

The Meixner process : theory and applications in finance

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The Meixner Process: Theory and Applications

in Finance

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Abstract

The Meixner process is a special type of Lévy process which originates from the theory of orthogonal polynomials. It is related to the Meixner-Pollaczek polynomials by a martingale relation. We discuss several properties of the Meixner process.

We apply the Meixner process to financial data. First, we show that the Normal distribution is a very poor model to fit log-returns of financial assets like stocks or indices. In order to achieve a better fit we replace the Normal distribution by the more sophisticated Meixner distribution, taking into account, skewness and excess kurtosis. We show that the underlying Meixner distribution allows a much better fit to the data by performing a number of statistical tests. Secondly, we introduce stock price models based on the Meixner process in order to price financial derivatives. A first significant improvement can be achieved with respect to the famous Black-Scholes model (BS-model) by replacing its Brownian motion by the more flexible Meixner process. However, there still is a discrepancy between market and theoretical prices. The main feature which these Lévy models are missing is the fact that volatility or more generally the environment is changing stochastically over time. By making business time stochastic, an idea which was developed in [9], one can incorporate these stochastic volatility effects. The resulting option prices can be calibrated almost perfectly to empirical prices.

1 Introduction

Financial mathematics has recently enjoyed considerable prestige on account of its impact on the finance industry. In parallel, the theory of Lévy processes has also seen exciting developments in recent years [2] [4] [26]. The fusion of these two fields of mathematics has provided new applied modeling perspectives within the context of finance and further stimulus for deep and intrinsically interesting problems within the context of Lévy processes.

The Meixner distribution belongs to the class of the infinitely divisible distributions and as such give rise to a Lévy process: The Meixner process. The Meixner process is very flexible, has a simple structure and leads to analytically and numerically tractable formulas. It was introduced in [27] (see also [28]) and originates from the theory of orthogonal polynomials and was proposed to serve as a model of financial data in [15].

We will apply the Meixner distribution and the Meixner process in the context of mathematical finance. More precisely, we will use the process to model the stochastic behaviour of financial assets like stocks or indices. The most famous continuous-time model for stock prices or indices is the celebrated Black-Scholes model [6]. It uses the Normal distribution to fit the log-returns of the underlying: the price process of the underlying is given by the geometric Brownian Motion

$$S_t = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right),$$

where $\{B_t, t \geq 0\}$ is standard Brownian motion, i.e. B_t follows a Normal distri-

bution with mean 0 and variance t . It is well known however that the log-returns of most financial assets have an actual kurtosis that is higher than that of the Normal distribution. In this paper we therefore propose another model which is based on the Meixner distribution.

In the late 1980s and in the 1990s several other similar process models were proposed. Madan and Seneta [19] have proposed a Lévy process with Variance Gamma distributed increments. We mention also the Hyperbolic Model [11] proposed by Eberlein and Keller and their generalizations [23]. In the same year Barndorff-Nielsen proposed the Normal Inverse Gaussian Lévy process [1]. Recently the CGMY model was introduced [8]. All models give a much better fit to the data and lead to an improvement with respect to the Black-Scholes model. In this paper we provide statistical evidence that the Meixner model performs also significantly better than the Black-Scholes Model.

A second application can be found in the same context: the pricing of financial derivatives. First we will try to price derivatives using a model where the Brownian motion of the BS-model is just replaced by a Lévy process. Although there is a significant improvement in accuracy with respect to the BS-model, there still is a discrepancy between model prices and market prices. The main feature which these Lévy models are missing, is the fact that the volatility or more general the environment is changing stochastically over time. In order to deal with this problem, we make (business) time stochastic as proposed in [9]. In this paper we show that by following the procedure of [9], we can almost perfectly calibrate model prices of the Meixner model with stochastic business

time, also called the Meixner Stochastic Volatility model (Meixner-SV model) to market prices.

Throughout this paper we make use of two data sets. The first data set consists of the log-returns of the Nikkei-225 Index during a period of three years. We show that the Meixner distribution can be fitted much more accurate to this set than the Normal distribution. A second data set consists of the mid-prices of a set of European call and put options on the SP500-index at the close of the market on the 4th of December 2001. At this day the SP500 closed at 1144.80. We will calibrate different models to this set.

This paper is organized as follows: we first introduce the Meixner distribution and the Meixner Process in Section 2. Next, in Section 3, we will apply the Meixner Process and its underlying distribution in the context of finance: we will fit the Meixner distributions to our data set of the Nikkei-255 log-returns and we perform a number of statistical test in order to proof the high accuracy of the fit and we will calibrate models based on the Meixner process to our set of option prices. We will show that the Meixner-SV model leads to option prices which can be calibrated almost perfectly to the market prices.

2 The Meixner Process

The density of the Meixner distribution (Meixner(a, b, d, m)) is given by

$$f(x; a, b, m, d) = \frac{(2 \cos(b/2))^{2d}}{2a\pi\Gamma(2d)} \exp\left(\frac{b(x-m)}{a}\right) \left| \Gamma\left(d + \frac{i(x-m)}{a}\right) \right|^2,$$

where $a > 0$, $-\pi < b < \pi$, $d > 0$, and $m \in \mathbb{R}$.

Moments of all order of this distribution exist. Next, we give some relevant quantities; similar, but more involved, expressions exist for the moments and the skewness.

	Meixner (a, b, d, m)	Normal (μ, σ^2)
mean	$m + ad \tan(b/2)$	μ
variance	$\frac{a^2 d}{2} (\cos^{-2}(b/2))$	σ^2
kurtosis	$3 + \frac{3-2 \cos^2(b/2)}{d}$	3

One can clearly see that the kurtosis of the Meixner distribution is always greater than the Normal kurtosis.

The characteristic function of the Meixner(a, b, d, m) distribution is given by

$$E[\exp(iuM_1)] = \left(\frac{\cos(b/2)}{\cosh \frac{au-b}{2}} \right)^{2d} \exp(imu)$$

Suppose $\phi(u)$ is the characteristic function of a distribution. If moreover for every positive integer n , $\phi(u)$ is also the n th power of a characteristic function, we say that the distribution is infinitely divisible. One can define for every such an infinitely divisible distribution a stochastic process, $X = \{X_t, t \geq 0\}$, called Lévy process, which starts at zero, has independent and stationary increments and such that the distribution of an increment over $[s, s + t]$, $s, t \geq 0$, i.e. $X_{t+s} - X_s$, has $(\phi(u))^t$ as characteristic function.

Clearly, the Meixner(a, b, d, m) distribution is infinitely divisible and we can associate with it a Lévy process which we call the Meixner process. More precisely, a Meixner process $\{M_t, t \geq 0\}$ is a stochastic process which starts at zero, i.e. $M_0 = 0$, has independent and stationary increments, and where the distribution of M_t is given by the Meixner distribution Meixner(a, b, dt, mt).

In general a Lévy process consists of three independent parts: a linear deterministic part (drift), a Brownian part, and a pure jump part. It is easy to show that our Meixner process $\{M_t, t \geq 0\}$ has no Brownian part and a pure jump part governed by the Lévy measure

$$\nu(dx) = d \frac{\exp(bx/a)}{x \sinh(\pi x/a)} dx.$$

The Lévy measure $\nu(dx)$ dictates how the jumps occur. Jumps of sizes in the set A occur according to a Poisson Process with parameter $\int_A \nu(dx)$. Because $\int_{-\infty}^{+\infty} |x| \nu(dx) = \infty$ it follows from standard Lévy process theory [4] [26], that our process is of infinite variation.

A number of stylized features of observational series from finance are discussed in [3]. One of this features is the semihaviness of the tails. Our Meixner(a, b, d, m) distribution has semiheavy tails [16]. This means that the tails of the density function behave as

$$f(x, a, b, d, m) \sim C_- |x|^{\rho_-} \exp(-\sigma_- |x|) \quad \text{as } x \rightarrow -\infty$$

$$f(x, a, b, d, m) \sim C_+ |x|^{\rho_+} \exp(-\sigma_+ |x|) \quad \text{as } x \rightarrow +\infty,$$

for some $\rho_-, \rho_+ \in \mathbb{R}$ and $C_-, C_+, \sigma_-, \sigma_+ \geq 0$. In case of the Meixner(a, b, d, m),

$$\rho_- = \rho_+ = 2d - 1, \quad \sigma_- = (\pi - b)/a, \quad \sigma_+ = (\pi + b)/a.$$

The Meixner process originates from the theory of orthogonal polynomials: The Meixner($1, 2\zeta - \pi, d, 0$) distribution is the measure of orthogonality of the Meixner-Pollaczek polynomials $\{P_m(x; d, \zeta), m = 0, 1, \dots\}$. Moreover the monic Meixner-Pollaczek polynomials $\{\tilde{P}_m(x; d, \zeta), m = 0, 1, \dots\}$ [17] are martingales

for the Meixner process ($a = 1$, $m = 0$, $d = 1$, $\zeta = (b + \pi)/2$):

$$E[\tilde{P}_m(M_t; t, \zeta) \mid M_s] = \tilde{P}_m(M_s; s, \zeta)$$

Note the similarity with the classical martingale relation between standard Brownian motion $\{W_t, \geq 0\}$ and the Hermite Polynomials $\{H_m(x; \sigma), m = 0, 1, \dots\}$ [28]:

$$E \left[\tilde{H}_m(W_t; t) \mid W_s \right] = \tilde{H}_m(W_s; s)$$

The Meixner distribution is a special case of the Generalized z -distributions: The Generalized z -distribution (GZ) [16] is defined through the characteristic function:

$$\phi_{GZ}(z; a, b_1, b_2, d, m) = \left(\frac{B(b_1 + \frac{iaz}{2\pi}, b_2 - \frac{iaz}{2\pi})}{B(b_1, b_2)} \right)^{2d} \exp(imz),$$

where $a, b_1, b_2, d > 0$ and $m \in \mathbb{R}$.

This distribution is infinitely divisible and we can associate with it a Lévy process, such that its time t distribution has characteristic function $\phi_{GZ}(z; a, b_1, b_2, dt, mt)$.

The Lévy measure for this GZ-Process is given by

$$\nu_{GZ}(dx) = \begin{cases} \frac{2d \exp(2\pi b_1 x/a)}{|x|(1 - \exp(2\pi x/a))} dx & x < 0 \\ \frac{2d \exp(2\pi b_2 x/a)}{x(1 - \exp(-2\pi x/a))} dx & x > 0 \end{cases}.$$

For

$$b_1 = \frac{1}{2} + \frac{b}{2\pi} \text{ and } b_2 = \frac{1}{2} - \frac{b}{2\pi},$$

we obtain the Meixner Process. Note that the Generalized z -distributions and the Generalized Hyperbolic distribution [11] [23] are non-intersecting sets.

The Meixner Process is also related to the process studied by Biane, Pitman and Yor [5] (see also [22]):

$$C_t = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\Gamma_{n,t}}{(n - \frac{1}{2})^2},$$

for a sequence of independent Gamma Processes $\Gamma_{n,t}$, i.e. Lévy process with $E[\exp(i\theta\Gamma_{n,t})] = (1 - i\theta)^{-t}$.

In [5] one shows that C_t has Laplace transform

$$E[\exp(-uC_t)] = \left(\frac{1}{\cosh \sqrt{2u}} \right)^t$$

This means that the Brownian time change B_{C_t} has characteristic function

$$E[\exp(iuB_{C_t})] = \left(\frac{1}{\cosh u} \right)^t,$$

or equivalently B_{C_t} follows a Meixner(2, 0, t , 0) distribution.

3 Applications

3.1 Fitting the log-returns of financial assets

As a first application of the Meixner Process, we try to fit its distribution to our data set of daily log-returns of the Nikkei 225 Index in the period from 01-01-1997 until 31-12-1999. The data set consists of the 737 daily-log-returns of the index during the mentioned period. The mean of this data set is equal to 0.00036180, while its standard deviation equals 0.01599747. In [29] one can find similar analyses for other indices during the same period.

To estimate the Meixner distribution we assume independent observations and use moments estimators. In the particular case of the Nikkei-225 Index, the

result of the estimation procedure is given by

$$\hat{a} = 0.02982825, \quad \hat{b} = 0.12716244, \quad \hat{d} = 0.57295483, \quad \hat{m} = -0.00112426$$

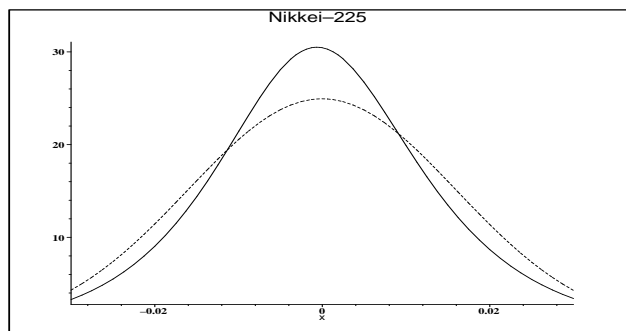


Figure 1: Meixner density (solid) versus Normal density (dashed)

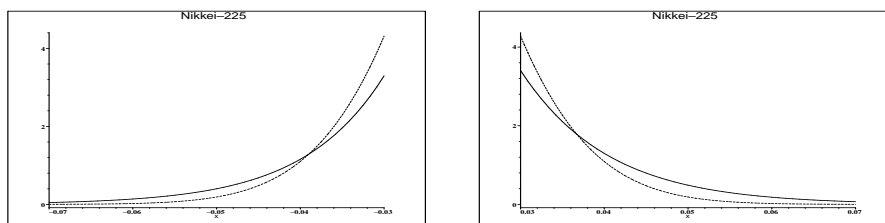


Figure 2: Meixner density tails (solid) versus Normal density tails (dashed)

From Figure 1, it is clear that there is considerably more mass around the center than the Normal distribution can provide. Figure 2 zooms in at the tails. As can be expected from the semiheavyness of the tails, the Meixner distribution has significant fatter tails than the Normal distribution. This is in correspondence with empirical observations, see e.g. [11].

We use different tools for testing the goodness of fit: QQ-plots and χ^2 -

tests. It will be shown that we obtain an almost perfect fit. So we arrive at the conclusion that the daily log-returns of the asset can be modeled very accurately by the Meixner distribution.

3.1.1 QQ-plots

The first evidence is provide by a graphical method: the quantile-quantile plot (QQ-plot). It is a qualitative yet very powerful method for testing the goodness of fit. A QQ-plot of a sample of n points plots for every $j = 1, \dots, n$ the empirical $(j - (1/2))/n$ -quantile of the data against the $(j - (1/2))/n$ -quantile of the fitted distribution. The plotted points should not deviated to much from a straight line.

For the classical model based on the Normal distribution, the deviation from the straight line and thus the Normal density is clearly seen from the next QQ-plot in Figure 3.

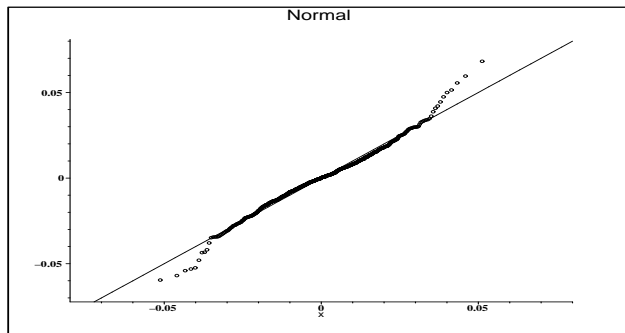


Figure 3: Normal QQ-plot

It can be seen that there is a severe problem in the tails if we try to fit the data with the Normal distribution. This problem almost completely disappears

when we use the Meixner distribution to fit the data, as can be seen in Figure 4.

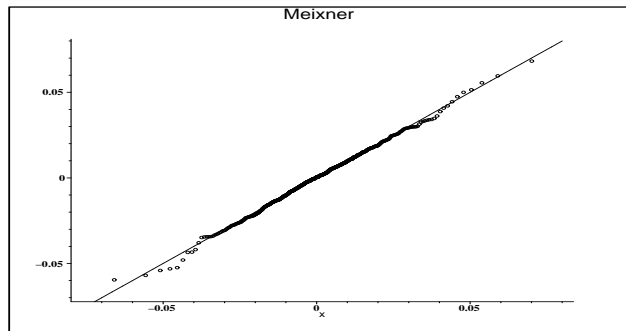


Figure 4: Meixner QQ-plot

The Meixner density shows much better fit. It indicates a strong preference for the Meixner model over the classical Normal one.

3.1.2 χ^2 -tests

The χ^2 -test counts the number of sample points falling into certain intervals and compares them with the expected number under the null hypothesis. We consider classes of equal width as well of equal probability. We take $N = 32$ classes of equal width. If necessary we collapse outer cells, such that the expected value of observations becomes greater than five. In our Nikkei-225 Index-example, we choose $-0.0225 + (j - 1) * (0.0015)$, $j = 1, \dots, N - 1$, as the boundary points of the classes.

We consider also the case with $N = 28$ classes of equal probability, the class boundaries are now given by the i/N -quantiles $i = 1, \dots, N - 1$ of the fitting distribution.

Because we have to estimate for the Normal distribution two parameters we taken in this case $N - 3$ degrees of freedom. In the Meixner case, there has to be estimated 4 parameters, so we take in this case $N - 5$ degrees of freedom.

Table 1 shows the values of the χ^2 -test statistic with equal width for the Normal null hypotheses and the Meixner null hypotheses and different quantiles of the χ_{29}^2 and χ_{27}^2 distributions.

Table 2 shows the values of the χ^2 -test statistic with equal probability for the Normal null hypotheses and the Meixner null hypotheses and different quantiles of the χ_{23}^2 and χ_{25}^2 distributions.

In Tables 1 and 2 we also give the so-called P -values of the test-statistics. The P -value is the probability that values are even more extreme (more in the tail) than our test-statistic. It is clear that very small P -values lead to a rejection of the null hypotheses, because they are themselves extreme. P -values not close to zero indicate that the test statistic is not extreme and lead to acceptance of the hypothesis. To be precise we reject if the P -value is less than our level of significance, which we take 0.05, and accept otherwise.

χ_{Normal}^2	$\chi_{29,0.95}^2$	$\chi_{29,0.99}^2$	P_{Normal} -value
47.45527092	42.55696780	49.58788447	0.01672773
$\chi_{Meixner}^2$	$\chi_{27,0.95}^2$	$\chi_{27,0.99}^2$	$P_{Meixner}$ -value
29.21660289	40.11327207	46.96294212	0.35047500

Table 1: χ_1^2 test-statistics and P -values (equal width)

χ_{Normal}^2	$\chi_{25,0.95}^2$	$\chi_{25,0.99}^2$	P_{Normal} -value
47.87381276	37.65248413	44.31410490	0.00386153
$\chi_{Meixner}^2$	$\chi_{23,0.95}^2$	$\chi_{23,0.99}^2$	$P_{Meixner}$ -value
20.44369064	35.17246163	41.63839812	0.61502001

Table 2: χ_2^2 test-statistics and P -values (equal probability)

We see that the Normal hypotheses is in both cases clearly rejected, whereas the Meixner hypotheses is accepted and yields a very high P -value.

3.2 Pricing of Derivatives

Throughout the text we will denote by r the daily interest rate. We assume our market consist of one riskless asset (the bond) with price process given by $B_t = e^{rt}$ and one risky asset (the stock) with price process S_t . Given our market model, let $G(\{S_u, 0 \leq u \leq T\})$ denote the payoff of the derivative at its time of expiry T .

According to the fundamental theorem of asset pricing (see [10]) the arbitrage free price V_t of the derivative at time $t \in [0, T]$ is given by

$$V_t = E_Q[e^{-r(T-t)}G(\{S_u, 0 \leq u \leq T\})|\mathcal{F}_t],$$

where the expectation is taken with respect to an equivalent martingale measure Q and $\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is the natural filtration of $S = \{S_t, 0 \leq t \leq T\}$. An equivalent martingale measure is a probability measure which is equivalent (it has the same null-sets) to the given (historical) probability measure and under

which the discounted process $\{e^{-rt}S_t\}$ is a martingale. Unfortunately for most models, in particular the more realistic ones, the class of equivalent measures is rather large and often covers the full no-arbitrage interval. In this perspective the Black-Scholes model, where there is an unique equivalent martingale measure, is very exceptional. Models with more than one equivalent measures are called incomplete.

3.2.1 The Meixner model

In real markets traders are well aware that the future probability distribution of the underlying asset may not be lognormal and they use a volatility smile adjustment. Typically the implicit volatility is higher in the money and out of the money. This smile-effect is decreasing with time to maturity. Moreover, smiles are frequently asymmetric. Instead of using one volatility parameter σ for the stock, one is thus using for every strike K and for every maturity T another parameter σ . This is completely wrong since this implies that only one underlying stock/index is modeled by a number of completely different stochastic processes.

Another more natural method is by replacing the Brownian motion in the BS-model, by a more sophisticated Lévy process. The model which produces exactly Meixner(a, b, d, m) daily log-returns for the stock is given by

$$S_t = S_0 \exp(M_t).$$

Next, we have to choose a equivalent martingale measure. Following Gerber and Shiu ([13] and [14]) we can by using the so-called Esscher transform easily

find an equivalent martingale measure. With the Esscher transform our equivalent martingale measure Q follows a Meixner($a, a\theta + b, d, m$) distribution (see also [15]), where θ is given by

$$\theta = \frac{-1}{a} \left(b + 2 \arctan \left(\frac{-\cos(a/2) + \exp((m-r)/(2d))}{\sin(a/2)} \right) \right)$$

Although the Esscher-transform is easy to obtain it is not clear that in reality the market chooses this kind of (exponential) transform. Another way to obtain an equivalent martingale measure Q is by mean correcting the exponential of a Lévy process. In this case the risk-neutral process is given by

$$S_t^{riskneutral} = S_0 \exp(X_t) \frac{\exp(rt)}{E[\exp(X_t)]}$$

For the Meixner process, Q follows now follows a Meixner($a, b, d, m_{riskneutral}$), with

$$m_{riskneutral} = r - 2d \log \left(\frac{\cos(b/2)}{\cos((a+b)/2)} \right).$$

If the payoff is only depending on the value of the asset at its time of expiry T , i.e. $G(S_t, 0 \leq t \leq T) = G(S_T)$, then the price of the derivative can be obtained by the Feynman-Kac Formula or by a Fourier inversion method.

If the price $V(t, M_t)$ at time t of the such a derivative satisfies some regularity conditions (i.e. $V(t, x) \in C^{(1,2)}$ (see [21])) it can be obtained by solving a partial differential integral equation (PDIE) with a boundary condition:

$$\begin{aligned} rV(t, x) &= \gamma \frac{\partial}{\partial x} V(t, x) + \frac{\partial}{\partial t} V(t, x) \\ &\quad + \int_{-\infty}^{+\infty} \left(V(t, x+y) - V(t, x) - y \frac{\partial}{\partial x} V(t, x) \right) \nu_Q(dy) \\ V(T, x) &= F(x), \end{aligned}$$

where $\nu_Q(dy)$ and γ is the Lévy measure and the drift of the risk-neutral distribution. This PDIE is the analogue of the Black-Scholes partial differential equation and follows from the Feynman-Kac formula for Lévy Processes [21].

A more explicit pricing method which can be applied in general when the characteristic function of the risk-neutral stock price process is known, was developed by Carr and Madan [7] for the classical vanilla options. More precisely, let $C(K, T)$ be the price of a European call option with strike K and maturity T . Let α be a positive constant such that the α th moment of the stock price exists. Carr and Madan then showed that

$$C(k, T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} e^{-iv \log(K)} \psi(v) dv$$

where

$$\psi(v) = \frac{e^{-rT} E[\exp(i(v - (\alpha + 1)i) \log(S_t))]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}.$$

The Fast Fourier Transform can be used to invert the generalized Fourier transform of the call price. Put options can be priced using the put-call parity. This Fourier-method was generalized to other types of options, like power options and self-quanto options in [25].

Typically Lévy models, including the Meixner case, incorporate by themselves a smile effect [29] [12]. However, if we estimate the model parameters by minimizing the root mean square error between market close prices and model prices, we can observe still a significant, although smaller than for the BS-model, difference as can be seen in Figure 5 for the SP500-index options.

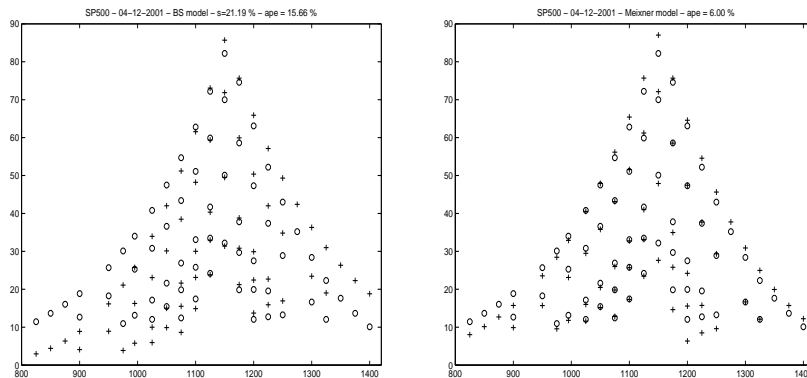


Figure 5: Black Scholes ($\sigma = 0.2119$) and Meixner ($a = 0.1277, b = -1.8742, d = 2.2603$) calibration on SP500 options (o's are market prices, +'s are model prices)

3.2.2 The Meixner-SV model

The main feature which the above described Lévy models are missing is the fact that volatility or more generally the environment is changing stochastically over time. It has been observed that the volatilities estimated (or more general the parameters or uncertainty) change stochastically over time and are clustered as can be seen in Figure 6, were the log-returns of the SP500-index over a period of 32 years is plotted. One clearly sees that there are periods with high absolute log-returns and periods with lower absolute log-returns.

In order to incorporate such an effect Carr, Madan, Geman and Yor [9] proposed the following: One increase or decrease the level of uncertainty by speeding up or slowing down the rate at which time passes. Moreover, in order to build clustering and to keep time going forward one employs a mean-reverting positive process as a measure of the local rate of time change. They use as the

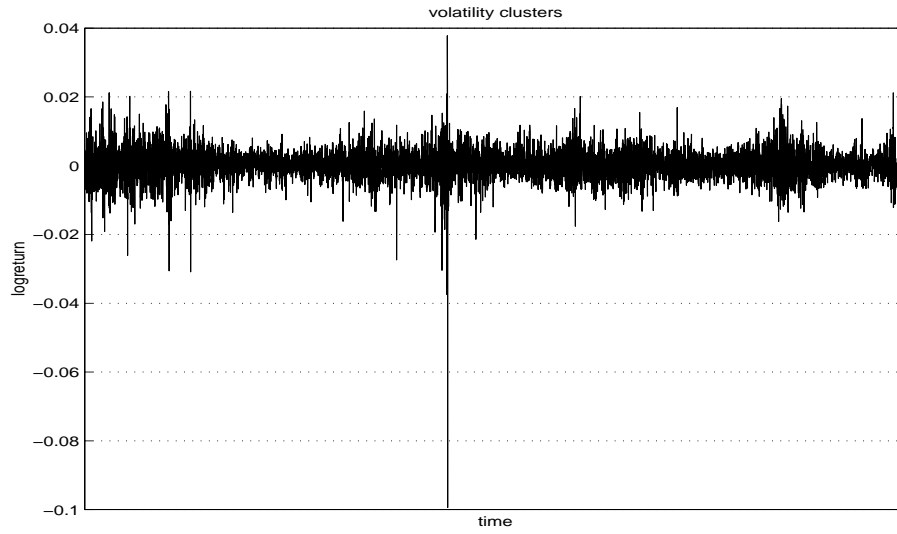


Figure 6: Volatility clusters: log-returns SP500-index between 1970 and 2001

rate of time change the classical example of a mean-reverting positive stochastic process: the CIR process $y(t)$ that solves the SDE

$$dy_t = \kappa(\eta - y_t)dt + \lambda y_t^{1/2}dW_t.$$

The economic time elapsed in t units of calendar time is then given by $Y(t)$

where

$$Y(t) = \int_0^t y_s ds$$

The characteristic function of $Y(t)$ is explicitly known:

$$\phi(u, t) = \frac{\exp(\kappa^2 \eta t / \lambda^2) \exp(2y(0)iu / (\kappa + \gamma \coth(\gamma t / 2)))}{(\cosh(\gamma t / 2) + \kappa \sinh(\gamma t / 2) / \gamma)^{2\kappa \eta / \lambda^2}},$$

where

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 i u}$$

The (risk-neutral) price process S_t is now modeled as follows:

$$S_t = S_0 \frac{\exp(rt)}{E[\exp(X_{Y(t)})]} \exp(X_{Y(t)}),$$

where X_t is a Lévy process with

$$E[\exp(iuX_t)] = \exp(t\psi_X(u)).$$

The characteristic function for the log of our stock price is given by:

$$E[\exp(iu \log(S_t))] = \exp(iu(rt + \log S_0)) \frac{\phi(-i\psi_X(u), t)}{\phi(-i\psi_X(-i), t)^{iu}}.$$

The model parameters can be estimated by minimizing the root mean square error between market close prices and model option prices and this over all strikes and maturities. An almost perfect calibration can be done. In Figure 7, one can see that the Meixner-SV model is a considerable improvement over the Meixner model, which on his turn was a considerable improvement over the BS-model.

For comparative purposes, we compute for the Black-Scholes model, the Meixner model and the Meixner SV model, the average absolute error as a percentage of the mean price. This statistic, which we will denote with *ape*, is an overall measure of the quality of fit. We have

$$ape_{BS} = 15.66\%, \quad ape_{Meixner} = 6.00\%, \quad ape_{MeixnerSV} = 2.10\%.$$

Once you have calibrated the model to a basic set of options, you can price other (exotic) options using the available pricing techniques or by Monte-Carlo simulations. Moreover, after obtaining the risk-neutral parameters, a compar-

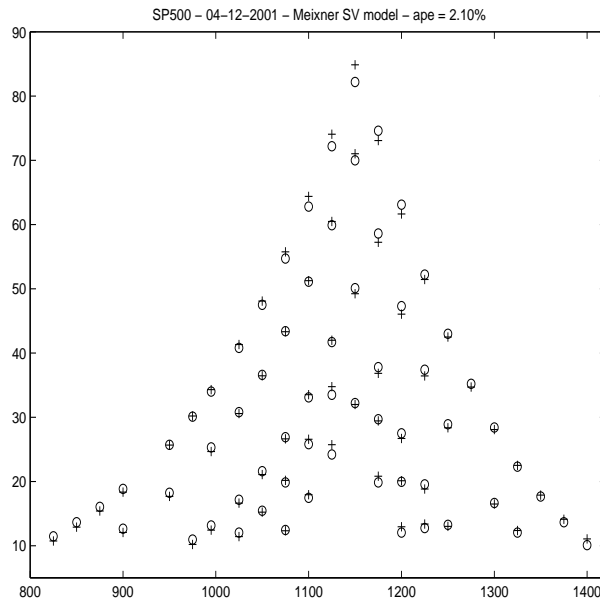


Figure 7: Meixner-SV calibration ($a = 0.0279, b = -0.1708, d = 22.0914, \kappa = 7.7859, \eta = 3.6548, \lambda = 12.2254, y_0 = 6.7871$) on SP500 options (o's are market prices, +'s are model prices)

ison with the statistical measure can lead to a better understanding of the measure change.

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