On the rate of convergence to optimality of the LPT rule - postscript

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by

J.B.G. Frenk
A.H.G. Rinnooy Kan

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ON THE RATE OF CONVERGENCE TO OPTIMALITY OF THE LPT RULE - POSTSCRIPT

J.B.G. Frenk
(Technische Hogeschool Eindhoven)

A.H.G. Rinnooy Kan
(Econometric Institute, Erasmus Universiteit Rotterdam)

Abstract

This postscript contains the proofs of two results listed in the paper 'On the rate of convergence to optimality of the LPT rule'.

The purpose of this postscript is to document brief proofs of two results listed in the paper 'On the rate of convergence to optimality of the LPT rule' by the same authors [1]. We refer to [1] for the problem setting and the notation. We shall prove that Theorems 1 and 2 of [1] can be extended to the case that, for $x \in [0,\varepsilon)$ $(\varepsilon>0)$,

$$Lx^a \leq F(x) \leq Ux^a \tag{1}$$

with $0 < L \leq U < \infty$.

<u>Theorem 1a.</u>

$$\lim\ \sup_{n\to\infty}\left(\frac{n}{\log_2 n}\right)^{1/a}\left(\underline{Z}_n^{(m)}(\mathrm{LPT}) - \underline{Z}_n^{(m)}(\mathrm{OPT})\right) < \infty \quad \text{(a.s.)} \tag{2}$$

<u>Proof</u>: As in [1], we consider

$$\underline{D}_n(\alpha) = \max_{1\leq k\leq n}\{\underline{p}_{k:n} - \frac{1}{\alpha}\Sigma_{j=1}^k\ \underline{p}_{j:n}\} \tag{3}$$

and distinguish between the case that
$k \in \{1,\ldots,[\varepsilon n]\}$ and $k \in \{[\varepsilon n]+1,\ldots,n\}$.
With respect to the latter range, we showed in [2] that, for every sequence $d(n) \uparrow \infty$,

$$\lim_{n\to\infty} d(n)\ \max_{[\varepsilon n]<k\leq n}\{p_{k:n} - \frac{1}{\alpha}\Sigma_{j=1}^k p_{j:n}, 0\} = 0 \text{ (a.s.)} \tag{4}$$

With respect to the former range, we have that for every $D > 0$, $\varepsilon \in (0,1)$,

$$\Pr\left\{\max_{1\leq k\leq[\varepsilon n]}\{\underline{p}_{k:n} - \frac{1}{\alpha}\Sigma_{j=1}^k\ \underline{p}_{j:n}\} \geq \left(\frac{D\log_2 n}{n}\right)^{1/a}\right\}$$

$$= \Pr\left\{\max_{1\leq k\leq[\varepsilon n]}\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\Sigma_{j=1}^k\ F^{-1}(\underline{U}_{j:n})\} \geq \left(\frac{D\log_2 n}{n}\right)^{1/a}\right\}$$

$$\leqq \Pr\left\{\underline{U}_{[\varepsilon n]:n} \leq 2\varepsilon,\ \max_{1\leq k\leq[\varepsilon n]}\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\Sigma_{j=1}^k\ F^{-1}(\underline{U}_{j:n})\} \geq \left(\frac{D\log_2 n}{n}\right)^{1/a}\right\}$$

$$+ \Pr\left\{\underline{U}_{[\varepsilon n]:n} > 2\varepsilon\right\}$$

$$\leq \Pr\{\underline{U}_{[\varepsilon n]:n} \leq 2\varepsilon,\ \max_{1 \leq k \leq [\varepsilon n]} \{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha} \Sigma_{j=1}^{k} F^{-1}(\underline{U}_{j:n})\} \geq (\frac{D\log_2 n}{n})^{1/a}\}$$

$$+ e^{-\frac{\varepsilon}{4}n}.$$

Now (1) implies that the first term on the right hand side is bounded by

$$\Pr\{\max_{1 \leq k \leq [\varepsilon n]} \{(\underline{U}_{k:n})^{1/a} - \frac{1}{\alpha^*} \Sigma_{j=1}^{k} (\underline{U}_{j:n})^{1/a}\} \geq (\frac{D^*\log_2 n}{n})^{1/a}\}$$

$$\leq \Pr\{\max_{1 \leq k \leq n} \{(\underline{U}_{k:n})^{1/a} - \frac{1}{\alpha^*} \Sigma_{j=1}^{k} (\underline{U}_{j:n})^{1/a}\} \geq (\frac{D^*\log_2 n}{n})^{1/a}\} \qquad (6)$$

where $\alpha^* = \alpha\bar{U}/\bar{L}$ and $D^* = D/\bar{U}^{1/a}$, with

$$0 < \bar{L}x^{1/a} \leq F^{-1}(x) \leq \bar{U}x^{1/a} < \infty \qquad (7)$$

for x sufficiently small. But with (6), we are essentially back in the situation analysed in [1, Section 2], and we can copy the arguments there and use (4) to prove Theorem 1a.

Theorem 2a. If

$$E\underline{p}^{q(1+b)+1} < \infty, \text{ where } b = 1/a \qquad (8)$$

then

$$\limsup_{n\to\infty} (n+1)^{qb} E((\underline{Z}_n^{(m)}(\mathrm{LPT}) - \underline{Z}_n^{(m)}(\mathrm{OPT}))^q) < \infty \qquad (9)$$

Proof: For every $q > 0$ and $\varepsilon \in (0,1)$,

$$E((\max_{1 \leq k \leq n} \{\underline{p}_{k:n} - \frac{1}{\alpha} \Sigma_{j=1}^{k} \underline{p}_{j:n}\})^q)$$

$$\leq E((\max(\max_{1 \leq k \leq [\varepsilon n]} \{\underline{p}_{k:n} - \frac{1}{\alpha} \Sigma_{j=1}^{k} \underline{p}_{j:n}\}, \underline{p}_{n:n} - \frac{1}{\alpha} \Sigma_{j=1}^{[\varepsilon n]} \underline{p}_{j:n}))^q)$$

$$\leq E((\max_{1 \leq k \leq [\varepsilon n]} \{\underline{p}_{k:n} - \frac{1}{\alpha} \Sigma_{j=1}^{k} \underline{p}_{j:n}\})^q) +$$

$$E((\max\{\underline{p}_{n:n} - \frac{1}{\alpha} \Sigma_{j=1}^{[\varepsilon n]} \underline{p}_{j:n}, 0\})^q) \qquad (10)$$

The first term on the right hand side of (10) can be bounded by

$$E\Big((\max_{1\le k\le[\epsilon n]}\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\Sigma_{j=1}^{k}F^{-1}(\underline{U}_{j:n})\})^{q}I_{\{\underline{U}_{[\epsilon n]:n}\ge 2\epsilon\}}\Big)$$

$$+\ E\Big((\max_{1\le k\le[\epsilon n]}\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\Sigma_{j=1}^{k}F^{-1}(\underline{U}_{j:n})\})^{q}I_{\{\underline{U}_{[\epsilon n]:n}< 2\epsilon\}}\Big)$$

$$\le E\Big((F^{-1}(\underline{U}_{[\epsilon n]}))^{q}I_{\{\underline{U}_{[\epsilon n]:n}\ge 2\epsilon\}}\Big) +$$

$$E\Big((\max_{1\le k\le[\epsilon n]}\{F^{-1}(\underline{U}_{k:n}) - \frac{1}{\alpha}\Sigma_{j=1}^{k}F^{-1}(\underline{U}_{j:n})\}^{q}I_{\{\underline{U}_{[\epsilon n]:n}< 2\epsilon\}}\Big) \qquad (11)$$

As in the previous proof, (1) implies that the second term is $O(n^{-qb})$. The first term can easily be seen to be $O(n^{-qb})$.

The second term on the right hand side of (10) can be bounded by conditioning on $\underline{p}_{n:n}$ being smaller or greater than $\beta n > 0$. In the former case, the conditional expectation can be seen to be bounded by

$$\Pr\Big\{\frac{1}{\alpha}\,\Sigma_{j=1}^{[\epsilon n]}\underline{p}_{j:n}\le \beta n\Big\}.O(n^{q}) \qquad (12)$$

In the latter case, it is bounded by

$$E(\underline{p}_{n:n}^{q}\ I_{\underline{p}_{n:n}\ge\beta n}) \le n\int_{\beta n}^{\infty} y^{q}F(dy) \qquad (13)$$

Now (1) implies that, for an appropriate choice of β, the probability in (12) decreases to 0 exponentially fast. The remaining term (13) then implies the need for (8) to hold for the theorem to be satisfied.

References

[1] J.B.G. Frenk, A.H.G. Rinnooy Kan, 'On the rate of convergence to optimality of the LPT rule', Technical Report, Econometric Institute, Erasmus University Rotterdam.

[2] J.B.G. Frenk, A.H.G. Rinnooy Kan, 'The asymptotic optimality of the LPT rule', Technical Report, Econometric Institute, Erasmus University Rotterdam.