills, steel mills, and chemical plants, two common types of operations can be identified, which are main processing operations, and finishing or converting operations. Main processing operations consist of converting the raw material into a product. For example, in paper, and steel mills the machines take in the raw material, which is wooden pulp or iron, and produce rolls of paper, or steel. Afterwards these products are handled and transported with specialized material handling equipment. In continuous manufacturing the machines that do the main processing operations typically have very high startup and shutdown costs and usually work around the clock. A machine in the continuous process industries also incurs a high changeover cost when it has to switch over from one product to another. Finishing or converting operations usually involve cutting of the material, bending, folding and possibly painting or printing. These operations often are focused on the commodity of the purchasing client of the producer. For example, a finishing operation in the paper industry may produce cut size paper from the rolls that are manufactured by a paper mill. The products in finishing operations are usually made for a fixed order, for storage (future orders) or in combination of order and storage.

In discrete manufacturing industries, such as automotive or semiconductor industry, three common types of operations can be identified. These operations are primary converting operations, main production operations, and assembly operations. Primary converting operations are similar to the finishing operations of the continuous processes as they may typically include stamping, cutting, and bending processes. Some examples of these types of operations for discrete manufacturing are stamping facilities that produce body parts for cars and facilities that produce epoxy boards of various sizes for the plants that manufacture printed circuit boards. The main production operations are those multiple different operations by different machine tools that are required to be performed on a product or on its parts before it is ready for assembly. Generally a product may have to follow a certain route through the facility going through various work centers. Here the main investment of a manufacturer is made on various types of machine tools like lathes, mills, chip fabrication equipment, etc. It is often the case that certain operations have to be performed repeatedly and that some orders have to visit certain work centers in the facility several times. For example, in the semiconductor industry wafers typically have to undergo hundreds of steps. These operations include oxidation, deposition, metallization, lithography, etching, ion implantation, photoresist stripping, inspection and measurements. The assembly operations serve the purpose of putting different parts together. Typically during these operations the shape or form of any of the individual parts is not altered. Assembly operations usually do not require major investments in machine tools as in main operations, but do require investments in material handling systems. An assembly operation may be organized in work cells, in assembly lines, or according to a mixture of work cells and assembly lines.

This introductory description of manufacturing networks and their processes could not be complete without mentioning control policies that manage the operation in manufacturing network processes.

1.2 Manufacturing control policies

Nowadays, human need, mentioned in the definition of manufacturing objective in the previous section, is becoming more difficult to satisfy. Market competition in brands and varieties in type of products, the complexity of production processes as well as production costs play a very significant role in manufacturing processes. Thus manufacturing industries are confronted with a problem of how to keep efficient track of their product demand while also minimizing costs of production. The problem is very complex as product demand and costs of production are strongly related. An incorrect forecast as well as reduction of production costs without proper knowledge of demand may result in production losses or customer dissatisfaction. Thus control mechanisms as well as effective methods for their performance evaluation with regards to a manufacturing network, where such mechanisms are applied, are required. In the context of this thesis we define a performance evaluation method as one that can easily show if and how a manufacturing network operated under a certain policy is capable to satisfy its production demand while keeping efficiency in distribution of the raw material throughout the network and with what level of accuracy.

Currently there is a substantial literature on manufacturing control policies and their performance. A lot of valuable approaches including queuing theory (see e.g. Govil and Fu (1999), Shanthikumar et al. (2007)), Petri nets (see e.g. Zhou and Venkatesh (1999), Kahraman and Tüysüz (2010)), dynamic and linear programming (see e.g. Sodhi and Tang (2009)), were proposed and some of them have been implemented.

A number of classifications of manufacturing control policies were introduced by various authors. Usually policies are divided into three types that are push, pull and a combination of both also called hybrid. A push-based policy releases the material to the network at a constant rate or according to a predefined schedule (e.g. Master Production Schedule, ERP). A pull-based policy authorizes the release of material based on the system status (see e.g., Hopp and Spearman (2008)). A typical example of a pull policy is Kanban presented in (González et al. (2012)) as a simple pull-based paradigm that allows a job to be processed only if there is demand for it, in order to avoid unnecessary products in the network. It is also interesting to note that in control theory the push policy could be associated with a feedforward control and pull with feedback tracking control.

From the research conducted by Bonney et al. (1999) most manufacturing control systems usually combine a mix of push and pull principles. For instance a Conwip policy is a classical example of push/pull mix, where material is released into a network in a pull manner and processed by the network in a push manner. For mode details on Conwip and other hybrid policies, see Appendix K.

In this thesis, we follow the policy classification based on its triggering mechanism introduced in Gershwin (2000), which puts the control policies into three categories: token-based, time-based and surplus-based, respectively. In token-based approaches, so called tokens are generated and utilized in order to trigger certain events occurring in the manufacturing system. The most famous examples of such a policy are Kanban (Rees et al. (1987)), Conwip (Spearman et al. (1990), Ioannidis and Kouikoglou (2008)) and Basestock (Silver et al. (1998)). In time-based approaches, the control decisions depend on the time when a certain operation should take place, i.e. Material Resource Planning, Least Stack and Earliest Due Date strategies (see, e.g., Burgess and Passino (1997)). In the surplus approach, control decisions are made based on the production error, which is the difference between the cumulative demand and the cumulative output of the system. Some examples of this approach can be found in Bielecki and Kumar (1988); Bonvik et al. (1997); Lefebvre (1999); Gershwin (2000); Quintana (2002); Kogan and Perkins (2003); Boukas (2006); Stockton et al. (2007); Subramaniam et al. (2009); Savkin and Evans (2002); Nilakantan (2010) and references therein.

1.3 Objective and contributions

This PhD research concerns the performance analysis of manufacturing networks operated under surplus-based control. In surplus-based control, decisions are made based on the demand tracking error, which is the difference between the cumulative demand and the cumulative output of the system, see, e.g., Bielecki and Kumar (1988); Gershwin (2000); Nilakantan (2010) and references therein. Thus the performance of manufacturing networks is studied within the scope of demand-driven manufacturing control problems.

The main research objective is to apply classical tools from control theory in order to evaluate the performance of any given demand-driven manufacturing network. In line with this objective, the following goals are considered:

- Give a general mathematical interpretation to the existing surplus-based methodologies and identify important performance indicators for a manufacturing network under the aforementioned policies;
- Evaluate surplus-based control performance by analytical and experimental means for several commonly used network topologies.

Production and inventory control of manufacturing networks are often characterized by discrete-event behavior, which makes it difficult to construct a proper controller, especially for large networks. However, if a network is approximated by a continuous or discrete flow model, then standard control theory can be applied to control the production and inventory of the system, see e.g., Dashkovskiy et al. (2011), van den Berg et al. (2008), Lefeber et al. (2005), Huang et al. (2009). In discrete time the cumulative number of produced products in time k for a simple manufacturing machine can be described as the sum of its production rates at each time step till time k. Alternatively, in continuous time a simple manufacturing machine can be interpreted as an integrator, where the cumulative number of finished products is the integral of the production rate. The content of the inventory (buffer) between two machines is given by the difference between the total number of products produced by the upstream machine and the total number of products produced by the downstream machine. The buffer content can never be negative considering that the downstream machine cannot produce more than the upstream one. In our study these notions are extended to investigate the behavior of the following network topologies: manufacturing line, multi-product manufacturing line, re-entrant machine and re-entrant line. The production flow dynamics of manufacturing networks are described by discrete time flow models. The performance is considered in terms of production demand tracking accuracy and inventory levels, which are evaluated by means of a Lyapunov function-based approach, see e.g., Khalil (2002). Simulation-based analysis is conducted in order to evaluate the accuracy of the theoretical results obtained for the discrete time models on the event-based models of the studied network topologies. The performance of three surplus-based control mechanisms is compared for manufacturing line and re-entrant network configurations. Alternatively, continuous-time flow model of a manufacturing line operated by an observer-based surplus control is introduced. Lyapunov function-based and frequency domain-based approaches are exploited for the network's stability and performance evaluation, respectively.

The main contributions of the research presented in this thesis are:

- Flow models for commonly used network topologies such as: a single machine, a line, a multi-product machine and line, and re-entrant networks are derived.
- The surplus-based production policy is interpreted in terms of a variable structure controller and applied to the above mentioned network topologies.
- Analytical, simulation-based and experimental analysis on performance of this strategy are conducted for the network topologies.
- Networks with limited and unlimited buffers are considered.
- A liquid based experimental hardware for education and research is designed, assembled and serving its purpose. The purpose of the platform is to emulate production processes of manufacturing networks.

1.4 Outline of the thesis

This thesis is organized as follows.

- Chapter 2 presents a discrete time-based flow model of a single manufacturing machine producing one product type. For this model the optimal control strategy is obtained in terms of a surplus-based control. The tracking accuracy of a single manufacturing machine operated under the optimal surplus-based controller is studied. The performance of the closed-loop machine is illustrated by computer simulations. In addition a discrete time model is compared to a more accurate discrete event approximation of the manufacturing machine.
- Chapter 3 contains an extension of the flow model introduced in the previous chapter for a manufacturing line with unlimited and limited inventory operated under the surplus-based control. The analysis of the production error of each machine with respect to the inventory level of the network is developed. The performance issues of the closed-loop flow models are illustrated

in numerical simulations. A comparative study of discrete time and discrete event models of a manufacturing line is presented. A simulation-based performance analysis of three surplus-based production policies is conducted for a manufacturing line of 4 machines and 3 buffers.

- Chapter 4 introduces an extension of the flow models of a manufacturing machine and a line of Chapters 2 and 3 to a manufacturing line that produces several product types. The surplus-based controller is implemented in both networks and a detailed analysis on the production error tracking accuracy and inventory levels are presented. Performance issues of the closed-loop flow models are illustrated in numerical simulations.

- Chapter 5 is subdivided in two parts. In the first part a flow model of one manufacturing machine with surplus-based control is presented. A detailed analysis of production error trajectories is developed in this section. An analytical relation is obtained between the steady state production demand tacking accuracy and the intermediate base stock levels in the network. The reliability of the analytical results obtained for the discrete time model is also tested on a discrete event model. A simulation-based performance analysis of three surplus-based production policies is conducted for a re-entrant machine of 3 production stages and 2 buffers. In the second part, the flow model of a re-entrant manufacturing line with surplus-based control is obtained. Here necessary conditions are derived to guarantee the ultimate boundedness of the demand tracking error trajectories of each machine in the network. Steady state bounds on demand tracking errors and its relation to a base stock level in the inventory are shown. Performance issues of the closed-loop flow models are illustrated in numerical simulations. Also a simulation-based performance analysis of three surplus-based production policies is conducted for a re-entrant line of 3 machines of 2 production stages and 5 buffers.

- Chapter 6 presents a prototype developed for education and research purposes. This prototype is a liquid-based emulator of manufacturing network processes called Liquitrol. The liquid-based emulator consists of several electrical pumps and liquid reservoirs. This chapter presents a detailed description of the Liquitrol, subdivided in mechanical and electrical specifications. Flow models of the platform for 3 selected network configurations are detailed. Three types of practical experimentations for educational purposes are presented.

- Chapter 7 deals with the problem of controlling a tandem line of manufacturing machines such that an unknown production demand is tracked with a desired accuracy. To solve the problem in case of unknown demand rate, a combination of a feedforward-feedback controller with a reduced-order observer is proposed. This surplus-based controller is studied for one machine as well as for a line of machines in continuous time-domain and in frequency-domain representations.

- Chapter 8 contains the conclusions of the thesis and recommendations for future research.

- Appendixes A to J contain the proofs of the presented theorems of this thesis and Appendix K presents the models of manufacturing networks that were utilized for the simulation-based performance analysis presented in this thesis.

All the above presented chapters contain an abstract, an introduction to the presented research and conclusions.

1.5 Summary of publications

This thesis is mostly based on conference and journal papers, either published or accepted. The present section summarizes the relationship between the papers and the chapters in this thesis. Note that some papers are used in more than one chapter.

Chapters 2 and 3 contain results presented in:

Starkov, K., Pogromsky, A., Rooda, J., 2010a. Production error analysis for a line of manufacturing machines, variable structure control approach. In: *APMS: International Conference on Advances in Production Management Systems.* Como, Italy, CD-ROM.

Starkov, K., Pogromsky, A., Rooda, J., 2010b. Towards a sustainable control of complex manufacturing networks. In: *Sustainable development: industrial practice, education and research.* Monopoli, Italy, pp. 174-179.

Starkov, K., Pogromsky, A., Rooda, J., 2010c. Variable structure control of a line of manufacturing machines. In: *IFAC on Intelligent Manufacturing Systems.* Lisboa, Portugal, pp. 277-282.

Starkov, K., Feoktistova, V., Pogromsky, A., Matveev, A., Rooda, J., 2011a. Optimal production control method for tandem manufacturing lines. In: *5th International Scientific Conference on Physics and Control.* Leon, Spain.

Starkov, K., Pogromsky, A., Rooda, J., 2011d. Performance analysis for tandem manufacturing lines under variable structure production control method. *International Journal of Production Research*, 13 (8), 2363-2375.

Starkov, K., Feoktistova, V., Pogromsky, A., Matveev, A., Rooda, J., 2012a. Performance analysis of a manufacturing line operated under optimal surplus-based production control. *Mathematical Problems in Engineering*, DOI: 10.1155/2012/602094.

Chapter 4 contains results presented in:

Starkov, K., Pogromsky, A., Adan, I., Rooda, J., 2011b. Performance analysis of a flexible manufacturing line operated under surplus-based production control. In: *ICACE: International Conference on Automation and Control Engineering.* Venice, Italy, CD-ROM.

Chapter 5 contains results presented in:

Starkov, K., Pogromsky, A., Adan, I., Rooda, J., 2011c. Performance analysis of re-entrant manufacturing networks under surplus-based production control. In: *Recent Advances in Manufacturing Engineering.* Barcelona, Spain, pp. 152-161.

Starkov, K., Pogromsky, A., Adan, I., Rooda, J., 2012c. Performance analysis of re-entrant manufacturing networks under surplus-based production control. *International Journal of Production Research*, accepted.

Chapter 6 contains results presented in:

Starkov, K., Kamp, H., Pogromsky, A., Adan, I., 2012b. Design and implementation of a water-based emulator of manufacturing processes. In: *9th IFAC Symposium on Advances in Control Education.* Nizhny Novgorod, Russia, CD-ROM.

Chapter 7 contains results presented in:

Andrievsky, B., Starkov, K., Pogromsky, A., 2012. Frequency and time domain analysis on performance of a production line operated by observer based distributed control. *International Journal of Systems Science*, DOI: 10.1080/00207721.2012.670300.

2
Single machine

This chapter is partly based on Starkov et al. (2010b), Starkov et al. (2010c), Starkov et al. (2011a), Starkov et al. (2012d) and Starkov et al. (2012a).

Abstract | In this chapter, we examine optimality and performance of a singe manufacturing machine driven by a surplus-based decentralized production control strategy. The main objective of this type of production strategies is to guarantee that the cumulative number of produced products follows the cumulative production demand on the output of the given network. The basic idea of surplus-based control strategy is presented for the case of one manufacturing machine. We prove that this strategy is optimal and present accuracy bounds on demand tracking error of a single machine operated under this strategy. The analytical results of this chapter are supported by discrete time and discrete event simulation examples.

2.1 Introduction

Nowadays the highly dynamic market environment requires that production control policies implemented in manufacturing industries should be capable of providing quick and accurate responses to constant and rapid changes in the production demand. This strongly shifts the interests of manufacturers to the need of theoretical analysis of the currently existing policies, i.e. the study of conformity between the production output and the demand of their product, as well as the relation between stock level (buffer content) and the production surplus of the manufacturing network.

Currently there is a substantial literature on manufacturing control policies and their performance. A number of classifications of these policies were introduced by various authors. In this chapter, we follow the classification introduced in Gershwin (2000), which puts the control policies into three categories: token-based, time-based and surplus-based, respectively (for more details see Chapter 1, Section 1.2). In the surplus approach, control decisions are based on the demand tracking[1] error, which is the difference between the cumulative demand and cumulative output of the system. For extensive surveys and further details concerning, in particular, production line control mechanisms, we refer the reader to Bonvik et al. (1997), Gershwin (2000), Ortega and Lin (2004), Sarimveis et al. (2008).

In the afore-mentioned literature, considerable research effort has been invested into the issue of optimality. In Bielecki and Kumar (1988), the authors showed that for unreliable manufacturing systems with parameters from a certain domain, the zero-inventory policies are optimal even if there is an uncertainty in the manufacturing capacity. This was demonstrated under the assumption that the demand rate is constant and the production rate can be adjusted. In Perkins and Kumar (1995), the optimality of pull controlled flow shop was established in the case where the performance index encompasses buffer holding costs and system shortfall/inventory costs. This research treated the production demand and processing rates as deterministic.

In Martinelli et al. (2001), a broad class of dynamic scheduling problems associated with single-server, multi-class, continuous-flow, flexible manufacturing systems was considered. The objective was to minimize the integral of an instantaneous cost function defined on the inventory/backlog state of the system. The authors provide sufficient conditions under which the optimal solution comes by implementing of the myopic scheduling policy. The paper also presents examples and counterexamples that explicitly illustrate the behavior and limitations of the myopic scheduling policies.

In Laumanns and Lefeber (2006) a model based on stochastic discrete-time controlled dynamical systems was developed in order to derive optimal policies for controlling the material flow in supply networks. Contrary to most studies in this area, which typically assume a given (parameterized) control strategy and analyze how the dynamics depends on the parameters, the authors do not assume a certain family of strategies a priori, thus allowing any control law in the form of a function of the current state of the system. The individual nodes are controlling their inflows in a decentralized fashion by placing orders to their immediate suppliers. An explicit optimal state-feedback control is derived with respect to the cost functional that typically takes into account both inventory holding costs and ordering costs.

Extended reviews on optimal control in production networks can be found in, e.g., Riddalls et al. (2000), Ortega and Lin (2004), and Sarimveis et al. (2008).

This chapter concerns the performance analysis of a single manufacturing machine. We find the optimal control policy that minimizes the mismatch between the cumulative production output and cumulative demand. Then we evaluate the system dynamics under this policy and obtain the steady state bounds on the tracking accuracy of its production demand.

Thus, we first tackle the problem of optimal control for one manufacturing machine

[1]In this thesis by the term tracking we refer to the control goal of a system (e.g., manufacturing network), which consists in keeping close the output of this system to its reference signal (e.g., cumulative demand). For more information see for example Khalil (2002), Chapter 12, page 474.

that nominally produces products in lots of a given size and is controlled by carrying out only two commands ('on' and 'off'), unlike Laumanns and Lefeber (2006), where the size of the lot was controllable and the set of feasible controls was a polygon. Another difference is that we deal with a deterministic model of the system. Similarly to Laumanns and Lefeber (2006) and contrary to the bulk of the literature in the area, we do not limit the class of the strategies a priori and take into account all control laws that are fed by the currently available data, which concerns not only the current system state but also the past states. The system is influenced by uncertain disturbances that are due to both market fluctuations in demand rate and random fluctuations in the production rate of the machine. By assuming known bounds on these disturbances and by applying the min-max dynamical programming, we prove that surplus-based pull control policy is optimal.

Further, we apply classical tools from the control theory in order to evaluate the tracking accuracy of a single machine operated under this optimal controller.

The production flow process is described by means of difference equations and in order to analyze performance, an approach based on Lyapunov theory is employed (see, e.g., Khalil (2002) and references therein).

The chapter is organized as follows. In Section 2.2, a general open-loop discrete time flow model of a single manufacturing machine is presented and the control objective is introduced. The assumptions imposed on the model are formulated in Section 2.3 and an optimal control policy is derived under those assumptions. Section 2.4 is devoted to analysis of the production demand tracking accuracy for a single manufacturing machine operated under the optimal surplus-based controller. Performance of the closed-loop system is illustrated by computer simulations in Section 2.5. Discrete time model is compared to discrete event model of a single machine in Section 2.6. Finally, Section 2.7 presents the conclusions.

2.2 Flow model

By pursuing the research objectives of Chapter 1 we start our analysis of manufacturing network topologies from the most basic configuration, which is a single machine. The machine presents an infinite raw material supply and it produces one product type. In Section 1.3 several approaches are mentioned for modeling such a system. In this chapter as well as in Chapters 3 till 5 the discrete time flow models are used to describe the product flow dynamics for the studied manufacturing networks.

The flow model of one manufacturing machine in discrete time is defined as

$$y(k+1) \quad = \quad y(k) + u(k) + f(k), \tag{2.1}$$

where all the events within the model occur at given time instances and k represents the current time so that the time step between all the events is constant. Here $y(k) \in \mathbb{R}$ is the cumulative output of the machine in time k, $f(k) \in \mathbb{R}$ is an external disturbance in time k and $u(k) \in \mathbb{R}$ is the control input which action can only be executed every time k. The model (2.1) describes the future cumulative output $y(k+1)$ of a single machine by its current cumulative output $y(k)$, its current controlled production rate $u(k)$ and its current uncontrolled production rate $f(k)$. Note that for this research it is assumed that $u(k) + f(k) \geq 0$ for all k, which implies that the machine can only produce products or be in its idle state. The equation (2.1) presents a general model that can describe a product flow for a wide range of machines. Further specific assumptions on system (2.1) are given in Sections 2.3 and 2.4.

Under the assumption that there is always sufficient quantity of the raw material to feed the machine, the control aim is to track the non-decreasing cumulative production demand. We define the cumulative production demand by using $y_d(k) \in \mathbb{R}$ given by

$$y_d(k) = y_{d0} + v_d k + \varphi(k), \tag{2.2}$$

where y_{d0} is a positive constant that represents the initial production demand, v_d is a positive constant that defines the average desired demand rate, and $\varphi(k) \in \mathbb{R}$ is the bounded fluctuation that is imposed on the linear demand $v_d k$. Further specific assumptions on the production demand model (2.1) are given in Sections 2.3 and 2.4.

2.3 Optimal control strategy

To efficiently fulfill the control aim, presented in the previous section, we are going to minimize the demand tracking error $\varepsilon(k) = y_d(k) - y(k)$ in the class of control strategies fed by available data:

$$u(k) = U_k[y(0), \ldots, y(k), y_d(0), \ldots, y_d(k)] \in \{0; 1\}\,. \tag{2.3}$$

It follows that, in time step k the control input u is limited to taking the value of 1 or 0. Note that from (2.1) this means that we define the nominal production rate of our machine as of 1 lot per time unit. The selected value serves here as a theoretical example and modifying it by another desired value won't modify the essence of the later presented results.

It follows from (2.1) and (2.2) that the increment of the demand tracking error given by $\varepsilon(k+1)) - \varepsilon(k)$ obeys the following equation:

$$\varepsilon(k+1) = \varepsilon(k) - u(k) + v_d + \Delta\varphi(k) - f(k), \tag{2.4}$$

where $\Delta\varphi(k) = \varphi(k+1) - \varphi(k)$ is the deviation of the demand rate fluctuations.

Denoting by $\xi(k) := v_d + \Delta\varphi(k) - f(k)$ we introduce the following condition (also known as capacity condition)

$$0 < \xi(k) < 1, \qquad \forall k \in \mathbb{N}. \tag{2.5}$$

The physical meaning of this condition is twofold :

1. The production capacity is always bigger than the demand rate;
2. The rate of cumulative demand can only be positive.

Note that in reality it is a common situation when the demand rate exceeds the production capacity of a machine. If the capacity of a machine is always lower than its demand rate, then the cumulative output of this system is unable to follow its cumulative production demand. In consequence such a system cannot fulfill the control objective. On the other hand, if the capacity of a machine is on average higher than its demand rate, then by (2.5) we merely assume that for discrete time model of such machine a time step can be selected in a way that the capacity condition is always fulfilled. For the rest of this thesis it is assumed that the time step is fixed and defined in such a way that the capacity of the network is always sufficient to fulfill its demand.

Given the system (2.4), (2.3) with (2.5) as the assumption on its operation, we examine the following two optimization problems:

$$J_T = \sup_{\xi(0),\ldots,\xi(T-1)} \sum_{k=0}^{T} |\varepsilon(k)|^p \to \min_U, \tag{2.6}$$

$$J_\infty = \limsup_{k\to\infty} \sup_{\xi(\cdot)} |\varepsilon(k)|^p \to \min_U . \tag{2.7}$$

Here sup is taken over all $\xi(\cdot)$ satisfying (2.5), $U = \{U_k(\cdot)\}_{k=0}^{\infty}$ is the set of all possible control strategies given by (2.3). By performance criterion (2.6) we want to select a control strategy that minimizes the value of the demand tracking error influenced by the demand rate v_d and feasible perturbations $\varphi(k), f(k)$ on a finite and given time horizon T. While by performance criterion (2.7) we desire to select such a control strategy that minimizes the value of the demand tracking error influenced by demand rate and perturbations after a certain large enough (infinite) time horizon. Though the control objectives in (2.6) and (2.7) are different, the main result of this section (Theorem 1) shows that they are achieved by a common control strategy.

In (2.6) and (2.7), larger values of the parameter p are used to penalize large demand tracking errors more severely. In the literature, the most popular values are $p = 1, 2$. We consider arbitrary $p \in [1, +\infty)$ partly in order to show that the solution of the optimization problem is not influenced by the value of p. This is presumably due to the use of discrete controls $u = 0, 1$; if the set of feasible controls is not finite, the solution typically depends on p.

The following theorem is the main result of this section.

Theorem 1. *For system (2.1) the following control strategy*

$$u(k) = sign_+\big(\varepsilon(k)\big) \tag{2.8}$$

is optimal with respect to the performance index (2.6) for any given T, as well as with respect to the performance criterion (2.7). This is true irrespective of the choice of $p \in [1, +\infty)$.

Proof. The proof of Theorem 1 is given in Appendix A. □

Here the control action is given by

$$u = \operatorname{sign}_+(\varepsilon) := \begin{cases} 1 & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon < 0 \\ 0,1 & \text{if } \varepsilon = 0 \end{cases} .$$

Basically this controller works as a pull control that authorizes the machine to produce if the product surplus is negative, stops the machine if the surplus is positive, and arbitrary selects between the two previous decisions if the surplus is zero.

2.4 Tracking accuracy

In this section the production demand tracking accuracy of a single manufacturing machine is analyzed in close-loop with surplus-based tracking controller. The production flow model (2.1) is now simplified to

$$y(k+1) \quad = \quad y(k) + \beta(k)u(k), \tag{2.9}$$

where $\beta(k) = \mu + f(k)$ with μ as a positive constant representing the nominal production rate of the machine. Note that in this setting the perturbation $f(k)$ influences the nominal production rate μ only when the machine is operational. Also for the sake of definiteness, in this section the control input (2.8) is adjusted to

$$u(k) = \operatorname{sign}_+\big(\varepsilon(k)\big), \tag{2.10}$$

where

$$u = \operatorname{sign}_+(\varepsilon) := \begin{cases} 1 & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon \leq 0 \end{cases} .$$

Unlike in the prior definition of $u(k)$ (see Section 2.2 equation (2.8)), in this case the control action authorizes the machine to produce at its processing speed of $\mu + f(k)$ [lots/time unit] only if the demand tracking error $\varepsilon(k) \in \mathbb{R}$ is positive. The production demand tracking error is given by $\varepsilon(k) = y_d(k) - y(k)$, where $\varepsilon(k+1) - \varepsilon(k)$, along the solutions of $\varepsilon(k)$, is defined as:

$$\varepsilon(k+1) - \varepsilon(k) = v_d + \Delta\varphi(k) - (\mu + f(k))\operatorname{sign}_+(\varepsilon(k)), \tag{2.11}$$

with $\Delta\varphi(k) = \varphi(k+1) - \varphi(k)$.

It follows from (2.11) that in order to guarantee a proper demand trajectory tracking the product demand cannot be higher than the machine processing speed, which in this case is μ lots per time unit. Thus, let us assume that all machine perturbations $W(k) = \Delta\varphi(k) - f(k)$ from (2.11) are bounded by

$$\alpha_1 < W(k) < \alpha_2, \quad \forall k \in \mathbb{N}, \tag{2.12}$$

where α_1, α_2 are some constants that satisfy

$$\alpha_2 \quad < \quad \mu - v_d, \tag{2.13}$$

$$\alpha_1 \quad > \quad -v_d. \tag{2.14}$$

By (2.13) and (2.14) we state that the production demand rate can never be faster than the speed of the machine and that considering the presence of perturbations bounded by (α_1, α_2) the demand rate can only de positive, respectively. In practice it can be rather unnatural to set bounds on market fluctuations together with the machine perturbations. Thus starting from Chapter 4 of this book, where the results on multi-product manufacturing line are discussed, we evaluate system perturbations independently from the demand rate fluctuations[2].

From (2.12), (2.13), and (2.14) the following condition (also known as capacity condition) holds

$$0 < v_d + W(k) < \mu. \tag{2.15}$$

Note that the physical meaning of the above mentioned assumptions is similar to the one discussed in detail in the pervious section.

In order to follow the product demand, the variable structure controller $\text{sign}_+(\varepsilon(k))$ is included in the flow model (2.9) of a single machine. The demand tracking error of a single machine is defined as the difference between the cumulative demand and the cumulative number of products produced up to this moment.

Now the following result regarding the performance of system (2.11) can be stated in the form of a theorem.

Theorem 2. *Assume that the discrete time system defined by (2.9) with control input (2.10) satisfies condition (2.15). Then all solutions of (2.11) are uniformly ultimately bounded by*

$$\limsup_{k\to\infty} \varepsilon(k) \leq v_d + \alpha_2, \tag{2.16}$$
$$\liminf_{k\to\infty} \varepsilon(k) \geq v_d + \alpha_1 - \mu. \tag{2.17}$$

The bounds (2.16) and (2.17) specify that, in steady state, the demand tracking error of a single machine will not grow further than the maximal demand rate of one time step and will not present a bigger product excess than the one given by de difference between the minimal demand rate and the maximal production rate of one time step, respectively.

Proof. The detailed proof of Theorem 2 is provided in the Appendix B.
Without loss of generality, the proof can be graphically summarized in Figure 2.1. Given the function $V(\varepsilon(k))$ as shown in Figure 2.1 we evaluate the increment $\Delta V(\varepsilon(k))$ of $V(\varepsilon(k))$ along the solutions of (2.11). From Figure 2.1 it can be observed that all the trajectories of $\varepsilon(k)$ regardless their initial conditions are attracted to the bounded zero set of function $V(\varepsilon(k))$ also known as Lyapunov function. The bounds of the zero set are shown with arrows in Figure 2.1, which correspond to the bounds (2.16) and (2.17). □

[2]Please note that the result of Theorem 2 of this section can be similarly deduced from Theorem 7 of Chapter 4, which is obtained for multi-product manufacturing line. This can made if in Theorem 7 a number of machines and stages in multi-product line is reduced to a single machine with only one production stage.

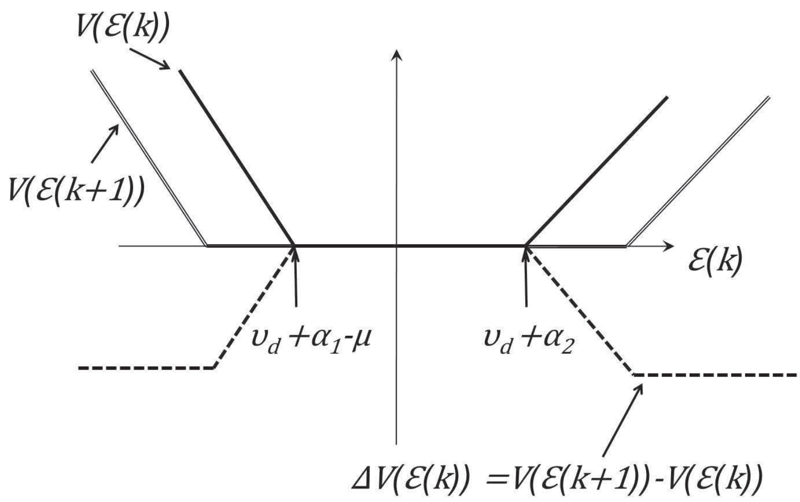


Figure 2.1: Lyapunov function-based analysis

2.5 Discrete time simulations

Simulation results for one manufacturing machine, driven by the surplus-based regulator (2.10), are presented in this section. Two simulation examples are introduced in order to verify the effectiveness of the analytical results from previous section. The examples are simulated using discrete time (DT) model given by 2.11 and 2.10.

For both examples the processing speed of the machine is fixed to 5 lots per time unit. In the first example (see the outcome in Figures 2.2 and 2.3) the initial product demand $y_{d0} = 20$ lots, the demand increment $\Delta y_d(k)$ is 4 lots per time unit and the initial production output $y(0) = 0$ lots. The market fluctuations as well as the external perturbations are set to the zero value. Results from Figures 2.2 and 2.3 show that the output $y(k)$ follows the product demand $y_d(k)$ with tracking error $\varepsilon(k)$ bounded by $0 \leq \varepsilon(k) \leq 4$ lots, which satisfy (2.16), (2.17) from where $-1 \leq \varepsilon(k) \leq 4$. It can be observed from Figure 2.2 and 2.3 that after the transient behavior the demand tracking error trajectory reaches its zero value in 20 time steps. Then, at time step 21 the demand signal $y_d(k)$ is gown 4 lots higher than the output $y(k)$. The controller $u(k)$ responds with the authorization for 5 lots, which are produced in the next time step while the cumulative demand increases by another 4 lots. Thus, the cumulative output at time step 22 is 3 lots lower than the cumulative demand. For the next 3 time steps the machine keeps producing lots in batches of 5 until the production demand is reached again at time step 25 with $y_d(k)$ of 120 lots. Now during the next time step, the machine remains idle while the demand grows to 124 lots. Thus, the production cycle from the 21st time step till the 25th is repeated.

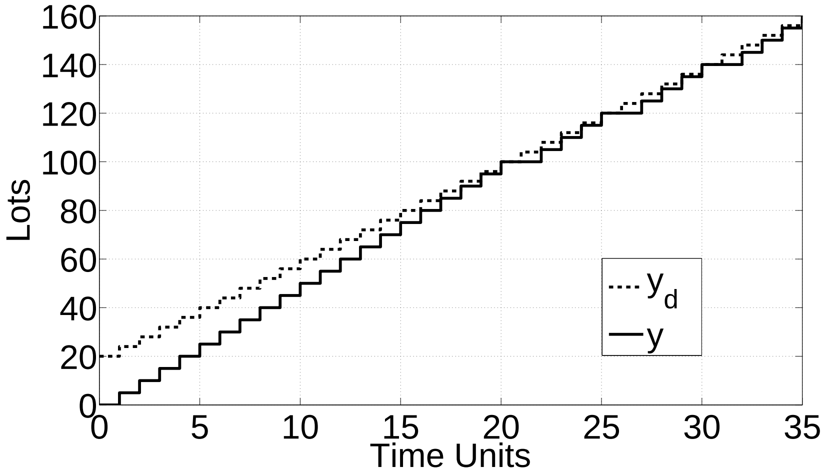


Figure 2.2: Demand vs Output, $v_d = 4$ and $y_{d0} = 20$.

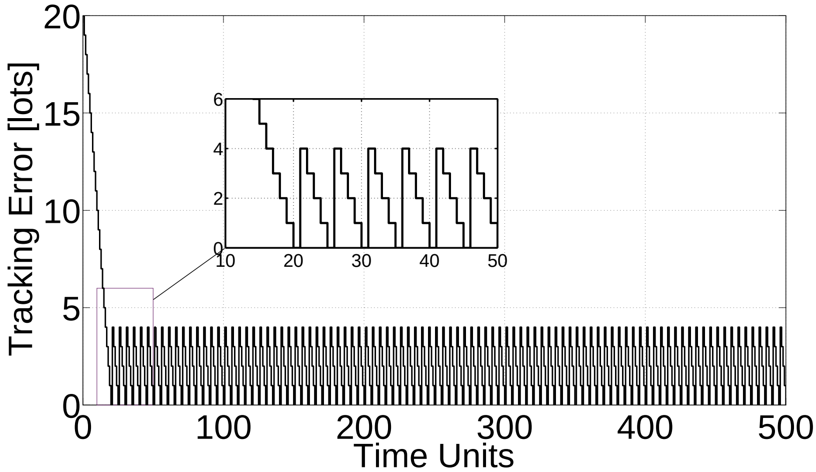


Figure 2.3: Demand Tracking Error, $v_d = 4$ and $y_{d0} = 20$.

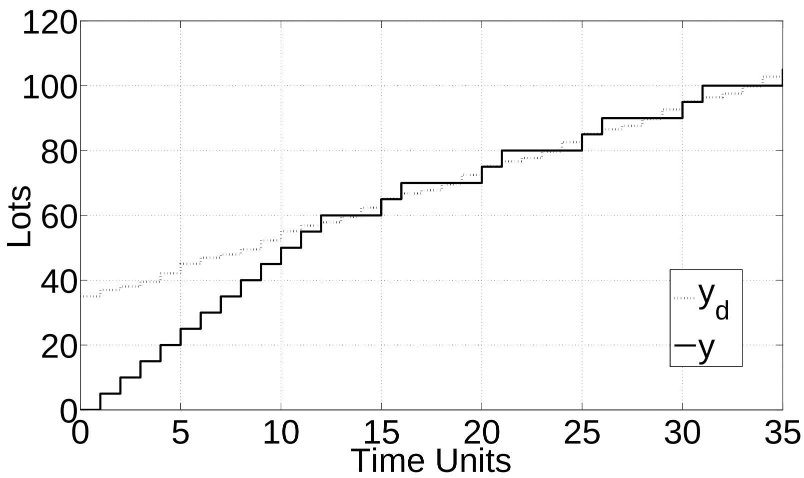


Figure 2.4: Demand vs. Output, with $y_{d0} = 35$, $v_d = 2$ and $\varphi(k) = 0.9\sin(5k)$.

For the second example the product demand rate is $v_d + \Delta\varphi(k) = 2 + 0.9(\sin(5(k+1)) - \sin(5(k)))$ [3] lots per time unit with the initial demand $y_{d0} = 35$ lots. Figures 2.4 and 2.5 show the output response of the machine to the nonlinear demand growth and the demand tracking error, respectively. From Figure 2.4 it can be observed that for given initial demand of 35 lots and initial output of 0 lots the machine manages to reach the cumulative demand trajectory in 12 time steps. Figure 2.5 shows that the resulting demand tracking error is bounded by $-3.9 \leq \varepsilon(k) \leq 2.9$ lots, which are identical to the theoretical results of (2.16) and (2.17).

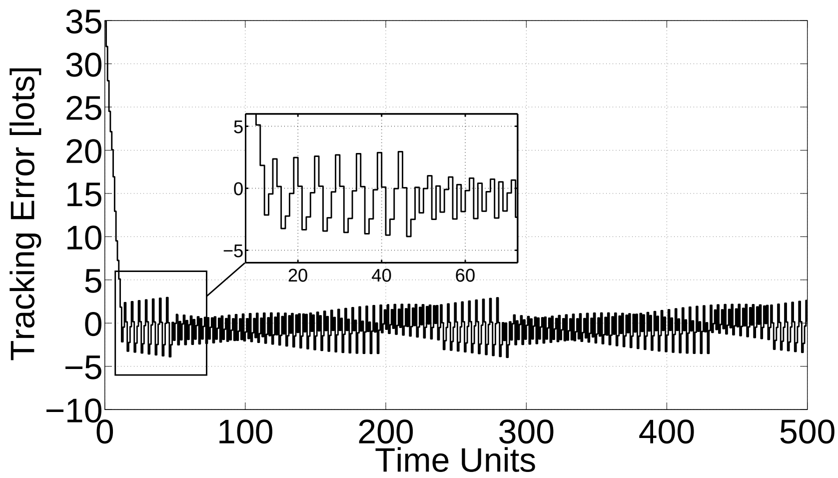


Figure 2.5: Demand Tracking Error, with $y_{d0} = 35$, $v_d = 2$ and $\varphi(k) = 0.9\sin(5k)$.

[3] In some simulation examples of this book in order to verify the effectiveness of the analytical bounds on demand tracking error accuracy we introduce periodic functions to model perturbations and market fluctuations in a manufacturing network. Note that these bounds are obtained for the worst case behavior of a product flow in a network. Thus, as long as all the assumptions on the flow model of a manufacturing network are satisfied, no relevant difference with respect to the accuracy of the analytical bounds can be expected in case an alternative function or a stochastic model is used to describe perturbations and fluctuations in the network.

It can be concluded that the simulation results reflect the expected flow model behavior for all the product demands, which are given in this section. These product demands were selected in order to test the model behavior inside the boundary of the given capacity condition (2.15).

2.6 Discrete event simulations

In discrete event (DE) models, the operation of a system is represented as a chronological sequence of events. Each event occurs at an instant in time and marks a change of state in the system (see Robinson (2004)). Thus the event occurrence is not limited to the fixed time step as in the previously presented discrete time (DT) model. Consequently the DE models can describe the product flow in a more accurate manner.

In this section the accuracy of the obtained demand tracking error bounds for the DT model is tested by means of two simulation examples on the DE model of a single manufacturing machine that produces one product type in one at a time manner. The DE model was built by using the specification language called χ (see Beek et al. (2006)) developed at the Eindhoven University of Technology[4].

Figure 2.6 shows the diagram of the DE model of one manufacturing machine operated under a surplus-based control. The circles represent the processes. The wide and the thin arrows indicate the lot and the information flow directions, respectively.

The machine is described by process M. It exchanges information with the controller (process C) that authorizes the machine to produce based on the value of the demand tracking error ε. The demand tracking error consists of the difference between integer values of the cumulative demand y_d and the cumulative number of produced products y. If $\varepsilon \geq 1$, then the controller C sends the authorization, e.g., by means of a token u send to M. If $\varepsilon \leq 0$, then no authorization is given till the value of y_d is updated. Process C recalculates the value of ε every time it receives y_d from the demand generating process D or the update on the value of y, e.g., in form of a token[5] y_f, which is send from process M to C.

Process D recalculates the cumulative demand value with preselected accuracy. Every time the demand value is incremented by one lot, process D sends the updated y_d value to process C. Process G functions as an infinite raw material generator, i.e. it can instantly send a product to process M. Process E mimics an infinite storage of finished products, i.e. it can instantly receive a product from process M. If authorization u is granted and no lots are presently processed in M, then machine process M is provided with one lot from G. After a delay, which occurs due to the lot processing, the product from M is send to the exit process E and the token y_f is send to process C, which increments by one its counter of y.

The following two examples are analyzed. In the first example (see the outcome on Figure 2.7) for event-based system the production speed of the machine (μ) is fixed

[4]For the details on implementation see Rooda and J.Vervoort (2007)

[5]Every time the machine produces one product the token y_f is send to the process C, which increments its counter of y by one.

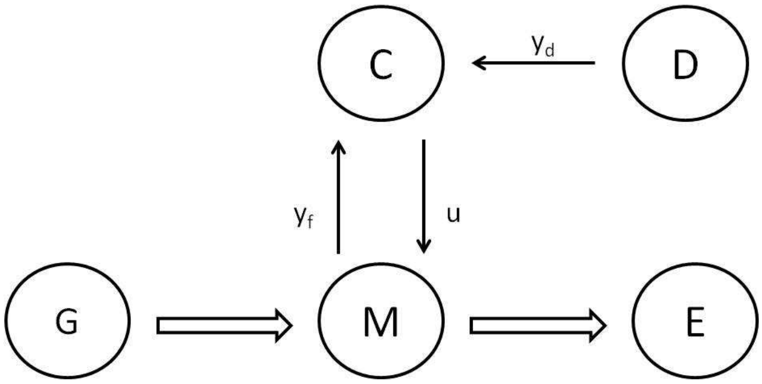


Figure 2.6: Schematic of DE model of a machine operated under surplus-based control

to 1 lot per time unit with zero initial output. The production demand consists of its initial value y_{d0} of 20 lots and the constant rate v_d of 1 lot every 1.111 units of time. The DE system is approximated by the DT model with the same production speed and demand rate v_d of 0.9 lots per time unit. The simulation run time was set to 500 time units.

The outcome of the experiment:

- From Figure 2.7, the duration of the transient behavior of the DE system is similar to that of the DT system.
- The steady state demand tracking error of DE model is bounded by $0 \leq \varepsilon \leq 1$ lots. The steady state demand tracking error of DT model is bounded by $-0.1 \leq \varepsilon \leq 0.9$ lots, which satisfies (2.16) and (2.17). The difference in the steady state bounds of ε of the two models comes from to their distinct interpretation of the demand arrival. In DE model the demand value is rounded to the closest integer, i.e. the product order that arrives to the system is for an integer number of products (1 lot every 1.111 time units). The DT model approximates the demand arrival differently. The quantity of 0.9 lots is added every fixed time step of 1 time unit to the cumulative demand signal y_d. For this example the bounds of DT model can be translated to DE model bounds as the ceil value of (2.16) and the floor value of (2.17).

The second example, see the outcome on Figure 2.8, differs form Figure 2.7 in that market perturbations φ are introduced in the demand variable. For DE model the constant demand rate is of 1 lot every 1.428 time units with market fluctuations of $\Delta\varphi(t) = 0.1\sin(5t)$, where t is a current time of experiment. Thus the product demand rate is oscillating between one lot every 1.250 and 1.666 time units. For the DT model this behavior can be approximated by product demand rate $v_d + \Delta\varphi(k) = 0.7 + 0.1\sin(5k)$ lots per time unit. Here k is the time step. The run time of the experiment was set to 500 time units.

The outcome of the experiment:

- From Figure 2.8 it can be observed that the duration of the transient behavior of the DE system is similar to that of the DT system.

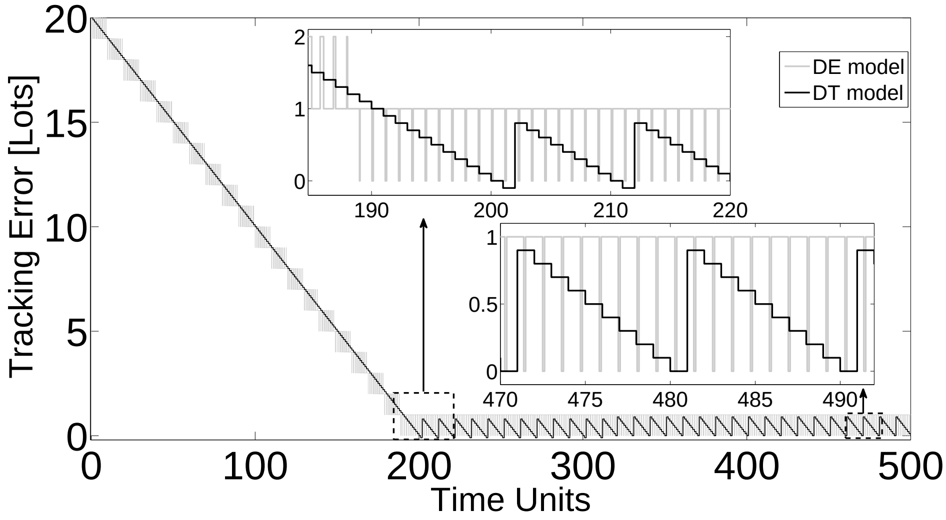


Figure 2.7: Demand Tracking Error for $v_d = 0.9$ and $y_{d0} = 20$.

- The steady state demand tracking error of the DE model is bounded by $0 \leq \varepsilon \leq 1$ lots. The steady state demand tracking error of the DT model is bounded by $-0.367 \leq \varepsilon \leq 0.787$ lots, which satisfies 0.8 lot bound from (2.16) and -0.4 lot bound from (2.17). The difference in the steady state bounds of ε of the two models is due to their distinct interpretations of the demand arrival. In the DE model the demand value is rounded to the closest integer, i.e. the product order of 1 lot that arrives to the system varies in its arrival time between 1.250 and 1.666 time units. The DT model approximates the demand arrival differently. The value of $0.7 + 0.1\sin(5(k))$ lots is added every fixed time step of 1 time unit to the cumulative demand signal y_d. In this example the bounds of the DT model can be also translated to the DE model bounds as the ceil value of (2.16) and the floor value of (2.17).

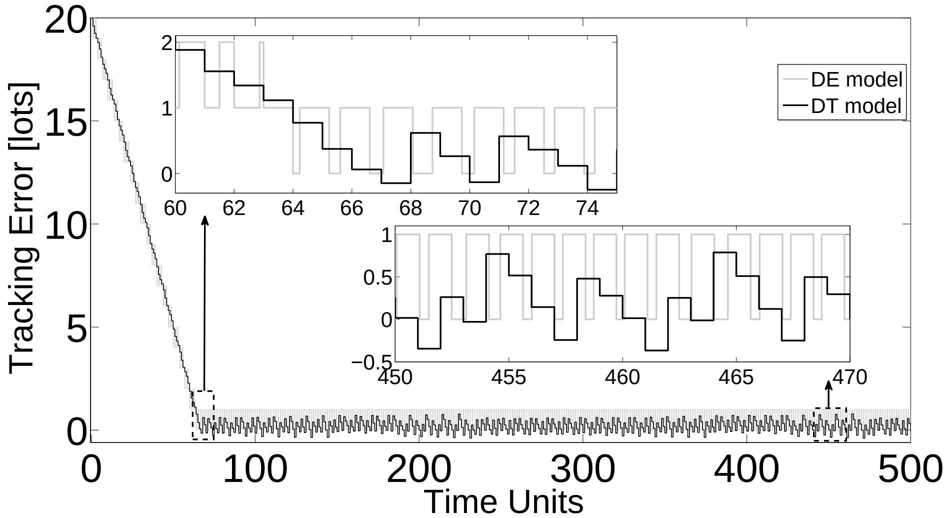


Figure 2.8: Demand Tracking Error for $v_d = 0.7$, $\Delta\varphi(k) = 0.1\sin(5t)$ and $y_{d0} = 20$.

2.7 Conclusions

The results on the performance analysis of a single manufacturing machine operated under a surplus-based control were presented in this chapter. First, for a single manufacturing machine operated under the influence of bounded but unknown market fluctuations and a production rate perturbation, the surplus-based controller was proven to be optimal for the cumulative demand tracking problem. Then, by means of the Lyapunov function presented in Appendix B, the closed-loop flow model was proven to be uniformly ultimately bounded. The bounds on the steady state demand tracking accuracy for a single machine were presented. The simulation results with the DT model of the product flow dynamics reflected the accuracy of the obtained demand tracking error bounds. The interpretation of these DT bounds for the DE model of a manufacturing machine was given for two selected simulation examples. It was observed that the DE model described the production flow process in a more accurate manner than the DT model, but, given an adequate interpretation of the obtained DT bounds, they can be used as a reference tool for the demand tracking error evaluation of the DE model.

3

Manufacturing lines

This chapter is partly based on Starkov et al. (2010a), Starkov et al. (2012d), Starkov et al. (2011a) and Starkov et al. (2012a).

Abstract | In this chapter the analysis of the surplus-based control is extended to a line of N manufacturing machines. The general idea of this control method was presented in Chapter 2 for the case of one manufacturing machine. In that chapter the implemented surplus-based control strategy was proven to be optimal. This chapter is focused on the performance of a manufacturing line under the surplus-based control. Particulary the relation between the product demand tracking accuracy and the base stock of the intermediate product in a line operated under the extended surplus-based controller of Chapter 2 is evaluated. The flow model of a production line is studied first under the assumption of the existence of infinite buffers and then under the assumption that the buffers are finite. All the details on the used procedure as well as the obtained bounds on the demand tracking error accuracy are given in the chapter. The production flow process is described by means of difference equations and in order to analyze the performance, an approach based on Lyapunov theory is exploited. Furthermore the proposed closed-loop flow model of a line in its discrete time representation is evaluated with respect to an event-based production model. Finally, simulation based comparative performance analysis is conducted between three selected surplus-based controllers for a time-based model of the line of 4 machines and 3 intermediate buffers.

3.1 Introduction

In the modern market keeping high competition in brands and varieties in type of products is the way for survival of manufacturing industries. Therefore production control methods with capabilities of quick responses to rapid changes in the demand and efficient distribution of the raw material throughout the network are of a big importance among leading manufacturers. Nowadays, the production control problem has been widely studied and a lot of valuable approaches including queuing theory, Petri nets, dynamic programming, linear programming, hybrid systems were proposed and some of them are implemented. Though up to this moment many methods have been developed, the factory performance remains unpredictable.

Some examples of implemented strategies are pull, push, and their combinations (see, e.g. Hopp and Spearman (2008); Deleersnyder et al. (1992); Ahn and Kaminsky (2005) and references therein). These control strategies, together with integrated computer-based systems, such as Manufacturing Resource Planning (MRP II), and Enterprise Resource Planning (ERP) (Vollmann et al. (2004)), are widely used by industries to control product flows in networks as well as inventory levels. Though these policies are conceptually simple, their response to disturbances and market fluctuations is not always fast enough for challenging requirements. Another often used control strategy is based on model predictive control (MPC) (see, e.g. Song et al. (2002); Doganis et al. (2007); Camacho and Bordons (2004) and references therein). This is a robust method that is capable of providing solutions for a production demand tracking problem. MPC is able to take into account hard constraints, such as the maximum production speed of the machine and buffer capacity restrictions. MPC is an effective method that can be used for real-time control of manufacturing systems, but it has two main drawbacks. The first one is that the optimization problem to be solved is generally very cumbersome, so it requires a lot of computational efforts, especially if the model presents some stochasticity in its behavior. The second one is that MPC strategy requires the knowledge of the future demand within a certain large enough finite control horizon. However, it is known that while dealing with demand planning the future is difficult to predict, even to the next time step. Thus, it may occur that the forecast presents certain inaccuracy which may result in production losses or backlog.

In this chapter we tackle the problem of performance analysis (see, e.g., Ruifeng and Subramaniam (2011)) for a surplus-based approach in control of manufacturing networks within the scope of demand driven manufacturing control problems. In the surplus approaches, control decisions are made based on the demand tracking error, which is the difference between the cumulative demand and the cumulative output of the system (for more details see Chapter 1). Some references for these strategies are presented later in this section. Each machine in the manufacturing system coordinates its individual production with those of the rest of the system. Its primary objective may be viewed as manufacturing of a sufficient quantity of parts to satisfy the demand of its immediate downstream machine and some desired amount for the purpose of back-up material storage in its downstream buffer. The proposed methodology is reformulated in terms of variable structure control. The production flow process is described by means of difference equations and in order to analyze performance, approach using Lyapunov theory is exploited (see, e.g., Khalil (2002) and references there in).

The novelty of our results, concerning the surplus-based approach (see, e.g. Bielecki and Kumar (1988); Bonvik et al. (1997); Lefebvre (1999); Gershwin (2000);

Quintana (2002); Kogan and Perkins (2003); Boukas (2006); Stockton et al. (2007); Subramaniam et al. (2009); Savkin and Evans (2002); Nilakantan (2010) and references there in), can be summarized as follows. The proposed tandem production model is considered in discrete time. The production speed of each machine is defined as deterministic with bounded perturbations. The future production demand is assumed to be unknown and with bounded fluctuations. As a result, strict, so called "worst case" bounds on the content of intermediate buffers and demand tracking errors for a unidirectional tandem production line are obtained.

In particular, this chapter contains an explicit description of a flow model of a manufacturing line (see, e.g. Dallery and Gershwin (1992); Alvarez-Vargas et al. (1994); Pogromsky et al. (2009) and references therein). Here, surplus-based control[1] is introduced as a manufacturing control technique in order to give a solution to the demand tracking problem. Special attention is paid to the constraints presented in the network, such as machine capacity and buffer limitations. Each machine in the network can produce a restricted number of products in a fixed period of time, known as the capacity constraint. The content of the buffer between two machines is given by the difference between the total number of products produced by the upstream machine and the total number of products produced by the downstream machine. Considering that a manufacturing line has a unidirectional product flow implies that the buffer content can never be negative, e.g. the downstream machine cannot produce more than the upstream one.

The chapter is organized as follows. First, in Section 3.2 the flow model of a manufacturing line with unlimited inventory operated under the surplus-based control is presented. The analysis of the demand tracking error of each machine with respect to the inventory level of the network is developed. Then, in Section 3.3, the flow model of a manufacturing line with limited inventory operated under the surplus-based control is presented. Similarly to Section 3.2, the analysis of the demand tracking error of each machine with respect to the inventory level of the network is presented. In Section 3.4, performance issues of the closed-loop flow models are illustrated in numerical simulations. Section 3.5 presents a comparative study of discrete time and discrete event models of a manufacturing line. In Section 3.6 simulation-based performance analysis is conducted for a manufacturing line of 4 machines and 3 buffers operated under three surplus-based production policies. The performance is evaluated in terms of the steady state demand tracking errors and inventory levels of the network. Finally, Section 3.7 contains the conclusions of the chapter.

3.2 A line of machines with unlimited buffers

This section presents the discrete time-based flow model and the results on performance of a manufacturing line with unlimited capacity intermediate buffers. The manufacturing line in question is operated under an extended version of the surplus-based control policy introduced in Chapter 2.

[1] In control theory this type of control is also know as a variable structure tracking control (see, e.g. Utkin (1983); Khalil (2002))

3.2.1 Flow model

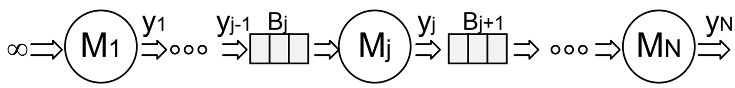


Figure 3.1: Schematics of a line of N manufacturing machines.

The flow model of a manufacturing line with unlimited intermediate inventories is presented in this section. Figure 3.1 presents the schematics of a line of N manufacturing machines with machines M_j, buffers B_j, and infinite product supply. Here the control strategy for one machine, introduced in the pervious chapter, is modified with respect to the number of buffers and machines present in the line. A new limitation such as desired buffer content and minimal buffer level are considered in the model.

The flow model of the manufacturing line[2] is defined by the following equations

$$\begin{aligned} y_1(k+1) &= y_1(k) + \beta_1(k)u_1(k), && (3.1)\\ y_j(k+1) &= y_j(k) + \beta_j(k)u_j(k)\mathrm{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k)), && (3.2)\\ &\quad \forall j = 2, \ldots, N, \end{aligned}$$

where all the events within the model occur at given time instances and k represents the current time so that the time step between all the events is constant. Here $y_j(k)$ is the cumulative output of machine M_j in time k, $w_j(k) = y_{j-1}(k) - y_j(k)$ is the buffer content of buffer B_j, $\beta_j(k) = \mu_j + f_j(k)$, $\forall j = 1, \ldots, N$, μ_j is the processing speed of machine j, f_j is the external disturbance affecting machine M_j (e.g. production speed variations, delay), u_j is the control input of machine M_j and $\mathrm{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k)) = 1$, if $w_j(k) - \beta_j(k) \geq 0$ and 0, otherwise. This last function describes the minimal buffer level that is needed in order for machine j to start its production, which will be discussed in detail further in this section. The equation (3.1) and (3.2) present a general model that can describe a product flow for a wide range of manufacturing lines. Further specific assumptions on system (3.1) and (3.2) are given in this section.

In order to give a solution to the production demand tracking problem introduced in Chapter 2 of this thesis we consider the following control inputs:

$$\begin{aligned} u_j(k) &= \mathrm{sign}_+(\varepsilon_{j+1}(k) + (w_{d_{j+1}} - w_{j+1}(k))), && (3.3)\\ &\quad \forall j = 1, \ldots, N-1,\\ u_N(k) &= \mathrm{sign}_+(y_d(k) - y_N(k)), && (3.4) \end{aligned}$$

where $w_{d_{j+1}}$ is the desired buffer level (base stock) of buffer B_{j+1}, ε_{j+1} is the demand tracking error of machine M_{j+1} and $y_d(k)$ is the cumulative production

[2]For details on discrete time flow model of manufacturing machine see Chapter 2, Section 2.2.

demand given by (2.2). The demand tracking error of each machine is given by:

$$\begin{aligned} \varepsilon_j(k) &= \varepsilon_{j+1}(k) + (w_{d_{j+1}} - w_{j+1}(k)), \qquad (3.5)\\ &\forall j = 1, \ldots, N-2, \\ \varepsilon_{N-1}(k) &= \varepsilon_N(k) + (w_{d_N} - w_N(k)), \qquad (3.6)\\ \varepsilon_N(k) &= y_d(k) - y_N(k). \qquad (3.7) \end{aligned}$$

It follows from (3.7) that the demand tracking error of machine M_N is defined exactly as for the single machine case. The buffer restriction, as seen from (3.2), is the only difference in the flow model of machine M_N with the flow model of (2.1). For (3.5), (3.6) new considerations are applied for the demand tracking error of each machine M_j, where $j = 1, ..., N-1$. Thus, the demand tracking error $\varepsilon_j(k)$ depends on the number of produced products $y_j(k)$ with respect to current demand $y_d(k)$ and desired buffer content $w_{d_{j+1}}$ of each downstream buffer. This means that every upstream machine needs to supply $w_{d_{j+1}}$ lots more than the downstream one. The constant parameter w_d is introduced in order to prevent downstream machines from lot starvation, e.g. in case of a sudden growth of the product demand.

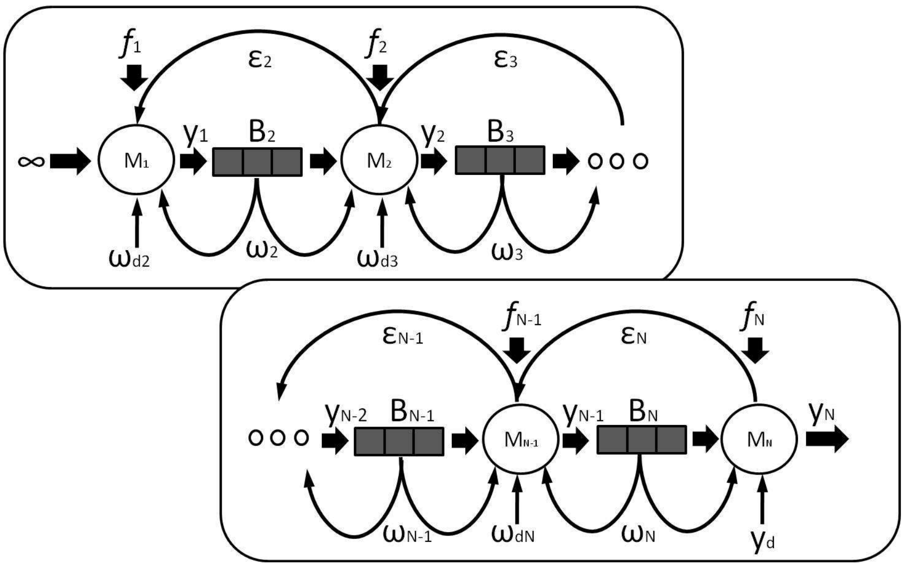


Figure 3.2: Flow model diagram for a line of N manufacturing machines.

Basically, model (3.1), (3.2) describes the product flow through the line of N manufacturing machines (see Figure 3.2). The first machine described by (3.1) is considered to have always access to raw material and there is always sufficient raw material. The administration of this raw material to machine M_1 is decided by the control input (3.3). Here we consider that our control input is acting as an authorizing switch, which turns on M_1 if its demand tracking error (3.5) is positive and turns M_1 off if its demand tracking error is negative or zero. Tracking error $\varepsilon_1(k)$, see (3.5), consists of the difference between what has been done ($w_2(k)$), what has to be done ($\varepsilon_2(k)$) and what has to be always in the buffer (w_{d_2}). It can be seen from (3.5), (3.6) and (3.3) that the same demand tracking error and control logics were applied to the rest of the machines till machine M_{N-1}. As for the last machine in the line, which is machine M_N (Figure 3.2), the control action is still based on the authorizing switch. However, the difference consists in the logic that triggers this switch. We expect that on the output of machine M_N the cumulative product demand is followed by the cumulative production of this machine. The control switch activates or deactivates the machine based directly on the production demand status (3.7). This control logic is the same as in the one machine case presented in the previous section. The difference in models for the rest of the machines from machine one can be seen through the general flow model (3.2). Here, for each machine in the line we introduce an extra restriction on the buffer content of each upstream buffer. Function $\text{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k))$ is acting as an extra authorization together with the control input. Thus, any machine M_j, with $j = 2, \ldots, N$, is activated only if two authorizations are given. The first authorization comes from the control input $(u_j(k))$ of the machine, which is based on the current demand tracking error status of this machine $(\varepsilon_j(k))$. The second authorization comes from the buffer content restriction which is granted if the buffer contains at least the minimal number of products required $(\beta_j(k))$ in order for the machine M_j to start its work. By $\text{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k))$ we imply that in order to start producing machine j is restricted to take a certain nonzero amount of products from its upstream buffer. This amount is defined by the processing speed of the machine from time k till time $k+1$, which for this authorizing action is assumed to be known in advance. In reality the interpretation of function $\text{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k))$ may vary depending on the physical system and its limitations. For example the restrictive action, given by this function, may already be implicit in a system, i.e. a system can be stopped if no product is detected on its input, or a system can be stopped unless some fixed quantity of material, e.g., a batch of products, is present on its input. In its essence the function $\text{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k))$ is imposed so to prevent the model of the network from describing active manufacturing processes while having insufficient product content in its buffers in order to initiate these processes. Thus, though the interpretation of function $\text{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k))$ may vary, the essence of its restrictive action is preserved for any buffered manufacturing network. It is also important to take into account that the control actions are decentralized throughout the network. In other words, the control action of each machine in the line depends only on the demand tracking error of its neighboring downstream machine (except for machine M_N, which depends directly on cumulative demand input) and the current buffer content of its upstream buffer (see Figure 3.2). This gives our flow model an extra robustness with respect to undesired events such as temporal machine setup or breakdown.

For further analysis, let us rewrite flow model (3.1), (3.2) in a closed-loop with (3.3), (3.4) as

$$\Delta\varepsilon_1(k) = v_d + \Delta\varphi(k) - \beta_1(k)\text{sign}_+(\varepsilon_1(k)), \quad (3.8)$$

$$\Delta\varepsilon_j(k) = v_d + \Delta\varphi(k) - \beta_j(k)\text{sign}_+(\varepsilon_j(k))\text{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k)), \quad (3.9)$$

where $\Delta\varepsilon_j(k) = \varepsilon_j(k+1) - \varepsilon_j(k)$, $\forall j = 1, \ldots, N$.

Here we assume that system (3.8), (3.9) satisfies the following assumptions.

Assumption 1. *(Boundedness of perturbations) There are constants* α_1, α_2 *and*

α_3 such that $W_j(k) = \Delta\varphi(k) - f_j(k)$ satisfies

$$\alpha_1 < W_j(k) < \alpha_2,\ \forall k \in \mathbb{N},\ j = 1, \ldots, N, \tag{3.10}$$

and $f_j(k)$ satisfies

$$f_j(k) \leq \alpha_3,\ \forall k \in \mathbb{N},\ j = 1, \ldots, N. \tag{3.11}$$

Assumption 2. *(Capacity condition) Constants α_1, α_2 satisfy the following inequalities*[3]

$$\alpha_2 \ < \ \mu_j - v_d,\ \forall j = 1, \ldots, N, \tag{3.12}$$

$$\alpha_1 \ > \ -v_d. \tag{3.13}$$

Thus, from (3.10), (3.12), and (3.13) the following condition holds

$$0 < v_d + W_j(k) < \mu_j,\ \forall j = 1, \ldots, N. \tag{3.14}$$

Note that the physical meaning of the above mentioned assumptions is similar to the one discussed in detail in Section 2.3 of Chapter 2.

It is important to notice that each M_j machine in the line has a processing speed of μ_j lots per time unit, which can differ from the rest of the machines, and the buffer condition is considered as

$$w_j(k) \geq \beta_j(k),\ \forall j = 2, \ldots, N. \tag{3.15}$$

Thus, from (3.5), (3.6) and (3.15) the following demand tracking error condition holds

$$\varepsilon_j(k) \geq \beta_j(k) - w_{d_j} + \varepsilon_{j-1}(k),\ \forall j = 2, \ldots, N,$$

where w_{d_j} satisfy the following assumption.

Assumption 3. *(Desired buffer content condition) The constants w_{d_j} comply with the following inequality*

$$w_{d_j} \geq \mu_j + \mu_{j-1} + \alpha_3 + \alpha_2 - \alpha_1,$$

from where it follows that

$$w_{d_j} \geq \beta_j(k) + \mu_{j-1} + \alpha_2 - \alpha_1,\ k \in \mathbb{N}, j = 2, \ldots, N. \tag{3.16}$$

If condition (3.15) is not satisfied, then

$$\varepsilon_{j-1}(k) \overset{(3.5,3.6,3.16)}{>} \mu_{j-1} + \alpha_2 - \alpha_1 + \varepsilon_j(k),\ \forall j = 2, \ldots, N. \tag{3.17}$$

[3] Note that in practice it can be rather unnatural to set bounds on market fluctuations together with a machine perturbations. Thus starting from Chapter 4 of this book, where the results on multi-product manufacturing line are discussed, we evaluate system perturbations independently from the demand rate fluctuations. The result of Theorem 3 of this section can be similarly deduced from Theorem 7 of Chapter 4, which are obtained for multi-product manufacturing line. This can be done if in Theorem 7 the production stages of a multi-product line are reduced to one production stage per machine.

3.2.2 Performance analysis

In this section we present results respecting the demand tracking error trajectories behavior of the flow model (3.8), (3.9).

Theorem 3. *Assume that the discrete time system defined by (3.8), (3.9) satisfies Assumptions 1, 2, and 3. Then all solutions of (3.8) and (3.9) are uniformly ultimately bounded by*

$$\limsup_{k\to\infty} \varepsilon_j(k) \le v_d + \alpha_2, \tag{3.18}$$
$$\liminf_{k\to\infty} \varepsilon_j(k) \ge v_d + \alpha_1 - \mu_j. \tag{3.19}$$

Proof. The proof of Theorem 3 is provided in Appendix C. □

As a consequence, for the buffer content $w_j(k)$ of each buffer B_j defined by (3.5), (3.6), considering the obtained demand tracking error bounds (3.18), (3.19) and Assumption 3, it holds that

$$\limsup_{k\to\infty} w_j(k) \quad \le \quad \mu_{j-1} + \alpha_2 - \alpha_1 + w_{d_j}.$$

Now, in order to support the present development let us extend our analysis of a manufacturing line to the limited buffers case.

3.3 A line of machines with limited buffers

This section presents the discrete time-based flow model and the results on performance of a manufacturing line with limited capacity of the intermediate buffers. The manufacturing line in question is operated under an extended version of the surplus-based control policy introduced in Chapter 2.

3.3.1 Flow Model

The flow model of the manufacturing line shown in Figure 3.1 with limited intermediate inventory is defined as

$$\begin{aligned} y_1(k+1) &= y_1(k) + \beta_1(k)u_1(k)\mathrm{sign}_-(w_2(k) - \gamma_2), && (3.20)\\ y_j(k+1) &= y_j(k) + \beta_j(k)u_j(k)\mathrm{sign}_{\mathit{Buff}}(w_j(k) - \beta_j(k)) && (3.21)\\ &\times\ \mathrm{sign}_-(w_{j+1}(k) - \gamma_{j+1}),\ j = 2, \ldots, N-1, \\ y_N(k+1) &= y_N(k) + \beta_N(k)u_N(k)\mathrm{sign}_{\mathit{Buff}}(w_N(k) - \beta_N(k)), && (3.22) \end{aligned}$$

where $y_j(k)$ is the cumulative output of machine M_j in time k, $w_j(k) = y_{j-1}(k) - y_j(k)$ is the content of buffer B_j, $\beta_j(k) = \mu_j + f_j(k)$, $\forall j = 1, \ldots, N$, μ_j is the constant processing speed of machine j, f_j is the external disturbance affecting machine M_j (e.g., production speed variations, delay or setup time), u_j is the control input of machine M_j, $\text{sign}_{Buff}(x) = (1, \text{ if } x \geq 0 \,|\, 0, \text{ otherwise})$, $\text{sign}_{-}(x) = (1, \text{ if } x \leq 0 \,|\, 0, \text{ otherwise})$ and γ_{j+1} is the threshold value of the buffer content w_{j+1}. The equation (3.20), (3.22) and (3.21) present a general model that can describe a product flow for a wide range of manufacturing lines. Further specific assumptions on system (3.20), (3.22) and (3.21) are given in this section.

In order to give a solution to the demand tracking problem we use the control inputs given by (3.3), (3.4), where the demand tracking errors are given by (3.5), (3.6) and (3.7).

Thus for each machine we introduce an extra restriction on production that is based on the buffer content of its upstream and downstream buffer. The functions $\text{sign}_{Buff}(w_j(k) - \beta_j(k))$ and $\text{sign}_{-}(w_{j+1}(k) - \gamma_{j+1})$ together with the control input $u_j(k)$ are acting as restrictions that are imposed on the production of M_j. Thus, any machine M_j, with $j = 2, \ldots, N-1$, is activated only if three authorizations are given. The first authorization comes from control input of M_j, which is based on the current value of the demand tracking error of this machine ($\varepsilon_j(k)$). The second authorization comes from the restriction on the upstream buffer content that is granted if the buffer contains at least the minimal number of products required ($\beta_j(k)$) in order for the machine M_j to start its work. Finally, the third authorization comes from the downstream buffer of given machine. This authorization is possible only if the downstream buffer have sufficient storage in order to accept incoming production.
Note that we could easily associate this control algorithm with the Basestock policy (see Bonvik et al. (1997) and references therein) as well as with the Hedging Point policy (see Gershwin (2000) and references therein). From (3.5) the demand tracking error of each intermediate machine can be interpreted as $\varepsilon_j(k) = y_d(k) - y_j(k) + w_{d_{j+1}} + \ldots + w_{d_N}$, $j = 1, \ldots, N-2$.

This means that each machine is keeping track of the current demand as well as of its Hedging point or its Basestock level, which in this case is the sum of the desired buffer contents of all the downstream buffers of M_j. Also due to our intermediate buffer content limitation (γ_j) this policy could be associated to a Kanban or a combination of local Conwip controllers (one for each intermediate machine) and a surplus-based pull control (for the output machine M_N).
It is also important to take into account that the control actions are decentralized throughout the network. In other words, the control action of each machine in the line depends only on the demand tracking error of its neighboring downstream machine (except for machine M_N, which control action depends directly on cumulative demand input) and the current buffer content of its upstream and downstream buffer (see Figure 3.2). This gives our flow model extra robustness with respect to undesired events such as temporal machine setup or breakdown.
For further analysis, let us rewrite flow model (3.20), (3.21), and (3.22) in feedback

with (3.3), (3.4) as

$$\begin{aligned}
\Delta\varepsilon_1(k) &= v_d + \Delta\varphi(k) && (3.23)\\
&- \beta_1(k)\mathrm{sign}_+(\varepsilon_1(k))\mathrm{sign}_-(w_2(k)-\gamma_2), \\
\Delta\varepsilon_j(k) &= v_d + \Delta\varphi(k) - \beta_j(k)\mathrm{sign}_+(\varepsilon_j(k)) && (3.24)\\
&\times\ \mathrm{sign}_{Buff}(w_j(k)-\beta_j(k))\mathrm{sign}_-(w_{j+1}(k)-\gamma_{j+1}), \\
\Delta\varepsilon_N(k) &= v_d + \Delta\varphi(k) - \beta_N(k)\mathrm{sign}_+(\varepsilon_N(k)) && (3.25)\\
&\times\ \mathrm{sign}_{Buff}(w_N(k)-\beta_N(k)),
\end{aligned}$$

where $\Delta\varepsilon_j(k) = \varepsilon_j(k+1) - \varepsilon_j(k)$.
Assumptions 1,2 and 3 also hold for system (3.23), (3.24), (3.25).

It is important to notice that each machine M_j in the line has a processing speed of μ_j lots per time unit, which can differ from the rest of the machines.
The buffer content condition is considered as

$$\beta_j(k) \leq w_j(k) < \gamma_j,\ \forall j = 2, \ldots, N. \tag{3.26}$$

Note that the physical restriction on buffer content is given as

$$0 \leq w_j(k) \leq \gamma_j + \mu_{j-1} + \alpha_3,\ \forall j = 2, \ldots, N. \tag{3.27}$$

Here, $\gamma_j = \mu_j + \alpha_2 - \alpha_1 + w_{d_j}$. Thus from (3.5), (3.6) and (3.26) the following demand tracking error condition holds

$$\varepsilon_j(k) \geq \beta_j(k) - w_{d_j} + \varepsilon_{j-1}(k),\ \forall j = 2, \ldots, N,$$

where w_{d_j} satisfies Assumption 3.

One can note the following relation between the buffer content and the demand tracking errors of its neighboring machines. That is, if the first part of inequality (3.26) is not satisfied, i.e., the buffer content is lower than the minimum, then

$$\varepsilon_{j-1}(k) \overset{(3.5,3.6,3.16)}{>} \mu_{j-1} + \alpha_2 - \alpha_1 + \varepsilon_j(k). \tag{3.28}$$

In case the second part of (3.26) is unsatisfied, i.e., the buffer is full, then

$$\varepsilon_j(k) \overset{(3.5,3.6)}{\geq} \mu_j + \alpha_2 - \alpha_1 + \varepsilon_{j-1}(k). \tag{3.29}$$

3.3.2 Performance Analysis

In this section we present the obtained results on the demand tracking error trajectories behavior of flow model (3.23), (3.24), (3.25).

Theorem 4. *Assume that the discrete time system defined by (3.23), (3.24), (3.25) satisfies Assumptions 1, 2, and 3. Then all solutions of (3.23), (3.24), (3.25) are uniformly ultimately bounded by*

$$\limsup_{k\to\infty} \varepsilon_j(k) \leq v_d + \alpha_2, \tag{3.30}$$
$$\liminf_{k\to\infty} \varepsilon_j(k) \geq v_d + \alpha_1 - \mu_j. \tag{3.31}$$

Proof. The proof of Theorem 4 is given in Appendix D. □

Theorem 4 states that, given that the assumptions 1, 2, 3 are satisfied, then the steady state demand tracking errors will remain within the same tracking bounds as the ones specified in Theorem 3, regardless initial conditions and capacity limitations of the line. Logically, starvation in a manufacturing line can be avoided by selecting a sufficiently large amount of a base stock (w_{dj}). What happens if this base stock is low or even zero? The answer to this question is presented in the following theorem.

Theorem 5. *Assume that the discrete time system defined by (3.23), (3.24), (3.25) satisfies Assumptions 1 and 2. Then all solutions of (3.23), (3.24), (3.25) are uniformly ultimately bounded by*

$$\limsup_{k\to\infty} \varepsilon_j(k) \leq v_d + \alpha_2 + x_j, \tag{3.32}$$
$$\liminf_{k\to\infty} \varepsilon_j(k) \geq v_d + \alpha_1 - \mu_j, \tag{3.33}$$

where $x_1 = 0$ *and* $x_j = \sum_{i=2}^{j} \max\left(\mu_{i-1} - \alpha_1 + \alpha_2 - w_{di} + \mu_i + \alpha_3, 0\right)$ *for all* $j = 2, \ldots, N$.

Proof. The proof of Theorem 5 is given in Appendix E. □

The inequalities (3.32) and (3.33) of Theorem 5 present a full behavioral overview of the steady state demand tracking error trajectories. A strong influence of the intermediate safe stock level w_{dj} on a tracking accuracy of each machine in a line can be observed from (3.32). For low w_{dj} (less than specified in (3.16)) the tracking inaccuracy of the downstream machine is directly influenced by the difference in the selected inventory level and its threshold value specified by (3.16). The further the machine is located from M_1, the bigger its tracking inaccuracy is. At the same time, if w_{dj} satisfies Assumption 3 for all j, then inequality (3.32) takes the form of (3.30), where the tracking inaccuracy purely depends on the demand rate and production speed perturbations. It can be observed that in all the presented results on performance (see Theorems 2, 3, 4 and 5) the worst case scenario bound on a positive surplus value at each machine (see, e.g., (3.33)) is kept unaltered.

From (3.23), (3.24), (3.25) it can be seen that the worst case bound on the positive surplus value[4] is given by the difference between the lowest possible demand growth and the highest possible production speed, i.e. $v_d + \alpha_1 - \mu_j$. This value represents the worst case bound of overproduction for each machine in the line operated under a surplus-based control in a period of one time step. If this value of overproduction is reached in the machine, then it will be stopped by its controller. Thus, along the time line, the overproduction value can never grow further from its value that is reached in one time step.

Note that the result of Theorem 5 also applies to the system defined by (3.8), (3.9).

Now, in order to support the proposed development, let us supplement our analysis by a simulation examples.

3.4 Discrete time simulations

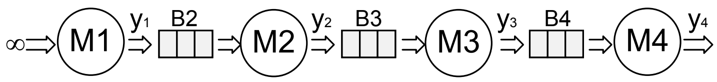


Figure 3.3: Schematics of a line of 4 manufacturing machines.

Simulation results for a line of 4 manufacturing machines M_j with 3 buffers B_j (see Figure 3.3), driven by the surplus-based regulators (3.3) and (3.4), are presented in this section[5]. In Sections 3.4.1 and 3.4.2 two simulation examples are introduced in order to verify the effectiveness of the analytical results from Sections 3.2.2 and 3.3.2, respectively. The examples of Section 3.4.1 are simulated using the discrete time (DT) model of a line with unbounded buffers, which is presented in Section 3.2.1. For the examples of Section 3.4.2 the DT model of a line is considered with limited buffer content as introduced in Section 3.3.1.

3.4.1 Unlimited buffers

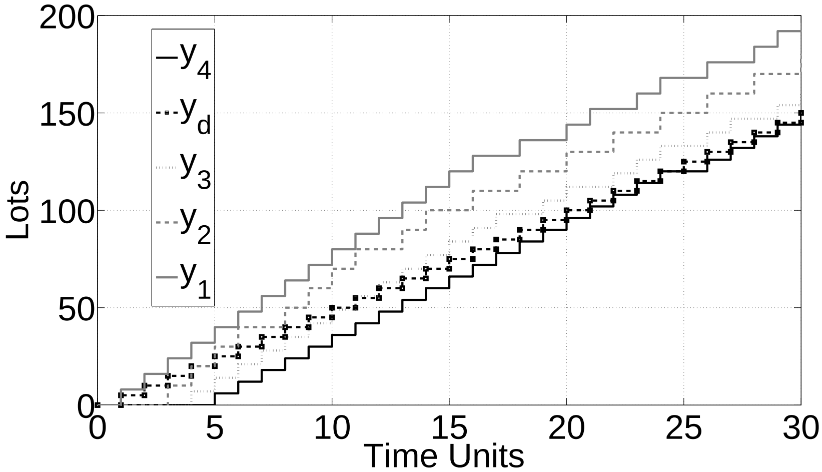
Simulation results for a line of 4 manufacturing machines (see Figure 3.3), driven by surplus-based controllers (3.3), and (3.4) are presented in this section. For all the examples, the processing speed of each machine is set to $\mu_j = (8, 10, 7, 6)$ lots

[4]Note that in this thesis the demand tracking error of a machine is defined as the difference between the cumulative values of products needed and products that are done. Thus the term *positive surplus* value refers to the negative demand tracking error value and the term *negative surplus* value refers to the positive demand tracking error value.

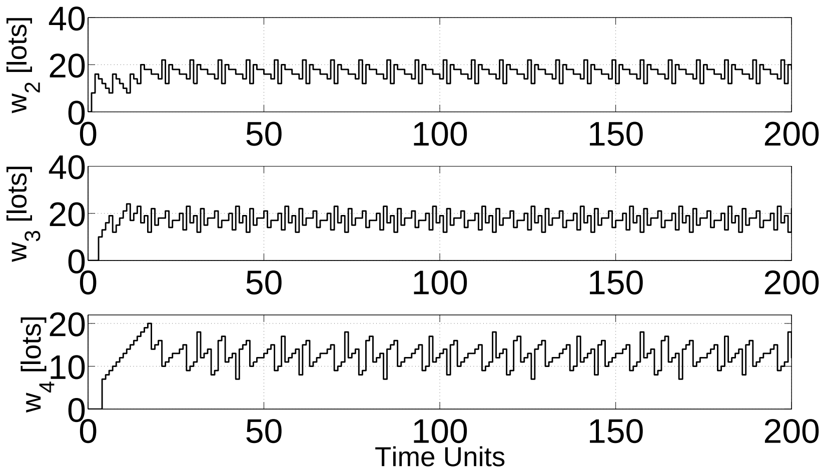
[5]In some simulation examples of this book in order to verify the effectiveness of the analytical bounds on demand tracking error accuracy we introduce periodic functions to model perturbations and market fluctuations in a manufacturing network. Note that these bounds are obtained for the worst case behavior of a product flow in a network. Thus, as long as all the assumptions of the flow model of a manufacturing network are satisfied, no relevant difference with respect to the accuracy of the analytical bounds can be expected in case an alternative function or a stochastic model is used to describe perturbations and fluctuations in the network.

per time unit, with $j = 1, .., 4$, and the desired buffer content of each buffer is selected considering (3.16) as $w_{d_j} = (20, 18, 14)$ lots, with $j = 2, .., 4$.

Example 1: The demand tracking error of each machine in the line is depicted in Figure 3.5. Here the initial conditions (y_{d0}, $y_1(0)$, $y_2(0)$, $y_3(0)$, $y_4(0)$) are set to the zero value. After the first 24 time steps, as it is shown in Figures 3.4(a) and 3.5, the system reaches its steady state. Demand tracking errors are maintained inside [-2,5] lots for machine M_1, [-4,1] lots for machine M_2, [-1,5] lots for machine M_3, and [0,5] lots for machine M_4, which satisfy (3.18) and (3.19). From Figure 3.4(b) it can be observed that the inventory level of each buffer satisfies the upper bound restriction (3.20).



(a)



(b)

Figure 3.4: Outputs $y_j(k)$ vs. Demand $yd(k)$ (a) and Buffer Content $w_j(k)$ (b), with $v_d = 5$.

Example 2: The output response and the demand tracking error of each machine are depicted in Figures 3.6 and 3.7(a) for the same initial conditions as in the previous example. Here it is noticeable that after the first 7 time steps the output

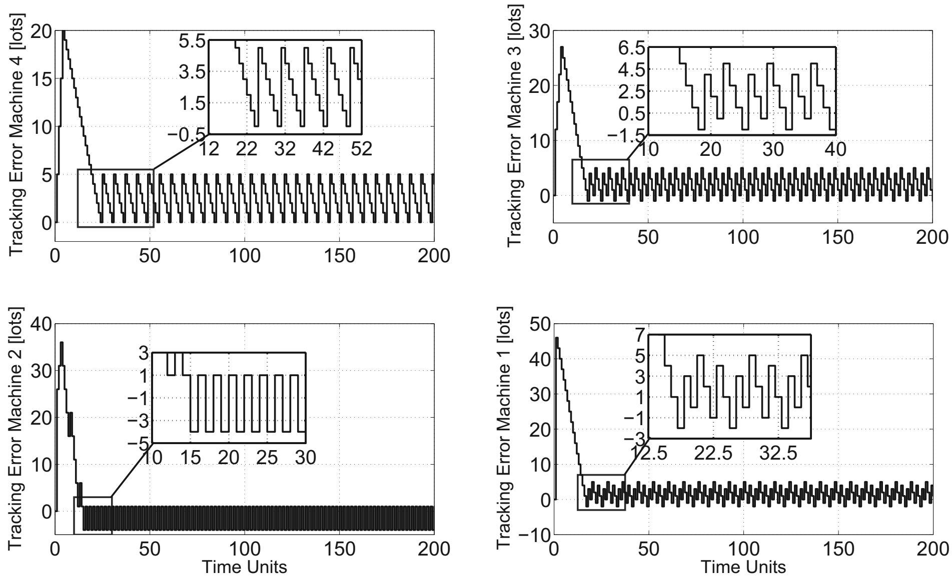


Figure 3.5: Demand Tracking Error $\varepsilon_j(k)$, with $v_d = 5$.

of machine M_4 reaches the current demand trajectory. Demand tracking errors are maintained inside [-6,4] lots for machine M_1, [-8,4] lots for machine M_2, [-5,4] lots for machine M_3, and [-4,4] lots for machine M_4, which are identical to the analytical bounds of (3.18), (3.19). The inventory level of each buffer is depicted in Figure 3.7(b). We indicate that the buffer content of each buffer satisfies the upper bound restriction (3.20), if given desired inventory level of each buffer satisfies (3.16).

Finally, the presented simulation results on the selected examples reflect the desired flow model behavior. All technical conditions proposed in this section correspond to the analytical results described in Section 3.2.

3.4.2 Limited buffers

Example 1: Consider the following example of a production line of 4 manufacturing machines (see Figure 3.3) operating under surplus-based regulators (3.3) and (3.4). The processing speed for each machine is set to $\mu_j = 6$ lots per time unit $\forall j = 1, 3$ and $\mu_j = 4$ lots per time unit $\forall j = 2, 4$, the desired inventory level of each buffer is selected considering (3.16) as $w_{d_j} = 12$ (lots), with $j = 2, .., 4$ and the mean demand rate $v_d = 3.5$ lots per time unit with fluctuation rate of $\Delta\varphi(k) = 0.2 \sin(5k)$. The demand tracking error of each machine in the line is depicted in Figure 3.5. Here the initial conditions $(y_{d0}, y_1(0), y_2(0), y_3(0), y_4(0))$ are set to the zero value. After the first 40 time steps, as it is shown in Figures 3.8 and 3.10, the system reaches its steady state. Demand tracking errors (see the dashed lines of Figure 3.10) are

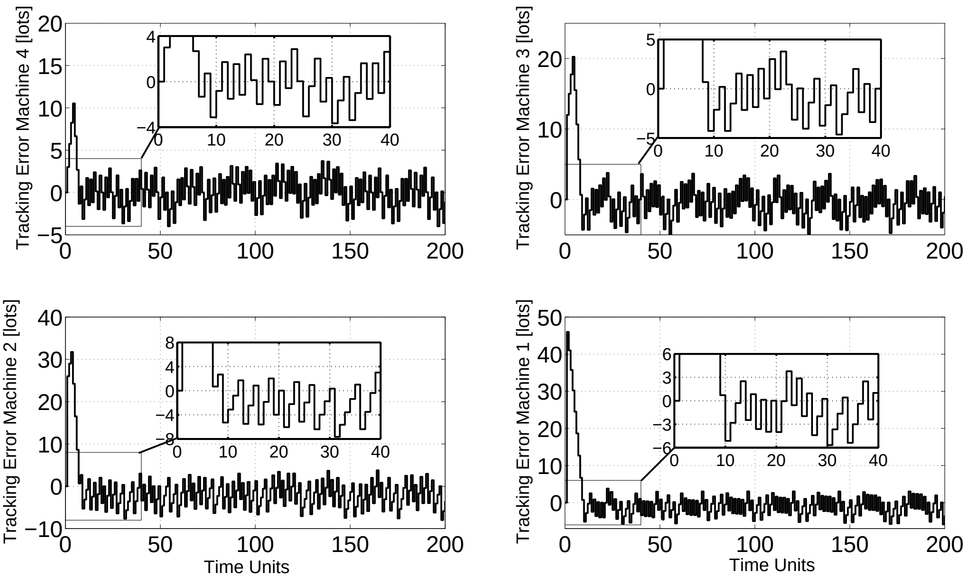
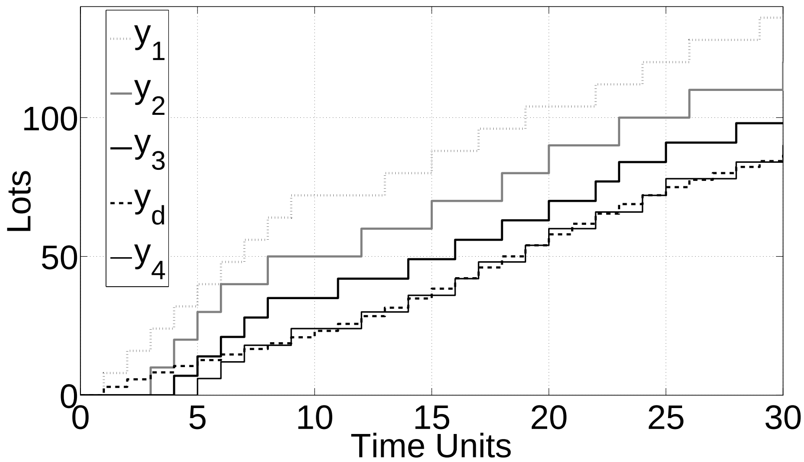


Figure 3.6: Demand Tracking Error $\varepsilon_4(k)$, with $v_d = 3$ and $\Delta\varphi(k) = \sin(50k)$.

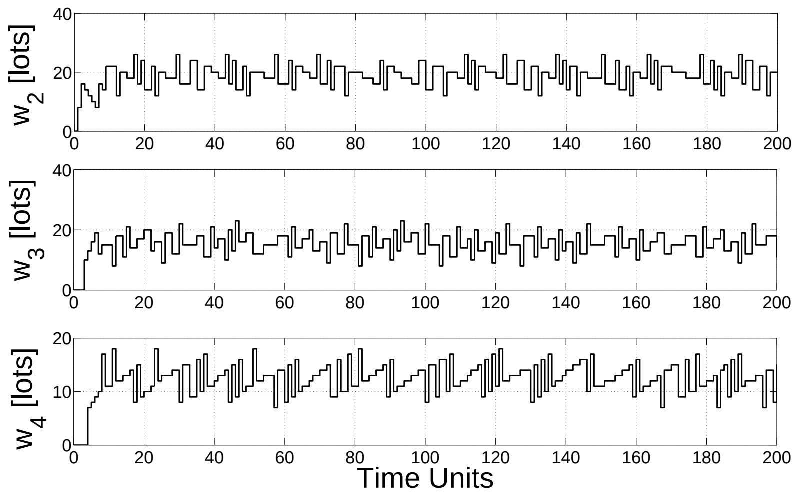
maintained inside [-2.7,3.7] lots for machine M_1, [-0.7,3.7] lots for machine M_2, [-2.7,3.7] lots for machine M_3, and [-0.7,3.7] lots for machine M_4, which satisfy the bounds (3.30) and (3.31). From Figure 3.9 it can be observed that the inventory level of each buffer satisfies the buffer limit given by the second part of inequality (3.27) and the capacity condition (3.26) is sometimes violated due to the discrete nature of the model. Here $\gamma_2 = 14.8$ lots, $\gamma_3 = 16.8$ lots, $\gamma_4 = 14.8$ lots.

Example 2: Now let us show the effectiveness of Theorem 5 by means of the following example. Consider a similar production line of 4 manufacturing machines (see Figure 3.3) operating under surplus-based regulators (3.3) and (3.4). The nominal speed for each machine is $\mu_1 + f_1(k) = 10.56 + 0.5\sin(180k)$, $\mu_2 + f_2(k) = 10.7 + 0.5\sin(20k)$, $\mu_3 + f_3(k) = 15.5 + 0.5\sin(45k)$ and $\mu_4 + f_4(k) = 20.5 + 0.5\sin(90k)$ lots per time unit. The desired inventory level of each buffer is selected as $w_{d_j} = a\mu_j$ lots, with $j = 2, .., 4$ and a is a constant. The experiment is executed 37 times. Thus the value of constant a is modified 37 times as well, starting from $a = 1$ and with increments of 0.25 units till $a = 10$. The demand rate is selected as $v_d + \Delta\varphi(k) = 10 + 0.05\cos(45k)$ lots per time unit. Here each γ_j is selected according to (3.26). For this experiment the initial conditions (y_{d0}, $y_1(0)$, $y_2(0)$, $y_3(0)$, $y_4(0)$) are set to the zero value.

The relation between the maximal value of steady state demand tracking errors of machines M_2, M_3, M_4 (see Figure 3.3) and the desired inventory levels of there upstream buffers is depicted in Figure 3.11. Each graphic of Figure 3.11 shows the maximal value of each ε_j in steady state with its values from simulation and from the analytical result given by (3.32). It can be observed that our analytical result describes in a rather accurate manner the tradeoff between the desired inventory level and the accuracy of the production demand tracking. It is important to notice that the precision of the demand tracking error is limited. Thus for each of 3 the



(a)



(b)

Figure 3.7: Outputs $y_j(k)$ vs. Demand $yd(k)$ (a) and Buffer Content $w_j(k)$ (b), with $v_d = 3$ and $\Delta\varphi(k) = \sin(50k)$.

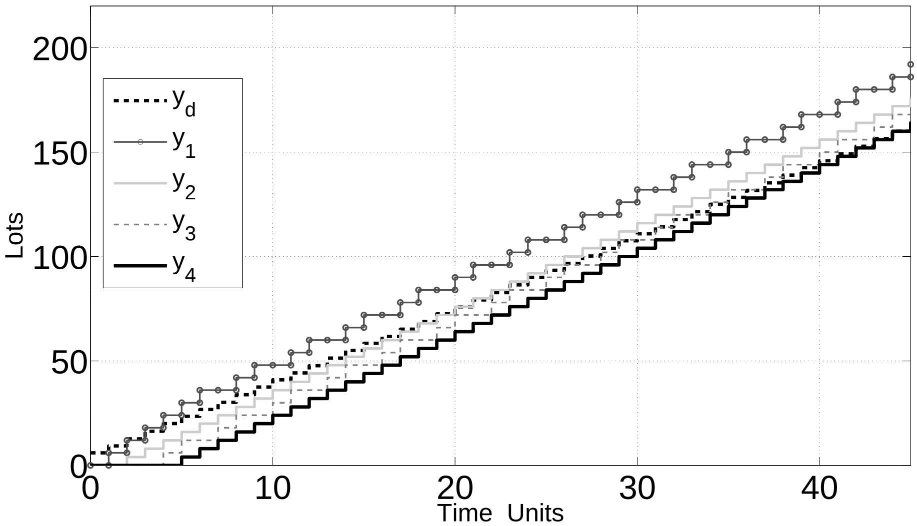


Figure 3.8: Outputs $y_j(k)$ vs. Demand $yd(k)$.

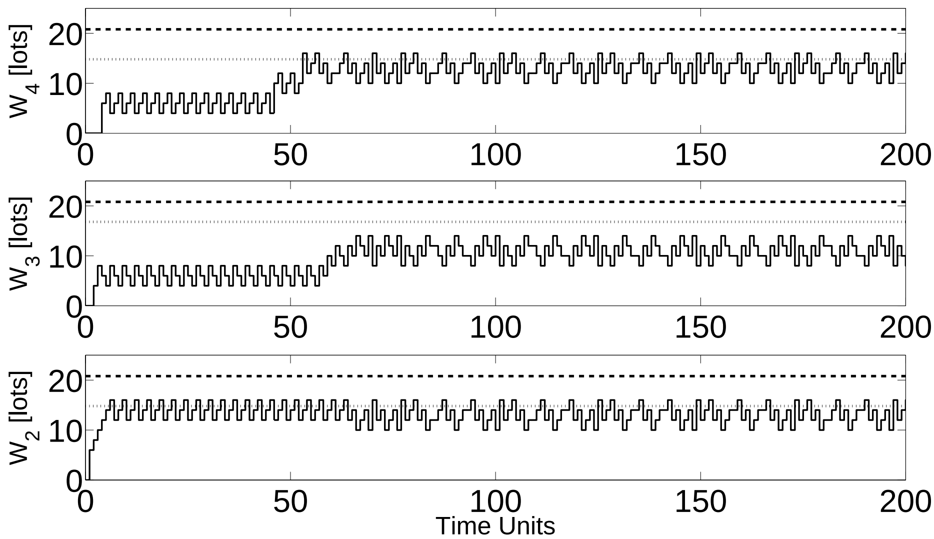


Figure 3.9: Buffer Content $w_j(k)$.

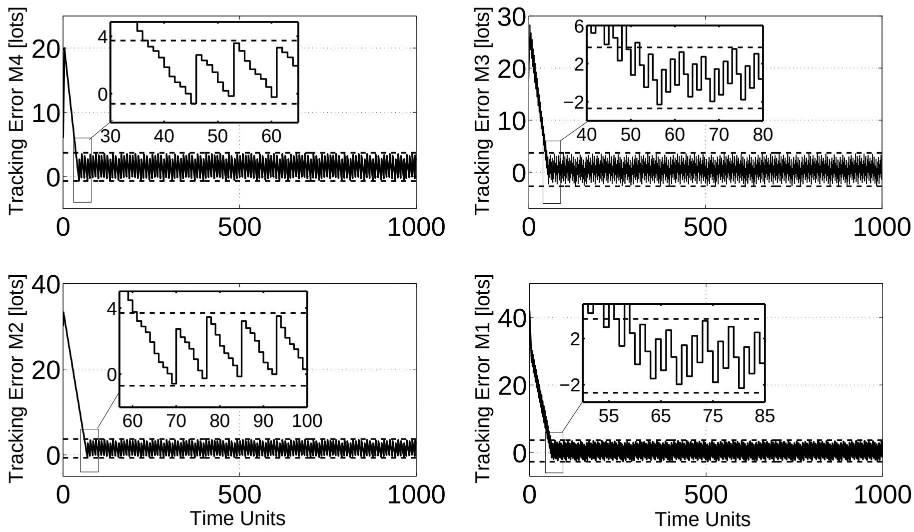


Figure 3.10: Tracking Error $\varepsilon_j(k)$.

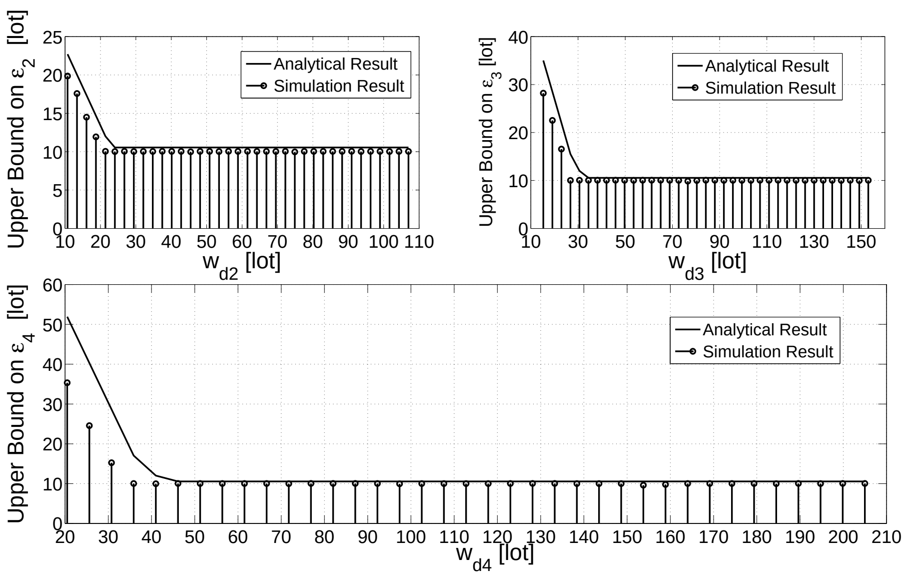


Figure 3.11: Demand Tracking Errors vs. Desired Inventory Levels.

machines, the production surplus can be decreased by incrementing the desired inventory level till some threshold value, which is given by (3.16), after which the bound on the surplus value remains constant. Doing so improves the service level of the network but increases the inventory costs. At the same time, lowering the inventory level further than the threshold decreases the service level of the network. This tradeoff relationship plays an important role in the decision making process of a production line manager. By using the relation provided beforehand in Theorem 5, the number of scheduling related decisions can be significantly reduced, which in consequence can decrease the computational time spent on planning for an efficient distribution of resources in a production line.

In conclusion, the presented simulation results reflect the desired flow model behavior, i.e., all technical conditions proposed in this section correspond to the analytical results described in Section 3.3. Also the result shown in Figure 3.11 underlines the practical importance of the obtained theoretical results.

3.5 Discrete event simulations

In this section the accuracy of the obtained demand tracking error bounds for the DT model is tested by means of two simulation examples on the DE model of a manufacturing line. The line shown in Figure 3.3, consists of 4 manufacturing machines and 3 buffers, and it produces one product type in one at a time manner. The DE model was built by using the specification language called χ (see Beek et al. (2006)) developed at the Eindhoven University of Technology[6].

Figure 3.12 shows the diagram of the DE model of one manufacturing machine operated under a surplus-based control. The circles represent the processes. The wide and the thin arrows indicate the lot and the information flow directions, respectively. This model is the extension of the DE model of a single machine

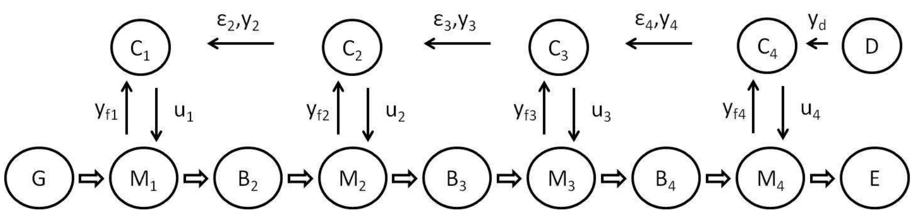


Figure 3.12: Diagram of DE model of a line of 4 manufacturing machines.

introduced in Section 2.6. Processes G, E and D are identical to the ones explained in Section 2.6. Processes M_j and C_j, shown on Figure 3.12, are similar to processes M and C of Section 2.6. Every process M_j, where index $j = 1, 2, 3, 4$, exchanges information with its controller (process C_j) that authorizes the machine to produce based on the value of the demand tracking error ε_j . Note that if process C_j authorizes the production for M_j, it does not imply that M_j can immediately start producing. Process C_j does not check the availability of products in buffer B_j. The demand tracking error in C_j, with $j = 1, 2, 3$, consists of the difference

[6]For the details on implementation see Rooda and J.Vervoort (2007)

between the sum value of the demand tracking error of the downstream machine ε_{j+1} and the desired constant base stock w_{dj+1} of the downstream buffer against the current product content $w_{j+1} = y_{j+1} - y_j$ of the downstream buffer, i.e. $\varepsilon_j = \varepsilon_{j+1} + w_{dj+1} - w_{j+1}$. The last machine tracks the demand y_d on its output. Thus for C_4 the demand tracking error $\varepsilon_4 = y_d - y_4$. Process C_j recalculates the value of ε_j every time it receives an updated information from the following sources:

- The downstream tracking error ε_{j+1}, in case of $j = 1, 2, 3$, from process C_{j+1}.
- The cumulative output of the downstream machine y_{j+1}, in case of $j = 1, 2, 3$, from process C_{j+1}.
- The cumulative demand value y_d, in case $j = 4$, from the demand generating process D.
- The update on the value of y_j, for all $j = 1, 2, 3, 4$, e.g., in form of a token[7] y_{fj}, which is send from process M_j to C_j.

Process B_j receives and stores the products from M_{j-1}. If the stored content is positive, then it tries to send products to its adjacent process M_j. Thus if the control authorization is given, e.g., by means of a token u_j send from C_j to M_j, and buffer B_j contains at least one product and process M_j is not busy working on another product, then M_j instantly takes one product form B_j.

The following two examples are analyzed. In the first example (see the outcome on Figures 3.13-3.16) for event-based system the production speed of each machine (μ_j) is fixed to 1 lot per time unit. The initial output $y_j(0) = 0$ for all j. The production demand consists of its initial value y_{d0} of 10 lots and the constant demand rate v_d of 1 lot every 1.25 units of time. The DE system is approximated by DT model with the same production speed and demand rate v_d of 0.8 lots per time unit. The simulation run time is set to 500 time units. Several simulation runs with different base stock values have been executed and two cases are presented. One setting the base stock levels $w_{dj} = 0$ for all $j = 2, 3$ and another setting $w_{dj} = 3$ lots for all $j = 2, 3$. This is done in order to test the upper bound 3.32 in relation to the base stock level in the network.

The outcome of the experiment is as follows :

- From Figures 3.13 and 3.15, showing the demand tracking error trajectories for the settings with $w_{dj} = 0$ and $w_{dj} = 3$ of DT and DE models, it can be observed that the duration of the transient behavior for DE system is similar to that of DT system.
- Also from these figures it can be observed that the steady state demand tracking error trajectories of DT model satisfy the theoretical bounds (3.32), and (3.33). Similarly to the conclusion of Section 2.6, here the theoretical bounds for DT model require an adequate interpretation for the DE model. The difference in the steady state trajectories of ε_j of the two models origins

[7] Every time the machine produces one product the token y_{fj} is send to the process C_j, which increments its counter of y_j by one.

from their distinct interpretation of the demand arrival. In DE model the demand value is rounded to the closest integer, i.e. the product order that arrives to the system is for an integer number of products, i.e., 1 lot every 1.25 time units. The DT model approximates the demand arrival differently. The quantity of 0.8 lots is added every fixed time step of 1 time unit to the cumulative demand signal y_d. Thus for this example the bounds of DT model can be translated to DE model bounds as the ceil values of (3.32) and of (3.33).

- The contents of the buffers for the settings with $w_{dj} = 0$ and $w_{dj} = 3$ are shown in Figures 3.14 and 3.16, respectively.
- In DT model, the maximal content and the general pattern of product content variation in each buffer is shown to be similar to the one of DE model, but the DE model describes the buffer content variations in a more detailed manner than the DT model. Note that in the DT model, the buffer content is calculated based on the difference between the upstream and the downstream cumulative outputs of the machines surrounding the buffer. Thus a product location is undistinguished from a buffer or its adjacent machine. Differently from DT in DE model, the products from the buffer content are distinguished from the products in the machine.

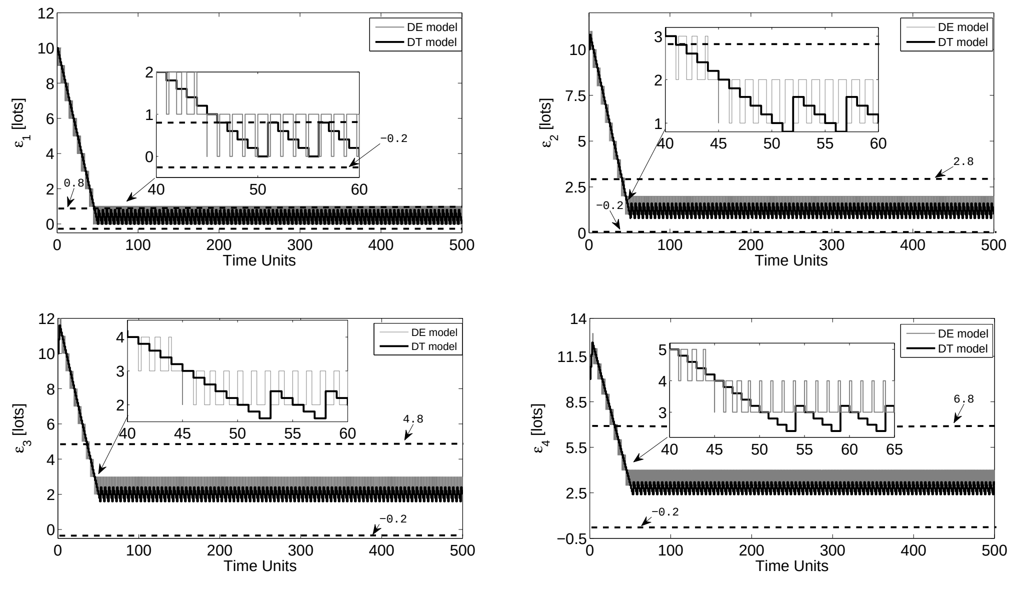


Figure 3.13: First example: Demand tracking errors ε_j, with bounds (- - -), for $w_{dj} = 0$.

The second example, see the outcome on Figures 3.17-3.20, differs form the previous one in that market perturbations φ are introduced in the demand variable and production speed perturbations f_j are introduced in the network model. For DE model, the constant demand rate is 1 lot every 1.25 time units with market fluctuations of $\varphi(t) = 0.049\cos(3t)$ (t is a current time of experiment). Thus the product demand rate is oscillating between one lot every 1.1086 and 0.9107 time units. Each

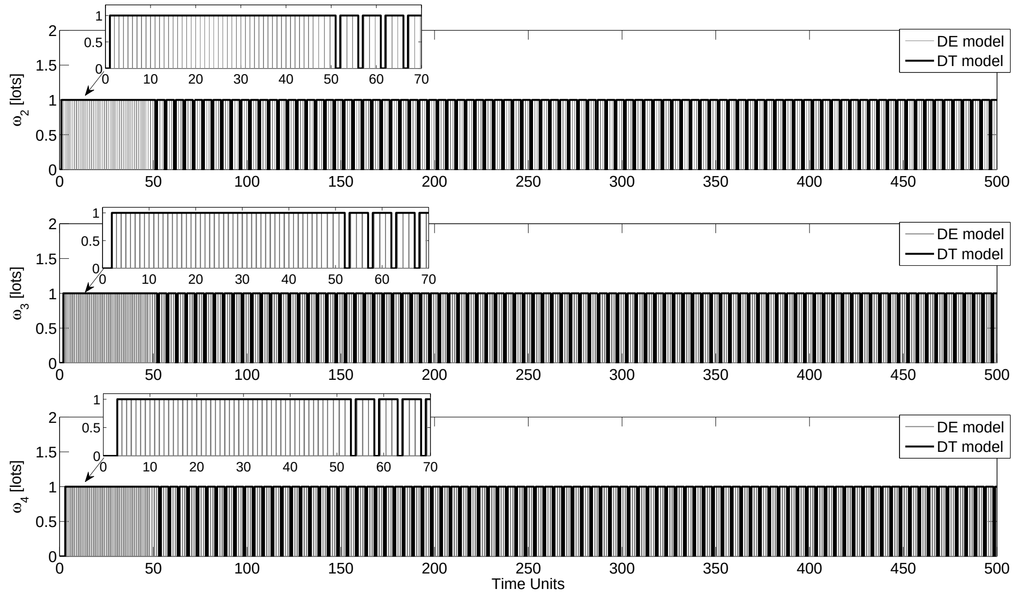


Figure 3.14: First example: Buffer content w_j, for $w_{dj} = 0$.

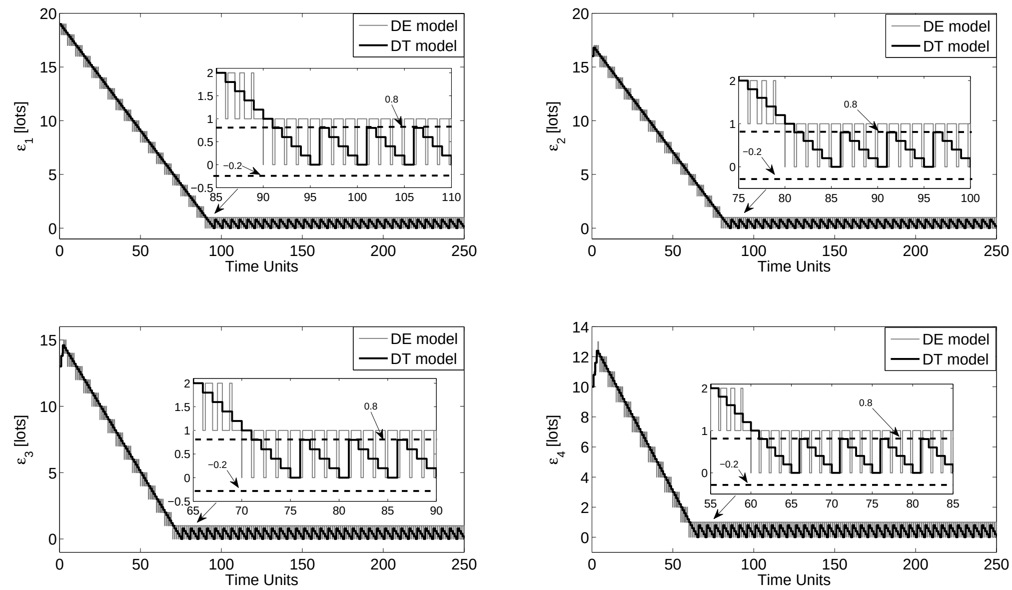


Figure 3.15: First example: Demand tracking errors ε_j, with bounds (- - -), for $w_{dj} = 3$.

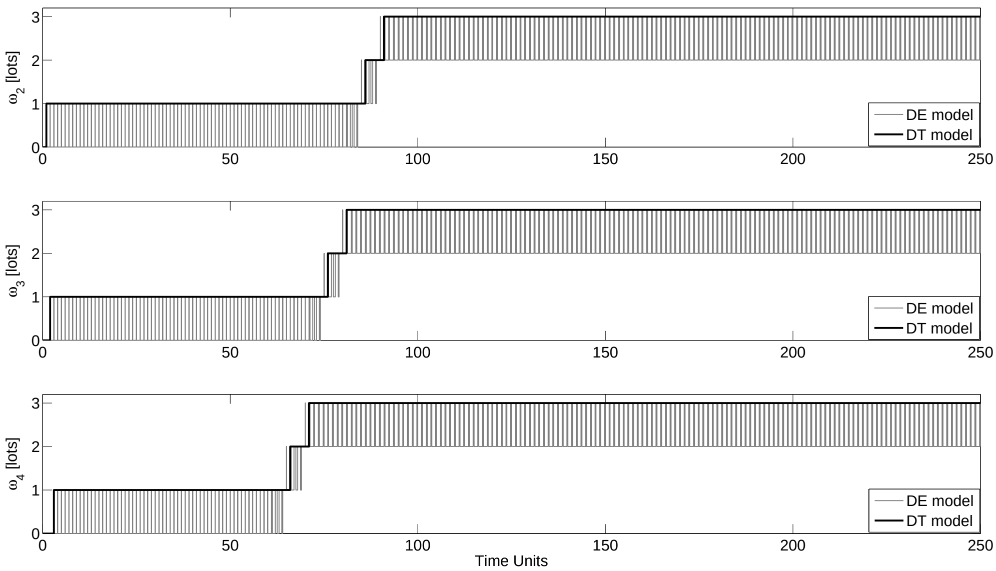


Figure 3.16: First example: Buffer content w_j, for $w_{dj} = 3$.

machine in the network produces one lot in $\frac{1}{\mu_j + f_j(t)}$ time units with $\mu_j = 1$ lot for all $j = 1, 2, 3, 4$ and $f_1(t) = 0.1\sin(1.5t)$, $f_2(t) = 0.1\sin(t)$, $f_3(t) = 0.1\sin(3t)$, and $f_4(t) = 0.1\sin(2.5t)$ lots. For the DT model, this behavior can be approximated by product demand rate $v_d + \Delta\varphi(k) = 0.8 + 0.049(\cos(3k+3) - \cos(3k))$ lots per time unit and demand rates $\mu_j + f_j(k)$ lots per time unit. Here k is the time step. The run time of the experiment is set to 500 time units. Several simulation runs with different base stock values are executed and two cases are presented. In one setting the base stock levels are $w_{dj} = 0$ for all $j = 2, 3, 4$ and in another setting $w_{dj} = 3$ lots for all $j = 2, 3, 4$. This is done in order to test the upper bound (3.32) in relation to the base stock level in the network.

The outcome of the experiment is as follows:

- From Figures 3.17 and 3.19, it can be observed that the duration of the transient behavior for the DE system is similar to that of the DT system.
- From these figures it can be seen that the steady state demand tracking error trajectories of DT model satisfy the theoretical bounds (3.32), and (3.33). Similarly to the conclusion of Section 2.6, here the theoretical bounds for DT model require an adequate interpretation for the DE model. The difference in the steady state trajectories of ε_j of the two models comes from their distinct interpretation of the demand arrival. In the DE model the demand value is rounded to the closest integer, i.e. the product order that arrives to the system is for an integer number of products, i.e., 1 lot every $\frac{1}{0.8+0.049(\cos(3k+3)-\cos(3k))}$ time units. The DT model approximates the demand arrival differently. The quantity of $0.8 + 0.049(\cos(3k+3) - \cos(3k))$ lots is added to the cumulative demand y_d every fixed time step.
- For $w_{dj} = 0$ the bounds obtained for DT model also satisfy the DE model

demand tracking error trajectories behavior.

- For $w_{dj} = 3$ the bounds for the DT model can be translated to bounds for the DE model as the ceil values of (3.32) and of (3.33).
- The content of the buffers for the experimental settings with $w_{dj} = 0$ and $w_{dj} = 3$ is shown in Figures 3.18 and 3.20, respectively.
- The variation of the product content of the DT model differs from the one of the DE model, but the control objective of maintaining the specified base stock levels is satisfied in both models. Note that in the DT model, the buffer content is calculated based on the difference between the upstream and the downstream cumulative outputs of the machines surrounding the buffer. Thus a product location is undistinguished from a buffer or its adjacent machine. Differently from the DT, in the DE model the products from the buffer content are distinguished from the products in the machine.

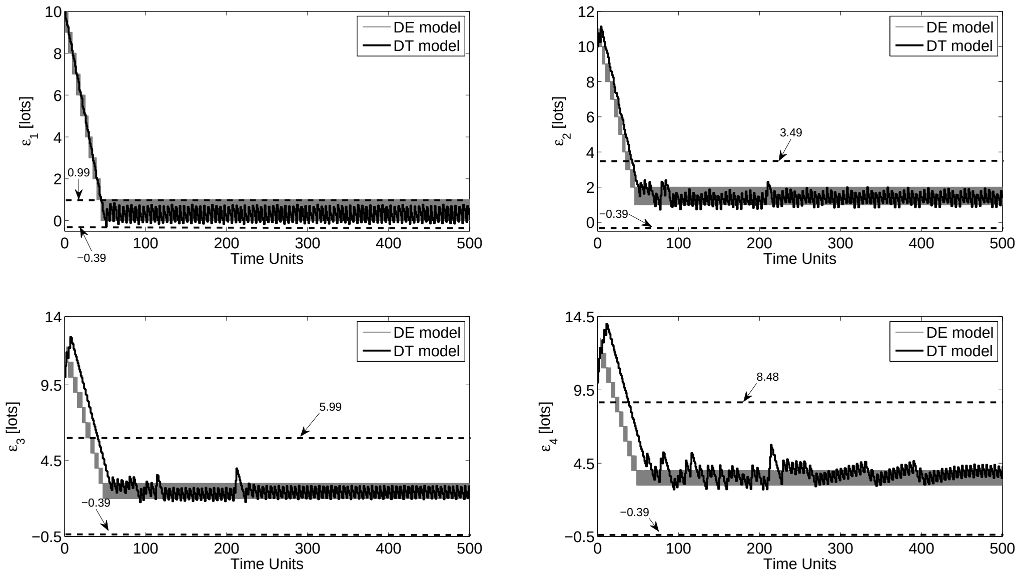


Figure 3.17: Second example: Demand tracking errors ε_j, with bounds (- - -), for $w_{dj} = 0$.

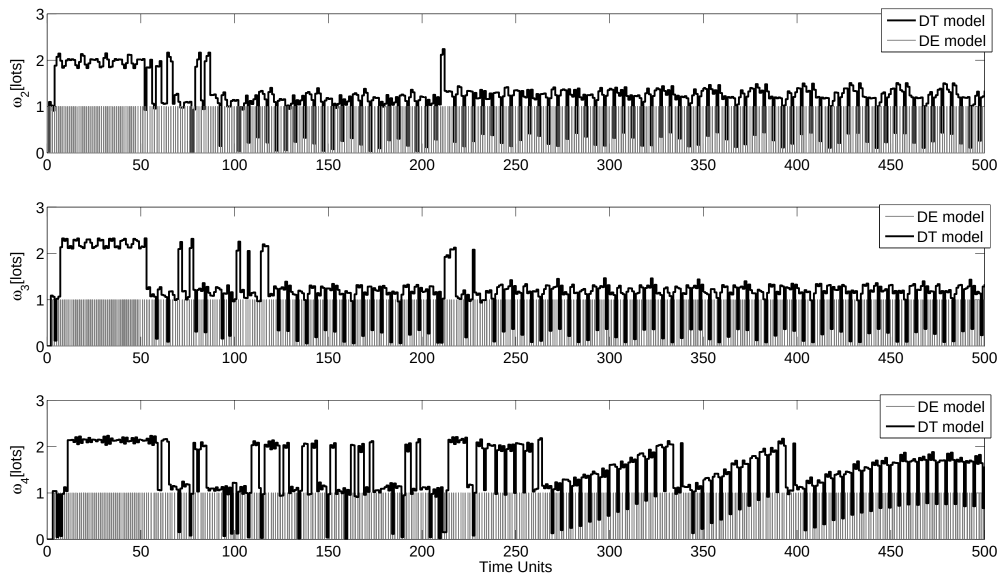


Figure 3.18: Second example: Buffer content w_j, for $w_{dj} = 0$.

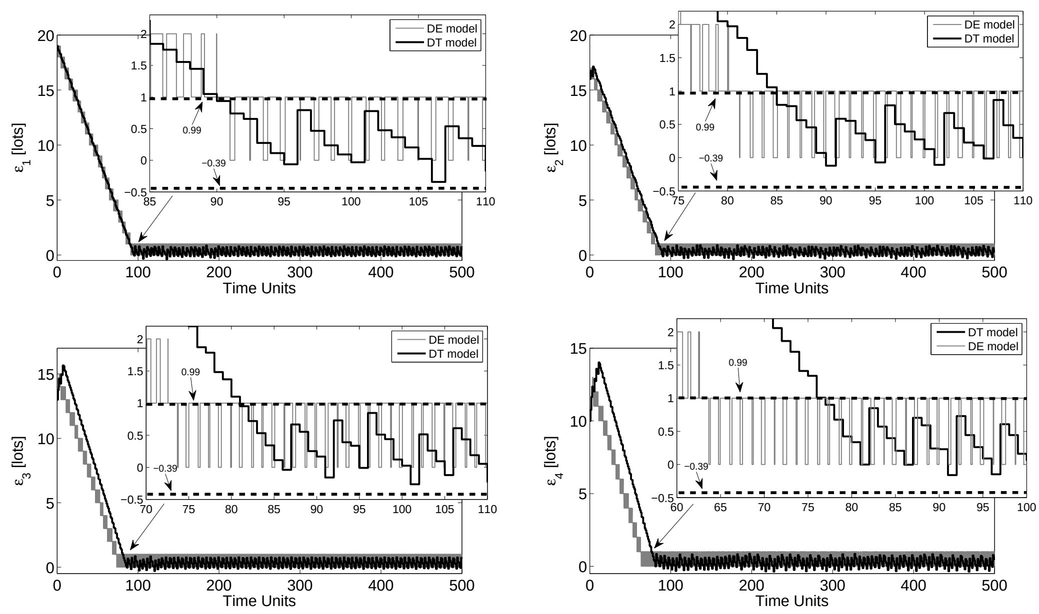


Figure 3.19: Second example: Demand tracking errors ε_j, with bounds (- - -), for $w_{dj} = 3$.

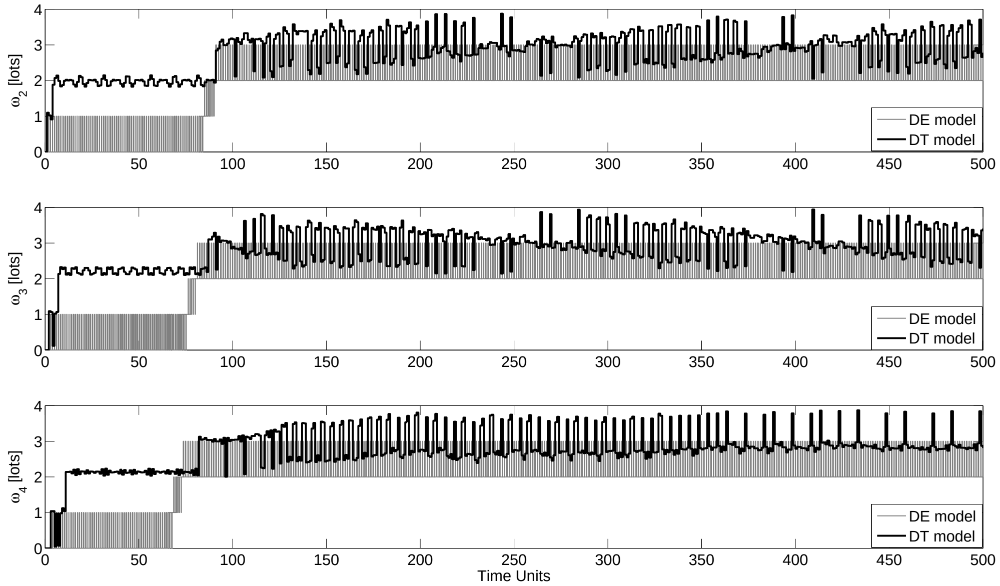


Figure 3.20: Second example: Buffer content w_j, for $w_{dj} = 3$.

3.6 Simulation based performance analysis

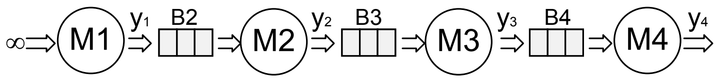


Figure 3.21: Schematics of a line of 4 manufacturing machines.

In this section we present the results of the simulation-based comparison of 3 particular surplus-based production control strategies applied to a network consisting of a line of 4 manufacturing machines ($j = 1, \ldots, 4$) with 3 intermediate buffers (B_2, B_3, B_4) as shown in Figure 3.21. The selected strategies are particular cases of Hedging Point Policy (HPP), Conwip (CWIP), and Base Stock Policy (BSP)(for details see Appendix K). The performance criteria upon which these 3 policies will be compared are the steady state maximum and minimum values of production demand tracking error of the last machine M_4 and the distribution of intermediate inventory (i.e, $w_2 + w_3 + w_4$) in the line.

Description of the experiment

The following common assumptions are made for all the implemented policies (see Appendix K for the details on the used flow models):

- The manufacturing line produces a single part type.
- It is assumed that the machines are never blocked and M_1 is never starved, i.e., $w_1(k) > \beta_1(k)$ for all k.

- For each machine the control action is executed every time step k.
- The operation of each machine M_j is managed through the corresponding control input $u_j(k)$.
- For all the policies, the control action of the output machine (M_4) is based on the demand tracking error $\varepsilon_4(k)$, which is given by (3.7).
- No setup times nor delays are considered in the models

The demand tracking errors and the operation principal of HPP can be described by (3.5)-(3.7) and (3.3), (3.4) respectively.
Under the CWIP policy, the first machine M_1 limits the number of products in the network while M_2 and M_3 produce products in a Push manner, i.e., M_2 and M_3 are always ready to produce as long as their adjacent upstream buffers contain sufficient material content. Thus $\varepsilon_1(k) = w_{d_{total}} - w_{total}(k)$, $\varepsilon_2(k) = w_2(k)$ and $\varepsilon_3(k) = w_3(k)$. For a fair comparison with the other two policies, $w_{d_{total}}$ can be interpreted as $w_{d_{total}} = w_{d_2} + w_{d_3} + w_{d_4}$ and $w_{total}(k) = y_1(k) - y_4(k)$. The control inputs $u_2(k) = 1$ and $u_3(k) = 1$ only if $w_2(k) \geq \beta_2(k)$ and $w_3(k) \geq \beta_3(k)$, respectively. As well as in HPP, the control action of M_4 is aimed at the production demand tracking.
The BSP is a commonly used scheduling policy, usually applied in warehouses for inventory control purposes. Frequently in the literature (see Silver et al. (1998), Bonvik et al. (1997), Duri et al. (2000), Karaesmen and Dallery (2000), González et al. (2012) and references therein) one can find BS as a policy under which all the machines in the network keep track of the product demand, while at the same time maintaining a base stock level of the immediate downstream inventory. In the present comparative study, this BS notion is slightly modified, i.e., only the last machine (M_4) is tracking the product demand while the rest keep the upstream inventory at its base stock level (see Appendix K for details). Thus for fair comparison with HPP and CWIP, the demand tracking errors of BSP policy are selected as $\varepsilon_j(k) = w_{d_j} - w_j(k)$ for all $j = 1, 2, 3$, while $\varepsilon_4(k) = y_d(k) - y_4(k)$.

Results of comparison

For all the three strategies the details of the conducted experiment can be summarized as follows:

- All the models are described by difference equations and the simulations were executed in $Matlab^{®}$.
- Each simulation run is set to 15000 steps with initial demand $y_{d0} = 500$ [lots] and $y_j(0) = 0$ for all j.
- Three simulation scenarios are tested: balanced line, M_1 fast and M_4 slow, and M_1 slow and M_4 fast.
- For all the three policies, the comparative study is focused on the steady state demand tracking accuracy of the output machine ($\varepsilon_4(k)$) and the inventory levels of the intermediate buffers ($w_j(k)$).

- Due to similarity of obtained results for all the three scenarios, only the outcome of the balanced line case is presented.
- The production speeds of the balanced line are given in Table 3.1.
- Each model is tested under 50%, 75% and 95% of the maximal production demand rate, such that it satisfies condition (3.14)(see Table 3.1 for details).
- For each demand rate, 8 simulation runs are executed. Each run with different w_{dj} in case of HPP and BSP, or $w_{dtotal} = w_{d2} + w_{d3} + w_{d4}$ in case of CWIP policy.
- The desired product content of a buffer is selected in multiples of the maximal production speed of its immediate downstream machine, from 0 till 20 times the value. Thus $w_{d_j} = a(\mu_j + c_4)$ lots, where the value of a is given in Tables 3.2-3.4.
- For each buffer in the network, its steady state mean buffer content value is shown in Figure 3.22.
- Each bar in Figure 3.22 stands for the steady state mean buffer content value for the selected amount of base stock level w_{d_j} or w_{dtotal} in case of CWIP, i.e., each bar represent the steady state $\bar{w}_j$ value of B_j for each value of a, for each production demand rate and for each policy, as it is shown for B_2.

Parameters of the manufacturing line				
j	$\mu_j + f_{jk}$	%	v_d	$\Delta\varphi_k$
1	$5 + 0.5\sin(10k)$	50	2.25	$0.2\cos(180k)$
2	$5 + 0.5\sin(20k)$	75	3.375	$0.2\cos(180k)$
3	$5 + 0.5\sin(45k)$	95	4.275	$0.2\cos(180k)$
4	$5 + 0.5\sin(90k)$			

Table 3.1: Production and demand rates in lot per time unit

Tables 3.2, 3.3 and 3.4 show the obtained steady state values of the demand tracking error of the output stage for 95%, 75% and 50% of the average production demand rate v_d. It is observed that for all the three policies, increasing the desired intermediate inventory level further than three times the minimal buffer content value ($a > 3$) has no significant influence on the steady state tracking accuracy of the network. For HPP, this buffer level effect is clearly reflected in the obtained demand tracking error bound (3.32). One can note that if inequality (3.16) of Assumption 3 is satisfied for all the intermediate inventories of the network, then in (3.32) the term $X_j = 0$, and the value of every demand tracking error bound becomes independent from the influence of w_{dj}. For the low inventory, the decrease of tracking accuracy of the line under HPP could be also observed from (3.32). If inequality (3.16) of Assumption 3 is not satisfied, then the upper bound on the demand tracking error (3.32) is influenced by x_j term that can drastically decrease the tracking accuracy of the system. The extreme case of HPP behavior, i.e. $w_{d_j} = 0$ for all j, is reported in Tables 3.2-3.4 as well as shown in Figure 3.22, where very low inventory levels and higher demand tracking inaccuracy can be observed for the network operated under the pure demand tracking Pull strategy. In this case, no desired inventories are specified for our HPP model and the control goal of every production stage consists in pure cumulative product demand tracking. Thus HPP with $w_{dj} = 0$ can also be classified as a basic Pull policy.

Different from HPP, the CWIP policy forces the line to constantly maintain a certain non zero inventory level in its buffers. This can be observed from Tables 3.2, 3.3 and 3.4, where the production line under CWIP is able to track the production demand starting from $a = 2$. Also from the tables, it can be seen that for $a \geq 2$, the tracking accuracy of CWIP is identical to that of HPP. From Figure 3.22 it can be observed that the amount of w_{dtotal} for CWIP policy is mostly accumulated in buffer B_4, which is the last buffer in the line. This material distribution can have positive as well as negative effects on the network performance. A positive effect, for example, is reflected in the tracking accuracy and low standard deviation (σ) from the mean demand tracking error value ($\bar{\varepsilon}_4(k)$). In other words the manufacturing network reacts faster on the rapid production demand changes if its inventory levels are high. At the same time, keeping high inventories may impose unnecessary storage costs, specially if the product demand is low.

For BSP the inventory levels and its distribution through the network are very similar to the ones in HPP (see Figure 3.22). Excluding the low inventory cases, where BSP is unable to perform due to its policy limitation, the demand tracking error accuracy (see Tables 3.2-3.4) of BSP is shown to be almost identical to HPP. Both policies permit an independent (distributed) control of intermediate inventory levels of the network. Thus for a given setting, BSP can be adjusted to perform as CWIP and HPP as pure demand tracking Pull, but not the other way around. This makes these two policies to stand out among the three. Also the obtained theoretical values on steady state $\max \varepsilon_4(k)$ and $\min \varepsilon_4(k)$ values for the HPP policy (see Tables 3.2-3.4 under HPP (theory) reference) show the accurate lower and upper bound evaluation. For HPP (theory) the calculations of the bounds of ε_4 were based on the expressions (3.32) and (3.33). Note that the theoretical bounds are obtained for the worst case scenario of the demand tracking error accuracy, thus in HPP (theory) row in Tables 3.2-3.4 bigger values can be noticed for the low inventory levels in the line, i.e, when $a = 0, 1, 2$. Note that differently from HPP, the analytical bounds on the production tracking accuracy of a manufacturing line under a BS and CWIP policies have not yet been reported in the current literature.

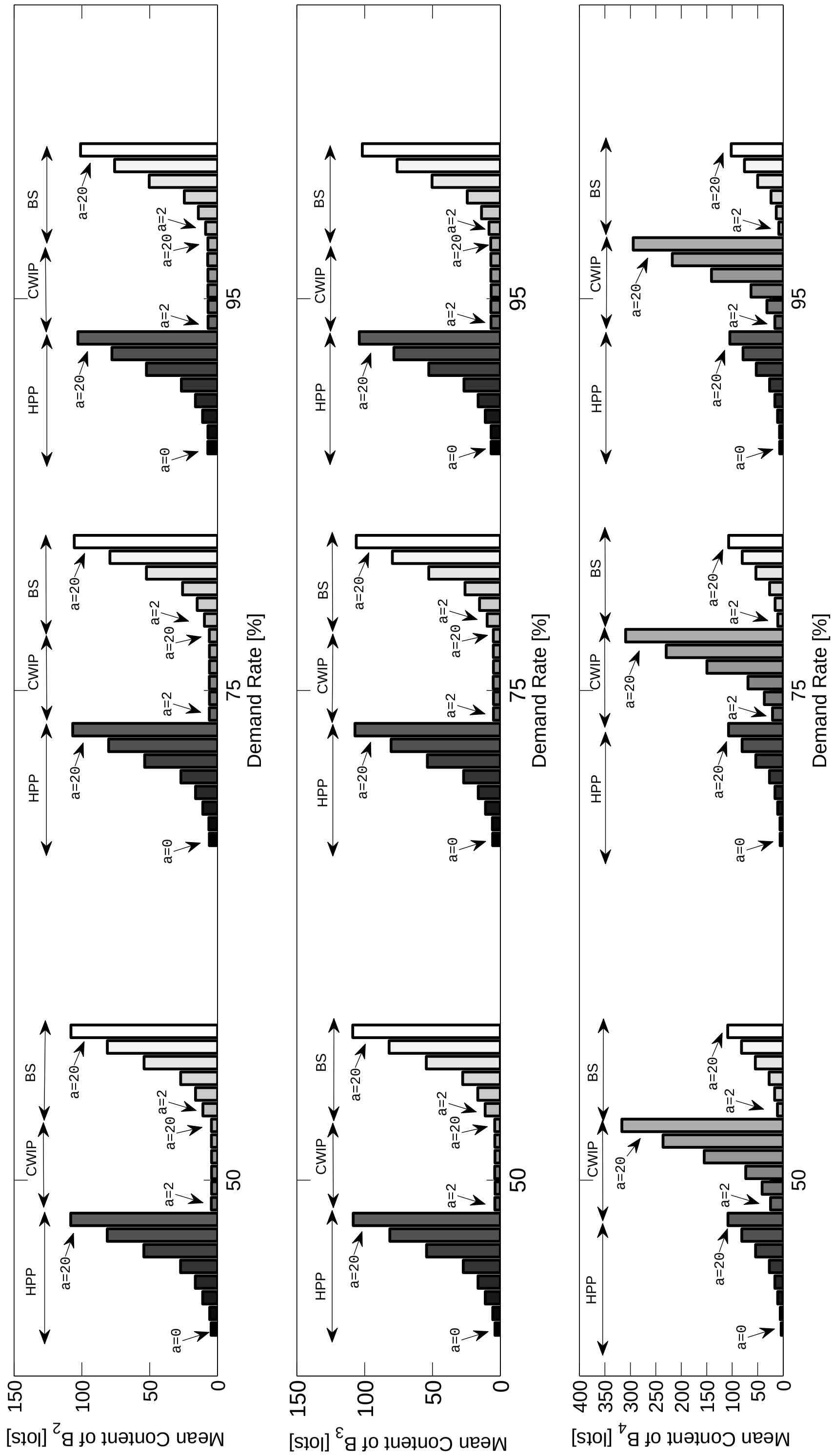


Figure 3.22: Intermediate buffer content

a	0	1	2	3,5,10,15,20
HPP $\bar{\varepsilon}_4(k)$ [lot]	23.35	6.96	1.99	1.99
CWIP $\bar{\varepsilon}_4(k)$ [lot]	∞	∞	1.99	1.99
BS $\bar{\varepsilon}_4(k)$ [lot]	∞	∞	4.27	1.99
HPP σ_{ε_4} [lot]	2.54	2.44	1.45	1.45
CWIP σ_{ε_4} [lot]	_	_	1.45	1.45
BS σ_{ε_4} [lot]	_	_	1.57	1.45
HPP (theory)max $\varepsilon_4(k)$ [lot]	38.08	21.57	5.07	4.47
HPP max $\varepsilon_4(k)$ [lot]	31.89	14.85	4.41	4.41
CWIP max $\varepsilon_4(k)$ [lot]	_	_	4.41	4.41
BS max $\varepsilon_4(k)$ [lot]	_	_	9.91	4.41
HPP (theory) min $\varepsilon_4(k)$ [lot]	-1.625	-1.625	-1.625	-1.625
HPP min $\varepsilon_4(k)$ [lot]	14.86	-0.937	-0.43	-0.43
CWIP min $\varepsilon_4(k)$ [lot]	_	_	-0.43	-0.43
BS min $\varepsilon_4(k)$ [lot]	_	_	-1.36	-0.43

Table 3.2: Comparison for demand rate of 95 %

a	0	1	2	3,5,10,15,20
HPP $\bar{\varepsilon}_4(k)$ [lot]	18.58	4.192	0.52	0.52
CWIP $\bar{\varepsilon}_4(k)$ [lot]	∞	∞	0.52	0.52
BS $\bar{\varepsilon}_4(k)$ [lot]	∞	∞	0.52	0.52
HPP σ_{ε_4} [lot]	2.577	2.188	1.54	1.54
CWIP σ_{ε_4} [lot]	_	_	1.54	1.54
BS σ_{ε_4} [lot]	_	_	1.54	1.54
HPP (theory)max $\varepsilon_4(k)$ [lot]	36.17	20.67	4.17	3.38
HPP max $\varepsilon_4(k)$ [lot]	26.91	10.65	3.36	3.36
CWIP max $\varepsilon_4(k)$ [lot]	_	_	3.36	3.36
BS max $\varepsilon_4(k)$ [lot]	_	_	3.36	3.36
HPP (theory) min $\varepsilon_4(k)$ [lot]	-2.52	-2.52	-2.52	-2.52
HPP min $\varepsilon_4(k)$ [lot]	10.25	-2.27	-2.32	-2.32
CWIP min $\varepsilon_4(k)$ [lot]	_	_	-2.32	-2.32
BS min $\varepsilon_4(k)$ [lot]	_	_	-2.32	-2.32

Table 3.3: Comparison for demand rate of 75 %

a	0	1	2	3,5,10,15,20
HPP $\bar{\varepsilon}_4(k)$ [lot]	13.95	1.51	-0.458	-0.458
CWIP $\bar{\varepsilon}_4(k)$ [lot]	∞	∞	-0.458	-0.458
BS $\bar{\varepsilon}_4(k)$ [lot]	∞	∞	-0.458	-0.458
HPP σ_{ε_4} [lot]	2.47	1.64	1.46	1.46
CWIP σ_{ε_4} [lot]	_	_	1.46	1.46
BS σ_{ε_4} [lot]	_	_	1.46	1.46
HPP (theory)$\max \varepsilon_4(k)$ [lot]	36.05	19.02	3.05	2.37
HPP $\max \varepsilon_4(k)$ [lot]	23.03	6.45	2.36	2.36
CWIP $\max \varepsilon_4(k)$ [lot]	_	_	2.36	2.36
BS $\max \varepsilon_4(k)$ [lot]	_	_	2.36	2.36
HPP (theory) $\min \varepsilon_4(k)$ [lot]	-3.65	-3.65	-3.65	-3.65
HPP $\min \varepsilon_4(k)$ [lot]	4.87	-3.43	-3.28	-3.28
CWIP $\min \varepsilon_4(k)$ [lot]	_	_	-3.28	-3.28
BS $\min \varepsilon_4(k)$ [lot]	_	_	-3.28	-3.28

Table 3.4: Comparison for demand rate of 50 %

3.7 Conclusions

The surplus-based control strategy, introduced in the previous chapter, was extended to a tandem production line. Flow models of a line with unbounded and bounded buffers were presented and their performance was evaluated. The performance of the closed-loop systems was addressed in the form of bounds on the demand tracking errors that occur for each machine in the line, respectively. The analytical results describing the tradeoff relationship between the demand tracking errors and the inventory level in the manufacturing line were obtained. All theoretical results were illustrated and confirmed by computer simulation. By means of simulation, the obtained relation on the performance for the DT model was also tested on the DE model. The interpretation of DT analytical bounds on the demand tracking errors for the DE model was given for selected modeling examples. It was observed that though the DT approximation is less accurate than the DE approximation of the production line, the DT analytical results on performance still hold for DE model. Further, simulation-based comparison of three selected surplus-based control strategies was conducted for a line of 4 machines and 3 buffers. The performance criterions were the maximum and minimum values of the steady state demand tracking errors and intermediate inventory levels, which were compared for the line operated under particular HPP, BS and CWIP policies. The obtained results showed similar behavior of all the three policies.

4
Multi-product manufacturing lines

This chapter is based on Starkov et al. (2011b).

Abstract | In this chapter results on the performance analysis of a multi-product manufacturing line are presented. The influence of external perturbations, intermediate buffer content and the number of manufacturing stages on the demand tracking error of each machine in the multi-product line operated under a surplus-based production control policy is studied. Starting with the analysis of a single machine with multiple production stages (one for each product type), bounds on the demand tracking error of each stage are provided. Then the analysis is extended to a line of multi-stage machines, where similarly, bounds on each demand tracking error for each product type, as well as buffer content are obtained. Details on performance of the closed-loop flow line model are illustrated in numerical simulations.

4.1 Introduction

A manufacturing network consisting of workstations interconnected in a tandem manner, where at each station one machine serves several buffers (i.e., flexible machine), can be frequently encountered as a part of an industrial production process. For example, in case of semiconductor manufacturing it is typical to observe that at some stages the machines are working with multiple product types. In order to produce a wafer several layers of semiconductor material have to be put together, which implies that the product (wafer) has to undergo several times (some wafers more that others) through the same process before it is finally ready (see, e.g., Montoya-Torres (2006)). In this case manufacturing machines work with intermediate products (wafers) of different processing stages. Another example of flexible manufacturing lines can be observed in the automotive industry (see, e.g., Li et al. (2009)).

Analysis on control and performance of networks, which present flexible behavior in the production process, has always attracted much attention of manufacturers, as well as of researchers. Thus control problems of flexible manufacturing lines are widely studied and a lot of valuable approaches including queuing theory, Petri nets, dynamic programming, linear programming, hybrid systems were proposed and some of them are implemented (for surveys see, e.g., Gershwin (2000); Ortega and Lin (2004); Sarimveis et al. (2008)).

In this chapter we focus on the performance analysis of a flexible production line controlled by a surplus-based[1] decentralized production control (see e.g., Bonvik et al. (1997)). Specifically, given the presence of unknown but bounded production speed perturbations, as well as demand rate fluctuations, we investigate how close the cumulative production output of the network follows its cumulative production demand under this control policy.

In order to achieve our goal we use classical tools from control theory. The production flow process is described by means of difference equations and in order to analyze its performance, an approach based on Lyapunov theory is exploited (see, e.g., Khalil (2002) and references therein). Each machine in the network is responsible for several production stages. At each stage the machine coordinates its individual production with those of the rest of the system. While working at one stage the machine does not switch to another one unless the primary control objective at this stage is fulfilled or product starvation occurs. The primary objective of each production stage may be viewed as manufacturing a sufficient quantity of parts to satisfy the demand of its immediate downstream production stage (belonging to the downstream machine) and some desired amount as back-up material storage in its downstream buffer. The production strategy itself is intuitive and it can be associated with a wide range of existing techniques such as Basestock policy (see, e.g., Bonvik et al. (1997)), Hedging Point policy (see, e.g., Gershwin (2000)), and Clearing policy (see, e.g., Perkins et al. (1994)).

To the best of our knowledge, concerning the previous results on performance analysis of surplus-based approaches (see, e.g., Gershwin (2000); Ortega and Lin (2004); Sarimveis et al. (2008); Somlo (2004); Lu and Kumar (1991); Quintana (2002); Subramaniam et al. (2009); Savkin and Evans (2002), the novelty of our results can be summarized as follows. The proposed production model is considered in discrete time. The production speed of each machine is defined as deterministic with bounded perturbations. The future production demands are assumed to be unknown and with bounded fluctuations. As a result, for one flexible manufacturing machine of N production stages, strict, so-called “worst case” bounds on the demand tracking error for each product type are obtained. Extending this strategy to a network of P machines with N production stages each, we present the results regarding the bounds on the demand tracking errors and buffer contents for each machine and its buffers. Furthermore, we show that, though the analysis given in this chapter is focused on multi-product manufacturing lines, the obtained results can be easily extended to re-entrant configurations with one product type demand.

[1] In the surplus-based control, decisions are made based on the demand tracking error, which is the difference between the cumulative demand and the cumulative output of the system.

The chapter is organized as follows. First, in Section 4.2 the flow model of one manufacturing machine with surplus-based pull control is presented. The detailed analysis of demand tracking error trajectories is developed in this section. Then the flow model of a flexible manufacturing line with surplus-based pull control is analyzed in Section 4.3. Here sufficient conditions are derived to guarantee the uniform ultimate boundedness of the demand tracking error trajectories of each machine. Performance and robustness issues of the closed-loop flow models are illustrated in numerical simulations in Section 4.3.3. Finally, Section 4.4 contains conclusions of the chapter.

4.2 Single multi-product machine

Figure 4.1 shows the schematics of one machine M with N production stages, which directly correspond to the number of product types that it can serve. The machine is interconnected with N buffers $B_1 \dots B_N$, each containing its infinite product supply of corresponding product type. One way of describing the product flow

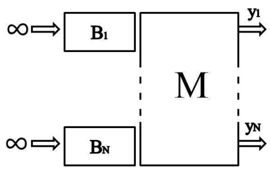


Figure 4.1: Schematics of one flexible manufacturing machine M.

through such a system is by means of discrete time flow model. The discrete time model of a single manufacturing machine with one production stage is described in detail in Chapter 2 of this thesis. In this section we will extend the model of Chapter 2 to a single manufacturing machine with multiple production stages.

4.2.1 Flow model

The flow model of each production stage of one flexible machine (see Figure 4.1) in discrete time is defined as

$$y_j(k+1) = y_j(k) + \beta_j(k)u_j(k), \ \forall k \in \mathbb{N}, j = 1, \dots, N, \tag{4.1}$$

where all the events within the model occur at given time instances and k represents the current time so that the time step between all the events is constant. Here $y_j(k) \in \mathbb{R}$ is the cumulative output of the machine for product type j in time k, $u_j(k) \in \mathbb{R}$ is the control input of the machine in processing stage of product type j and $\beta_j(k) = \mu_j + f_j(k)$ where μ_j is a positive constant that represents the processing speed of the machine for servicing the product type j and $f_j(k) \in \mathbb{R}$ is an unknown external disturbance affecting the performance of the machine at stage j. The equation (4.1) present a general model that can describe a product flow for a wide range of production stages of manufacturing networks. Further specific assumptions on system (4.1) are given in this section.

Under the assumption that there is always sufficient raw material to feed every stage of the machine, the control aim is to track the non-decreasing cumulative production demand of each product type j on its output. We define the cumulative production demand by using $y_{dj}(k) \in \mathbb{R}$ given by

$$y_{dj}(k) = y_{dj0} + v_{dj}k + \varphi_j(k), \ \forall j = 1, \ldots, N, \tag{4.2}$$

where y_{dj0} is a positive constant that represents the initial production demand of product j, v_{dj} is a positive constant that defines the average desired demand rate of product j, and $\varphi_j(k) \in \mathbb{R}$ is the bounded fluctuation that is imposed on the linear demand $v_{dj}k$.

In order to give a solution to this production demand tracking problem we consider the controller based on the demand tracking error of each product type. The machine can only work at one stage at a time (i.e., at a time step k). The controller alternatively selects the stage at which the machine must work [2], from those where production is needed. The machine works at this stage till its product demand is satisfied. Then the controller again selects a stage for the machine to work at. In case the product demand of all product types are satisfied, the controller idles the machine. The above mentioned can be formulated by following control algorithm (see next paragraph for a summary):

$$\begin{aligned}
&\{q(k) = B_j\}\\
&\textbf{if}\, \varepsilon_j(k) > 0\, \textbf{then}\\
&u_j(k) = 1,\\
&u_s(k) = 0, \ \forall s \neq j, \ s, j = 1, \ldots, N,\\
&q(k+1) = B_j,\\
&\textbf{end}\\
&\textbf{if}\, \varepsilon_j(k) \leq 0\, \textbf{and}\, \exists s \neq j : \varepsilon_s(k) > 0\, \textbf{then}\\
&u_j(k) = 0,\\
&u_s(k) = 1,\\
&q(k+1) = B_s,\\
&\textbf{end}\\
&\textbf{if}\, \varepsilon_s(k) \leq 0, \ \forall s\, \textbf{then}\\
&u_j(k) = 0,\\
&u_s(k) = 0, \ \forall s \neq j, \ s, j = 1, \ldots, N,\\
&q(k+1) = 0\\
&\textbf{end}
\end{aligned} \tag{4.3}$$

[2]The results presented in this chapter only concentrate on the steady state behavior of a multi-product network. In consequence the scheduling problem within the machines play no significant role as long as within a certain period of time all the production stages of each machine in the network are served. Thus in algorithm 4.3 a policy to select a production stage s in case their exist several stages with $\varepsilon_s(k) > 0$ is not detailed. Note that the priority rule for the production stage selection for each of the multi product manufacturing machines in the line is also one of the challenging problems in this type of systems (for more details see Winands et al. (2011) and references therein).

where $q(k)$ is the internal variable that specifies the buffer that machine M is processing, $\varepsilon_j(k) \in \mathbb{R}$ is the demand tracking error at stage j. Note that all B_j buffers are considered to always have sufficient raw material.
Summarizing (4.3), the machine can only work on one buffer (product type) at a time. The control input $u_j(k)$ of each production stage j can only take the value of 0 (stop) or 1 (produce). The $u_j(k)$ receives the value of 1 only if production stage j needs to produce ($\varepsilon_j(k) > 0$). The machine will remain at its current state ($q(k) = B_j$) while all the conditions of the state are satisfied. The value of 0 is given to the control input of stage j if at least one of the conditions of the current state $q(k) = B_j$ is unsatisfied. The change in the value of the control signal of a stage j also implies a change in the machine's state $q(k)$. The machine has $N + 1$ states. This is due to that N is the total number of processing stages (product types) that M can be working in, which directly relate to the states of the machine, plus the idle state ($q(k) = 0$).

The demand tracking error at each stage of M is given by:

$$\varepsilon_j(k) = y_{dj}(k) - y_j(k), \ \forall k \in \mathbb{N}. \tag{4.4}$$

For further analysis, let us rewrite flow model (4.1) in a closed-loop with (4.3) in terms of demand tracking errors as

$$\Delta\varepsilon_j(k) = v_{dj} + \Delta\varphi_j(k) - \beta_j(k)u_j(k), \tag{4.5}$$

where for all $j = 1, \ldots, N$, $\Delta\varepsilon_j(k) = \varepsilon_j(k+1) - \varepsilon_j(k)$ and $\Delta\varphi_j(k) = \varphi_j(k+1) - \varphi_j(k)$. Here we assume that system (4.5) satisfies the following assumptions.

Assumption 4. *(Boundedness of perturbations) There are constants c_1, c_2, c_3 and c_4 such that*

$$c_1 < \Delta\varphi_j(k) < c_2, \ \forall k, \ j = 1, \ldots, N, \tag{4.6}$$
$$c_3 < f_j(k) < c_4, \ \forall k, \ j = 1, \ldots, N. \tag{4.7}$$

From *Assumption 1*, it follows that $W_j(k) = \Delta\varphi_j(k) - f_j(k)$ satisfies

$$\alpha_1 < W_j(k) < \alpha_2, \ \forall k, \ j = 1, \ldots, N, \tag{4.8}$$

with $\alpha_1 = c_1 - c_4$ and $\alpha_2 = c_2 - c_3$.

Assumption 5. *(Capacity condition) Constants c_1, c_2, c_3 and c_4 satisfy the following inequalities*

$$c_1 \ > \ -v_{dj}, \ \forall j = 1, \ldots, N, \tag{4.9}$$
$$\alpha_2 \ < \ \mu_j - v_{dj}, \ \forall j = 1, \ldots, N, \tag{4.10}$$

and the following condition (also know as capacity condition) holds

$$0 < \sum_{j=1}^{N} \frac{v_{dj} + \Delta\varphi_j(k)}{\mu_j + f_j(k)} < 1, \ \forall k. \tag{4.11}$$

By (4.9), (4.10), and (4.11) we state that, in the presence of market fluctuations bounded by (c_1, c_2), the demand rate for each product type can only be positive, the production speed at each manufacturing stage of the machine is always faster than the demand rate of its product and in general the processing speed of the machine is faster than its demand rate, respectively.

It is important to notice that machine M at each process step j has a processing speed of $\mu_j + f_j(k)$ lots per time unit, which can differ from the other processing steps.

4.2.2 Results on performance

In this section we present the results respecting the demand tracking error trajectories behavior of flow model (4.5).

Theorem 6. *Assume that the discrete time system defined by (4.5) satisfies Assumptions 4 and 5. Then all solutions of (4.5) are ultimately bounded by*

$$\limsup_{k\to\infty} \sum_{j=1}^{N} \frac{\varepsilon_j(k) - v_{dj} - \alpha_2}{\mu_j + c_3} \leq 0, \tag{4.12}$$

$$\liminf_{k\to\infty} \varepsilon_j(k) \geq v_{dj} + \alpha_1 - \mu_j. \tag{4.13}$$

Proof. see Appendix F. □

The obtained bounds can be visualized through a phase portrait of the demand tracking error trajectories shown in Figure 4.2, which was made for a single machine producing 2 product types. The product demand rate $v_{dj} = 0.99$ [lots/time unit]

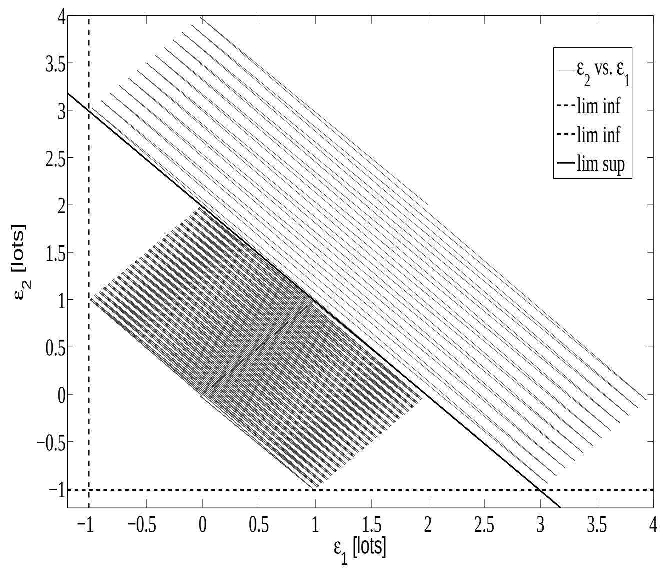


Figure 4.2: Demand Tracking Errors $\varepsilon_2(k)$ vs. $\varepsilon_1(k)$, with $v_{dj} = 0.99$ [lots/time unit] and μ_j=2[lot/ time unit].

and the production rate at each stage μ_j=2[lot/ time unit]. Here the experiment starts with initial demand tracking errors $\varepsilon_1(0) = 2$ [lots] and $\varepsilon_2(0) = 2$ [lots]. It can be observed that first the controller activates stage 1 of M. The machine works with this stage till $\varepsilon_1(k) \leq 0$ and then switches to stage 2. Eventually the trajectories of the demand tracking errors enter the zone depicted by the rectangular triangle, where they remain for the rest of the experiment. The legs of this triangle are given by (4.13) and the hypotenuse by (4.12).

4.3 Multi-product flow line

Figure 4.3 shows the schematics of a flexible manufacturing line consisting of P machines $M_1, \ldots, M_P$ with N production stages each. Each machine M_i receives its intermediate products from N upstream buffers $B_{i,1}, \ldots, B_{i,N}$. The products flow through the network in the unidirectional manner.

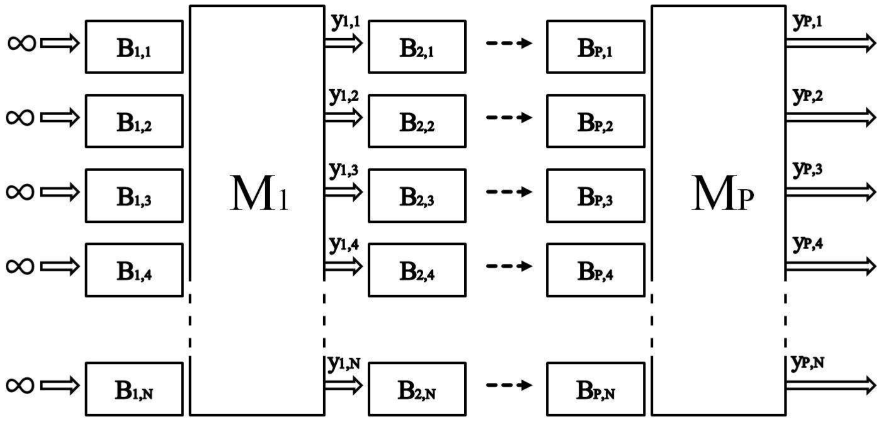


Figure 4.3: Schematic of a flexible production line

4.3.1 Flow model

The flow model of each production stage of a flexible line (Figure 4.3) in discrete time is defined as

$$y_{i,j}(k+1) = y_{i,j}(k) + \beta_{i,j}(k)u_{i,j}(k), \ \forall k, i, j, \tag{4.14}$$

where $i = 1, .., P$ is the machine number, $j = 1, ..., N$ is the processing stage (product type) number of machine i, $y_{i,j}(k) \in \mathbb{R}$ is the cumulative output of machine i in processing stage j in time k, $u_{i,j}(k) \in \mathbb{R}$ is the control input of machine i in processing stage j and $\beta_{i,j}(k) = \mu_{i,j} + f_{i,j}(k)$ where $\mu_{i,j}$ is a positive constant that represents the processing speed of the machine i at its stage j and $f_{i,j}(k) \in \mathbb{R}$ is an unknown external disturbance affecting the performance of the ith machine at its stage j. The equation (4.14) present a general model that can describe a product flow for a wide range of production stages of manufacturing networks. Further specific assumptions on system (4.14) are given in this section.

Under the assumption that there is always sufficient raw material to feed the input buffers $B_{1,j}$, the control aim is to track the non-decreasing cumulative production demands given by (4.2) on each output of the multi-product manufacturing line.

In order to give a solution to this tracking problem we consider the following control algorithm:

$$
\begin{array}{l}
\{q_i(k) = B_{i,j}\} \\
\textbf{if } \varepsilon_{i,j}(k) > 0 \textbf{ and } w_{i,j}(k) \geq \beta_{i,j}(k) \textbf{ then} \\
u_{i,j}(k) = 1, \\
u_{i,s}(k) = 0, \forall s \neq j,\ s, j = 1 \ldots, N, \\
q_i(k+1) = B_{i,j}, \\
\textbf{end} \\
\textbf{if } (\varepsilon_{i,j}(k) \leq 0 \textbf{ or } w_{i,j}(k) < \beta_{i,j}(k)) \textbf{ and} \\
\exists s \neq j : \varepsilon_{i,s}(k) > 0 \textbf{ and } w_{i,s}(k) \geq \beta_{i,s}(k) \textbf{ then} \\
u_{i,j}(k) = 0, \\
u_{i,s}(k) = 1, \\
q_i(k+1) = B_{i,s}, \\
\textbf{end} \\
\textbf{if } (\varepsilon_{i,s}(k) \leq 0 \textbf{ or } w_{i,s}(k) < \beta_{i,s}(k)),\ \forall s \textbf{ then} \\
u_{i,j}(k) = 0, \\
u_{i,s}(k) = 0, \forall s \neq j,\ s, j = 1 \ldots, N, \\
q_i(k+1) = 0 \\
\textbf{end}
\end{array}
\tag{4.15}
$$

where $q_i(k)$ is the internal variable representing the current buffer that M_i is processing, $w_{i,j}(k)$ is the buffer content of $B_{i,j}$. For the current time step $\beta_{i,j}(k)$ is the minimal raw material content in buffer $B_{i,j}$, such that machine M_i is able to process if required at this stage. Note that $B_{1,j}$ is considered to always contain sufficient raw material. Thus the buffer content condition $w_{1,j}(k) \geq \beta_{1,j}(k)$ is assumed to be always satisfied. The demand tracking error for each product type j at each stage of M_i is given by:

$$
\varepsilon_{i,j}(k) = \varepsilon_{i+1,j}(k) + w_{di+1,j} - w_{i+1,j}(k), \tag{4.16}
$$

$$
\varepsilon_{P,j}(k) = y_{dj}(k) - y_{P,j}(k), \tag{4.17}
$$

where $i = 1, \ldots, P-1$, and $j = 1, \ldots, N$. Here $w_{i+1,j}(k) = y_{i,j}(k) - y_{i+1,j}(k)$ is the buffer content of buffer $B_{i+1,j}$ and $w_{d_{i+1,j}}$ is the constant that represents the desired buffer level (extra stock) of buffer $B_{i+1,j}$.
For further analysis, let us rewrite flow model (4.14) in a closed-loop with (4.15) in terms of demand tracking errors as

$$
\begin{aligned}
\Delta\varepsilon_{i,j}(k) &= v_{dj} + \Delta\varphi_j(k) - \beta_{i,j}(k)u_{i,j}(k), \\
&\quad \forall j = 1, \ldots, N,\ i = 1, \ldots, P
\end{aligned}
\tag{4.18}
$$

where $\Delta\varepsilon_{i,j}(k) = \varepsilon_{i,j}(k+1) - \varepsilon_{i,j}(k)$.

Notice that machine M_i operates at each production step j under a processing speed of $\mu_i + f_i(k)$ lots per time unit, which is the same for each production stage of the machine, but it can differ from the other machines in the network.

For system (4.18) the following assumptions are made.

Assumption 6. *(Boundedness of perturbations) There are constants c_1, c_2, c_3 and c_4 such that*

$$c_1 < \Delta\varphi_j(k) < c_2,\ \forall k, j = 1, \ldots, N \tag{4.19}$$

$$c_3 < f_i(k) < c_4\ \forall k,\ i = 1, \ldots, P. \tag{4.20}$$

From *Assumption 3*, it follows that $W_{i,j}(k) = \Delta\varphi_j(k) - f_i(k)$ satisfies

$$\alpha_1 < W_{i,j}(k) < \alpha_2,\ \forall k, \tag{4.21}$$

with $\alpha_1 = c_1 - c_4$ and $\alpha_2 = c_2 - c_3$.

Assumption 7. *(Capacity condition) Constants c_1, c_2, c_3 and c_4 satisfy the following inequalities*

$$c_1 > -v_{dj},\ \forall j = 1, \ldots, N, \tag{4.22}$$

$$\alpha_2 < \mu_i - v_{dj},\ \forall i = 1, \ldots, P, \tag{4.23}$$

and the following condition (Capacity Condition) holds for each M_i in the network

$$0 < \frac{1}{\mu_i + f_i(k)} \sum_{j=1}^{N} (v_{dj} + \Delta\varphi_j(k)) < 1,\ \forall k, i. \tag{4.24}$$

Note that the physical meaning of the above mentioned assumptions is similar to the one discussed in detail in Chapter 2, Section 2.3.

One of the important physical limitations in the network is the buffer content restriction. In our model, in order for the positive control action ($u_{i,j}(k) = 1$) of the selected production stage ($B_{i,j}$) of M_i to take place, the buffer of this stage must satisfy the following condition on its content[3]

$$w_{i,j}(k) \geq \beta_{i,j}(k),\ \forall i = 2, \ldots, P,\ j = 1, \ldots, N. \tag{4.25}$$

Thus, from (4.16) and (4.25), the following demand tracking error condition holds

$$\varepsilon_{i+1,j}(k) \geq \beta_{i+1,j}(k) - w_{d_{i+1,j}} + \varepsilon_{i,j}(k),$$

where $i = 1, \ldots, P-1$, $j = 1, \ldots, N$, and $w_{d_{i,j}}$ satisfies the following assumption:

Assumption 8. *(Desired buffer content condition) The constants $w_{d_{i,j}}$ comply with the following inequality*

$$\begin{aligned} w_{d_{i,j}} \geq\ & \mu_{i,j} + N\mu_{i-1,j} + (N+1)c_4 \\ & +(N-1)(c_2 - c_1), \end{aligned} \tag{4.26}$$

From (4.26) it follows that $w_{d_{i,j}} > \beta_{i,j}(k)$, $\forall k$.

[3]By (4.25) we imply that in order to start producing each production stage j of machine i is restricted to take a certain nonzero amount of products from its upstream buffer. This amount is defined by the processing speed of the machine from the current time k till time $k+1$, which exclusively for the buffer content restriction of (4.15) is assumed to be known in advance. For more discussion on buffer content restriction see Chapter 3, Section 3.2.1.

4.3.2 Results on performance

In this section we present the results respecting the demand tracking error trajectories behavior of flow model (4.18).

Theorem 7. *Assume that the discrete time system defined by (4.18) satisfies Assumptions 6, 7, and 8. Then all solutions of (4.18) are ultimately bounded by*

$$\limsup_{k\to\infty} \sum_{j=1}^{N} (\varepsilon_{i,j}(k) - v_{dj} - \alpha_2) \leq 0, \tag{4.27}$$

$$\liminf_{k\to\infty} \varepsilon_j(k) \geq v_{dj} + \alpha_1 - \mu_j. \tag{4.28}$$

Proof. see Appendix G. □

Note that by replacing $v_{d_j} + \Delta\varphi_j(k)$ by $v_d + \Delta\varphi(k)$ this result can be also extended to a re-entrant production line serving one product type. From (4.27), and (4.28) it can be deduced that for the buffer content $w_{i,j}(k)$ of each buffer $B_{i,j}$ defined by (4.16), it holds that

$$\begin{aligned}\limsup_{k\to\infty} w_{i,j}(k) \quad &\leq \quad (N-1)\mu_i + N(\alpha_2 - \alpha_1) \\ &\qquad + \mu_{i-1} + w_{d_{i,j}}, \forall i = 2, \ldots, P. \end{aligned} \tag{4.29}$$

Now, in order to support the present development let us present simulation results.

4.3.3 Simulation examples

Consider the following example of a flexible production line consisting of 2 manufacturing machines with 2 production stages each (see Figure 4.3). The line is operating under surplus-based regulators (4.15). The processing speed of each machine is set to $\mu_i + f_i(k) = (10, 5)$ (lots per time unit), the desired buffer content of each buffer is selected considering (4.26) as $w_{d_2} = (w_{d_{2,1}}, w_{d_{2,2}}) = (26, 26)$ (lots), and the mean demand rate for each product type $v_{dj} = 2$ (lots per time unit) with fluctuation rate of $\Delta\varphi_j(k) = 0.4\sin(90k)$. The demand tracking error of each machine in the line is depicted in Figure 4.4. Here the initial conditions ($y_{1,1}(0)$, $y_{1,2}(0)$, $y_{2,1}(0)$, $y_{2,2}(0)$) are set to the zero value and $y_{d0} = 100$ (lots). After the first 245 time steps for product type 1 and 241 time steps for product type 2 , as it is shown in Figures 4.4 and 4.5, the system reaches its steady state. Demand tracking errors are maintained inside $[-8.4, 13.2]$ lots for M_1, and $[-3.4, 8.2]$ lots for M_2 (see the dashed lines of Figure 4.4), which satisfy the bounds given by (4.27) and (4.28). Figure 4.5 shows the buffer content of each $B_{i,j}$ in the network. After some transient behavior the inventory level of each buffer is maintained inside the obtained bound (4.29).
Another experimental result is presented in Figure 4.6. This two graphics show the relation between the upper bound on the demand tracking errors $\varepsilon_{2,1}(k)$ and $\varepsilon_{2,2}(k)$, and the desired buffer content of the network from the previous example. Here it can be observed that the amount of extra storage for intermediate products has only limited influence on the tracking precision of the network and the

threshold value of this influence is given by (4.26). In conclusion, the presented simulation results reflect the desired flow model behavior, i.e., all the values assigned to the parameters used in this section are consistent with the assumptions of Section 4.3.1 and the outcome of the simulation examples satisfy the theoretical results.

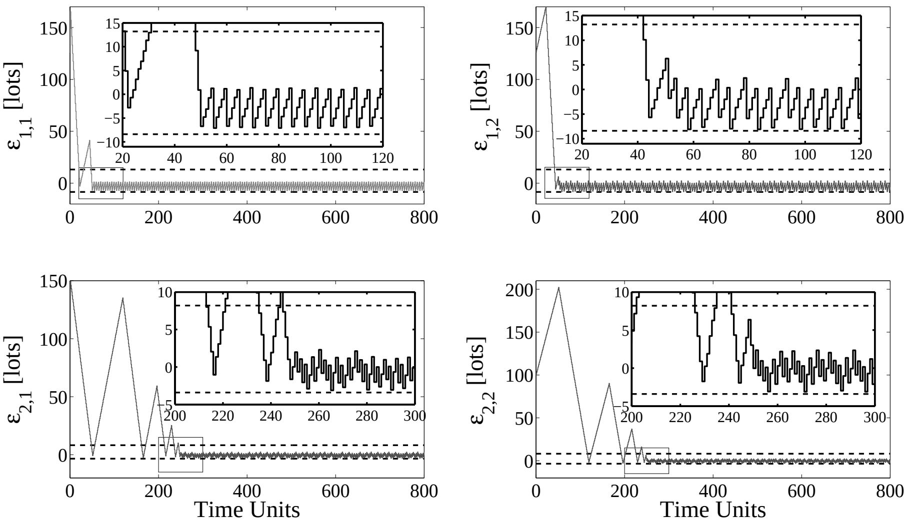


Figure 4.4: Demand tracking errors and the obtained bounds (dotted lines), with $v_{dj} = 2$, $\Delta\varphi_j(k) = 0.4\sin(90k)$,$\forall j = 1,2$, $w_{d_2} = (26,26)$, $\mu_i + f_i(k) = (10,5)$, and $y_{d0} = 100$.

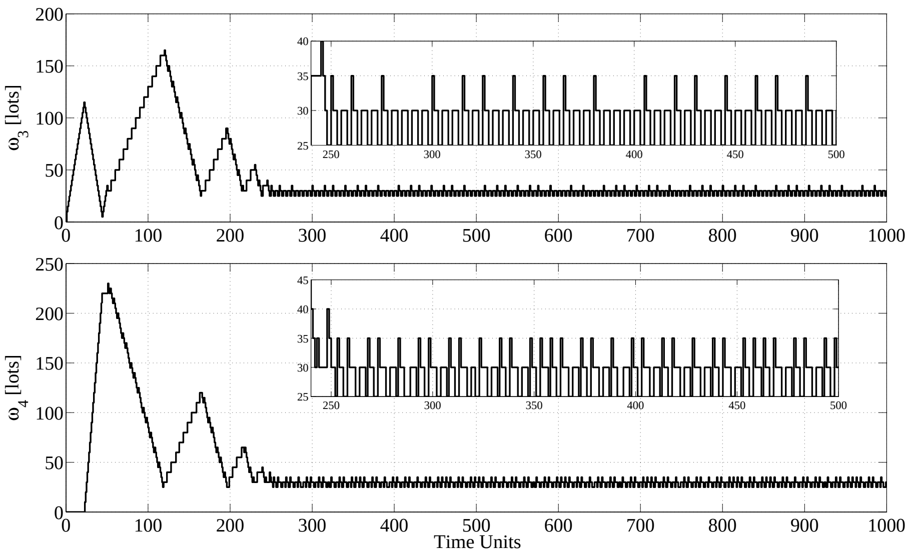


Figure 4.5: Buffer contents, with $v_{dj} = 2$, $\Delta\varphi_j(k) = 0.4\sin(90k)$, $\forall j = 1,2$, $w_{d_2} = (26,26)$, $\mu_i + f_i(k) = (10,5)$ and $y_{d0} = 100$.

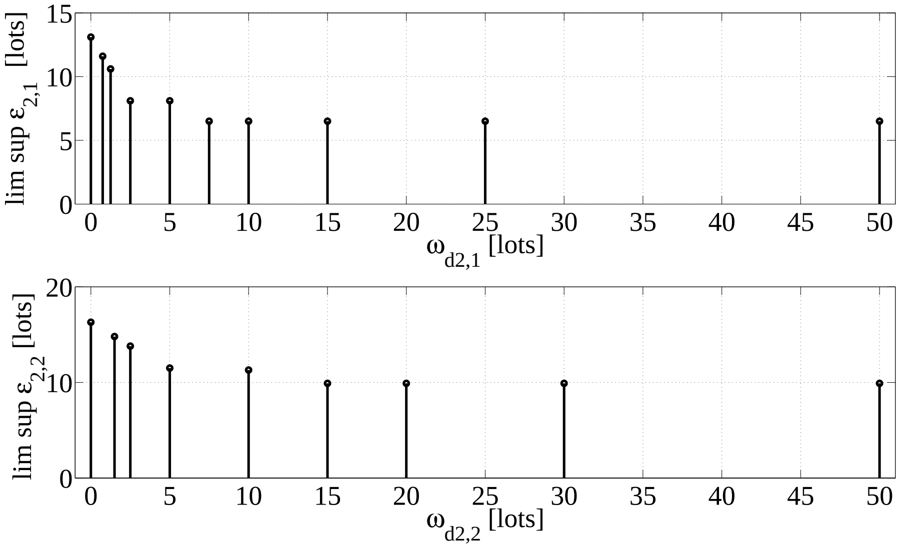


Figure 4.6: Upper bound on output demand tracking errors vs. desired buffer contents, with $v_{dj} = 3.3$ [lots/time unit], $\mu_{1,j} + f_{1,j}(k) = (10, 5)$ [lots/time unit], $\mu_{2,j} + f_{2,j}(k) = (5, 10)$ [lots/time unit].

4.4 Conclusions

The performances of a multi-product manufacturing network operated under surplus-based pull control has been studied. Developed results show boundedness for trajectories of each demand tracking error for one flexible machine considering that each production stage has a variable processing speed. Also bounds on the demand tracking error of each stage of a multi-product manufacturing line were presented. For a line it was considered that each production machine has a variable processing speed. Simulation examples were presented and discussed in order to illustrate and support analytical results. One of the important outcomes of these examples is the relation between the amount of extra intermediate product storage and the demand tracking error. It was shown that extra storage capacity has a limited influence on the demand tracking error. The threshold value on the desired capacity for each buffer content was provided in *Assumption 5* of the flow model analysis.

5

Re-entrant manufacturing networks

This chapter is partly based on Starkov et al. (2011c) and Starkov et al. (2012c).

Abstract | Motivated by the problem of scheduling in large semiconductor manufacturing facilities we study the performance of multi re-entrant production networks. In this chapter we present results on the influence of perturbations, buffer inventory levels and the number of manufacturing stages on the demand tracking error of each machine in the network operated under a surplus-based production control policy. First the performance of a single machine with multiple production stages is analyzed. As a result we provide bounds on the steady-state production demand tracking accuracy of each stage as well as bounds on the content of each intermediate buffer. Furthermore a detailed dependency relation between the efficiency of the production tracking and the intermediate inventory levels of a re-entrant machine is obtained. Then our analysis is extended to a line of multi re-entrant machines. For this network structure accuracy bounds on demand tracking error of each stage as well as intermediate inventory levels are obtained. Finally by means of simulation examples we show that the obtained results can be used as a reference tool in practice.

5.1 Introduction

A manufacturing network consisting of workstations interconnected in a non-acyclic[1] manner, where at each station one machine serves several buffers, can be frequently encountered as a part of an industrial production process. For example, in case of semiconductor manufacturing it is typical to observe that a production network exhibits a multi re-entrant behavior. In order to produce a wafer several layers of

[1]Here the term non-acyclic refers to the part flow route in the network, which is re-entrant (see Kumar and Seidman (1990))

semiconductor material have to be put together, which implies that the product (wafer) has to undergo several times through the same process before it is finally ready (see, e.g., Montoya-Torres (2006)).
Analysis on control and performance of networks that present re-entrant behavior in their production process has always attracted much attention of manufacturers, as well as of researchers. Thus re-entrant networks are widely studied and a lot of valuable control and performance evaluation approaches including queuing theory, Petri nets, dynamic programming, linear programming, hybrid systems were proposed and some of them have been implemented (for surveys see, e.g., Gershwin (2000), Ortega and Lin (2004), Sarimveis et al. (2008), Danping and Lee (2011), and Scholz-Reiter et al. (2011)).

Re-entrant networks are commonly associated with semiconductor manufacturing processes, where performance factors such as throughput, cycle time, utilization, work in progress (WIP) are of a great importance. It is also important to notice that in the current market manufacturers try to interact more closely with customers, which means that customer satisfaction is also an important performance indicator to be considered. In order to do that, in our research the analysis on performance of a re-entrant network is based on evaluation of the accuracy of production tracking policies. Specifically, given the presence of unknown but bounded production speed perturbations as well as demand rate fluctuations, we investigate how close the cumulative production output of a re-entrant network follows its cumulative production demand under a surplus-based[2] pull control policy. Also, by means of analytical tools, we investigate the relation between the production demand tracking accuracy[3] and the intermediate product inventory level of a re-entrant network.

The motivation for trying to find this relation lies in the lack of an analytical link between the production surplus at each stage and several important factors such as: base stock level, production speed, number of stages in each machine, number of buffers and machines in the re-entrant network. For implementation purposes it is important to keep this relation as general as possible, for example in terms of demand tracking accuracy bounds with respect to base stock levels of the network. Thus use of this relation as a reference tool can reduce the need for simulation based performance analysis and consequently simplify factory planning, as well as scheduling processes.

In order to achieve our goal we use classical tools from control theory. The production flow process is described by means of difference equations and in order to analyse its performance, an approach using Lyapunov theory is exploited (see, e.g., Khalil (2002), Dashkovskiy et al. (2011), Starkov et al. (2012d), and references there in).
Each machine in a re-entrant network is responsible for several production stages. At each stage the machine coordinates its individual production with those of

[2] In the surplus-based control, decisions are made based on the demand tracking error, which is the difference between the cumulative demand and the cumulative output of the system (see e.g., Bonvik et al. (1997)).

[3] Note that what we call production demand tracking accuracy is closely related to a commonly used service level term (see, e.g., Bonvik et al. (1997), González et al. (2012))

the rest of the system. While the machine is controlled by a surplus-based pull controller, the objective of each of its production stage may be viewed as manufacturing a sufficient quantity of parts to satisfy the demand of its immediate downstream production stage (not necessarily belonging to the same machine) and some extra amount as back-up material storage in its downstream buffer. The production strategy itself is intuitive and it can be associated with a wide range of existing techniques such as basestock policy (see, e.g., Bonvik et al. (1997)), hedging point policy (see, e.g., Gershwin (2000)), and clearing policy (see, e.g., Kumar and Seidman (1990)).
To the best of our knowledge, concerning the previous results on performance analysis of surplus-based approaches (see, e.g., Kumar (1993); Lu and Kumar (1991); Quintana (2002); Subramaniam et al. (2009); Nilakantan (2010); Boukas (2006); Sarimveis et al. (2008); Kogan and Perkins (2003); Savkin and Evans (2002); Starkov et al. (2012d), the novelty of our results can be summarized as follows. The proposed production model is considered in discrete time. The production speed of each machine is defined as deterministic with bounded perturbations. The future production demand is assumed to be unknown and with bounded fluctuations. As a result, for a single re-entrant manufacturing machine of N production stages strict, so-called "worst case" bounds on demand tracking errors and inventory content of the unlimited capacity intermediate buffers are obtained. It is also shown that the obtained bounds on demand tracking errors are satisfied even when the intermediate buffers of the re-entrant machine is of limited capacity. Further, the relation between the demand tracking error accuracy on the output of each stage of a re-entrant machine and its inventory level is investigated and derived in the form of worst case bounds. Extending this strategy to a network of P machines with N production stages each, we investigate the existence of the accuracy bounds on the demand tracking error and inventory levels for each stage of each machine and its neighboring buffers, respectively.
This chapter presents our results from Starkov et al. (2011c) and Starkov et al. (2012c) on the analysis of surplus-based pull policies that concern a single machine and a line of re-entrant machines with both being driven by the cumulative production demand. This chapter is organized as follows. First, in Section 5.2 the flow model of one manufacturing machine with surplus-based pull control is presented. The detailed analytical and simulation-based analysis on performance is developed in this section. Then the flow model of a re-entrant manufacturing line with surplus-based pull control is analyzed in Section 5.3. In this section analytical as well as simulation-based performance analysis is provided for a re-entrant manufacturing line. Finally, Section 5.4 contains conclusions.

5.2 Single re-entrant machine

Figure 5.1 shows a schematic picture of one re-entrant machine M with N production stages. The output of each stage is given by y_1 till y_N, respectively. The machine is interconnected by $N-1$ buffers, i.e., $B_2 \dots B_N$. Each buffer stores the intermediate product that is produced by the upstream stage of the machine. It is assumed that the machine has always sufficient supply of raw material on its input, which is the reason why the buffer of stage 1 is omitted in Figure 5.1.

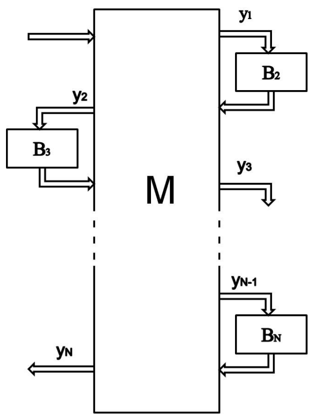


Figure 5.1: One re-entrant machine M with $N-1$ buffers $B_2, \ldots, B_N$.

5.2.1 Flow model

The discrete time model of a simple manufacturing machine with one production stage is described in detail in Chapter 2 of this thesis. This model can be extended for each production stage of one re-entrant machine (see Figure 5.1) as

$$y_j(k+1) \quad = \quad y_j(k) + \beta_j(k)u_j(k), \tag{5.1}$$

where all the events within the model occur at given time instances and k represents the current time so that the time step between all the events is constant. Here $y_j(k) \in \mathbb{R}$ is the cumulative output of the machine in processing stage j in time k, $u_j(k) \in \mathbb{R}$ is the control input of the machine in processing stage j and $\beta_j(k) > 0$ for all k is given by $\beta_j(k) = \mu_j + f_j(k)$. The positive constant μ_j represents the processing speed of the machine at the stage j and $f_j(k)$ is an unknown external disturbance affecting the performance of the machine at stage j.The equation (5.1) present a general model that can describe a product flow for a wide range of production stages of manufacturing networks. Further specific assumptions on system (5.1) are given in this section.

Under the assumption that there is always sufficient raw material to feed the machine, the control aim is to track the non-decreasing cumulative production demand on its output. We define the cumulative production demand by using $y_d(k) \in \mathbb{R}$ given by

$$y_d(k) = y_{d0} + v_d k + \varphi(k), \tag{5.2}$$

where y_{d0} is a positive constant that represents the initial production demand, v_d is a positive constant that defines the average desired demand rate, and $\varphi(k) \in \mathbb{R}$ is the bounded fluctuation that is imposed on the linear demand $v_d k$.

In order to give a solution to this tracking problem we consider a controller based on the demand tracking error of each stage. The machine can only work at one stage at a time. The controller arbitrary selects the stage at which the machine must work[4], from those where production is needed. The machine works at this stage till its product demand is satisfied. Then the controller again selects a stage for the machine to work at. In case the product demand is satisfied at all stages, the controller idles the machine.

The above mentioned strategy can be formulated by the following control algorithm (see next paragraph for a summary):

$$
\begin{aligned}
&\{q(k) = B_j\},\\
&\textbf{if}\, \varepsilon_j(k) > 0 \,\textbf{and}\, w_j(k) \geq \beta_j(k) \,\textbf{then}\\
&u_j(k) = 1,\\
&u_s(k) = 0,\ \forall s \neq j,\ s, j = 1, \ldots, N,\\
&q(k+1) = B_j,\\
&\textbf{end}\\
&\textbf{if}\ (\varepsilon_j(k) \leq 0 \,\textbf{or}\, w_j(k) < \beta_j(k))\,\textbf{and}\\
&\exists\, s \neq j : (\varepsilon_s(k) > 0 \,\textbf{and}\, w_s(k) \geq \beta_s(k))\ \textbf{then}\\
&u_j(k) = 0,\\
&u_s(k) = 1,\\
&q(k+1) = B_s,\\
&\textbf{end}\\
&\textbf{if}\ (\varepsilon_s(k) \leq 0 \,\textbf{or}\, w_s(k) < \beta_s(k))\,,\ \forall s \,\textbf{then}\\
&u_j(k) = 0,\\
&u_s(k) = 0,\ \forall s \neq j,\ s, j = 1, \ldots, N,\\
&q(k+1) = 0,\\
&\textbf{end}
\end{aligned}
\qquad (5.3)
$$

where $q(k)$ is the internal variable that specifies the buffer that machine M is processing, $\beta_j(k)$ is the minimal raw material content in buffer B_j, such that at time k machine M is able to process if required, $\varepsilon_j(k)$ is the current demand tracking error at stage j, and $w_j(k)$ is the current buffer content of B_j. Note that B_1 is considered to always have sufficient raw material, which is the reason why it is not present in the algorithm.

Algorithm (5.3) can be summarized as follows:

- The machine can only work on one buffer at a time.
- The control input $u_j(k)$ of each production stage j can only take the value of 0 (stop) or 1 (produce).

[4]The results presented in this chapter only concentrate on the steady state behavior of a re-entrant network. In consequence the scheduling problem within the machines play no significant role as long as within a certain period of time all the production stages of each machine in the network are served. Thus in algorithm (5.3) a policy to select a production stage s in case their exist several stages with $\varepsilon_s(k) > 0$ is not detailed.

- The $u_j(k)$ receives the value of 1 only if production stage j needs to produce ($\varepsilon_j(k) > 0$) and its buffer is not empty ($w_j(k) \geq \beta_j(k)$).

- The machine will remain at its current state $q(k) = B_j$ while all the conditions of the state are satisfied, i.e., while $\varepsilon_j(k) > 0$ and $w_j(k) \geq \beta_j(k)$.

- The value of 0 is given to the control input of stage j if at least one of the above mentioned conditions for the current state $q(k) = B_j$ is unsatisfied.

- The change in the value of the control signal of stage j also implies a change in machine's state $q(k)$.

- The machine has $N + 1$ states. This is due to that N corresponds to the total number of production stages that M can be working in, which directly relate to the states of the machine, plus the idle state ($q(k) = 0$).

Note that algorithm (5.3) does not consider limitations of buffer sizes in the network. These buffer restrictions can be easily included in (5.3) and they are also considered in Appendix H, Case 1, Subcase 3.
The production error at each stage of machine M is given by:

$$\begin{aligned} \varepsilon_j(k) &= \varepsilon_{j+1}(k) + (w_{d_{j+1}} - w_{j+1}(k)), \quad (5.4) \\ &\quad \forall j = 1, \ldots, N-2, \\ \varepsilon_{N-1}(k) &= \varepsilon_N(k) + (w_{d_N} - w_N(k)), \quad (5.5) \\ \varepsilon_N(k) &= y_d(k) - y_N(k). \quad (5.6) \end{aligned}$$

Here $w_{j+1}(k) = y_j(k) - y_{j+1}(k)$ is the buffer content of buffer B_{j+1} and $w_{d_{j+1}}$ is the constant that represents the desired inventory level (base stock level) of buffer B_{j+1}. For further analysis, let us rewrite flow model (5.1) in a closed-loop with (5.3) in terms of demand tracking errors as

$$\Delta\varepsilon_j(k) = v_d + \Delta\varphi(k) - \beta_j(k)u_j(k), \quad (5.7)$$

where for all $j = 1, \ldots, N$, $\Delta\varepsilon_j(k) = \varepsilon_j(k+1) - \varepsilon_j(k)$ and $\Delta\varphi(k) = \varphi(k+1) - \varphi(k)$. We assume that system (5.7) satisfies the following conditions.

Assumption 9. *(Boundedness of perturbations)*
There are constants c_1, c_2, c_3 and c_4 such that

$$c_1 < \Delta\varphi(k) < c_2, \ \forall k \in \mathbb{N}, \quad (5.8)$$
$$c_3 < f_j(k) < c_4, \ \forall k \in \mathbb{N}, \ j = 1, \ldots, N. \quad (5.9)$$

From Assumption 9, it follows that $W_j(k) = \Delta\varphi(k) - f_j(k)$ satisfies

$$\alpha_1 < W_j(k) < \alpha_2, \ \forall k \in \mathbb{N}, \ j = 1, \ldots, N, \quad (5.10)$$

with $\alpha_1 = c_1 - c_4$ and $\alpha_2 = c_2 - c_3$.

Assumption 10. *(Capacity condition)*
The constants c_1, c_2, c_3 and c_4 in 5.8 and 5.9 satisfy the following inequalities

$$c_1 > -v_d, \tag{5.11}$$
$$\alpha_2 < \mu_j - v_d, \ \forall j = 1, \ldots, N, \tag{5.12}$$

and the following condition (also know as capacity condition) holds

$$0 < (v_d + \Delta\varphi(k)) \sum_{j=1}^{N} \frac{1}{\mu_j + f_j(k)} < 1, \ \forall k. \tag{5.13}$$

By (5.11), (5.12), and (5.13) we state that, in the presence of market fluctuations bounded by (c_1, c_2), the production demand rate can only be positive, the production speed at each manufacturing stage of the machine is always faster than the demand rate of its product and in general the processing speed of the machine is faster than its demand rate, respectively. It is important to notice that machine M at each process step j has a processing speed of $\mu_j + f_j(k)$ lots per time unit, which can differ from the rest of the processing steps. The physical meaning of the above mentioned assumptions is similar to the one discussed in detail in Section 2.3.
The buffer content condition[5] is

$$w_j(k) \geq \beta_j(k), \ \forall j = 2, \ldots, N. \tag{5.14}$$

Thus, from (5.4) and (5.14), the following demand tracking error condition holds

$$\varepsilon_j(k) \geq \beta_j(k) - w_{d_j} + \varepsilon_{j-1}(k), \ \forall j = 2, \ldots, N,$$

where w_{d_j} is assumed to satisfy the following condition.

Assumption 11. *(Desired buffer content condition)*
The constants w_{d_j} comply with the following inequality

$$w_{d_j} \geq \mu_j + c_4.$$

From *Assumption 11* it follows that for all k,

$$w_{d_j} > \beta_j(k), \forall j = 2, \ldots, N. \tag{5.15}$$

If condition (5.14) is not satisfied, then

$$\varepsilon_{j-1}(k) \overset{(5.4,5.15)}{>} \varepsilon_j(k), \ \forall j = 2, \ldots, N. \tag{5.16}$$

[5]This condition refers to a minimal buffer inventory level w_j that is needed in order to start the manufacturing process of M at stage j. For more details see Chapter 3, Section 3.2.1 and Chapter 4 Section 4.3.1

5.2.2 Results on performance

In this section we present results on the behavior of the demand tracking error trajectories of flow model (5.7).

Theorem 8. *Assume that the discrete time system defined by (5.7) satisfies Assumptions 9, 10, and 11. Then all solutions of (5.7) are ultimately bounded by*

$$\limsup_{k\to\infty} \sum_{j=1}^{N} \frac{\varepsilon_j(k) - v_d - \alpha_2}{\mu_j + c_3} \leq 0, \tag{5.17}$$

$$\liminf_{k\to\infty} \varepsilon_j(k) \geq v_d + \alpha_1 - \mu_j. \tag{5.18}$$

Proof. see Appendix H. □

Note that in Case 1, Subcase 3 of Appendix H we also show that Theorem 8 holds for networks with limited capacity buffers.
The obtained bounds can be visualized through a phase portrait of the demand tracking error trajectories shown in Figure 5.2, which was made for a single machine producing 2 product types. In this example, the product demand rate $v_{dj} = 0.49$ [lots/time unit] and the production rate at each stage μ_j=1[lot/ time unit]. Here the experiment starts with initial demand tracking errors $\varepsilon_1(0) = 2$ [lots] and $\varepsilon_2(0) = 1$ [lots]. It can be observed that first the controller activates stage 1 of M. The machine works at this stage till $\varepsilon_1(k) \leq 0$ and then switches to stage 2. Eventually the trajectories of the demand tracking errors enter the zone depicted by the rectangular triangle, where they remain for the rest of the experiment. The legs of this triangle are given by (5.18) and the hypotenuse by (5.17).

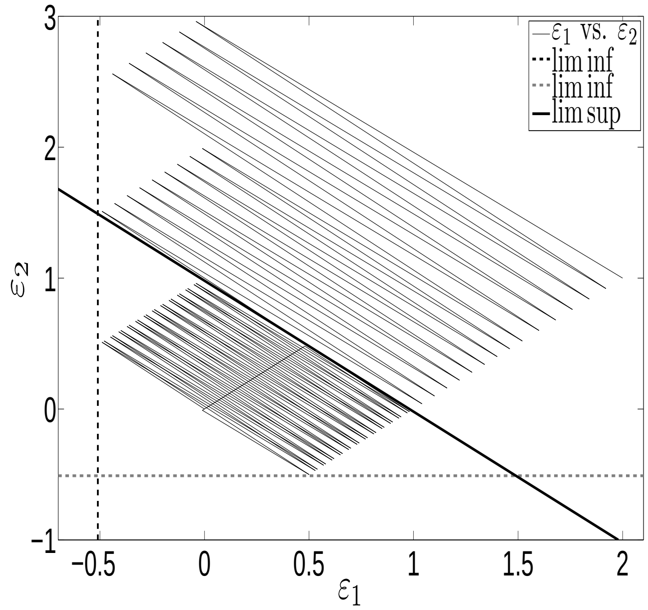


Figure 5.2: Demand Tracking Errors $\varepsilon_1(k)$ vs. $\varepsilon_2(k)$, with $v_d = 0.49$ [lots/time unit] and μ_j=1[lot/ time unit].

From (5.17) and (5.18), it can be deduced that for buffer content $w_j(k)$ of each buffer B_j defined by (5.4) it holds that

$$\begin{aligned}\limsup_{k\to\infty} w_j(k) \;\; &\leq \;\; (\mu_{j+1}+c_3)\sum_{s=1}^{N}\frac{\mu_s-\alpha_1+\alpha_2}{\mu_s+c_3} \\ &+\mu_j+w_{dj+1}+\alpha_2-\alpha_1. \qquad (5.19)\end{aligned}$$

Note that the bounds for the demand tracking error from Theorem 8 are also valid in case the intermediate buffers are of a limited capacity (see Appendix H, Case 1, Subcase 3), i.e. each buffer content is limited by

$$0 \leq w_j(k) \leq \gamma_j,$$

for all k. Here $j = 2, \ldots, N$, $\gamma_j = w_{dj} + \mu_j + c_3$, which is a constant representing an upper bound on the inventory level of B_j. Theorem 8 states that, given the assumptions 9, 10, and 11 are satisfied, then the steady-state demand tracking errors will remain within the same tracking bounds as the ones specified by (5.17) and (5.18) regardless of the initial conditions and capacity limitations of a line. Clearly starvation in the manufacturing line can be avoided by selecting a sufficiently large amount of base stock (w_{dj}). What happens if the base stock is low or even zero? The answer to this question is presented in the following theorem.

Theorem 9. *Assume that the discrete time system defined by (5.7) satisfies Assumptions 9, and 10. Then all solutions of (5.7) are ultimately bounded by*

$$\limsup_{k\to\infty}\sum_{j=1}^{N}\frac{\varepsilon_j(k)-v_d-\alpha_2-X_j}{\mu_j+c_3} \leq 0, \qquad (5.20)$$

$$\liminf_{k\to\infty}\varepsilon_j(k) \geq v_d+\alpha_1-\mu, \qquad (5.21)$$

where $X_j = \sum_{s=2}^{j}\max(\mu_s + c_4 - w_{ds}, 0)$ *for all* $j = 2, \ldots, N$ *and* $X_1 = 0$.

Proof. see Appendix I. □

In contrast with Theorem 8, Theorem 9 shows the complete relation between the base stock level and the demand tacking accuracy of the network. From (5.20) it can be observed that the tracking accuracy drastically decreases by decreasing the base stock level further than specified in (5.14).
Note that the upper bound from (5.20) can be also defined as

$$\begin{aligned}\limsup_{k\to\infty}\varepsilon_j(k) \leq \;\; &(\mu_j+c_3)\sum_{s=1}^{N}\frac{\mu_s-\alpha_1+\alpha_2+X_s}{\mu_s+c_3} \\ &+v_d+\alpha_2+X_j, \qquad (5.22)\end{aligned}$$

where for selected index j, index s takes any value from 1 till N except for the selected value of j. It is important to note that though bound (5.22) is simpler to interpret, it is less accurate than bound (5.20).

From (5.22) and (5.21), it can be deduced that for buffer content $w_j(k)$ of each buffer B_j defined by (5.4) it holds that

$$\begin{aligned}\limsup_{k\to\infty} w_j(k) &\leq (\mu_j + c_3)\sum_{s=1}^{N}\frac{\mu_s - \alpha_1 + \alpha_2 + X_s}{\mu_s + c_3} \\ &\quad + X_j + \mu_{j-1} + w_{dj} + \alpha_2 - \alpha_1, \qquad (5.23)\end{aligned}$$

where $j = 2, \ldots, N$ and $s \neq j$.

5.2.3 Discrete event simulation

In this section the accuracy of the obtained demand tracking error bounds for the DT model is tested by means of a simulation example on the DE model of a re-entrant machine. The DE model was built by using the specification language called χ (see Beek et al. (2006)) developed at Eindhoven University of Technology[6].

Figure 5.3 shows the schematics of the DE model of one re-entrant machine operated under a surplus-based control. The circles represent the processes: raw material supply (G), product demand (D), controller (C), buffers for intermediate products (B) and buffer for final product (E). The rectangle represents the process M, which is the manufacturing machine of 3 stages. The wide and the thin arrows indicate the lot and the information flow directions, respectively. The re-entrant machine consists of 3 manufacturing stages and 2 buffers, and it produces one product type. Note that only one stage can be operational at a time.

Processes G, E and D are identical to the ones explained in Section 2.6. Process B is explained in Section 3.5. Process C uses a different control algorithm (see control algorithm (5.3)). Process M introduced in Section 2.6 is extended to a machine of 3 stages. From Figure 5.3, controller C sends the authorization to M through the index j to which the operating stage number is assigned. Every time the j^{th} stage of M produces a product, it is send to the corresponding buffer B and the stage number is sent to controller C. The controller increments a cumulative output counter of a stage every time it receives the j value from M. It recalculates the demand tracking error values for each stage using an event based algorithm, which is similar to (5.3).

The following example is analyzed. It is considered that a single manufacturing machine of 3 stages produces products based on a cumulative production demand value. For the event-based system, the production speed of each stage $j = 1, 2, 3$ is fixed to 1 lot per $\frac{1}{\mu_1+f_1} = \frac{1}{4+0.02\sin(0.25t)}$ time units, $\frac{1}{\mu_2+f_2} = \frac{1}{2+0.02\sin(1.5t)}$ time units and $\frac{1}{\mu_2+f_2} = \frac{1}{1+0.02\sin(3t)}$ time units. Here t represents the current value of time of the experiment. The initial output $y_j(0) = 0$ for all j. The production demand consists of its initial value y_{d0} of 10 lots and the demand rate of 1 lot every $\frac{1}{v_d+\Delta\varphi(t)} = \frac{1}{0.5+0.04\cos(0.5t)}$ units of time. The base stock levels w_{dj} are selected as $w_{d2} = 3$ and $w_{d3} = 2$ lots for all $j = 2, 3$. This is done in order to test the upper bound on steady-state demand tracking errors (5.22) when the Assumption 11 on buffer content is satisfied.

[6] For the details on implementation see Rooda and J.Vervoort (2007)

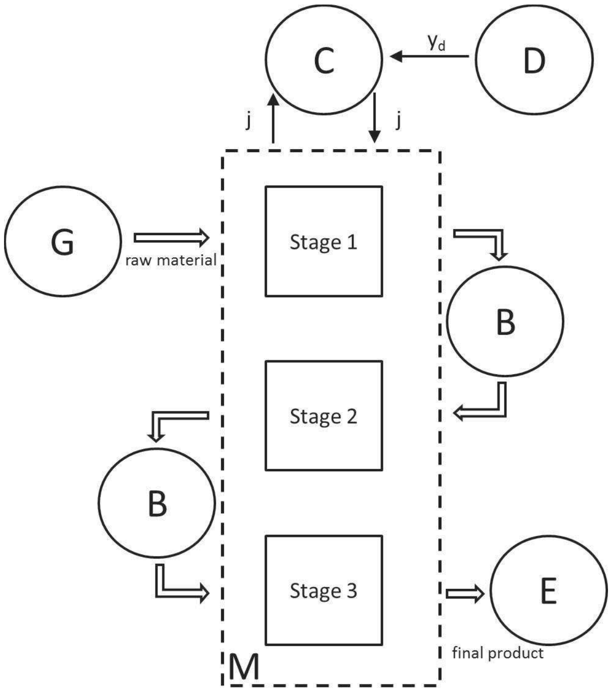


Figure 5.3: Schematics of the DE model of one re-entrant machine M with 3 stages.

Two DT model approximations of DE model are considered in this example. First the DE model is approximated by DT model with the production speeds of $\mu_j + f_j(k)$ lots per time unit and demand rate $v_d + \Delta\varphi(k)$ lots per time unit. The simulation run time is set to 500 time units.

The outcome of the experiment is as follows:

- From Figures 5.4 and 5.5 it can be observed that the duration of the transient behavior for the DE system is similar to that of the DT system.
- From Figure 5.4 it can be observed that the steady state demand tracking error trajectories of the DT and DE models are different, but they satisfy the theoretical bounds (5.17), and (5.18).
- This DT approximation of the DE model is inaccurate.
- The contents of the buffers for the settings with $w_{d2} = 3$ and $w_{d3} = 2$ lots are shown in Figure 5.5.
- In the DT model, the maximal content and the general pattern of product content variation in each buffer differs from the one of the DE model. The DE model describes the buffer content variations in a more detailed manner than the DT model. Note that in the DT model, the buffer content is calculated based on the difference between the upstream and the downstream cumulative outputs of the machines surrounding the buffer. Thus a product location is undistinguished from a buffer or its adjacent machine. Differently from DT, in the DE model the products from the buffer content are distinguished from the products in the machine.

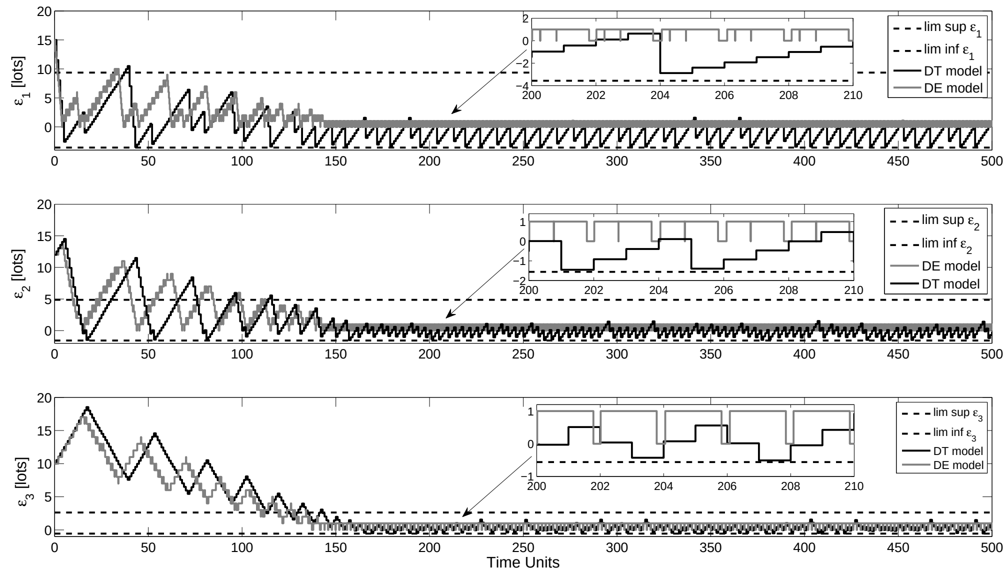


Figure 5.4: Demand Tracking Errors

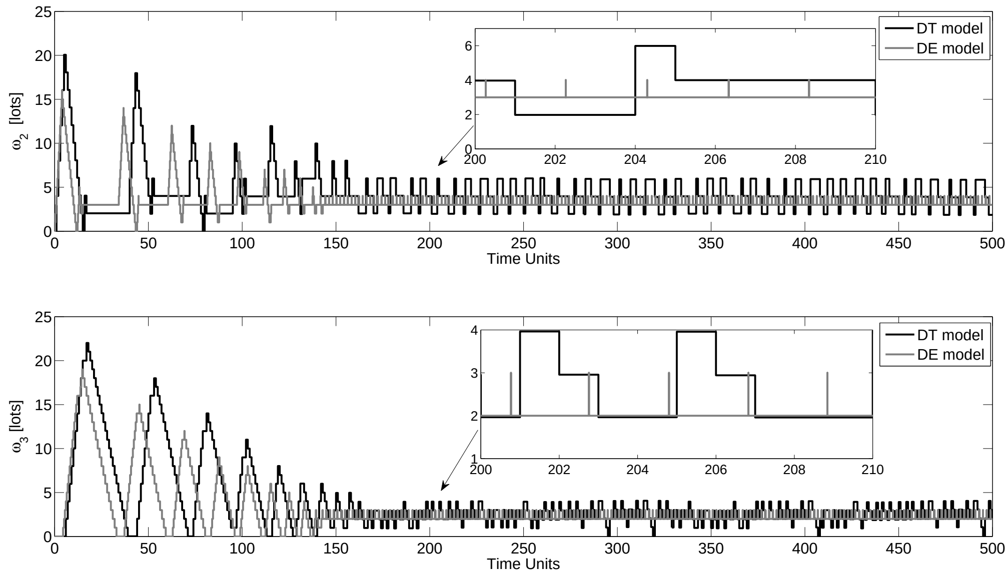


Figure 5.5: Buffer contents

Secondly, the same DE model is approximated by a DT model with different time scale. The time scale for the DT model is now selected as $k = \min_j \frac{1}{\mu_j + c_3} = \frac{1}{4.02}$. The simulation run time is set to 2010 time steps, which is equivalent to 500 time units of the previous experiment.

The outcome of the experiment is as follows:

- From Figures 5.6 and 5.7 it can be observed that the duration of the transient behavior for the DE system is identical to that of the DT system.

- From Figure 5.6 it can be noticed that the demand tracking error trajectories of the DT and DE models have a similar behavior. The theoretical bounds (5.22) and (5.21) are satisfied for the DT model, but require a different interpretation for the DE model.

- Similarly to the manufacturing line of Section 3.6, in this example the bounds of the DT model can be translated to the DE model bounds as the ceil values of (5.22) and (5.21).

- This DT approximation of the DE model is much more accurate than the one from the fist example.

- The contents of the buffer for the settings with $w_{d2} = 3$ and $w_{d3} = 2$ are shown in Figure 5.7.

- In the DT model, the maximal content and the general pattern of product content variation in each buffer is very similar to the one of the DE model. But the DE model describes the buffer content variations in a more detailed manner than the DT model. Note that in the DT model the buffer content is calculated based on the difference between the upstream and the downstream cumulative outputs of the machines surrounding the buffer. Thus a product location is undistinguished from a buffer or its adjacent machine. Differently from the DT in DE model the products from the buffer content are distinguished from the products in the machine.

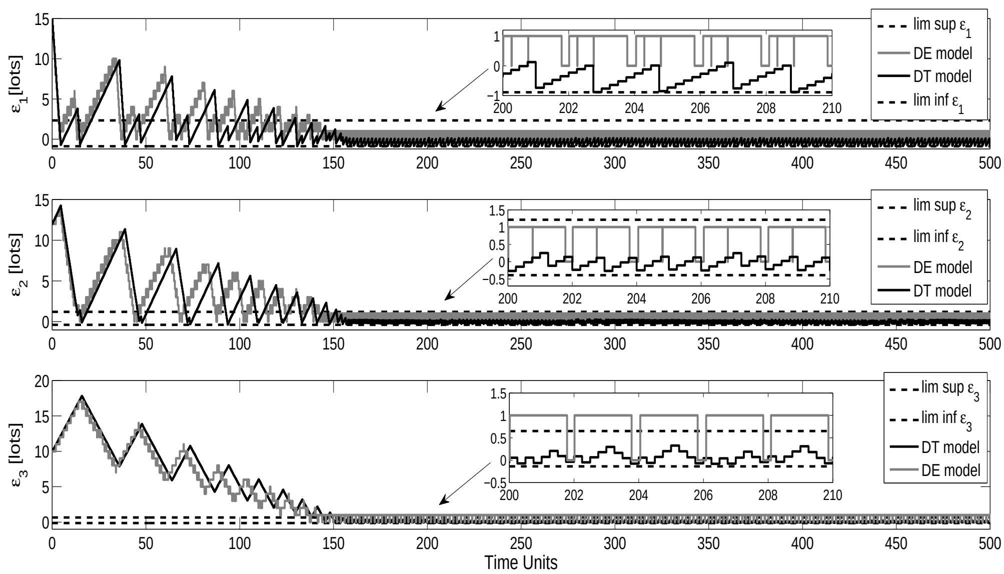


Figure 5.6: Demand Tracking Errors

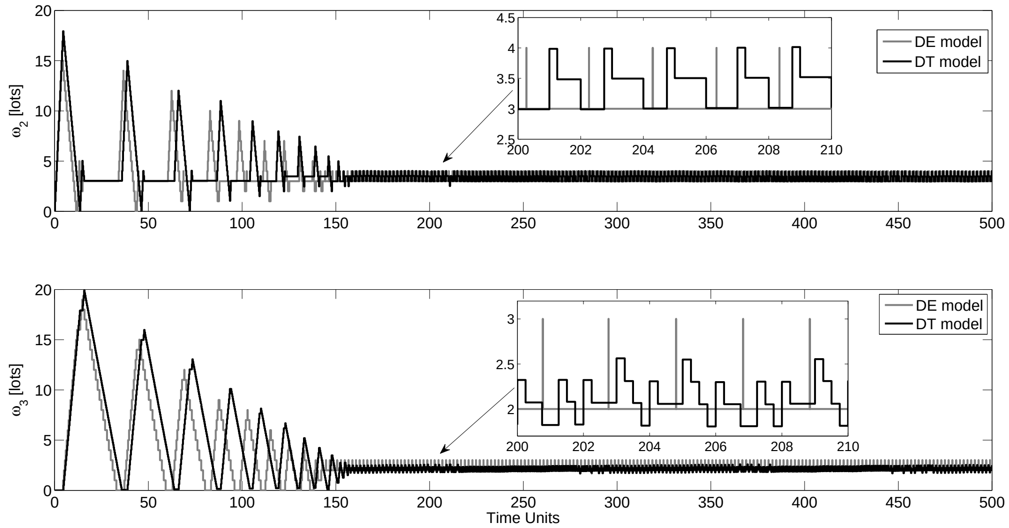


Figure 5.7: Buffers content

5.2.4 Simulation based performance analysis

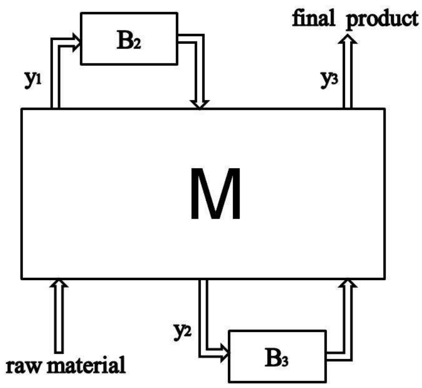


Figure 5.8: Schematics of one re-entrant machine M with 3 stages and 2 buffers.

In this section we present the results on the simulation-based comparison of 3 particular cases of surplus-based production control strategies applied to a network consisting of a single re-entrant manufacturing machine composed of 3 production stages ($j \in 1, 2, 3$) and 2 intermediate buffers (B_2, B_3) as shown in Figure 5.8. The selected strategies are particular cases of Hedging Point Policy (HPP), Conwip (CWIP) ,and modified Base Stock Policy (BSP) (for details see Appendix K). The performance criteria upon which these 3 policies will be compared are the steady state maximum and minimum values of demand tracking error of the output stage (i.e., $j = 3$) and the intermediate inventory level (i.e, $w_2 + w_3$) in the machine.

Description of the experiment

The following common assumptions are made for all the implemented policies (for more details on used flow models see Appendix K):

- The machine M produces a single part type.
- It is assumed that there is always sufficient raw material (see Figure 5.8) and the machine is never blocked on its final product output (y_3).
- The machine can only work with one buffer at a time.
- The control input u_j of each production stage can only take the value of 0 (stop) or 1 (produce).
- The control actions are executed every time step k.
- The controller takes the value of 1 only if the selected production stage needs to produce and its buffer is not empty.
- The value of 0 is given to the control input of stage j if its demand tracking error $\varepsilon_j(k) \leq 0$ or its adjacent buffer is empty.
- Once the production stage is selected, the machine will operate at this stage till its control input changes its value to 0.
- For all the policies, the demand tracking error of the output stage ($j = 3$) of the machine is given by (5.6).
- No setup times are considered in the models.

The demand tracking errors and the operation principle of HPP are described by (5.4)-(5.6) and (5.3), respectively (for more details see Appendix K).
Under CWIP policy the controller of the first stage ($j = 1$) limits the number of products in the network, the second stage produces products in a clearing manner and the output stage ($j = 3$) keeps track of a production demand on its output. Thus $\varepsilon_1(k) = w_{d_{total}} - w_{total}(k)$, $\varepsilon_2(k) = w_2(k)$ and $\varepsilon_3(k) = y_d(k) - y_3(k)$. For a fair comparison between the three policies, the desired inventory level in the network $w_{d_{total}}$ is interpreted as $w_{d_{total}} = w_{d_2} + w_{d_3}$ and $w_{total}(k) = y_1(k) - y_3(k)$.
Given $q(k) = B_2$ the control input $u_2(k) = 1$ only if $w_2(k) \geq \beta_2(k)$. As well as in HPP, the control action of the output stage $j = 3$ aims at production demand tracking.
In this comparative study, each stage of the re-entrant machine operated under BSP is tracking its own reference signal. The first and second stages try to maintain constant their immediate downstream buffer content while the third stage tracks the production demand. Thus the demand tracking errors of each stage operated by BSP were selected as $\varepsilon_j(k) = w_{d_j} - w_j(k)$ for all $j = 1, 2$, and $\varepsilon_3(k) = y_d(k) - y_3(k)$.

Results of comparison

The experimental details for all the policies can be summarized as follows:

- All the models are described by difference equations and the simulations are executed in *Matlab*®.
- Each simulation run is set to 15000 steps with initial demand $y_{d0} = 500$ [lots] and $y_j(0) = 0$ for all j.
- For all the simulations, the production speed of every stage is set according to the values listed in Table 5.1.
- Each model is tested under 50%, 75% and 95% of the maximal production demand rate, such that it satisfies the capacity condition (5.13). The details are shown in Table 5.1.
- For each demand rate, 10 simulation runs are executed. Each run with different w_{dj} in case of HPP and BSP, or $w_{dtotal} = w_{d2} + w_{d3}$ in case of CWIP.
- For each simulation run, the desired product content of a buffer is incremented in multiples of the maximal production speed of its immediate downstream stage, from 1 till 10 times the value. Thus $w_{d_j} = a(\mu_j + c_4)$ [lots], where a=1,. . . ,10. Note that for the CWIP policy, the total desired inventory level in the network is selected as $w_{dtotal} = a(\mu_2 + c_4) + a(\mu_3 + c_4)$.
- Each bar in Figure 5.9 stands for the steady-state mean buffer content value for the selected amount of base stock level w_{d_j} or w_{dtotal} in case of CWIP, i.e., each bar represents the steady-state $\bar{w}_j$ value of B_j for each value of a, for each production demand rate and for each policy.

Parameters for re-entrant machine					
j	μ_j	f_{jk}	%	v_d	$\Delta\varphi_k$
1	10	$0.5\sin(10k)$	50	0.6870	$0.5\cos(90k)$
2	5	$0.5\sin(70k)$	75	1.0305	$0.3\cos(90k)$
3	3	$0.5\sin(45k)$	95	1.3053	$0.04\cos(90k)$

Table 5.1: Production and demand rates in lot per time unit

We focus our performance comparison on the demand tracking error of the output stage $\varepsilon_3(k)$ and the inventory levels of intermediate buffers $w_j(k)$. Tables 5.2, 5.3 and 5.4 show the obtained steady-state values of the demand tracking error of the output stage for 50%, 75% and 95% of the linear production demand rate v_d. It is observed that for all the three policies, increasing desired intermediate inventory level has no significant influence on the steady-state ε_3. Thus each parameter value reported in Tables 5.2, 5.3 and 5.4 corresponds to all the tested inventory level values, i.e. for $a = 1, 2, 3, \ldots, 10$. For HPP, the observed steady-state behavior of ε_3 is clearly reflected in the obtained demand tracking error bound (5.22). One can note that if inequality (5.15) of Assumption 11 is satisfied for all the intermediate inventories of the network, then in (5.22) the term $X_j = 0$, and the value of every demand tracking error bound becomes independent form the influence of w_{dj}. The low inventory and high tracking inaccuracy effect of the machine under HPP is clearly reflected in the obtained bound (5.22). One can note that if inequality (5.15) of Assumption 11 is not satisfied then the upper bound on the demand tracking error (5.22) is influenced by the X_s and X_j terms that can drastically decrease the tracking accuracy of the system. This extreme case of re-entrant machine behavior under HPP , i.e. when $w_{d_j} = 0$ for all j, is reported in Tables 5.2-5.4 under the Pull column, where very low inventory levels and higher demand tracking inaccuracy can be observed for the network operated by a pure demand tracking error-based Pull strategy. No desired inventories are specified for our Pull model and the control goal of every production stage consists in pure cumulative product demand tracking. Note that this Pull policy can also be classified as a $w_{dj} = 0$ case of HPP. Thus, in this comparative study, it is not considered as a fourth policy.

From the model description it is known that in order to operate, CWIP policy forces the re-entrant machine to constantly maintain a certain non zero inventory level in the system. Thus the machine under CWIP can not operate with zero desired inventory levels as HPP. From Tables 5.2-5.4 it can be seen that under CWIP the machine presents tracking accuracy similar to HPP for a demand rate of 50%, while at 75 and 95%, the mean accuracy loss doubles the tracking accuracy of HPP. This accuracy loss with CWIP can be attributed to the switching priorities between the stages, which in steady-state under CWIP are from stage 3 to stage 1 to stage 2 and back to stage 3. This circular switching delays the output stage reaction time. The extreme steady-state demand tracking error values of CWIP are very similar to HPP for all the tested demand rates. Also in case of CWIP the w_{dtotal} amount is mostly accumulated in in the last buffer B_3, which can have positive as well as negative effects on the network. A positive effect, for example, is reflected in the tracking accuracy and low standard deviation (σ) from the demand tracking error mean value ($\bar{\varepsilon}_3(k)$). In other words the network reacts faster on the rapid production demand changes if product starvation never occurs, i.e network's inventory levels are high. At the same time keeping high inventories may impose unnecessary storage costs, specially if the product demand is low.

For BSP, the inventory quantity and its distribution through the network are quiet similar to the one in HPP (see Figure 5.9), while the demand tracking error accuracy (see Tables 5.2-5.4) of BSP is higher than in HPP. Both policies permit an independent (distributed) control of intermediate inventory levels of the network. Thus for a given setting, BSP can be adjusted to perform as CWIP and HPP as pure demand tracking Pull, but not the other way around. This makes these two policies to stand out among the three. Also the obtained theoretical values on $\max \varepsilon_3(k)$ and $\min \varepsilon_3(k)$ for the HPP policy (see Tables 5.2-5.4 under HPP Theory) show an accurate lower bound, but a bit conservative upper bound. This accuracy loss is due to the fact that the calculations are based on the upper bound expression (5.22), which gives a less accurate result than (5.20).

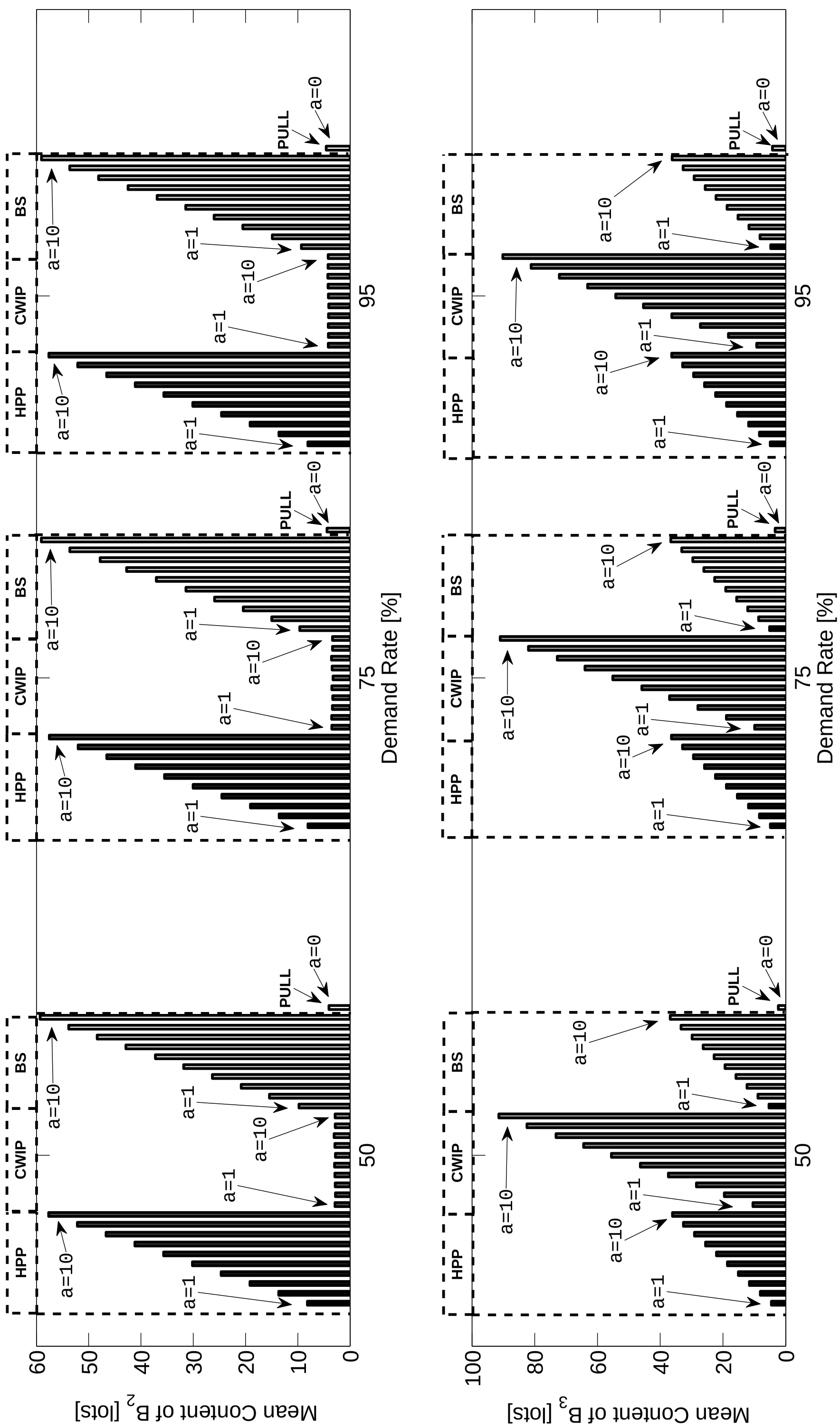


Figure 5.9: Intermediate buffer content

Policies	HPP Theory	HPP	CWIP	BS	Pull
$\bar{\varepsilon}_3(k)$ [lot]		-0.6	-0.7	-1	3
σ_{ε_3} [lot]		1	1	0.9	1.9
$\max \varepsilon_3(k)$	8.73	2	1.7	1.13	9
$\min \varepsilon_3(k)$	-3.31	-3.31	-3.3	-3.3	-2.8

Table 5.2: Comparison for demand rate of 50 %

Policies	HPP Theory	HPP	CWIP	BS	Pull
$\bar{\varepsilon}_3(k)$ [lot]		0.17	0.4	-0.46	4.5
σ_{ε_3} [lot]		1.1	1	1	2.1
$\max \varepsilon_3(k)$	8.55	3	3.4	1.8	11
$\min \varepsilon_3(k)$	-2.77	-2.7	-2.7	-2.7	-1.6

Table 5.3: Comparison for demand rate of 75 %

Policies	HPP Theory	HPP	CWIP	BS	Pull
$\bar{\varepsilon}_3(k)[lot]$		0.85	1.5	0.25	7
σ_{ε_3} [lot]		1.25	1.3	1	2.3
$\max \varepsilon_3(k)$	8.13	4	5	2.6	14
$\min \varepsilon_3(k)$	-2.23	-2.1	-2.2	-2.2	-0.5

Table 5.4: Comparison for demand rate of 95 %

Demand rate	HPP	CWIP	BS	Pull
50 %	3.34	3.0	3.35	3.1
75 %	3.5	3.5	3.6	3.4
95 %	3.5	3.7	3.7	3.6

Table 5.5: Standard deviation σ_{ω_2} of lots in B_2

Demand rate	HPP	CWIP	BS	Pull
50 %	1.89	3.2	1.95	2.1
75 %	2.07	3.3	2.09	2.87
95 %	2.28	3.5	2.25	3.18

Table 5.6: Standard deviation σ_{ω_3} of lots in B_3

5.3 Re-entrant line

In this section we extend our analysis of surplus-based control of a single machine to a re-entrant line of P machines. Each product has to go several times through the entire manufacturing line before it is ready. In the previous section we analyzed the performance of one machine composed of N stages that under application of surplus-based control can only operate on one stage at a time. Extending the control strategy to a line composed of these machines we study its demand tracking accuracy, as well as the influence of the number of machines and production stages on the demand tracking error.

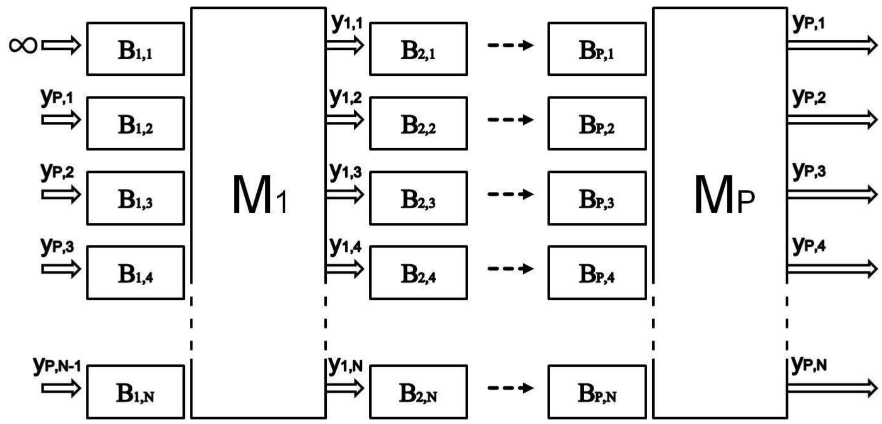


Figure 5.10: A re-entrant line composed of P machines $M_1, \ldots, M_P$.

Figure 5.10 shows a schematic picture of a re-entrant network of P machines and N production stages at each machine. The machines are depicted as big rectangles with inside label M_j, where M stands for machine and j for its number. Each machine has a number of buffers, i.e. small rectangles, connected to it. Each of these buffers has a label $B_{i,j}$, where B stands for the buffer, i is the machine number and j is the stage number of the machine i. Each buffer retains the intermediate product to be processed on the production stage of its machine. The production flow of the network is indicated by the dashed and bold arrows.

5.3.1 Flow model

The flow model of each production stage of a re-entrant line (Figure 5.10) in discrete time is defined as

$$y_{i,j}(k+1) = y_{i,j}(k) + \beta_{i,j}(k)u_{i,j}(k), \tag{5.24}$$

where all the events within the model occur at given time instances and k represents the current time so that the time step between all the events is constant. Here i is the machine number ($i = 1, .., P$), j is the processing stage number ($j = 1, ..., N$) of machine i, $y_{i,j}(k) \in \mathbb{R}$ is the cumulative output of machine i in processing stage j in time k, $u_j(k) \in \mathbb{R}$ is the control input of machine i in processing stage j and $\beta_{i,j}(k) = \mu_{i,j} + f_{i,j}(k)$ with $\mu_{i,j}$ as a positive constant that represents the processing speed of the machine i at its stage j and $f_{i,j}(k) \in \mathbb{R}$ is an unknown external disturbance affecting the performance of the ith machine at its stage j.The equation (5.24) present a general model that can describe a product flow for a wide range of production stages of manufacturing networks. Further specific assumptions on system (5.24) are given in this section. Under the assumption that there is always sufficient raw material to feed the input buffer $B_{1,1}$, the control aim is to track the non-decreasing cumulative production demand given by (5.2) on the output of this multi re-entrant network.

In order to give a solution to this tracking problem we consider the following control algorithm:

$$
\begin{aligned}
&\{q_i(k) = B_{i,j}\} \\
&\textbf{if}\, \varepsilon_{i,j}(k) > 0 \textbf{ and } w_{i,j}(k) \geq \beta_{i,j}(k) \textbf{ then} \\
&u_{i,j}(k) = 1, \\
&u_{i,s}(k) = 0,\ \forall s \neq j,\ s,j = 1\ldots,N, \\
&q_i(k+1) = B_{i,j}, \\
&\textbf{end} \\
&\textbf{if}\, (\varepsilon_{i,j}(k) \leq 0 \textbf{ or } w_{i,j}(k) < \beta_{i,j}(k)) \textbf{ and} \\
&\exists\, s \neq j :\ (\varepsilon_{i,s}(k) > 0 \textbf{ and } w_{i,s}(k) \geq \beta_{i,s}(k)) \textbf{ then} \\
&u_{i,j}(k) = 0, \\
&u_{i,s}(k) = 1, \\
&q_i(k+1) = B_{i,s}, \\
&\textbf{end} \\
&\textbf{if}\ (\varepsilon_{i,s}(k) \leq 0 \textbf{ or } w_{i,s}(k) < \beta_{i,s}(k))\,,\ \forall s \textbf{ then} \\
&u_{i,j}(k) = 0, \\
&u_{i,s}(k) = 0,\ \forall s \neq j,\ s,j = 1\ldots,N, \\
&q_i(k+1) = 0, \\
&\textbf{end}
\end{aligned} \tag{5.25}
$$

where $q_i(k)$ is the internal variable representing the current buffer that M_i is processing, for the current time step $\beta_{i,j}(k)$ is the minimal content of raw material in buffer $B_{i,j}$, such that machine M_i is able to process if required at stage j. Note that $B_{1,1}$ is assumed to always contain sufficient raw material. The demand tracking error of each stage of M_i is given by:

$$
\begin{aligned}
\varepsilon_{P,N}(k) &= y_d(k) - y_{P,N}(k), && (5.26) \\
\varepsilon_{i,j}(k) &= \varepsilon_{r,s}(k) + (w_{i,j} - w_{di,j}(k)), \forall j = 1\ldots,N, && (5.27)
\end{aligned}
$$

where the constants r and s can be selected through the following relation

$$
\begin{aligned}
&\textbf{if}\, i > 1 \textbf{ then} \\
&r = i-1,\ s = j \\
&\textbf{else} \\
&r = P,\ s = j-1.
\end{aligned} \tag{5.28}
$$

Here $w_{i,j+1}(k) = y_{i,j}(k) - y_{i,j+1}(k)$ is the buffer content of buffer $B_{i,j+1}$ and $w_{d_{i,j+1}}$ is the constant that represents the desired inventory level of buffer $B_{i,j+1}$.
Figure 5.11 shows a schematic picture of the information flow throughout the re-entrant line. The dashed rectangles represent the multi-stage machines, with inside numbered circles denoting the production stages and outside short black arrows with white fill denoting the external perturbations (f_i), which are affecting its production rates. Each machine has a set of buffers connected to it, each one denoted by 3 joined squares. The product flow directions are denoted by solid black bold arrows. The transfer of the demand tracking error ($\varepsilon_{i,j}$) information is shown by arched, dashed arrows. For each stage the upstream and downstream inventory level ($w_{i,j}$) information transfer is depicted by a straight solid arrow with hooklike ends and the input on the desired downstream inventory level is shown by a short black arrow pointing to each production stage.

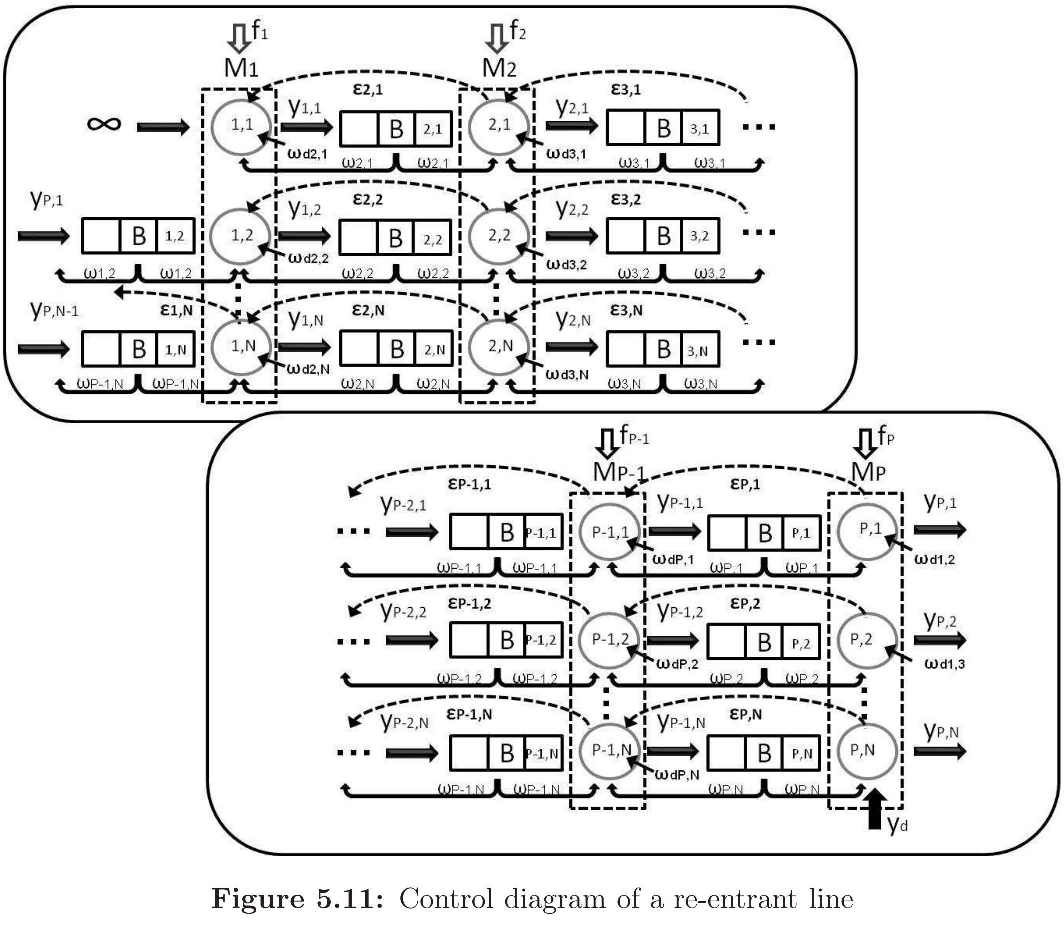


Figure 5.11: Control diagram of a re-entrant line

Basically, model (5.24) describes the product flow for each of the N stages of P manufacturing machines combined into a re-entrant line configuration. The first machine M_1 is considered to have always access to the raw material at its stage (1,1) (see Figure 5.11) and there is always sufficient raw material. The administration of this raw material to machine M_1 is decided by the control input (5.25). For every machine M_i we consider that our control input is acting as an authorizing switch, which turns on M_i if the demand tracking error $\varepsilon_{i,j}(k)$ of at least one of its N stages is positive and turns M_i off if all the demand tracking errors of this machine are zero or negative, an also if the intermediate inventory levels are insufficient for the machine to produce a product. Each machine can only work at one stage at a time. The control logic is such that once the production stage is selected, machine M_i remains working at this stage till the demand tracking error $\varepsilon_{i,j}(k) \leq 0$, or its buffer $B_{i,j}$ gets empty (i.e.,$w_{i,j}(k) < \beta_{i,j}(k)$). This approach is similar to clearing policies, but based on the demand tracking error and not on the buffer inventory level.

Each demand tracking error $\varepsilon_{r,s}(k)$, see (5.27) and Figure 5.11, consists of the difference between what is done ($w_{i,j}(k)$), what has to be done ($\varepsilon_{i,j}(k)$) and what has to be always in the downstream buffer ($w_{d_{i,j}}$). It can be seen from (5.27) that the same demand tracking error logic is applied to every production stage of every machine till stage $(P, N-1)$. As for the last production stage (P, N) of the last machine in the line, which is machine M_P (Figure 5.11), we expect that on its output the cumulative product demand is followed by the cumulative production of this machine. Thus control activates or deactivates this stage based directly on the production demand status (5.26), as well as its upstream buffer content (see Figure 5.11).

Note that the control actions are decentralized throughout the network. In other words, the control action of each stage of every machine in the re-entrant line depends only on the demand tracking error of its neighboring downstream stage (except for the stage P, N, which depends directly on the cumulative demand input) and the current buffer content of its upstream buffer (Figure 5.11). This gives our flow model an extra robustness with respect to undesired events such as temporal machine setup or breakdown.

For further analysis, let us rewrite flow model (5.24) in a closed-loop with (5.25) in terms of demand tracking errors as

$$\begin{aligned} \Delta\varepsilon_{i,j}(k) &= v_d + \Delta\varphi(k) - \beta_{i,j}(k)u_{i,j}(k), \\ &\forall j = 1, \dots, N,\ i = 1, \dots, P \end{aligned} \tag{5.29}$$

where $\Delta\varepsilon_{i,j}(k) = \varepsilon_{i,j}(k+1) - \varepsilon_{i,j}(k)$, $\Delta\varphi(k) = \varphi(k+1) - \varphi(k)$.

Assume that system (5.29) satisfies the following requirements.

Assumption 12. *(Production speed limitations)*
Each machine M_i operates at each production step j under a processing speed of $\beta_{i,j}(k) = \mu_i + f_i(k)$ lots per time unit, which is the same for each production stage of the machine, but can differ from the other machines in the network. Thus $\beta_{i,j}(k) = \beta_{i,s}(k)$ for all k, where $j, s = 1, \dots, N$ and $j \neq s$.

Assumption 13. *(Boundedness of perturbations)*
There are constants c_1, c_2, c_3 and c_4 such that

$$c_1 < \Delta\varphi(k) < c_2,\ \forall k \in \mathbb{N}, \tag{5.30}$$

$$c_3 < f_i(k) < c_4\ \forall k \in \mathbb{N},\ i = 1, \dots, P. \tag{5.31}$$

From *Assumption 13*, it follows that $W_{i,j}(k) = \Delta\varphi(k) - f_i(k)$ satisfies

$$\alpha_1 < W_i(k) < \alpha_2,\ \forall k \in \mathbb{N},\ i = 1, \dots, P, \tag{5.32}$$

with $\alpha_1 = c_1 - c_4$ and $\alpha_2 = c_2 - c_3$.

Assumption 14. *(Capacity condition)*
The constants c_1, c_2, c_3 and c_4 in (5.30) and (5.31) satisfy the following inequalities

$$c_1 > -v_d, \tag{5.33}$$
$$\alpha_2 < \mu_i - v_d, \ \forall i = 1, \ldots, P, \tag{5.34}$$

and the following condition (Capacity Condition) holds for each M_i in the network

$$0 < (v_d + c_1)\frac{N}{\mu_i + c_4} < (v_d + \Delta\varphi(k))\frac{N}{\mu_i + f_i(k)} < (v_d + c_2)\frac{N}{\mu_i + c_3} < 1, \tag{5.35}$$

for all k. Note that the physical meaning of the above mentioned assumptions is similar to the one discussed in detail in Section 2.3 of Chapter 2.

It is important to notice that machine M_i at each process step j has a processing speed of $\mu_i + f_i(k)$ lots per time unit, which can differ from the rest of the machines in the network. One of the important physical limitations in the network is the buffer content restriction. In our model, in order for the control action $(u_{i,j})$ of the selected production stage $(b_{i,j})$ of M_i to take place, the buffer of this stage must satisfy the following condition on its content

$$w_{i,j}(k) \geq \beta_{i,j}(k), \ \forall j = 2, \ldots, N. \tag{5.36}$$

Thus, from (5.27) and (5.36), the following demand tracking error condition holds

$$\begin{aligned} \varepsilon_{i,j}(k) \geq \quad & \beta_{i,j}(k) - w_{d_{i,j}} + \varepsilon_{r,s}(k), \\ & \forall i, r = 1, \ldots, P, \ j, s = 1, \ldots, N, \end{aligned}$$

where $w_{d_{i,j}}$ is assumed to satisfy the following inequality:

Assumption 15. *(Desired buffer content condition)*
The constants $w_{d_{i,j}}$ comply with the following inequality

$$\begin{aligned} w_{d_{i,j}} \geq \quad & \mu_{i,j} + N\,\mu_{r,s} + (N+1)c_4 \\ & +(N-1)(c_2 - c_1). \end{aligned} \tag{5.37}$$

From (5.37) it follows that for all k

$$w_{d_{i,j}} > \beta_{i,j}(k), \ \forall i = 1, \ldots, P, \ j = 1, \ldots, N.$$

5.3.2 Results on performance

In this section we present the results regarding the demand tracking error trajectories behavior of flow model (5.29).

Theorem 10. *Assume that the discrete time system defined by (5.29) satisfies Assumptions 12-15. Then all solutions of (5.29) are ultimately bounded by*

$$\limsup_{k\to\infty} \sum_{j=1}^{N} \varepsilon_{i,j}(k) - v_d - \alpha_2 \leq 0, \tag{5.38}$$
$$\liminf_{k\to\infty} \varepsilon_{i,j}(k) \geq v_d + \alpha_1 - \mu_i. \tag{5.39}$$

Proof. see Appendix J. □

Note that the upper bound from (5.38) can be also defined as

$$\limsup_{k\to\infty} \varepsilon_j(k) \leq \quad (N-1)(\mu_i - \alpha_1 + \alpha_2) + v_d + \alpha_2. \tag{5.40}$$

It is important to note that bound (5.40) is simpler to implement, but is less precise than the bound (5.38).

From (5.38) and (5.39) it can be deduced that for the buffer content $w_{i,j}(k)$ of each buffer $B_{i,j}$ defined by (5.27) it holds that

$$\limsup_{k\to\infty} w_{i,j}(k) \leq (N-1)(\mu_i) + N(\alpha_2 - \alpha_1) + \mu_r + w_{d_{i,j}}. \tag{5.41}$$

Now, in order to support the present development we present simulation results in the next section.

5.3.3 Discrete time simulation

Consider the following example of a production line of 2 manufacturing machines with 2 production stages each (see Figure 5.10) operating under surplus-based regulators (5.25). The processing nominal speeds are set as $\mu_{i,j} = 5$ lots per time unit and assume that the perturbations are selected in such a way that $f_{i,j}(k) = 5$ lots per time unit at stage $j = 1$ of M_1 and stage $j = 2$ of M_2. For the rest of the stages $\mu_{i,j} = 5$ and $f_{i,j}(k) = 0$ lots per time unit. The desired buffer content of each buffer is selected considering (5.37) as $w_{d_{i,j}} = (w_{d_{1,2}}, w_{d_{2,1}}, w_{d_{2,2}}) = (26, 16, 21)$ lots, and the mean demand rate $v_d = 2$ lots per time unit [7] with fluctuation rate of $\Delta\varphi(k) = 0.4\sin(90k)$. The demand tracking error of each machine in the line is depicted in Figure 5.12. Here the initial conditions $(y_{1,1}(0), y_{1,2}(0), y_{2,1}(0), y_{2,2}(0))$ are set to zero and $y_{d0} = 100$ lots. After the first 250 time steps, as it is shown in Figures 5.12 and 5.13, the system reaches its steady-state. Demand tracking errors (see the dashed lines of Figure 5.12) are maintained inside [-8.4, 13.2] lots for M_1, and [-3.4, 8.2] lots for M_2, which satisfy the bounds given by (5.38) and (5.39). Figure 5.13 shows the buffer content of each $B_{i,j}$ in the network. After some transient behavior the inventory level of each buffer is maintained inside the obtained bound (5.41).
Another experimental result is shown in Figure 5.14. This graph shows the relation between the upper bound on the demand tracking error $\varepsilon_{2,2}$ versus the desired buffer content of the network from the previous example. Here it can be observed that the amount of extra storage for intermediate products has only limited influence on the tracking accuracy of the network and the threshold value of this

[7] Note that this v_d value was selected as an 80% of a maximal allowed demand rate value, which according to (5.35) is less then 2.5 lots per time unit. In (5.35) the lower bound on the production speed $\mu_{i,j} + c_3 = 5$ lots.

influence is given by (5.37). In Figure 5.14 one can observe that maintaining the inventory level of $B_{2,2}$ at more than 10 lots has no influence on the demand tracking error of the output stage ($\varepsilon_{2,2}$). Thus doing so may only result in extra inventory costs. In conclusion, the presented simulation results reflect the desired flow model behavior, i.e., all the values assigned to the parameters utilized in this section are consistent with the assumptions of Section 5.3.1 and the outcome of the simulation example satisfies the theoretical results.

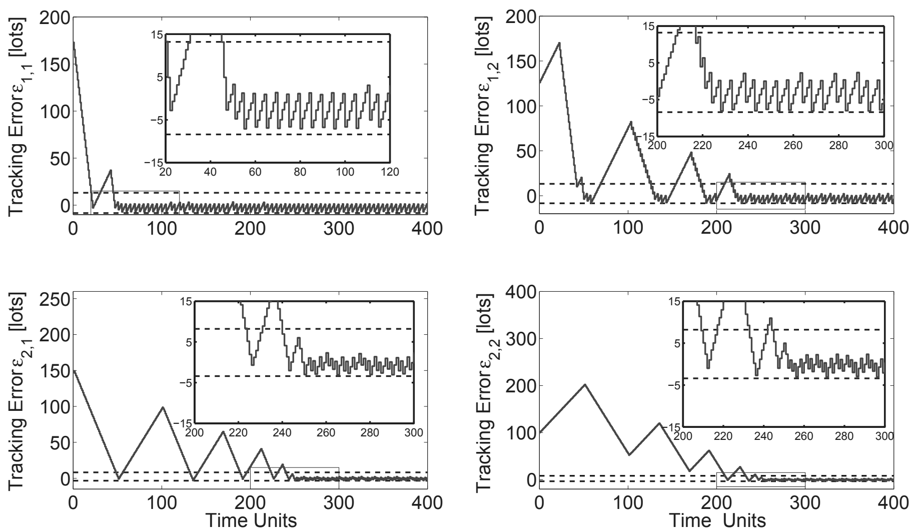


Figure 5.12: Demand tracking errors and their steady-state bounds (dotted lines), with $v_d = 2$, $\Delta\varphi(k) = 0.4\sin(90k)$, $\mu_{i,j} = 5$, $f_{i,j}(k) \in 0, 5$, $w_{d_{i,j}} = (26, 16, 21)$ and $y_{d0} = 100$.

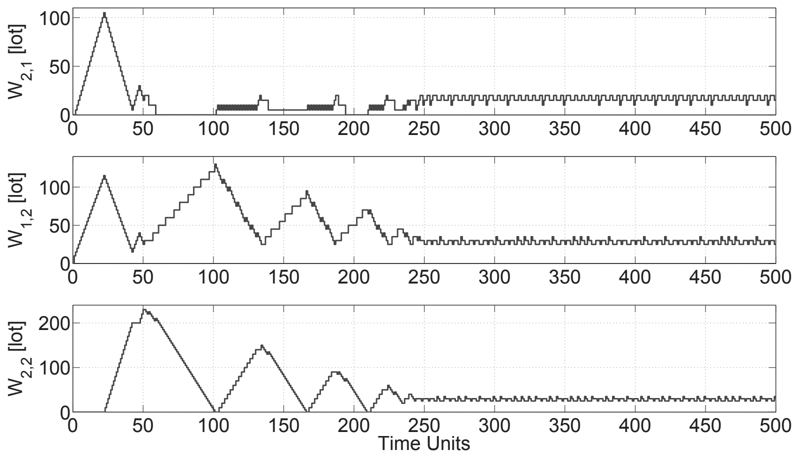


Figure 5.13: Buffer contents, with $v_d = 2$, $\Delta\varphi(k) = 0.4\sin(90k)$, $\mu_{i,j} = 5$, $f_{i,j}(k) \in 0, 5$, $w_{d_{i,j}} = (26, 16, 21)$ and $y_{d0} = 100$.

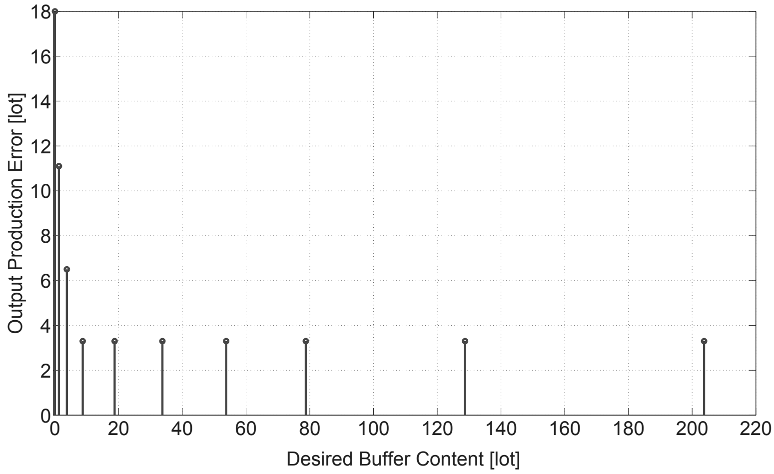


Figure 5.14: Upper bound on demand tracking error $\varepsilon_{2,2}(k)$ vs. desired buffer content $w_{d2,2}$.

5.3.4 Simulation based performance analysis

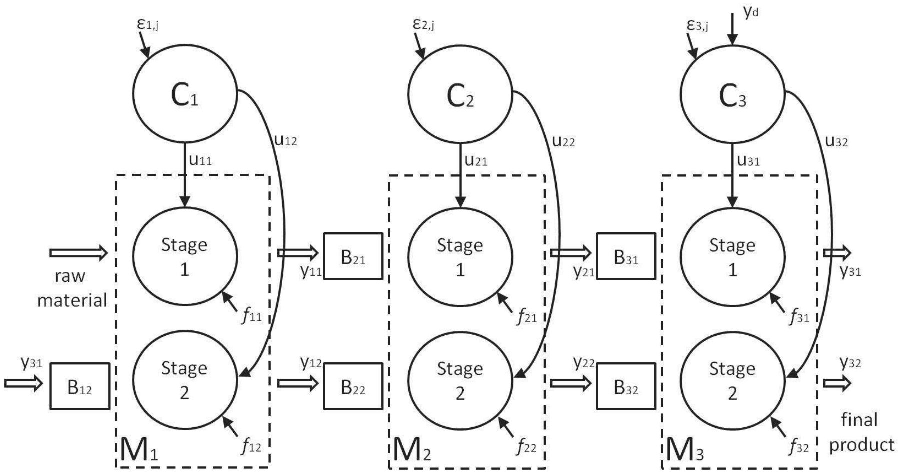


Figure 5.15: Schematics of one re-entrant line with 3 machines M_i of 2 stages and 5 buffers $B_{i,j}$.

In this section we present results on the simulation-based comparison of 3 particular cases of surplus-based production control strategies applied to a re-entrant line. Figure 5.15 shows the re-entrant line consisting of 3 manufacturing machines composed of 2 production stages each ($i = 1, 2, 3$ and $j = 1, 2$) and 5 intermediate buffers $B_{i,j}$. Similarly to the comparative analysis presented for one re-entrant machine in Section 5.2.4, in this section the performance of the re-entrant line is tested under particular cases of HPP, CWIP and BSP strategies. The performance criteria upon which these three policies will be compared are the steady state maximum and minimum values of demand tracking error of the output stage $j = 2$ of machine $i = 3$ and the steady state intermediate inventory level (i.e,

$w_{2,1} + w_{3,1} + w_{1,2} + w_{2,2} + w_{3,2}$) in the re-entrant line.

Description of experiment

The following common assumptions were made for all the implemented policies (see more details in Appendix K, Sections K.1.2, K.2.2 and K.3.2 on a single re-entrant machine for more details on the used flow models):

- In the line, each machine M_i produces a single part type.
- Each M_i can only work with one buffer at a time.
- The control input $u_{i,j}$ of M_i at each production stage can only take the value of 0 (stop) or 1 (produce).
- The control actions are executed every time step k.
- The control input $u_{i,j}$ of M_i at stage j takes the value of 1 only the production stage j needs to produce (based on the operating policy) and its buffer $B_{i,j}$ is not empty.
- The value of 0 is given to the control input $u_{i,j}$ of M_i at stage j if its demand tracking error $\varepsilon_{i,j}(k) \leq 0$, its adjacent buffer is empty or the machine is currently working with another stage.
- Once the production stage is selected, the machine will operate at this stage till its control input changes its value to 0.
- The demand tracking error of the output stage $j = 2$ of the last machine M_3 is given by $\varepsilon_{3,2}(k) = y_d(k) - y_{3,2}(k)$.
- No setup times are considered in the models

For HPP the demand tracking errors and the operation principal are described by (5.27), (5.26) and (5.25), respectively.
For CWIP policy the first stage $j = 1$ of M_1 limits the number of products in the network and the rest of the stages till $j = 2$ of M_3 produce products in a clearing manner. Thus $\varepsilon_{1,1}(k) = w_{d_{total}} - w_{total}(k)$, $\varepsilon_{i,j}(k) = w_{i,j}(k)$ for $i, j = (1, 2; 2, 1; 2, 2; 3, 1)$ and $\varepsilon_{3,2}(k) = y_d(k) - y_{3,2}(k)$. For fair comparison with HPP and BSP, $w_{d_{total}}$ can be interpreted as $w_{d_{total}} = \sum_{i=1}^{3} w_{d_{i,1}} + w_{d_{i,2}}$[8] and $w_{total}(k) = y_{1,1}(k) - y_{3,2}(k)$. As well as in HPP, the control action of the output stage $j = 2$ of M_3 aims at production demand tracking.
The operational principal of BSP is presented in Section K.2.2. The demand tracking errors of this policy were selected as $\varepsilon_{r,s}(k) = w_{d_{i,j}} - w_{i,j}(k)$ for all $i = 1, 2, 3$ and $j = 1, 2$ except for the demand tracking error of the output stage of the last machine $\varepsilon_{3,2}(k) = y_d(k) - y_{3,2}(k)$. The constants r and s are given by (5.28).

[8] It is assumed that $B_{1,1}$ is never empty. Thus $w_{d_{1,1}} = 0$.

Results of comparison

For all the three strategies the details of the conducted experiment can be summarized as follows:

- All the models are described by difference equations and the simulations are executed in $Matlab^{®}$.
- Each simulation run is set to 20000 steps with initial demand $y_{d0} = 500$ [lots] and $y_{i,j}(0) = 0$ for all i,j.
- For all the simulations the production speed of every stage is set according to the value given in Table 5.7.
- Each model is tested under 50%, 75% and 95% of the maximal production demand rate, such that it satisfies condition (5.35). See Table 5.7 for details.
- The steady-state demand tracking error values are reported in Tables 5.8-5.10.
- For each demand rate, 8 simulation runs are executed. Each run with a different value of $w_{di,j}$ in case of HPP and BSP, or $w_{d_{total}} = \sum_{i=1}^{3} w_{d_{i,1}} + w_{d_{i,2}}$ in case of CWIP.
- The desired product content of a buffer is selected in multiples of the maximal production speed of its downstream stage, from 1 till 20 times the value. Thus $w_{d_{i,j}} = a(\mu_{i,j} + c_4)$ [lots], where a=1,. . . ,20 as also shown in Tables 5.8-5.10.
- For each buffer in the network, its steady-state mean buffer content value is shown in Figures 5.16-5.18.
- Each bar in Figures 5.16-5.18 stands for the steady-state mean buffer content value for the selected amount of base stock level $w_{d_{i,j}}$ or w_{dtotal} in case of CWIP, i.e., each bar represents the steady-state $\bar{w}_{i,j}$ value of $B_{i,j}$ for each value of a, for each production demand rate and for each policy as it is shown in the figures.

Parameters for re-entrant machine				
i,j	$\mu_{i,j} + f_{i,jk}$	%	v_d	$\Delta\varphi_k$
1,1	$5 + 0.5\sin(10k)$	50	1.125	$\cos(3k)$
1,2	$5 + 0.5\sin(10k)$	75	1.6875	$0.5\cos(3k)$
2,1	$6 + 0.5\sin(3k)$	95	2.1375	$0.1\cos(3k)$
2,2	$6 + 0.5\sin(3k)$			
3,1	$5 + 0.5\sin(1.5k)$			
3,2	$5 + 0.5\sin(1.5k)$			

Table 5.7: Production and demand rates in lot per time unit

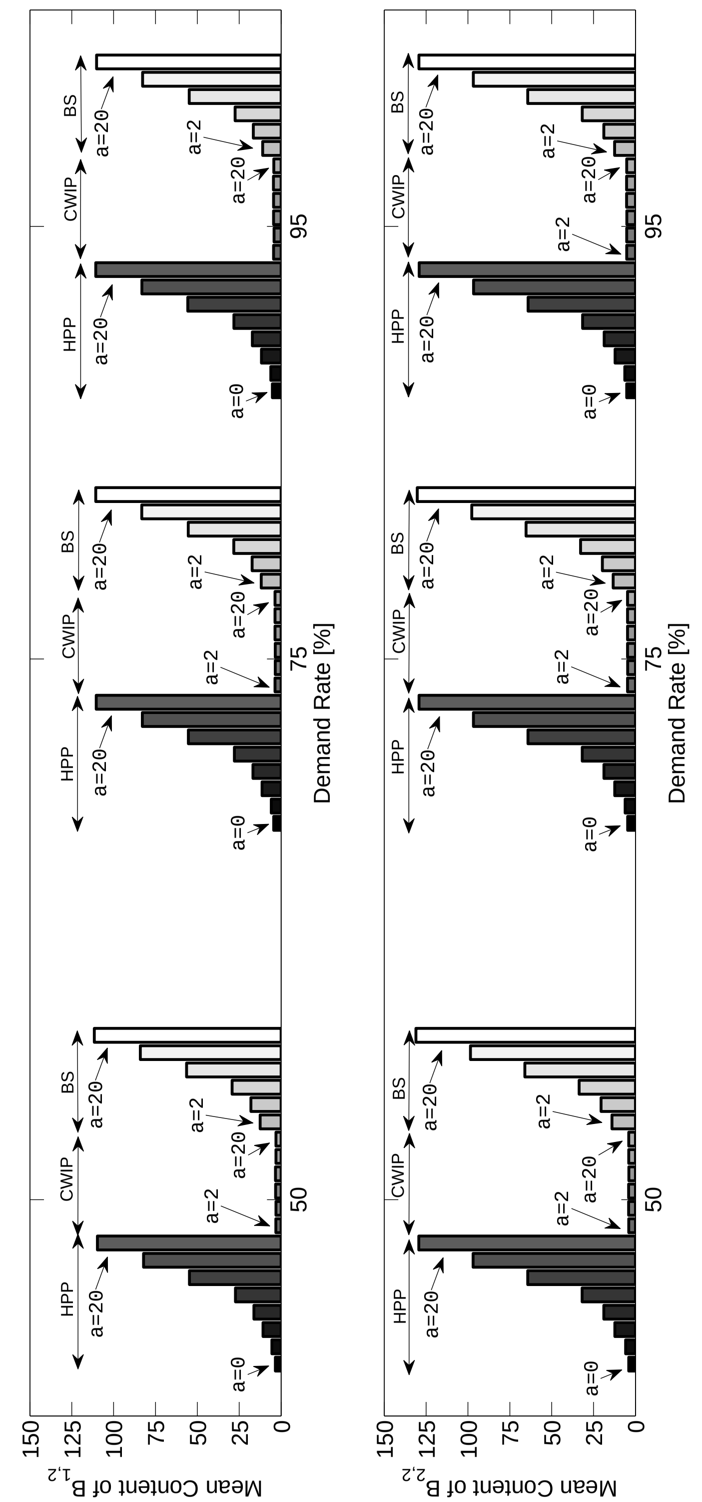


Figure 5.16: Content of intermediate buffers $B_{1,2}$ and $B_{2,2}$

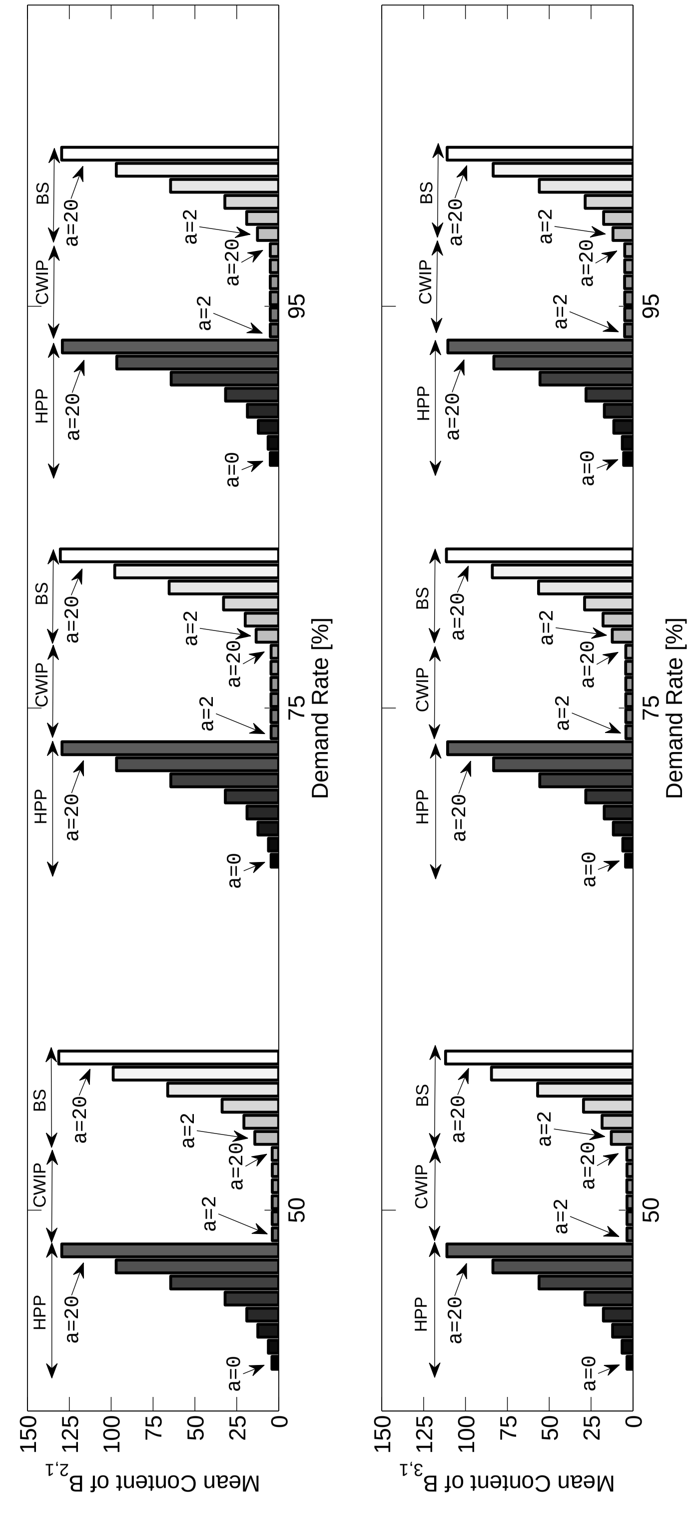


Figure 5.17: Content of intermediate buffers $B_{2,1}$ and $B_{3,1}$

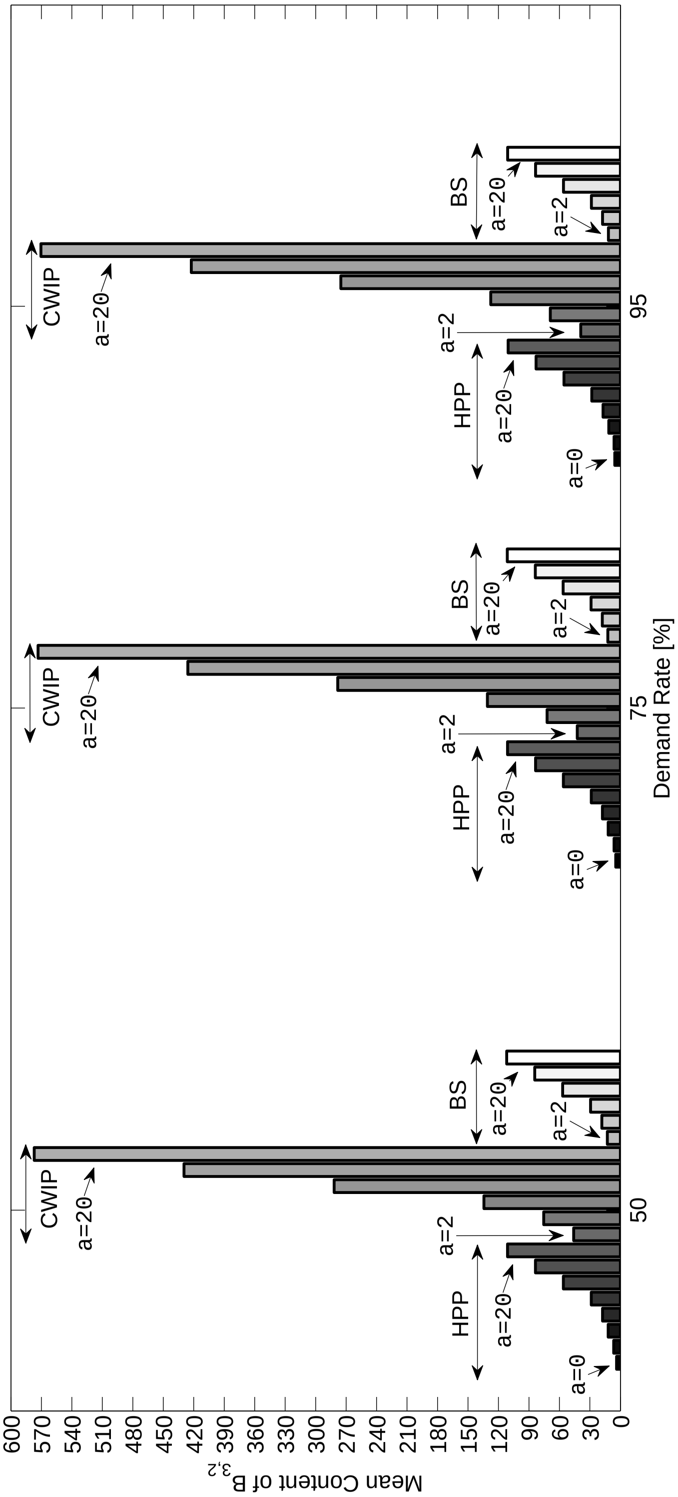


Figure 5.18: Content of intermediate buffer $B_{3,2}$

As well as in Section 5.2.4, here we focus our performance comparison on the steady-state demand tracking error of the output stage $\varepsilon_{3,2}(k)$ and the inventory levels of intermediate buffers $w_{i,j}(k)$.

For BSP the amount of inventory and its distribution through the network are quiet similar to the one in HPP, but different from CWIP. The mean inventory level could be observed in Figures 5.16-5.18 and their variation $\sigma_{w_{i,j}}$ in Tables 5.8-5.10. The demand tracking error accuracy (see Tables 5.8-5.10) of BSP and CWIP is lower then in HPP. As also observed in Section 5.3, HPP and BSP policies permit an independent (distributed) control of intermediate inventory levels of the network, which makes these two policies easier to adapt to any kind of network configuration. CWIP policy maintains the inventory level close to w_{dtotal} amount, which is mostly accumulated in the last buffer of the network $B_{3,2}$. This inventory distribution (see Figures 5.16-5.18) can have positive as well as negative effects on the network. A positive effect, for example, is reflected in the tracking accuracy, which is similar to the one reached with BSP, but with less control effort. This could be noticed from $\sigma_{\varepsilon_{3,2}(k)}$ and $\bar{\varepsilon}_{3,2}(k)$ of Tables 5.8-5.10. The network operated by CWIP is able to react faster on the rapid production demand changes due to its high inventory of the last buffer. Thus the chance of starvation for stage 2 of M_3 is low. At the same time keeping high inventories may impose unnecessary storage costs, specially if the product demand is low.

The superiority in demand tracking accuracy of the re-entrant line under HPP respecting CWIP and BSP is more pronounced for 75% and 95% of the product demand value. From Tables 5.9 and 5.10, it can be observed that CWIP and BSP present a similar demand tracking accuracy that is half of the preciseness reached under HPP. The zero base stock case of the network's behavior under HPP, i.e. $w_{d_{i,j}} = 0$ for all i, j, is reported in tables Tables 5.8-5.10, where very low inventory levels and higher demand tracking inaccuracy can be observed for the network operated by the pure demand tracking Pull strategy. No desired inventories are specified and the control goal of every production stage consists in pure cumulative production demand tracking. This type of policy can be classified as simple Pull or as a $w_{d_{i,j}} = 0$ case of HPP. The obtained theoretical values on $\max \varepsilon_3(k)$ and $\min \varepsilon_3(k)$ for the HPP policy (see Tables 5.8-5.10 under HPP (theory)) show a tight lower bound, but a bit conservative upper bound. This conservatism is due to the upper bound expression (5.40) that was used to obtain the numerical value, and which gives a less accurate result than (5.38).

a	0	1	2	3	5	10,15,20
HPP $\bar{\varepsilon}_{3,2}(k)$ [lot]	16.97	-0.96	-1.48	-1.49	-1.48	-1.52
CWIP $\bar{\varepsilon}_{3,2}(k)$ [lot]	∞	-0.81	-0.72	-0.71	-0.62	-0.63
BS $\bar{\varepsilon}_{3,2}(k)$ [lot]	∞	-0.87	-0.38	-0.56	-0.32	-0.55
HPP $\sigma_{\varepsilon_{3,2}}$ [lot]	3.32	1.53	1.50	1.52	1.52	1.52
CWIP $\sigma_{\varepsilon_{3,2}}$ [lot]	_	1.64	1.64	1.64	1.64	1.64
BS $\sigma_{\varepsilon_{3,2}}$ [lot]	_	1.59	1.63	1.59	1.62	1.62
HPP (theory)$\max \varepsilon_{3,2}(k)$ [lot]					7.62	7.62
HPP $\max \varepsilon_{3,2}(k)$ [lot]	27.97	3.36	2.3	2.26	2.28	2.24
CWIP $\max \varepsilon_{3,2}(k)$ [lot]	_	3.73	3.88	3.95	4.13	4.08
BS $\max \varepsilon_{3,2}(k)$ [lot]	_	3.6	4.58	4.22	4.69	4.73
HPP (theory) $\min \varepsilon_{3,2}(k)$ [lot]					-5.37	-5.37
HPP $\min \varepsilon_{3,2}(k)$ [lot]	5.98	-5.29	-5.28	-5.24	-5.24	-5.29
CWIP $\min \varepsilon_{3,2}(k)$ [lot]	_	-5.36	-5.32	-5.36	-5.37	-5.36
BS $\min \varepsilon_{3,2}(k)$ [lot]	_	-5.36	-5.34	-5.35	-5.35	-5.35
HPP $\sigma_{w_{3,2}}$ [lot]	2.75	2.47	2.42	2.43	2.43	2.42
CWIP $\sigma_{w_{3,2}}$ [lot]	_	3.55	3.65	3.56	3.62	3.54
BS $\sigma_{w_{3,2}}$ [lot]	_	2.62	2.65	2.63	2.64	2.65
HPP $\sigma_{w_{2,2}}$ [lot]	2.72	2.47	2.43	2.42	2.46	2.43
CWIP $\sigma_{w_{2,2}}$ [lot]	_	2.70	2.73	2.67	2.68	2.72
BS $\sigma_{w_{2,2}}$ [lot]	_	2.72	2.77	2.75	2.78	2.77
HPP $\sigma_{w_{1,2}}$ [lot]	2.56	2.18	2.16	2.15	2.14	2.15
CWIP $\sigma_{w_{1,2}}$ [lot]	_	2.42	2.49	2.46	2.46	2.45
BS $\sigma_{w_{1,2}}$ [lot]	_	2.51	2.55	2.52	2.58	2.55
HPP $\sigma_{w_{3,1}}$ [lot]	2.69	2.31	2.26	2.23	2.27	2.24
CWIP $\sigma_{w_{3,1}}$ [lot]	_	2.68	2.69	2.68	2.69	2.69
BS $\sigma_{w_{3,1}}$ [lot]	_	2.77	2.76	2.76	2.74	2.75
HPP $\sigma_{w_{2,1}}$ [lot]	2.70	2.34	2.32	2.35	2.29	2.28
CWIP $\sigma_{w_{2,1}}$ [lot]	_	2.70	2.71	2.70	2.71	2.73
BS $\sigma_{w_{2,1}}$ [lot]	_	2.76	2.78	2.85	2.83	2.84

Table 5.8: Comparison for demand rate of 50 %

a	0	1	2	3	5	10,15,20
HPP $\bar{\varepsilon}_{3,2}(k)$ [lot]	22.27	1.61	-0.38	-0.4	-0.39	-0.42
CWIP $\bar{\varepsilon}_{3,2}(k)$ [lot]	∞	0.99	1.18	1.13	1.12	1.05
BS $\bar{\varepsilon}_{3,2}(k)$ [lot]	∞	0.79	1.16	1.08	1.25	1.23
HPP $\sigma_{\varepsilon_{3,2}}$ [lot]	3.44	1.86	1.57	1.57	1.57	1.57
CWIP $\sigma_{\varepsilon_{3,2}}$ [lot]	_	1.93	1.93	1.93	1.92	1.93
BS $\sigma_{\varepsilon_{3,2}}$ [lot]	_	1.75	1.94	1.94	1.94	1.94
HPP (theory)$\max \varepsilon_{3,2}(k)$ [lot]					7.68	7.68
HPP $\max \varepsilon_{3,2}(k)$ [lot]	33.48	7.48	3.4	3.35	3.4	3.4
CWIP $\max \varepsilon_{3,2}(k)$ [lot]	_	6.28	6.65	6.54	6.55	6.41
BS $\max \varepsilon_{3,2}(k)$ [lot]	_	5.83	6.62	6.44	6.76	6.75
HPP (theory) $\min \varepsilon_{3,2}(k)$ [lot]					-4.31	-4.31
HPP $\min \varepsilon_{3,2}(k)$ [lot]	11.07	-4.24	-4.18	-4.15	-4.19	-4.25
CWIP $\min \varepsilon_{3,2}(k)$ [lot]	_	-4.29	-4.28	-4.28	-4.30	-4.31
BS $\min \varepsilon_{3,2}(k)$ [lot]	_	-4.24	-4.29	-4.27	-4.26	-4.29
HPP $\sigma_{w_{3,2}}$ [lot]	3.22	2.75	2.47	2.48	2.49	2.44
CWIP $\sigma_{w_{3,2}}$ [lot]	_	3.67	3.71	3.51	3.60	3.61
BS $\sigma_{w_{3,2}}$ [lot]	_	2.94	2.95	2.98	2.96	2.97
HPP $\sigma_{w_{2,2}}$ [lot]	2.97	2.67	2.43	2.44	2.45	2.45
CWIP $\sigma_{w_{2,2}}$ [lot]	_	2.93	2.94	2.97	2.95	2.95
BS $\sigma_{w_{2,2}}$ [lot]	_	2.97	3.19	3.19	3.19	3.19
HPP $\sigma_{w_{1,2}}$ [lot]	3.09	2.28	2.14	2.15	2.12	2.19
CWIP $\sigma_{w_{1,2}}$ [lot]	_	2.77	2.73	2.75	2.69	2.76
BS $\sigma_{w_{1,2}}$ [lot]	_	2.68	2.78	2.79	2.91	2.85
HPP $\sigma_{w_{3,1}}$ [lot]	2.95	2.47	2.31	2.29	2.29	2.29
CWIP $\sigma_{w_{3,1}}$ [lot]	_	2.97	2.98	2.97	2.97	2.98
BS $\sigma_{w_{3,1}}$ [lot]	_	2.94	2.97	2.97	2.96	2.98
HPP $\sigma_{w_{2,1}}$ [lot]	2.89	2.38	2.29	2.30	2.26	2.29
CWIP $\sigma_{w_{2,1}}$ [lot]	_	2.94	2.94	2.93	2.92	2.93
BS $\sigma_{w_{2,1}}$ [lot]	_	3.00	3.21	3.24	3.23	3.20

Table 5.9: Comparison for demand rate of 75 %

a	0	1	2	3	5	10,15,20
HPP $\bar{\varepsilon}_{3,2}(k)$ [lot]	28.76	4.95	0.41	0.42	0.42	0.41
CWIP $\bar{\varepsilon}_{3,2}(k)$ [lot]	∞	5.23	3.5	3.54	3.41	3.26
BS $\bar{\varepsilon}_{3,2}(k)$ [lot]	∞	∞	3.45	3.3	3.57	3.53
HPP $\sigma_{\varepsilon_{3,2}}$ [lot]	3.55	2.48	1.58	1.66	1.64	1.65
CWIP $\sigma_{\varepsilon_{3,2}}$ [lot]	_	2.58	2.5	2.49	2.48	2.45
BS $\sigma_{\varepsilon_{3,2}}$ [lot]	_	_	2.56	2.61	2.58	2.58
HPP (theory)$\max \varepsilon_{3,2}(k)$ [lot]					7.73	7.73
HPP $\max \varepsilon_{3,2}(k)$ [lot]	40.83	13.3	4.27	4.27	4.27	4.27
CWIP $\max \varepsilon_{3,2}(k)$ [lot]	_	13.91	10.45	10.51	10.24	9.92
BS $\max \varepsilon_{3,2}(k)$ [lot]	_	_	10.24	10.05	10.57	10.44
HPP (theory) $\min \varepsilon_{3,2}(k)$ [lot]					-3.46	-3.46
HPP $\min \varepsilon_{3,2}(k)$ [lot]	16.68	-3.38	-3.44	-3.42	-3.43	-3.46
CWIP $\min \varepsilon_{3,2}(k)$ [lot]	_	-3.44	-3.43	-3.43	-3.41	-3.4
BS $\min \varepsilon_{3,2}(k)$ [lot]	_	_	-3.33	-3.44	-3.42	-3.36
HPP $\sigma_{w_{3,2}}$ [lot]	3.74	3.33	2.46	2.47	2.41	2.51
CWIP $\sigma_{w_{3,2}}$ [lot]	_	3.66	3.68	3.70	3.76	3.69
BS $\sigma_{w_{3,2}}$ [lot]	_	3.11	3.09	3.09	3.07	3.07
HPP $\sigma_{w_{2,2}}$ [lot]	3.18	3.09	2.62	2.56	2.57	2.58
CWIP $\sigma_{w_{2,2}}$ [lot]	_	3.08	3.08	3.15	3.11	3.05
BS $\sigma_{w_{2,2}}$ [lot]	_	3.02	3.65	3.78	3.80	3.75
HPP $\sigma_{w_{1,2}}$ [lot]	3.45	2.77	2.36	2.38	2.37	2.38
CWIP $\sigma_{w_{1,2}}$ [lot]	_	2.97	2.86	3.03	3.04	2.94
BS $\sigma_{w_{1,2}}$ [lot]	_	2.86	2.98	3.04	2.91	3.05
HPP $\sigma_{w_{3,1}}$ [lot]	3.53	2.86	2.27	2.30	2.30	2.31
CWIP $\sigma_{w_{3,1}}$ [lot]	_	3.20	3.17	3.22	3.20	3.20
BS $\sigma_{w_{3,1}}$ [lot]	_	3.01	3.11	3.08	3.09	3.04
HPP $\sigma_{w_{2,1}}$ [lot]	3.08	2.64	2.38	2.37	2.38	2.39
CWIP $\sigma_{w_{2,1}}$ [lot]	_	3.10	3.15	3.12	3.11	3.12
BS $\sigma_{w_{2,1}}$ [lot]	_	3.13	3.53	3.70	3.65	3.73

Table 5.10: Comparison for demand rate of 95 %

5.4 Conclusions

The performances of a multi re-entrant manufacturing network under surplus-based pull control has been studied. The developed results are presented in a form of inequalities that express upper and lower steady-state bounds on the production surplus.

First, these bounds are presented for every output of each production stage of a single re-entrant machine of N stages. It is considered that each production stage has an independent and variable processing speed. The bounds shown in Theorem 8 can be quite useful for network design as they clearly reflect the influence of the number of production stages on the demand tracking error accuracy. For more demanding manufacturing environments, where the size of the intermediate inventory can not be neglected, it was shown that the bounds on the production surplus also extend to the case when the machine has a limited intermediate buffer capacity. For the single re-entrant machine the relation between production surplus and the inventory level of every stage was presented in Theorem 9. We consider this last result to be of valuable importance for managers of such a network, which is due to the following. The upper bound on the demand tracking error given by (5.20) in Theorem 9 describes the relation between the demand tracking accuracy and the desired inventory (base stock) level of the manufacturing machine for any given number of production stages. Thus without any simulation-based analysis, a manager can use this relation as a reference tool for production control related decision making.

Then, the steady-state bounds on the demand tracking errors for each stage of a multi re-entrant manufacturing line were introduced. These bounds as well show the relation between the production surplus of each stage and several important factors such as: desired inventory level, production speed, demand rate and number of stages in each machine. It was assumed that for the re-entrant line each production machine has a variable processing speed while it is the same for all the stages in the machine. Simulation examples were presented and discussed in order to illustrate and support analytical results. One of the important outcomes of these examples is the relation between the amount of desired buffer content and the demand tracking error of the output stage of the re-entrant production line of 2 machines with 2 stages each. By simulation it was shown that the base stock levels have a limited influence on the demand tracking errors, which explains *Assumption 15*. This assumption on buffer content, eliminates the dependency of the demand tracking error bounds on the buffer levels of the network (see Theorem 10).

By means of simulation, the obtained theoretical results for the DT model are also tested on the DE model. The interpretation of the DT analytical bounds on demand tracking errors for the DE model is given for the selected examples on a single re-entrant machine. It was observed that, though the DT approximation is less accurate than DE approximation, the DT analytical results on performance are valid for the DE model as well. Further, simulation-based comparison of three particular cases of the selected surplus-based control strategies was conducted for a single re-entrant machine of 3 stages and a re-entrant line of 3 machines with 2 stages each. The maximum and minimum values of the steady-state demand tracking errors and intermediate inventory levels of the networks were evaluated under particular cases of HPP, BSP and CWIP policies. The obtained results show the dominance of HPP over the other two policies.

6

Liquitrol experimental platform

This chapter is based on Starkov et al. (2012b).

Abstract | In this chapter we present a prototype developed for education and research purposes. The prototype is a liquid based emulator of manufacturing network processes. At its core, the liquid-base emulator consists of several electrical pumps and liquid reservoirs. The electrical pumps emulate manufacturing machine behavior, while the liquid reservoirs serve as intermediate product storages also called buffers. In the platform, pumps and tanks can be interconnected in a flexible manner. In that way the prototype permits an easy and intuitive way of studying manufacturing control techniques and performance for several network topologies. This chapter contains a detailed system description and its application. Several network configurations and experimentations are presented and discussed.

6.1 Introduction

Year round, hundreds of student from Eindhoven University of Technology (TUE) in The Netherlands study multiple subjects on control and performance of production networks offered by the Manufacturing Networks group (MN) of the department of Mechanical Engineering. Though learning these subjects from theory and simulation is sufficient to prepare Bachelor or Master students for their future carrier, the presence of an experimental tool, where all studied theoretical phenomena can be visualized, gives the students an extra assurance of the obtained knowledge as well as facilitates their learning process. Furthermore, hand-on experiments, where students acquire the ability to solve problems using real equipment, form a fundamental part of their education as engineers.
In case of the MN group of the TUE, some of its education related activities consist

in lecturing and supervising students during their laboratory works, e.g. analysis of manufacturing networks, embedded systems, hybrid dynamics, and control of manufacturing systems courses. This in addition to research activities, which also include experimental studies, that are performed by its integrants. In general, research and teaching activities that are commonly practiced in engineering departments of higher education centers, commonly involve some practical tasks. Thus having an experimental tool can be of a great help for their completion.
In this chapter we present the experimental platform that was recently designed by the MN group and presently developed by Wafer Based Solutions company[1]. The prototype serves as a liquid-based emulator of manufacturing networks processes. This liquid-based emulator consists of several electrical pumps and liquid reservoirs. The electrical pumps emulate manufacturing machine behavior, while the liquid reservoirs serve as intermediate product storages also called buffers. These pumps and tanks can be interconnected in any desired or possible configuration. In that way the prototype permits an easy and intuitive way of studying the effects of manufacturing control techniques on the performance of several network topologies.

Similar prototypes can be frequently encountered in studies on control of fluid dynamics, and industrial processes, e.g., chemical reactants level and flow control. Fang et al. (2009) applied a fuzzy decision making mechanism to a part of an industrial system, the behavior of which was mimicked by a coupled water tank system developed by Quanser[2]. Another application of Quanser's coupled water tank system was presented in Pan et al. (2005), where the authors were motivated by a desire to provide precise liquid level control, developed a set of nonlinear backstepping techniques for the state-coupled, two-tank, liquid level system dynamics. Two coupled tanks CE105 prototype of TecQuipment [3] was used in Boubakir et al. (2009). In this paper the authors develop a neuro-fuzzy-sliding mode controller with a nonlinear sliding surface for a coupled tank system. Presented experimental results show that the suggested approach has considerable advantages compared to the classical sliding mode control. A quadruple tank prototype (see Johansson (2000)) was developed motivated by the necessity of demonstration tool for multivariable zero location and direction in an illustrative way for a frequency domain analysis of dynamical systems. This prototype is used in several courses in the control education, e.g., at LTH, Lund, Sweden, and at KTH, Stockholm, Sweden. Currently all the above mentioned prototypes serve a great purpose in education and research. However, there does not seem to exist any laboratory tool that can emulate the dynamics of several manufacturing networks in an illustrative and simple way. This was one of the main motivations for developing of Liquitrol platform. The name of the platform comes from combining the words liquid and control, which describe the basic operational principal and the purpose of this setup, respectively.

The remainder of the chapter is structured as follows. Section 6.2 presents a detailed system description subdivided in the mechanical and electrical specification. In Section 6.3 the dynamical model of the platform behavior for 3 selected network

[1] http://waferbasedsolutions.com
[2] Quanser Control Challenges at http://www.quanser.com
[3] http://www.tecquipment.com/Control/Control-Engineering/CE105.aspx

configurations are detailed. Section 6.4 presents 3 types of piratical experimentations for students. Conclusions and future work are given in Section 6.5.

6.2 System description

In this section we present a detailed Liquitrol platform description.

6.2.1 Mechanical specification

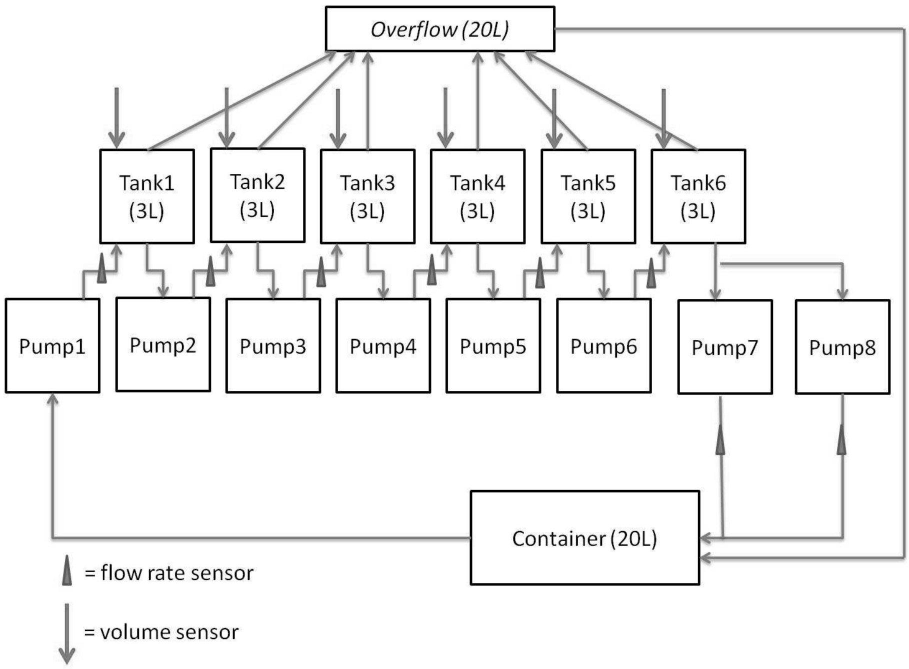


Figure 6.1: Liquitrol Block Diagram

Figure 6.1 shows a block diagram of the platform configured as a flow line. The platform consists of 6 water tanks, made of acrylate material, and 8 voltage driven water pumps of 12 volts each, with 3 liters per minute as the maximal speed of water transfer. A flow rate sensor, indicated by the thin blue triangle, is located on the output of every water pump. Each water tank is of 3 liter capacity. The water level of each tank can be measured by a pressure sensor depicted by the red arrow in Figure 6.1. There are two 20 liter containers in the platform. One container supplies the liquid into the system, which is returned to this container once it circulates through the system. The second container called *overflow sink* is directly connected to the supply container. It is installed under the water related equipment for protection, e.g. in case of a sudden tank overflow or undesired water leakage of the platform in general.

As can be seen on Figure 6.2, each of the 3 liter water tanks present 4 bottom connections: one for water inlets, two for water outlets and one for the pressure measurement. In the middle, each tank is equipped with an overflow tube made

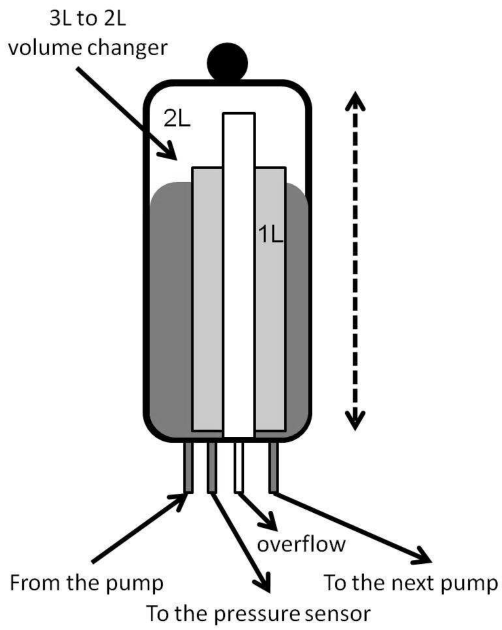


Figure 6.2: Liquitrol Liquid Reservoir

of thick pom material. In Figure 6.2, this tube is depicted in a light blue color. Once installed over the overflow tube, the cylinder acts as a volume reducer. Thus permitting a better visualization and a faster occurrence of certain phenomenons, e.g. instability in the Lu-Kumar network (Dai et al. (2004)), for its presentation during the lectures.

The outcome of the Liquitrol platform design, provided by Wafer Based Solutions company, can be seen in Figure 6.3, which shows the front view of the emulator. The platform is mounted on a four wheel table, the dimensions of which can be found in Table 1. The pipe switch board with 36 leak free copper connectors, whose purpose is described in the Figure 6.3, gives the platform flexibility in its configuration. Several network topologies can be emulated given the proper tubing connections.

Table 6.1: Liquitrol Dimensions

Platform	mm	Tank	mm
hight	900	outside diameter	120
width	630	inside diameter	114
length	990	hight	320

6.2.2 Electrical specification

All the electrical devices of the platform are controlled through the Beckhoff EtherCAT[4]. The Ethercat consists of several blocks such as: potential supply terminals,

[4]Beckhoff Automation EtherCAT at http://beckhoff.com

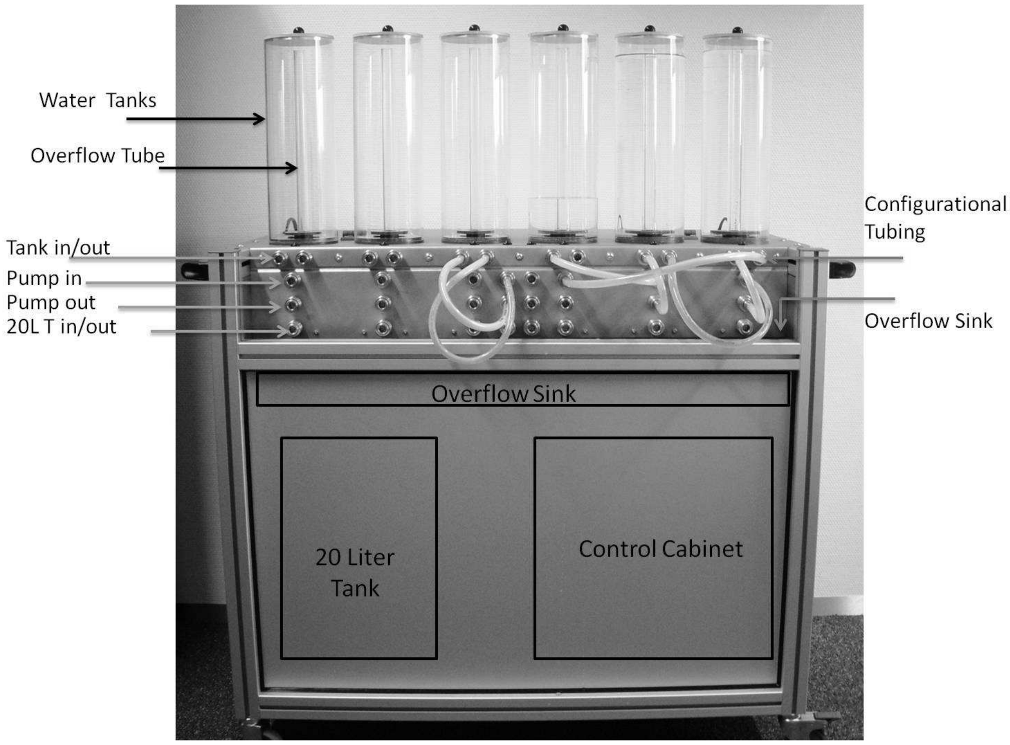


Figure 6.3: Liquitrol Platform

DC motor output, coupler, and AD/DA signal input/output terminals. All these blocks are mounted inside the control cabinet. The content of control cabinet can be graphically visualized by the block diagram of Figure 6.4. All the data acquisition and control signals to the Liquitrol are managed through an ethernet cable. The systems presents 3 electrically isolated power supplies; 2 of 12 volts and 1 of 24 volts. A more detailed power distribution diagram can be observed in Figure 6.5.

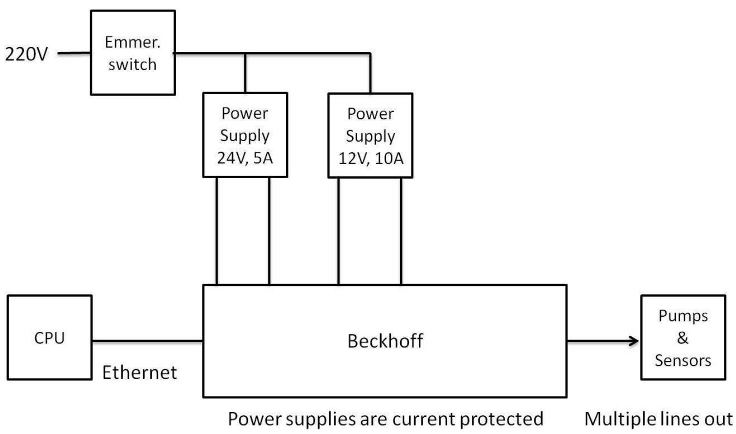


Figure 6.4: Liquitrol Electrical Block Diagram

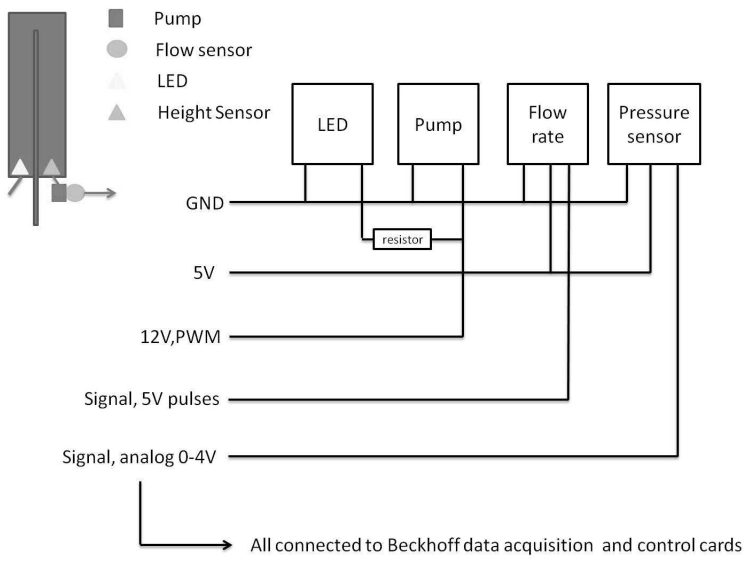


Figure 6.5: Liquitrol Electrical Block Diagram

6.3 Dynamical model

In this section the dynamical models for some of the selected configurations of the platform are described. It is shown how the liquid flow dynamics in the platform can be associated with flow models of manufacturing networks.

6.3.1 Single manufacturing machine configuration

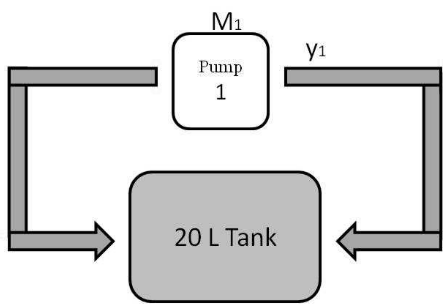


Figure 6.6: Schematics of a manufacturing machine configuration.

In discrete time, the cumulative number of produced products (pumped liquid) in time k for one manufacturing machine (water pump) can be described as the sum of its production rates at each time step till time k. Thus the flow model of one manufacturing machine (one water pump) in discrete time is defined as

$$y(k+1) \quad = \quad y(k) + u(k) + f(k), \tag{6.1}$$

where $y(k) \in \mathbb{R}$ is the cumulative output of the machine in time k, $u(k) \in \mathbb{R}$ is the control signal, and $f(k) \in \mathbb{R}$ is an unknown external disturbance. Under the assumption that there is always sufficient raw material (liquid) to feed the machine (pump), we can pose the control aim as to track the non-decreasing cumulative production demand. We can define the production demand by using $y_d(k) \in \mathbb{R}$ given by

$$y_d(k) = y_{d0} + v_d k + \varphi(k), \tag{6.2}$$

where y_{d0} is a positive constant that represents the initial production demand, which is the initial quantity of liquid required to be pumped, v_d is a positive constant that defines the average desired demand rate (desired flow rate), and $\varphi(k) \in \mathbb{R}$ is the bounded fluctuation that is imposed on the linear demand $v_d k$. In order to give a solution to this tracking problem, the following surplus-based Pull controller (Starkov et al. (2012a))

$$u(k) = \mu \mathrm{sign}_+(\varepsilon(k)) \tag{6.3}$$

is considered. Here μ is a positive constant that represents the processing speed of the machine (constant flow rate of the pump), step function $\mathrm{sign}_+(\varepsilon(k)) =(1$, if $\varepsilon(k) > 0|0$, otherwise), and $\varepsilon(k) \in \mathbb{R}$ is the output tracking error with respect to the demand. This demand tracking error is given by $\varepsilon(k) = y_d(k) - y(k)$, where $\varepsilon(k+1) - \varepsilon(k)$ along the solutions of $\varepsilon(k)$ is given by:

$$\varepsilon(k+1) - \varepsilon(k) = v_d + \Delta\varphi(k) - \mu \mathrm{sign}_+(\varepsilon(k)) - f(k), \tag{6.4}$$

with $\Delta\varphi(k) = \varphi(k+1) - \varphi(k)$. It follows from (6.4) that in order to guarantee proper demand trajectory tracking, the product demand cannot be higher than the machine processing speed, which in this case is μ lots per time unit. Thus, let us assume that all machine (pump) perturbations $W(k) = \Delta\varphi(k) - f(k)$ from (6.4) are bounded by

$$\alpha_1 < W(k) < \alpha_2, \quad \forall k \in \mathbb{N}, \tag{6.5}$$

where α_1, α_2 are some constants that satisfy

$$\alpha_2 \;<\; \mu - v_d, \tag{6.6}$$
$$\alpha_1 \;>\; -v_d. \tag{6.7}$$

By (6.6) and (6.7) we state that the machine (the pump) can never produce products (transfer fluids) faster than its maximal speed and that considering the presence of perturbations bounded by (α_1, α_2) the demand rate can only be positive, respectively. Thus, by this last condition we assume that the pump transfers the liquid in only one direction. From (6.5), (6.6), and (6.7) the following condition (also known as capacity condition) holds

$$0 < v_d + W(k) < \mu. \tag{6.8}$$

In other words, the flow rate in the system is limited by the speed of the pump. Basically, in order to follow the product demand, variable structure controller $\mu \mathrm{sign}_+(\varepsilon(k))$ is included in the flow model of one machine. The demand tracking error of a single machine is defined as the difference between the cumulative demand and the cumulative number of products produced (liquid pumped) up to this moment. More detailed theoretical and simulation results on this topic can be found in Chapter 2.

6.3.2 Line of manufacturing machines configuration

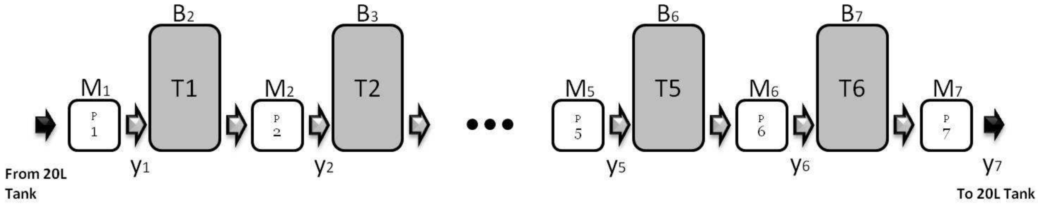


Figure 6.7: Schematics of a line configuration of 7 manufacturing machines.

The flow model of a manufacturing line is presented in this section. Figure 6.7 shows the schematics of a manufacturing line configuration for the Liquitrol platform. The small white rectangles represent the 7 water pumps, the big gray rectangles depict the 6 water tanks and the arrows indicate the liquid flow direction. On top of each white and gray rectangles, manufacturing machine and buffer labels are denoted by M_j and B_j, respectively. The letter M stands for machine with its number given by a constant $j = 1, ..., 7$ and the letter B stands for buffer with $j = 2, ..., 7$. For M_1 the raw material (liquid) is provided from the main 20 liter deposit.

The flow model of the manufacturing line is defined as

$$
\begin{aligned}
y_1(k+1) &= y_1(k) + \beta_1(k)\mathrm{sign}_-(w_2(k) - \gamma_2), && (6.9)\\
y_j(k+1) &= y_j(k) + \beta_j(k)\mathrm{sign}_{Buff}(w_j(k) - \beta_j(k)) && (6.10)\\
&\times\ \mathrm{sign}_-(w_{j+1}(k) - \gamma_{j+1}),\ j = 2, \ldots, 6, &&\\
y_7(k+1) &= y_7(k) + \beta_7(k)\mathrm{sign}_{Buff}(w_7(k) - \beta_7(k)), && (6.11)
\end{aligned}
$$

where $y_j(k)$ is the cumulative output of the machine M_j in time k, $w_j(k) = y_{j-1}(k) - y_j(k)$ is the content of the buffer (tank) B_j, $\beta_j(k) = u_j(k) + f_j(k)$, $\forall j = 1, \ldots, 7$, f_j is the external disturbance affecting machine M_j (e.g., production speed variations, undesired delay or setup time), u_j is the control input of machine M_j, $\mathrm{sign}_{Buff}(x) = (1,\text{ if } x \geq 0 \,|\, 0, \text{ otherwise})$, $\mathrm{sign}_-(x) = (1, \text{ if } x \leq 0 \,|\, 0,$ otherwise), and γ_{j+1} is the threshold value of the buffer (tank) content w_{j+1}.

In order to give a solution to the demand tracking problem we propose the following control inputs:

$$
\begin{aligned}
u_j(k) &= \mu_j\mathrm{sign}_+(\varepsilon_{j+1}(k) + w_{d_{j+1}} - w_{j+1}(k)), && (6.12)\\
&\quad \forall j = 1, \ldots, 6, &&\\
u_7(k) &= \mu_7\mathrm{sign}_+(y_d(k) - y_7(k)), && (6.13)
\end{aligned}
$$

where μ_j is the constant processing (pumping) speed of the machine (pump) j, $w_{d_{j+1}}$ is the constant that represents the desired inventory level (liquid level) of buffer (tank) B_{j+1} and ε_{j+1} is the demand tracking error of machine M_{j+1}. Here the value of sign_+ function is same as was defined in the previous subsection. The demand tracking error of each machine is given by:

$$\varepsilon_j(k) = \varepsilon_{j+1}(k) + (w_{d_{j+1}} - w_{j+1}(k)), \quad \forall j = 1, \ldots, 6, \tag{6.14}$$

$$\varepsilon_7(k) = y_d(k) - y_7(k). \tag{6.15}$$

It follows from (6.15) that the demand tracking error of machine M_N is defined exactly as for the single machine case. The water tank capacity restriction, as seen from (6.11), is the only difference in the flow model of machine M_7 with the flow model of (6.1). In (6.14) new considerations are applied to the demand tracking error of each machine M_j, where $j = 1, ..., 6$. Here $\varepsilon_j(k)$ depends on number of produced products (amount of transferred liquid) $y_j(k)$ with respect to current demand $y_d(k)$ (desired amount of liquid) and desired buffer content (desired tank fluid level) $w_{d_{j+1}}$ of each downstream buffer. This means that every upstream machine needs to produce $w_{d_{j+1}}$ lots more than the downstream one. Constant parameter w_d is introduced in order to prevent downstream machines from starvation, e.g., in case of a sudden growth of the product demand. More detailed theoretical results on this topic can be found in Chapter 3.

6.3.3 Re-entrant network configuration

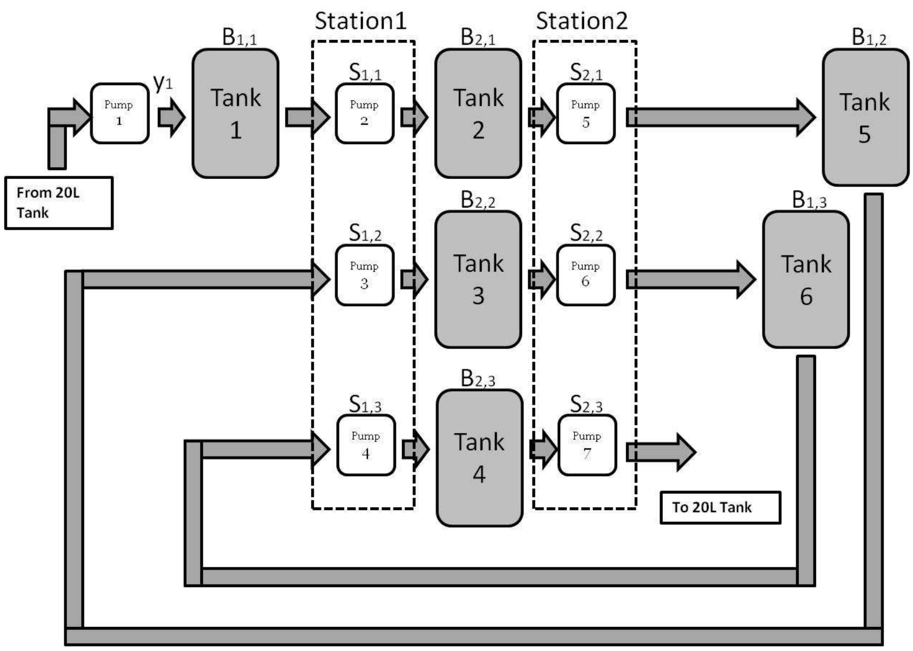


Figure 6.8: Schematics of a re-entrant line configuration of 2 manufacturing stations.

Figure 6.7 shows the schematics of a re-entrant manufacturing network configuration for the Liquitrol platform. The small white rectangles represent the 7 water pumps, where pump 1 acts as the arrival rate generator and the rest of the pumps as the production stages of 2 station network. Each station, given by a dashed transparent rectangle, contains 3 water pumps that supply the liquid to 3 liter water tanks denoted by the gray rectangles. The liquid flow through the system is indicated by the gray arrows. On top of each white and gray rectangle, a manufacturing stage label and buffer label is denoted by numbers, respectively. The letter S stands for the word stage with the first subindex $i = 1, 2$ indicating the station number and the second subindex $j = 1, 2, 3$ denoting the stage number, which is similar for the buffers $B_{i,j}$. For $B_{1,1}$, raw material (liquid) is provided from the main 20 liter deposit through the arrival rate generating pump.

The flow model of arrival rate (generated by pump 1 of Figure 6.8) and each production stage of a re-entrant line in discrete time is defined as

$$\begin{aligned} y_1(k+1) &= y_1(k) + (v_d k + \varphi(k))\text{sign}_-(w_{1,1}(k) - \gamma_{1,1}), \\ y_{i,j}(k+1) &= y_{i,j}(k) + \beta_{i,j}(k)u_{i,j}(k)\text{sign}_-(w_{r,s}(k) - \gamma_{r,s}), \end{aligned} \tag{6.16}$$

where $y_1(k)$ is the cumulative output of pump 1 in time k, $v_d k + \varphi(k)$ is the arrival rate of raw material to S_1 (similar to (6.2)), $y_{i,j}(k) \in \mathbb{R}$ is the cumulative output of station i in processing stage j in time k, $u_j(k) \in \mathbb{R}$ is the control input of station i in processing stage j and $\beta_{i,j}(k) = \mu_{i,j} + f_{i,j}(k)$ with $\mu_{i,j}$ as a positive constant that represents the processing speed (pump speed) of the machine i at its stage j and $f_{i,j}(k) \in \mathbb{R}$ is an unknown external disturbance affecting the performance of the ith station at its stage j. The variable $w_{r,s}(k) = y_{i,j}(k) - y_{r,s}(k)$ is the buffer content of buffer $B_{r,s}$ and $\gamma_{r,s}$ is the constant that represents the threshold inventory level of buffer $B_{r,s}$. The constants r and s can be selected as follows. If $i = 1$ then $r = 2$, and $s = j$. If $i = 2$ then $r = 1$, and $s = j + 1$.

One of the possible control aims of such a system could be to manage a stable production flow on the output of this network given a certain raw material arrival rate on its input.

In order to give a solution to this production problem we consider the following clearing policy (see Kumar and Seidman (1990)):

$\{q_i(k) = B_{i,j}\}$
if $w_{i,j}(k) \geq \beta_{i,j}(k)$ **then**
$u_{i,j}(k) = 1,$
$u_{i,s}(k) = 0, \forall s \neq j,\ s, j = 1 \ldots, 7,$
$q_i(k+1) = B_{i,j},$
end
if $w_{i,j}(k) < \beta_{i,j}(k)$ **and**
$\exists\, s \neq j : w_{i,s}(k) \geq \beta_{i,s}(k)$ **then**
$u_{i,j}(k) = 0,$
$u_{i,s}(k) = 1,$
$q_i(k+1) = B_{i,s},$
end

$$
\begin{aligned}
&\textbf{if } w_{i,s}(k) < \beta_{i,s}(k), \forall s \textbf{ then}\\
&u_{i,j}(k) = 0,\\
&u_{i,s}(k) = 0, \forall s \neq j,\ s, j = 1 \ldots, 7,\\
&q_i(k+1) = 0\\
&\textbf{end}
\end{aligned} \tag{6.17}
$$

where $q_i(k)$ is the internal variable representing the current buffer (tank) that S_i is processing, for the current time step $\beta_{i,j}(k)$ is the minimal raw material content in buffer $B_{i,j}$ (minimal amount of liquid that the pump is able to transfer without running dry), such that station S_i is able to process if required at its stage j. More detailed theoretical results on this topic can be found e.g. in Kumar (1993), and Laumanns and Lefeber (2006).

6.3.4 Network Configurations

More complex network configurations can also be emulated on Liquitrol platform. Some examples of these networks are shown in Figures 6.9, 6.10, 6.11, and 6.19. The details on dynamical models and control applications for these networks can be found in Lefeber et al. (2011), Savkin and Evans (2002), Banks and Dai (1997), and Kumar (1993), respectively. Due to extensive technical details we omit the dynamical models and control applications for the depicted networks of this subsection and only restrict ourselves by providing the bibliographical references for an interested reader.

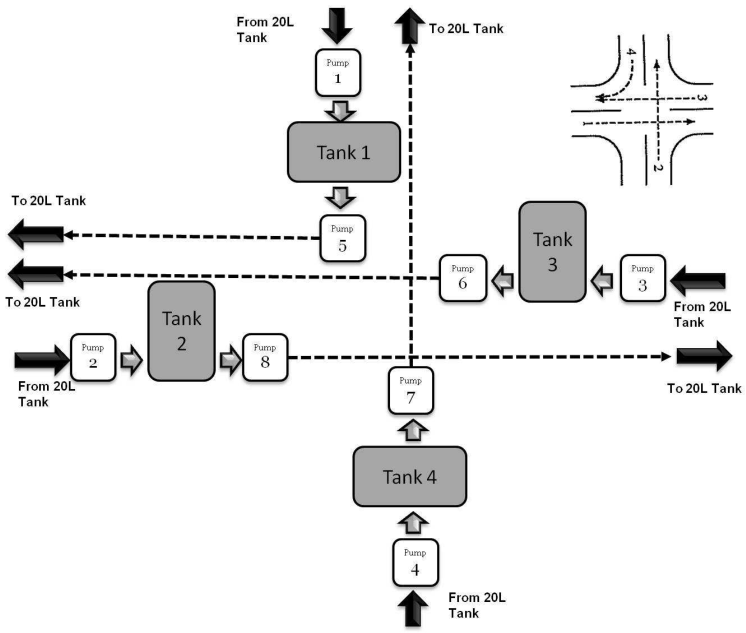


Figure 6.9: Road intersection for traffic control, Lefeber et al. (2011).

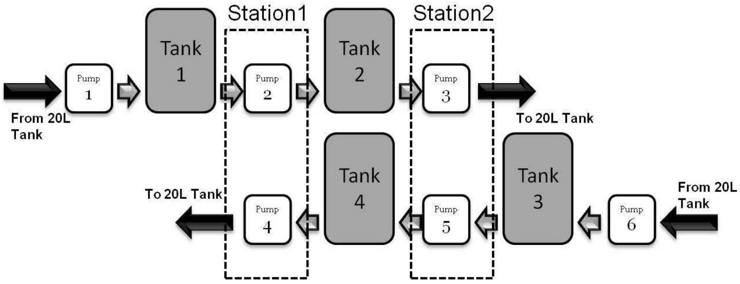


Figure 6.10: Flexible manufacturing network, Savkin and Evans (2002).

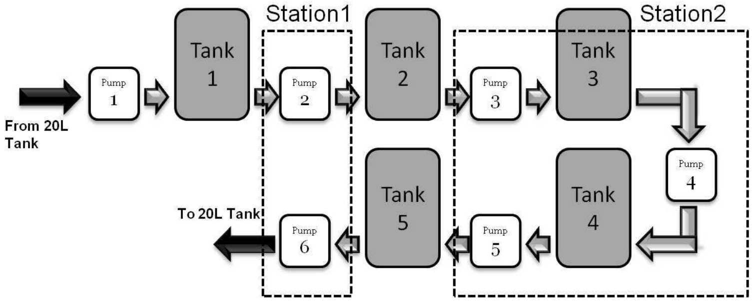


Figure 6.11: Bramson type re-entrant network, Banks and Dai (1997).

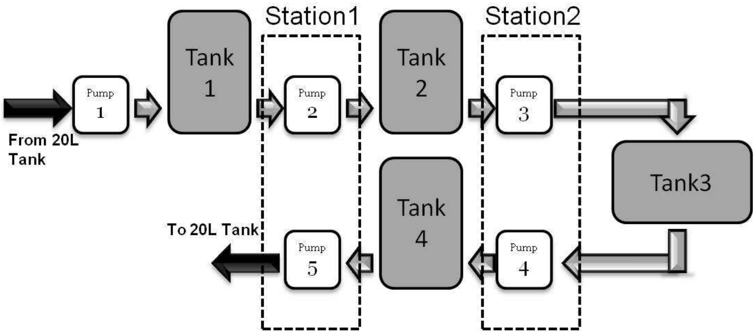


Figure 6.12: Kumar-Seidman type re-entrant network, Kumar (1993).

6.4 Experiment types

Several experiments can be executed on Liquitrol by students during their practice sessions.

6.4.1 Get familiarized with Liquitrol

The Beckhoff TwinCAT 2.11 software gives a student the opportunity to get familiarized with the platform. Simple tasks such as pumping the liquid from the main container to one of the tanks, reading the data and calibrating pressure and flow rate sensors, can be performed. The screen shot of such an experiment is shown on Figure 6.13. Two graphs and two tables can be observed in the picture. The upper

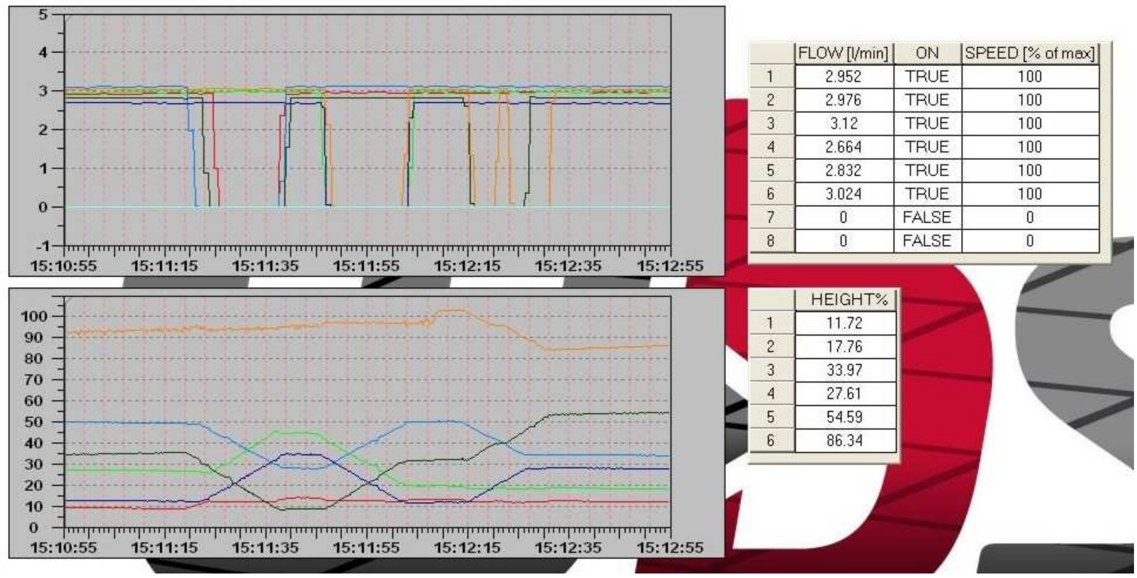


Figure 6.13: Testing the platform

graph presents the flow rate measurements, in liters per minute, of 6 pumps and the lower graph depicts the volumes, in percentage, of the 6 tanks. The pumps and tanks were arranged in a line configuration (see Figure 6.7). The experiment lasts for 2 minutes and the last measured values are reported in the tables of Figure 6.13.

6.4.2 Queuing theory based experiments

EtherLab, an open source toolkit for rapid real time code generation under Linux using Simulink®, permits the student to implement control models on the platform[5]. For example, given the selected network topology, the arrival rate of the liquid (deterministic or stochastic), and production rates of the pumps (deterministic or stochastic) a student can perform the following tasks: identification of a bottleneck machine, visualize the dynamics of buffer levels over time, calculation of mean throughput and utilization, work in process (WIP) identification, test and comparison of Push, Pull, and Conwip strategies.

6.4.3 Production Control

More complicated tasks can be given to the Master student. Such as comparative performance analysis of a certain network topology operated under different production control strategies, identification of the optimal production control strategy given the network and their operational characteristics, or evaluation and visualization of network performance under influence of delays and set-up times.

[5]For Windows users, with some additional work, the interface with the platform can be managed through TwinCAT 2.11

6.5 Experiments

This section presents the results of selected experiments made with the Liquitrol. The purpose of these experiments is to show the capability and flexibility of the platform.

6.5.1 Unidirectional line

The experiment of this section is based on the first example of Section 3.4.2.

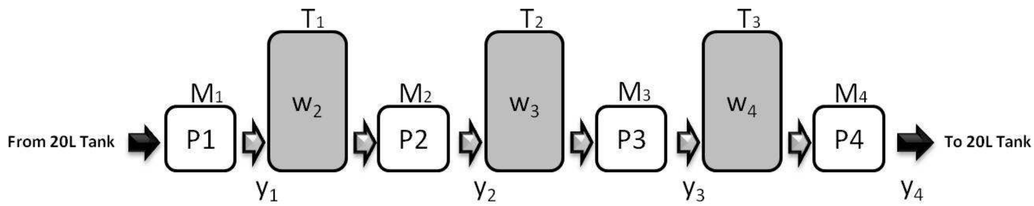


Figure 6.14: Manufacturing line on Liquitrol

In that example a production line of 4 manufacturing machines (see Figure 6.14) operated under surplus-based regulators (3.3) and (3.4) is studied. The processing speed of each machine is set to $\mu_j = 6$ lots per time unit $\forall j = 1, 3$ and $\mu_j = 4$ lots per time unit $\forall j = 2, 4$, the desired inventory level of each buffer is selected considering (3.16) as $w_{d_j} = 12$ (lots), with $j = 2, .., 4$ and the mean demand rate $v_d = 3.5$ lots per time unit with fluctuation rate of $\Delta\varphi(k) = 0.2 \sin(5k)$.

Similarly in our experiment we consider a line of 4 pumps and 3 tanks (see Figure 3.3) operated under regulators (3.3) and (3.4). Every millisecond, in our experimental setting, the values of all the sensors (pressure and flow rate) are updated and new control actions are taken. The value of 100 milliliters is set in correspondence to one lot with the example of Section 3.4.2. The processing speed of each pump is set to $\mu_j = 2.75$ liters per minute $\forall j = 1, 3$ and $\mu_j = 1.83$ liters per minute $\forall j = 2, 4$, which corresponds to 6 and 4 lots per time step respectively, with time step k of 13.09 seconds. The desired inventory level of each buffer is selected as in the example of Section 3.4.2. The mean demand rate is given by $v_d = 1.6$ liters per minute with fluctuation rate of $\Delta\varphi(k) = 0.0917 \sin(5k)$ liters per minute. Similarly to the discrete time (DT) simulation example, each pump in the line cannot start its production if its immediate downstream buffer contains less liquid than the pump can produce in one time step. The difference of this example with the one from Section 3.4.2 is that the control action as well as the growth of the production demand signal are not restricted by the time step of 13.09 seconds, but by one millisecond, which is the cycle time of the experimental setup. It is important to notice that, different from the DT model of a manufacturing machine, the water pump, if turned on, will provide liquid in a continuous manner. Thus, if some pump in the line has a lower processing rate then its upstream pump, then it may start its operation earlier than assumed in our DT model of the manufacturing line.

The cumulative demand and output trajectories are shown in Figure 6.15. The demand tracking error of each machine is depicted in Figure 6.16. The buffer contents are shown in Figure 6.17. The similarity in behavioral patterns can be observed with the DT simulation results of Figures 3.8, 3.9 and 3.10 of Section 2.5, respectively. Also, similarly to the second example of Section 2.5, the demand tracking error dependency on the base stock levels in the manufacturing network is depicted in Figure 6.18. The experimental results of Figure 6.18 show similarity in behavioral pattern with the results presented in Figure 3.11 of Section 2.5.

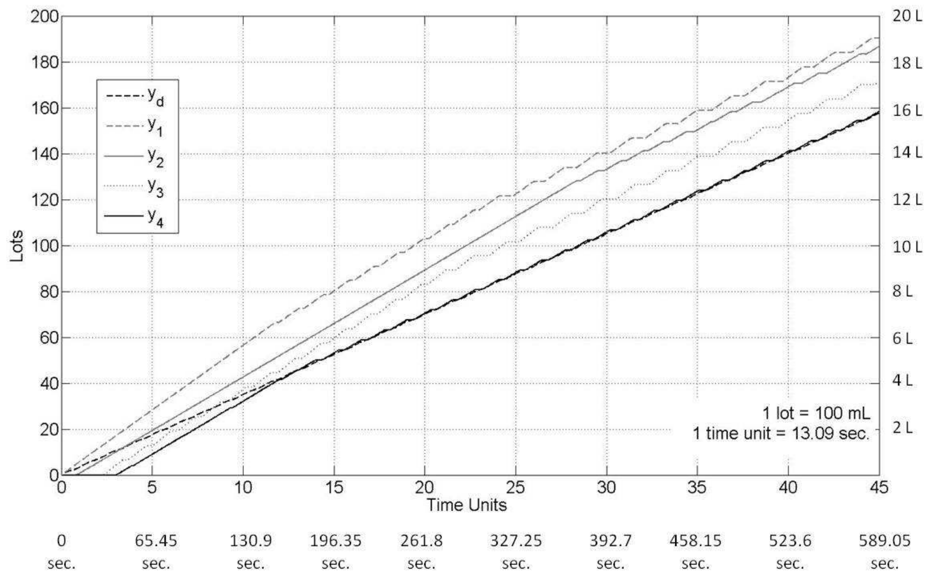


Figure 6.15: Manufacturing line on Liquitrol: Outputs y_j vs. Demand y_d.

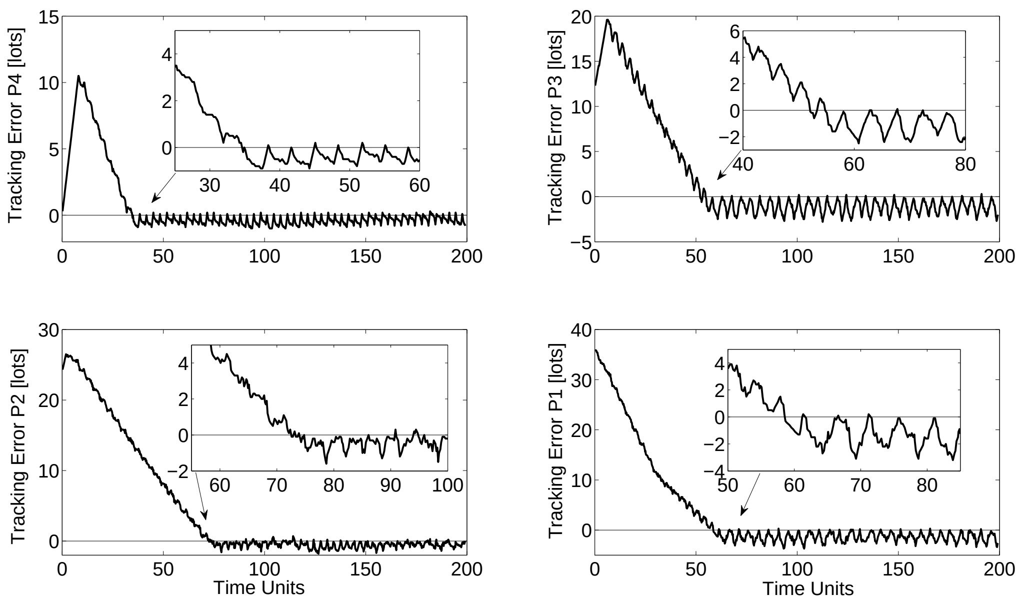


Figure 6.16: Manufacturing line on Liquitrol: Demand Tracking Errors.

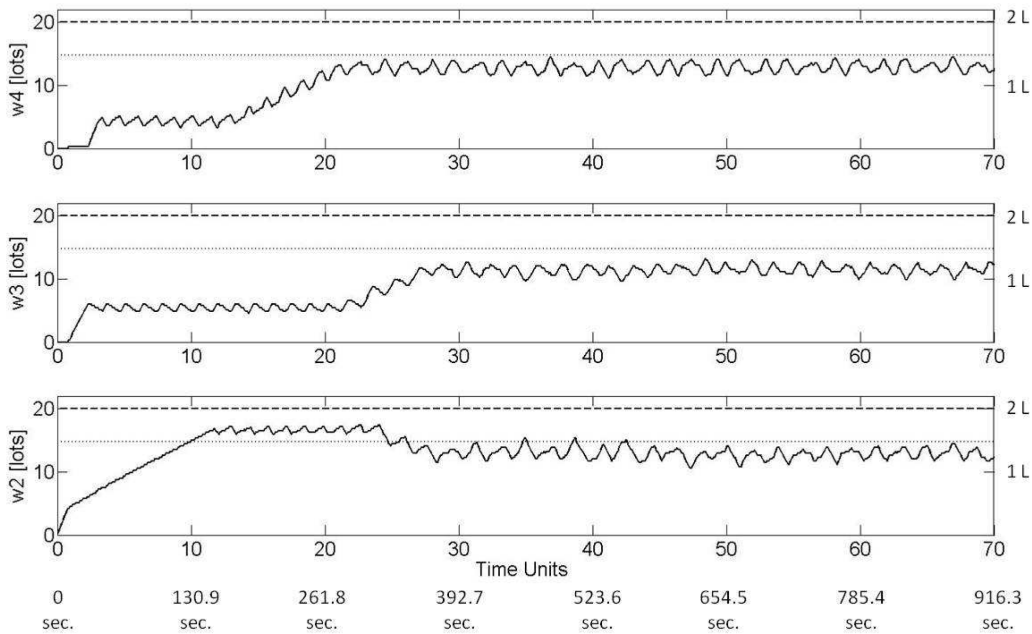


Figure 6.17: Manufacturing line on Liquitrol: Buffer Content $w_j(k)$.

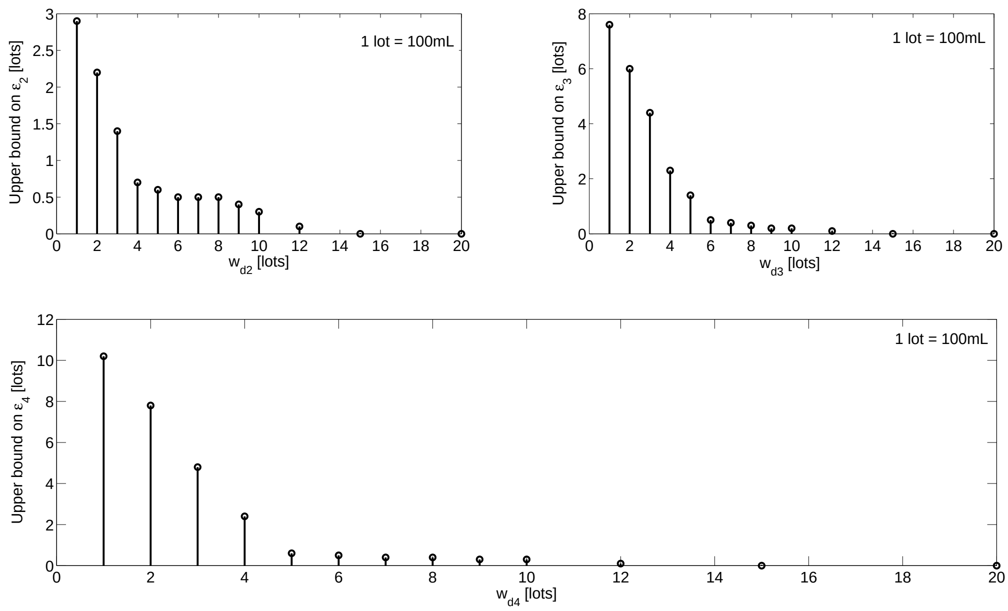


Figure 6.18: Manufacturing line on Liquitrol: Demand Tracking Errors vs. Desired Inventory Levels.

6.5.2 Re-entrant line: Kumar-Seidman instability example

A Kumar-Seidman instability example (see Kumar (1993)) is tested on our setup. In this setting we shall say that the manufacturing network is unstable if the buffer level trajectories of the network are unbounded. In the example, a re-entrant

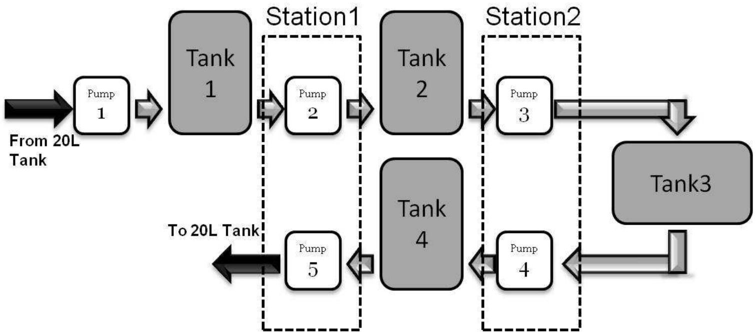


Figure 6.19: Kumar-Seidman type re-entrant network

network (see Figure 6.19) of 2 stations of 2 stages (pumps) each, is operated under a buffer clearing policy. Thus assuming a constant arrival rate of raw material (water provided by Pump 1) to the first buffer (Tank 1), the goal of each stage of every station is to process all the products of its adjacent buffer, i.e clear its buffer, in one at a time manner. In other words, in each station only one stage can perform its buffer clearing action at a time and only after the buffer of operational stage is empty, another stage is selected to perform its clearing action. In the example the arrival rate is selected in accordance to the *capacity condition*. This condition delimits the product arrival rate to the production speed of the slowest station in the network. From Figure 6.19, Pump 2 and 4 operate at 2.7 [liters/min], and Pump 3 and 5 at 1.35 [liters/min]. The arrival rate of liquid to Pump 2 is generated by Pump 1 at 0.8 [liters/min]. In the example of Kumar (1993) it was

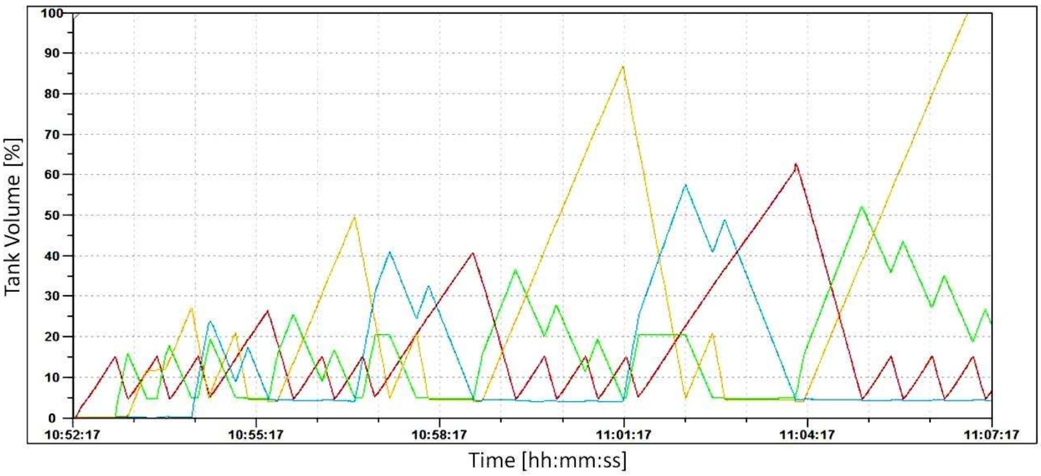


Figure 6.20: Kumar-Seidman example on Liquitrol. Volume of tank 1 till 4 is depicted by red, green, yellow and blue lines, respectively.

shown that, although the *capacity condition* for the network is satisfied, under a certain selection of production speed values, starvation phenomena occur in the network, which consequently lead the instability of the system. The products are accumulated in some intermediate buffers while in others there is none. Similar to the result in Kumar (1993), the growing oscillatory behavior of the trajectories of

4 buffer contents of the re-entrant network of Figure 6.19 is shown in Figure 6.20. More detailed theoretical results on this topic can be found e.g. in Kumar (1993). Note that several control solutions are given to the instability problem of re-entrant networks; see for example “buffer regulator” of Humes (1994), “stream modifier” of Burgess and Passino (1997) and “controlled buffer technique” of Somlo et al. (2004).

6.6 Conclusions

In this chapter a detailed description of a manufacturing network emulating tool, called Liquitrol, was presented. The tool was developed by the Manufacturing Networks Group so that students can visualize and control the production flow for several manufacturing network topologies. With Liquitrol, students can deepen in basic as well as advanced concepts of manufacturing network dynamics and control. Some of these control concepts were mentioned in Sections 6.3, 6.4, and 6.5 of this chapter. Furthermore, from topologies shown in Section 6.3.4, one can observe that the Liquitrol is not only restricted to manufacturing networks, but it can also be useful in the analysis of traffic control problems. Currently the platform is located in the department of Mechanical Engineering of Eindhoven University of Technology, where it serves its purpose for the Bachelor and Master students of the Manufacturing Networks Group.

Future work will include the implementation of laboratory sessions with Liquitrol via internet. In that way, students can create their control models and test them on the platform via a limited access webpage containing the real live video image of the prototype and the necessary tools to upload and test their assignments on the platform.

7

Observer-based approach

This chapter is based on Andrievsky et al. (2012).

Abstract | This chapter deals with the problem of controlling a tandem line of manufacturing machines such that an unknown production demand is tracked with a desired accuracy. To study this problem, a manufacturing machine is approximated by an integrator which is subject to input saturation as a result of the finite capacity of the machine. To solve the problem in case of unknown demand rate, a combination of feedforward-feedback controller with a reduced-order observer is proposed. The decentralized feedback control strategy for a line of machines is proposed and studied both in continuous time-domain and in frequency-domain representation.

7.1 Introduction

The production control of manufacturing systems, i.e. how to control the production rates of machines such that the system tracks a certain customer demand while keeping a low inventory level, has been a field of interest for several decades. Early control strategies based on simple push and pull concepts, such as material requirements planning (MRP), enterprize resources planning (ERP), and just-in-time (JIT), see e.g. (Hopp and Spearman, 2000), can provide an adequate solution if the system requirements are not very strict and a fast reaction to possible disturbances/failures is not required (e.g. since such disturbances/failures hardly occur). However, as manufacturing systems become more complex and the system's performance must constantly improve in order to stay competitive in today's global economy, these control strategies become less effective.

A possible way to tackle the problem is to describe manufacturing systems using so-called flow models, see e.g. (Alvarez-Vargas et al., 1994). These models, which are based on ordinary differential/difference equations (ODEs), or sometimes partial differential equations (PDEs), see e.g. (van den Berg et al., 2008; Lefeber et al., 2005), form a continuous approximation of the discrete-event manufacturing systems and therefore result in a simpler control problem. Moreover, various (advanced) control theories are already available for ODEs, which makes these models attractive to work with. Some of the valuable developments in control of manufacturing systems by means of ODEs or PDEs are mentioned in the following literature review.

Feng and Yan (2000) considered a production control problem in a manufacturing system with a failure-prone machine and a stochastic demand. The case of the exogenous demand forming a homogeneous Poisson flow is studied. It is shown that a near-optimal production control remains a *threshold production* control type in the sense of (Kimemia and Gershwin, 1983). The explicit form of the production control policy and the objective functions are provided.

Boukas and Liu (2001) considered the production and maintenance control problem of a manufacturing system with part deterioration. By using the stochastic dynamic programming technique, the production and the maintenance rates are optimized.

Hong and Prabhu (2003) present differential equation-based models for distributed arrival time control of parallel dissimilar machines. It is shown that the behavior of general systems under distributed arrival time control was predictable. Convergence properties of the resulting nonlinear systems established using the theory of discontinuous differential equations.

Braun et al. (2003) developed the translation of the supply chain problem into a formulation amenable to MPC implementation. The Intel demand network problem is used to evaluate the relative merits of various information sharing strategies between controllers in the network. The material flows from the Factory to the Retailer and on to the Customer are modeled with a mass balance. Results show the potential of Model Predictive Control as an integral component of a hierarchical, enterprize-wide planning tool that functions on a real-time basis, supports varying levels of information sharing and centralization/decentralization, and relies on combined feedback-feedforward control action to enhance the performance and robustness of demand networks.

In Surana et al. (2005) the treatment of supply chains as a kind of complex adaptive systems is given. The authors demonstrate how tools and techniques based on the fields of nonlinear dynamics, statistical physics and information theory can be exploited to characterize and model supply-chain networks.

Huang et al. (2009) considered the closed-loop supply chain model under uncertainty of time-delay in re-manufacturing and returns, uncertainty of system cost parameters and uncertainty of customers' demand disturbances. The robust operations in the closed-loop supply chains are studied and relative strategies with robust H_∞ control methods are presented.

Most control strategies proposed in literature that use flow models to describe the manufacturing system, are based on the assumption that an estimate of nominal or the future demand is known, and use some optimization algorithm to find a suitable control signal, see e.g. (Gershwin, 1989; Sharifnia, 1994; Vargas-Villamil et al., 2003; Savkin, 1998; Bauso et al., 2006) and references therein. In the ODE models, a manufacturing machine is usually interpreted as an integrator, where the cumulative number of finished products is the integral of the production rate. Bounds on the production rate, due to the finite capacity of the machine, are then taken into account in the optimization problem. Disadvantages of these control strategies are that they depend on future demands (which are hard to predict and therefore often inaccurate) and that in general the optimization problem requires much computational effort.

In this chapter, a different strategy is employed for the control of manufacturing machines, which does not depend on future demands and requires less computational effort. For this control strategy, the manufacturing machines are still approximated by an integrator, but the bounds on the production rate are interpreted as a saturation function. This approach is based on the previous results (van den Bremer et al., 2008) where a simple PI-controller is proposed to solve the problem. Due to saturation in the control loop, an anti-windup compensator is required in (van den Bremer et al., 2008) to avoid undesirable oscillations in the presence of disturbances. In this chapter we implement a simpler approach: under assumptions that the nominal demand rate is constant (as well as possible disturbances) we design an observer to estimate this rate to utilize in the control algorithm. In case of uncertainty in the demand, caused, for example, by seasonal fluctuations in the market, we study if the closed loop system affected by this fluctuations is convergent. This idea results in a simple control algorithm that can be also used in more complex manufacturing lines where additional constraints on buffer content are also imposed. It is worth mentioning that this control algorithm can be classified as a *surplus-based production control* (Gershwin, 2000), where the control decisions are based on the demand tracking error which is the difference between the cumulative demand and the cumulative output of the system (see, e.g. (Bielecki and Kumar, 1988; Bonvik et al., 1997; Lefebvre, 1999; Gershwin, 2000; Quintana, 2002; Kogan and Perkins, 2003; Boukas, 2006; Stockton et al., 2007; Subramaniam et al., 2009; Nilakantan, 2010) and references there in). Detailed literature surveys specifically on the production line control mechanisms can be found in, e.g., (Bonvik et al., 1997), (Gershwin, 2000), (Ortega and Lin, 2004), (Sarimveis et al., 2008).

To analyze performance of the production control strategy we utilize a frequency domain based approach. The implementation of this approach is possible due to the convergency property of the system. That is, if the system is convergent for a class of a given periodic input signals, then there exists a unique steady-state periodic solution of this system (Pavlov et al., 2005). This property gives us the option of conducting a performance analysis by means of generalized sensitivity and complementary sensitivity functions. Though the frequency domain analysis can be considered as a rather classical control notion, the usage of this tool for the performance analysis of nonlinear production models is to the best of our knowledge and experience known as a novel application. Further details on this topic are presented in the Section 7.5.

7.2 Problem statement and guideline of presented research

Following (van den Berg et al., 2006; Andrievsky et al., 2009; Kommer et al., 2009), let us use a continuous approximation of the single discrete-event manufacturing machine. Namely, consider a manufacturing machine that produces items with a *production rate* $u_p(t) \in \mathbb{R}$, $t \in \mathbb{R}$. Assume that there is always sufficient raw material to feed the machine. The total amount of items produced by the machine is denoted by $y(t) \in \mathbb{R}$ and is related to production rate $u_p(t)$ by the following equation

$$\dot{y}(t) = u_p(t) + f(t), \tag{7.1}$$

where $f(t) \in \mathbb{R}$ stands for an unknown *external disturbance*. This term may describe manufacturing losses, or variations of the machine capacity, for example. The production rate u_p is considered to be positively valued and has a certain upper bound $u_{p,\max}$, caused by the machine capacity limitation. Therefore, the following bounds are valid for the production rate:

$$0 \le u_p \le u_{p,\max}. \tag{7.2}$$

Assuming that the production capacity of the machine is limited, the inequality (7.2) introduces a *saturation* in the control loop. The saturation effect complicates the design of the control law and the system performance analysis.

Here the control aim is to track the non-decreasing production demand $y_d(t)$. In what follows we assume that $y_d(t)$ may be modeled as

$$y_d(t) = y_{d,0} + v_d\, t + \varphi(t), \tag{7.3}$$

where $y_{d,0}$ denotes the desired production at $t = 0$, v_d is a constant that represents the average desired production rate, $\varphi(t)$ is a bounded function, describing fluctuation of the desired production from the linearly increasing time-varying demand, caused by market (e.g. seasonal) fluctuation. It is natural to suppose that the following inequalities (also known as capacity condition) are satisfied

$$0 \le v_d \le u_{p,\max}. \tag{7.4}$$

Thus the production rate is positive and on average the machine is capable to satisfy the product demand. It may be also assumed that $\varphi(t)$ has a "zero mean" in some sense, because its averaged value may be considered as part of $y_{d,0}$.

Though the general form of the production dynamics is given, the future demand value in most of the practical cases is not always know in advance. Here we consider to give a solution to the production demand tracking problem for a unidirectional manufacturing line under assumption that the future production demand is given by some unknown but bounded function as described in (7.3).

First, to tackle the above mentioned problem, the PI-controller with an anti-windup control strategy was proposed and thoughtfully studied in van den Berg et al.

(2006). This controller ensures an asymptotically vanishing demand tracking error $e(t) = y_d(t) - y(t)$ for constant $\varphi(t)$, $f(t)$ and independence of the asymptotic system behavior of the initial conditions if fluctuations and disturbances are present (the so called "*convergence property*", see Pavlov et al. (2004); van den Berg et al. (2006); Pavlov et al. (2006) for details).

In this chapter we propose an alternative control law, which consists in a combined feedforward–feedback control strategy. Since the integral of the demand tracking error is not used in the proposed controller, the anti-windup compensator in the controller is no longer needed.

The chapter has the following structure. In Section 7.3 we present an analysis on performance of one manufacturing machine operated under feedforward–feedback production controller. Here in Section 7.3.1 we present the control input for our machine and deepen to the systems behavior for the case when the future demand is given by a known linear function. Then in Section 7.3.2 we extend our analysis of this system. First, we consider that the future demand is an unknown but bounded linear function with a constant fluctuation parameter. Thus we design an observer (estimator) for the nominal demand estimation purpose. Furthermore we show by means of incremental stability property that the demand tracking error of our closed-loop system converges to zero value disregarding its initial value. Secondly we assume that the fluctuation parameter is now given by some unknown bounded function. Here by means of Lyapunov function based analysis (see e.g., Khalil (2002)) we show that, given the presence of market fluctuations in our demand model, the demand tracking error dynamics are uniformly ultimately bounded.

In Section 7.4, the observer-based production control is extended to the case of a unidirectional manufacturing line. First, in Section 7.4.1 we present the model of the network. Followed by the control design in Section 7.4.2. Finalizing with Section 7.4.3 where we analyse the stability of the close-loop N machine production line. Furthermore in Section 7.5, we utilize the convergency property in order to introduce an attractive tool for the production network performance evaluation. In Section 7.5.1 we provide an illustrating example of a 4 machine production line behavior in case of control saturation and buffer exhaustion. In Section 7.5.2 we show that for any closed-loop, nonlinear, continuous-time flow model of a given network, which satisfies the convergency property, we can obtain their generalized sensitivity and complimentary sensitivity functions. Here we present 2 approaches for the analysis of these functions. We choose a numerical approach in order to illustrate the capabilities of this tool on a production line of 4 machines. Finally conclusions on the presented research are given in Section 7.6.

7.3 Combined observer-based control of a manufacturing machine

7.3.1 Combined control law

At the beginning, we assume that the variables $y(t)$, $f(t)$ and the value of v_d may be measured and used to form the control action $u_p(t)$. Let us take the control law in the following feedforward–feedback form:

$$u_p(t) = \mathrm{sat}_{[0,u_{p,\max}]}\big(k_p e(t) + v_d - f(t)\big), \tag{7.5}$$

where $e(t) = y_d(t) - y(t)$ denotes the demand tracking error, k_p is the controller parameter (a *proportional gain*), $\mathrm{sat}_[a,b](z)$ denotes the *saturation function*,

$$\begin{aligned}\mathrm{sat}_{[a,b]}(z) &= \min\big(b, \max(a,z)\big)\\ &= \begin{cases} b & \text{if } z \geq b,\\ a & \text{if } z \leq a,\\ z & \text{otherwise,}\end{cases} \qquad (b > a).\end{aligned} \tag{7.6}$$

Equations (7.1), (7.5) describe the closed-loop manufacturing system model for time-varying demand $y_d(t)$ given by (7.3). Performance analysis for the case of linear time-varying demand under the valid assumption $0 < -f(t) + v_d < u_{p,\max}$ for all t shows that the close-loop system, expressed in terms of demand tracking error,

$$\dot{e}(t) = -u_p(t) - f(t) + v_d \tag{7.7}$$

presents an asymptotically stable behavior for its dynamics, which is finite time reachable, regardless of the initial conditions; see Chapter 14 of Khalil (2002).

7.3.2 Observer-based feedback controller

In the above sections it was assumed that the average rate v_d and the disturbance $f(t)$ are known (measured) signals. This assumption is rather unpractical. From now on, we assume that only the error signal $e(t)$ can be measured and used to form the control action. Let us replace the signals v_d, f in the control law (7.5) by their estimates $\hat{v}_d(t)$, $\hat{f}(t)$ provided by the *observer* (state estimator) which uses only available signals $e(t) = y_d(t) - y(t)$ and $u_p(t)$.

Since the observer design is based on modeling the external signals as outputs of some dynamical system, let us assume at this stage that both v_d and f are constants and that the reference signal is strictly linear: $y_d(t) = y_{d,0} + v_d t$. Differentiating the demand tracking error $e(t) = y_d(t) - y(t)$ w.r.t. t we obtain from (7.1) the error model given by (7.7).

From (7.7) it is clear that the signals $f(t)$ and v_d can not be estimated separately based on the measurements of $e(t)$. It is possible to estimate the joint signal $r(t) = -f(t) + v_d$ only. Using the above notation and assumptions, we obtain from (7.7) the following *extended plant model*:

$$\begin{cases}\dot{e}(t) = -u_p(t) + r(t),\\ \dot{r}(t) = 0.\end{cases} \tag{7.8}$$

Luenberger's design method (Luenberger, 1964, 1971) leads to the following *reduced-order observer*

$$\begin{cases}\dot{\sigma}(t) = -\lambda\sigma(t) - \lambda^2 e(t) + \lambda u_p(t)\\ \hat{r}(t) = \sigma(t) + \lambda e(t),\end{cases} \tag{7.9}$$

where $\hat{r}(t)$ denotes the estimate of the signal $r(t)$, produced by observer (7.9), $\lambda > 0$ is the observer parameter (observer gain), setting the transient time for the estimation procedure.

Let us use the estimate $\hat{r}(t)$ in the control law (7.5) instead of $v_d - f(t) = r(t)$. Then the control action $u_p(t)$ takes the form

$$u_p(t) = \mathrm{sat}_{[0,u_{p,\max}]}\big(k_p e(t) + \hat{r}(t)\big), \tag{7.10}$$

where $e(t) = y_d(t) - y(t)$, $\hat{r}(t)$ is governed by (7.9). Equations (7.9), (7.10) describe the first-order feedback controller. The control signal $u(t) = k_p e(t) + \hat{r}(t)$ is calculated based on the error $e(t)$ measurement only. The gains $k_p > 0$ and $\lambda > 0$ are the controller parameters.

The closed-loop system (7.1), (7.9), (7.10) performance differs from that of the system with the controller (7.5), described in Sec. 7.3.1 due to the estimation error $\varepsilon_r(t) = r(t) - \hat{r}(t)$. This error is caused by the difference in the initial conditions of the external and estimated signals, variations of the estimated variable $r(t)$ (due to fluctuation $\varphi(t)$ of $y_d(t)$ and inconstancy of $f(t)$), and the measurement errors. To find the estimation error let us write down the plant–observer model taking into account representation (7.3) for the reference signal $y_d(t)$. We get the following equations

$$\begin{cases} \dot{y}(t) = u_p(t) + f(t), \\ e(t) = y_{d,0} + v_d t + \varphi(t) - y(t), \\ \hat{r}(t) = \sigma(t) + \lambda e(t), \\ \dot{\sigma}(t) = -\lambda\sigma(t) - \lambda^2 e(t) + \lambda u_p(t). \end{cases} \tag{7.11}$$

where $u_p(t)$, $f(t)$, $y_{d,0}$, v_d, $\varphi(t)$ are external inputs. After simple algebra we obtain from (7.11) the following equation for the estimation error $\varepsilon_r(t)$:

$$\begin{aligned} &\varepsilon_r(t) = \mu(t) - \xi(t), \\ &\dot{\mu}(t) + \lambda\mu(t) = \lambda\xi(t), \quad \mu(0) = \mu_0, \end{aligned} \tag{7.12}$$

where $\mu(t) = vd(1 - \lambda t) - \sigma(t) - \lambda y_{d,0} + \lambda y(t)$ and $\xi(t) = f(t) + \lambda\varphi(t)$. We see that the error $\varepsilon_r(t)$ is independent of $u_p(t)$, $y_{d,0}$, v_d in (7.11).

Taking into account the estimation error $\varepsilon_r(t)$, the control law (7.5) reads as

$$u_p(t) = \mathrm{sat}_{[0,u_{p,\max}]}\big(k_p e(t) + v_d - f(t) - \varepsilon_r(t)\big), \tag{7.13}$$

and the closed-loop system dynamics is described by (7.1), (7.12), (7.13).

If there exists a fluctuation of the desired production $\varphi(t) \neq \text{const}$, convergence of the demand tracking error $e(t)$ to zero can not be attained. In such a case it is advisable, both for simplification of the analysis and for system industrial applicability, to assure convergence of the system trajectories to each other for different initial conditions. This property is represented by the notion of the *incremental stability* (Angeli, 2002). To verify incremental stability of the considered system let us rewrite (7.1), (7.5) in the transformed coordinates. Introduce a new variable z as $z(t) = y(t) - y_{d,0} - v_d t$. Differentiating $z(t)$ w.r.t. t and taking in view (7.1) we obtain that $\dot{z}(t) = u_p(t) - v_d + f(t)$. Then the demand tracking error $e(t)$ reads as

$$e(t) = y_d(t) - y(t) = y_{d,0} + v_d t + \varphi(t) - y(t) = \varphi(t) - z(t).$$

Let us rewrite expression (7.5) for the control signal u_p using the standard (symmetric) saturation function $\mathrm{sat}_m(x) = \min\big(m, \max(-m, x)\big)$. Introducing the "unsaturated" control signal as $u(t) = k_p e(t) + v_d - f(t)$ we obtain that $u_p = \mathrm{sat}_{[0,u_{p,\max}]}(u) = \mathrm{sat}_m(k_p e + v_d - f - m) + m$. This leads to the following closed-loop system model in the transformed coordinates:

$$\dot{z}(t) = \bar{u}(t) + \eta(t), \tag{7.14}$$

$$\begin{cases} \bar{u}(t) = \mathrm{sat}_m\big(k_p e(t) + \hat{r}(t) - m\big), \\ e(t) = \varphi(t) - z(t), \end{cases} \tag{7.15}$$

$$\begin{cases} \hat{r}(t) = \sigma(t) + \lambda e(t), \\ \dot{\sigma}(t) = -\lambda\sigma(t) - \lambda^2 e(t) + \lambda(\bar{u}(t) + m), \end{cases} \tag{7.16}$$

where $\eta(t) = f(t) + m - v_d$, $m = u_{p,\max}/2$.

Recall that *convergent systems* being excited by a bounded input have a unique bounded globally asymptotically stable steady-state solution (Pavlov et al., 2004, 2006). Therefore, for any given input, the solutions of the convergent system, independently of the initial, conditions converge to the uniquely defined limit solution.

Let us check if this property is valid for system (7.14)–(7.16), forced by the reference signal $\varphi(t)$. Let us apply the general results presented in Andrievsky et al. (2009). For doing that, rewrite the closed-loop system model in the following state-space form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\,\mathrm{sat}_m(u) + Fw(t), \\ u(t) &= Cx(t) + Dw(t), \quad z(t) = Hx(t) \end{aligned} \tag{7.17}$$

To this end, introduce the variables $\psi(t) = \sigma(t) - m$ and $\rho(t) = \lambda e(t) + \sigma(t) - m$. In the new notation, (7.14)–(7.16) read as

$$\begin{aligned} \dot{z}(t) &= \bar{u}(t) + \eta(t), \\ \bar{u}(t) &= \mathrm{sat}_m\big(k_p e(t) + \rho(t)\big), \\ e(t) &= \varphi(t) - z(t), \\ \rho(t) &= \psi(t) + \lambda e(t), \\ \dot{\psi}(t) &= -\lambda\psi(t) - \lambda^2 e(t) + \lambda\bar{u}(t), \\ \eta(t) &= f(t) + m - v_d. \end{aligned} \tag{7.18}$$

Introducing the state-space vector $x(t) \in \mathbb{R}^2$ as $x = [x_1, x_2]^{\mathrm{T}}$ where $x_1(t) = z(t)$, $x_2(t) = \psi(t)$ and the external input vector $w(t) \in \mathbb{R}^2$ as $w = [w_1, w_2]^{\mathrm{T}}$, where $w_1(t) = \varphi(t)$, $w_2(t) = \eta(t)$ we obtain the system model in the form (7.17) with the following matrices:

$$\begin{aligned} &A = \begin{bmatrix} 0 & 0 \\ \lambda^2 & -\lambda \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{bmatrix}, \\ &C = \big[-(k_p + \lambda), 1\big], \quad D = \big[k_p + \lambda, 0\big], \quad H = [1, 0]. \end{aligned} \tag{7.19}$$

Let us introduce the following assumptions.

Assumption 16. *There exists a number* κ, $0 < \kappa < \frac{u_{p,\max}}{2}$, *s.t. for all* t *the following inequalities*

$$\kappa \le v_d - f(t) \le u_{p,\max} - \kappa.$$

are satisfied. Thus the product demand rate together with external perturbations lie within the admissible bounds of the machine production rate with some margin κ.

Assumption 17. *Function* $\varphi(t)$ *is bounded, i.e. there exists a constant* $C_\varphi > 0$ *s.t.* $|\varphi(t)| \le C_\varphi$ *for all* t, *and uniformly continuous on* $(-\infty, \infty)$.

The convergent property of system (7.14)–(7.16) is given by the following Theorem.

Theorem 11. *System* (7.14)–(7.16) *is convergent in the following sense: for all* $k_p > 0$, $\lambda > 0$ *and for any reference signal* (7.3), *satisfying Assumptions 16 and 17 there is unique solution* $\mathrm{col}(\bar z(t), \bar\sigma(t))$ *bounded on* $(-\infty, +\infty)$ *and this solution is globally asymptotically stable uniformly in* t_0 *and uniformly w.r.t initial conditions from any compact set.*

Proof. At first let us show that the considered system is uniformly ultimately bounded. Perform a similarity transformation $\bar x = Tx$ with the following nonsingular matrix

$$T = \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}.$$

In the transformed coordinates the state-space system representation has the following matrices:

$$\bar A = TAT^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & -\lambda \end{bmatrix}, \quad \bar B = TB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar F = TF = \begin{bmatrix} 0 & 1 \\ -\lambda^2 & -\lambda \end{bmatrix},$$
$$\bar C = CT^{-1} = [-k_p, 1], \quad \bar D = D, \quad \bar H = H.$$

Then the state-space system equations in expanded form with respect to the transformed variables are as follows:

$$\begin{cases} \dot{\bar x}_1(t) = \mathrm{sat}_m(u) + \eta(t), \\ \dot{\bar x}_2(t) = -\lambda \bar x_2(t) - \lambda^2 \varphi(t) \end{cases} \qquad (7.20)$$
$$u(t) = -k_p \bar x_1(t) + \bar x_2(t) + (k_p + \lambda)\varphi(t),$$

where $\eta(t) = f(t) + m - v_d(t)$. The second equation of system (7.20) represents asymptotically stable LTI system (recall that $\lambda > 0$), forced by the bounded input $\varphi(t)$. Then there exists $\limsup_{t\to\infty} |\bar x_2(t)|$. Consider the first equation of (7.20) and introduce the Lyapunov function $V(\bar x_1) = \bar x_1^2$. The time derivative of $V(\bar x_1)$ along the solutions of (7.20) is given by

$$\dot V = 2\bar x_1 \big(\mathrm{sat}_m\big(-k_p \bar x_1 + \bar x_2 + (k_p + \lambda)\varphi\big) + \eta(t)\big) \qquad (7.21)$$

Since $|\eta(t)| \le m - \kappa < m$ by Assumption 16, and $\varphi(t)$ is bounded by Assumption 17 it follows that $\mathrm{sat}_m\big(- k_p\bar{x}_1 + \bar{x}_2 + (k_p + \lambda))\varphi = m \cdot \mathrm{sign}(\bar{x}_1)$ if $k_p|\bar{x}_1| > |\bar{x}_2| + (k_p + \lambda)|\varphi| + m$ and therefore

$$\dot{V} \le -2\kappa|\bar{x}_1| < 0 \quad \text{if} \quad k_p|\bar{x}_1| > |\bar{x}_2| + (k_p + \lambda)|\varphi| + m. \tag{7.22}$$

Implying that system (7.20) is uniformly ultimately bounded.

The proof now can be continued along the lines of proof of Theorem 11 of (Andrievsky et al., 2009). □

Remark. Assumption 16 is not necessary for ensuring boundedness of the system error. This assumption may be weakened by permission for $v_d - f(t)$ to leave the interval $(0, u_{p,\max})$ for some "short periods", see van den Berg et al. (2006) for details.

7.4 Control of a line of manufacturing machines

7.4.1 Manufacturing line model

Consider a line of N manufacturing machines M_1, M_2, $\ldots M_N$, which are separated by buffers B_j, $j = 1, \ldots, N$ with infinite capacity, see Figure 7.1. The first machine M_1 is supplied by raw material, the Nth machine M_N produces finished product. Each machine M_j takes out a raw product from the corresponding input buffer B_j and puts a processed product to the output buffer B_{j+1}. In what follows suppose that there is always sufficient raw material to feed the first machine, i.e. that the buffer B_1 is never exhausted.

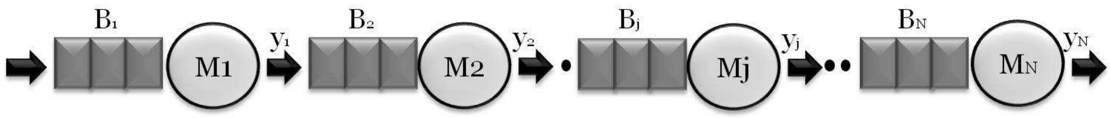


Figure 7.1: Schematics of a line of N manufacturing machines. $\mathbf{M}_j$ – machines, $\mathbf{B}_j$ – buffers, $j = 1, \ldots, N$.

Assume that a manufacturing machine produces items continuously in time $t \in \mathbb{R}$ with a certain *production rate* $u_j(t) \in \mathbb{R}$, where $j = 1, \ldots N$ is a number of the machine. The total amount of items produced by jth machine is described by a continuous variable $y_j(t) \in \mathbb{R}$. Interaction between the machines is described by the *buffer content* variables $w_j(t) = \max(y_{j-1} - y_j, 0)$, $j = 2, \ldots, N$. The case of $w_j(t) = 0$ means absence of the raw material in the input buffer of jth machine and, therefore, the machine M_j works at the rate of its raw material inflow (for more details see Fillipov (1988)). The above reasons lead to the following continuous model of the manufacturing machine:

$$\dot{y}_j(t) = \begin{cases} u_j(t) + f_j(t), & \text{if} \quad w_j(t) > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{7.23}$$

where $t \in \mathbb{R}$ stands for continuous time argument; $j = 1, \ldots, N$ is a machine number; variables $f_j(t) \in \mathbb{R}$ denote *external disturbances*. For example, f_j may describe manufacturing losses, or variations of the machine capacity. The production rates u_j are bounded by $u_{\max}$ due to machine capacity limitation. In the sequel we assume, without loss of generality, that all the machine capacities in the line have the same upper bound $u_{\max}$. Since the production rates u_j can not be negative, the following bounds are valid for $u_j(t)$:

$$0 \le u_j(t) \le u_{\max}, \; j = 1, \ldots, N, \; \forall t \ge 0. \tag{7.24}$$

Inequalities (7.24) lead to a *saturation effect* in the system. This effect restricts the production rate, and complicates design of the controller and the system performance analysis.

Summarizing, we obtain the following manufacturing line model

$$\begin{cases} \dot{y}_1(t) = u_1(t) + f_1(t), \\ \dot{y}_2(t) = \big(u_2(t) + f_2(t)\big) \cdot \mathrm{sign}(w_2(t)), \\ \cdots\cdots \\ \dot{y}_N(t) = \big(u_N(t) + f_N(t)\big) \cdot \mathrm{sign}(w_N(t)), \end{cases} \tag{7.25}$$

where $\mathrm{sign}(\theta) = \big(1, \text{ if } \theta > 0 \mid 0, \; \text{ otherwise}\big)$.

7.4.2 Decentralized control strategy

Let us extend the described above control strategy for a single-machine (7.9), (7.10) to control of a manufacturing line. The direct usage of (7.9), (7.10) for each machine is unsatisfactory, because it does not take into account the buffer contents, which leads to exhaustion of some buffers or, alternatively, to stacking in buffers an extra amount of material. Besides, from implementational reasons, it is desirable to organize interactions between the neighboring machines only and avoid transferring the reference signal to each machine. Due to these reasons, in this chapter we propose the modification of the control strategy (7.9), (7.10), intended to control of a manufacturing line.

Firstly, introduce the desirable constant level of the buffer contents $w_d > 0$. Add the "penalty" term $k_w\big(w_d - w_{j+1}(t)\big)$ (used to slow down the production in case the buffer content is bigger than the desired one), where $k_w > 0$ is a certain gain (designed parameter) to jth control action $u_j(t)$. Secondly, change the demand signal for jth machine to ensure equality $y_{j-1}(t) = y_j(t) + w_d$ in the steady-state nominal regime. Starting from these reasons, the following control strategy for a line of N machines is obtained. This strategy is described below in recursive form.

Take the control law for Nth machine in the form (7.9), (7.10), namely let the control signal $u_N(t)$ be calculated as

$$\begin{cases} u_N = \mathrm{sat}_{[0,u_{\max}]}\big(k_p \varepsilon_N + \hat{r}_N\big), \\ \hat{r}_N(t) = \sigma_N(t) + \lambda e(t), \\ \dot{\sigma}_N(t) = -\lambda \sigma_N(t) - \lambda^2 e(t) + \lambda u_N(t), \end{cases} \tag{7.26}$$

where $\varepsilon_N(t) \equiv e(t) = y_d(t) - y_N(t)$ is the reference error. Take the control law for machine M_j, $j=1,\ldots,N-1$ in the following form:

$$\begin{cases} u_j = \mathrm{sat}_{[0,u_{\max}]}\big(k_p\varepsilon_j + \hat{r}_j + k_w(w_d - w_{j+1})\big), \\ \varepsilon_j(t) = w_d + \varepsilon_{j+1}(t) - w_{j+1}(t) \\ \hat{r}_j(t) = \sigma_j(t) + \lambda\varepsilon_j(t), \\ \dot{\sigma}_j(t) = -\lambda\sigma_j(t) - \lambda^2\varepsilon_j(t) + \lambda u_j(t), \end{cases} \tag{7.27}$$

where $w_{j+1}(t) = y_j(t) - y_{j+1}(t)$; w_d is the buffer contents demand. Formulas (7.26), (7.27) recursively specify the distributed controller for a line of $N \geq 2$ manufacturing machines.

7.4.3 Nominal mode analysis

In what follows let us refer to an operating regime when no buffer is empty and all the control actions lie within the permissible interval (i.e. no saturation occurs) as a *nominal mode*. In this mode, the closed-loop manufacturing system with a controller may be modeled by following linear state-space equations:

$$\dot{x}(t) = Ax(t) + Bg(t) + B_f f(t), \quad y(t) = Cx(t), \tag{7.28}$$

where $x(t) = \big[y_1(t), \sigma_1(t), \ldots, y_j(t), \sigma_j(t), \ldots, y_N(t), \sigma_N(t)\big]^{\mathrm{T}} \in \mathbb{R}^{2N}$ is the system state-space vector, where y_j, σ_j are outputs of jth machine and state of jth observer, respectively; the column-vector $y(t) \in \mathbb{R}^N$ merges outputs of separate machines, $y(t) = \big[y_1(t), \ldots, y_N(t)\big]^{\mathrm{T}}$; $g(t) = \big[y_d(t), w_d\big]^{\mathrm{T}} \in \mathbb{R}^2$ is a column vector of the reference signals; $f(t) = \big[f_1(t), \ldots, f_N(t)\big]^{\mathrm{T}} \in \mathbb{R}^N$ denotes the column-vector of disturbances. After simple calculations the following matrices in (7.28) may be obtained:

$$A=\overbrace{\begin{bmatrix} -k_{wp\lambda} & 1 & k_w & 0 & 0 & \dots & 0 & 0\\ -\lambda k_{wp} & 0 & \lambda k_w & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & -k_{wp\lambda} & 1 & k_w & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & -k_{wp\lambda} & 1 & k_w & 0\\ 0 & 0 & 0 & 0 & -\lambda k_{wp} & 0 & \lambda k_w & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -k_{p\lambda} & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & -k_p\lambda & 0 \end{bmatrix}}^{\mathbf{2N}}$$

$$B_g=\begin{bmatrix} k_{p\lambda} & (N-1)k_{p\lambda}+k_w\\ \lambda k_p & \lambda\big((N-1)k_p+k_w\big)\\ \vdots & \vdots\\ k_{p\lambda} & (N-j)k_{p\lambda}+k_w\\ \lambda k_p & \lambda\big((N-j)k_p+k_w\big)\\ \vdots & \vdots\\ k_{p\lambda} & k_{wp\lambda}\\ \lambda k_p & \lambda k_{wp}\\ k_{p\lambda} & 0\\ \lambda k_p & 0 \end{bmatrix}, \quad b_{f_j}=\begin{bmatrix}0\\0\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\\0\\0\\0\end{bmatrix}\begin{matrix}\\ \\ \\ \}\,\mathbf{2j-1}\\ \}\,\mathbf{2j}\\ \\ \\ \\ \\ \\ \end{matrix}, \tag{7.29}$$

$$B_f=[b_{f_1},b_{f_2},\dots,b_{f_N}]\in\mathbb{R}^{2N\times N},$$

$$C=\{c_{i,j}\}\in\mathbb{R}^{N\times 2N},\quad c_{j,2j+1}=\begin{cases}1, & j=2i+1,\quad i=1,\dots,N\\ 0, & \text{otherwise,}\end{cases}$$

where the gains $k_{wp\lambda}=k_w+k_p+\lambda$, $k_{wp}=k_w+k_p$, $k_{p\lambda}=k_p+\lambda$ are introduced.

To evaluate the closed-loop system performance let us use the following *sensitivity* and *complimentary sensitivity functions* $S_i^j(s)$, $T_i^j(s)$:

$$T_{y_d}^{y_i}(s)=\frac{Y_i(s)}{Y_d(s)},\, S_{y_d}^{y_i}(s)=1-T_{y_d}^{y_i}(s),$$

where $Y_i(s)$, $Y_d(s)$ are the Laplace transforms of $y_i(t)$, $y_d(t)$, respectively, $i=1,\dots N$. Due to a tridiagonal form of the system matrix A, its characteristic polynomial $A(s)=\det(sI-A)$ (here I is $2N\times 2N$ identity matrix) can be easily found as

$$A(s)=(s+k_p)(s+\lambda)^N(s+k_{wp})^{N-1}. \tag{7.30}$$

Evidently, the closed-loop system (7.28) is asymptotically stable and its dynamics are defined by the controller/observer gains k_p, k_w, λ. The polynomial $A(s)$ is a common denominator of all transfer functions. From (7.28), (7.29) after simple algebra and cancelations the common poles and zeros, we obtain the following expressions for $T_{y_d}^{y_j}(s)$ and $S_{y_d}^{y_j}(s)$:

$$T_{y_d}^{y_j}(s)=\frac{k_{p\lambda}s+k_p\lambda}{(s+\lambda)(s+k_p)},\quad S_{y_d}^{y_j}(s)=\frac{s^2}{(s+\lambda)(s+k_p)}, \tag{7.31}$$

where $j = 1, \ldots, N$. Expressions (7.31) show that, in the nominal mode, all the machine outputs $y_j(t)$ track the reference signal $y_d(t)$ with the same dynamics, defined by the controller/observer gains k_p, λ. The gain k_w has no influence on this process. It is also clear that for linearly varying demand $y_d(t) = y_{d,0} + v_d t$ the asymptotic errors are zero i.e. $y_d(t) - y_j(t) \to 0$ as $t \to \infty$.

7.5 Numerical example

Consider the manufacturing line from $N = 4$ machines described by (7.25), (7.26), (7.27). Let us take the following numerical values of the system parameters (Andrievsky et al. (2009); Kommer et al. (2009)): $u_{\rm p,max} = 1.0$, $k_{\rm p} = 5$, $\lambda = 25$. Choose $k_w = 30$.

7.5.1 System behavior in the case of control saturation and buffer exhaustion

In case of control saturation and buffer exhaustion, the system nonlinearity is essential. Let us consider the system (7.25), (7.26), (7.27) behavior for a typical demand $y_d(t)$ given by (7.3). Let $w_d = 0.1$ be taken. The first example illustrates the stage of the system performance when some buffers are empty, therefore, the machines, fed by these buffers are not able to work. The initial conditions $y_1(0) = 1$, $y_2(0) = 4$, $y_3(0) = 3$, $y_4(0) = 2$ are taken. The reference signal parameters are following: $y_{d,0} = 5$, $v_d = 0.5$, $\varphi(t) = a \sin(\omega t)$, where $\omega = 1.0$ s^{-1}. Time histories of machine outputs $y_j(t)$ and the reference signal $y_d(t)$ for the case of $a = 0$ are plotted in Figure 7.2.

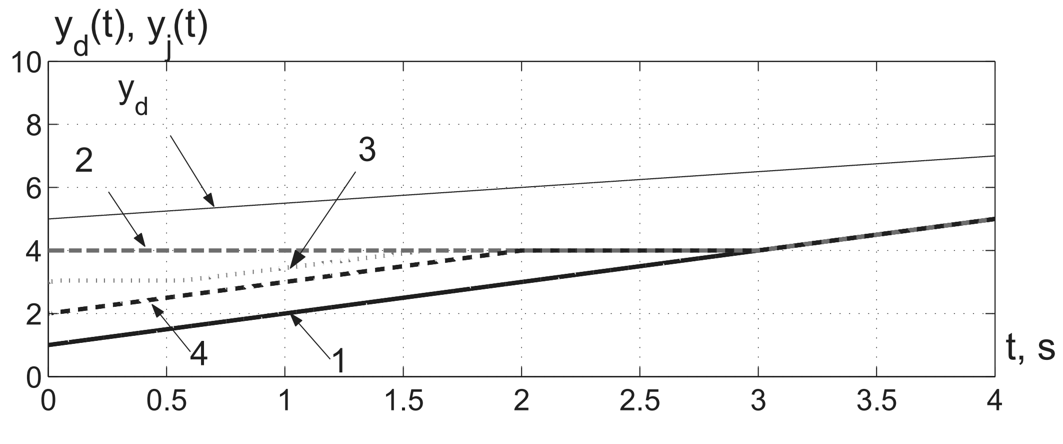


Figure 7.2: Time histories: $y_1(t)$ – solid line; $y_2(t)$ – dashed line; $y_3(t)$ – dotted line; $y_1(t)$ – dash-dot line; $y_d(t)$ – thin solid line. $a = 0$.

This plot shows that the machine M_3 does not work prior to its production requirement by the machine M_4. The machine M_2 is waiting for the raw product to arrive in the buffer B_2 from the machine M_1. The machine M_4 produces finished product while the buffer B_4 is not empty. It is seen that after the finite transient time $t_{\rm buf} = 3$ s all the buffers have positive amount of product and the machines work properly. Time histories of $y_j(t)$ for the same initial conditions, a fluctuation $\varphi(t)$

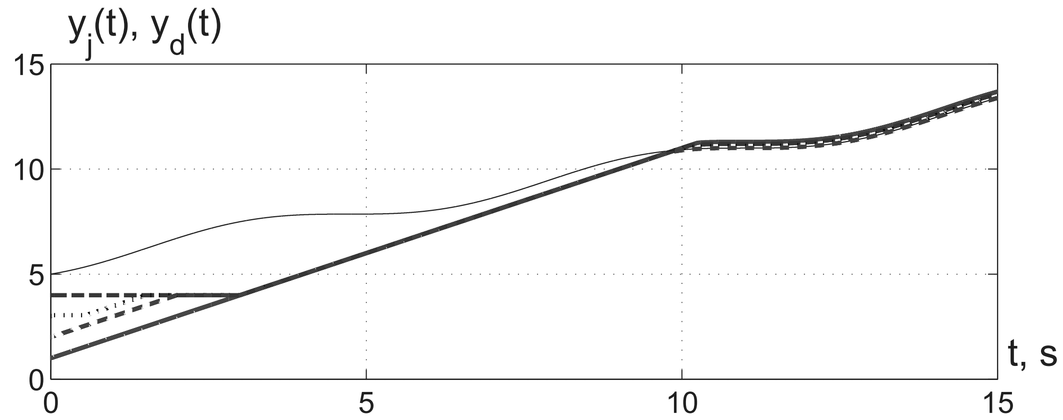


Figure 7.3: Time histories: $y_1(t)$ – solid line; $y_2(t)$ – dashed line; $y_3(t)$ – dotted line; $y_1(t)$ – dash-dot line; $y_d(t)$ – thin solid line. Fluctuation $\varphi(t)$ magnitude $a = 0.5$, frequency $\omega = 1.0$ s^{-1}.

magnitude $a = 0.5$ and a frequency $\omega = 1.0$ s^{-1} are plotted in Figure 7.3. It is seen that some controls are saturated during about 10 seconds duration time interval. After that the manufacturing line output $y_4(t)$ tracks the reference signal with a good accuracy. It should be noticed that in the steady-state mode the outputs of the machines differ among themselves on the prescribed value $w_d = 0.1$.

7.5.2 Frequency domain analysis of the manufacturing line

The performance analysis of linear control systems is essentially based on frequency domain characteristics such as *sensitivity function* $S(i\omega)$ and *complementary sensitivity function* $T(i\omega)$. The generalized versions of these functions are defined in (Pavlov et al., 2007) for nonlinear Lur'e systems, possessing convergence property. Let the nonlinear system be modeled as

$$\dot{x}(t) = f(x, r), \quad y(t) = h(x, r), \quad e(t) = r(t) - y(t), \tag{7.32}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $y(t) \in \mathbb{R}$ is the system output, $r(t)$ is the reference signal, $e(t)$ is the reference error. Assume that $r(t) = a\sin(\omega t)$, where a is the amplitude and ω is the frequency of the input (reference) signal, and that the system (7.32) is uniformly convergent in this class of inputs. Then there exists an unique steady-state $\frac{2\pi}{\omega}$-periodic solution $\bar{x}(t)$ of (7.32) with the corresponding response $\bar{y}(t)$ and $\bar{e}(t) = r(t) - \bar{y}(t)$ (Pavlov et al. (2005); van den Berg et al. (2006)).

Definition (Pavlov et al., 2007): The functions

$$\mathcal{S}(a, \omega) = \|\bar{e}\|_2 / \|r\|_2, \quad \mathcal{T}(a, \omega) = \|\bar{y}\|_2 / \|r\|_2$$

where $\|z\|_2 = \left(\frac{\omega}{2\pi}\int_0^{2\pi/\omega} z(\tau)^2\, d\tau\right)^{1/2}$, are called, respectively, the generalized sensitivity and the generalized complementary sensitivity functions of the convergent system (7.32). For the linear case, the functions $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ coincide with customary amplification frequency characteristics $|S(i\omega)|$, $|T(i\omega)|$, respectively. For nonlinear systems, the functions $\mathcal{S}(a,\omega)$ and $\mathcal{T}(a,\omega)$ depend not only on the excitation frequency ω, but also on the amplitude a. These generalized sensitivity functions may be evaluated numerically based on system (7.32) simulation with given harmonic input $r(t)$. Significant reduction of the computation costs may be achieved by using the generalization of the *describing functions* (harmonic linearization) method on nonautonomous convergent systems given in (van den Berg et al., 2007). In this thesis we limit our performance evaluation only to a numerically based approach and consider the system analysis by means of describing functions as part of our future research.

Thus, let us calculate the function $\mathcal{S}(a,\omega)$ numerically, based on the system simulation.

To rephrase (7.25), (7.26), (7.27) in the form of the system (7.32), excited by harmonic input signal, let us make the following coordinate transformation (Andrievsky et al., 2009). Define the *transformed coordinates* $z_j(t)$ as $z_j(t) = y_j(t) - y_{d,0} - v_d t$, $j = 1,\dots,4$, and introduce the vector $z(t) = [z_1(t),\dots,z_4(t)]^{\mathrm{T}} \in \mathbb{R}^4$. Then, taking into account (7.3), expressions for the reference errors $e_j(t) = y_d(t) - y_j(t)$ read as $e_j(t) = \varphi(t) - z_j(t)$ (recall that $\varphi(t)$ denotes the demand fluctuation about the linear trend).

The calculations were made for the sets of amplitudes $a \in \{0.025, 0.05, 0.1, 0.5\}$ and frequencies $\omega \in [0.1, 30]$ s^{-1} of the harmonic fluctuation $\varphi(t)$, and for the different values of the demand rate $v_d \in \{0.5, 0.7, 0.9\}$ s^{-1}. The simulations were made only for the cases of nondecreasing manufacturing demand $y_d(t)$, i.e. when the inequality $a \cdot \omega \le v_d$ was fulfilled. The results are depicted in Figure 7.4. It is seen that for the case of half-load line (i.e. when $v_d = 0.5 \cdot u_{\max} = 0.5$), the nominal mode is imposed in the manufacturing line and the sensitivity function $\mathcal{S}(a,\omega)$ coincides with that of the linear system. For greater values of the load (e.g. 70% or 90% from the maximal line capacity), the saturation takes effect on system performance for sufficiently large fluctuation magnitudes. The reference error magnitude achieves that of fluctuation, or even exceeds it. As an example, the time histories of $z_4(t)$, $e_4(t)$, $u_4(t)$ for $v_d = 0.9$ s^{-1}, $a = 0.5$, $\omega = 1.0$ s^{-1} are plotted in Figure 7.5. One should notice that even in this case, the manufacturing line is capable to work. This last is due to the fact that, after some short period of time, the relative error of the finished product is small in comparison with the product demand. Also it is worth to mention that the convergence property remains for the system, despite that the controls are periodically saturated during some of the time intervals. This statement may be rigorously proved applying the approach of van den Berg et al. (2006). As an illustration, a pencil of trajectories $z_4(t)$ for different initial conditions is plotted in Figure 7.6. The simulations were made for the same demand signal $y_d(t)$ as above and random initial conditions $y_j(0) \in [0, 3]$, $j = 1,\dots,4$.

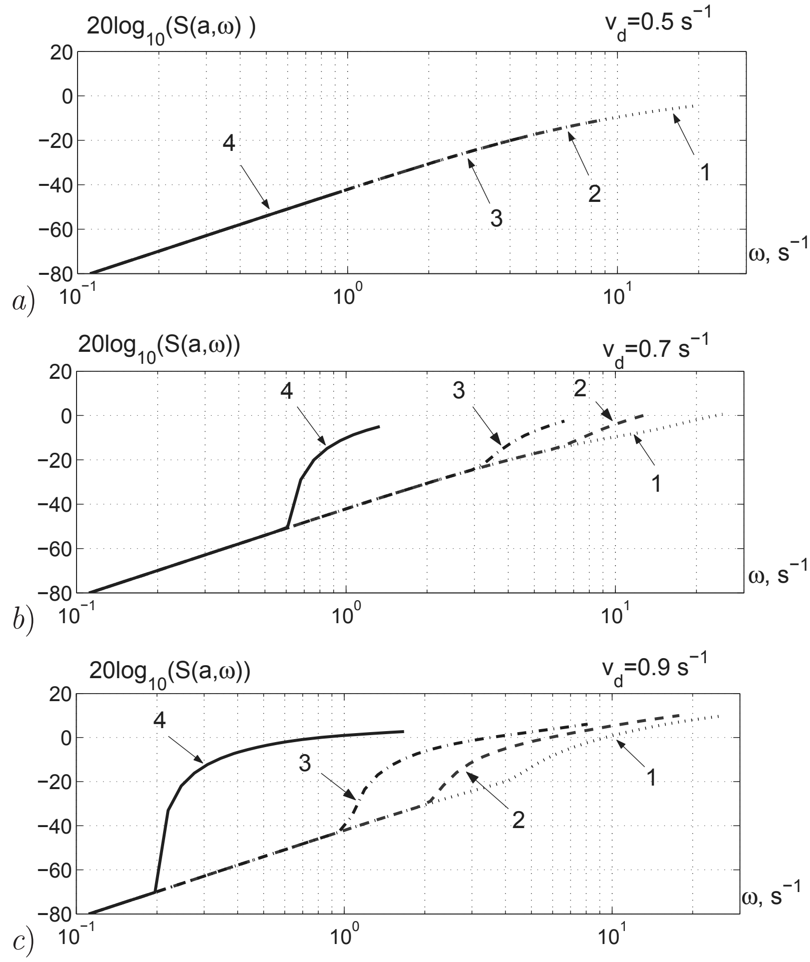


Figure 7.4: Nonlinear sensitivity function $\mathcal{S}(a,\omega) = \|\bar{e}_j\|_2/\|\varphi\|_2$. *a)* $v_d = 0.5$ s^{-1}, *b)* $v_d = 0.7$ s^{-1}, *c)* $v_d = 0.9$ s^{-1}. Fluctuation $\varphi(t) = a\sin(\omega t)$; 1) $a = 0.025$, 2) $a = 0.05$, 3) $a = 0.1$, 4) $a = 0.5$.

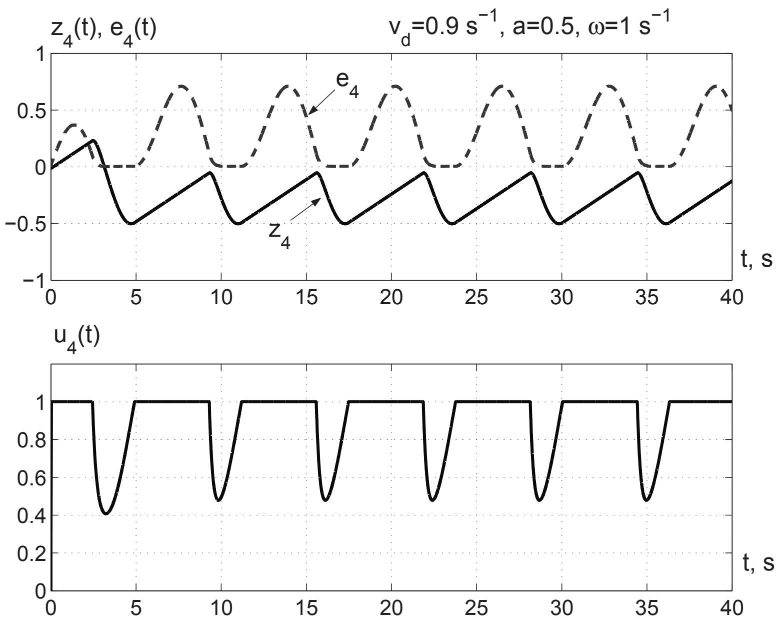


Figure 7.5: Time histories of $z_4(t)$, $e_4(t)$, $u_4(t)$ for $v_d = 0.9$ s^{-1}, $a = 0.5$, $\omega = 1.0$ s^{-1}.

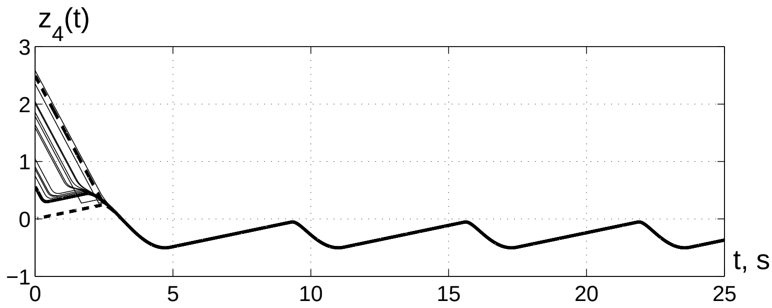


Figure 7.6: Pencil of trajectories $z_4(t)$; $v_d = 0.9\ \mathrm{s}^{-1}$, $a = 0.5$, $\omega = 1.0\ \mathrm{s}^{-1}$.

7.6 Conclusions

In this chapter the problem of controlling manufacturing machine such that a customer demand is tracked with a desired accuracy under assumptions that the nominal demand rate is constant (as well as possible disturbances) is studied. A combination of feedforward-feedback controller with a reduced-order observer is proposed and the system behavior in time domain and a steady-state performance of the system with periodic demand fluctuations are analyzed. Furthermore this control approach is extended to a tandem production line configuration. Here the results on nominal model analysis show the asymptotic stability of the closed-loop system. A numerical example was presented, where an unidirectional production line of 4 machines was tested under control saturation and buffer exhaustion conditions. The frequency domain tool for the analysis of nonlinear systems satisfying the convergency property, is introduced and shown to be an efficient method for the performance evaluation of production networks. In our future research we are interested in performing a comparison of the feedforward-feedback observer based production control using the sensitivity function-based analysis with existent production techniques such as *Kanban* and *Control Point Policy*.

8
Conclusions and future directions

Abstract | This chapter summarizes the conclusions from the research presented in the thesis. The main contributions are highlighted and future research directions are presented.

8.1 Conclusions

Motivated by the need of manufacturing industries to maintain high customer satisfaction and at the same time minimize their product inventories the following research objective was formulated:

- evaluate the performance of demand-driven control mechanisms for manufacturing networks by applying classical tools from control theory.

After an extensive review of existing literature, the suitable control methodology was identified as a surplus-based control and, in line with the main objective, the following two goals were posed:

- give a general mathematical interpretation to the surplus-based production control methodology and identify important performance indicators for a manufacturing network under the aforementioned control;
- evaluate surplus-based control performance by analytical and experimental means for several commonly used network topologies.

By reading through this thesis work it could be noticed that the above mentioned goals were pursued in every chapter of this book. Starting from Chapter 2 the research results reported in this thesis were subdivided based on the studied network topology and their models. The evaluated network topologies were

- a single machine serving one product type in Chapter 2 based on discrete time (DT) model and in Chapter 7 based on continuous time (CT) model
- a unidirectional manufacturing line serving one product type in Chapter 3 based on DT model and Chapter 7 based on CT model
- a single machine and a unidirectional manufacturing line serving multiple product types in Chapter 4 based on DT model
- a re-entrant machine and a re-entrant manufacturing line serving one product type in Chapter 5 based on DT model

The contributions of this research can be subdivided into theoretical, simulation-based and experimental parts.

8.1.1 Theoretical contributions

The main theoretical contributions of this thesis are:

- the obtained flow models for the above mentioned network topologies;
- the mathematical interpretation of the surplus-based production control methodology;
- the derived analytical relation between the production tracking accuracy and the base stock levels for the studied manufacturing networks.

Throughout the chapters of this thesis the complexity of performance analysis increments in line with the studied network configurations, starting from the basic configurations, such as a single machine and a manufacturing line serving one product type till more complex networks such as multi-product lines and re-entrant networks.

For example, the obtained bounds on demand tracking accuracy (2.16) and (2.17) of Chapter 2 can be interpreted through the idea that in the steady state the demand tracking error of a single machine will not grow further than the maximal product demand rate of one time step and will not present a bigger product excess than the one given by the difference between the minimal demand rate and the maximal production rate of one time step, respectively. Also similar bounds on demand tracking accuracy were obtained in case of a line, which were obtained under the assumption that sufficiently large base stock levels are kept in the network. Another theoretical result was obtained from the relation between the base stock levels and the production tracking accuracy in a line, which was reported in Theorem 5 of Chapter 3. This relation reflects the idea that increasing base stock levels in a manufacturing line will shorten the network's reaction time to the product demand growth, which, in consequence, will increase the tracking accuracy of such a network. Also this relation shows that, due to the network capacity limitation, there exists a certain base stock threshold value after which the demand tracking accuracy of the network will not change. Though these two ideas might seem qualitatively intuitive, the obtained estimate of this relation is not. Further results on a manufacturing line show that with or without limited inventory levels the production demand tracking accuracy of the network remains the same. Naturally, what differs is the time that is needed for a line with limited inventory to reach the steady state tracking accuracy level in comparison to a line with unlimited inventory. Thus further studies are needed in order to estimate the duration of the transient behavior in the network. This issue is approached further in the section on future directions of this chapter.

Chapter 4 concerns the performance analysis of more complicated network structure namely a multi-product manufacturing line. As the name suggests the study is evolved from one product serving manufacturing line into a line with the capability of serving several types of products. What complicates the analysis of such a network in comparison with a one product serving line is that each machine can receive products from different routings, but the products from only one of the routings can be served at a time. For the analysis of a single multi-product machine it was considered that the processing time may change depending on the product type, while for the line this assumption was omitted, but the processing time between the machines was considered to differ. For the single multi-product machine as well as for a line it was shown that the tracking accuracies for every product type are interrelated.

In Chapter 5 it was shown how the obtained results of Chapter 4 can be extended to a re-entrant machine followed by a re-entrant line. A complete analytical relation was obtained between the inventory levels and demand tracking accuracy of a single re-entrant machine with multiple production stages. Similarly to a multi-product machine it was considered that a re-entrant machine can have a different processing speed for each production stage. For the re-entrant line further studies are needed to obtain a similar relation.

In the aforementioned results the production control mechanism had no influence on the processing speed of a network, i.e. each machine was controlled by on and off actions of the controller. It is also important to consider such networks where a control action can also influence production speed of every machine in the network. For that purpose in Chapter 7 a continuous time flow models of a single machine and a line were derived. In previous works a simple PI-controller was proposed to solve the problem. Due to saturation in the control loop, an anti-windup compensator (see van den Bremer et al. (2008)) was required in order to avoid undesirable oscillations in the presence of disturbances. In the research reported in Chapter 7 a simpler approach was implemented. Under the assumptions that the nominal demand rate is constant as well as possible disturbances an observer was designed to estimate this rate, so it could be utilized in the control algorithm.

8.1.2 Simulation-based contributions

The simulation-based contributions of this thesis are

- the theoretical results on the performance obtained for the DT flow models of manufacturing networks where shown to be valid for the DE models of a single machine, a line and a re-entrant machine topologies;
- the performance in terms of the steady state demand tracking errors and inventory levels was tested for a line, a re-entrant machine and a re-entrant network operated under a particular case of Hedging Point, Conwip and Base Stock policies;
- the studied surplus-based production controller presented as a particular case of Hedging Point policy was shown to be superior in comparison with other two.

8.1.3 Experimental contributions

The experimental contributions of this thesis are obtained by means of a liquid-based emulator of manufacturing networks that consists of several electrical pumps and liquid reservoirs. The electrical pumps emulate a manufacturing machine behavior while the liquid reservoirs serve as the intermediate product storages also called buffers. In the platform, pumps and tanks can be interconnected in a flexible manner. In that way the prototype permits an easy and intuitive way of studying manufacturing control techniques and performance for several network topologies. The contributions are as follows:

- An experimental prototype is invented, designed and developed for education and research purposes.
- An accurate description and capabilities of this experimental hardware are presented.
- A similar behavioral pattern of manufacturing line under surplus-based control and a re-entrant network under clearing policy was shown on the liquid-based manufacturing networks emulator.

More detailed conclusions on the obtained results are presented at the end of each chapter of this book.

8.2 Future directions

The future directions of presented results can be subdivided into two main streams of research. The first stream concerns the future development of existing results and further analysis of performance techniques for manufacturing networks. The second stream refers to applications of existing results in other domains of interest such as irrigation networks, and offshore oil and gas production and supply networks.

8.2.1 Manufacturing Networks

With regards to the presented research results still quite a few open problems can be found on the performance analysis of evaluated demand-driven manufacturing networks. For the case of discrete time flow models the extension of results on performance presented in Chapter 5 for a single machine to a re-entrant line with bounded buffers could be considered. Currently the obtained bounds on demand tracking accuracy are limited to a certain non zero threshold value on inventory levels (see the Assumption 15) in the network as well as the assumption that production speed for every stage of each re-entrant machine in the network is the same. Further research on the complete relation between the production tracking accuracy and inventory level of multi-product network of Chapter 4 is also of interest. The solution to the above mentioned problems lies in the extension of the Lyapunov functions presented in the corresponding Appendices of Chapters 4 and 5.

Another open problem of interest, the solution of which can follow from the obtained results, concerns a demand planning mechanism. The current analysis presented in this work assumes that the production demand rate itself is unknown, but what is known is that the demand rate is lower than the network's production capacity. Thus it is assumed that product orders arriving to a network are distributed (e.g. by means of ERP, SAP tools) in such a manner that the network is able to fulfill them. It can be of interest to evaluate if the obtained results (Lyapunov functions) can be used to obtain the time at which the demand tracking errors of a network reach the steady state. This time to reach steady state can be used in demand planning in order to estimate an order completion time. In other words, by knowing the current number of produced products in the manufacturing network, production speeds and base stock levels, a new order from a customer can be received and by relying on the time to steady state the order completion time can be immediately provided without a need of simulation. Naturally, the accuracy of time to order completion will depend on the level of details included in the model of a manufacturing network. Thus it is also of importance to consider delays and setup times that may occur in a system. It is also of interest to consider such networks where the production speed of the machines could be adjusted to follow the production demand in a more accurate manner. Thus another line of future research concerns the further analysis and extension of feedforward-feedback controller presented in Chapter 7 for more complex network topologies.

8.2.2 Other Applications

Alternative application of the results of this thesis to irrigation networks, and oil and gas production is discussed in this section.

Irrigation networks

In irrigation networks, water is typically transported along the open-water channels under the power of gravity. The flow of water through the network is regulated by gates along the channels. The open-water channels in an irrigation network can be thought of as strings of pools linked by the regulating gates. Another possible, but more expensive, water transportation may consist in pipelines, reservoirs and water pumps. In terms of manufacturing network models, the gates or pumps can be associated to machines and pools or other type of reservoirs to buffers. A typical control problem in such systems is to maintain an efficient distribution of water throughout the network in order to satisfy its demand (see e.g. Cantoni et al. (2007), Negenborn et al. (2009) and references therein). Different from manufacturing networks, the water demand is commonly known in advance. Further studies are required to evaluate the applicability of our results to this important application.

Oil and gas production

Among the existing challenging problems in oil and gas production, two research directions, which are offshore production and petroleum products distribution, can be of interest for alternative application of the results of this thesis. In offshore oil and gas production, a mixture of water, gas and oil is extracted from underwater wells and transported by pipelines to the processing centers. Typically this, mixture is transported by its natural pressure and for the purpose of uniform flow, mechanical and electrical chokes and reservoirs are used along the pipeline. In other cases, due to the transportation distance or the density of the mixture, electrical submersible pumps are also used. The control problem typically consists in regulating the pressure and volumetric flow rate of the mixture's flow, so as to provide its uniform flow throughout the pipelines as well as to extend the lifetime of the distribution system itself (see e.g., Wartmann et al. (2008), Meglio et al. (2010), and references therein).
Another interesting research direction is towards a petroleum products distribution. Large volumes of different types of refined petroleum products are transported through multi-product pipelines from major supply sources to distribution centers near market areas. Large volumes of different petrol types (also called batches) are pumped back-to-back in the same pipeline for its delivery to distribution centers. By abstracting from the petrol distribution, operations can be seen as if different types of liquid are pumped through the same tube, which at the end has several cranes, one for each product type extraction, for example. Research concerning these type of applications (see e.g., Cafaro and Cerdá (2004), Cafaro and Cerdá (2012), Sasikumar et al. (1997) and references therein) also concerns the problem of market demand tracking, while satisfying many pipeline operational constraints. In this type of scheduling problems, the sequence and lengths of so called pumping runs are of a great importance. Association with demand driven manufacturing networks can be also found with respect to these applications. Further studies are required to evaluate the usefulness of our results and accuracy of our models with respect to these applications.

A
Proof of Theorem 1

Based on (2.4), it is easy to see that without any loss of generality, the class of admissible control strategies (2.3) can be reduced to those processing only the tracking errors:

$$u(k) = U_k[\varepsilon(0), \dots, \varepsilon(k)] \in \{0; 1\}.$$

We start with the problem (2.6). The proof is based on the min-max dynamic programming. So we first introduce the cost-to-go:

$$V_\tau(a) = \min_{U_\tau(\cdot),\dots,U_{T-1}(\cdot)} \sup_{\xi(\cdot)} \sum_{k=\tau}^{T} |\varepsilon(k)|^p, \quad V_T[a] := |a|^p, \tag{A.1}$$

where the minimum is over all functions $U_k(\varepsilon_k, \dots, \varepsilon_{T-1}) \in \{0; 1\}$, and $\varepsilon(k)$ is obtained from (2.4), where $k = \tau, \dots, T-1$ and $\varepsilon(\tau) = a$. This function satisfies the Bellman equation (Bertsekas (2005)):

$$V_{\tau-1}(a) = \min_{u=0;1} \sup_{\xi \in (0;1)} \{|a|^p + V_\tau[a - u + \xi]\}, \tag{A.2}$$

and the optimal strategy is given by $u(\tau - 1) = U^0_{\tau-1}[\varepsilon(\tau - 1)]$, where $U^0_{\tau-1}[a]$ is the point u furnishing the minimum in (A.2).

Lemma 1. *The cost-to-go (A.1) is the piece-wise smooth even function depicted in Figures A.1 and A.2, and*

$$U^0_\tau(a) = sign_+(a) \quad for \quad \tau = 0, \dots, T-1. \tag{A.3}$$

Proof. We first note that (A.2) can be shaped into

$$V_{\tau-1}(a) = \min \left\{ \overbrace{\sup_{\xi \in (0;1)} V_\tau[a + \xi]}^{S_0} ; \overbrace{\sup_{\xi \in (0;1)} V_\tau[a - \xi]}^{S_1} \right\} + |a|^p. \tag{A.4}$$

Here S_0 and S_1 correspond to $u = 0$ and $u = 1$, respectively. So $U^0_{\tau-1}(a) = \sigma_{\min}$, where $\sigma_{\min} = 0, 1$ is the index of the term S_σ furnishing the minimum in (A.4). We also note that since the function $a \mapsto |a|^p$ is even, simple induction on $\tau = T, \ldots, 0$ and the last equation from (A.1) show that $V_\tau[\cdot]$ is even for any τ. With this in mind, it becomes clear that firstly, $\sigma_{\max} = 0, 1$ for $a = 0$ and secondly, substitution $a := -a$ in (A.4) switches $\sigma_{\min}$ to the alternative value. This permits us to focus on $a > 0$ in the subsequent proof. For $a > 0$, formula (A.3) (to be justified) takes the form $U^0_\tau(a) = 1$.

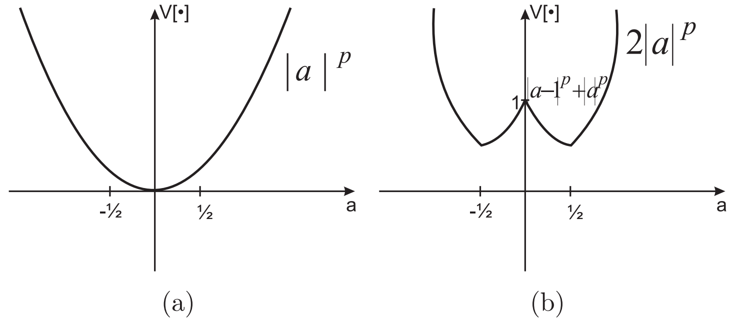


Figure A.1: (a):The graph of V_T; (b): The graph of V_{T-1}

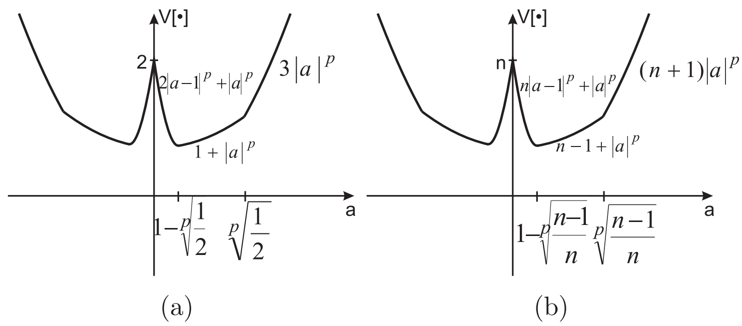


Figure A.2: (a):The graph of V_{T-2}; (b): The graph of V_{T-n} with $n \geq 3$

We proceed with immediate proof of the lemma, arguing by induction on $\tau = T - n, n = 0, 1, \ldots$.

$\boldsymbol{n = 0}$. The claim is immediate from the last equation in (A.1).

$\boldsymbol{n = 1}$.

- $\boldsymbol{a \geq \frac{1}{2}:}$ Then evidently, $S_1 = |a|^p$, and $S_0 = |a+1|^p > S_1$. So, due to (A.4), $V_{T-1}(a) = 2|a|^p$, as is depicted in Fig. A.1(b), and $U^0_\tau(a) = 1$.
- $\boldsymbol{0 < a < \frac{1}{2}:}$ Since $V_T(\cdot)$ is even, $S_1 = |a-1|^p < |a+1|^p = S_0$. So $V_{T-1} = |a-1|^p + |a|^p$, as is depicted in Fig. A.1(b), and $U^0_\tau(a) = 1$.

$\boldsymbol{n = 2}$

- $\boldsymbol{a \geq 1:}$ Similarly, in (A.4), the supremum S_0 is equal to $2|a+1|^p$, whereas $S_1 = 2|a|^p < S_0$.

- $\frac{1}{2} \le a < 1$ **:** $S_1 = \left\{ \begin{array}{ll} 2|a|^p & a > \sqrt[p]{\frac{1}{2}} \\ 1 & a < \sqrt[p]{\frac{1}{2}} \end{array} \right\} < 2|a+1|^p = S_0$

- $0 \le a < \frac{1}{2}$ **:** $S_1 = \left\{ \begin{array}{ll} 2|a-1|^p & a < 1 - \sqrt[p]{\frac{1}{2}} \\ 1 & a > 1 - \sqrt[p]{\frac{1}{2}} \end{array} \right\} < 2|a+1|^p = S_0$

  Thus $V_{T-2}(a) = \left\{ \begin{array}{ll} 3|a|^p & a \ge \sqrt[p]{\frac{1}{2}} \\ 1+|a|^p & 1-\sqrt[p]{\frac{1}{2}} \le a < \sqrt[p]{\frac{1}{2}} \\ 2|a-1|^p + |a|^p & a < 1 - \sqrt[p]{\frac{1}{2}} \end{array} \right.$, as is depicted in Figure A.2(a), so $U_\tau^0(a) = 1$.

Figure A.2(a) is a particular case of Figure A.2(b). So to complete the proof, it suffices to show that

C) Figure A.2(b) is correct and $U_{T-n}^0(a) = 1$

for $n = 2, 3, \ldots$, arguing by induction on n.

Suppose that **C)** is true for some $n \ge 2$. To compute $V_{T-n-1}(a)$, we consider separately several cases.

- $a \ge \sqrt[p]{\frac{n}{n+1}}$ **:** Here $\sqrt[p]{\frac{n}{n+1}} > \sqrt[p]{\frac{n-1}{n}}$. It follows that in (A.4), the supremum S_1 is attained at $\xi = 0$ and thus equals $(n+1)|a|^p$, whereas $S_0 = (n+1)|a+1|^p > S_1$. Thus **C)** does hold for $n := n+1$.
- $\sqrt[p]{\frac{n-1}{n}} \le a \le \sqrt[p]{\frac{n}{n+1}}$ **:** Then evidently $S_1 = n$, whereas $S_0 = (n+1)|a+1|^p > S_1$. Thus **C)** does hold for $n := n+1$.
- $1 - \sqrt[p]{\frac{n-1}{n}} \le a \le \sqrt[p]{\frac{n-1}{n}}$ **:** Since the left end $a-1$ of the interval $[a-1, a]$ is still to the right of the first fracture point of the graph from Figure A.2(b), the situation replicates the previous one.
- $1 - \sqrt[p]{\frac{n}{n+1}} \le a \le 1 - \sqrt[p]{\frac{n-1}{n}}$ **:** That end is to the left of the first fracture point. So either $S_1 = n$ (and is attained at the third fracture point) or $S_1 = (n+1)|a-1|^p$ (and is attained at $\xi = 1$). Elementary comparison shows that in fact $S_1 = n$, and so the situation still replicates the previous two ones.
- $0 \le a \le 1 - \sqrt[p]{\frac{n}{n+1}}$ **:** Then conversely, $S_1 = (n+1)|a-1|^p$, whereas $S_0 = (n+1)|a+1|^p > S_1$. Thus **C)** does hold for $n := n+1$, which completes the proof. □

For the performance index (2.6), Theorem 1 is straightforward from Lemma 1 and the dynamic programming principle (Bertsekas (2005)).

To deal with (2.7), we introduce the following intermediate performance criterion

$$J_{\text{av}} = \limsup_{T\to\infty} \sup_{\xi(0),\ldots,\xi(T-1)} \frac{1}{T} \sum_{k=0}^{T} |\varepsilon(k)|^p. \tag{A.5}$$

It is clear that

$$\inf_U J_{\text{av}} \geq \limsup_{T\to\infty} \frac{1}{T} \min_U J_T \overset{(A.1)}{=\!=} \limsup_{T\to\infty} \frac{V_0^T(a)}{T},$$

where the upper index ^{T} in V_τ^T underscores that the cost-to-go is computed for the time horizon $[0:T]$. As a result, Lemma 1 and the evident inequality $J_\infty \geq J_{\text{av}}$ imply the following lower estimates

$$\inf_U J_\infty \geq \inf_U J_{\text{av}} \geq \begin{cases} |a|^p & \text{if } |a| \geq 1 \\ 1 & \text{otherwise} \end{cases}.$$

Now we are going to show that this lower estimate of J_∞ is attained at the control strategy (2.8), which will complete the proof.

Let the system (2.4) be driven by the control law (2.8). By invoking (2.5), we conclude that

$$\varepsilon[k+1] \in \begin{cases} \big(\varepsilon[k]-1, \varepsilon[k]\big) & \text{if } \varepsilon(k) > 0 \\ \big(\varepsilon[k], \varepsilon[k]+1\big) & \text{if } \varepsilon(k) < 0 \\ \big(\varepsilon[k]-1, \varepsilon[k]+1\big) & \text{if } \varepsilon(k) = 0 \end{cases}.$$

Hence

$$f_-[\varepsilon(k)] \leq \varepsilon(k+1) \leq f_+[\varepsilon(k)], \qquad \text{where}$$
$$f_-(\varepsilon) := \min\{\varepsilon; -1\}, \qquad f_+(\varepsilon) := \max\{\varepsilon; 1\}.$$

It follows that

$$\varepsilon_-(k) \leq \varepsilon(k) \leq \varepsilon_+(k) \quad \forall k,$$

where $\varepsilon_-(k)$ and $\varepsilon_+(k)$ are the solutions of the following recursions

$$\varepsilon_\pm(k+1) = f_\pm[\varepsilon_\pm(k)], \qquad \varepsilon_\pm(0) = a.$$

It is evident that

$$\varepsilon_\pm(k) \in \Big[\min\{-|a|, -1\}; \max\{|a|; 1\}\Big],$$

which completes the proof.

B
Proof of Theorem 2

For (2.11) let us introduce the following Lyapunov function

$$\begin{aligned} V_k &= \max\left(\left|\varepsilon(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{1}{2}\right|\right. \\ &- \left.\frac{(\alpha_2-\alpha_1)}{2} - \frac{1}{2}, 0\right) > 0, \end{aligned} \tag{B.1}$$

where, for the sake of brevity, $V_k = V(\varepsilon(k))$, with $V_k = 0 \quad \forall \varepsilon(k) \in [v_d + \alpha_1 - 1, v_d + \alpha_2]$.

Here, ΔV_k along the solutions of $\varepsilon(k)$ is given by

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k, \\ \Delta V_k &= \max\left(\left|\varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2}\right.\right. \\ &- \left.\left.\operatorname{sign}_+(\varepsilon(k)) + \frac{1}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{1}{2}, 0\right) - V_k, \end{aligned} \tag{B.2}$$

where, omitting the detailed analysis, $\Delta V_k = 0 \quad \forall \varepsilon(k) \in [v_d + \alpha_1 - 1, v_d + \alpha_2]$.

In order to proof that $\Delta V_k < 0 \; \forall \varepsilon(k) \in \mathbb{R} - [v_d + \alpha_1 - 1, v_d + \alpha_2]$, let us subdivide the analysis in 4 cases.

Case 1:

Let $\varepsilon(k)$ satisfy the following inequality

$$\varepsilon(k) > 1 + \alpha_2 - W(k). \tag{B.3}$$

From (B.3) it follows that

$$\varepsilon(k) > 1 + \alpha_2 - W(k) \overset{(2.15)}{>} \alpha_2 + v_d \overset{(2.12,2.14)}{>} 0 \tag{B.4}$$

and therefore ΔV_k is defined by

$$\begin{aligned}\Delta V_k &= \max\left(\left|\varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2} - \frac{1}{2}\right|\right.\\ &- \left.\frac{(\alpha_2-\alpha_1)}{2} - \frac{1}{2}, 0\right) - V_k.\end{aligned} \tag{B.5}$$

Thus, it is also possible to state from (B.3) that

$$\begin{aligned}&\varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2} - \frac{1}{2} > 1 + \alpha_2 - \frac{(\alpha_2+\alpha_1)}{2} - \frac{1}{2}\\ &= \frac{(\alpha_2-\alpha_1)}{2} + \frac{1}{2} \overset{(2.12)}{>} 0,\end{aligned}$$

and therefore

$$\begin{aligned}&\left|\varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2} - \frac{1}{2}\right|\\ &= \varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2} - \frac{1}{2},\end{aligned}$$

which in turn implies from (B.5) that

$$\Delta V_k = \varepsilon(k) + W(k) - \alpha_2 - 1 - V_k. \tag{B.6}$$

From (B.1) if follows that

$$\begin{aligned}V_k &= \max\left(\left|\varepsilon(k) - v_d - \alpha_2 + \frac{(\alpha_2-\alpha_1)}{2} + \frac{1}{2}\right|\right.\\ &- \left.\frac{(\alpha_2-\alpha_1)}{2} - \frac{1}{2}, 0\right) \overset{(2.12,B.4)}{=} \varepsilon(k) - v_d - \alpha_2.\end{aligned} \tag{B.7}$$

Substituting (B.7) into (B.6) yields

$$\Delta V_k = v_d + W(k) - 1 \overset{(2.15)}{<} 0. \tag{B.8}$$

Case 2:

Let $\varepsilon(k)$ satisfy the following inequality

$$v_d + \alpha_2 < \varepsilon(k) \leq 1 + \alpha_2 - W(k). \tag{B.9}$$

From (2.15) it follows that $v_d + \alpha_2 < 1 + \alpha_2 - W(k)$ and hence the set defined by (B.9) is not empty. Inequalities (2.12) and (2.14) imply that $v_d + \alpha_2 > 0$ and therefore from (B.9), $\varepsilon(k) > 0$, so one can rewrite (B.2) as (B.5).

Let us prove that from (B.9) the following inequality holds

$$\left|\varepsilon(k) + W(k) - \frac{(\alpha_2 + \alpha_1)}{2} - \frac{1}{2}\right| \leq \frac{(\alpha_2 - \alpha_1)}{2} + \frac{1}{2}. \tag{B.10}$$

Indeed, from the second inequality of (B.9) it holds that

$$\begin{aligned} &\varepsilon(k) + W(k) - \frac{(\alpha_2 + \alpha_1)}{2} - \frac{1}{2} \leq 1 + \alpha_2 - \frac{(\alpha_2 + \alpha_1)}{2} - \frac{1}{2} \\ &= \frac{(\alpha_2 - \alpha_1)}{2} + \frac{1}{2}. \end{aligned} \tag{B.11}$$

Now, from the first inequality of (B.9) one concludes that

$$\begin{aligned} &\varepsilon(k) + W(k) - \frac{(\alpha_2 + \alpha_1)}{2} - \frac{1}{2} \\ &> v_d + W(k) + \alpha_2 - \frac{(\alpha_2 + \alpha_1)}{2} - \frac{1}{2} \\ &= v_d + W(k) + (\alpha_2 - \alpha_1) - \frac{(\alpha_2 - \alpha_1)}{2} - \frac{1}{2} \\ &\overset{(2.12)}{>} v_d + W(k) - \frac{(\alpha_2 - \alpha_1)}{2} - \frac{1}{2} \\ &\overset{(2.15)}{>} -\frac{(\alpha_2 - \alpha_1)}{2} - \frac{1}{2}. \end{aligned} \tag{B.12}$$

Combining (B.11) and (B.12) one concludes that (B.10) holds. Substituting (B.10) into (B.5) yields

$$\Delta V_k \quad = \quad -V_k. \tag{B.13}$$

Case 3:

Let $\varepsilon(k)$ satisfy the following inequality

$$-1 + \alpha_1 - W(k) \leq \varepsilon(k) < v_d + \alpha_1 - 1. \tag{B.14}$$

From (2.15) it follows that the set defined by (B.14) is not empty.
Here, from the second inequality of (B.14)

$$\varepsilon(k) < v_d + \alpha_1 - 1 \overset{(2.12)}{<} v_d + \alpha_2 - 1 \overset{(2.13)}{<} 0 \tag{B.15}$$

and, therefore, from (B.2) it follows that

$$\begin{aligned}\Delta V_k \;=\;& \max\left(\left|\varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{1}{2}\right|\right. \\ &- \left.\frac{(\alpha_2-\alpha_1)}{2} - \frac{1}{2}, 0\right) - V_k.\end{aligned} \tag{B.16}$$

Similarly to Case 2

$$\varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{1}{2} < \frac{(\alpha_2-\alpha_1)}{2} + \frac{1}{2} \tag{B.17}$$

and substituting (B.17) in (B.16) yields

$$\Delta V_k \;=\; -V_k. \tag{B.18}$$

Case 4:
Let $\varepsilon(k)$ satisfy the following inequality

$$\varepsilon(k) < -1 + \alpha_1 - W(k). \tag{B.19}$$

Considering (B.19) and (2.15) it follows that

$$\varepsilon(k) < v_d + \alpha_1 - 1 \overset{(2.12)}{<} 0 \tag{B.20}$$

and, hence, ΔV_k is given by (B.16).
Following the similar procedure as in Case 1, from (B.16) and (B.19) it yields that

$$\begin{aligned}\Delta V_k \;=\;& \max\left(-\varepsilon(k) - W(k) + \frac{(\alpha_2+\alpha_1)}{2} - \frac{1}{2} - \frac{(\alpha_2-\alpha_1)}{2}\right. \\ &- \left.\frac{1}{2}, 0\right) - V_k \overset{(B.19)}{=} -\varepsilon(k) - W(k) - 1 + \alpha_1 - V_k.\end{aligned} \tag{B.21}$$

Now, from (B.1) let us rewrite V_k as

$$V_k = \max\left(\left|\varepsilon(k) - v_d - \alpha_1 - \frac{(\alpha_2-\alpha_1)}{2} + \frac{1}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{1}{2}, 0\right) \overset{(2.12,B.20)}{=} -\varepsilon(k) + v_d - 1 + \alpha_1. \tag{B.22}$$

Combining (B.21) and (B.22) yields

$$\Delta V_k = -(v_d + W(k)) \overset{(2.15)}{<} 0. \tag{B.23}$$

Summarizing the 4 cases, we have shown that for the Lyapunov function given by (B.1) its ΔV_k is defined as:

$$\Delta V_k = \begin{cases} -1 + v_d + W(k) < 0, & \text{Case 1;} \\ -V_k, & \text{Case 2;} \\ -V_k, & \text{Case 3;} \\ -(v_d + W(k)) < 0, & \text{Case 4;} \\ -V_k, & \text{otherwise.} \end{cases}$$

Thus, we have proven the uniform ultimate boundedness for the solutions of (2.11), presented in (2.16) and (2.17).

C

Proof of Theorem 3

First, let us prove that Theorem 3 holds for a line of 2 manufacturing machines ($j = 1, 2$) defined by (3.8) and (3.9). With this goal, let us introduce the following Lyapunov function

$$V^{2M}(\varepsilon_1(k), \varepsilon_2(k)) \quad = \quad V_1(\varepsilon_1(k)) + V_2(\varepsilon_2(k)), \tag{C.1}$$

where

$$\begin{aligned}
V_1(\varepsilon_1(k)) &= \max\left(\left|\varepsilon_1(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_1}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right),\\
V_2(\varepsilon_2(k)) &= \frac{1}{n}\max\left(\left|\varepsilon_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}, 0\right),
\end{aligned}$$

with n be a positive constant. It will be defined later on in the proof.

Here for the sake of brevity $V^{2M}(\varepsilon_1(k), \varepsilon_2(k)) = V_k^{2M}$, $V_1(\varepsilon_1(k)) = V_{1k}$, and $V_2(\varepsilon_2(k)) = V_{2k}$, with $V_k^{2M} = 0 \quad \forall \varepsilon_j(k) \in [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$.

Thus, ΔV_k^{2M} along the solutions of $\varepsilon_1(k)$ and $\varepsilon_2(k)$ is given by

$$\Delta V_k^{2M} \quad = \quad V_{k+1}^{2M} - V_k^{2M} = \Delta V_{1k} + \Delta V_{2k}, \tag{C.2}$$

where

$$\begin{aligned}
\Delta V_{1k} &= \max\left(\left|\varepsilon_1(k) + W_1(k) - \frac{(\alpha_2+\alpha_1)}{2} - \mu_1 \mathrm{sign}_+(\varepsilon_1(k)) + \frac{\mu_1}{2}\right|\right.\\
&\quad \left. - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right) - V_{1k},\\
\Delta V_{2k} &= \frac{1}{n}\max\left(\left|\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2}\right.\right.\\
&\quad \left. - \mu_2 \mathrm{sign}_+(\varepsilon_2(k)) \mathrm{sign}_{\mathit{Buff}}(w_2(k) - \beta_2(k)) + \frac{\mu_2}{2}\right|\\
&\quad \left. - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}, 0\right) - V_{2k}.
\end{aligned}$$

In order to perform a more detailed analysis on ΔV_k^{2M}, let us divide this proof into 3 cases.

Case 1 (Insufficient Buffer Content and Unsatisfied Demand)

Here we analyse the case where the second machine needs to keep up with the demand, but its buffer content is not sufficient in order to start working. Suppose that $\varepsilon_2(k)$ and $w_2(k)$ satisfies the following inequality

$$0 \leq w_2(k) < \beta_2(k), \tag{C.3}$$
$$\varepsilon_2(k) > 0. \tag{C.4}$$

Considering the condition (C.4), from (3.17) it follows that

$$\varepsilon_1(k) > v_d + \alpha_2 + \varepsilon_2(k). \tag{C.5}$$

Now for (C.3), (C.4), and (C.5), let us rewrite ΔV_{1k} and ΔV_{2k} from (C.2) as

$$\begin{aligned}
\Delta V_{1k} &= \max\left(\left|\varepsilon_1(k) + W_1(k) - \frac{(\alpha_2+\alpha_1)}{2} - \frac{\mu_1}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right) - V_{1k},\\
\Delta V_{2k} &= \frac{1}{n}\max\left(\left|\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}, 0\right) - V_{2k}.
\end{aligned} \tag{C.6}$$

where V_{1k} and V_{2k} are defined by (C.1).

From the previous result (see (B.3),(B.9)), we know that for $\varepsilon_1(k)$ defined by (C.5) the increment ΔV_{1k} from (C.6) is given by:

$$\Delta V_{1k} = \begin{cases} -\varepsilon_1(k) + v_d + \alpha_2 < 0, & \text{if } v_d + \alpha_2 < \varepsilon_1(k) \leq \mu_1 + \alpha_2 - W_1(k);\\ -\mu_1 + v_d + W_1(k) < 0, & \text{if } \varepsilon_1(k) > \mu_1 + \alpha_2 - W_1(k). \end{cases} \tag{C.7}$$

As for ΔV_{2k}, let us subdivide the analysis of (C.6) into 3 subcases.

Case 1.1

Assume that $\varepsilon_2(k)$ satisfies the following inequality

$$0 < \varepsilon_2(k) \leq \alpha_2 - W_2(k), \tag{C.8}$$

where $\alpha_2 - W_2(k) \overset{(3.10)}{>} 0$.

Thus, from (C.8) and (C.5) the following inequality entailes

$$\varepsilon_1(k) > v_d + \alpha_2. \tag{C.9}$$

Let us prove that for $\varepsilon_2(k)$ defined by (C.8) the following inequality holds

$$\left|\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}\right| \leq \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}. \tag{C.10}$$

Indeed, from the second inequality of (C.8) it gives that

$$\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} \le \alpha_2 - W_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2},$$
$$\alpha_2 - W_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} = \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}.$$

Now from the first inequality of (C.8) it holds that

$$\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} > W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}$$
$$= \mu_2 + W_2(k) - \alpha_1 - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2} \overset{(3.10)}{>} -\frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}.$$

Thus, it follows that (C.10) is satisfied.

Now, let us prove that for (C.8) the following inequality holds

$$\left|\varepsilon_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}\right| < \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}. \tag{C.11}$$

From the second inequality of (C.8) it yields that

$$\varepsilon_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} \le \alpha_2 - W_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}$$
$$= \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2} - v_d - W_2(k) \overset{(3.10,3.14)}{<} \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}.$$

From the first inequality of (C.8) it entails that

$$\varepsilon_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} > -v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}$$
$$= -\frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2} + \mu_2 - v_d - \alpha_1 \overset{(3.10,3.12)}{>} -\frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}.$$

Thus, it follows that (C.11) holds.

Now substituting (C.10) and (C.11) into ΔV_{2k} of (C.6) yields $\Delta V_{2k} = 0$. As for $\varepsilon_1(k)$ defined by (C.9), the increment ΔV_{1k} is given by (C.7). Thus, it implies for (C.2) that $\Delta V_k^{2M} = \Delta V_{1k} < 0$.

Case 1.2

Suppose that $\varepsilon_2(k)$ satisfies the following inequality

$$\alpha_2 - W_2(k) < \varepsilon_2(k) \le \alpha_2 + v_d, \tag{C.12}$$

where $-W_2(k) \overset{(3.14)}{<} v_d$.

It follows from (C.12) and (C.5) that the following inequality holds

$$\varepsilon_1(k) > v_d + 2\alpha_2 - W_2(k). \tag{C.13}$$

Let us prove that for $\varepsilon_2(k)$ defined by (C.12) following inequality satisfies

$$\left|\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}\right| > \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}, \tag{C.14}$$

where

$$\alpha_2 - \alpha_1 \overset{(3.10)}{>} 0. \tag{C.15}$$

It follows that from the first inequality of (C.12)

$$\varepsilon_2(k) - W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} > \alpha_2 - W_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2},$$
$$\alpha_2 - W_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} = \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}.$$

Thus, considering (C.15), the latter inequality entails that inequality (C.14) holds. Now let us prove that for (C.12) the following inequality satisfies

$$\left|\varepsilon_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}\right| \le \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}. \tag{C.16}$$

Indeed, from the second inequality of (C.12) it follows that

$$\varepsilon_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} \le v_d + \alpha_2 - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} = \frac{(\alpha_2-\alpha_1)}{2} + \frac{\mu_2}{2}.$$

From the first inequality of (C.12) it yields that

$$\varepsilon_2(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2} > -v_d + \alpha_2 - W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}$$
$$= -v_d - W_2(k) + (\alpha_2-\alpha_1) - \frac{(\alpha_2+\alpha_1)}{2} - \frac{\mu_2}{2} + \mu_2 \overset{(3.14,3.10)}{>} -\frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}.$$

Thus, the inequality (C.16) holds.
Now substituting (C.14) and (C.16) into ΔV_{2k} of (C.6) yields

$$\Delta V_{2k} = \frac{1}{n}\left(\varepsilon_2(k) + W_2(k) - \alpha_2\right). \tag{C.17}$$

As for $\varepsilon_1(k)$ defined in (C.13) the increment ΔV_{1k} is given by (C.7). Thus, by combining the first part of (C.7) and (C.17) we obtain that

$$\begin{aligned}\Delta V_k^{2M} &= -\varepsilon_1(k) + v_d + \alpha_2 + \frac{1}{n}(\varepsilon_2(k) + W_2(k) - \alpha_2),\\ \Delta V_k^{2M} &\overset{(3.10)}{<} -\varepsilon_1(k) + v_d + \alpha_2 + \frac{1}{n}\varepsilon_2(k) \overset{(C.5)}{<} -\varepsilon_2(k) + \frac{1}{n}\varepsilon_2(k).\end{aligned}$$

Assuming that $n \geq 1$, we get that $\Delta V_k^{2M} < 0$.
Now combining the second part of (C.7) and (C.17) yields

$$\begin{aligned}\Delta V_k^{2M} &= -\mu_1 + v_d + W_1(k) + \frac{1}{n}(\varepsilon_2(k) + W_2(k) - \alpha_2),\\ \Delta V_k^{2M} &\overset{(C.12)}{\leq} -\mu_1 + v_d + W_1(k) + \frac{1}{n}(v_d + W_2(k)).\end{aligned}$$

Assuming that n satisfies

$$n > \frac{v_d + \alpha_2}{\mu_1 - v_d - \alpha_2} \tag{C.18}$$

and

$$n \geq 1, \tag{C.19}$$

we get that $\Delta V_k^{2M} < 0$.

Case 1.3

Let $\varepsilon_2(k)$ satisfy the following inequality

$$\varepsilon_2(k) > v_d + \alpha_2. \tag{C.20}$$

It follows from (C.20) and (C.5) that

$$\varepsilon_1(k) > 2(v_d + \alpha_2). \tag{C.21}$$

For $\varepsilon_2(k)$ given by (C.20) inequality (C.14) is satisfied as well. Now let us prove that in case (C.20) the inequality

$$\left|\varepsilon_2(k) - v_d - \frac{(\alpha_2 + \alpha_1)}{2} + \frac{\mu_2}{2}\right| > \frac{(\alpha_2 - \alpha_1)}{2} + \frac{\mu_2}{2}. \tag{C.22}$$

holds. Indeed, from (C.20) it follows that

$$\varepsilon_2(k) - v_d - \frac{(\alpha_2 + \alpha_1)}{2} + \frac{\mu_2}{2} > v_d + \alpha_2 - v_d - \frac{(\alpha_2 + \alpha_1)}{2} + \frac{\mu_2}{2} = \frac{(\alpha_2 - \alpha_1)}{2} + \frac{\mu_2}{2}.$$

Thus, using (C.15) the inequality (C.22) holds. Substituting (C.14) and (C.22) into ΔV_{2k} of (C.6) yields

$$\Delta V_{2k} = \frac{1}{n}(v_d + W_2(k)). \tag{C.23}$$

As for $\varepsilon_1(k)$ given by (C.21), the increment ΔV_{1k} is defined by (C.7). Thus, combining the first part of (C.7) and (C.23)

$$\begin{aligned}
\Delta V_k^{2M} &= -\varepsilon_1(k) + v_d + \alpha_2 + \frac{1}{n}(v_d + W_2(k)),\\
\Delta V_k^{2M} &\overset{(C.21)}{<} -2(v_d + \alpha_2) + v_d + \alpha_2 + \frac{1}{n}(v_d + W_2(k)),\\
\Delta V_k^{2M} &< -v_d - \alpha_2 + \frac{1}{n}(v_d + W_2(k)).
\end{aligned}$$

Assuming that n satisfies conditions (C.18), (C.19) we get that $\Delta V_k^{2M} < 0$. Combining the second part of (C.7) and (C.23) we obtain that

$$\Delta V_k^{2M} = -\mu_1 + v_d + W_1(k) + \frac{1}{n}(v_d + W_2(k)) \overset{(C.18,C.19)}{<} 0.$$

Summarizing the 3 subcases, we have shown that for the Lyapunov function given by (C.1), its increment ΔV_k^{2M} is given by:

$$\Delta V_k^{2M} = \begin{cases}
\text{Case 1.1;}\\
-\varepsilon_1(k) + v_d + \alpha_2 < 0,\\
-\mu_1 + v_d + W_1(k) < 0;\\
\text{Case 1.2}\\
-\varepsilon_1(k) + v_d + \alpha_2 + \frac{1}{n}(\varepsilon_2(k) + W_2(k) - \alpha_2) < 0,\\
-\mu_1 + v_d + W_1(k) + \frac{1}{n}(\varepsilon_2(k) + W_2(k) - \alpha_2) < 0;\\
\text{Case 1.3}\\
-\varepsilon_1(k) + v_d + \alpha_2 + \frac{1}{n}(v_d + W_2(k)) < 0,\\
-\mu_1 + v_d + W_1(k) + \frac{1}{n}(v_d + W_2(k)) < 0;
\end{cases}$$

provided (C.3) and (C.4) holds and n satisfies conditions (C.18), (C.19).

Case 2 (Insufficient Buffer Content and Satisfied Demand)

Let $\varepsilon_2(k)$ and $w_2(k)$ satisfy the following inequalities

$$0 \le w_2(k) < \beta_2(k), \tag{C.24}$$
$$\varepsilon_2(k) \le 0. \tag{C.25}$$

Omitting further irrelevant details for the bounds of $\varepsilon_1(k)$, let us assume that $\varepsilon_1(k) \in \mathbb{R}$. Then we can rewrite (C.2) as

$$\Delta V_k^{2M} = V_{k+1}^{2M} - V_k^{2M} = \Delta V_{1k} + \Delta V_{2k}, \tag{C.26}$$

with

$$\begin{aligned}
\Delta V_{1k} &= \max\left(\left|\varepsilon_1(k) + W_1(k) - \frac{(\alpha_2+\alpha_1)}{2} - \mu_1 \mathrm{sign}_+(\varepsilon_1(k)) + \frac{\mu_1}{2}\right|\right.\\
&\quad \left. -\frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right) - V_{1k},\\
\Delta V_{2k} &= \frac{1}{n}\max\left(\left|\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_2}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}, 0\right) - V_{2k}.
\end{aligned}$$

Here ΔV_{1k} and ΔV_{2k} can be analyzed independently. From the proof of Theorem 2 and assuming that $\varepsilon_1(k) \in \mathbb{R}$, we know that the analysis of ΔV_{1k} given by (C.26) is identical to ΔV_k from (B.5). Thus, ΔV_{1k} is given by

$$\Delta V_{1k} = \begin{cases}
-\mu_1 + v_d + W_1(k) < 0, & \text{if } \varepsilon_1(k) > \mu_1 + \alpha_2 - W_1(k);\\
-V_{1k}, & \text{if } v_d + \alpha_2 < \varepsilon_1(k) \le \mu_1 + \alpha_2 - W_1(k);\\
-V_{1k}, & \text{if } -\mu_1 + \alpha_1 - W_1(k) < \varepsilon_1(k)\\
& \text{and } \varepsilon_1(k) \le v_d + \alpha_1 - \mu_1;\\
-(v_d + W_1(k)) < 0, & \text{if } \varepsilon_1(k) < -\mu_1 + \alpha_1 - W_1(k);\\
-V_{1k}, & \text{if } v_d + \alpha_1 - \mu_1 \le \varepsilon_1(k) \le v_d + \alpha_2.
\end{cases} \tag{C.27}$$

Similarly, considering (C.25), ΔV_{2k} is deduced from the analysis of ΔV_k, see (B.5), and it is given by

$$\Delta V_{2k} = \begin{cases}
-V_{2k}, & \text{if } -\mu_2 + \alpha_1 - W_2(k) \le \varepsilon_2(k)\\
& \text{and } \varepsilon_2(k) < -\mu_2 + \alpha_1 + v_d;\\
-\frac{1}{n}(v_d + W_2(k)) < 0, & \text{if } \varepsilon_2(k) < -\mu_2 + \alpha_1 - W_2(k);\\
-V_{2k}, & \text{if } -\mu_2 + \alpha_1 + v_d < \varepsilon_2(k) \le 0.
\end{cases} \tag{C.28}$$

Case 3 (Sufficient Buffer Content)
Assume that $w_2(k)$ satisfies the following inequality

$$w_2(k) \ge \beta_2(k). \tag{C.29}$$

Omitting further irrelevant details for bounds of $\varepsilon_1(k)$ and $\varepsilon_2(k)$, let us consider that $\varepsilon_1(k) \in \mathbb{R}$ and $\varepsilon_2(k) \in \mathbb{R}$. Thus, we can rewrite (C.2) as

$$\Delta V_k^{2M} = V_{k+1}^{2M} - V_k^{2M} = \Delta V_{1k} + \Delta V_{2k}, \tag{C.30}$$

with

$$\begin{aligned}
\Delta V_{1k} &= \max\left(\left|\varepsilon_1(k) + W_1(k) - \frac{(\alpha_2+\alpha_1)}{2} - \mu_1 \mathrm{sign}_+(\varepsilon_1(k)) + \frac{\mu_1}{2}\right|\right.\\
&\quad \left. - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right) - V_{1k},\\
\Delta V_{2k} &= \frac{1}{n}\max\left(\left|\varepsilon_2(k) + W_2(k) - \frac{(\alpha_2+\alpha_1)}{2} - \mu_2 \mathrm{sign}_+(\varepsilon_2(k)) + \frac{\mu_2}{2}\right|\right.\\
&\quad \left. - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_2}{2}, 0\right) - V_{2k}.
\end{aligned}$$

Here ΔV_{1k} and ΔV_{2k} can be analyzed independently. Thus, using the proof of Theorem 2 we notice that ΔV_{1k} and ΔV_{2k} from (C.30) can be analyzed similarly to ΔV_k from (B.5). In this case ΔV_{1k} is defined by (C.27) as well and ΔV_{2k} results in

$$\Delta V_{2k} = \begin{cases}
-\frac{1}{n}(\mu_2 - v_d - W_2(k)) < 0, & \text{if } \varepsilon_2(k) > \mu_2 + \alpha_2 - W_2(k);\\
-V_{2k}, & \text{if } v_d + \alpha_2 < \varepsilon_2(k) \le \mu_2 + \alpha_2 - W_2(k);\\
-V_{2k}, & \text{if } -\mu_2 + \alpha_1 - W_2(k) < \varepsilon_2(k)\\
 & \text{and } \varepsilon_2(k) \le v_d + \alpha_1 - \mu_2;\\
-\frac{1}{n}(v_d + W_2(k)) < 0, & \text{if } \varepsilon_2(k) < -\mu_2 + \alpha_1 - W_2(k);\\
-V_{2k}, & \text{if } v_d + \alpha_1 - \mu_2 \le \varepsilon_2(k) \le v_d + \alpha_2.
\end{cases} \tag{C.31}$$

Thus, in case $N = 2$ we have proved the uniform ultimate boundedness for solutions of equations(3.8), (3.9).

Now let us prove the uniform ultimate boundedness for solutions of (3.8) and (3.9) for the case of $N > 2$; N is a number of machines. For that, let us introduce the following Lyapunov function

$$V_k^{NM} = V_{1k} + \sum_{j=2}^{N} V_{jk}, \tag{C.32}$$

where

$$\begin{aligned}
V_{1k} &= \max\left(\left|\varepsilon_1(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_1}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right),\\
V_{jk} &= \sum_{j=2}^{N} \frac{1}{n_j}\max\left(\left|\varepsilon_j(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_j}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_j}{2}, 0\right),
\end{aligned}$$

with $V_k^{NM} = 0 \quad \forall \varepsilon_j(k) \in [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$.
Here ΔV_k^{NM} is given by

$$\Delta V_k^{NM} \quad = \quad \Delta V_{1k} + \sum_{j=2}^{N} \Delta V_{jk}, \tag{C.33}$$

where

$$\begin{aligned}
\Delta V_{1k} &= \max\left(\left|\varepsilon_1(k) + W_1(k) - \frac{(\alpha_2+\alpha_1)}{2} - \mu_1 \mathrm{sign}_+(\varepsilon_1(k)) + \frac{\mu_1}{2}\right|\right. \\
&\quad \left. - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right) - V_{1k}, \\
\Delta V_{jk} &= \frac{1}{n_j} \max\left(\left|\varepsilon_j(k) + W_j(k) - \frac{(\alpha_2+\alpha_1)}{2}\right.\right. \\
&\quad \left. - \mu_j \mathrm{sign}_+(\varepsilon_j(k)) \mathrm{sign}_{Buff}(w_j(k) - \beta_j(k)) + \frac{\mu_j}{2}\right| \\
&\quad \left. - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_j}{2}, 0\right) - V_{jk}.
\end{aligned}$$

with n_j be a positive constant. It will be presented later on in this proof.
Let us subdivide the analysis of ΔV_k^{NM} into 4 cases. First, we analyse the simplest case, in which the capacity condition (3.15) for each buffer j is satisfied. As a result, each ΔV_{jk} can be considered independently. Then in Case 2N we will analyse ΔV_k^{NM} when the buffer content condition (3.15) is satisfied for every buffer j except for buffer i, where $i > 2$, and $\varepsilon_i(k) > 0$. In Case 3N, we will extend the analysis of ΔV_{jk} to the worst case scenario, in which the content condition for all buffers is not satisfied. At the same time the production demand is growing and $\varepsilon_j(k) > 0$, $\forall j > 2$, which means that the machines are starving. We conclude our analysis by Case 4N. In this case the content condition for all buffers is not satisfied and $\varepsilon_j(k) \leq 0$, for all $j \neq 1$. The latter inequality signifies that the customer demand is fulfilled.
Case 1N(Sufficient Buffer Content)
Assume that

$$w_j(k) \geq \beta_j(k),\ \forall j = 2, \ldots, N. \tag{C.34}$$

and suppose that $\varepsilon_j(k) \in \mathbb{R}$, $\forall j = 1, \ldots, N$. Then, from previous result (see Case 3 of $N = 2$) we know that for $\varepsilon_j(k) \in \mathbb{R}$ and $w_j(k)$ given by (C.34) the increment ΔV_{1k} of (C.33) is defined by (C.27) and ΔV_{jk} is given by

$$
\Delta V_{jk} = \begin{cases} -\frac{1}{n_j}(\mu_j - v_d - W_2(k)) < 0, & \text{if } \varepsilon_j(k) > \mu_j + \alpha_2 - W_j(k); \\ -V_{jk}, & \text{if } v_d + \alpha_2 < \varepsilon_j(k) \le \mu_j + \alpha_2 - W_j(k); \\ -V_{jk}, & \text{if } -\mu_j + \alpha_1 - W_j(k) < \varepsilon_j(k) \\ & \text{and } \varepsilon_j(k) \le v_d + \alpha_1 - \mu_j; \\ -\frac{1}{n_j}(v_d + W_j(k)) < 0, & \text{if } \varepsilon_j(k) < -\mu_j + \alpha_1 - W_j(k); \\ -V_{jk}, & \text{if } v_d + \alpha_1 - \mu_j \le \varepsilon_j(k) \le v_d + \alpha_2. \end{cases} \tag{C.35}
$$

Case 2N (Insufficient Content in One of the Buffers)

Let us analyse the case where

$$
0 \le w_i(k) < \beta_i(k), \tag{C.36}
$$

$$
\varepsilon_i(k) > 0. \tag{C.37}
$$

The constant i stands for a number of one machine and of its buffer in the line of N machines such that satisfies $i > 2$. We also consider that for the rest of machines the following conditions are satisfied

$$
w_j(k) \ge \beta_j(k) \; \forall j \ne i, j = 2 \ldots, N, \tag{C.38}
$$

$$
\varepsilon_j(k) \in \mathbb{R} \; \forall j \ne i-1, i, j = 1 \ldots, N. \tag{C.39}
$$

It is known from Case 1N that for any $\varepsilon_j(k)$ given by (C.39) the function ΔV_{jk} is defined by (C.35) provided (C.38) holds. It is also known from Case 1 that for any $\varepsilon_{i-1}(k)$ given by (3.17) the function ΔV_{i-1k} is given by

$$
\Delta V_{i-1k} = \begin{cases} -\frac{1}{n_{i-1}}(\varepsilon_{i-1}(k) - v_d - \alpha_2) < 0, \\ \text{if } v_d + \alpha_2 < \varepsilon_{i-1}(k) \le \mu_{i-1} + \alpha_2 - W_{i-1}(k); \\ -\frac{1}{n_{i-1}}(\mu_{i-1} - v_d - W_{i-1}(k)) < 0, \\ \text{if } \varepsilon_{i-1}(k) > \mu_{i-1} + \alpha_2 - W_{i-1}(k); \end{cases} \tag{C.40}
$$

provided that (C.38) holds. As for $\varepsilon_i(k)$ introduced by (C.37), using (C.36), we have that

$$
\Delta V_{ik} = \begin{cases} 0, & \text{if } 0 < \varepsilon_i(k) \le \alpha_2 - W_i(k); \\ \frac{1}{n_i}(\varepsilon_i(k) + W_i(k) - \alpha_2), & \text{if } \alpha_2 - W_i(k) < \varepsilon_i(k) \le v_d + \alpha_2; \\ \frac{1}{n_i}(v_d + W_i(k)), & \text{if } \varepsilon_i(k) > v_d + \alpha_2, \end{cases} \tag{C.41}
$$

see Case 1. Thus, let us analyse $\Delta V_{i-1k} + \Delta V_{ik}$ by following steps.

First step, it holds for $0 < \varepsilon_i(k) \leq \alpha_2 - W_i(k)$ and $\varepsilon_{i-1}(k) \overset{(3.17)}{>} v_d + \alpha_2$ that

$$\Delta V_{i-1k} + \Delta V_{ik} \overset{(C.40,C.41)}{=} \Delta V_{i-1k} < 0.$$

Second step, it holds for $\alpha_2 - W_i(k) < \varepsilon_i(k) \leq v_d + \alpha_2$ and $\varepsilon_{i-1}(k) \overset{(3.17)}{>} v_d + 2\alpha_2 - W_i(k)$ that

$$\begin{aligned}
\Delta V_{i-1k} + \Delta V_{ik} &\overset{(C.40,C.41)}{=} -\frac{1}{n_{i-1}}(\varepsilon_{i-1}(k) - v_d - \alpha_2) + \frac{1}{n_i}(\varepsilon_i(k) + W_i(k) - \alpha_2)\\
\Delta V_{i-1k} + \Delta V_{ik} &\leq -\frac{1}{n_{i-1}}\varepsilon_{i-1}(k) + \frac{1}{n_i}\varepsilon_i(k) < 0,
\end{aligned}$$

where

$$n_i \geq n_{i-1} \geq 1. \tag{C.42}$$

We obtain that

$$\begin{aligned}
\Delta V_{i-1k} + \Delta V_{ik} &\overset{(C.40,C.41)}{=} -\frac{1}{n_{i-1}}(\mu_{i-1} - v_d - W_{i-1}(k)) + \frac{1}{n_i}(\varepsilon_i(k) + W_i(k) - \alpha_2),\\
\Delta V_{i-1k} + \Delta V_{ik} &\overset{(3.10)}{<} -\frac{1}{n_{i-1}}(\mu_{i-1} - v_d - W_{i-1}(k)) + \frac{1}{n_i}(v_d + W_i(k)) < 0,
\end{aligned}$$

where

$$n_i > \frac{v_d + \alpha_2}{\mu_{i-1} - v_d - \alpha_2} n_{i-1}, \tag{C.43}$$

and condition (C.42) are satisfied.

Third step, it holds for $\varepsilon_i(k) > v_d + \alpha_2$ and $\varepsilon_{i-1}(k) \overset{(3.17)}{>} 2v_d + 2\alpha_2$ that

$$\begin{aligned}
\Delta V_{i-1k} + \Delta V_{ik} &\overset{(C.40,C.41)}{=} -\frac{1}{n_{i-1}}(\varepsilon_{i-1}(k) - v_d - \alpha_2) + \frac{1}{n_i}(v_d + W_i(k)),\\
\Delta V_{i-1k} + \Delta V_{ik} &\overset{(3.17)}{<} -\frac{1}{n_{i-1}}(v_d + \alpha_2) + \frac{1}{n_i}(v_d + W_i(k)) < 0,
\end{aligned}$$

where n_i and n_{i-1} satisfy (C.42) and (C.43).
We obtain that

$$\Delta V_{i-1k} + \Delta V_{ik} \overset{(C.40,C.41)}{=} -\frac{1}{n_{i-1}}(\mu_{i-1} - v_d - W_{i-1}(k)) + \frac{1}{n_i}(v_d + W_i(k)) < 0,$$

where n_i and n_{i-1} satisfy (C.42) and (C.43).

Case 3N(All Buffer Contents are Insufficient and Unsatisfied Demand)

Let us extend Case 2N to the following scenario where

$$0 \leq w_j(k) < \beta_j(k) \; \forall j = 2, \ldots, N, \tag{C.44}$$
$$\varepsilon_N(k) > 0. \tag{C.45}$$

We get from (3.5) that the inequality (3.17) is satisfied. Thus, we can rewrite ΔV_{1k} and ΔV_{jk} from (C.33) as

$$\begin{aligned}
\Delta V_{1k} &= \max\left(\left|\varepsilon_1(k) + W_1(k) - \frac{(\alpha_2+\alpha_1)}{2} - \frac{\mu_1}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right) - V_{1k},\\
\Delta V_{jk} &= \frac{1}{n_j}\max\left(\left|\varepsilon_j(k) + W_j(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_j}{2}\right|\right.\\
&\quad \left. - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_j}{2}, 0\right) - V_{jk}.
\end{aligned}$$

Here ΔV_{1k} is defined exactly as in (C.6) (see Case 1), which results in (C.7). As for ΔV_{jk} it can be easily deduced from (C.41) (see Case 2N) that

$$\begin{aligned}
&\Delta V_{Nk} = 0, \text{ if } 0 < \varepsilon_N(k) \leq \alpha_2 - W_N(k);\\
&\Delta V_{Nk} = \frac{1}{n_N}(\varepsilon_N(k) + W_N(k) - \alpha_2), \text{ if } \alpha_2 - W_N(k) < \varepsilon_N(k) \leq v_d + \alpha_2;\\
&\Delta V_{jk} = \frac{1}{n_j}(v_d + W_j(k)), \text{ if } \varepsilon_j(k) > v_d + \alpha_2.
\end{aligned}$$

Here $\varepsilon_j(k) \overset{(3.17)}{>} v_d + \alpha_2 \; \forall j = 2, \ldots, N-1$.

Thus, in order for $\Delta V_{1k} + \sum_{j-2}^{N} \Delta V_{jk}$ to satisfy $\Delta V_k^{NM} < 0$, the following conditions for n_j must be accomplished

$$\begin{aligned}
&n_2 > \frac{v_d + \alpha_2}{\mu_1 - v_d - \alpha_2}\\
&n_j > \frac{v_d + \alpha_2}{\mu_{j-1} - v_d - \alpha_2} n_{j-1} \; \forall j = 2, \ldots, N,\\
&n_j \geq n_{j-1} \geq 1 \; \forall j = 2, \ldots, N.
\end{aligned}$$

Case 4N (Satisfied Demand)

Let us assume that $\varepsilon_j(k) \leq 0$ and $w_j(k) < \beta_j(k)$, $\forall j = 2, \ldots, N$, and $\varepsilon_1(k) \in \mathbb{R}$. Then we remind that ΔV_{1k} is defined as ΔV_{1k} from (C.27). As for ΔV_{jk} we can extend the result from (C.28) (see Case 2) as

$$\Delta V_{jk} = \begin{cases} -V_{jk}, & \text{if } -\mu_j + \alpha_1 - W_j(k) \leq \varepsilon_j(k) < -\mu_j + \alpha_1 + v_d;\\ -\frac{1}{n_j}(v_d + W_j(k)) < 0, & \text{if } \varepsilon_j(k) < -\mu_j + \alpha_1 - W_j(k);\\ -V_{jk}, & \text{if } -\mu_j + \alpha_1 + v_d < \varepsilon_j(k) \leq 0. \end{cases}$$

Hence, $\Delta V_k^{NM} = \Delta V_{1k} + \sum_{j=2}^{N} \Delta V_{jk} < 0$.

Thus, in this proof we have analyzed the increment of the proposed Lyapunov function by means of selected cases. It can be shown that any particular unmentioned case is covered throughout the combination of given cases in this proof.

D
Proof of Theorem 4

Let us prove that Theorem 4 holds for a line of 2 manufacturing machines ($j = 1, 2$) defined by (3.23) and (3.25). With this goal, let us introduce the following Lyapunov function

$$V^{2M}(\varepsilon_1; \varepsilon_2) \quad = \quad \max\left\{V_1(\varepsilon_1), V_2(\varepsilon_2)\right\}, \tag{D.1}$$

where

$$\begin{aligned} V_j(\varepsilon_j) \quad &= \quad \max\left\{-\varepsilon_j - \mu_j + v_d + \alpha_1, \varepsilon_j - v_d - \alpha_2, 0\right\} \;> 0\;, \\ &\quad\;\; \forall\varepsilon_j \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2],\; j = 1, 2. \end{aligned} \tag{D.2}$$

Here for the sake of brevity $V^{2M}(\varepsilon_1(k), \varepsilon_2(k)) = V_k^{2M}$, $V_j(\varepsilon_j(k)) = V_{j,k}$, with $V^{2M} = 0 \quad \forall\varepsilon_j \in [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$.

Thus, ΔV_k^{2M} along the solutions of $\varepsilon_1(k)$ and $\varepsilon_2(k)$ is given by

$$\Delta V_k^{2M} \quad = \quad V_{k+1}^{2M} - V_k^{2M} = \max\{V_{1,k+1}, V_{2,k+1}\} + \min\{-V_{1,k}, -V_{2,k}\}, \tag{D.3}$$

where

$$V_{j,k+1} = \max\left\{\begin{array}{c} -\varepsilon_j(k) - W_j(k) + \alpha_1 - \mu_j + \mu_j\eta_{j,k}, \\ \varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j\eta_{j,k}, \\ 0 \end{array}\right\},\; j = 1, 2.$$

For the sake of brevity we introduce $\eta_{j,k}$ as

$$\eta_{1,k} \quad = \quad \text{sign}_+(\varepsilon_1(k))\text{sign}_-(w_2(k) - \gamma_2), \tag{D.4}$$

$$\eta_{2,k} \quad = \quad \text{sign}_+(\varepsilon_2(k))\text{sign}_{Buff}(w_2(k) - \beta_2(k)). \tag{D.5}$$

In order to perform a more detailed analysis on ΔV_k^{2M}, let us divide this proof into 3 cases.

Case 1 (Sufficient Buffer Content)

Suppose that $w_2(k)$ satisfies the following inequality

$$\beta_2(k) \leq w_2(k) < \gamma_2, \tag{D.6}$$

which means that machine M_2 has sufficient material in its buffer B_2 in order to start working and machine M_1 always has an access to the infinite raw material supply. Thus these machines have an independent behavior and it will be sufficient to analyse the increment of only one of the functions $V_{j,k}$ in order to determine the behavior of ΔV_k^{2M}.

Let us assume that $\varepsilon_j(k)$ satisfies the following condition

$$\varepsilon_j(k) > 0, \ \forall j = 1, 2 \tag{D.7}$$

and in consequence from (D.4) and (D.5) it follows that $\eta_{j,k} = 1$.

Then, $\Delta V_{j,k}$ along the solutions of $\varepsilon_j(k)$ is given by

$$\Delta V_{j,k} = \underbrace{\max \begin{Bmatrix} -\varepsilon_j(k) - W_j(k) + \alpha_1, \\ \varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j, \\ 0 \end{Bmatrix}}_{V_{j,k+1}} + \underbrace{\min \begin{Bmatrix} \varepsilon_j(k) + \mu_j - v_d - \alpha_1, \\ -\varepsilon_j(k) + v_d + \alpha_2, \\ 0 \end{Bmatrix}}_{-V_{j,k}},$$

with $j = 1, 2$.

From where with help of *Assumptions 1* and *2* it can be easily deduced that

$$\Delta V_{j,k} = \begin{cases} 0 & \text{if } \varepsilon_j(k) \leq v_d + \alpha_2, \\ -\varepsilon_j(k) + v_d + \alpha_2 < 0 & \text{if } v_d + \alpha_2 < \varepsilon_j(k) \leq \mu_j + \alpha_2 - W_j(k), \\ -\mu_j + v_d + W_j(k) < 0 & \text{if } \varepsilon_j(k) > \mu_j + \alpha_2 - W_j(k). \end{cases} \tag{D.8}$$

Now, suppose that for $\varepsilon_j(k)$ the following condition holds

$$\varepsilon_j(k) \leq 0, \tag{D.9}$$

and in consequence from (D.4) and (D.5) it yields that $\eta_{j,k} = 0$. Then $\Delta V_{j,k}$ along the solutions of $\varepsilon_j(k)$ is given by

$$\Delta V_{j,k} = \underbrace{\max \begin{Bmatrix} -\varepsilon_j(k) - W_j(k) + \alpha_1 - \mu_j, \\ \varepsilon_j(k) + W_j(k) - \alpha_2, \\ 0 \end{Bmatrix}}_{V_{j,k+1}} + \underbrace{\min \begin{Bmatrix} \varepsilon_j(k) + \mu_j - v_d - \alpha_1, \\ -\varepsilon_j(k) + v_d + \alpha_2, \\ 0 \end{Bmatrix}}_{-V_{j,k}},$$

where $j = 1, 2$.

Here with help of *Assumptions 1* and *2* it can be easily deduced that

$$\Delta V_{j,k} = \begin{cases} 0 & \text{if } v_d + \alpha_1 - \mu_j \leq \varepsilon_j(k) \leq 0, \\ -v_d - W_j(k) < 0 & \text{if } \varepsilon_j(k) < -\mu_j + \alpha_1 - W_j(k), \\ \varepsilon_j(k) + \mu_j - v_d - \alpha_1 < 0 & \text{if } -\mu_j + \alpha_1 + W_j(k) \leq \varepsilon_j(k) < v_d + \alpha_1 - \mu_j. \end{cases} \tag{D.10}$$

Summarizing, for conditions (D.6), (D.7), and (D.9) from (D.8) and (D.10) it holds that if $V_{j,k} > 0$ its increment $\Delta V_{j,k} < 0$. From the definition of min it yields that for ΔV_k^{2M} given by (D.3) the following inequality is satisfied

$$\Delta V_k^{2M} \leq V_{i,k+1} - V_{i,k} \leq 0, \tag{D.11}$$

where $i = \arg\max_{j=1,2}\{V_{j,k+1}\}$. Note that $\Delta V_{j,k} = V_{j,k+1} - V_{j,k} = 0$ only if either first condition of (D.8) or first condition of (D.10) is satisfied and for all $\varepsilon_j(k) \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$ it follows that $\Delta V_k^{2M} < 0$. Thus, in this case it holds that for $V_k^{2M} > 0$ given by (D.1) its increment $\Delta V_k^{2M} < 0$.

Case 2 (Insufficient Buffer Content)

Let us assume that $w_2(k)$ satisfies the following inequality

$$w_2(k) < \beta_2(k), \forall k \in \mathbb{N}, \tag{D.12}$$

and $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) \leq 0, \forall k \in \mathbb{N}. \tag{D.13}$$

Then from (3.17) it holds that

$$\varepsilon_1(k) > \mu_1 + \alpha_2 - \alpha_1 + \varepsilon_2(k). \tag{D.14}$$

Here similarly to Case 1 the behavior of these two machines can be considered independently. Thus, for $\varepsilon_1(k)$ satisfying (D.14) it holds that $\Delta V_{1,k}$ is given by (D.8) or (D.10) if $\varepsilon_2(k) < -\mu_1 - \alpha_2 + \alpha_1$ or $\Delta V_{1,k}$ is given by (D.8) if $-\mu_1 - \alpha_2 + \alpha_1 \leq \varepsilon_2(k) \leq 0$. For $\varepsilon_2(k)$ satisfying (D.13) the increment $\Delta V_{2,k}$ is given by (D.10). In consequence for ΔV_k^{2M} given by (D.3) the inequality (D.11) in this case is also satisfied.

Now, let us assume that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) > 0, \forall k \in \mathbb{N}. \tag{D.15}$$

Then from (3.17) it holds that $\varepsilon_1(k)$ is given by (D.14). In this case M_2 has a positive tracking error, but its buffer B_2 has insufficient raw material content (D.12) in order to start working ($\eta_{2,k} = 0$). Machine M_1 has a positive error as well, but due to its infinite raw material supply access it can immediately initiate its production process ($\eta_{1,k} = 1$). Thus, for (D.15) and (D.14) let us rewrite ΔV_k^{2M} from (D.3) as

$$\Delta V_k^{2M} = \underbrace{\max\left\{\begin{matrix} -\varepsilon_1(k) - W_1(k) + \alpha_1, \\ \varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1, \\ -\varepsilon_2(k) - W_2(k) + \alpha_1 - \mu_2, \\ \varepsilon_2(k) + W_2(k) - \alpha_2, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{2M}} + \underbrace{\min\left\{\begin{matrix} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ -\varepsilon_1(k) + v_d + \alpha_2, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ -\varepsilon_2(k) + v_d + \alpha_2, \\ 0 \end{matrix}\right\}}_{-V_k^{2M}}. \tag{D.16}$$

It follows from (3.10), (D.15) and (D.14) that ΔV_k^{2M} from (D.16) can be reduced to

$$\Delta V_k^{2M} = \underbrace{\max\left\{\begin{matrix}\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1,\\ \varepsilon_2(k)+W_2(k)-\alpha_2,\\ 0\end{matrix}\right\}}_{V_{k+1}^{2M}} + \underbrace{\min\left\{\begin{matrix}-\varepsilon_1(k)+v_d+\alpha_2,\\ -\varepsilon_2(k)+v_d+\alpha_2,\\ 0\end{matrix}\right\}}_{-V_k^{2M}}. \tag{D.17}$$

Now, let us prove that for $\varepsilon_1(k)$ given by (D.14) inequality

$$\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1 > \varepsilon_2(k)+W_2(k)-\alpha_2 \tag{D.18}$$

is satisfied.

Indeed, from condition (D.14) it yields that

$$\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1 > \varepsilon_2(k)+W_1(k)-\alpha_1 \overset{(3.10)}{>} \varepsilon_2(k)+W_2(k)-\alpha_2. \tag{D.19}$$

Thus, inequality (D.18) is satisfied. Also, from (D.19) it holds that

$$\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1 \overset{(3.10),(D.15)}{>} 0. \tag{D.20}$$

Now, considering (D.18) and (D.20) we can rewrite V_{k+1}^{2M} given by the first term of (D.17) as

$$V_{k+1}^{2M} = \varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1. \tag{D.21}$$

Let us prove that for $\varepsilon_1(k)$ given by (D.14) inequality

$$-\varepsilon_2(k)+v_d+\alpha_2 > -\varepsilon_1(k)+v_d+\alpha_2 \tag{D.22}$$

is satisfied. Here from condition (D.14) it yields that

$$\begin{aligned}&-\varepsilon_2(k)+v_d+\alpha_2 > -\varepsilon_1(k)+\mu_1+\alpha_2-\alpha_1+v_d+\alpha_2,\\ &-\varepsilon_1(k)+\mu_1+\alpha_2-\alpha_1+v_d+\alpha_2 \overset{(3.10),(3.13)}{>} -\varepsilon_1(k)+v_d+\alpha_2.\end{aligned} \tag{D.23}$$

Thus, inequality (D.22) is satisfied. From inequalities (D.14), (3.12) it follows that

$$-\varepsilon_1(k)+v_d+\alpha_2 < 0. \tag{D.24}$$

From (D.22), (D.24) we can rewrite V_k^{2M} given by the second term of (D.17) as

$$V_k^{2M} = \varepsilon_1(k)-v_d-\alpha_2 \overset{(D.24)}{>} 0. \tag{D.25}$$

Having V_{k+1}^{2M} given by (D.21) and V_k^{2M} given by (D.25), we can finally reduce ΔV_k^{2M} from (D.17) to

$$V_k^{2M} = -\mu_1+v_d+W_1(k) \overset{(3.14)}{<} 0. \tag{D.26}$$

Thus, for this Case it holds that for $V_k^{2M} > 0$ given by (D.1) its increment $\Delta V_k^{2M} < 0$.

Case 3 (Limited Buffer Content)

Suppose that $w_2(k)$ satisfies the following inequality

$$w_2(k) \geq \gamma_2, \forall k \in \mathbb{N}, \tag{D.27}$$

and let us first assume that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) \leq 0, \forall k \in \mathbb{N}. \tag{D.28}$$

Then from (3.29) it holds that

$$\varepsilon_1(k) \leq \varepsilon_2(k) - \mu_2 - \alpha_2 + \alpha_1 \tag{D.29}$$

where $-\mu_2 - \alpha_2 + \alpha_1 \overset{(3.10)}{<} 0$.

In this case machines M_1 and M_2 are not working ($\eta_{j,k} = 0$) and there behavior can be considered similar to the first part of Case 2. It follows that for (D.28) and (D.29) the increments $\Delta V_{1,k}$ and $\Delta V_{2,k}$ are given by (D.10) respectively. Thus, for ΔV_k^{2M} given by (D.3) the inequality (D.11) in this case is also satisfied.

Now, let us assume that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) > 0, \forall k \in \mathbb{N}. \tag{D.30}$$

In this case M_2 has sufficient material to start working ($\eta_{2,k} = 1$) and M_1 is stopped ($\eta_{1,k} = 0$) due to the limited capacity of its downstream buffer B_2. Thus, two situations may occur. First, consider that M_1 is stopped, but its tracking error $\varepsilon_1(k) \leq 0$. This may occur if $\varepsilon_2(k)$ satisfies $0 < \varepsilon_2(k) \overset{(D.29)}{\leq} \mu_2 + \alpha_2 - \alpha_1$. The behavior of these 2 machines can be considered independently and by following the procedure from Case 1 we arrive to the conclusion that for $V_k^{2M} > 0$ given by (D.1) its increment $\Delta V_k^{2M} < 0$.

In the second situation, consider that $\varepsilon_1(k)$ satisfies

$$\varepsilon_1(k) > 0, \forall k \in \mathbb{N}, \tag{D.31}$$

which by (D.29) implies that

$$\varepsilon_2(k) > \mu_2 + \alpha_2 - \alpha_1. \tag{D.32}$$

Then for (D.31) and (D.32) let us rewrite ΔV_k^{2M} from (D.3) as

$$\Delta V_k^{2M} = \underbrace{\max \begin{Bmatrix} -\varepsilon_1(k) - \mu_1 - W_1(k) + \alpha_1, \\ \varepsilon_1(k) + W_1(k) - \alpha_2, \\ -\varepsilon_2(k) - W_2(k) + \alpha_1, \\ \varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2, \\ 0 \end{Bmatrix}}_{V_{k+1}^{2M}} + \underbrace{\min \begin{Bmatrix} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ -\varepsilon_1(k) + v_d + \alpha_2, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ -\varepsilon_2(k) + v_d + \alpha_2, \\ 0 \end{Bmatrix}}_{-V_k^{2M}}. \tag{D.33}$$

It follows from (3.10) and (3.14) that ΔV_k^{2M} from (D.33) can be reduced to

$$\Delta V_k^{2M} = \underbrace{\max\left\{\begin{matrix}\varepsilon_1(k)+W_1(k)-\alpha_2,\\ \varepsilon_2(k)+W_2(k)-\alpha_2-\mu_2,\\ 0\end{matrix}\right\}}_{V_{k+1}^{2M}} + \underbrace{\min\left\{\begin{matrix}-\varepsilon_1(k)+v_d+\alpha_2,\\ -\varepsilon_2(k)+v_d+\alpha_2,\\ 0\end{matrix}\right\}}_{-V_k^{2M}}. \tag{D.34}$$

Now, let us derive from (D.29) that the following inequality

$$\varepsilon_2(k)+W_2(k)-\alpha_2-\mu_2 > \varepsilon_1(k)+W_1(k)-\alpha_2 \tag{D.35}$$

is satisfied. Indeed, from (D.29) it holds that

$$\begin{aligned}\varepsilon_2(k)+W_2(k)-\alpha_2-\mu_2 &\geq \varepsilon_1(k)+\mu_2+\alpha_2-\alpha_1+W_2(k)-\alpha_2-\mu_2,\\ \varepsilon_1(k)+\mu_2+\alpha_2-\alpha_1+W_2(k)-\alpha_2-\mu_2 &\overset{(3.10)}{>} \varepsilon_1(k)+W_1(k)-\alpha_2.\end{aligned} \tag{D.36}$$

Thus, inequality (D.35) is satisfied. From (D.32) and (3.10) it also holds that

$$\varepsilon_2(k)+W_2(k)-\alpha_2-\mu_2 > 0. \tag{D.37}$$

Considering (D.35) and (D.37) we can rewrite V_{k+1}^{2M} from the first part of (D.34)as

$$V_{k+1}^{2M} = \varepsilon_2(k)+W_2(k)-\alpha_2-\mu_2. \tag{D.38}$$

Let us show that from (D.29) the following inequality

$$-\varepsilon_1(k)+v_d+\alpha_2 > -\varepsilon_2(k)+v_d+\alpha_2 \tag{D.39}$$

is satisfied. Here, from condition (D.29) it yields that

$$\begin{aligned}-\varepsilon_1(k)+v_d+\alpha_2 &> -\varepsilon_2(k)+v_d+\alpha_2+\mu_2+\alpha_2-\alpha_1,\\ -\varepsilon_2(k)+v_d+\alpha_2+\mu_2+\alpha_2-\alpha_1 &\overset{(3.10),(3.13)}{>} -\varepsilon_2(k)+v_d+\alpha_2.\end{aligned} \tag{D.40}$$

Thus, inequality (D.39) is satisfied. From inequalities (D.29), (3.12) it follows that

$$-\varepsilon_2(k)+v_d+\alpha_2 < 0. \tag{D.41}$$

From (D.39), (D.41) we can rewrite V_k^{2M} given by the second part of (D.34)as

$$V_k^{2M} = \varepsilon_2(k)-v_d-\alpha_2 \overset{(D.41)}{>} 0. \tag{D.42}$$

Having V_{k+1}^{2M} given by (D.38) and V_k^{2M} given by (D.42), we can finally reduce ΔV_k^{2M} from (D.34) to

$$\Delta V_k^{2M} = -\mu_2+v_d+W_2(k) \overset{(3.14)}{<} 0. \tag{D.43}$$

Thus, for this Case it holds that for $V_k^{2M} > 0$ given by (D.1) its increment $\Delta V_k^{2M} < 0$. Summarizing for 3 cases, we have shown that for $V_k^{2M} > 0$ given by (D.1) its increment $\Delta V_k^{2M} < 0$ for all $\varepsilon_j(k) \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$ and $\Delta V_k^{2M} = 0$ for all $\varepsilon_j(k) \in [v_d + \alpha_1 - \mu_j, v_d + \alpha_2]$. Thus, $\limsup_{k\to\infty} V_k^{2M} = 0$, which completes our proof.

In this proof we have analyzed the increment of the proposed Lyapunov function by means of 3 cases. For a line of N manufacturing machines ($j = 1, ..., N$) defined by (3.23) and (3.24) the Lyapunov function (D.1) is extended to

$$V_k^{NM} = \max\left\{V_1(\varepsilon_1), ..., V_N(\varepsilon_N)\right\}.$$

Here the same reasoning is followed as for the proof for 2 machines. The analysis is subdivided into the same 3 cases. Case 1 (Sufficient Buffer Content), the first part of Case 2 ($w_j(k) < \beta_j(k)$ and $\varepsilon_j(k) \leq 0, \forall j = 2, \ldots, N$) and Case 3 (Limited Buffer Content) are solved identically to the proof for the line of 2 machines. For the second part of Case 2 the proof relies on the condition (3.28) and the assumption that machine M_1 has always an access to the infinite raw material supply. Due to the extensive technical details and similarity in the procedure we omit the complete analysis for a line of N machines and restrict ourselves by only giving this general description of the proof.

E

Proof of Theorem 5

Let us prove that Theorem 5 holds for a line of N manufacturing machines defined by (3.23) and (3.25). Assuming that now the safety stock level constants satisfy $w_{dj} \geq 0$ for all $j = 2, .., N$, the Lyapunov function is modified as follows. The function for $j = 1$ remains as in (D.2) and for $j = 2, .., N$ the function is now given by

$$V_j(\varepsilon_j) = \max \left\{ \begin{array}{c} -\varepsilon_j - \mu_j + v_d + \alpha_1, \\ \varepsilon_j - v_d - \alpha_2 - \underbrace{\sum_{i=2}^{j} \max\left(\mu_{i-1} - \alpha_1 + \alpha_2 - w_{di} + \underbrace{\mu_i + \alpha_3}_{\beta_i}, 0 \right)}_{x_j}, \\ 0 \end{array} \right\}$$

$$V_j(\varepsilon_j) > 0, \ \forall \varepsilon_j \notin [v_d + \alpha_1 - \mu_j, v_d + \alpha_2 + x_j], j = 2, .., N.$$

Thus, ΔV_k^{NM} along the solutions of $\varepsilon_j(k)$ is given by

$$\Delta V_k^{NM} = V_{k+1}^{NM} - V_k^{NM} = \max\{V_{1,k+1}, ..., V_{N,k+1}\} + \min\{-V_{1,k}, ..., -V_{N,k}\}, \tag{E.1}$$

where for $j = 2, .., N$

$$V_{j,k+1} = \max \left\{ \begin{array}{c} -\varepsilon_j(k) - W_j(k) + \alpha_1 - \mu_j + \mu_j \eta_{j,k}, \\ \varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j \eta_{j,k} - x_j, \\ 0 \end{array} \right\}.$$

Here $\eta_{j,k}$ is given by

$$\begin{aligned} \eta_{1,k} &= \text{sign}_+(\varepsilon_1(k))\text{sign}_-(w_2(k) - \gamma_2), \\ \eta_{j,k} &= \text{sign}_+(\varepsilon_j(k))\text{sign}_{Buff}(w_j(k) - \beta_j(k))\text{sign}_-(w_{j+1}(k) - \gamma_{j+1}), \\ \eta_{N,k} &= \text{sign}_+(\varepsilon_N(k))\text{sign}_{Buff}(w_N(k) - \beta_N(k)). \end{aligned}$$

The analysis of (E.1) is subdivided into the same 3 cases. Case 1 (Sufficient Buffer Content) and the first part of Case 2 ($w_j(k) < \beta_j(k)$ and $\varepsilon_j(k) \leq 0, \forall j = 2, \ldots, N$) are solved identically to the proof for the line of 2 machines. For the second part of Case 2 and for the Case 3 (just as in proof of Theorem 4) this proof relies on the condition (3.26) in combination with equations (3.5), (3.6). Due to the extensive technical details and the similarity of the procedure with the proof of Theorem 4 we omit the complete analysis for a line of N machines and restrict ourselves by only giving this general description of the procedure.

F

Proof of Theorem 6

Let us prove that Theorem 6 holds for one machine with $j = 1, \dots, N$ defined by (4.5). With this goal, let us introduce the following Lyapunov function

$$V_k^{B_N} = \max \left\{ \begin{array}{c} -\varepsilon_1(k) - \mu_1 + v_{d1} + \alpha_1, \\ \vdots \\ -\varepsilon_N(k) - \mu_N + v_{dN} + \alpha_1, \\ \underbrace{\sum_{j=1}^{N} \frac{\varepsilon_j(k) - v_{dj} - \alpha_2}{\mu_j + c_3}}_{X_k}, \\ 0 \end{array} \right\}. \tag{F.1}$$

Here for the sake of brevity $V_k^{B_N} = V^{B_N}(\varepsilon_1(k), ..., \varepsilon_N(k))$, with $V^{B_N} \geq 0$, for all $\varepsilon_j(k) \in \mathbb{R}$.

Thus, $\Delta V_k^{B_2}$ along the solutions of $\varepsilon_j(k)$ is given by

$$\Delta V_k^{B_N} = \underbrace{\max \left\{ \begin{array}{c} -\varepsilon_1(k) - \Delta\varphi_1(k) + \alpha_1 - \mu_1 + \beta_1(k)u_1(k), \\ \vdots \\ -\varepsilon_N(k) - \Delta\varphi_N(k) + \alpha_1 - \mu_N + \beta_N(k)u_N(k), \\ X_{k+1}, \\ 0 \end{array} \right\}}_{V_{k+1}^{B_N}}$$

$$+\underbrace{\min\left\{\begin{matrix}\varepsilon_1(k)+\mu_1-v_d-\alpha_1,\\ \vdots\\ \varepsilon_N(k)+\mu_N-v_{dN}-\alpha_1,\\ -X_k,\\ 0\end{matrix}\right\}}_{-V_k^{B_N}}, \tag{F.2}$$

where $X_{k+1}=\sum_{j=1}^{N}\frac{\varepsilon_j(k)+W_j(k)-\alpha_2-\beta_j(k)u_j(k)}{\mu_j+c_3}$.

In order to perform a more detailed analysis on $\Delta V_k^{B_N}$, let us divide this proof into 2 cases.

Case 1 ($q(k)=0$)

Suppose that $q(k)=0$, which from (4.3) implies that $u_{j,k}=0$, for all $j=1,...,N$. Then we can rewrite $\Delta V_k^{B_N}$ from (F.2) as

$$\begin{aligned}&\Delta V_k^{B_2}=\\ &\underbrace{\max\left\{\begin{matrix}-\varepsilon_1(k)-\Delta\varphi_1(k)+\alpha_1-\mu_1,\\ \vdots\\ -\varepsilon_N(k)-\Delta\varphi_N(k)+\alpha_1-\mu_N,\\ X_{k+1},\\ 0\end{matrix}\right\}}_{V_{k+1}^{B_N}}\\ &+\underbrace{\min\left\{\begin{matrix}\varepsilon_1(k)+\mu_1-v_{d1}-\alpha_1,\\ \vdots\\ \varepsilon_N(k)+\mu_N-v_{dN}-\alpha_1,\\ -X_k,\\ 0\end{matrix}\right\}}_{-V_k^{B_N}}.\end{aligned} \tag{F.3}$$

From (4.3), $\varepsilon_j(k)$ satisfies

$$\varepsilon_j(k)\le 0, \tag{F.4}$$

for all k and j. Then we can reduce $\Delta V_k^{B_N}$ from (F.3) to

$$\Delta V_k^{B_N} = \underbrace{\max\left\{\begin{matrix} -\varepsilon_1(k) - \Delta\varphi_1(k) + \alpha_1 - \mu_1, \\ \vdots \\ -\varepsilon_N(k) - \Delta\varphi_N(k) + \alpha_1 - \mu_N, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{B_N}} + \underbrace{\min\left\{\begin{matrix} \varepsilon_1(k) + \mu_1 - v_{d1} - \alpha_1, \\ \vdots \\ \varepsilon_N(k) + \mu_N - v_{dN} - \alpha_1, \\ 0 \end{matrix}\right\}}_{-V_k^{B_N}}. \tag{F.5}$$

Here, let us assume that for $V_{k+1}^{B_N}$ the maximum is reached in the j element of the function, i.e. $V_{k+1}^{B_N} = -\varepsilon_j(k) - \Delta\varphi_j(k) + \alpha_1 - \mu_j$. Then from the definition of min it holds that

$$\begin{aligned} \Delta V_k^{B_N} &\leq -\varepsilon_j(k) - \Delta\varphi_j(k) + \alpha_1 - \mu_j \\ &\quad +\varepsilon_j(k) + \mu_j - v_{dj} - \alpha_1, \\ \Delta V_k^{B_N} &\leq -v_{dj} - \Delta\varphi_j(k) \overset{(4.6,4.9)}{<} 0. \end{aligned} \tag{F.6}$$

For $V_{k+1}^{B_N}$ with maximum reached by its last element it holds that

$$\Delta V_k^{B_N} = -V_k^{B_N}. \tag{F.7}$$

Thus, for in this case for $V_k^{B_N} > 0$ given by (F.1) its increment $\Delta V_k^{BN} < 0$. This concludes the analysis of Case 1.

Case 2 $(q(k) = B_j)$

Suppose that $\varepsilon_j(k)$ satisfies

$$\varepsilon_j(k) > 0 \tag{F.8}$$

for all k. Thus, the machine is working with buffer B_j $(q(k) = B_j)$, which is assumed to always have a sufficient raw material. Without loss of generality let us assume for now that the value of ε_s for all $s \neq j$ is of arbitrary sign. Then we can rewrite $\Delta V_k^{B_N}$ from (F.3) as

$$\Delta V_k^{B_N} = \underbrace{\max\left\{\begin{array}{c} -\varepsilon_1(k) - W_1(k) + \alpha_1, \\ \vdots \\ -\varepsilon_N(k) - \Delta\varphi_N(k) + \alpha_1 - \mu_N, \\ X_{k+1}, \\ 0 \end{array}\right\}}_{V_{k+1}^{B_N}} + \underbrace{\min\left\{\begin{array}{c} \varepsilon_1(k) + \mu_1 - v_{dN} - \alpha_1, \\ \vdots \\ \varepsilon_N(k) + \mu_N - v_{dN} - \alpha_1, \\ -X_k, \\ 0 \end{array}\right\}}_{-V_k^{B_N}}. \tag{F.9}$$

Subcase 1: Let us first analyse (F.9) assuming that $\varepsilon_s(k)$ satisfies

$$\varepsilon_s(k) > 0 \tag{F.10}$$

for all k, $s \neq j$, and $s = 1, ..., N$. Then due to condition (F.8) and (F.10), the increment (F.9) satisfies

$$\Delta V_k^{B_N} \leq \underbrace{\max\left\{\begin{array}{c} \frac{\varepsilon_j(k)+W_j(k)-\alpha_2-\mu_j}{\mu_j+c_3} + \frac{\varepsilon_s(k)+\Delta\varphi_s(k)-\alpha_2}{\mu_s+c_3}, \\ 0 \end{array}\right\}}_{-V_{k+1}^{B_N *}} + \underbrace{\min\left\{\begin{array}{c} \frac{-\varepsilon_j(k)+v_{dj}+\alpha_2}{\mu_j+c_3} + \frac{-\varepsilon_s(k)+v_{ds}+\alpha_2}{\mu_s+c_3}, \\ 0 \end{array}\right\}}_{-V_k^{B_N *}}. \tag{F.11}$$

Consider now that for $V_{k+1}^{B_N *}$ the maximum is reached in its first element, i.e. $V_{k+1}^{B_N *} = \frac{\varepsilon_j(k)+W_j(k)-\alpha_2-\mu_j}{\mu_j+c_3} + \frac{\varepsilon_s(k)+\Delta\varphi_s(k)-\alpha_2}{\mu_s+c_3}$. Then from the definition of min it holds that

$$\Delta V_k^{B_N} \leq -\frac{\mu_j + f_j(k)}{\mu_j + c_3} + \frac{v_{dj} + \Delta\varphi(k)}{\mu_j + c_3} + \frac{v_{ds} + \Delta\varphi_s(k)}{\mu_s + c_3} \overset{(4.7,4.11)}{<} 0.$$

In case that for $V_{k+1}^{B_N *}$ given by (F.11) the maximum is reached in its second element, then from the definition of min

$$\Delta V_k^{B_N} = -V_k^{B_N}. \tag{F.12}$$

Thus, for this subcase for $V_k^{B_N} > 0$ given by (F.1) its increment $\Delta V_k^{B_N} < 0$.

Subcase 2: Now let us analyse (F.9) assuming that $\varepsilon_s(k)$ satisfies

$$\varepsilon_s(k) \leq 0 \tag{F.13}$$

for all k, $s \neq j$, $s = 1, ..., N$. In analogy with the procedure followed in previous subcase it is obtained that

$$\begin{cases} \Delta V_k^{B_N} \leq -v_{ds} - \Delta\varphi_s(k) \overset{(4.6,4.9)}{<} 0 \\ \text{if } V_{k+1}^{B_N} = -\varepsilon_s(k) - \Delta\varphi(s) + \alpha_1 - \mu_s, \\ \Delta V_k^{B_N} \leq -\frac{\mu_j + f_j(k)}{\mu_j + c_3} + \frac{v_{dj} + \Delta\varphi_j(k)}{\mu_j + c_3} + \frac{v_{ds} + \Delta\varphi_s(k)}{\mu_s + c_3} \overset{(4.7,4.11)}{<} 0 \\ \text{if } V_{k+1}^{B_N} = \frac{\varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j}{\mu_j + c_3} + \frac{\varepsilon_s(k) + \Delta\varphi_s(k) - \alpha_2}{\mu_s + c_3}, \\ \Delta V_k^{B_N} = -V_k^{B_N} \\ \text{if } V_{k+1}^{B_N} = 0. \end{cases}$$

Thus, for this subcase for $V_k^{B_N} > 0$ given by (F.1) its increment $\Delta V_k^{B_N} < 0$. This concludes the analysis of Case 2.
Summarizing for 2 cases, we have shown that for $V_k^{B_N} > 0$ given by (F.1) its increment $\Delta V_k^{B_N} < 0$. Thus, $\limsup_{k\to\infty} V_k^{B_N} = 0$, which completes our proof.

G
Proof of Theorem 7

Let us prove that Theorem 7 holds for the multi-product manufacturing line with P machines ($i = 1, \dots, P$) and each one with N production stages ($j = 1, \dots, N$) and $N \times P - 1$ intermediate buffers defined by (4.18).
Let us start by describe the general structure of our proof.

- First, we will show that Theorem 7 holds for a line of 2 machines with 2 stages each. For that we will analyse the increment of the proposed Lyapunov function, which will be given later in the proof. The analysis is subdivided into 4 Cases. As it can be observed form (4.15), each machine in the network has 2 modes, which are the working mode and the stand by (idle) mode. Thus if we have 2 machines in the network this network may present 4 working modes, which correspond to the number of cases in our analysis.
- Second, we will extend our proof to a complete $P \times N$ network. Given the proof for 2 machines we will describe the followed procedure for the proof of a complete line.

Let us begin our proof by showing first that Theorem 7 holds for a smaller network, which consists of 2 machines M_i ($i = 1, 2$) with 2 production stages each ($j = 1, 2$) and 2 intermediate buffers ($B_{2,1}, B_{2,2}$). For that purpose, let us introduce the following Lyapunov function

$$V_k^{FM} = \max\left\{V_k^{FM_1}, V_k^{FM_2}\right\}, \tag{G.1}$$

where

$$V_k^{FM_i} = \max\begin{Bmatrix} -\varepsilon_{i,1}(k) - \mu_i + v_{d1} + \alpha_1, \\ -\varepsilon_{i,2}(k) - \mu_i + v_{d2} + \alpha_1, \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - v_{d1} - N\alpha_2, \\ \varepsilon_{i,2}(k) - (N-1)(\mu_i - \alpha_1) - v_{d2} - N\alpha_2, \\ \varepsilon_{i,1}(k) + \varepsilon_{i,2}(k) - \sum_{j=1}^{N} v_{dj} - N\alpha_2 \\ 0 \end{Bmatrix}. \tag{G.2}$$

Here $N=2$, $V_k^{FM}=0,\ \forall\varepsilon_{i,j}(k)\in[v_{dj}+\alpha_1-\mu_i,(N-1)(\mu_i-\alpha_1)+v_{dj}+N\alpha_2]$. Thus, ΔV_k^{FM} along the solutions of $\varepsilon_{i,j}(k)$ is given by

$$\Delta V_k^{FM} \quad = \quad \max\left\{V_{k+1}^{FM_1},V_{k+1}^{FM_2}\right\}-\max\left\{V_k^{FM_1},V_k^{FM_2}\right\}, \tag{G.3}$$

where

$$V_{k+1}^{FM_i} = \tag{G.4}$$

$$\max\left\{\begin{array}{c} -\varepsilon_{i,1}(k)-\Delta\varphi_1(k)+\alpha_1-\mu_i+\beta_i(k)u_{i,1}(k),\\ -\varepsilon_{i,2}(k)-\Delta\varphi_2(k)+\alpha_1-\mu_i+\beta_i(k)u_{i,2}(k),\\ \varepsilon_{i,1}(k)-(N-1)(\mu_i-\alpha_1)-N\alpha_2+\Delta\varphi_1(k)-\beta_i(k)u_{i,1}(k),\\ \varepsilon_{i,2}(k)-(N-1)(\mu_i-\alpha_1)-N\alpha_2+\Delta\varphi_2(k)-\beta_i(k)u_{i,2}(k),\\ \varepsilon_{i,1}(k)+\varepsilon_{i,2}(k)+\sum_{j=1}^{N}\Delta\varphi_j(k)-N\alpha_2-\beta_i(k)u_{i,1}(k)-\beta_i(k)u_{i,2}(k),\\ 0 \end{array}\right\}.$$

In order to perform a more detailed analysis on ΔV_k^{FM}, let us divide this proof into 4 cases.

Case 1: All Machines Off ($q_1(k)=0$ and $q_2(k)=0$) From (4.15), it can be observed that in this case at least one of the following conditions must be satisfied

$$\varepsilon_{i,j}(k)\le 0,\ \forall i,j=1,2\,, \tag{G.5}$$

$$\varepsilon_{1,j}(k)\le 0 \text{ and } \omega_{2,j}(k)<\beta_{2,j}(k). \tag{G.6}$$

Note that we consider $B_{1,1}$ and $B_{1,2}$ to always contain sufficient amount of products. First, given the values of $q_1(k)$ and $q_2(k)$ let us rewrite ΔV_k^{FM} from (G.3) as

$$\Delta V_k^{FM} \quad = \quad \max\left\{V_{k+1}^{FM_1},V_{k+1}^{FM_2}\right\}+\min\left\{-V_k^{FM_1},-V_k^{FM_2}\right\}, \tag{G.7}$$

where

$$V_{k+1}^{FM_i} \quad = \quad \max\left\{\begin{array}{c} -\varepsilon_{i,1}(k)-\Delta\varphi_1(k)+\alpha_1-\mu_i,\\ -\varepsilon_{i,2}(k)-\Delta\varphi_2(k)+\alpha_1-\mu_i,\\ \varepsilon_{i,1}(k)-(N-1)(\mu_i-\alpha_1)-N\alpha_2+\Delta\varphi_1(k),\\ \varepsilon_{i,2}(k)-(N-1)(\mu_i-\alpha_1)-N\alpha_2+\Delta\varphi_2(k),\\ \varepsilon_{i,1}(k)+\varepsilon_{i,2}(k)+\sum_{j=1}^{N}\Delta\varphi(k)-N\alpha_2,\\ 0 \end{array}\right\} \tag{G.8}$$

and

$$-V_k^{FM_i} \quad = \quad \min\left\{\begin{array}{c} \varepsilon_{i,1}(k)+\mu_i-v_{d1}-\alpha_1,\\ \varepsilon_{i,2}(k)+\mu_i-v_{d2}-\alpha_1,\\ -\varepsilon_{i,1}(k)+(N-1)(\mu_i-\alpha_1)+v_{d1}+N\alpha_2,\\ -\varepsilon_{i,2}(k)+(N-1)(\mu_i-\alpha_1)+v_{d2}+N\alpha_2,\\ -\varepsilon_{i,1}(k)-\varepsilon_{i,2}(k)+\sum_{j=1}^{N}v_{dj}+N\alpha_2\\ 0 \end{array}\right\}. \tag{G.9}$$

Second, let us assume that the maximum of V_{k+1}^{FM} is reached in $V_{k+1}^{FM_i}$. Thus from the definition of minimum it is possible to state that ΔV_k^{FM} satisfies the following inequality

$$\Delta V_k^{FM} \quad \le \quad V_{k+1}^{FM_i}-V_k^{FM_i}. \tag{G.10}$$

Third, let us show that $V_{k+1}^{FM_i} - V_k^{FM_i} < 0$. For that purpose we assume that the maximum of $V_{k+1}^{FM_i}$ is reached by one of its first 2 elements, i.e., $V_{k+1}^{FM_i} = -\varepsilon_{i,j}(k) - \Delta\varphi_j(k) + \alpha_1 - \mu_i$. Then from the definition of minimum it is possible to state that ΔV_k^{FM} satisfies

$$\begin{aligned} \Delta V_k^{FM} &\leq -\varepsilon_{i,j}(k) - \Delta\varphi_j(k) + \alpha_1 - \mu_i + \varepsilon_{i,j}(k) + \mu_i - v_{dj} - \alpha_1, \\ \Delta V_k^{FM} &\leq -v_{dj} - \Delta\varphi_j(k) \overset{(4.19,4.22)}{<} 0. \end{aligned} \tag{G.11}$$

Now suppose that the maximum of $V_{k+1}^{FM_i}$ is reached by its sixth element, i.e., $V_{k+1}^{FM_i} = 0$. Then from the definition of minimum it is valid to state that ΔV_k^{FM} satisfies

$$\Delta V_k^{FM} \leq -V_k^{FM_i} < 0. \tag{G.12}$$

One from the rest of the elements of $V_{k+1}^{FM_i}$ could be selected as its maximum only if the tracking errors of that element are positive enough. But, from (G.6), (4.16) and (4.26) we know that

$$\varepsilon_{i,j}(k) < \beta_{i,j}(k) - w_{d_{i,j}} + \varepsilon_{i-1,j}(k) < 0, \ \forall i = 2, j = 1, \ldots, 2. \tag{G.13}$$

Thus the previous statement does not hold for this case and inequality $V_{k+1}^{FM_i} - V_k^{FM_i} < 0$ holds true. In consequence for Case 1 we have shown that given $V_k^{FM} > 0$ its increment satisfies $\Delta V_k^{FM} < 0$.

Case 2: All Machines On ($q_1(k) = B_{1,j}$ and $q_2(k) = B_{2,n}$) Here j and n are not necessarily equal, though they both belong to the set $\{1, 2\}$. From (4.15) we know that in order for this case to occur the following conditions must be satisfied for at least one of the production steps at each machine M_i

$$\varepsilon_{i,j}(k) > 0, \text{ and } \omega_{i,j}(k) \geq \beta_{i,j}(k). \tag{G.14}$$

Given the conditions (G.14), (4.19), (4.21), (4.22) and (4.23), we can rewrite $V_{k+1}^{FM_i}$ form (G.4) as

$$V_{k+1}^{FM_i} = \max \left\{ \begin{array}{c} -\varepsilon_{i,n}(k) - \Delta\varphi_n(k) + \alpha_1 - \mu_i, \\ \varepsilon_{i,j}(k) - 2\mu_i + \alpha_1 - 2\alpha_2 + \Delta\varphi_j(k) - f_i(k), \\ \varepsilon_{i,j}(k) + \varepsilon_{i,n}(k) + \sum_{j=1}^{2} \Delta\varphi_j(k) - 2\alpha_2 - \mu_i - f_i(k), \\ 0 \end{array} \right\}, \tag{G.15}$$

where $j \neq n$. Now, let us assume that the maximum of V_{k+1}^{FM} is reached in one of the elements of (G.15). Then from the definition of minimum it holds that

$$\Delta V_k^{FM} \leq V_{k+1}^{FM_i} - V_k^{FM_i}, \tag{G.16}$$

where $-V_k^{FM_i}$ is given by (G.9). Thus let us prove that $V_{k+1}^{FM_i} - V_k^{FM_i} < 0$. Assume that the maximum value of $V_{k+1}^{FM_i}$ is reached by its first element. Then inequality (G.16) takes the following form

$$\begin{aligned} \Delta V_k^{FM} &\leq -\varepsilon_{i,n}(k) - \Delta\varphi_n(k) + \alpha_1 - \mu_i + \varepsilon_{i,n}(k) + \mu_i - v_{dn} - \alpha_1, \\ \Delta V_k^{FM} &\leq -v_{dn} - \Delta\varphi_n(k) \overset{(4.19,4.22)}{<} 0. \end{aligned} \tag{G.17}$$

Consider now that the maximum value of $V_{k+1}^{FM_i}$ is reached by its second element. Then inequality (G.16) is reduced to

$$\begin{aligned}
\Delta V_k^{FM} &\leq \varepsilon_{i,j}(k) - 2\mu_i + \alpha_1 - 2\alpha_2 + \Delta\varphi_j(k) - f_i(k) \\
&\quad -\varepsilon_{i,j}(k) + \mu_i + v_{dj} - \alpha_1 + 2\alpha_2, \\
\Delta V_k^{FM} &\leq -\mu_i + v_{dj} + \Delta\varphi_j(k) - f_i(k) \overset{(4.23)}{<} 0. \qquad \text{(G.18)}
\end{aligned}$$

In case the maximum value of $V_{k+1}^{FM_i}$ is reached by its third element, the inequality (G.16) is reduced to

$$\begin{aligned}
\Delta V_k^{FM} &\leq \varepsilon_{i,j}(k) + \varepsilon_{i,n}(k) + \sum_{j=1}^{2} \Delta\varphi_j(k) - 2\alpha_2 - \mu_i - f_i(k) \\
&\quad -\varepsilon_{i,j}(k) - \varepsilon_{i,n}(k) + \sum_{j=1}^{2} v_{dj} + 2\alpha_2, \\
\Delta V_k^{FM} &\leq -\mu_i - f_i(k) + \sum_{j=1}^{2} (v_{dj} + \Delta\varphi_j(k)) \overset{(4.24)}{<} 0. \qquad \text{(G.19)}
\end{aligned}$$

Finally, if the maximum value of $V_{k+1}^{FM_i}$ is reached by its fourth element then inequality (G.16) is reduced to

$$\Delta V_k^{FM} \leq -V_k^{FM_i} < 0. \qquad \text{(G.20)}$$

Thus we have proven that the inequality $V_{k+1}^{FM_i} - V_k^{FM_i} < 0$ holds true. In consequence for Case 2 we have shown that given $V_k^{FM} > 0$ its increment satisfies $\Delta V_k^{FM} < 0$.

Case 3: Some Machines On ($q_1(k) = 0$ and $q_2(k) = B_{2,j}$)

For this case to occur (see (4.15)) the following conditions must be satisfied

$$\varepsilon_{1,n}(k) \leq 0, \ \forall n = 1, 2, \qquad \text{(G.21)}$$

$$\varepsilon_{2,j}(k) > 0 \text{ and } \omega_{2,j}(k) \geq \beta_{2,j}(k), \qquad \text{(G.22)}$$

where $j \in \{1, 2\}$. With out further analysis one can note that the conditions of this case were already analyzed in the previous Cases. Recall that in Case 1 for this value of $q_1(k)$ with condition (G.21), as well as in Case 2 for the present value of $q_2(k)$ with condition (G.22), we had proven that the increment of our Lyapunov function is decreasing. In consequence for Case 3 we can also conclude that given $V_k^{FM} > 0$ its increment satisfies $\Delta V_k^{FM} < 0$.

Case 4: Some Machines On ($q_1(k) = B_{1,j}$ and $q_2(k) = 0$) For this case to occur (see (4.15)), the following conditions must be satisfied

$$\varepsilon_{1,j}(k) > 0, \qquad \text{(G.23)}$$

$$\varepsilon_{2,n}(k) \leq 0, \text{ or } \omega_{2,n}(k) < \beta_2(k), \qquad \text{(G.24)}$$

where $j, n \in \{1, 2\}$. Recall that in Case 1 for the current value of $q_2(k)$ with the first inequality of (G.24), as well as in Case 2 for the current value of $q_1(k)$ with

the condition (G.23), we have proven that the increment of our Lyapunov function is decreasing.
As for the second part of the condition (G.24), which was also analyzed in Case 1, influence on the increment ΔV_k^{FM} may differ, i.e., for this case the second part of inequality (G.13) may not be satisfied. Thus let us emphasize our analysis of ΔV_k^{FM}, where the value of $V_{k+1}^{FM_2}$ is influenced by the second part of the condition (G.24) and the value of $V_{k+1}^{FM_1}$ is influenced by the condition (G.23). For that let us rewrite ΔV_k^{FM} of (G.3) as

$$\Delta V_k^{FM} = \max\left\{V_{k+1}^{FM_1}, V_{k+1}^{FM_2}\right\} + \min\left\{-V_k^{FM_1}, -V_k^{FM_2}\right\}, \tag{G.25}$$

where

$$V_{k+1}^{FM_1} = \max\left\{\begin{array}{c} -\varepsilon_{1,n}(k) - \Delta\varphi_n(k) + \alpha_1 - \mu_1, \\ \varepsilon_{1,j} - 2\mu_1 + \alpha_1 - 2\alpha_2 + \Delta\varphi_j(k) - f_1(k), \\ \varepsilon_{1,j}(k) + \varepsilon_{1,n}(k) + \sum_{s=1}^{2}\Delta\varphi_s(k) - 2\alpha_2 - \mu_1 - f_1(k), \\ 0 \end{array}\right\}. \tag{G.26}$$

and

$$V_{k+1}^{FM_2} = \max\left\{\begin{array}{c} -\varepsilon_{2,1}(k) - \Delta\varphi_1(k) + \alpha_1 - \mu_2, \\ -\varepsilon_{2,2}(k) - \Delta\varphi_2(k) + \alpha_1 - \mu_2, \\ \varepsilon_{2,1} - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi_1(k), \\ \varepsilon_{2,2} - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi_2(k), \\ \varepsilon_{2,1}(k) + \varepsilon_{2,2}(k) + \sum_{s=1}^{2}\Delta\varphi_s(k) - 2\alpha_2, \\ 0 \end{array}\right\}. \tag{G.27}$$

Here $j \neq n, \forall j, n \in \{1, 2\}$.

Let us assume that the maximum value of V_{k+1}^{FM} is reached by one of the element of $V_{k+1}^{FM_2}$. Then it follows that for ΔV_k^{FM} the following inequality is satisfied

$$\Delta V_k^{FM} \leq V_{k+1}^{FM_2} - V_k^{FM_2}. \tag{G.28}$$

From Case 1 we know that $\Delta V^{FM}(k) < 0$ if the maximum of $V_{k+1}^{FM_2}$ is reached by one of its first two elements or by the last one. Now, let us verify this assumption in case one from the rest of the elements of $V_{k+1}^{FM_2}$ is selected as its maximum. For that we can now reduce $V_{k+1}^{FM_2}$ from (G.27) to

$$V_{k+1}^{FM_2} = \max\left\{\begin{array}{c} \varepsilon_{2,1}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi_1(k), \\ \varepsilon_{2,2}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi_2(k), \\ \varepsilon_{2,1}(k) + \varepsilon_{2,2}(k) + \sum_{s=1}^{2}\Delta\varphi_s(k) - 2\alpha_2), \\ 0 \end{array}\right\}. \tag{G.29}$$

If $\varepsilon_{2,1}(k) \leq 0$ and $\varepsilon_{2,2}(k) \leq 0$ then the following inequalities hold

$$\varepsilon_{2,1}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi_1(k) \overset{(4.19,4.21,4.22,4.23)}{<} 0,$$
$$\varepsilon_{2,2}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi_2(k) \overset{(4.19,4.21,4.22,4.23)}{<} 0,$$
$$\varepsilon_{2,1}(k) + \varepsilon_{2,2}(k) + \sum_{s=1}^{2} \Delta\varphi_s(k) - 2\alpha_2 \overset{(4.19,4.21,4.22,4.23)}{<} 0,$$

which leads to

$$\Delta V_k^{FM} \leq V_{k+1}^{FM_2} - V_k^{FM_2} < 0. \tag{G.30}$$

Thus let us analyse the first, the second and the third element of (G.29) under the assumption that $\varepsilon_{2,1}(k) > 0$ and $\varepsilon_{2,2}(k) > 0$. For that we will involve $V_{k+1}^{FM_1}$ from (G.26) and analyze the relation between $\varepsilon_{1,1}(k)$ and $\varepsilon_{2,1}(k)$, as well as between $\varepsilon_{1,2}(k)$ and $\varepsilon_{2,2}(k)$. Note that the present condition on $w_{2,n}$ is given by (G.24). Thus we can rewrite the first part of inequality (G.13) as

$$\varepsilon_{2,1}(k) < \beta_{2,1}(k) - w_{d_{2,1}} + \varepsilon_{1,1}(k), \tag{G.31}$$
$$\varepsilon_{2,2}(k) < \beta_{2,2}(k) - w_{d_{2,2}} + \varepsilon_{1,2}(k), \tag{G.32}$$
$$\varepsilon_{2,1}(k) \overset{(4.26)}{<} \varepsilon_{1,1}(k) - 2\mu_1 - 2c_4 + c_1 - c_2, \tag{G.33}$$
$$\varepsilon_{2,2}(k) \overset{(4.26)}{<} \varepsilon_{1,2}(k) - 2\mu_1 - 2c_4 + c_1 - c_2, \tag{G.34}$$
$$\varepsilon_{1,1}(k) - 2\mu_1 - 2c_4 + c_1 - c_2 \overset{(4.19,4.20)}{<} \varepsilon_{1,1}(k) - 2\mu_1 + \alpha_1 - \Delta\varphi_1(k) - f_1(k), \tag{G.35}$$
$$\varepsilon_{1,2}(k) - 2\mu_1 - 2c_4 + c_1 - c_2 \overset{(4.19,4.20)}{<} \varepsilon_{1,2}(k) - 2\mu_1 + \alpha_1 - \Delta\varphi_2(k) - f_1(k). \tag{G.36}$$

From inequalities (G.33) and (G.34) it follows that the third and in consequence the second, and the first elements of (G.29) are of a less value then the first or the second element of (G.26) (depending on the value of $q_1(k)$, i.e., $q_1(k) = B_{1,1}$ or $q_1(k) = B_{1,2}$). Thus if (G.29) is selected as a maximal element of V_{k+1}^{FM} then ΔV_k^{FM} will result in (G.17) or in (G.18), respectively. This concludes our proof of Case 4, which is the last case.

Summarizing for 4 cases we have shown that given the Lyapunov function (G.1) that satisfies $V_k^{FM} > 0$, its increment given by (G.3) satisfies $\Delta V_k^{FM} < 0$. Thus $\limsup_{k\to\infty} V_k^{FM} = 0$, which completes our proof for a line of 2 machines with 2 stages.

In the above presented proof, we have analyzed the increment of the proposed Lyapunov function by means of 4 cases. Now for a line of P manufacturing machines each with N production stages defined by (4.18) the Lyapunov function (G.1) is extended to

$$V_k^{FM} = \max\left\{V_k^{FM_1}, ..., V_k^{FM_P}\right\}$$

where $V_k^{FM_i}$ is given by

$$V_k^{FM_i} = \max \left\{ \begin{array}{c} -\varepsilon_{i,1}(k) - \mu_i + v_{d1} + \alpha_1, \\ \vdots \\ -\varepsilon_{i,N}(k) - \mu_i + v_{dN} + \alpha_1, \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - v_{d1} - N\alpha_2, \\ \vdots \\ \varepsilon_{i,N}(k) - (N-1)(\mu_i - \alpha_1) - v_{dN} - N\alpha_2, \\ \Sigma_{j=1}^{2}(\varepsilon_{i,j}(k) - v_{dj}) - (N-2)(\mu_i - \alpha_1) - N\alpha_2, \\ \vdots \\ \Sigma_{j=N-1}^{N}(\varepsilon_{i,j}(k) - v_{dj}) - (N-2)(\mu_i - \alpha_1) - N\alpha_2, \\ \vdots \\ \Sigma_{j=1}^{N-1}(\varepsilon_{i,j}(k) - v_{dj}) - (\mu_i - \alpha_1) - N\alpha_2, \\ \Sigma_{j=2}^{N}(\varepsilon_{i,j}(k) - v_{dj}) - (\mu_i - \alpha_1) - N\alpha_2, \\ \Sigma_{j=1}^{N}(\varepsilon_{i,j}(k) - v_{dj}) - N\alpha_2, \\ 0 \end{array} \right\} . \quad (G.37)$$

Here similar reasoning is followed as for the proof of 2 machines.

"All Machines Off" Case as well as "All Machines On" Case are solved identically as in the proof for 2 machines. As for the cases of "Some Machines On" , the analysis is based on pure evaluation of the dependencies that are formed between the tracking errors of the network. As it was shown in Case 4, this dependencies are reflected through the two possible conditions $\omega_{i,j}(k) < \beta_{i,j}(k)$ and $\omega_{i,j}(k) \geq \beta_{i,j}(k)$, which are imposed on each intermediate buffer content. Thus in "Some Machines On" cases two situations may occur:

1. All the errors of the non-working machine are negative
   - This situation was already solved in "All Machines Off" case
2. All or some of the errors of the non-working machine are positive, which consequently mean that the buffer content of the non-working stage is less than the minimal required, i.e., $\omega_{i,j}(k) < \beta_{i,j}(k)$
   - First, use inequality $\omega_{i,j}(k) < \beta_i(k)$ to obtain the relation between the $\varepsilon_{i,j}(k)$ and its upstream production stage error $\varepsilon_{i-1,j}(k)$. If the upstream production stage error is positive then $\varepsilon_{i,j}(k)$ can not grow bigger than $\varepsilon_{i-1,j}(k)$. Being more precise, $\varepsilon_{i-1,j}(k)$ is separated from $\varepsilon_{i,j}(k)$ by $w_{di,j} - \beta_{i,j}(k)$ implying that the positive element, which contains $\varepsilon_{i,j}(k)$, of $V_{k+1}^{FM_i}$ is less than the positive element of $V_{k+1}^{FM_{i-1}}$, which contains $\varepsilon_{i-1,j}(k)$, of $V_{k+1}^{FM_r}$. Now, if the upstream production stage error is negative then $\varepsilon_{i,j}(k)$ must be negative as well.
   - Then, by relying on the obtained production tracking errors relations evaluate V_{k+1}^{FM} and deduce the possible candidates for its maximal value
   - Finally, use the definition of minimum and verify that the following inequality $\Delta V_k^{FM} \leq V_{k+1}^{FM_i} - V_k^{FM_i} < 0$ holds for all the candidates.

Due to the extensive technical details and similarity in the procedure we omit the explicit analysis for a line of N machines and restrict ourselves by the above presented logic that lies behind the solution.

H

Proof of Theorem 8

Let us prove that Theorem 8 holds for one re-entrant machine with one buffer ($j = 1, 2$) defined by (5.7). With this goal, let us introduce the following Lyapunov function

$$V^{B_2}(\varepsilon_1(k), \varepsilon_2(k)) = \max \left\{ \begin{array}{c} -\varepsilon_1(k) - \mu_1 + v_d + \alpha_1, \\ -\varepsilon_2(k) - \mu_2 + v_d + \alpha_1, \\ \frac{\varepsilon_1(k) - v_d - \alpha_2}{\mu_1 + c_3} + \frac{\varepsilon_2(k) - v_d - \alpha_2}{\mu_2 + c_3} \\ 0 \end{array} \right\}. \tag{H.1}$$

Here for the sake of brevity $V^{B_2}(\varepsilon_1(k), \varepsilon_2(k)) = V_k^{B_2}$, with $V^{B_2} \geq 0$ for all $\varepsilon_j(k) \in \mathbb{R}$ with $j = 1, 2$.

Thus, $\Delta V_k^{B_2}$ along the solutions of $\varepsilon_1(k)$ and $\varepsilon_2(k)$ is given by

$$\Delta V_k^{B_2} = \underbrace{\max \left\{ \begin{array}{c} -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1 + \beta_1(k) u_1(k), \\ -\varepsilon_2(k) - \Delta\varphi(k) + \alpha_1 - \mu_2 + \beta_2(k) u_2(k), \\ \frac{\varepsilon_1(k) + \Delta\varphi(k) - \alpha_2 - \beta_1(k) u_1(k)}{\mu_1 + c_3} + \frac{\varepsilon_2(k) + \Delta\varphi(k) - \alpha_2 - \beta_2(k) u_2(k)}{\mu_2 + c_3}, \\ 0 \end{array} \right\}}_{V_{k+1}^{B_2}}$$

$$+ \underbrace{\min \left\{ \begin{array}{c} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ \frac{-\varepsilon_1(k) + v_d + \alpha_2}{\mu_1 + c_3} + \frac{-\varepsilon_2(k) + v_d + \alpha_2}{\mu_2 + c_3}, \\ 0 \end{array} \right\}}_{-V_k^{B_2}}. \tag{H.2}$$

In order to perform a more detailed analysis on $\Delta V_k^{B_2}$, let us divide this proof into 3 cases.

Case 1 ($q(k) = 0$)

Suppose that $q(k) = 0$.

Then we can rewrite $\Delta V_k^{B_2}$ from (H.2) as

$$\Delta V_k^{B_2} = \underbrace{\max\left\{\begin{matrix} -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ -\varepsilon_2(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ \frac{\varepsilon_1(k)+\Delta\varphi(k)-\alpha_2}{\mu_1+c_3} + \frac{\varepsilon_2(k)+\Delta\varphi(k)-\alpha_2}{\mu_2+c_3}, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{B_2}} + \underbrace{\min\left\{\begin{matrix} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ \frac{-\varepsilon_1(k)+v_d+\alpha_2}{\mu_1+c_3} + \frac{-\varepsilon_2(k)+v_d+\alpha_2}{\mu_2+c_3}, \\ 0 \end{matrix}\right\}}_{-V_k^{B_2}}. \tag{H.3}$$

Subcase 1: Let us first assume that $\varepsilon_j(k)$ satisfies

$$\varepsilon_j(k) \leq 0, \tag{H.4}$$

for all k and $j = 1, 2$. Then by relying on (5.13) we can reduce $\Delta V_k^{B_2}$ from (H.3) to

$$\Delta V_k^{B_2} = \underbrace{\max\left\{\begin{matrix} -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ -\varepsilon_2(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{B_2}} + \underbrace{\min\left\{\begin{matrix} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ 0 \end{matrix}\right\}}_{-V_k^{B_2}}. \tag{H.5}$$

Here, let us assume that for $V_{k+1}^{B_2}$ the maximum is reached in the first element of the function, i.e. $V_{k+1}^{B_2} = -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1$. Then from the definition of min it holds that

$$\begin{aligned} \Delta V_k^{B_2} &\leq -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1 + \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \Delta V_k^{B_2} &\leq -v_d - \Delta\varphi(k) \overset{(5.8,5.11)}{<} 0. \end{aligned} \tag{H.6}$$

Similarly, assuming that the maximum of $V_{k+1}^{B_2}$ is reached in the second element of the function, it holds that from the definition of min

$$\Delta V_k^{B_2} \leq -v_d - \Delta\varphi(k) \overset{(5.8,5.11)}{<} 0. \tag{H.7}$$

Finally, when the maximum of $V_{k+1}^{B_2}$ is reached by the third element of the function it holds that

$$\Delta V_k^{B_2} = -V_k^{B_2}. \tag{H.8}$$

Thus, for this subcase, for $V_k^{B_2} > 0$ given by (H.1) its increment $\Delta V_k^{B_2} < 0$.

Subcase 2: Now let us assume that $\varepsilon_1(k)$, $\varepsilon_2(k)$ and $w_2(k)$ satisfy

$$\varepsilon_1(k) \leq 0, \tag{H.9}$$
$$\varepsilon_2(k) > 0, \tag{H.10}$$
$$w_2(k) < \beta_2(k), \tag{H.11}$$

for all k. From (5.16) we can rewrite condition (H.11) as

$$\varepsilon_1(k) \overset{(5.4,5.15)}{>} \varepsilon_2(k). \tag{H.12}$$

Thus, for the case $q(k) = 0$, subcase 2 is not possible.

Subcase 3: Assume that buffer B_2 has a limited capacity. Thus

$$w_2(k) > \gamma_2, \tag{H.13}$$

and given the state $q(k) = 0$ it follows that

$$\varepsilon_2(k) \leq 0, \tag{H.14}$$

for all k. Here $\gamma_2 = w_{d2} + \mu_2 + c_4$, which is a constant representing an upper bound on the product content of B_2.

We can rewrite condition (H.13) as $\varepsilon_2(k) - \varepsilon_1(k) + w_{d2} > \gamma_2$ from where

$$\varepsilon_1(k) < \varepsilon_2(k) - \mu_2 - c_4. \tag{H.15}$$

For (H.13) and (H.15) the analysis on $\Delta V_k^{B_2}$ was presented in Subcase 1. This concludes the analysis of Case 1.

Case 2 ($q(k) = B_1$)

Suppose the machine is working with buffer B_1 ($q(k) = B_1$), which is assumed to never starve. Thus, $\varepsilon_1(k)$ satisfies

$$\varepsilon_1(k) > 0 \tag{H.16}$$

for all k. Without loss of generality let us assume for now that $\varepsilon_2(k) \in \mathbb{R}$. Then we can rewrite $\Delta V_k^{B_2}$ from (H.3) as

$$\begin{aligned} \Delta V_k^{B_2} \;=\; & \underbrace{\max\left\{\begin{array}{c} -\varepsilon_1(k) - W_1(k) + \alpha_1, \\ -\varepsilon_2(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ \frac{\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1}{\mu_1+c_3} + \frac{\varepsilon_2(k)+\Delta\varphi(k)-\alpha_2}{\mu_2+c_3}, \\ 0 \end{array}\right\}}_{V_{k+1}^{B_2}} \\ & + \underbrace{\min\left\{\begin{array}{c} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ \frac{-\varepsilon_1(k)+v_d+\alpha_2}{\mu_1+c_3} + \frac{-\varepsilon_2(k)+v_d+\alpha_2}{\mu_2+c_3}, \\ 0 \end{array}\right\}}_{-V_k^{B_2}}. \end{aligned} \tag{H.17}$$

Here due to condition (H.16) the increment $\Delta V_k^{B_2}$ can be immediately reduced to

$$\Delta V_k^{B_2} = \underbrace{\max\left\{\begin{matrix} -\varepsilon_2(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ \frac{\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1}{\mu_1+c_3} + \frac{\varepsilon_2(k)+\Delta\varphi(k)-\alpha_2}{\mu_2+c_3}, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{B_2}} + \underbrace{\min\left\{\begin{matrix} \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ \frac{-\varepsilon_1(k)+v_d+\alpha_2}{\mu_1+c_3} + \frac{-\varepsilon_2(k)+v_d+\alpha_2}{\mu_2+c_3}, \\ 0 \end{matrix}\right\}}_{-V_k^{B_2}}. \tag{H.18}$$

Subcase 1: Let us first analyse (H.18) assuming that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) > 0 \tag{H.19}$$

for all k. Then by relying on (5.13) the increment (H.18) of $V_k^{B_2}$ can de reduced to

$$\Delta V_k^{B_2} = \underbrace{\max\left\{\begin{matrix} \frac{\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1}{\mu_1+c_3} + \frac{\varepsilon_2(k)+\Delta\varphi(k)-\alpha_2}{\mu_2+c_3}, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{B_2}} + \underbrace{\min\left\{\begin{matrix} \frac{-\varepsilon_1(k)+v_d+\alpha_2}{\mu_1+c_3} + \frac{-\varepsilon_2(k)+v_d+\alpha_2}{\mu_2+c_3}, \\ 0 \end{matrix}\right\}}_{-V_k^{B_2}}. \tag{H.20}$$

Consider now that the maximum of $V_{k+1}^{B_2}$ is reached in its first element, i.e. $V_{k+1}^{B_2} = \frac{\varepsilon_1(k)+W_1(k)-\alpha_2-\mu_1}{\mu_1+c_3} + \frac{\varepsilon_2(k)+\Delta\varphi(k)-\alpha_2}{\mu_2+c_3}$. Then from the definition of min it holds that

$$\begin{aligned} \Delta V_k^{B_2} &\le \frac{\varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1}{\mu_1 + c_3} + \frac{\varepsilon_2(k) + \Delta\varphi(k) - \alpha_2}{\mu_2 + c_3} \\ &+ \frac{-\varepsilon_1(k) + v_d + \alpha_2}{\mu_1 + c_3} + \frac{-\varepsilon_2(k) + v_d + \alpha_2}{\mu_2 + c_3}, \\ \Delta V_k^{B_2} &\le -\frac{\mu_1 + f_1(k)}{\mu_1 + c_3} + \frac{v_d + \Delta\varphi(k)}{\mu_1 + c_3} + \frac{v_d + \Delta\varphi(k)}{\mu_2 + c_3} \overset{(5.9,5.13)}{<} 0. \end{aligned}$$

In case that the maximum of $V_{k+1}^{B_2}$ given by (H.20) is reached in its second element, then from the definition of min it holds that

$$\Delta V_k^{B_2} = -V_k^{B_2}. \tag{H.21}$$

Thus, for this subcase, for $V_k^{B_2} > 0$ given by (H.1) its increment $\Delta V_k^{B_2} < 0$.

Subcase 2: Now let us analyse (H.18) assuming that $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) \le 0 \tag{H.22}$$

for all k. In analogy with the procedure followed in Subcase 1 it is obtained that

$$\begin{cases} \Delta V_k^{B_2} & \leq -v_d - \Delta\varphi(k) \overset{(5.8,5.11)}{<} 0 \\ & \text{if } V_{k+1}^{B_2} = -\varepsilon_2(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ \Delta V_k^{B_2} & \leq -\frac{\mu_1 + f_1(k)}{\mu_1 + c_3} + \frac{v_d + \Delta\varphi(k)}{\mu_1 + c_3} + \frac{v_d + \Delta\varphi(k)}{\mu_2 + c_3} \overset{(5.9,5.13)}{<} 0 \\ & \text{if } V_{k+1}^{B_2} = \frac{\varepsilon_1(k) + W_1(k) - \alpha_2 - \mu_1}{\mu_1 + c_3} + \frac{\varepsilon_2(k) + \Delta\varphi(k) - \alpha_2}{\mu_2 + c_3}, \\ \Delta V_k^{B_2} & = -V_k^{B_1} \\ & \text{if } V_{k+1}^{B_2} = 0. \end{cases} \tag{H.23}$$

Thus, for this subcase for $V_k^{B_2} > 0$ given by (H.1) its increment $\Delta V_k^{B_2} < 0$. This concludes the analysis of Case 2.

Case 3 $(q(k) = B_2)$

Suppose the machine is working with buffer B_2 $(q(k) = B_2)$. Thus, $\varepsilon_2(k)$ satisfies

$$\varepsilon_2(k) > 0 \tag{H.24}$$

for all k. Without loss of generality let us assume for now that value of $\varepsilon_1(k)$ can take any sign. Then we can rewrite $\Delta V_k^{B_2}$ from (H.3) as

$$\begin{aligned} \Delta V_k^{B_2} &= \underbrace{\max\left\{\begin{matrix} -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ -\varepsilon_2(k) - W_2(k) + \alpha_1, \\ \frac{\varepsilon_1(k) + \Delta\varphi(k) - \alpha_2}{\mu_1 + c_3} + \frac{\varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2}{\mu_2 + c_3}, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{B_2}} \\ &\quad + \underbrace{\min\left\{\begin{matrix} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ \frac{-\varepsilon_1(k) + v_d + \alpha_2}{\mu_1 + c_3} + \frac{-\varepsilon_2(k) + v_d + \alpha_2}{\mu_2 + c_3}, \\ 0 \end{matrix}\right\}}_{-V_k^{B_2}}. \end{aligned} \tag{H.25}$$

Here due to condition (H.24) the increment $\Delta V_k^{B_2}$ can be immediately reduced to

$$\begin{aligned} \Delta V_k^{B_2} &= \underbrace{\max\left\{\begin{matrix} -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ \frac{\varepsilon_1(k) + \Delta\varphi(k) - \alpha_2}{\mu_1 + c_3} + \frac{\varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2}{\mu_2 + c_3}, \\ 0 \end{matrix}\right\}}_{V_{k+1}^{B_2}} \\ &\quad + \underbrace{\min\left\{\begin{matrix} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \frac{-\varepsilon_1(k) + v_d + \alpha_2}{\mu_1 + c_3} + \frac{-\varepsilon_2(k) + v_d + \alpha_2}{\mu_2 + c_3}, \\ 0 \end{matrix}\right\}}_{-V_k^{B_2}}. \end{aligned} \tag{H.26}$$

By conducting the similar analysis as the one presented in case 2 we obtain that

$$\begin{cases} \Delta V_k^{B_2} & \leq -v_d - \Delta\varphi(k) \overset{(5.8,5.11)}{<} 0 \\ & \text{if } V_{k+1}^{B_2} = -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ \Delta V_k^{B_2} & \leq -\frac{\mu_2 + f_2(k)}{\mu_2 + c_3} + \frac{v_d + \Delta\varphi(k)}{\mu_1 + c_3} + \frac{v_d + \Delta\varphi(k)}{\mu_2 + c_3} \overset{(5.9,5.13)}{<} 0 \\ & \text{if } V_{k+1}^{B_2} = \frac{\varepsilon_1(k) + \Delta\varphi(k) - \alpha_2}{\mu_1 + c_3} + \frac{\varepsilon_2(k) + W_2(k) - \alpha_2 - \mu_2}{\mu_2 + c_3}, \\ \Delta V_k^{B_2} & = -V_k^{B_2} \\ & \text{if } V_{k+1}^{B_2} = 0, \end{cases} \tag{H.27}$$

which completes Case 3. In this proof we have analyzed the increment of the proposed Lyapunov function by means of 3 cases. Now for machine M serving N number of buffers ($j = 1, \ldots, N$) defined by (5.7) the function (H.1) is extended to

$$V_k^{B_N} = \max \left\{ \begin{array}{c} -\varepsilon_1 - \mu_1 + v_d + \alpha_1, \\ \vdots \\ -\varepsilon_N - \mu_N + v_d + \alpha_1, \\ \sum_{j=1}^{N} \frac{\varepsilon_j - v_d - \alpha_2}{\mu_j + c_3}, \\ 0 \end{array} \right\}.$$

Here the same reasoning is followed as for the proof for 2 machines. Summarizing the 3 cases, we have shown that for $V_k^{B_2} > 0$ given by (H.1) its increment $\Delta V_k^{B_2} < 0$. Thus $\limsup_{k\to\infty} V_k^{B_2} = 0$, which completes our proof.

I

Proof of Theorem 9

Due to the similarity of the procedure with the proof of Theorem 8 we omit the explicit analysis of this theorem. Instead we present all the necessary tools and the logic that lies behind the solution. The Lyapunov function that leads to its proof is given by

$$V_k^{B_N} = \max \left\{ \begin{array}{c} -\varepsilon_1 - \mu_1 + v_d + \alpha_1, \\ \vdots \\ -\varepsilon_N - \mu_N + v_d + \alpha_1, \\ \sum_{j=1}^{N} \frac{\varepsilon_j - v_d - \alpha_2 - X_j}{\mu_j + c_3}, \\ 0 \end{array} \right\}. \tag{I.1}$$

where $X_j = \sum_{s=2}^{j} (\max(\mu_s + c_4 - w_{ds}, 0))$ for all $j = 2, \ldots, N$ and $X_1 = 0$. The proof can be subdivided in to the same 3 cases as for Theorem 8. The procedure that was followed in all those cases, except for Subcase 2 of Case 1, are also valid for this proof. Contrary to the result of Subcase 2 of Case 1 of the proof of Theorem 8 the conditions (H.9), (H.10), and (H.11) can occur given that $w_{dj} < \beta_j(k)$. In this case the inequality (H.12) is substituted by

$$\varepsilon_j(k) \overset{(5.4)}{<} \sum_{s=2}^{j} (\beta_s(k) - w_{ds}) \overset{(5.9)}{<} X_j, \ \forall j = 2, \ldots, N. \tag{I.2}$$

Relying on the above mentioned inequality it can be easily deduced that for $V_k^{B_N} > 0$ given by (I.1) its increment $\Delta V_k^{B_N} < 0$. For the rest of the cases, the obtained results are similar to the once presented in the proof of Theorem 8.

J

Proof of Theorem 10

Let us prove that Theorem 10 holds for the re-entrant network with P machines $(i = 1, \ldots, P)$ each one with N production stages $(j = 1, \ldots, N)$ and $N \times P - 1$ intermediate buffers defined by (5.29).
Let us start by describing the general structure of our proof.

- First, we will show that Theorem 10 holds for a 2 machine re-entrant network. For that we will analyse the increment of the proposed Lyapunov function, which will be given later in the proof. The analysis is subdivided into 4 Cases. As it can be observed form (5.25), each machine in the network has 2 modes, which are the working mode and the stand by (idle) mode. Thus if we have 2 machines in the network this network may present 4 working modes, which correspond to the number of cases in our analysis.
- Second, we extend our proof to a complete $P \times N$ network. Given the proof for 2 machines now we can describe the resulting 2^P cases and explain the obtained results.

Let us begin our proof by showing that Theorem 10 holds for a smaller network, which consists of 2 machines M_i $(i = 1, 2)$ with 2 production stages each $(j = 1, 2)$ and 3 intermediate buffers $(B_{1,2}, B_{2,1}, B_{2,2})$. For that purpose, let us introduce the following Lyapunov function

$$V_k^M = \max\left\{V_k^{M_1}, V_k^{M_2}\right\} \tag{J.1}$$

where

$$V_k^{M_i} = \max\left\{\begin{array}{c} -\varepsilon_{i,1}(k) - \mu_i + v_d + \alpha_1, \\ -\varepsilon_{i,2}(k) - \mu_i + v_d + \alpha_1, \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \varepsilon_{i,2}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \varepsilon_{i,1}(k) + \varepsilon_{i,2}(k) - N(v_d + \alpha_2) \\ 0 \end{array}\right\}. \tag{J.2}$$

Here $N = 2$, $V_k^M = 0,\ \forall \varepsilon_{i,j}(k) \in [v_d + \alpha_1 - \mu_i, (N-1)(\mu_i - \alpha_1) + v_d + N\alpha_2]$. Thus, ΔV_k^M along the solutions of $\varepsilon_{i,j}(k)$ is given by

$$\Delta V_k^M = \max\left\{V_{k+1}^{M_1}, V_{k+1}^{M_2}\right\} - \max\left\{V_k^{M_1}, V_k^{M_2}\right\}, \tag{J.3}$$

where

$$V_{k+1}^{M_i} = \max\left\{\begin{array}{c} -\varepsilon_{i,1}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i + \beta_i(k)u_{i,1}(k), \\ -\varepsilon_{i,2}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i + \beta_i(k)u_{i,2}(k), \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - N\alpha_2 + \Delta\varphi(k) - \beta_i(k)u_{i,1}(k), \\ \varepsilon_{i,2}(k) - (N-1)(\mu_i - \alpha_1) - N\alpha_2 + \Delta\varphi(k) - \beta_i(k)u_{i,2}(k), \\ \varepsilon_{i,1}(k) + \varepsilon_{i,2}(k) + N(\Delta\varphi(k) - \alpha_2) - \beta_i(k)u_{i,1}(k) - \beta_i(k)u_{i,2}(k), \\ 0 \end{array}\right\}. \tag{J.4}$$

In order to perform a more detailed analysis on ΔV_k^M, let us divide this proof into 4 cases.

Case 1: All Machines Off ($q_1(k) = 0$ and $q_2(k) = 0$)

As we know from (5.25), in this case at least one of the following conditions must be satisfied

$$\varepsilon_{i,j}(k) \le 0,\ \forall i, j = 1, 2, \tag{J.5}$$

$$\varepsilon_{1,1}(k) \le 0 \text{ and } \omega_{i,j}(k) < \beta_i(k). \tag{J.6}$$

Note that we assume $B_{1,1}$ to always contain a sufficient amount of products.

First, given the states of $q_1(k)$ and $q_2(k)$ let us rewrite ΔV_k^M from (J.3) as

$$\Delta V_k^M = \max\left\{V_{k+1}^{M_1}, V_{k+1}^{M_2}\right\} + \min\left\{-V_k^{M_1}, -V_k^{M_2}\right\}, \tag{J.7}$$

where

$$V_{k+1}^{M_i} = \max\left\{\begin{array}{c} -\varepsilon_{i,1}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i, \\ -\varepsilon_{i,2}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i, \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - N\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{i,2}(k) - (N-1)(\mu_i - \alpha_1) - N\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{i,1}(k) + \varepsilon_{i,2}(k) + N(\Delta\varphi(k) - \alpha_2), \\ 0 \end{array}\right\} \tag{J.8}$$

and

$$-V_k^{M_i} = \min\left\{\begin{array}{c} \varepsilon_{i,1}(k) + \mu_i - v_d - \alpha_1, \\ \varepsilon_{i,2}(k) + \mu_i - v_d - \alpha_1, \\ -\varepsilon_{i,1}(k) + (N-1)(\mu_i - \alpha_1) + v_d + N\alpha_2, \\ -\varepsilon_{i,2}(k) + (N-1)(\mu_i - \alpha_1) + v_d + N\alpha_2, \\ -\varepsilon_{i,1}(k) - \varepsilon_{i,2}(k) + N(v_d + \alpha_2) \\ 0 \end{array}\right\}. \tag{J.9}$$

Second, let us assume that the maximum of V_{k+1}^{M} is reached in $V_{k+1}^{M_i}$. Thus from the definition of minimum it is possible to state that ΔV_k^M satisfies the following inequality

$$\Delta V_k^M \quad \leq \quad V_{k+1}^{M_i} - V_k^{M_i}. \tag{J.10}$$

Third, let us show that $V_{k+1}^{M_i} - V_k^{M_i} < 0$. For that purpose we assume that the maximum of $V_{k+1}^{M_i}$ is reached by one of its first 2 elements, i.e. $V_{k+1}^{M_i} = -\varepsilon_{i,j}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i$. Then from the definition of minimum it is possible to state that ΔV_k^M satisfies

$$\begin{aligned} \Delta V_k^M &\leq -\varepsilon_{i,j}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i + \varepsilon_{i,j}(k) + \mu_i - v_d - \alpha_1, \\ \Delta V_k^M &\leq -v_d - \Delta\varphi(k) \overset{(5.30,5.33)}{<} 0. \end{aligned} \tag{J.11}$$

Now suppose that the maximum of $V_{k+1}^{M_i}$ is reached by its sixth element, i.e. $V_{k+1}^{M_i} = 0$. Then from the definition of the minimum it is valid to state that ΔV_k^M satisfies

$$\Delta V_k^M \quad \leq \quad -V_k^{M_i} < 0. \tag{J.12}$$

One of the other elements of $V_{k+1}^{M_i}$ could be selected as its maximum only if the tracking errors of this element is positive enough. But, from (J.6), (5.27) and (5.37) we know that

$$\varepsilon_{i,j}(k) < \beta_{i,j}(k) - w_{d_{i,j}} + \varepsilon_{r,s}(k) < 0, \; \forall i,j = 1,\ldots,2, \tag{J.13}$$

where r and s are given by (5.28). Thus the previous statement does not hold for this case and inequality $V_{k+1}^{M_i} - V_k^{M_i} < 0$ holds true. Hence, for Case 1 we have shown that given $V_k^M > 0$ its increment satisfies $\Delta V_k^M < 0$.

Case 2: All Machines On ($q_1(k) = B_{1,j}$ and $q_2(k) = B_{2,n}$)

Here j and n are not necessarily equal, though they both belong to the set $\{1,2\}$. From (5.25) we know that in order for this case to occur, the following conditions must be satisfied for at least one of the production steps at each machine M_i

$$\varepsilon_{i,j}(k) > 0, \text{ and } \omega_{i,j}(k) \geq \beta_i(k). \tag{J.14}$$

Given the conditions (J.14), (5.30), (5.32), (5.33) and (5.34), we can rewrite $V_{k+1}^{M_i}$ form (J.4) as

$$V_{k+1}^{M_i} \quad = \quad \max \left\{ \begin{array}{c} -\varepsilon_{i,n}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i, \\ \varepsilon_{i,j}(k) - 2\mu_i + \alpha_1 - 2\alpha_2 + \Delta\varphi(k) - f_i(k), \\ \varepsilon_{i,j}(k) + \varepsilon_{i,n}(k) + 2(\Delta\varphi(k) - \alpha_2) - \mu_i - f_i(k), \\ 0 \end{array} \right\}, \tag{J.15}$$

where $j \neq n$. Now, let us assume that the maximum of V_{k+1}^{M} is reached in one of the elements of (J.15). Then from the definition of minimum it holds that

$$\Delta V_k^M \quad \leq \quad V_{k+1}^{M_i} - V_k^{M_i}, \tag{J.16}$$

where $-V_k^{M_i}$ is given by (J.9). Thus let us prove that $V_{k+1}^{M_i} - V_k^{M_i} < 0$. Assume that the maximum value of $V_{k+1}^{M_i}$ is reached by its first element. Then inequality (J.16) takes the following form

$$\begin{aligned} \Delta V_k^M &\leq -\varepsilon_{i,n}(k) - \Delta\varphi(k) + \alpha_1 - \mu_i + \varepsilon_{i,n}(k) + \mu_i - v_d - \alpha_1, \\ \Delta V_k^M &\leq -v_d - \Delta\varphi(k) \overset{(5.30,5.33)}{<} 0. \end{aligned} \tag{J.17}$$

Consider now that the maximum value of $V_{k+1}^{M_i}$ is reached by its second element. Then inequality (J.16) is reduced to

$$\begin{aligned} \Delta V_k^M &\leq \varepsilon_{i,j}(k) - 2\mu_i + \alpha_1 - 2\alpha_2 + \Delta\varphi(k) - f_i(k) \\ &\quad -\varepsilon_{i,j}(k) + \mu_i + v_d - \alpha_1 + 2\alpha_2, \\ \Delta V_k^M &\leq -\mu_i + v_d + \Delta\varphi(k) - f_i(k) \overset{(5.34)}{<} 0. \end{aligned} \tag{J.18}$$

In case the maximum value of $V_{k+1}^{M_i}$ is reached by its third element, the inequality (J.16) is reduced to

$$\begin{aligned} \Delta V_k^M &\leq \varepsilon_{i,j}(k) + \varepsilon_{i,n}(k) + 2(\Delta\varphi(k) - \alpha_2) - \mu_i - f_i(k) \\ &\quad -\varepsilon_{i,j}(k) - \varepsilon_{i,n}(k) + 2(v_d + \alpha_2), \\ \Delta V_k^M &\leq -\mu_i - f_i(k) + 2(v_d + \Delta\varphi(k)) \overset{(5.35)}{<} 0. \end{aligned} \tag{J.19}$$

Finally, if the maximum value of $V_{k+1}^{M_i}$ is reached by its fourth element, then inequality (J.16) reduces to

$$\Delta V_k^M \leq -V_k^{M_i} < 0. \tag{J.20}$$

Thus we have proven that the inequality $V_{k+1}^{M_i} - V_k^{M_i} < 0$ holds true. Hence, for Case 2 we have shown that given $V_k^M > 0$ its increment satisfies $\Delta V_k^M < 0$.

Case 3: Some Machines On ($q_1(k) = 0$ and $q_2(k) = B_{2,j}$)

For this case to occur (see (5.25)), the following conditions must be satisfied

$$\varepsilon_{1,n}(k) \leq 0, \text{ or } \varepsilon_{1,1}(k) \leq 0 \text{ and } \omega_{1,2}(k) < \beta_1(k), \tag{J.21}$$

$$\varepsilon_{2,j}(k) > 0 \text{ and } \omega_{2,j}(k) \geq \beta_2(k), \tag{J.22}$$

where $j, n \in \{1, 2\}$. Recall that in Case 1 for this value of $q_1(k)$ with the first inequality of (J.21), as well as in Case 2 for the present value of $q_2(k)$ with the condition (J.22), we had proven that the increment of our Lyapunov function is decreasing. In this case the second part of the condition (J.21), which was also analyzed in Case 1, may have a different influence on the increment ΔV_k^M, i.e. for this case the second part of inequality (J.13) may not always hold true. Thus let us analyze ΔV_k^M, where the value of $V_{k+1}^{M_1}$ is influenced by the second part of the condition (J.21) and the value of $V_{k+1}^{M_2}$ is influenced by the condition (J.22). For that let us rewrite ΔV_k^M of (J.3) as

$$\Delta V_k^M = \max\left\{V_{k+1}^{M_1}, V_{k+1}^{M_2}\right\} + \min\left\{-V_k^{M_1}, -V_k^{M_2}\right\}, \tag{J.23}$$

where

$$V_{k+1}^{M_1} = \max \left\{ \begin{matrix} -\varepsilon_{1,1}(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ -\varepsilon_{1,2}(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ \varepsilon_{1,1}(k) - \mu_1 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{1,2}(k) - \mu_1 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{1,1}(k) + \varepsilon_{1,2}(k) + 2(\Delta\varphi(k) - \alpha_2), \\ 0 \end{matrix} \right\} \tag{J.24}$$

and

$$V_{k+1}^{M_2} = \max \left\{ \begin{matrix} -\varepsilon_{2,n}(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ \varepsilon_{2,j}(k) - 2\mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k) - f_2(k), \\ \varepsilon_{2,j}(k) + \varepsilon_{2,n}(k) + 2(\Delta\varphi(k) - \alpha_2) - \mu_2 - f_2(k), \\ 0 \end{matrix} \right\}. \tag{J.25}$$

Here $j \neq n$, $\forall j, n \in \{1, 2\}$.

Let us assume that the maximum value of V_{k+1}^{M} is reached by one of the element of $V_{k+1}^{M_1}$. Then it follows that for ΔV_k^M the following inequality is satisfied

$$\Delta V_k^M \leq V_{k+1}^{M_1} - V_k^{M_1}. \tag{J.26}$$

From Case 1 we know that $\Delta V_k^M < 0$ if the maximum of $V_{k+1}^{M_1}$ is reached by one of its first two elements or by the last one. Now, let us verify this assumption in case one from the rest of the elements of $V_{k+1}^{M_1}$ is selected as its maximum value. For that we can now reduce $V_{k+1}^{M_1}$ from (J.24) to

$$V_{k+1}^{M_1} = \max \left\{ \begin{matrix} \varepsilon_{1,1}(k) - \mu_1 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{1,2}(k) - \mu_1 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{1,1}(k) + \varepsilon_{1,2}(k) + 2(\Delta\varphi(k) - \alpha_2), \\ 0 \end{matrix} \right\}. \tag{J.27}$$

Here the first element of (J.27) can be excluded due to the condition $\varepsilon_{1,1}(k) \leq 0$ (see (J.21)) and the inequality

$$\varepsilon_{1,1}(k) - \mu_1 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k) \overset{(5.30, 5.32, 5.33, 5.34)}{<} 0.$$

If $\varepsilon_{1,2}(k) \leq 0$ then the second and the third element of $V_{k+1}^{M_1}$ can be excluded as well. Thus we must focus on the analysis of the second and third elements of (J.27) under the assumption that $\varepsilon_{1,2}(k) > 0$. For that we will involve $V_{k+1}^{M_2}$ from (J.25) and analyze the following 2 scenarios.
In the first scenario we assume that in (J.25), $j = 1$ and $n = 2$. Thus let us analyze the relation between $\varepsilon_{1,2}(k)$ and $\varepsilon_{2,1}(k)$. From the second inequality of (J.21) we can rewrite the first part of (J.13) as

$$\varepsilon_{1,2}(k) < \beta_{1,2}(k) - w_{d_{1,2}} + \varepsilon_{2,1}(k), \tag{J.28}$$

$$\varepsilon_{1,2}(k) \overset{(5.37)}{<} \varepsilon_{2,1}(k) - 2\mu_2 - 2c_4 + c_1 - c_2, \tag{J.29}$$

$$\varepsilon_{2,1}(k) - 2\mu_2 - 2c_4 + c_1 - c_2 \overset{(5.30, 5.31)}{<} \varepsilon_{2,1}(k) - 2\mu_2 + \alpha_1 - \Delta\varphi(k) - f_2(k). \tag{J.30}$$

From inequality (J.30), it follows that the third and in consequence the second element of (J.27) are of the less value then the second element of (J.25), which if selected as the maximal value element of V_{k+1}^{M} will result in (J.18). Thus in the first scenario we have shown that in $V_{k+1}^{M_1}$ given by (J.27) its maximal value can not be reached in its second, nor in its third element.
In the second scenario we assume that in (J.25) $j = 2$ and $n = 1$. Thus let us now analyze the relation between $\varepsilon_{1,2}(k)$ and $\varepsilon_{2,1}(k)$. The present condition on $w_{2,1}(k)$ is given by (J.21). Thus the inequality (J.29) remains the same and the result of the first scenario applies here as well. This concludes the proof of Case 3. In this case we have shown that given $V_k^M > 0$ its increment satisfies $\Delta V_k^M < 0$.

Case 4: Some Machines On ($q_1(k) = B_{1,j}$ and $q_2(k) = 0$)

For this case to occur (see (5.25)), the following conditions must be satisfied

$$\varepsilon_{1,j}(k) > 0 \text{ and } \omega_{1,j}(k) \geq \beta_1(k), \tag{J.31}$$

$$\varepsilon_{2,n}(k) \leq 0, \text{ or } \omega_{2,n}(k) < \beta_2(k), \tag{J.32}$$

where $j, n \in \{1, 2\}$. Recall that in Case 1 for the current value of $q_2(k)$ with the first inequality of (J.32), as well as in Case 2 for the current value of $q_1(k)$ with the condition (J.31), we have proven that the increment of our Lyapunov function is decreasing.
As for the second part of the condition (J.32), which was also analyzed in Case 1, influence on the increment ΔV_k^M may differ, i.e. for this case the second part of inequality (J.13) may not be satisfied. Thus let us analyze of ΔV_k^M, where the value of $V_{k+1}^{M_2}$ is influenced by the second part of the condition (J.32) and the value of $V_{k+1}^{M_1}$ is influenced by the condition (J.31). For that let us rewrite ΔV_k^M of (J.3) as

$$\Delta V_k^M = \max\left\{V_{k+1}^{M_1}, V_{k+1}^{M_2}\right\} + \min\left\{-V_k^{M_1}, -V_k^{M_2}\right\}, \tag{J.33}$$

where

$$V_{k+1}^{M_1} = \max\left\{\begin{array}{c} -\varepsilon_{1,n}(k) - \Delta\varphi(k) + \alpha_1 - \mu_1, \\ \varepsilon_{1,j} - 2\mu_1 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k) - f_1(k), \\ \varepsilon_{1,j}(k) + \varepsilon_{1,n}(k) + 2(\Delta\varphi(k) - \alpha_2) - \mu_1 - f_1(k), \\ 0 \end{array}\right\}. \tag{J.34}$$

and

$$V_{k+1}^{M_2} = \max\left\{\begin{array}{c} -\varepsilon_{2,1}(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ -\varepsilon_{2,2}(k) - \Delta\varphi(k) + \alpha_1 - \mu_2, \\ \varepsilon_{2,1} - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{2,2} - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{2,1}(k) + \varepsilon_{2,2}(k) + 2(\Delta\varphi(k) - \alpha_2), \\ 0 \end{array}\right\}. \tag{J.35}$$

Here $j \neq n$, $\forall j, n \in \{1, 2\}$.

Let us assume that the maximum value of V_{k+1}^{M} is reached by one of the element of $V_{k+1}^{M_2}$. Then it follows that for ΔV_k^M the following inequality is satisfied

$$\Delta V_k^M \leq V_{k+1}^{M_2} - V_k^{M_2}. \tag{J.36}$$

From Case 1 we know that $\Delta V^M(k) < 0$ if the maximum of $V_{k+1}^{M_2}$ is reached by one of its first two elements or by the last one. Now, let us verify this assumption in case one of the other elements of $V_{k+1}^{M_2}$ is selected as its maximum. For that we can now reduce $V_{k+1}^{M_2}$ from (J.35) to

$$V_{k+1}^{M_2} = \max \begin{Bmatrix} \varepsilon_{2,1}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{2,2}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k), \\ \varepsilon_{2,1}(k) + \varepsilon_{2,2}(k) + 2(\Delta\varphi(k) - \alpha_2), \\ 0 \end{Bmatrix}. \tag{J.37}$$

If $\varepsilon_{2,1}(k) \leq 0$ and $\varepsilon_{2,2}(k) \leq 0$ then the following inequalities hold

$$\begin{aligned} \varepsilon_{2,1}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k) &\overset{(5.30,5.32,5.33,5.34)}{<} 0, \\ \varepsilon_{2,2}(k) - \mu_2 + \alpha_1 - 2\alpha_2 + \Delta\varphi(k) &\overset{(5.30,5.32,5.33,5.34)}{<} 0, \\ \varepsilon_{2,1}(k) + \varepsilon_{2,2}(k) + 2(\Delta\varphi(k) - \alpha_2) &\overset{(5.30,5.32,5.33,5.34)}{<} 0, \end{aligned}$$

which leads to

$$\Delta V_k^M \leq V_{k+1}^{M_2} - V_k^{M_2} < 0. \tag{J.38}$$

Thus let us analyse the first, the second and the third element of (J.37) under the assumption that $\varepsilon_{2,1}(k) > 0$ and $\varepsilon_{2,2}(k) > 0$. For that we will involve $V_{k+1}^{M_1}$ from (J.34) and analyze the relation between $\varepsilon_{1,1}(k)$ and $\varepsilon_{2,1}(k)$, as well as between $\varepsilon_{1,2}(k)$ and $\varepsilon_{2,2}(k)$. Note that the present condition on $w_{2,n}$ is given by (J.32). Thus we can rewrite the first part of inequality (J.13) as

$$\varepsilon_{2,1}(k) < \beta_{2,1}(k) - w_{d_{2,1}} + \varepsilon_{1,1}(k), \tag{J.39}$$
$$\varepsilon_{2,2}(k) < \beta_{2,2}(k) - w_{d_{2,2}} + \varepsilon_{1,2}(k), \tag{J.40}$$
$$\varepsilon_{2,1}(k) \overset{(5.37)}{<} \varepsilon_{1,1}(k) - 2\mu_1 - 2c_4 + c_1 - c_2, \tag{J.41}$$
$$\varepsilon_{2,2}(k) \overset{(5.37)}{<} \varepsilon_{1,2}(k) - 2\mu_1 - 2c_4 + c_1 - c_2, \tag{J.42}$$
$$\varepsilon_{1,1}(k) - 2\mu_1 - 2c_4 + c_1 - c_2 \overset{(5.30,5.31)}{<} \varepsilon_{1,1}(k) - 2\mu_1 + \alpha_1 - \Delta\varphi(k) - f_1(k), \tag{J.43}$$
$$\varepsilon_{1,2}(k) - 2\mu_1 - 2c_4 + c_1 - c_2 \overset{(5.30,5.31)}{<} \varepsilon_{1,2}(k) - 2\mu_1 + \alpha_1 - \Delta\varphi(k) - f_1(k). \tag{J.44}$$

From inequalities (J.41) and (J.42) it follows that the third and in consequence the second, and the first elements of (J.37) are of a less value then the first or the second element of (J.34) (depending on $q_1(k)$ value), which if selected as a maximal element of V_{k+1}^M will result in (J.17) or (J.18), respectively. This concludes our proof of Case 4, which is the last case.

Summarizing for 4 cases we have shown that given the Lyapunov function (J.1) that satisfies $V_k^M > 0$, its increment given by (J.3) satisfies $\Delta V_k^M < 0$. Thus $\limsup_{k\to\infty} V_k^M = 0$, which completes our proof.
In this proof, we have analyzed the increment of the proposed Lyapunov function by means of 4 cases. Now for a line of P manufacturing machines, each with N production stages defined by (5.29), the Lyapunov function (J.1) is extended to

$$V_k^M = \max\left\{V_k^{M_1}, ..., V_k^{M_P}\right\},$$

where

$$V_k^{M_i} = \max\left\{\begin{array}{c} -\varepsilon_{i,1}(k) - \mu_i + v_d + \alpha_1, \\ \vdots \\ -\varepsilon_{i,N}(k) - \mu_i + v_d + \alpha_1, \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \vdots \\ \varepsilon_{i,N}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \Sigma_{j=1}^{2}\varepsilon_{i,j}(k) - (N-2)(\mu_i - \alpha_1) - 2v_d - N\alpha_2, \\ \vdots \\ \Sigma_{j=N-1}^{N}\varepsilon_{i,j}(k) - (N-2)(\mu_i - \alpha_1) - 2v_d - N\alpha_2, \\ \vdots \\ \Sigma_{j=1}^{N-1}\varepsilon_{i,j}(k) - (\mu_i - \alpha_1) - (N-1)v_d - N\alpha_2, \\ \Sigma_{j=2}^{N}\varepsilon_{i,j}(k) - (\mu_i - \alpha_1) - (N-1)v_d - N\alpha_2, \\ \Sigma_{j=1}^{N}\varepsilon_{i,1}(k) - N(v_d + \alpha_2), \\ 0 \end{array}\right\} \quad (J.45)$$

for all $i = 1, ..., P$. Here similar reasoning is followed as in the proof of 2 machines.

The "All Machines Off" Case as well as the "All Machines On" Case are solved identically as in the proof for 2 machines. As for the cases of "Some Machines On", the analysis is based on pure evaluation of the dependencies that are formed between the tracking errors of the network. As it was shown in Case 3 and 4, these dependencies are reflected through the 2 possible conditions ($\omega_{i,j}(k) < \beta_i(k)$ and $\omega_{i,j}(k) \geq \beta_i(k)$), which are imposed on each intermediate buffer content. Thus in the "Some Machines On" cases two situations may occur:

1. All the errors of the non-working machine are negative.
   - This situation was already solved in "All Machines Off" case.
2. All or some of the errors of the non-working machines are positive, which consequently means that the buffer content of the non-working stages is less than the minimal required.
   - First, use inequality $\omega_{i,j}(k) < \beta_i(k)$ to obtain a relation between the $\varepsilon_{i,j}(k)$ and its upstream production stage error $\varepsilon_{r,s}(k)$. If the upstream production stage error is positive, then $\varepsilon_{i,j}(k)$ can not grow bigger than $\varepsilon_{r,s}(k)$. Being more precise, $\varepsilon_{r,s}(k)$ is separated from $\varepsilon_{i,j}(k)$ by $w_{di,j} - \beta_i(k)$ implying that the positive element, which contains $\varepsilon_{i,j}(k)$, of $V_{k+1}^{M_i}$

is less than the positive element, which contains $\varepsilon_{r,s}(k)$, of $V_{k+1}^{M_r}$. Now, if the upstream production stage error is negative, then $\varepsilon_{i,j}(k)$ must be negative as well.

- Then, by relying on the obtained production tracking errors relations evaluate V_{k+1}^{M} and deduce the possible candidates for its maximal value.
- Finally, use the definition of the minimum and verify that the following inequality $\Delta V_k^M \leq V_{k+1}^{M_i} - V_k^{M_i} < 0$ holds.

Due to the extensive technical details we omitted the explicit analysis for a line of N machines and restricted ourselves to only explaining the logic that lied behind the solution.

K
Models of manufacturing networks

The reader is invited to use this appendix as a reference on flow models of Sections 3.6 and 5.2.4 on simulation-based performance analysis. This appendix presents a detailed description of discrete time flow models for a manufacturing line and a re-entrant machine operated under three surplus-based production controllers. The production controllers also known as policies are Hedging Point, Base Stock and Conwip.

The appendix is subdivided in three sections. In each section introduces one of the above mentioned policies first fore a manufacturing line of 4 machines and 3 buffers and then for a single re-entrant machine of 3 sages and 2 buffers.

K.1 Hedging Point Policy

In this thesis we study the performance of several network topologies operated under a production demand oriented policy that is introduced under the general name of surplus-based control. In the literature their exist several variants of surplus-based controllers. Thus the controllers studied in this thesis are presented under the name of the specific policy called Hedging Point. The Hedging Point Policy (HPP) was first introduced by Design and Operation of Manufacturing System Group of MIT headed by Stanley B. Gershwin (see Kimemia and Gershwin (1983), Perkins (2004) and references therein). The aim of this policy is to provide an efficient production demand tracking while also limiting the network's inventory. The interpretation of the studied surplus-based controllers in terms of a Hedging Point Policy will be given in this section. In the following subsections the close-loop models for a manufacturing line and a re-entrant machine operated under HPP are presented.

K.1.1 Manufacturing line

Recall from Chapter 3 that in discrete time the flow model of the manufacturing line is defined as

$$y_1(k+1) = y_1(k) + \beta_1(k), \tag{K.1}$$
$$y_j(k+1) = y_j(k) + \beta_j(k)\,\mathrm{sign}_{Buff}(w_j(k) - \beta_j(k)), \tag{K.2}$$

where $\beta_j(k) = u_j(k) + f_j(k)$, with f_j as the external disturbance affecting machine M_j, u_j is the control input of machine M_j and $w_j(k) = y_{j-1}(k) - y_j(k)$ is the content of buffer B_j. Note that for further consistency with the flow model used in Section 3.6 in this section the machine and buffer index j is limited to $j = 1, \ldots, 4$. The HPP controller u_j is given by

$$u_j(k) = \mu_j \mathrm{sign}_+(\varepsilon_{j+1}(k) + w_{d_{j+1}} - w_{j+1}(k)), \quad \forall j = 1, \ldots, 3, \tag{K.3}$$
$$u_4(k) = \mu_4 \mathrm{sign}_+(y_d(k) - y_4(k)), \tag{K.4}$$

where the tracking error of each machine is given by

$$\varepsilon_j(k) = \varepsilon_{j+1}(k) + (w_{d_{j+1}} - w_{j+1}(k)), \quad \forall j = 1, \ldots, 3, \tag{K.5}$$
$$\varepsilon_4(k) = y_d(k) - y_4(k). \tag{K.6}$$

Here $w_{d_{j+1}}$ represents a constant buffer level that is desired to maintain in each B_{j+1} buffer of the line.

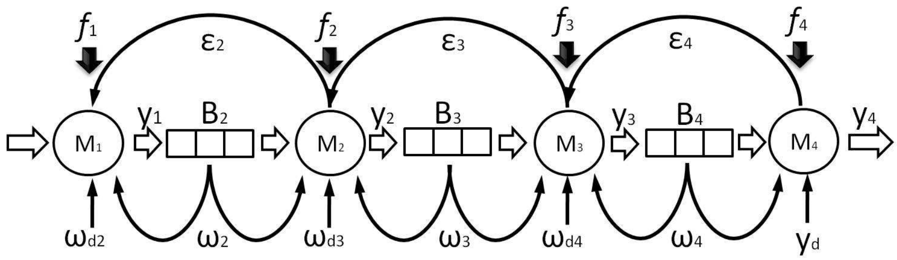


Figure K.1: Control diagram of a line of 4 machines under HPP.

Figure K.1 presents the control diagram of a manufacturing line composed of 4 machines M and 3 unbounded buffers B operated under HPP. The circles represent the manufacturing machines each with inside label M_j and outside short thick black arrows denote the external perturbations f_i, which are affecting its production rates. Each machine (except for M_1) have a buffer connected to it, each one denoted by 3 joined squares. The product flow directions are denoted by a thick white arrows with a black frame. The transferring of information on the production tracking error ε_j is shown by arched black arrows, going from one machine to another in upstream manner. For each machine the upstream and downstream inventory level w_j information transfer is depicted by a curved black arrow coming

from each buffer and the desired downstream buffer inventory level w_{dj} is shown by a short thin black arrow pointing to each machine.
Note that in this HPP implementation the control actions are decentralized throughout the network. In other words the control action of each machine in the line depends only on the production error of its neighboring downstream machine (except for machine M_4, which control action depends directly on cumulative demand input) and the current buffer content of its upstream and downstream buffer (see Fig. K.1).

From the definition of $w_j(k)$ and equations(K.5) and (K.6) the production error of each machine can be also presented in the following form

$$\begin{aligned} \varepsilon_j(k) &= y_d(k) - y_j(k) + w_{d_{j+1}} + \ldots + w_{d_N}, \qquad (K.7) \\ & \forall j = 1, \ldots, 3, \\ \varepsilon_4(k) &= y_d(k) - y_4(k), \qquad (K.8) \end{aligned}$$

which is commonly found in the existent literature on HPP (see e.g. Gershwin (2000)). In this interpretation each machine in the line is keeping track of the current demand as well as of its hedging point (also known as the base stock level), which in this case is given by the sum of the desired buffer contents $w_{d_{j+1}}$ of all the downstream buffers in front of M_j.

K.1.2 Re-entrant machine

In Chapter 5 the discrete time flow model of each production stage of one re-entrant machine was defined as

$$y_j(k+1) = y_j(k) + \beta_j(k)u_j(k), \qquad (K.9)$$

where $y_j(k) \in \mathbb{R}$ is the cumulative output of the machine in processing stage j in time k, $u_j(k) \in \mathbb{R}$ is the control input of the machine in processing stage j and $\beta_j(k) = \mu_j + f_j(k)$ where μ_j is a positive constant that represents the processing speed of the machine at the stage j and $f_j(k) \in \mathbb{R}$ is an unknown external disturbance affecting the performance of the machine at stage j. Note that for further consistency with the model used in Section 5.2.4 in this section the machine and buffer index j is limited to $j = 1, \ldots, 3$.

Figure K.2 shows a control diagram for one re-entrant machine M (dashed line rectangle) with 3 production stages. The output of each stage is indicated by variables y_1, y_2 and y_3, respectively. The machine is interconnected with 2 buffers B_2, B_3. Each buffer stores the intermediate product that is produced by the upstream stage of the machine. It is assumed that the machine has always sufficient raw material on its input, which is the reason why the buffer of stage 1 (B_1) is omitted in this figure.

The HPP controller u_j is given by the following algorithm

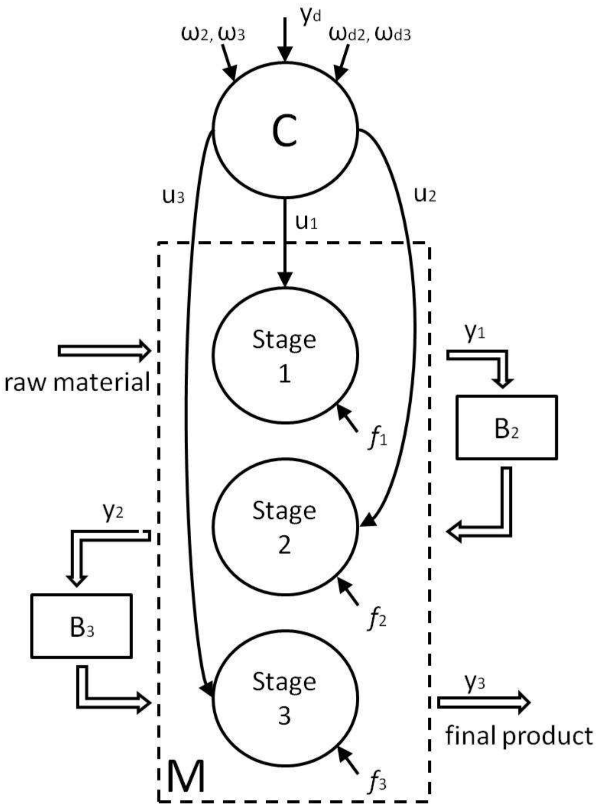


Figure K.2: Control diagram of a re-entrant machine of 3 stages operated under HPP.

$$
\begin{aligned}
&\{q(k) = B_j\},\\
&\textbf{if}\, \varepsilon_j(k) > 0\, \textbf{and}\, w_j(k) \geq \beta_j(k)\, \textbf{then}\\
&u_j(k) = 1,\\
&u_s(k) = 0,\ \forall s \neq j,\ s, j = 1, \ldots, 3,\\
&q(k+1) = B_j,\\
&\textbf{end}\\
&\textbf{if}\ (\varepsilon_j(k) \leq 0\, \textbf{or}\, w_j(k) < \beta_j(k))\, \textbf{and}\\
&\exists\, s \neq j : (\varepsilon_s(k) > 0\, \textbf{and}\, w_s(k) \geq \beta_s(k))\ \textbf{then}\\
&u_j(k) = 0,\\
&u_s(k) = 1,\\
&q(k+1) = B_s,\\
&\textbf{end}\\
&\textbf{if}\ (\varepsilon_s(k) \leq 0\, \textbf{or}\, w_s(k) < \beta_s(k))\,,\ \forall s\, \textbf{then}\\
&u_j(k) = 0,\\
&u_s(k) = 0,\ \forall s \neq j,\ s, j = 1, \ldots, N,\\
&q(k+1) = 0,\\
&\textbf{end}
\end{aligned}
\qquad (K.10)
$$

Summarizing for (K.10), the machine can only work on one buffer at a time. The control input $u_j(k)$ of each production stage j can only take the value of 0 (stop) or 1 (produce). The $u_j(k)$ receives the value of 1 only if j production stage needs to produce ($\varepsilon_j(k) > 0$) and its buffer is not empty ($w_j(k) \geq \beta_j(k)$). The machine will remain at its current state ($q(k) = B_j$) while all the conditions of the state are satisfied. The value of 0 is given to the control input of stage j if at least one of the conditions of the current state $q(k) = B_j$ is unsatisfied. The change in the value of the control signal of a stage j also implies a change in machine's state $q(k)$. The machine has 4 states. This is due to that 3 is a total number of processing stages that M can be working in, which directly relate to the states of the machine, plus the idle state ($q(k) = 0$). The cumulative production error at each stage of M and the buffer content are defined in a similar way to the above provided line model as

$$\varepsilon_j(k) = \varepsilon_{j+1}(k) + (w_{d_{j+1}} - w_{j+1}(k)), \quad \forall j = 1, \ldots, 2, \tag{K.11}$$

$$\varepsilon_3(k) = y_d(k) - y_3(k) \tag{K.12}$$

and $w_{j+1}(k) = y_j(k) - y_{j+1}(k)$, respectively.

K.2 Base Stock Policy

Base Stock Policy (BSP) is a commonly utilized scheduling policy usually applied in warehouses for inventory control purposes. Frequently in the literature (see Silver et al. (1998), Bonvik et al. (1997), Duri et al. (2000), Karaesmen and Dallery (2000), González et al. (2012) and references therein) one can find BS as a policy under which all the machines in the network keep track of the product demand while at the same time maintaining a base stock (also called safety stock) level of the immediate downstream inventory. In our comparative study this BS notion is slightly modified. In case of a line only the last machine (M_4) is tracking the product demand while the rest keep the upstream inventory at its base stock level. In case of a re-entrant machine only the last stage ($j = 3$) is tracking the product demand while the rest keep the upstream inventory at its base stock level.

K.2.1 Manufacturing line

In this section BSP is introduced for a line of 4 manufacturing machines and 3 buffers previously described by the flow model (K.1), (K.2). For BSP the control input u_j is given by

$$u_j(k) = \mu_j \text{sign}_+(\varepsilon_j(k)), \quad \forall j = 1, \ldots, 3, \tag{K.13}$$

$$u_4(k) = \mu_4 \text{sign}_+(\varepsilon_4(k)). \tag{K.14}$$

and tracking error of each machine is defined as

$$\varepsilon_j(k) = w_{d_{j+1}} - w_{j+1}(k), \quad \forall j = 1, \ldots, 3, \tag{K.15}$$

$$\varepsilon_4(k) = y_d(k) - y_4(k). \tag{K.16}$$

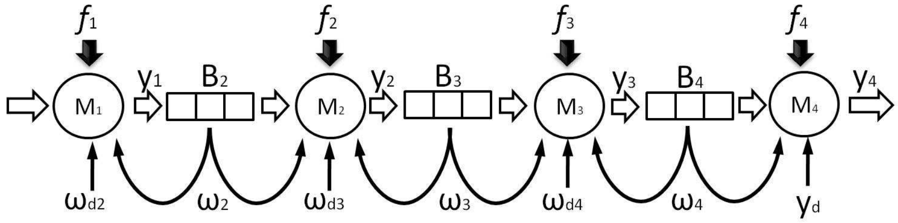


Figure K.3: Control diagram of a line of 4 machines under BSP.

Figure K.3 presents the control diagram of a manufacturing line composed of 4 machines M and 3 unbounded buffers B operated under BSP. The circles represent the manufacturing machines each with inside label M_j and outside short thick black arrows denote the external perturbations (f_i), which are affecting its production rates. Each machine (except for M_1) has a buffer connected to it, each one denoted by 3 joined squares. The product flow directions are denoted by a thick white arrows with a black frame. For each machine the upstream and downstream inventory level w_j information transfer is depicted by a curved black arrow coming from each buffer and the desired downstream buffer inventory level w_{dj} is shown by a short thin black arrow pointing to each machine.

Note that the control actions are decentralized throughout the network. The control action of each machine, except for M_4, depends on a product availability of its upstream buffer and the base stock level of its downstream buffer (see Fig. K.3). In case of M_4 its control action also depends on a product availability, but unlike the rest of the machines the production process of M_4 depends on the value of the cumulative product demand y_d (see K.18).

K.2.2 Re-entrant machine

In this section BSP is introduced for a re-entrant machine of 3 stages and 2 buffers previously described by (K.9). The structure of BS controller is also defined by (K.10). The difference of BSP with respect to HPP lies in the interpretation of the production errors ε_j.

For BSP the tracking error for each stage j in the machine is defined as

$$\begin{aligned} \varepsilon_j(k) &= w_{d_{j+1}} - w_{j+1}(k), \quad &\text{(K.17)} \\ &\quad \forall j = 1, \ldots, 2, \\ \varepsilon_3(k) &= y_d(k) - y_3(k), \quad &\text{(K.18)} \end{aligned}$$

where $w_{j+1}(k) = y_j(k) - y_{j+1}(k)$. Summarizing for the control action of BSP, the last stage (Stage 3 of Figure K.2) of the re-entrant machine is keeping track of the demand while the rest of the stages maintain the basestock level of their immediate downstream inventory.

K.3 Conwip Policy

From (Framinan et al. (2003)) Constant Work In Process (Conwip) production refer to any system maintaining constant the maximum amount of Work In Process (WIP). Originally Conwip was introduced by (Spearman et al. (1990)) as a an alternative pull type mechanism to the Kanban system. Although the control mechanism under the name Conwip was first presented in Spearman (1988), its basics principal was already described in (Jackson (1963)). Other identical or similar systems have been proposed by different authors, such as the Workload control system by (Bertrand (1983)), the C-WIP system by Glassey and Resende (1988), the Long pull system by Lambrecht and Segaert (1990), the Globally flexible line by So (1990), and the single stage Kanban system, studied, among others, by Spearman (1992), Mascolo et al. (1996), Karaesmen and Dallery (2000), and Tardif and Maaseidvaag (2001). Although these systems have some differences among them, there common similarity is in maintaining a constant WIP.

Note that in this thesis Conwip is addressed as CWIP policy.

K.3.1 Manufacturing line

In this section CWIP policy is introduced for a line of 4 manufacturing machines and 3 buffers described by (K.1) and (K.2). For CWIP the control input u_j is given by

$$u_1(k) = \mu_1 \text{sign}_+(\varepsilon_1(k)), \tag{K.19}$$
$$u_j(k) = \mu_j \ \forall j = 2, 3, \tag{K.20}$$
$$u_4(k) = \mu_4 \text{sign}_+(\varepsilon_4(k)). \tag{K.21}$$

The tracking errors of machines M_1 and M_4 are defined as

$$\varepsilon_1(k) = w_{d_{total}} - w_{total}(k), \tag{K.22}$$
$$\varepsilon_4(k) = y_d(k) - y_4(k). \tag{K.23}$$

The constant $w_{d_{total}}$ represents the total desired inventory level in the network. In Section 3.6 for the fair comparison with HPP and BSP this constant is defined as $w_{d_{total}} = w_{d_2} + w_{d_3} + w_{d_4}$. The total inventory level in the system in time k is given by $w_{total}(k) = y_1(k) - y_4(k)$.

Figure K.4 presents the control diagram of a manufacturing line composed of 4 machines M and 3 unbounded buffers B operated under CWIP. The circles represent the manufacturing machines each with inside label M_j and outside short thick black arrows denote the external perturbations (f_i), which are affecting its production rates. Each machine (except for M_1) have a buffer connected to it, each one denoted by 3 joined squares. The product flow directions are given by a thick white arrows with a black frame. For each machine the downstream inventory level w_j and w_{total} information transfer is depicted by a curved black arrow and the desired total inventory level w_{dtotal} is shown by a short thin black arrow pointing to M_1. The machine M_1 controls the product arrival into the whole network (WIP level). The intermediate machines M_2 and M_3 produce based on the product availability

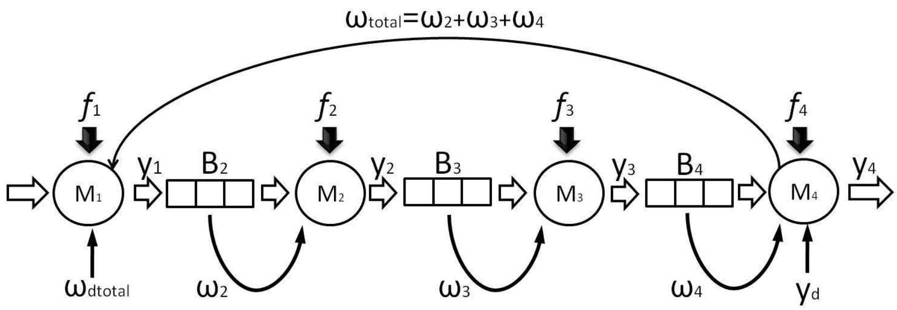


Figure K.4: Control diagram of a line of 4 machines under CWIP.

in their upstream buffer and M_4 tracks the cumulative production demand $y_d(k)$ on its output.

K.3.2 Re-entrant machine

In this section CWIP policy is introduced for a re-entrant machine of 3 stages and 2 buffers. The network is depicted in Figure K.2 and its flow model is given by (K.9). The structure of CWIP controller is defined in a similar way as in (K.10). The machine can only work on one buffer at a time. The control input $u_j(k)$ of each production stage j can only take the value of 0 (stop) or 1 (produce). The $u_j(k)$ with $j = 1, 3$ receives the value of 1 only if Stage j needs to produce and its buffer is not empty, i.e.$\varepsilon_j(k) > 0$ and $w_j(k) \geq \beta_j(k)$. The $u_2(k)$ acts as a Push controller, it receives the value of 1 only if the buffer B_2 is not empty ($w_2(k) \geq \beta_2(k)$). In order to keep the same structure of the control algorithm as the one given by (K.10) it can be assumed that $\varepsilon_2(k) = c$ for all k, where c can be given by any positive constant. Thus $u_2(k)$ receives the value of 1 only if $\varepsilon_2(k) > 0$ and $w_2(k) \geq \beta_j(k)$. The machine will remain at its current state ($q(k) = B_j$ with $j = 1, 2, 3$) while all the conditions of this state are satisfied. The value of 0 is given to the control input of stage j if at least one of the conditions of the current state $q(k) = B_j$ is unsatisfied. The change in the value of the control signal of a stage j also implies a change in machine's state $q(k)$. The machine has 4 states. This is due to that 3 is a total number of processing stages that M can be working in, which directly relate to the states of the machine, plus the idle state ($q(k) = 0$). Thus with CWIP policy the fist production stage maintains a constant inventory in the network while the last stage keeps the track of a current production demand and the intermediate stage simply push the products along the network. It follows that the production errors of stages 1, 2 and 3 can be defined as

$$\varepsilon_1(k) = w_{d_{total}} - w_{total}(k), \tag{K.24}$$

$$\varepsilon_2(k) = c, \tag{K.25}$$

$$\varepsilon_3(k) = y_d(k) - y_3(k). \tag{K.26}$$

The constant $w_{d_{total}}$ represents the total desired inventory level in the network. In Section 5.2.4 for the fair comparison of CWIP with HPP and BSP this constant is defined as $w_{d_{total}} = w_{d_2} + w_{d_3}$. The total inventory level in the system in time k is given by $w_{total}(k) = y_1(k) - y_3(k)$.

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Curriculum Vitae

Konstantin Konstantinovich Starkov was born in Moscow, Russia, in 1983. He graduated as engineer in electronics at Autonomous University of Baja California of Tijuana, Mexico in 2006. In 2008 he finished his master studies on Automatic Control of Underactuated Systems from National Polytechnical Institute in Tijuana and decided to continue his education and research in Eindhoven, The Netherlands. Since October of 2008 he was actively involved in FP7 research project on Control for Coordination of Distributed Systems. His doctoral studies and research interests at Eindhoven University of Technology concerned the topic of control and performance analysis of manufacturing networks. In his research Konstantin used control theory methods to obtain novel results for important industrial problems. In March of 2012 he organized and participated in engineering project on design and assembly of the experimental tool (servo system) for research and education, which now serves as a production process emulator in Mechanical Engineering Department of TU/e. He is graduating in November of 2012.

$$V_{1k} = \max\left(\left|\varepsilon_1(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_1}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_1}{2}, 0\right),$$

$$V_{jk} = \sum_{j=2}^{N} \frac{1}{n_j} \max\left(\left|\varepsilon_j(k) - v_d - \frac{(\alpha_2+\alpha_1)}{2} + \frac{\mu_j}{2}\right| - \frac{(\alpha_2-\alpha_1)}{2} - \frac{\mu_j}{2}, 0\right),$$

$$V_k^{B_N} = \max\left\{\begin{array}{c} -\varepsilon_1 - \mu_1 + v_d + \alpha_1, \\ \vdots \\ -\varepsilon_N - \mu_N + v_d + \alpha_1, \\ \sum_{j=1}^{N} \frac{\varepsilon_j - v_d - \alpha_2 - X_j}{\mu_j + c_3}, \\ 0 \end{array}\right\}$$

$$\Delta V_{j,k} = \underbrace{\max\left\{\begin{array}{c} -\varepsilon_j(k) - W_j(k) + \alpha_1, \\ \varepsilon_j(k) + W_j(k) - \alpha_2 - \mu_j, \\ 0 \end{array}\right\}}_{V_{j,k+1}} + \underbrace{\min\left\{\begin{array}{c} \varepsilon_j(k) + \mu_j - v_d - \alpha_1, \\ -\varepsilon_j(k) + v_d + \alpha_2, \\ 0 \end{array}\right\}}_{-V_{j,k}}.$$

$$\begin{aligned} \Delta V_k &= \max\left(\left|\varepsilon(k) + W(k) - \frac{(\alpha_2+\alpha_1)}{2} + \frac{1}{2}\right|\right. \\ &\quad - \left.\frac{(\alpha_2-\alpha_1)}{2} - \frac{1}{2}, 0\right) - V_k. \end{aligned}$$

$$V_j(\varepsilon_j) =$$

$$\max\left\{\begin{array}{c} -\varepsilon_j - \mu_j + v_d + \alpha_1, \\ \varepsilon_j - v_d - \alpha_2 - \underbrace{\sum_{i=2}^{j} \max\left(\mu_{i-1} - \alpha_1 + \alpha_2 - w_{di} + \underbrace{\mu_i + \alpha_3}_{\beta_i}, 0\right)}_{x_j}, \\ 0 \end{array}\right\}$$

$$V_k^{M_i} = \max\left\{\begin{array}{c} -\varepsilon_{i,1}(k) - \mu_i + v_d + \alpha_1, \\ \vdots \\ -\varepsilon_{i,N}(k) - \mu_i + v_d + \alpha_1, \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \vdots \\ \varepsilon_{i,N}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \Sigma_{j=1}^{2}\varepsilon_{i,j}(k) - (N-2)(\mu_i - \alpha_1) - 2v_d - N\alpha_2, \\ \vdots \\ \Sigma_{j=N-1}^{N}\varepsilon_{i,j}(k) - (N-2)(\mu_i - \alpha_1) - 2v_d - N\alpha_2, \\ \vdots \\ \Sigma_{j=1}^{N-1}\varepsilon_{i,j}(k) - (\mu_i - \alpha_1) - (N-1)v_d - N\alpha_2, \\ \Sigma_{j=2}^{N}\varepsilon_{i,j}(k) - (\mu_i - \alpha_1) - (N-1)v_d - N\alpha_2, \\ \Sigma_{j=1}^{N}\varepsilon_{i,1}(k) - N(v_d + \alpha_2), \\ 0 \end{array}\right\}$$

$$\Delta V_k^{B_2} =$$

$$\underbrace{\max\left\{\begin{array}{c} -\varepsilon_1(k) - \Delta\varphi(k) + \alpha_1 - \mu_1 + \beta_1(k)u_1(k), \\ -\varepsilon_2(k) - \Delta\varphi(k) + \alpha_1 - \mu_2 + \beta_2(k)u_2(k), \\ \frac{\varepsilon_1(k) + \Delta\varphi(k) - \alpha_2 - \beta_1(k)u_1(k)}{\mu_1 + c_3} + \frac{\varepsilon_2(k) + \Delta\varphi(k) - \alpha_2 - \beta_2(k)u_2(k)}{\mu_2 + c_3}, \\ 0 \end{array}\right\}}_{V_{k+1}^{B_2}}$$

$$\underbrace{+\min\left\{\begin{array}{c} \varepsilon_1(k) + \mu_1 - v_d - \alpha_1, \\ \varepsilon_2(k) + \mu_2 - v_d - \alpha_1, \\ \frac{-\varepsilon_1(k) + v_d + \alpha_2}{\mu_1 + c_3} + \frac{-\varepsilon_2(k) + v_d + \alpha_2}{\mu_2 + c_3}, \\ 0 \end{array}\right\}}_{-V_k^{B_2}}.$$

$$V_k^{M_i} = \max\left\{\begin{array}{c} -\varepsilon_{i,1}(k) - \mu_i + v_d + \alpha_1, \\ -\varepsilon_{i,2}(k) - \mu_i + v_d + \alpha_1, \\ \varepsilon_{i,1}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \varepsilon_{i,2}(k) - (N-1)(\mu_i - \alpha_1) - v_d - N\alpha_2, \\ \varepsilon_{i,1}(k) + \varepsilon_{i,2}(k) - N(v_d + \alpha_2) \\ 0 \end{array}\right\}$$