A-Posteriori Symbol Probabilities and Log-Likelihood Ratios for Coherently Detected $\frac{\pi}{4}$-DE-QPSK

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Abstract—In this letter, coherent detection of $\frac{\pi}{4}$-DE-QPSK is considered, but our analysis also holds for common DE-QPSK. It is shown that maximum a-posteriori (MAP) sequence detection can be regarded as an approximation, based on selecting dominant exponentials, of MAP symbol detection. A better approximation, relying on piecewise-linear fitting of the logarithm of the hyperbolic cosine, is proposed. This approximation results in a performance very close to optimal symbol detection. For the case where the symbols are produced by convolutional encoding and Gray mapping, the log-likelihood ratios are investigated. Again a simple approximation based on selecting dominant exponentials and an improved approximation relying on piecewise-linear fits, are discussed. As in the uncoded case the improved approximation gives a performance quite close to ideal. While the particular examples considered show modest gains in performance, this letter provides a way of improving performance when needed.

Index Terms—Differentially encoded phase-shift keying, coherent detection, MAP detection, log-likelihood ratios.

I. Introduction

IN this letter, we discuss *a-posteriori* (AP) symbol-probabilities and optimal symbol-detection, but also log-likelihood ratios (LLRs) for $\frac{\pi}{4}$-DE-QPSK systems when coherent detection is performed. Our analysis carries over to common DE-QPSK. We assume that the detector has perfect knowledge of the carrier frequency. The carrier phase is known up to an ambiguity of a multiple of $\frac{\pi}{2}$.

Colavolpe [1] mentioned that for differentially encoded phase-shift keying, *maximum a-posteriori* (MAP) sequence detection has a symbol-error performance *practically identical* to MAP symbol detection. We will demonstrate here that MAP sequence detection can be regarded as a straightforward approximation of MAP symbol detection. We will also discuss a better approximation of MAP symbol detection, which outperforms MAP sequence detection. This improvement is based on a piecewise-linear approximation of the logarithm of the hyperbolic cosine.

In the second part of this letter we will focus on coded systems and investigate a-posteriori log-likelihood ratios (LLRs), and some approximations for these ratios. Although the performance gain is small we show that practically optimal performance can be achieved again using the piecewise-linear approximation mentioned above.

For ease of comparison, we will follow the notation of Colavolpe [1], with $M = 4$.

II. A-Posteriori Probabilities

In the sequence $\boldsymbol{c} = c_0, c_1, \ldots, c_N$ of transmitted $\frac{\pi}{4}$-DE-QPSK code symbols, the symbols $c_n \in \mathcal{X}_e = \left\{ e^{j\pi l/2}, l = 0, 1, 2, 3 \right\}$ for n even and the symbols $c_n \in \mathcal{X}_o = \left\{ e^{j\pi l/2 + \pi/4}, l = 0, 1, 2, 3 \right\}$ for n odd. Furthermore, these symbols are determined by the differential encoding rule

$$c_n = a_n c_{n-1}, \quad \text{for } n = 1, 2, \ldots, N, \tag{1}$$

where the first symbol $c_0 \in \mathcal{X}_e$, with $\Pr\{c_0 = e^{j\pi l/2}\} = 1/4$, for $l = 0, 1, 2, 3$.

As in Colavolpe [1] we consider the case where the information symbols $\boldsymbol{a} = a_1, a_2, \ldots, a_N$ are independent of each other and uniformly distributed (*iud*) over $\mathcal{X} = \left\{ e^{j\pi l/2 + \pi/4}, l = 0, 1, 2, 3 \right\}$. Denoting the received sequence by $\mathbf{x} = x_0, x_1, \ldots, x_N$, we can write for channel output x_n for $n = 0, 1, \cdots, N$

$$x_n = c_n + w_n, \tag{2}$$

where w_n is circularly symmetric complex white Gaussian noise with variance σ^2 per component.

Now, as in (13) in [1], the a-posteriori probability (AP) of an iud $\frac{\pi}{4}$-DE-QPSK information symbol a_n for $n = 1, 2, \cdots, N$ can be expressed as

$$\Pr\{a_n | \mathbf{x}\} \propto \tag{3}$$

$$\begin{cases} \displaystyle\sum_{c_{n-1} \in \mathcal{X}_e} \exp\left(\frac{1}{\sigma^2} \Re\left\{ c_{n-1}^* \left[x_n a_n^* + x_{n-1} \right] \right\}\right), & n \text{ odd}, \\ \displaystyle\sum_{c_{n-1} \in \mathcal{X}_o} \exp\left(\frac{1}{\sigma^2} \Re\left\{ c_{n-1}^* \left[x_n a_n^* + x_{n-1} \right] \right\}\right), & n \text{ even}. \end{cases}$$

which results in the MAP symbol decision rule

$$\hat{a}_n = \arg\max_{a_n} \Pr\{a_n | \mathbf{x}\}. \tag{4}$$

For MAP sequence detection in the case of iud information symbols (i.e. maximum-likelihood (ML) detection), the decision rule is, see (17) in [1],

$$\hat{a}_n = \hat{c}_n \hat{c}_{n-1}^*, \tag{5}$$

with

$$\hat{c}_n = \arg\min_{c_n} |x_n - c_n|^2. \tag{6}$$

Now consider (3) for $n - 1$ is even and for some fixed $a_n \in \mathcal{X}$. If we define

$$v_n \triangleq x_n a_n^* + x_{n-1}, \tag{7}$$

we obtain

$$
\begin{aligned}
\Pr\{a_n|\mathbf{x}\} \quad &\propto \quad \exp\left(\frac{1}{\sigma^2}\Re(v_n)\right) + \exp\left(-\frac{1}{\sigma^2}\Im(v_n)\right) \\
&\qquad + \exp\left(-\frac{1}{\sigma^2}\Re(v_n)\right) + \exp\left(\frac{1}{\sigma^2}\Im(v_n)\right) \\
&= \quad 2\cosh\left(R\right) + 2\cosh\left(I\right),
\end{aligned}
\tag{8}
$$

with $R \triangleq \frac{\Re(v_n)}{\sigma^2}$ and $I \triangleq \frac{\Im(v_n)}{\sigma^2}$. Note that for odd $n-1$ we get an identical expression for $\Pr\{a_n|\mathbf{x}\}$ if we define

$$
v_n \triangleq (x_n a_n^* + x_{n-1})\, e^{j\frac{\pi}{4}}.
\tag{9}
$$

A. First approximation

From (8) we may conclude that the AP of an iud information symbol is proportional to the sum of two cosh-functions. To avoid underflow, calculations are often carried out in the logarithmic domain, and therefore instead of (8) the metric

$$
m_0(a_n) \triangleq \ln\left(2\cosh\left(R\right) + 2\cosh\left(I\right)\right),
\tag{10}
$$

can be applied. As approximation for $m_0(a_n)$ we can now use

$$
m_1(a_n) \triangleq \ln\left(\max\left(e^{|R|}, e^{|I|}\right)\right) = \max(|R|, |I|).
\tag{11}
$$

Note that this approximation avoids calculating exponentials and logarithms. To demonstrate that this approximation leads to a performance identical to that of MAP sequence detection, note that for iud information symbols, the MAP sequence decision rule (5) can be written as:

$$
\begin{aligned}
\hat{a}_n &= \quad \arg\min_{a_n}\min_{c_{n-1}}(|x_n - c_{n-1}a_n|^2 + |x_{n-1} - c_{n-1}|^2) \\
&= \quad \arg\max_{a_n}\max_{c_{n-1}}\frac{1}{\sigma^2}\Re\left\{c_{n-1}^*\left(x_n a_n^* + x_{n-1}\right)\right\}.
\end{aligned}
\tag{12}
$$

If we now call

$$
m_{\mathrm{seq}}(a_n) \triangleq \max_{c_{n-1}}\frac{1}{\sigma^2}\Re\left\{c_{n-1}^*\left(x_n a_n^* + x_{n-1}\right)\right\},
\tag{13}
$$

then, for $n-1$ even and all a_n we obtain

$$
\begin{aligned}
m_{\mathrm{seq}}(a_n) &= \quad \max\left(\frac{\Re\{v_n\}}{\sigma^2}, -\frac{\Im\{v_n\}}{\sigma^2}, -\frac{\Re\{v_n\}}{\sigma^2}, \frac{\Im\{v_n\}}{\sigma^2}\right) \\
&= \quad \max\left(|R|, |I|\right),
\end{aligned}
\tag{14}
$$

which is identical to our first approximation given by (11). As a consequence, our first approximation will result in the same symbol estimates as MAP sequence detection.

In the next subsection we discuss a better approximation for the AP symbol metric $m_0(a_n)$ than $m_1(a_n)$.

B. Second approximation

To improve approximation $m_1(a_n)$ of $m_0(a_n)$, we propose

$$
m_2(a_n) \triangleq f\left(\frac{R+I}{2}\right) + f\left(\frac{R-I}{2}\right) + 2\ln 2,
\tag{15}
$$

where we used the identity

$$
\cosh\left(R\right) + \cosh\left(I\right) = 2\cosh\left(\frac{R+I}{2}\right)\cosh\left(\frac{R-I}{2}\right),
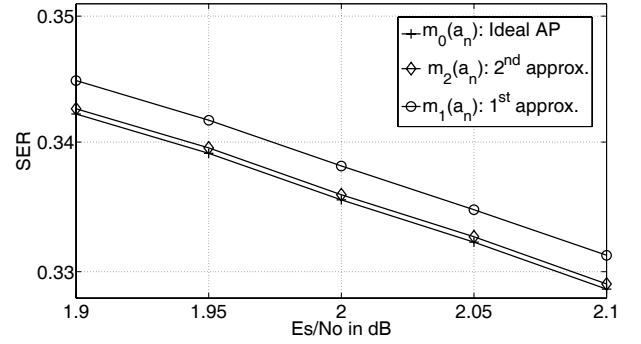\tag{16}
$$

Fig. 1. SER-performance for ideal and approximated versions of the APs.

and approximated the $\ln\left(\cosh(g)\right)$ by the piecewise-linear function:

$$
f(g) = \begin{cases} |g| - \ln 2, & |g| > \ln 2 \\ 0, & |g| \le \ln 2. \end{cases}
\tag{17}
$$

To see that this is reasonable note that for large $|g|$ one of the exponentials in $\cosh(|g|)$ dominates, which results in linearity.

C. Simulations

We have simulated the first and second approximation in terms of the uncoded Symbol Error Rate (SER) versus the signal-to-noise ratio $E_s/N_0 = \frac{1}{2\sigma^2}$, where E_s is the received signal energy of a code symbol and $N_0/2$ the two-sided power spectral density of the noise. The results of these simulations can be found in Fig. 1, together with the optimum symbol-detection results, i.e. the results based on metrics $m_0(\cdot)$.

It can be observed that optimal symbol detection and our second approximation outperform MAP sequence detection (first approximation). In contrast to the other two methods, MAP sequence detection requires no knowledge of the noise variance however.

III. LOG-LIKELIHOOD RATIOS

In a coded situation the detector is followed by a Viterbi-decoder which needs soft-information about the coded bits. The desired metrics for transmission n, i.e. the LLRs [2], in the case of Gray encoding, see [3], can be expressed as

$$
\lambda_n^1 = \ln\left(\frac{e^{m(\frac{3\pi}{4})} + e^{m(\frac{5\pi}{4})}}{e^{m(\frac{\pi}{4})} + e^{m(\frac{7\pi}{4})}}\right), \lambda_n^2 = \ln\left(\frac{e^{m(\frac{5\pi}{4})} + e^{m(\frac{7\pi}{4})}}{e^{m(\frac{\pi}{4})} + e^{m(\frac{3\pi}{4})}}\right),
\tag{18}
$$

where λ_n^1 corresponds to the first bit and λ_n^2 to the second bit. Ideally the symbol-metric $m(\cdot)$ should be $m_0(\cdot)$ but to reduce complexity we could also use an approximated version, i.e. $m_1(\cdot)$ or $m_2(\cdot)$.

A. Third approximation

To avoid arithmetic based on exponential and logarithmic functions, we introduce a third approximation, which just as our first approximation, selects the dominant exponential. Observe that a LLR is a difference of the logarithm of a numerator and the logarithm of a denominator. Both the numerator
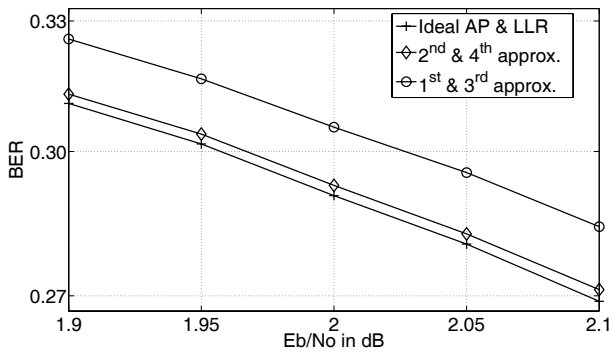
Fig. 2.   BER-performance for ideal and approximated versions of the LLRs.

and denominator consist of two exponentials. Therefore we make the following approximation

$$\ln\left(e^{m(a)} + e^{m(a')}\right) \approx \max\left(m(a), m(a')\right). \qquad (19)$$

Note, that in this approximation we can use the ideal value $m_0(\cdot)$ for $m(\cdot)$, but also one of its approximations $m_1(\cdot)$ or $m_2(\cdot)$. If we take $m_1(\cdot)$ we get an approximation which is identical to the approximation (10) by Bottomley *et al.* [4].

B. Fourth approximation

Motivated by the fact that our first approximation could be improved, we propose a fourth approximation, similar to the second one. Based on the identity

$$e^{m(a)} + e^{m(a')} = 2e^{\frac{m(a)+m(a')}{2}} \cosh\left(\frac{m(a)-m(a')}{2}\right), \quad (20)$$

and (17), we can approximate

$$\ln\left(e^{m(a)} + e^{m(a')}\right) \qquad\qquad (21)$$
$$\approx \quad \frac{m(a)+m(a')}{2} + f\left(\frac{m(a)-m(a')}{2}\right) + \ln 2.$$

Again we could use the ideal symbol metrics $m_0(\cdot)$ for $m(\cdot)$, but also the approximations $m_1(\cdot)$ or $m_2(\cdot)$.

C. Simulations

Fig. 2 shows simulation results when we use the ideal symbol-metrics $m_0(\cdot)$ with the ideal LLR computations as in (18), but also for the case where we use the approximated symbol-metrics $m_1(\cdot)$ in combination with LLR-approximation three (19), and for the case in which we use the improved symbol-metrics $m_2(\cdot)$ together with the improved fourth approximation (21). The Bit Error Rate (BER) versus the signal-to-noise ration $E_b/N_0 = \frac{1}{2\sigma^2}$ is shown, where E_b is the received signal energy of an information bit.

We used, conform [3], the de-facto industry standard $R_c = \frac{1}{2}$, $K = 7$, convolutional code with generator polynomials $g_0 = 133$ and $g_1 = 171$. Its output is randomly bit-interleaved and differentially encoded after Gray mapping. The BER simulations are performed with the Viterbi-algorithm for decoding the convolutional code. Note that even if we do ideal LLR-computations based on ideal symbol-metrics $m_0(\cdot)$,

the concatenated demodulator/decoder can only be close to optimal. As expected the ideal symbol-metrics $m_0(\cdot)$ together with ideal LLR-computation result in the best (ideal) results, and using approximated symbol-metrics $m_1(\cdot)$ together with the LLR-approximation three yields the worst results. Using symbol-metrics $m_2(\cdot)$ together with LLR-approximation four achieves results that are only slightly worse than the best results. Note again that the combination of the second and fourth approximation requires knowledge of the noise variance.

IV. CONCLUSIONS

We have shown that a straightforward approximation of MAP symbol detection is actually identical to MAP sequence detection, which is suboptimal for coherent symbol detection of $\frac{\pi}{4}$-DE-QPSK. This approximation is based on selecting dominant exponentials. Simulations showed that MAP symbol detection outperforms MAP sequence detection, but the difference is small. In this way we have made the statement of Colavolpe [1] more precise. Moreover we have proposed an improved approximation of the symbol-metrics that results in a symbol-error performance close to optimal. The resulting arithmetic is based on a piecewise-linear function.

In the coded case LLRs must be computed. Apart from exact computation of these LLRs, we have considered an approximation based again on selecting dominant exponentials, i.e. Bottomley's [4] approximation, but also a better approximation based on the piecewise-linear function mentioned before. Improved LLR-approximation combined with improved symbol-metric approximation gives a performance close to that of exact LLR-computation based on exact symbol metrics.

Selecting dominant exponentials is similar to max-log-MAP detection applied in turbo-decoders, see Robertson [5]. Our improved approximations are based on a piecewise-linear fit for the logarithm of a hyperbolic cosine and can be shown to be similar to approximations of the Jacobian proposed by Talakoub *et al.* in [6] for turbo decoding. While the particular examples of our considered approximations show modest gains in performance, the letter provides a way of improving performance when needed.

REFERENCES

[1] G. Colavolpe, "Classical coherent receivers for differentially encoded M-PSK are optimal," *IEEE Commun. Lett.*, vol. 8, no. 4, pp. 211–213, Apr. 2004.

[2] J. Hagenauer and P. Hoeher, "A Viterbi algorithm with soft-decision outputs and its applications," in *Proc. GLOBECOM '89*, vol. 3, pp. 1680–1686, Nov. 1989.

[3] European Standard (Telecommunications series), "Radio broadcasting systems; digital audio broadcasting (DAB) to mobile, portable and fixed receivers," ETSI EN 300 401 V1.4.1 (2006-06), June 2006, v1.4.1.

[4] G. Bottomley, H. Arslan, R. Ramesh, and G. Brismark, "Coherent MAP detection of DQPSK bits," *IEEE Commun. Lett.*, vol. 4, no. 11, pp. 354–356, Nov. 2000.

[5] P. Robertson, E. Villebrun, and P. Hoeher, "A comparison of optimal and sub-optimal MAP decoding algorithms operating in the log domain," in *Proc. ICC '95*, vol. 2, pp. 1009–1013, June 1995.

[6] S. Talakoub, L. Sabeti, B. Shahrrava, and M. Ahmadi, "An improved Max-Log-MAP algorithm for turbo decoding and turbo equalization," *IEEE Trans. Instrum. Meas.*, vol. 56, no. 3, pp. 1058–1063, June 2007.