

## Solution to Problem 74-20\*: Gravitational attraction

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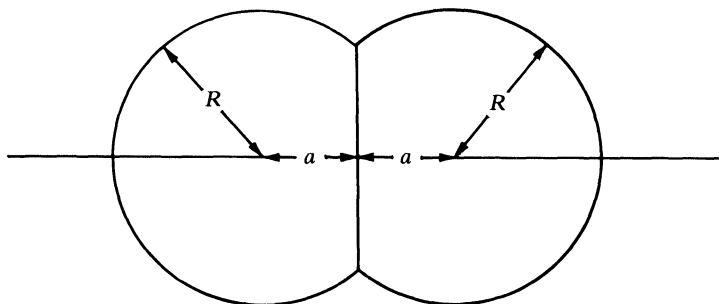
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*Problem 74-20\**, *Gravitational Attraction*, by H. R. AGGARWAL (NASA, Ames Research Center).

Determine explicitly the mutual force of gravitational attraction between two congruent spherical segments forming a “dumb-bell” shaped body whose central cross section is given by the figure.



Solution by C. J. BOUWKAMP, Technological University, Eindhoven, the Netherlands.

Let units be so chosen that the force between two unit point masses with interdistance  $r$  equals  $1/r^2$ , and assume that the body has unit mass density. Further, set  $p = a/R$  and let  $F$  be the mutual force of gravitational attraction between the two spherical segments. Then

$$(1) \quad F/(4\pi^2 R^4) = G(p) = \int_0^\infty (f(t, p))^2 t^{-1} dt,$$

in which

$$(2) \quad f(t, p) = \int_0^{1+p} e^{-tu} J_1(t(1 - (u - p)^2)^{1/2})(1 - (u - p)^2)^{1/2} du,$$

where  $J_1$  denotes the Bessel function of order one.

To prove this, I first determine the mutual potential energy  $V dz_1 dz_2$  between two thin coaxial disks. The first disk lies in the left segment, between  $z_1$  and  $z_1 + dz_1$ , with distance  $z_1$  to the midplane of the body; the second disk lies in the right segment, between  $z_2$  and  $z_2 + dz_2$ , with distance  $z_2$  to the same plane. Their axial interdistance is  $z = z_1 + z_2$ , while their radii are  $R_i = (R^2 - (z_i - a)^2)^{1/2}$ ,  $i = 1, 2$ . In each disk, polar coordinates are introduced; one of the two angular integrations can be carried out because of rotational symmetry, and we are left with

$$(3) \quad V = 2\pi \int_0^{R_1} r_1 dr_1 \int_0^{R_2} r_2 dr_2 \int_0^{2\pi} d\theta (z^2 + r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta)^{-1/2}.$$

This integral is easy to transform by means of Bessel-function techniques. With

$$(z^2 + r^2)^{-1/2} = \int_0^\infty e^{-zt} J_0(rt) dt,$$

the addition theorem of Bessel functions, and

$$\int_0^x t J_0(t) dt = x J_1(x),$$

it is found that

$$(4) \quad V = 4\pi^2 R_1 R_2 \int_0^\infty e^{-zt} J_1(R_1 t) J_1(R_2 t) t^{-2} dt.$$

The force between the two disks is  $-(\partial V / \partial z) dz_1 dz_2$ , in which the dependency of  $R_i$  on  $z_i$  is irrelevant. Since this force is in the axial direction, all forces can be added in scalar fashion. Thus

$$(5) \quad F = 4\pi^2 \int_0^\infty \frac{dt}{t} \left[ \int_0^{R+a} R_1 J_1(R_1 t) e^{-tz_1} dz_1 \right] \left[ \int_0^{R+a} R_2 J_1(R_2 t) e^{-tz_2} dz_2 \right].$$

Upon substituting  $z_i = Ru$ , the two inner integrals are seen to be identical and equal to  $R^2 f(Rt, p)$ , where  $f$  is defined in (2). The substitution  $Rt \rightarrow t$  finally gives (1).

Let me first remark that for the "dumb-bell" shaped body the parameter  $p$  lies in the closed interval  $0 \leq p \leq 1$ , the endpoints corresponding to the cases of two touching half-spheres and two touching full spheres, respectively.

For two spheres in contact the force of attraction is known by elementary methods. This comes down to  $G(1) = \frac{1}{9}$ . I do not know whether the other limiting case was ever treated before, but I am able to prove that  $G(0) = \frac{1}{12}$ . Secondly, (1) and (2) are meaningful for  $-1 \leq p < 0$ , and then they describe the force between two segments each smaller than a half-sphere. If  $G(p)$  denotes the force between the two spherical segments of the figure, then  $G(-p)$  is precisely the force between their complements, the two "flat" segments that form the cut of the two full spheres. Of course,  $G(-1) = 0$ .

To further discuss (2), I substitute  $u = p + \cos \theta$ , so that

$$(6) \quad f(t, p) = e^{-pt} \int_0^\alpha e^{-t \cos \theta} J_1(t \sin \theta) \sin^2 \theta d\theta,$$

$$\alpha = \cos^{-1}(-p) = \pi - \cos^{-1}(p),$$

where the principal value of the  $\cos^{-1}$  function is implied. One has  $\alpha = 0, \pi/2, \pi$  as  $p = -1, 0, 1$ , in that order. It is not too difficult to show that

$$(7) \quad \int_0^\pi e^{-t \cos \theta} J_1(t \sin \theta) \sin^2 \theta d\theta = \frac{2}{3} t,$$

but this is left to the reader. Thus  $f(t, 1) = \frac{2}{3} t \exp(-t)$ , and then  $G(1) = \frac{1}{9}$  follows from (1).

The case  $p = 0$  is not as nice, since

$$(8) \quad f(t, 0) = \int_0^{\pi/2} e^{-t \cos \theta} J_1(t \sin \theta) \sin^2 \theta d\theta$$

$$= t \left\{ \frac{1}{3} - \int_0^t x^{-2} J_2(x) dx \right\} = t \int_t^\infty x^{-2} J_2(x) dx,$$

which may be left unproved here. By using the last expression of (8) and manipulating with integrals in (1), I obtain  $G(0) = \frac{1}{12}$  as announced.

The two special cases above are included in the general equation

$$(9) \quad f(t, p) = t e^{-pt} \left\{ \frac{1}{3} + \frac{1}{2}p - \frac{1}{6}p^3 - (1 - p^2) \int_0^t e^{px} J_2(x\sqrt{1 - p^2}) x^{-2} dx \right\},$$

first proved by my coworker D. L. A. Tjaden of the Philips Research Laboratories. If  $p$  is nonpositive, an alternative equation is

$$(10) \quad f(t, p) = t e^{-pt} (1 - p^2) \int_t^\infty e^{px} J_2(x\sqrt{1 - p^2}) x^{-2} dx.$$

Several trials to use (9) or (10) in (1) have only led to unwieldy expressions, integrals of elliptic integrals, as might have been expected from (4), which is known to be expressible in terms of elliptic integrals. It does not seem possible to express  $F$  in terms of complete elliptic integrals, as I originally hoped in view of various similar problems encountered before.

A last resort to an "explicit" solution is numerical integration. With the exception of  $p = 1$ , the integral (1) is badly convergent; the integrand is oscillating

TABLE I

$p$	$G(p)$	$G(-p)$	$p$	$G(p)$	$G(-p)$
0.00	0.0833333	0.0833333	0.50	0.1453554	0.0178390
0.02	0.0864331	0.0802335	0.52	0.1464798	0.0161248
0.04	0.0895273	0.0771391	0.54	0.1474414	0.0145013
0.06	0.0926106	0.0740553	0.56	0.1482337	0.0129693
0.08	0.0956770	0.0709873	0.58	0.1488503	0.0115294
0.10	0.0987209	0.0679402	0.60	0.1492849	0.0101817
0.12	0.1017364	0.0649187	0.62	0.1495313	0.0089263
0.14	0.1047175	0.0619278	0.64	0.1495831	0.0077628
0.16	0.1076582	0.0589720	0.66	0.1494345	0.0066906
0.18	0.1105525	0.0560558	0.68	0.1490793	0.0057088
0.20	0.1133941	0.0531836	0.70	0.1485116	0.0048161
0.22	0.1161769	0.0503596	0.72	0.1477259	0.0040109
0.24	0.1188944	0.0475878	0.74	0.1467164	0.0032910
0.26	0.1215405	0.0448722	0.76	0.1454778	0.0026543
0.28	0.1241086	0.0422165	0.78	0.1440047	0.0020980
0.30	0.1265924	0.0396242	0.80	0.1422923	0.0016188
0.32	0.1289854	0.0370987	0.82	0.1403356	0.0012131
0.34	0.1312811	0.0346432	0.84	0.1381302	0.0008769
0.36	0.1334728	0.0322607	0.86	0.1356718	0.0006055
0.38	0.1355542	0.0299540	0.88	0.1329565	0.0003937
0.40	0.1375185	0.0277259	0.90	0.1299809	0.0002358
0.42	0.1393593	0.0255786	0.92	0.1267417	0.0001253
0.44	0.1410700	0.0235144	0.94	0.1232366	0.0000551
0.46	0.1426439	0.0215352	0.96	0.1194636	0.0000172
0.48	0.1440746	0.0196430	0.98	0.1154217	0.0000023
0.50	0.1453554	0.0178390	1.00	0.1111111	0.0000000