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Citation for published version (APA):

Bhongale, S. G., & Kokkelmans, S. J. J. M. F. (2006). BCS-BEC crossover in the strongly correlated regime of ultra-cold Fermi gases. *arXiv*, 1/4. Article 0607782.

Document status and date:

Published: 01/01/2006

Document Version:

Accepted manuscript including changes made at the peer-review stage

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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BCS-BEC Crossover in the Strongly Correlated Regime of ultra-cold Fermi gases

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(Dated: 4th February 2008)

We study BCS-BEC crossover in the strongly correlated regime of two component rotating Fermi gases. We predict that the strong correlations induced by rotation will have the effect of modifying the crossover region relative to the non-rotating situation. We show via the two particle correlation function that the crossover smoothly connects the s -wave paired fermionic fractional quantum Hall state to the bosonic Laughlin state.

PACS numbers: 03.75.-b,73.43.-f,71.27.+a,71.10.Ca

In recent years techniques based on Feshbach scattering resonances [1, 2] in ultra-cold atomic gases have allowed the study of condensation in a Fermi system [3, 4, 5]. For condensation to occur, one can distinguish two distinct physical mechanisms: (1) formation of bound pairs of fermionic atoms (molecules) which are composite bosons and hence undergo Bose-Einstein condensation (BEC), and (2) condensation of Bardeen-Cooper-Schrieffer (BCS) pairs in analogy with low temperature superconductivity. In separate publications [6, 7] both Eagles and Leggett argued that these scenarios were limiting cases of a more general theory, the so-called BCS-BEC crossover. It was only recently that this crossover phenomenon was observed in rotating trap experiments via the use of a Feshbach scattering resonance. A vortex lattice generated in the molecular BEC phase was observed to persist into the BCS paired phase as the interaction is adiabatically tuned from repulsive to attractive across the Feshbach resonance [8].

While developments such as above have allowed us to enhance our understanding of numerous many body effects, applications of trapped atomic systems to study strong correlation effects such as those responsible for the fractional quantum Hall (FQH) effect are limited. However, there have been theoretical proposals to configure the ultra-cold atomic system in the FQH regime [9, 10]. These proposals are based on rotating the trap at frequencies close to the trapping frequency, ω . This allows us to draw a phase diagram of experimental parameters (rotational frequency Ω and the Feshbach tuning parameter) and identify separate regions corresponding BEC, BCS and FQH as shown in Fig. 1. The region below the dot-dashed line can be quite successfully described at the mean field level and hence we will refer to it as the mean field regime. The vertical shaded area in the mean field regime represents the BCS-BEC crossover where $|k_F a| > 1$ (k_F is the Fermi wave vector and a is the two body scattering length). While the crossover has been experimentally explored only in the mean field regime, on general grounds one would expect it to ex-

ist even in the strongly correlated regime. However, in this regime the correlations will induce modifications to the mean field crossover physics discussed above. This is in part due to the global vortex lattice structure imprinted on the system due to rotations, which gives rise to an emergent length scale corresponding to the vortex radius. While this effect may not be significant in the mean-field-vortex-lattice (hashed) region of the phase diagram, at high rotations, where the number of particles is comparable to the number of vortices, implications to the crossover physics may be drastic. At the same time, very recently FQH states have attracted special attention due to their possible use in topological schemes of quantum computation. While manipulating interaction has remained a major difficulty, possibility of crossover in the FQH regime of atomic ensembles may turn out to be of significant importance in such schemes.

Therefore, the goal of this letter is to investigate the implications of strong correlations due to rotations on the BCS-BEC crossover. We also argue that the crossover is expected to be smooth for s -wave interactions. We will verify this by considering a specific Cooper paired

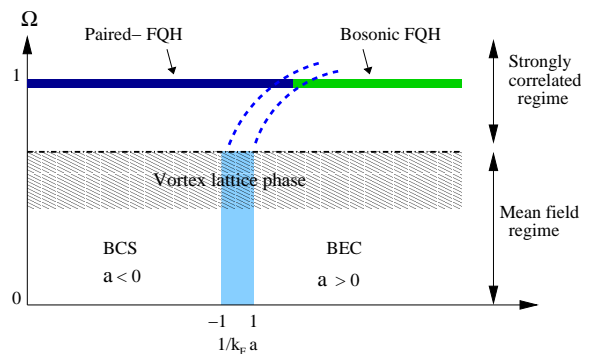


Figure 1: Schematic of the zero temperature phase diagram of the two component Fermi gas with Feshbach tuning parameter $1/k_F a$ along the horizontal axis and rotational frequency Ω (in units of trapping frequency ω) along the vertical axis. a represents the s -wave scattering length.

FQH state on the BCS side. Finally we establish the nature of the strongly correlated state on the BEC side of the crossover by a numerical study of the two-particle correlation functions.

We consider a two component Fermi system consisting of a 50%-50% mixture of fermionic atoms in different hyperfine states represented by $|\uparrow\rangle$ and $|\downarrow\rangle$ confined by a 2D rotating harmonic trap. The Hamiltonian for this system in the FQH regime ($\Omega - \omega \rightarrow 0^-$) in the absence of interactions is given by

$$H = \sum_{\sigma} \int \phi_{\sigma}^{\dagger}(\mathbf{r}) \frac{1}{2m} [\mathbf{p} + \mathbf{A}(\mathbf{r})]^2 \phi_{\sigma}(\mathbf{r}) d\mathbf{r}, \quad (1)$$

with m the mass and $\mathbf{A} = m\omega y\hat{x} - m\omega x\hat{y}$ is analogous to the vector potential associated with the magnetic field in the electronic FQH effect, and ϕ_{σ} is the annihilation operator representing fermionic atoms with spin σ . Now in order to simplify the above Hamiltonian we choose to work in a frame where the vector potential \mathbf{A} is gauged out. This is done by performing the Chern-Simons transformation [11] by attaching gauge field $a_{\alpha}(\mathbf{r}) = -\nu\hbar \sum_{\sigma} \int d^2\mathbf{r}' \epsilon_{\alpha\beta} \rho_{\sigma}(\mathbf{r})(\mathbf{r} - \mathbf{r}')_{\beta} / |\mathbf{r} - \mathbf{r}'|^2$ to each bare particle resulting in

$$H = \int \varphi_{\sigma}^{\dagger}(\mathbf{r}) \frac{1}{2m} [\mathbf{p} + \mathbf{A}(\mathbf{r}) + \mathbf{a}(\mathbf{r})]^2 \varphi_{\sigma}(\mathbf{r}) d\mathbf{r}, \quad (2)$$

where φ_{σ} is the annihilation operator and ρ_{σ} is the density of composite fermions of spin σ and $1/\nu$ is the fraction of the FQH effect. The transformation is such that the average gauge field $\bar{\mathbf{a}}$ [12] cancels the external field, thus $\mathbf{A}(\mathbf{r}) + \mathbf{a}(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \bar{\mathbf{a}}(\mathbf{r}) + \delta\mathbf{a}(\mathbf{r}) = \delta\mathbf{a}(\mathbf{r})$. We can now write the interaction part of the above Hamiltonian as $H_{\text{int}} = (1/2m) \sum_{\sigma} \int d^2\mathbf{r} \varphi_{\sigma}^{\dagger}(\mathbf{r}) [2\mathbf{p}\delta\mathbf{a}(\mathbf{r}) + \delta\mathbf{a}(\mathbf{r})^2] \varphi_{\sigma}(\mathbf{r}) = H_1 + H_2$, where

$$H_1 = -\frac{\nu\hbar}{m} \sum_{\sigma\sigma'} \int \int d^2\mathbf{r} d^2\mathbf{r}' \varphi_{\sigma}^{\dagger}(\mathbf{r}) p_{\alpha} \varphi_{\sigma}(\mathbf{r}) \epsilon_{\alpha\beta} \times \frac{(\mathbf{r} - \mathbf{r}')_{\beta}}{|\mathbf{r} - \mathbf{r}'|^2} \delta\rho_{\sigma'}(\mathbf{r}'), \quad (3)$$

$$H_2 = \frac{\nu^2\hbar^2}{2m} \sum_{\sigma\sigma'\sigma''} \int \int \int d^2\mathbf{r} d^2\mathbf{r}' d^2\mathbf{r}'' \rho_{\sigma}(\mathbf{r}) \times \frac{(\mathbf{r} - \mathbf{r}')_{\beta} (\mathbf{r} - \mathbf{r}'')_{\beta}}{|\mathbf{r} - \mathbf{r}'|^2 |\mathbf{r} - \mathbf{r}''|^2} \delta\rho_{\sigma'}(\mathbf{r}') \delta\rho_{\sigma''}(\mathbf{r}''), \quad (4)$$

and $\rho_{\sigma}(\mathbf{r}) = \varphi_{\sigma}^{\dagger}(\mathbf{r})\varphi_{\sigma}(\mathbf{r})$ and $\delta\rho_{\sigma}(\mathbf{r}) = \rho_{\sigma}(\mathbf{r}) - \bar{\rho}$, with $\bar{\rho}$ the average density. Thus we see that even in the absence of interactions between the bare particles, the Chern-Simons transformation gives rise to a two-body (contained in H_1 and H_2) and three-body contribution (only contained in H_2). However, H_1 represents a coupling of the density to the current $j_{\alpha}(\mathbf{r}) = \varphi_{\sigma}^{\dagger}(\mathbf{r})p_{\alpha}\varphi_{\sigma}(\mathbf{r})$ which is believed to be important only near the boundary of the sample. It can be shown that the two-body part

of H_2 is logarithmic and has been attributed to have important consequences for the formation of pairing in the electronic FQH effect [12]. The three-body part will be neglected in the following, while the two-body part can be taken as an additional contribution to the two-particle atomic interaction.

Let us now assume that there is a Feshbach resonance in the s -wave interaction between atoms in the up and down states. We argue in this letter that this can generate a crossover in the FQH regime between two different types of strongly correlated many-body states. The resonant interaction persists even in the gauge transformed composite particle picture [13]. However, the additional two-body contributions from Eq. (4), which have the character of a repulsive logarithmic potential, can strongly modify the resonance properties. In a single-channel picture of the Feshbach mechanism, which is accurate for broad resonances, the additional potential will shift the bound states in the potential to higher energies, and therefore also shift the crossover region as shown in Fig. 1. In a two-channel picture of the Feshbach mechanism (see Fig. 2), valid for narrow resonances, the same shift of the molecular levels in the so-called closed channels will take place. However, the scattering as a whole could be more involved as the scattering properties of the open channel can be affected as well. Thus the strong correlations induce modifications to the crossover physics and can be treated systematically within a Chern-Simons composite particle picture once the details of the two-body inter atomic potential are known.

We will now consider a specific FQH state to investigate the nature of the crossover. If we are on the weakly attractive side, then in the FQH regime the situation is analogous to the one described by the Haldane-Rezayi (HR) paired wavefunction [14]

$$\Psi_{HR} = \det(\Xi) \prod_{i<j} (z_i - z_j)^2 \prod_{i<j} (\xi_i - \xi_j)^2 \prod_{i,j} (z_i - \xi_j)^2 \times \exp \left[-\sum_k |z_k|^2/4 - \sum_k |\xi_k|^2/4 \right], \quad (5)$$

where z and ξ are scaled in units of harmonic oscillator length l_0 and represent the complex coordinate of the spin up and spin down components respectively, and Ξ represents pairing between the up and down particles. Apart from the pairing part, the above wavefunction for our atomic system is completely justified due to the short range repulsive piece of the interatomic interaction. The anti-symmetrization under exchange of identical spin components is taken care of by the determinant of Ξ .

The form of Ξ will turn out to be important in understanding the nature of the crossover. For this purpose we again work in the Chern-Simons gauge transformed picture with $\nu = 2$ for the HR state of Eq. (5). By virtue of this transformation the fermionic atoms are transformed to free interacting composite particles (bound

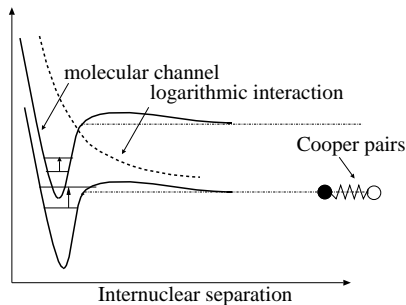


Figure 2: Schematic showing the modification of the interatomic potential by an additional logarithmic contribution coming from the global vortex structure. Figure is not drawn to scale.

state composed of a fermion and 2-vortices) possessing Fermi statistics. We can therefore write the Hamiltonian for the composite fermion (CF) system in the standard BCS form

$$H_{CF} = \sum_{\sigma, \mathbf{k}} (\epsilon_{\mathbf{k}} - \mu) a_{\sigma, \mathbf{k}}^{\dagger} a_{\sigma, \mathbf{k}} + \sum_{\mathbf{q}, \mathbf{k}, \mathbf{k}'} U_{\mathbf{q}, \mathbf{k}, \mathbf{k}'} a_{\uparrow \mathbf{q}/2 + \mathbf{k}}^{\dagger} a_{\downarrow \mathbf{q}/2 - \mathbf{k}}^{\dagger} a_{\downarrow \mathbf{q}/2 - \mathbf{k}'} a_{\uparrow \mathbf{q}/2 + \mathbf{k}'}. \quad (6)$$

Here $a_{\sigma, \mathbf{k}}$ and $a_{\sigma, \mathbf{k}}^{\dagger}$ are annihilation and creation operators for composite fermions with momentum \mathbf{k} and spin σ respectively, μ is the chemical potential and U represents the effective attractive inter composite particle interaction. We diagonalize the above Hamiltonian via Bogoliubov transformations $\gamma_{\mathbf{k}\uparrow} = u_{\mathbf{k}} a_{\mathbf{k}\uparrow} - v_{\mathbf{k}} a_{-\mathbf{k}\downarrow}^{\dagger}$ and $\gamma_{-\mathbf{k}\downarrow}^{\dagger} = u_{\mathbf{k}} a_{-\mathbf{k}\downarrow}^{\dagger} + v_{\mathbf{k}} a_{\mathbf{k}\uparrow}$, $H_{CF} = \sum_{\mathbf{k}, \sigma} E_{\mathbf{k}} \gamma_{\mathbf{k}\sigma}^{\dagger} \gamma_{\mathbf{k}\sigma}$ with $E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta_{\mathbf{k}}^2}$, $u_{\mathbf{k}}^2 = (1/2)(1 + (\epsilon_{\mathbf{k}} - \mu)/E_{\mathbf{k}})$, and $v_{\mathbf{k}}^2 = (1/2)(1 - (\epsilon_{\mathbf{k}} - \mu)/E_{\mathbf{k}})$. If we now define $g_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$, then the configuration space first quantized wavefunction for $2N = N_{\uparrow} + N_{\downarrow}$ composite fermions can be written as $\Psi_{CF}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2N}) = \langle 0 | \hat{\psi}(\mathbf{x}_{2N}) \dots \hat{\psi}(\mathbf{x}_2) \hat{\psi}(\mathbf{x}_1) | G \rangle = \mathcal{A}(\phi_{11}, \phi_{22}, \dots, \phi_{NN'})$, where $|G\rangle$ is the variational second quantized BCS wavefunction, primed and unprimed indexes correspond to different spin components and $\phi_{jj'} = \sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{x}_j - \mathbf{x}_{j'})}$, and the anti-symmetrization is separately performed over up and down spins [15]. Thus one can identify Ξ in Eq. (5) with the product of ϕ 's over different pairs, and the anti-symmetrization operator \mathcal{A} with the determinant.

In the BCS theory $E_{\mathbf{k}}$ and $\Delta_{\mathbf{k}}$ are found self consistently from the gap and the number equation. Here we will not do such a calculation, however only focus on the nature of pairing phases. In the $\mathbf{k} \rightarrow 0$ limit, the weak pairing phase corresponds to $\epsilon_{\mathbf{k}} - \mu < 0$ where $|u_{\mathbf{k}}| \rightarrow 0$ and $|v_{\mathbf{k}}| \rightarrow 1$. Thus the leading behavior of $g_{\mathbf{k}}$ goes as $1/u_{\mathbf{k}} \propto 1/\Delta_{\mathbf{k}}$. However in the BCS phase $\Delta_{\mathbf{k}}$ is significant only in the immediate vicinity of the Fermi surface, $\phi_{jj'}$ acquires a long range exponential tail. Thus it is reasonable to assume an exponentially decay-

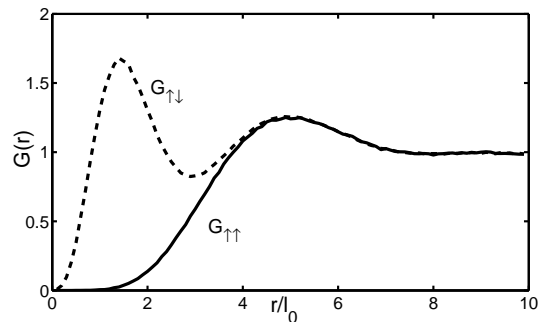


Figure 3: Figure shows the two particle correlation functions $G_{\uparrow\uparrow}(\mathbf{r})$ and $G_{\uparrow\downarrow}(\mathbf{r})$ for $N_{\uparrow} = N_{\downarrow} = 100$ for $\eta = l_0$. For large r/l_0 , $G_{\uparrow\uparrow}(\mathbf{r}) - G_{\uparrow\downarrow}(\mathbf{r}) \rightarrow 0$.

ing form $\phi_{jj'} = e^{-|z_j - z_{j'}|/\eta}$ for the pairing function where $\eta \equiv \hbar v_f / (\pi \Delta_0)$ is the BCS coherence length and v_f is the Fermi velocity.

Now as the strength of interaction is increased by tuning towards resonance resulting in stronger pairing, the gap Δ_0 increases exponentially and one may argue that the BCS description is no longer valid. However as mentioned before we are only concerned about the form of the pairing wavefunction. Let us therefore consider the s -wave T matrix instead which in the $k \rightarrow 0$ limit is a smooth function of a_s , the s -wave scattering length [16]. We would like to point out here that even though FQH effect exists in 2D systems, ultra-cold atomic systems under extreme rotations can be considered to be quasi-2D. Quasi here means that the confinement in the third dimension is strong compared to the remaining two. Therefore the scattering can still be considered to be in 3D justifying the use of the particular T matrix above. Thus we notice that even near the Feshbach resonance, the functional form of the T matrix and hence the gap $\Delta_{\mathbf{k}}$ will remain unchanged hinting a smooth crossover. Thus we will parametrize the crossover by the ratio η/l_0 . This is quite different from the case where the pairing interaction is p -wave where $\Delta_{\mathbf{k}} = \tilde{\Delta}(k_x + ik_y)$. There even if the functional form of the T matrix remains unchanged, the extra phase associated with the $\Delta_{\mathbf{k}}$ can give a totally different behavior for $g_{\mathbf{k}}$ in the strong and weak pairing limits. As argued by Read and Green, the p -wave paired FQH state in the strong and weak pairing limits is separated by a second order phase transition [17].

Having obtained the form of the paired FQH state as a function of η we can directly calculate the two particle correlation function $G(\mathbf{r}_1 - \mathbf{r}_2) = \int \dots \int d^2 \mathbf{r}_3 \dots d^2 \mathbf{r}_N |\Psi_{HR}|^2$ for different values of η by using a metropolis Monte-Carlo algorithm with $N_{\uparrow} = N_{\downarrow} = 100$. In Fig. 3, we plot both $G_{\uparrow\uparrow}(\mathbf{r})$ and $G_{\uparrow\downarrow}(\mathbf{r})$ for $\eta/l_0 = 1$. We see that $G_{\uparrow\downarrow}(\mathbf{r})$ shows a peaked behavior for small r that is absent in $G_{\uparrow\uparrow}(\mathbf{r})$. At the same time for large r , $G_{\uparrow\uparrow}(\mathbf{r}) - G_{\uparrow\downarrow}(\mathbf{r}) \rightarrow 0$ implying the existence of a sum rule special to the HR state valid throughout the region of our current interest.

The crossover behavior is clear from Fig. 4, which shows that as η becomes small compared to l_0 , $G_{\uparrow\uparrow}(\mathbf{r})$ gets modified continuously and tends towards a limiting form. However the most important point to note is that the limiting form of $G_{\uparrow\uparrow}(\mathbf{r})$ is exactly that of the $G(\mathbf{r})$ for the (1/8)-FQH state given by the Laughlin form [18]

$$\Psi_{1/8} = \prod_{i < j} (z_i - z_j)^8 \exp \left[- \sum_k |z_k|^2 / 2 \right]. \quad (7)$$

One way to understand this transition is as follows. In the $\nu = 2$ HR state each composite fermion is associated with two vortices (quanta). Therefore a molecule formed out of two composite fermions will consist of four quantas. Moreover, since the molecule has twice the mass the molecular harmonic oscillator length is $l_0/\sqrt{2}$ and hence eight quanta are required and therefore the fraction is 1/8 for the bosonic Laughlin state. Even though our calculations are for the particular HR state, it is justified to expect that similar behavior will be obtained for other strongly correlated states that occur at slightly lower rotational frequencies between the FQH and the vortex lattice phase.

In conclusion, we have shown that the strong correlations associated with rapid rotations can cause strong modifications to the crossover, for example shift the crossover towards the BEC side relative to the non-rotating case. Using the example of the HR wavefunction we have shown that the crossover is smooth and the paired FQH state of fermions smoothly goes over to 1/8 bosonic FQH state of molecules when one goes across the Feshbach resonance so that $\eta \ll l_0$.

A detailed calculation of the crossover physics of this region will require an elaborate treatment with the effect of rotations included by an effective potential in addition to the actual multichannel interatomic potential. Within such an effective picture Nozières-Schmitt-Rink calculations of the crossover region [19] can be carried out. Also these calculations can be extended to situations with p - and d - wave pairing schemes in ultra-cold Fermi gases. These scenarios while having close resemblance with, for example, the 5/2 FQH effect, will be extremely useful and will be dealt with in a future publication.

At the same time paired FQH states such as 5/2 are known to possess exotic non-abelian quasi particles excitation. While the existence of non-abelian statistics is the basis for topological scheme of implementing quantum logic in a quantum computer, the 5/2 state is known to be computationally non-universal. However, there have been proposals [20] in which this symptom can be remedied by dynamically tuning-in additional non-topological interactions. Dynamic control while hard in the solid state configurations of the FQH effect, controlled transitions between different FQH states like the one discussed here may be extremely useful for implementing such topological schemes.

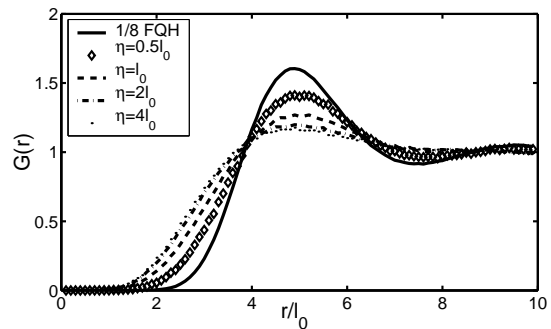


Figure 4: Figure shows the comparison between the correlation function $G_{\uparrow\uparrow}(\mathbf{r})$ in the strong pairing limit and the $G(\mathbf{r})$ for the 1/8 FQH wavefunction Eq. (7).

We thank Anil Shaji and Maikel Goosen for useful discussions. SB acknowledges financial support from the ONR, Contract No. N00014-03-1-0508. SK acknowledges financial support from the NWO.

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