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Abstract

A uniparametric 4-DOF family of perturbed Hamiltonian oscillators in 1:1:1:1 resonance is studied as a generalization for several models for perturbed Keplerian systems. Normalization by Lie-transforms (only first order is considered here) as well as geometric reduction related to the invariants associated to the symmetries is used based on previous work of the authors. A description is given of the lower dimensional relative equilibria in such normalized systems. In addition bifurcations of relative equilibria corresponding to three dimensional tori are studied in some particular cases where we focus on Hamiltonian Hopf bifurcations and bifurcations in the 3-D van der Waals and Zeeman systems.

Contents

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1 Introduction

Continuing previous work [9] and [10], [11] on perturbed isotropic oscillators in four dimensions (other authors refer to them as perturbed harmonic oscillators in 1:1:1:1 resonance), we consider in $\mathbb{R}^8$, the symplectic form $\omega = d\mathbf{Q} \wedge \mathbf{dq}$, and the uniparametric family of Hamiltonian systems defined by

$$\mathcal{H}_\beta(\mathbf{Q}, \mathbf{q}) = \mathcal{H}_2 + \varepsilon \mathcal{H}_6 \tag{1}$$

where

$$\mathcal{H}_2 = \frac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2) \tag{2}$$

is the isotropic oscillator,

$$\mathcal{H}_6(\mathbf{Q}, \mathbf{q}) = \left(q_1^2 + q_2^2 + q_3^2 + q_4^2\right)\left(\beta^2\left(q_1^2 + q_2^2 - q_3^2 - q_4^2\right)^2 + 4\left(q_1^2 + q_2^2\right)\left(q_3^2 + q_4^2\right)\right)$$

and ε is an small parameter $\varepsilon \ll 1$. Without loss of generality ε can be scaled to 1. The ε is only used to indicate a small perturbation. The system defined by Hamiltonian function Eq. (1) has two first integrals in involution given by

$$\Xi = q_1Q_2 - Q_1q_2 + q_3Q_4 - Q_3q_4, \quad L_1 = q_3Q_4 - Q_3q_4 - q_1Q_2 + Q_1q_2, \tag{3}$$

associated to which we have rotational symmetries. We use the same notation as in [10].

Let $\bar{\mathcal{H}}_\beta(\mathbf{Q}, \mathbf{q}) = \mathcal{H}_2 + \varepsilon \bar{\mathcal{H}}_6$ denote the normal form of the system (1) with respect to $\mathcal{H}_2$ which is truncated after terms of order ε (or order 6 if one wishes to put ε equal to 1). The normalized truncated system is an integrable system with integrals $\bar{\mathcal{H}}_\beta$, $\mathcal{H}_2$, Ξ, and L_1. When this system is reduced with respect to the symmetries given by $\mathcal{H}_2$ and Ξ, and one considers the reduced phase space given by $\Xi = 0$, then this reduced phase space is isomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$ and by combining the results in [10] and [5] on sees that, as a symmetric Poisson system, the system is equivalent to a regularized perturbed Keplerian system in normal form. More precisely the system under consideration then is equivalent to the model for the hydrogen atom subject to a generalized van der Waals potential (see [12], [15] and references therein). For $\beta = 0$ this system reduces to the model for the

quadratic Zeeman effect. When $\beta = \sqrt{2}$ we have the Van der Waals sytem [1]. For this reason we propose to name our system as the *generalized Van der Waals 4-D oscillator*. The generalized van der Waals system has, as a perturbed Keplerian system, been subject of many studies concerning bifurcations and integrability (see [12], [15], [16], [14]).

This paper will concentrate on several particular aspects concerning relative equilibria of this generalized 4-D Van der Waals oscillator.

In Section 2 the Hamiltonian (1) is put into normal form with respect to $\mathcal{H}_2$. Considering the truncated normal form a system is obtained that is invariant under the the $\mathbb{S}^1$-actions corresponding to $\mathcal{H}_2$, Ξ, and L_1. These three actions together generate a $\mathbb{T}^3$-action. Reduction with respect to this $\mathbb{T}^3$-action will then give a one-degree-of-freedom system. In the subsequent sections 2, 3, 4 a constructive geometric reduction in stages will be performed. Constructive because the orbit spaces and reduced phase spaces will actually be constructed using invariants and orbit maps. In stages because the reduction will be performed by three consecutive reductions with respect to one $\mathbb{S}^1$-action at a time. In this paper only the first order normalization will be considered. For most results this will be sufficient. Only when degeneracies occur a higher order normalization will be needed. The first reduction is the reduction with respect to the H_2 symmetry. This is a regular reduction and the reduced phase space is isomorphic to $\mathbb{CP}^3$(see [26]). However, we will not use the standard invariants and therefore obtain a slightly different Lie-Poisson system rather than the common one based on the well known invariants of the isotropic oscillator (see [6]).

In Section 3 we carry out a second reduction associated to the $\mathbb{S}^1$-action generated by the Hamiltonian flow defined by Ξ. The resulting orbit space is stratified with reduced phase spaces of which the regular ones are four dimensional and isomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$. However, there are also two singular strata corresponding to two dimensional reduced phase spaces isomorphic to $\mathbb{S}^2$.

In Section 4, we make use of the third reduction, this time with respect to the integral L_1, which reduces our system to a one degree of freedom system on the *thrice reduced phase space*. The regular reduced phase spaces are isomorphic to a 2-sphere which is a symplectic leaf for the Poisson structure on the orbit space. There are singular reduced phase spaces which are homomorphic to a 2-sphere, and which contain one or two singular points. These reduced phase spaces are build from two or three symplectic leafs. Besides these there are singular reduced phase spaces consisting of a single point. (see figure (1) and [10]).

At each reduction step we will compute some stationary points of the reduced system which of course correspond to relative equilibria of our system. It turns out that the computable relative equilibria coincide with those implicated by the symmetry group. In section 5 we will further concentrate on relative equilibria as singular points of the energy-moment map. We will show that the lower dimensional relative equilibria, i.e. those that correspond to invariant $\mathbb{S}^1$ or $\mathbb{T}^2$, are given by the singularity of the moment map for the $\mathbb{T}^3$-symmetry group, and can be described by a moment polytope. The relation between toric fibrations of phase space and moment polytopes has recently also been considered in [29] for systems with less degrees of freedom. The edges and faces of the moment polytope can be considered as a generalization of the idea of a normal mode in systems with two degrees of freedom. At particular points of these sets we may then find bifurcations of

families of relative equilibria corresponding to $\mathbb{T}^3$. The regular $\mathbb{T}^3$ relative equilibria will correspond to stationary points of the trice reduced system on the regular parts of the reduced phase spaces.

The final sections are devoted to studying some particular bifurcations. In section 6 we set $\beta = 0$ obtaining a generalized Zeeman problem, because the model describing the Zeeman effect is obtained by setting $\xi = 0$. In this case the bifurcation of relative equilibria can completely be described. In section 7 we will show that for $\Xi = \xi$, $L_1 = l$, $\xi = l$ Hamiltonian Hopf bifurcation are present for particular values of λ. Similar arguments can be used for $\xi = -l$. In Section 8 we consider the generalized 3D Van der Waals model which is obtained from our model by setting $\xi = 0$. This problem was considered earlier in [13]. Our approach allows us to give a more refined description as the one presented in [13].

2 Normalization and reduction with respect to the oscillator symmetry H_2

2.1 The first reduced phase space

In order to normalize the system defined by (1) with respect to H_2, and reduce the normalized system we compute the invariants for the H_2 action. There are 16 quadratic polynomials in the variables $(\mathbf{Q}, \mathbf{q})$ that generate the space of functions invariant with respect to the action given by the flow of H_2. Explicitly they are

$$\begin{array}{llll}
\pi_1 = Q_1^2 + q_1^2 & \pi_2 = Q_2^2 + q_2^2 & \pi_3 = Q_3^2 + q_3^2 & \pi_4 = Q_4^2 + q_4^2 \\
\pi_5 = Q_1Q_2 + q_1q_2 & \pi_6 = Q_1Q_3 + q_1q_3 & \pi_7 = Q_1Q_4 + q_1q_4 & \pi_8 = Q_2Q_3 + q_2q_3 \\
\pi_9 = Q_2Q_4 + q_2q_4 & \pi_{10} = Q_3Q_4 + q_3q_4 & \pi_{11} = q_1Q_2 - q_2Q_1 & \pi_{12} = q_1Q_3 - q_3Q_1 \\
\pi_{13} = q_1Q_4 - q_4Q_1 & \pi_{14} = q_2Q_3 - q_3Q_2 & \pi_{15} = q_2Q_4 - q_4Q_2 & \pi_{16} = q_3Q_4 - q_4Q_3
\end{array} \tag{4}$$

The invariants are obtained using canonical complex variables (see [9] for more details). Expressing the H_2 normal form up to first order in ε for (1) in those invariants we have

$$\overline{\mathcal{H}} = \mathcal{H}_2 + \varepsilon\overline{\mathcal{H}}_6 \tag{5}$$

where

$$\mathcal{H}_2 = \frac{1}{2}\left(\pi_1 + \pi_2 + \pi_3 + \pi_4\right) = n \tag{6}$$

and

$$\begin{aligned}\overline{\mathcal{H}}_6 = \frac{1}{2}\big[&n\,(1-4\beta^2)(\pi_{15}^2+\pi_{14}^2+\pi_{13}^2+\pi_{12}^2)\\ &+2(\beta^2-1)(\pi_{11}^2(\pi_4+\pi_3)-\pi_{16}^2(\pi_3+\pi_4))\\ &+5n\,(1-\beta^2)(\pi_9^2+\pi_8^2+\pi_7^2+\pi_6^2)+\beta^2 n\,(5n^2-3\pi_{11}^2)+n(\beta^2-4)\pi_{16}^2\big]\end{aligned}$$

The reduction is now performed using the orbit map

$$\rho_\pi : \mathbb{R}^8 \to \mathbb{R}^{16}; (q,Q) \to (\pi_1, \cdots, \pi_{16})\ .$$

The image of this map is the orbit space for the H_2-action. The image of the level surfaces $H_2(q,Q) = n$ under ρ_π are the reduced phase spaces. These reduced phase spaces are isomorphic to $\mathbb{CP}^3$. The normalized Hamiltonian can be expressed in the invariants and therefore naturally lifts to a function on $\mathbb{R}^{16}$, which, on the reduced phase spaces, restricts to the reduced Hamiltonian.

However, in the following we will not use the invariants π_i as is done in [10], but instead use the (K_i, L_j, J_k) invariants as introduced in [9]. That is we replace the generating invariants π_i by the following set of invariants which is actually a linear coordinate transformation on the image of the orbit map. By this change of coordinates the integral are now among the invariants defining the image.

$$\begin{array}{llll} H_2 = \frac{1}{2}(\pi_1+\pi_2+\pi_3+\pi_4) & K_2 = \pi_8-\pi_7 & L_2 = \pi_{12}+\pi_{15} & K_3 = -\pi_6-\pi_9 \\ K_1 = \frac{1}{2}(-\pi_1-\pi_2+\pi_3+\pi_4 & J_3 = \pi_8+\pi_7 & J_7\ = \pi_{12}-\pi_{15} & J_6 = \pi_6-\pi_9 \\ J_1 = \frac{1}{2}(\pi_1-\pi_2-\pi_3+\pi_4) & J_4 = \pi_5+\pi_{10} & L_3\ = \pi_{14}-\pi_{13} & \Xi = \pi_{16}+\pi_{11} \\ J_2 = \frac{1}{2}(\pi_1-\pi_2+\pi_3-\pi_4) & J_5 = \pi_5-\pi_{10} & J_8 = \pi_{14}+\pi_{13} & L_1 = \pi_{16}-\pi_{11} \end{array} \tag{7}$$

The normal form is in these invariants

$$\begin{aligned}\overline{\mathcal{H}}_\Xi = \tfrac{1}{2}\big[n\ &(5\,{K_2}^2+5\,{K_3}^2+2\,{L_1}^2+{L_2}^2+{L_3}^2+\beta^2(5\,{K_1}^2+{L_2}^2+{L_3}^2)\,)\\ &-((4+\beta^2)\,(K_2\,L_2+K_3\,L_3)+(2+3\,\beta^2)K_1L_1)\ \xi\big]\end{aligned} \tag{8}$$

The reduction of the H_2 action may now be performed through the orbit map

$$\rho_{K,L,J} : \mathbb{R}^8 \to \mathbb{R}^{16}; (q,Q) \to (H_2, \cdots, J_8)\ .$$

Note that on the orbit space we have the reduced symmetries due to the reduced actions given by the reduced flows of X_Ξ and X_{L_1}. The orbit space is defined by the following relations (9) and (10). These relations can be obtained by applying (7) to the 36 relation among the generators π_i as given in [9], [10].

$$
\begin{aligned}
K_1\,L_1 + K_2\,L_2 + K_3\,L_3 - \Xi\,n = 0, \quad & J_6\,L_1 + J_4\,L_2 - J_1\,L_3 - J_8\,n = 0 \\
J_3\,L_1 - J_2\,L_2 - J_5\,L_3 + J_7\,n = 0, \quad & J_8\,K_3 + J_2\,L_1 + J_3\,L_2 + J_1\,\Xi = 0 \\
J_7\,K_3 - J_4\,L_1 + J_6\,L_2 - J_5\,\Xi = 0, \quad & J_8\,K_2 - J_5\,L_1 - J_3\,L_3 - J_4\,\Xi = 0 \\
J_7\,K_2 - J_1\,L_1 - J_6\,L_3 - J_2\,\Xi = 0, \quad & J_5\,K_2 - J_2\,K_3 + J_8\,L_1 - J_6\,n = 0 \\
J_1\,K_2 + J_4\,K_3 + J_7\,L_1 + J_3\,n = 0, \quad & J_8\,K_1 + J_5\,L_2 - J_2\,L_3 - J_6\,\Xi = 0 \\
J_7\,K_1 + J_1\,L_2 + J_4\,L_3 + J_3\,\Xi = 0, \quad & J_6\,K_1 + J_4\,K_2 - J_1\,K_3 - J_8\,\Xi = 0 \\
J_5\,K_1 + J_3\,K_3 - J_8\,L_2 + J_4\,n = 0, \quad & J_4\,K_1 - J_6\,K_2 - J_7\,L_3 + J_5\,n = 0 \\
J_3\,K_1 - J_2\,K_2 - J_5\,K_3 + J_7\,\Xi = 0, \quad & J_2\,K_1 + J_3\,K_2 + J_8\,L_3 + J_1\,n = 0 \\
J_1\,K_1 + J_6\,K_3 - J_7\,L_2 + J_2\,n = 0, \quad & J_6\,J_7 + J_3\,J_8 + K_3\,L_2 - K_2\,L_3 = 0 \\
J_5\,J_7 - J_1\,J_8 + K_3\,\Xi - L_3\,n = 0, \quad & J_4\,J_7 - J_2\,J_8 - K_3\,L_1 + K_1\,L_3 = 0 \\
J_3\,J_7 - J_6\,J_8 - K_1\,\Xi + L_1\,n = 0, \quad & J_2\,J_7 + J_4\,J_8 + K_2\,\Xi - L_2\,n = 0 \\
J_1\,J_7 + J_5\,J_8 - K_2\,L_1 + K_1\,L_2 = 0, \quad & J_3\,J_5 + J_1\,J_6 - K_1\,K_3 + L_1\,L_3 = 0 \\
J_3\,J_4 + J_2\,J_6 - L_3\,\Xi + K_3\,n = 0, \quad & J_2\,J_4 - J_1\,J_5 - J_3\,J_6 - J_7\,J_8 = 0 \\
J_1\,J_4 - J_2\,J_5 - K_2\,K_3 + L_2\,L_3 = 0, \quad & J_2\,J_3 - J_4\,J_6 - K_1\,K_2 + L_1\,L_2 = 0 \\
J_1\,J_3 - J_5\,J_6 - L_2\,\Xi + K_2\,n = 0, \quad & J_1\,J_2 + J_4\,J_5 - L_1\,\Xi + K_1\,n = 0,
\end{aligned} \tag{9}
$$

joint with

$$
\begin{aligned}
{K_1}^2 + {K_2}^2 + {K_3}^2 + {L_1}^2 + {L_2}^2 + {L_3}^2 - \Xi^2 - n^2 = 0 \\
{J_7}^2 + {J_8}^2 - {L_1}^2 - {L_2}^2 - {L_3}^2 + \Xi^2 = 0 \\
{J_4}^2 - {J_5}^2 + {J_6}^2 - {J_8}^2 - {K_3}^2 + {L_3}^2 = 0 \\
{J_3}^2 + {J_6}^2 - {K_2}^2 - {K_3}^2 - {L_1}^2 + \Xi^2 = 0 \\
{J_2}^2 + {J_5}^2 - {J_6}^2 + {J_8}^2 + {K_2}^2 + {K_3}^2 + {L_1}^2 - n^2 = 0 \\
{J_1}^2 + {J_5}^2 + {K_3}^2 + {L_1}^2 + {L_2}^2 - n^2 = 0
\end{aligned} \tag{10}
$$

and

$$
H_2 = n. \tag{11}
$$

The last relation $H_2 = n$ defines the symplectic leaves for the induced Poisson structure on this orbit space which are the reduced phase spaces. Let $B_{K,L,J}$ denote the structure matrix for induced Poisson structure$\{\ ,\ \}_{(K,L,J)}$. This matrix is given in table 1.

Note that the motivation for this choice of invariants is that the reduced Ξ invariants are the (K_i, L_j), which makes that the second reduction is easily obtained (see section 3).

2.2 Relative equilibria on $\mathbb{CP}^3$.

A relative equilibrium for a Hamiltonian system with respect to a symmetry group G is an orbit which is a solution of the system and simultaneously an orbit of the group. In our case the relative equilibria are therefore orbits of $X_{\bar{H}}$ as well as orbits of X_{H_2}, where $\bar{H}$ denotes the first order normal form for H. These relative equilibria correspond to

$\{,\}$	H_2	K_1	J_1	J_2	K_2	J_3	J_4	J_5	K_3	J_6	Ξ	L_1	L_2	J_7	L_3	J_8
H_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
K_1	0	0	0	0	$-2L_3$	$-2J_8$	0	0	$2L_2$	$-2J_7$	0	0	$-2K_3$	$2J_6$	$2K_2$	$2J_3$
J_1	0	0	0	0	0	0	$-2L_1$	2Ξ	$-2J_7$	$2L_2$	$-2J_5$	$2J_4$	$-2J_6$	$2K_3$	0	0
J_2	0	0	0	0	$-2J_8$	$-2L_3$	2Ξ	$-2L_1$	0	0	$-2J_4$	$2J_5$	0	0	$2J_3$	$2K_2$
K_2	0	$2L_3$	0	$2J_8$	0	0	$-2J_7$	0	$-2L_1$	0	0	$2K_3$	0	$2J_4$	$-2K_1$	$-2J_2$
J_3	0	$2J_8$	0	$2L_3$	0	0	0	$-2L_2$	0	-2Ξ	$2J_6$	0	$2J_5$	0	$-2J_2$	$-2K_1$
J_4	0	0	$2L_1$	-2Ξ	$2J_7$	0	0	0	0	$2L_3$	$2J_2$	$-2J_1$	0	$-2K_2$	$-2J_6$	0
J_5	0	0	-2Ξ	$2L_1$	0	$2L_2$	0	0	$-2J_8$	0	$2J_1$	$-2J_2$	$-2J_3$	0	0	$2K_3$
K_3	0	$-2L_2$	$2J_7$	0	$2L_1$	0	0	$2J_8$	0	0	0	$-2K_2$	$2K_1$	$-2J_1$	0	$-2J_5$
J_6	0	$2J_7$	$-2L_2$	0	0	2Ξ	$-2L_3$	0	0	0	$-2J_3$	0	$2J_1$	$-2K_1$	$2J_4$	0
Ξ	0	0	$2J_5$	$2J_4$	0	$-2J_6$	$-2J_2$	$-2J_1$	0	$2J_3$	0	0	0	$2J_8$	0	$-2J_7$
L_1	0	0	$-2J_4$	$-2J_5$	$-2K_3$	0	$2J_1$	$2J_2$	$2K_2$	0	0	0	$-2L_3$	0	$2L_2$	0
L_2	0	$2K_3$	$2J_6$	0	0	$-2J_5$	0	$2J_3$	$-2K_1$	$-2J_1$	0	$2L_3$	0	0	$-2L_1$	0
J_7	0	$-2J_6$	$-2K_3$	0	$-2J_4$	0	$2K_2$	0	$2J_1$	$2K_1$	$-2J_8$	0	0	0	0	2Ξ
L_3	0	$-2K_2$	0	$-2J_3$	$2K_1$	$2J_2$	$2J_6$	0	0	$-2J_4$	0	$-2L_2$	$2L_1$	0	0	0
J_8	0	$-2J_3$	0	$-2K_2$	$2J_2$	$2K_1$	0	$-2K_3$	$2J_5$	0	$2J_7$	0	0	-2Ξ	0	0

Table 1: Poisson brackets structure in the K, L, J invariants

stationary points of the reduced system obtained from $X_{\bar{H}}$ after reduction with respect to the X_{H_2}-action, i.e. the action of the one-parameter group given by the flow of X_{H_2}.

The reduced system on $\mathbb{R}^{16}$ is given by the differential equations

$$\frac{dz}{dt} = \{z, \bar{H}(z)\}_{(K,L,J)} =< z, B_{(K,L,J)} D\bar{H}(z) > , \tag{12}$$

with $z = (H_2, K_1, J_1, J_2, K_2, J_3, J_4, J_5, K_3, J_6, \Xi, L_1, L_2, J_7, L_3, J_8)$, which on the reduced phase spaces restrict to a Hamiltonian system.

For computing the H_2 relative equilibria it is sufficient to compute the stationary points of (12) on the reduced phase space, that is, these stationary points should also fulfill relations (9) to (11). Which gives a total of 52 nonlinear equations to be solved for 15 unknowns, taking into account that $H_2 = n$ is given. Now on the reduced phase space we still have the T^2-action induced by the two integrals Ξ and L_1. Let G_Ξ denote the one-parameter group given by the action of X_Ξ. Similarly introduce G_{L_1} and G_{Ξ,L_1}. Let $Fix_{\mathbb{CP}^3}(G_{\Xi,L_1})$ denote subspace of the reduced phase space which is the fixed point space for the actions of Ξ and L_1. Furthermore let $Fix_{\mathbb{CP}^3}(G_\Xi)$ be the fixed point space for the Ξ-action and let $Fix_{\mathbb{CP}^3}(G_{L_1})$ be the fixed point space for the L_1-action. Any stationary point belonging to $Fix_{\mathbb{CP}^3}(G_{\Xi,L_1})$ is an isolated relative equilibrium. A stationary point belonging to either $Fix_{\mathbb{CP}^3}(G_\Xi)$ or $Fix_{\mathbb{CP}^3}(G_{L_1})$ will belong to a circle of stationary points. Finally stationary points belonging to none of these fixed point spaces will be rotated by both actions and therefore fill a two-torus of stationary points.

From the bracket table for the (K, L, J) variables we see that the action of Ξ on $\mathbb{R}^{16}$ consists of four harmonic oscillators in 1:1 resonance simultaneously rotating in the (J_1, J_5), (J_2, J_4), (J_3, J_6) and (J_7, J_8) planes. Similarly the L_1 action on $\mathbb{R}^{16}$ consists of four harmonic oscillators in 1:1 resonance simultaneously rotating in the (J_1, J_4), (J_2, J_5), (K_2, K_3) and (L_2, L_3) planes.

Using these specific actions in (K, L, J) coordinates any stationary point, not equal to the origin, in $Fix_{\mathbb{CP}^3}(G_{\Xi,L_1})$ will have $H_2 = n$, and $J_1 = J_2 = J_3 = J_4 = J_5 = J_6 = J_7 = J_8 = K_2 = K_3 = L_2 = L_3 = 0$. A straightforward computation solving the set of equations using Mathematica or Maple gives the four isolated stationary points $(H_2, \Xi, K_1, L_1) = (n, n, n, n)$, $(n, n, -n, -n)$, $(n, -n, n, -n)$ or $(n, -n, -n, n)$, with all the other variables equal to zero.

For stationary points in $Fix_{\mathbb{CP}^3}(G_{\Xi})$, we need to have $H_2 = n$ and $J_1 = J_2 = J_3 = J_4 = J_5 = J_6 = J_7 = J_8 = 0$. Again solving the equations using Mathematica or Maple gives the isolated stationary points already found and the following invariant sets of stationary points

$$K_2^2 + K_3^2 = n^2\ ,\ L_2^2 + L_3^2 = n^2\ ,\ K_2 = L_2\ ,\ H_2 = \Xi = n\ , \tag{13}$$
$$K_2^2 + K_3^2 = n^2\ ,\ L_2^2 + L_3^2 = n^2\ ,\ K_2 = -L_2\ ,\ H_2 = -\Xi = n\ , \tag{14}$$
$$K_2^2 + K_3^2 = n^2\ ,\ L_2^2 + L_3^2 = n^2\ ,\ K_3 = L_3\ ,\ H_2 = \Xi = n\ , \tag{15}$$
$$K_2^2 + K_3^2 = n^2\ ,\ L_2^2 + L_3^2 = n^2\ ,\ K_3 = -L_3\ ,\ H_2 = -\Xi = n\ . \tag{16}$$

Note that the variables not mentioned are equal to zero. It describes the reduced X_{L_1} orbit with initial value $(H_2, \Xi, K_2, K_3, L_2, L_3) = (n, \pm n, n, 0, n, 0)$ and the other variables zero.

For stationary points in $Fix_{\mathbb{CP}^3}(G_{L_1})$, we need to have $H_2 = n$ and $J_1 = J_2 = K_2 = J_4 = J_5 = K_3 = L_2 = L_3 = 0$. Again solving the equations gives the isolated stationary points already found and the following invariant sets of stationary points

$$J_3^2 + J_6^2 = n^2\ ,\ J_7^2 + J_8^2 = n^2\ ,\ H_2 = n\ ,\ L_1 = \pm n\ . \tag{17}$$

Note that the variables not mentioned are equal to zero. Thus we obtain again two invariant "circles" of stationary points. These are reduced X_{Ξ} orbits with initial values for instance $(H_2, L_1, J_3, J_6, J_7, J_8) = (n, n, n, 0, n, 0)$, $(n, -n, n, 0, n, 0)$ and other variables zero.

Finding invariant stationary sets which are fixed by neither the action of Ξ nor the action of L_1 is much harder because we have to solve the full set of equations. Therefore we will restrict to some examples which do not form an exhaustive list.

Set $H_2 = n, K_2 = K_3 = J_3 = J_6 = L_2 = L_3 = J_7 = J_8 = 0$. Then we obtain the invariant T^2 with a basis given by the "circles"

$$J_1^2 + J_5^2 = n^2\ ,\ J_2^2 + J_4^2 = n^2\ , \text{and } J_1^2 + J_4^2 = n^2\ ,\ J_2^2 + J_5^2 = n^2\ ,\ H_2 = n\ ,\ K_1 = \pm n\ . \tag{18}$$

This set is obtained by rotating the stationary point $(H_2, K_1, J_1, J_2, J_4, J_5) = (n, \pm n, n, n, 0, 0)$, (other variables zero),with the Ξ and L_1 action.

When we set $H_2 = n$, $K_1 = L_1 = \Xi = K_2 = K_3 = J_3 = J_6 = 0$. Then we obtain the torus with basic "circles"

$$\begin{aligned} &J_7^2 + J_8^2 = n^2 \ , J_1^2 + J_5^2 = n^2 - J_7^2 \ , \ J_2^2 + J_4^2 = n^2 \ , \text{and} \\ &L_2^2 + L_3^2 = n^2 \ , \ J_1^2 + J_4^2 = n^2 - L_2^2 \ , \ J_2^2 + J_5^2 = n^2 \ , \ H_2 = n \ . \end{aligned} \tag{19}$$

So the precise choice of basic "circles" depends on the initial stationary point. The precise nature of this set is still to be investigated.

Similar sets can be found when other combinations of variables are set equal to zero.

3 Further reduction with respect to the rotational symmetry Ξ

3.1 The second reduced phase space $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$

The rotational symmetry Ξ reduces $\mathbb{CP}^3$ to a variety made of strata of dimension 4, and two strata of dimension 2. In order to see that, we fix $\Xi = \xi$ and consider $\mathbb{CP}^3/S^1$ where S^1 is the action generated by the symmetry Ξ. We perform this reduction by expressing the second reduced system in the 8 invariants defined by this action

$$\begin{aligned} H_2 &= \tfrac{1}{2}(\pi_1 + \pi_2 + \pi_3 + \pi_4) & \Xi &= \pi_{16} + \pi_{11} \\ K_1 &= \tfrac{1}{2}(\pi_3 + \pi_4 - \pi_1 - \pi_2) & L_1 &= \pi_{16} - \pi_{11} \\ K_2 &= \pi_8 - \pi_7 & L_2 &= \pi_{12} + \pi_{15} \\ K_3 &= -(\pi_6 + \pi_9) & L_3 &= \pi_{14} - \pi_{13} \end{aligned}$$

This, in turn, leads us to the orbit mapping

$$\rho_2 : \mathbb{R}^{16} \to \mathbb{R}^8; \ \ (\pi_1, \cdots, \pi_{16}) \to (K_1, K_2, K_3, L_1, L_2, L_3, \mathcal{H}_2, \Xi)$$

The orbit space $\rho_2(\mathbb{CP}^3)$ is defined as a six dimensional algebraic variety in $\mathbb{R}^8$ by the two relations

$$K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 = H_2^2 + \Xi^2 \ , \ K_1L_1 + K_2L_2 + K_3L_3 = H_2\Xi \ . \tag{20}$$

the reduced phase spaces are obtained by setting

$$\Xi = \xi \ , \ H_2 = n \ .$$

Thus there are $2+2$ relations defining the second reduced space with $n \geq 0$. The reduced phase spaces can now be represented as, in general, four dimensional varieties in $\mathbb{R}^6$, with the variables $(K_1, K_2, K_3, L_1, L_2, L_3)$, given by the relations

$$\begin{aligned} K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 &= n^2 + \xi^2, \\ K_1L_1 + K_2L_2 + K_3L_3 &= n\xi. \end{aligned} \tag{21}$$

$\{\cdot,\cdot\}_2$	K_1	K_2	K_3	L_1	L_2	L_3
K_1	0	$-2L_3$	$2L_2$	0	$-2K_3$	$2K_2$
K_2	$2L_3$	0	$-2L_1$	$2K_3$	0	$-2K_1$
K_3	$-2L_2$	$2L_1$	0	$-2K_2$	$2K_1$	0
L_1	0	$-2K_3$	$2K_2$	0	$-2L_3$	$2L_2$
L_2	$2K_3$	0	$-2K_1$	$2L_3$	0	$-2L_1$
L_3	$-2K_2$	$2K_1$	0	$-2L_2$	$2L_1$	0

Table 2: Bracket relations for the (K, L) variables.

Introducing a new set of coordinates $(\sigma_1, \sigma_2, \sigma_3, \delta_1, \delta_2, \delta_3)$ by the relations $\sigma_i = K_i + L_i$ and $\delta_i = L_i - K_i$ with $i = 1, 2, 3$ we obtain

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = (n+\xi)^2$$
$$\delta_1^2 + \delta_2^2 + \delta_3^2 = (n-\xi)^2$$

Thus (21) is isomorphic to $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$. Note that when $\xi = 0$ the second reduced space is isomorphic to $\mathbb{S}^2_n \times \mathbb{S}^2_n$. This space, as we know, may be obtained when normalizing perturbed Keplerian systems by immersion in a space of dimension 4 by means of regularization by the Kustaanheimo-Stiefel transformation [25] or by Moser regularization [26]. When $n = \xi$ or $n = -\xi$ we obtain singular symplectic leaves of dimension two in stead of four dimensional reduced phase spaces.

Note that in the following we will refer to the second reduced phase spaces as $S^2_{n+\xi} \times S^2_{n-\xi}$ although strictly speaking in (K, L) coordinates the reduced phase spaces are only isomorphic to this representation.

Brackets for the invariants $(K_1, K_2, K_3, L_1, L_2, L_3)$ defining the second reduced phase spaces $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ are given in table (2). Moreover the second reduced Hamiltonian up to first order, modulo a constant takes the form

$$\begin{aligned}\overline{\mathcal{H}}_\Xi =& \frac{1}{2}\Big[n\,\left(5\,{K_2}^2 + 5\,{K_3}^2 + 2\,{L_1}^2 + {L_2}^2 + {L_3}^2 + \beta^2\,(5\,{K_1}^2 + {L_2}^2 + {L_3}^2)\right) \\ & - \left((4+\beta^2)\,(K_2\,L_2 + K_3\,L_3) + (2+3\,\beta^2)K_1L_1\right)\,\xi\Big] \end{aligned} \tag{22}$$

Thus $(\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}, \{\cdot,\cdot\}_2, \overline{\mathcal{H}}_\Xi)$ is a Lie-Poisson system. Identifying $\mathbb{R}^6$ with $\mathfrak{so}(4)^*$, the linear coordinate change from (K, L) to (ρ, δ) is precisely the Lie algebra isomorphism between $\mathfrak{so}(4)^*$ and $\mathfrak{so}(3)^* \times \mathfrak{so}(3)^*$. The regular reduced phase spaces can be considered as co-adjoint orbits of $SO(3) \times SO(3)$ on the dual of its Lie algebra. The symplectic form is the standard Lie Poisson structure [30]. The dynamics in $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ is given by the following set of equations

$$\frac{d\mathbf{K}}{dt} = \{\mathbf{K}, \overline{\mathcal{H}}_\Xi\}_2 \tag{23}$$

with $\mathbf{K}$ =$(K_1, K_2, K_3, L_1, L_2, L_3)$. Explicitly, the equations (23) are

$$\begin{cases} \dfrac{dK_1}{dt} = 2n(4-\lambda)\,(L_2K_3 - L_3K_2)\,, \\ \dfrac{dK_2}{dt} = 2n(4\lambda-1)L_3K_1 + 2\xi(1-\lambda)(L_3L_1 + K_3K_1) - 6nL_1K_3, \\ \dfrac{dK_3}{dt} = 2\,n(1-4\lambda)L_2K_1 - 2(1-\lambda)\,\xi L_2L_1 + 6\,nL_1K_2 + 2(\lambda-1)\xi K_2K_1, \\ \dfrac{dL_1}{dt} = 0, \\ \dfrac{dL_2}{dt} = 2\,(1-\lambda)\,(-5\,nK_1K_3 + \xi K_1L_3 + \xi K_3L_1 + nL_3L_1)\,, \\ \dfrac{dL_3}{dt} = -2\,(1-\lambda)\,(\,\xi K_1L_2 + \,\xi K_2L_1 + nL_2L_1 - 5\,nK_1K_2)\,, \end{cases} \tag{24}$$

where, in what follows, $\lambda = \beta^2$.

Remark 3.1 Note that when $\lambda = 1$ the normal form approximation has L_1 and L_2 as additional integrals. When $\lambda = 4$ one obtains K_1 is an additional integral for the normal form approximation. As we will see, in the latter case, the fully reduced Hamiltonian is just a function of K_1.

3.2 Relative equilibria in $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$

Equilibria $\mathbf{z}_e = (K_1^e, K_2^e, K_3^e, L_1^e, L_2^e, L_3^e)$ are obtained using the Lagrange multiplier procedure to determine the points where the reduced Hamiltonian is tangent to the reduced phase space. We get the equations

$$\mathbf{d}\overline{\mathcal{H}}_\Xi + \alpha_1\,\mathbf{d}f_1 + \alpha_2\,\mathbf{d}f_2 = \mathbf{0}\ ,\ f_1 = 0\ ,\ f_2 = 0\ . \tag{25}$$

where

$$\begin{aligned} f_1 &= K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 - (n^2 + \xi^2)\ , \\ f_2 &= K_1L_1 + K_2L_2 + K_3L_3 - n\xi\ , \end{aligned} \tag{26}$$

and $\alpha_1,\ \alpha_2 \in \mathbb{R}$. The search for roots in that system leads us to the following set of solutions

$$\begin{aligned} &\mathbf{z}_e = (n,0,0,\xi,0,0)\ ,\ \mathbf{z}_e = (-n,0,0,-\xi,0,0)\ , \\ &\mathbf{z}_e = (\xi,0,0,n,0,0)\ ,\ \mathbf{z}_e = (-\xi,0,0,-n,0,0)\ . \end{aligned} \tag{27}$$

In the $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ representation these points correspond to the poles on the σ_1- and δ_1-axis.

Furthermore we obtain the following 1-dimensional manifolds of stationary solutions

$$\begin{aligned} &K_1 = 0\ ,\ L_1 = 0\ ,\ nK_2 = \xi L_2\ ,\ nK_3 = \xi L_3\ ,\ L_2^2 + L_3^2 = n^2 \\ &\text{and} \\ &K_1 = 0\ ,;L_1 = 0\ ,\ \xi K_2 = nL_2\ ,\ \xi K_3 = nL_3\ ,\ L_2^2 + L_3^2 = \xi^2\ . \end{aligned} \tag{28}$$

When, on the second set, we pick one of the points

$$\mathbf{z}_e = (0, 0, \pm n, 0, 0, \pm\xi)$$

and we make the reconstruction of this relative equilibrium to the full phase space, we obtain a 2-torus, consequently reconstruction of the full set given by the second equation in (28) gives a 3-torus of periodic orbits.

For the very particular case $\xi = 0$, the above relative equilibria in 27 correspond to the poles on $\mathbb{S}^2_n \times \mathbb{S}^2_n$ on the σ_1- and δ_1-axis. The relative equilibria in 28 collapse onto one set corresponding to the equators on $\mathbb{S}^2_n \times \mathbb{S}^2_n$ around the σ_1- and δ_1-axis. The solutions are

$$\begin{array}{ll} \mathbf{z}_e = (\pm n, 0, 0, 0, 0, 0)\ , & \mathbf{z}_e = (0, 0, 0, \pm n, 0, 0) \text{ and} \\ \{(0, 0, 0, 0, L_2, L_3) | L_2^2 + L_3^2 = n^2\}\ , & \{(0, 0, 0, 0, K_2, K_3) | K_2^2 + K_3^2 = n^2\}\ . \end{array}$$

The above computed relative equilibria are all induced by the symmetry of the problem. There are relative equilibria that involve the specific Hamiltonian, these can be computed using more complicated numerical calculations. These solutions then may depend on the parameter λ.

4 Further reduction with respect to the rotational symmetry L_1.

4.1 The third reduced space $V_{n\,\xi\,l}$

As we said in the Introduction the process of reduction of the 4-D isotropic oscillator with the symmetries Ξ and L_1 has already been reported in [9] and [10]. In order to make the paper self contained, we will reproduce the main aspects contained in those references.

To further reduce from $\mathbb{S}^2_{n+\xi} \times \mathbb{S}^2_{n-\xi}$ to $V_{n\,\xi\,l}$ one divides out the S^1-action generated by L_1 and fixes $L_1 = l$. The 8 invariants for the L_1 action on $\mathbb{R}^8$ are

$$\begin{array}{l} \mathcal{H}_2\ ,\ \Xi\ ,\ L_1\ ,\ K = K_1\ , \\ M = \frac{1}{2}\left(K_2^2 + K_3^2\right) + \frac{1}{2}\left(L_2^2 + L_3^2\right)\ ,\ N = \frac{1}{2}\left(K_2^2 + K_3^2\right) - \frac{1}{2}\left(L_2^2 + L_3^2\right)\ , \\ Z = K_2 L_2 + K_3 L_3\ ,\ S = K_2 L_3 - K_3 L_2\ . \end{array} \tag{29}$$

There are $3 + 3$ relations defining the third reduced phase space

$$\begin{array}{l} K^2 + L_1^2 + 2M = \mathcal{H}_2^2 + \Xi^2 \\ K L_1 + Z = \mathcal{H}_2 \Xi \\ M^2 - N^2 = Z^2 + S^2 \\ L_1 = l,\ \Xi = \xi,\ \mathcal{H}_2 = n \end{array}$$

Consequently we may represent the third reduced phase space $V_{n\,\xi\,l}$ in (K, N, S)-space by the equation

$$(n^2 + \xi^2 - l^2 - K^2)^2 - 4(n\xi - lK)^2 = 4N^2 + 4S^2. \tag{30}$$

If we set

$$f(K) = (n^2+\xi^2-l^2-K^2)^2 - 4(n\xi - lK)^2 = [(n+\xi)^2-(K+l)^2][(n-\xi)^2-(K-l)^2]$$

then our reduced phase space is a surface of revolution obtained by rotating $\phi(K) = \sqrt{f(K)}$ around the K-axis.

Remark 4.1 The reduced phase spaces as well as the Hamiltonian are invariant (see Eq. (32)) under the discrete symmetry $S \to -S$. Thus all critical points of the reduced Hamiltonian on the reduced phase space will be in the plane $S = 0$

The shape of the reduced phase space is determined by the positive part of $f(K)$. The polynomial $f(K)$ can be written as

$$f(K) = (K+n+\xi+l)(K-n-\xi+l)(K-n+\xi-l)(K+n-\xi-l),$$

thus, the four zeroes of $f(K)$ are given by

$$K_1 = -l-n-\xi\ ,\ K_2 = l+n-\xi\ ,\ K_3 = l-n+\xi\ ,\ K_4 = -l+n+\xi\ .$$

So $f(K)$ is positive (or zero) in the subsequent intervals of K:

$$\begin{array}{llll} l<\xi\,,\,-l<\xi & K_1<K_3<K_2<K_4 & K\in[K_3,K_2] \\ l>\xi\,,\,-l<\xi & K_1<K_3<K_4<K_2 & K\in[K_3,K_4] \\ l<\xi\,,\,-l>\xi & K_3<K_1<K_2<K_4 & K\in[K_1,K_2] \\ l>\xi\,,\,-l>\xi & K_3<K_1<K_4<K_2 & K\in[K_1,K_4] \end{array} \tag{31}$$

When we have a simple root of $f(K)$ which belongs to one of the above intervals, we have that the intersection of the reduced phase space with the K-axis is smooth. $f(K)$ has four different roots in the following two cases: (i) $l \neq \xi$ and $\xi, l \neq 0$; (ii) $l \neq \xi$ and $\xi = 0$ or $l = 0$. In these cases the reduced phase space is diffeomorphic to a sphere. A point on this sphere corresponds to a three-torus in original phase space.

To find the the double zeroes of $f(K)$ we compute the discriminant of $f(K) = 0$. It is

$$(l-n)^2(l+n)^2(l-\xi)^2(l+\xi)^2(n-\xi)^2(n+\xi)^2\ .$$

Thus there are double zeroes at $l = \pm n$, $l = \pm\xi$ and $\xi = \pm n$. If we have just one double zero the reduced phase space is a sphere with one cone-like singularity at the intersection point given by the double root ($l = \pm\xi \neq 0$). If we have two double zeroes the reduced phase space is a sphere with two cone-like singularities at the intersection points given by the double roots ($l = \xi = 0$). In the other cases the reduced phase space is just one singular point. The singular points correspond to two-tori in original phase space.

Triple zeroes occur when $|l| = |\xi| = n$. The reduced phase space is just a point which corresponds to a circle in original phase space.

Quadruple zeroes only occur when $l = n = \xi = 0$, which corresponds to the origin in original phase space and is a stationary point. See figure (1). More details on this analysis can be found in [9].

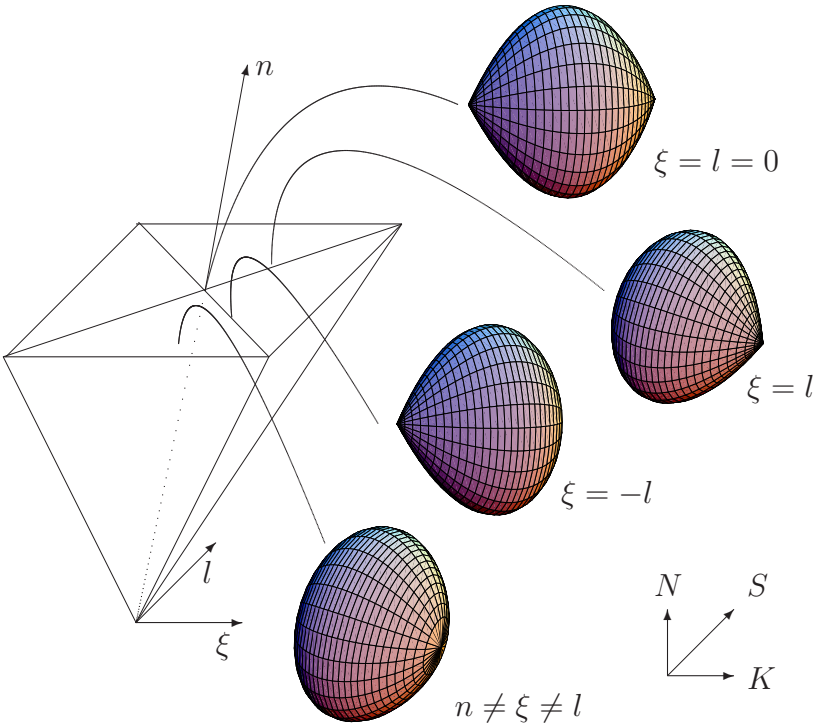

Figure 1: The thrice reduced phase space over the parameter space. K is the symmetry axis of each surface.(See [10])

$\{\cdot,\cdot\}_3$	M	N	Z	S	K	L_1
M	0	$4KS$	0	$-4KN$	0	0
N	$-4KS$	0	$-4L_1S$	$-4(KM-L_1Z)$	$4S$	0
Z	0	$4L_1S$	0	$-4L_1N$	0	0
S	$4KN$	$4(KM-L_1Z)$	$4L_1N$	0	$-4N$	0
K	0	$-4S$	0	$4N$	0	0
L_1	0	0	0	0	0	0

Table 3: Bracket relations for the M, N, Z, S, K, L_1 variables.

The cone-like singularities of the reduced phase space are candidates for the occurrence of Hamiltonian Hopf bifurcations. Hamiltonian Hopf bifurcations might therefore occur along the lines $l = \xi$ and $l = -\xi$ in parameter space.

The Poisson structure for M, N, Z, S, K, L_1 is given

The Hamiltonian on the third reduced phase space is

$$\begin{aligned}\overline{\mathcal{H}}_{\Xi,L_1} =& \frac{3n}{4}(3\lambda-2)K^2 + \xi l(1-\lambda)K + \frac{n}{2}(4-\lambda)N \\ &+ n^3(\frac{3}{2}+\frac{\lambda}{4}) - \left(l^2+\xi^2\right)(\frac{\lambda}{2}+1)\frac{n}{2}\end{aligned} \tag{32}$$

In (K, N, S)-space the energy surfaces are parabolic cylinders. The intersection with the reduced phase space give the trajectories of the reduced system. Tangency with the reduce phase spaces gives relative equilibria that generically will correspond to three dimensional tori in the original phase space.

Thus $(V_{n\,\xi\,l}, \{\cdot,\cdot\}_3, \overline{\mathcal{H}}_{\Xi,L_1})$ is a Lie-Poisson system. The corresponding dynamics is given by

$$\begin{aligned}\frac{dK}{dt} &= 2n(\lambda-4)\,S, \\ \frac{dN}{dt} &= 2[3n(3\lambda-2)K + 2\xi l(1-\lambda)]\,S, \\ \frac{dS}{dt} &= n(\lambda-4)(K^3 - (\xi^2+l^2+n^2)K) - (3\lambda-2)[6nKN + 4\xi l(\lambda-1)N + 2ln^2\xi].\end{aligned} \tag{33}$$

Remark 4.2 Note that for $\lambda = 2/3$, the function $\mathcal{H}$ is linear in the variable space (K, N, S). Likewise for $\lambda = 1$, we note that $\mathcal{H}$, modulo constants, is independent of ξ and l. Moreover when $\lambda = 4$, $\mathcal{H}$ is only a function of K.

Remark 4.3 It is easy to see that this system can be integrated by means of elliptic functions, but we do not plan to follow that path. We intend to classify the different types of flows as functions of the integrals and parameter of the system. Only then we will be ready for the integration of a specific initial value problem.

4.2 Equilibria in the thrice reduced space $V_{n\,\xi\,l}$

In order to search for equilibria we have to study the tangencies of (32) with the third reduced space $V_{n,\xi,l}$. This leads us to the following set of equations

$$\mathbf{d}\mathcal{H} + \alpha_1 \mathbf{d}g_1 = 0, \qquad g_1 = 0, \qquad \alpha_1 \in \mathbb{R} \tag{34}$$

where

$$g_1 = (n^2+\xi^2-l^2-K^2)^2 - 4(n\xi - lK)^2 - 4(N^2+S^2).$$

After some computations we arrive to the following equation

$$n(\lambda-4)(K^3 - (\xi^2+l^2+n^2)K + 2l\xi n) \pm \sqrt{f(K)}(4\xi l(\lambda-1) + 6n(2-3\lambda)K) = 0 \tag{35}$$

which, after some manipulations may be written as sixth degree polynomial

$$p(K) = a_0K^6 + a_1K^5 + a_2K^4 + a_3K^3 + a_4K^2 + a_5K + a_6 = 0 \tag{36}$$

with coefficients given by

$$a_0 = -20n^2\,(4\lambda - 1)\,(\lambda - 1)$$

$$a_1 = 12ln\xi\,(3\lambda - 2)\,(\lambda - 1)$$

$$a_2 = 4\,(\lambda - 1)\left(40l^2n^2\lambda + 40\xi^2n^2\lambda + 40n^4\lambda - \xi^2l^2\lambda - 10\xi^2n^2 + \xi^2l^2 - 10n^4 - 10n^2l^2\right)$$

$$a_3 = -4ln\xi(-238n^2\lambda - 3\xi^2\lambda + 68n^2 + 18\xi^2\lambda^2 - 30l^2\lambda + 179n^2\lambda^2 + 12\xi^2 + 12l^2 + 18l^2\lambda^2)$$

$$\begin{aligned} a_4 = {} & -20n^6 + 460\xi^2n^2\lambda^2l^2 + 100n^6\lambda - 728\xi^2n^2l^2\lambda - 232\xi^2n^4\lambda + 104\xi^2n^4 - 20\xi^4n^2 \\ & - 232l^2n^4\lambda - 80\xi^4n^2\lambda^2 + 164l^2n^4\lambda^2 + 104l^2n^4 - 80l^4n^2\lambda^2 \\ & + 100l^4n^2\lambda + 100\xi^4n^2\lambda + 8\xi^4l^2\lambda^2 - 16\xi^4l^2\lambda + 304\xi^2n^2l^2 \\ & - 16l^4\xi^2\lambda + 8\xi^4l^2 + 8l^4\xi^2\lambda^2 + 164\xi^2n^4\lambda^2 - 20l^4n^2 - 80n^6\lambda^2 + 8l^4\xi^2 \end{aligned}$$

$$\begin{aligned} a_5 = {} & 4ln\xi\,\big(38l^2n^2\lambda + 9\xi^4\lambda^2 - 10n^4 + 38\xi^2n^2\lambda - 15l^4\lambda + 9l^4\lambda^2 \\ & -28n^2l^2 + 46\xi^2l^2\lambda - 19\xi^2n^2\lambda^2 - 7n^4\lambda - 15\xi^4\lambda - 20\xi^2l^2 \\ & -26\xi^2l^2\lambda^2 + 6\xi^4 - 28\xi^2n^2 + 8n^4\lambda^2 + 6l^4 - 19l^2n^2\lambda^2\big) \end{aligned}$$

$$\begin{aligned} a_6 = {} & 4\,\xi^2l^2\,\big(15n^4 + 2\xi^2l^2 + 2\xi^2l^2\lambda^2 - 4\xi^2l^2\lambda - 6n^4\lambda - \xi^4\lambda^2 + 2\xi^4\lambda \\ & - 4\xi^2n^2\lambda + 2l^2n^2\lambda^2 - 4l^2n^2\lambda + 2\xi^2n^2 - \xi^4 + l^4 + 2n^2l^2 \\ & +2l^4\lambda + 2\xi^2n^2\lambda^2 - l^4\lambda^2\big) \end{aligned}$$

The general study of thrice reduced system, with four parameters: three integrals (n, ξ, l) and a physical parameter λ, will not be done here. However in the next section we will derive some general results concerning the relative equilibria in systems with symmetries generated by H_2, Ξ, and L_1. As for the Hamiltonian system under consideration we will satisfy ourselves with some particular scenarios. More precisely, in what follows we will consider in some detail three situations:

$$(i)\quad \lambda = 0, \qquad (ii)\quad \xi = l, \qquad (iii)\qquad \xi = 0$$

The first situation is a generalization of the Zeeman model in four dimensions. The second corresponds to the case where the thrice reduced space has singular points; we will show that there are Hamiltonian Hopf bifurcations related to those points. The third case is the generalized Van der Waals model in three dimensions. We will recover known results as well as clarify some aspects of the polar case of this system.

5 Relative equilibria and moment polytopes

In this section we will consider the relation between the relative equilibria and moment polytopes. This relation has recently also been observed by others (see [29]).

Recall that a relative equilibrium for a Hamiltonian system with respect to a symmetry group G is an orbit which is a solution of the system and simultaneously an orbit of the symmetry group. In the following we will use the equivalent definition that a relative equilibrium is a critical point of the energy-momentum map. For the normalized truncated system $X_{\bar{\mathcal{H}}}$ the energy-momentum map is

$$\mathcal{EM} : \mathbb{R}^8 \to \mathbb{R}^4; (q, Q) \to (\bar{\mathcal{H}}, H_2, \Xi, L_1) \ .$$

In this case the full symmetry group is G_{H_2,Ξ,L_1}, the group generated by the actions of H_2, Ξ, and L_1. Because the H_2 level surfaces are compact we know that, according to the Arnold-Liouville theorem, $\mathcal{EM}^{-1}(p)$, for p a regular value, will be a $\mathbb{T}^4$ or a disjoint union of $\mathbb{T}^4$. When p is not a regular value the counter images $\mathcal{EM}^{-1}(p)$ will be tori of lower dimension, which are group orbits and invariant under the dynamics of the system $X_{\bar{\mathcal{H}}}$. Thus for each point z_e on such a torus there exists a one-parameter subgroup g_t of G_{H_2,Ξ,L_1} such that $z_e(t) = g_t \cdot z_e$ is an orbit of $X_{\bar{\mathcal{H}}}$. That is z_e is a relative equilibrium (see [3]). Consider the moment map $\mathcal{J}_1 : \mathbb{CP}^3 \to (\Xi, L_1, H_2) \subset \mathbb{R}^3$ for the torus action G_{H_2,Ξ,L_1}. It is clear that any critical point for this moment map is also a critical point for $\mathcal{EM}$. Thus the critical points of $\mathcal{J}_1$ will describe relative equilibria for any Hamiltonian system with these symmetries.

According to a theorem by Atiyah [2] and Guillemin and Sternberg [18, 19] the image for a moment map for a torus action is a convex polytope.

Considering 4-DOF families of perturbed Hamiltonian oscillators in fourfold 1:1 resonance with integrals Ξ and L_1 we may, after H_2-reduction, introduce the moment map $\mathcal{J}_3 : \mathbb{CP}^3 \to (\Xi, L_1, K) \subset \mathbb{R}^3$. Because we have the inequalities

$$H_2 \pm \Xi = \tfrac{1}{2} \left((q_1 \pm Q_2)^2 + (q_2 \mp Q_1)^2 + (q_3 \pm Q_4)^2 + (q_4 \mp Q_3)^2 \right) \geq 0 \ ,$$

and

$$H_2 \pm L_1 = \tfrac{1}{2} \left((q_1 \mp Q_2)^2 + (q_2 \pm Q_1)^2 + (q_3 \pm Q_4)^2 + (q_4 \mp Q_3)^2 \right) \geq 0 \ .$$

and the for the thrice reduced phase space

$$\begin{array}{llll} L_1 < \Xi \, , \, -L_1 < \Xi & K_1 < K_3 < K_2 < K_4 & K \in [K_3, K_2] \\ L_1 > \Xi \, , \, -L_1 < \Xi & K_1 < K_3 < K_4 < K_2 & K \in [K_3, K_4] \\ L_1 < \Xi \, , \, -L_1 > \Xi & K_3 < K_1 < K_2 < K_4 & K \in [K_1, K_2] \\ L_1 > \Xi \, , \, -L_1 > \Xi & K_3 < K_1 < K_4 < K_2 & K \in [K_1, K_4] \end{array} \tag{37}$$

with

$$K_1 = -L - 1 - n - \Xi \ , \ K_2 = L_1 + n - \Xi \ , \ K_3 = L_1 - n + \Xi \ , \ K_4 = -L_1 + n + \Xi \ .$$

we obtain as the Delzant [7] polytope for the moment map $\mathcal{J}_3$ the tetrahedron given in figure (2).

The critical values of this map correspond to the vertices, edges, and faces of the tetrahedron. Note that each vertical line in this tetrahedron represents a reduced phase space. The K-action itself does not have any dynamical meaning for our system. Projecting in

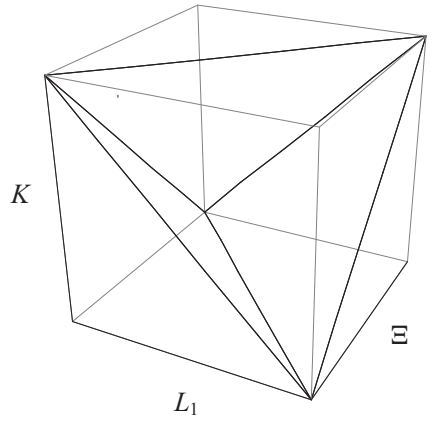

Figure 2: The image of $\mathcal{J}_3$.

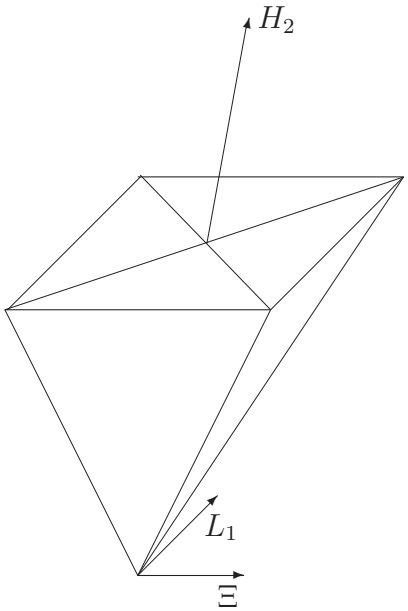

Figure 3: The image of $\mathcal{J}_1$.

the K-direction we obtain a square with corners (n,n), $(n,-n)$, $(-n,-n)$, and $(-n,n)$ as the image of the T^2 moment map $\mathcal{J}_2 : \mathbb{CP}^3 \to (\Xi, L_1) \subset \mathbb{R}^2$. The critical values of this moment map are the vertices, edges, and diagonals of the square. The latter being a simple example of a moment map of deficiency one [24] on $\mathbb{CP}^3$. We may also consider the moment map $\mathcal{J}_1 : \mathbb{CP}^3 \to (\Xi, L_1, H_2) \subset \mathbb{R}^3$. The image of $\mathcal{J}_1$ is given in figure (3). It is now clear how points in the image of $\mathcal{J}_1$ correspond to the different types of reduced phase spaces (see figure (1)). The critical values correspond to the edges, faces and diagonal surfaces of this infinite polytope.

Points in the fixed point space of a subgroup of the symmetry group G will have this subgroup as its isotropy subgroup. Consequently these points belong to lower dimensional group orbits. Thus fixed point spaces for subgroups of G are fibred with relative equilibria. To be more precise we have the following proposition (see [17])

Proposition 5.1 *Let M be a symplectic manifold and G be Lie a group acting symplectically on M. Let $\mathcal{H} : M \to \mathbb{R}$ be a G-invariant Hamiltonian and let $X_{\mathcal{H}}$ be the associated Hamltonian vector field. Then $X_{\mathcal{H}}$ leaves $Fix_M(G)$ invariant and $X_{\mathcal{H}}|Fix_M(G)$ is a Hamiltonian vector field with Hamiltonian $H|Fix_M(G)$.*

To illustrate this consider the action of $\pi_{16} = \frac{1}{2}(L_1 + \Xi)$. $G_{\pi_{16}}$ is a subgroup of G_{H_2,Ξ,L_1}, and $Fix_{\mathbb{R}^8}(G_{\pi_{16}}) = \{(q,Q) \in \mathbb{R}^8 | q_1 = Q_1 = q_2 = Q_2 = 0\}$ is an invariant space. Similarly for the action of $\pi_{11} = \frac{1}{2}(\Xi - L_1)$, $G_{\pi_{11}}$ is a subgroup of G_{H_2,Ξ,L_1}, and $Fix_{\mathbb{R}^8}(G_{\pi_{11}}) = \{(q,Q) \in \mathbb{R}^8 | q_3 = Q_3 = q_4 = Q_4 = 0\}$ is an invariant space.

Theorem 5.2 *$\mathcal{J}_1(Fix_{\mathbb{R}^8}(G_{\pi_{16}}))$ is the restriction of the image of $\mathcal{J}_1$ to the plane $\Xi = L_1$. $\mathcal{J}_1(Fix_{\mathbb{R}^8}(G_{\pi_{11}}))$ is the restriction of the image of $\mathcal{J}_1$ to the plane $\Xi = -L_1$. The fibration in each diagonal plane is equivalent to the fibration of the energy-moment map for the harmonic oscillator. Points in the interior correspond to a fibre topologically equivalent*

to $\mathbb{T}^2$, points on the edges correspond to a fibre topologically equivalent $\mathbb{S}^1$. A line with $H_2 = n$ corresponds to an invariant surface topologically equivalent to $\mathbb{S}^3$.

Proof : On $Fix_{\mathbb{R}^8}(G_{\pi_{16}})$ we have that $\bar{\mathcal{H}}$ has integrals $\tilde{H}_2 = H_2|Fix_{\mathbb{R}^8}(G_{\pi_{16}}) = \frac{1}{2}(q_3^2 + Q_3^2 + q_4^2 + Q_4^2)$ and $\pi_{16} = q_3Q_4 - q_4Q_3$. The associated moment map has image given by $\tilde{H}_2 \geqslant |\pi_{16}|$ which corresponds to the standard harmonic oscillator (see [28]). Because on this fixed point space $H_2 = K_1$ it follows from the relations that $\Xi = L_1$. The results now follow. For $Fix_{\mathbb{R}^8}(G_{\pi_{11}})$ we have the same but with $H_2 = -K_1$ and $\Xi = -L_1$. **q.e.d.**

Remark 5.3 The points in the interior of the diagonal planes correspond to the zero dimensional symplectic leaves of the final orbit space, that are the cone-like singularities in the singular reduced phase spaces.

The fixed point spaces corresponding to the actions of Ξ and L_1 are not so easy to characterize on $\mathbb{R}^8$ with the present choice of coordinates. However, they can easily be characterized on $\mathbb{CP}^3$ (see section 2.2).

Theorem 5.4 *$\mathcal{J}_1(Fix_{\mathbb{CP}^3}(G_\Xi))$ is the restriction of the image of $\mathcal{J}_1$ to the planes $\Xi = \pm H_2$. $\mathcal{J}_1(Fix_{\mathbb{CP}^3}(G_{L_1}))$ is the restriction of the image of $\mathcal{J}_1$ to the planes $L_1 = \pm H_2$. Points in the interior correspond to a fibre topologically equivalent to $\mathbb{T}^2$, points on the edges correspond to a fibre topologically equivalent $\mathbb{S}^1$.*

Proof : $Fix_{\mathbb{CP}^3}(G_\Xi)$ consists of those points on $\mathbb{CP}^3$ for which $J_i = 0$, $1 \leqslant i \leqslant 8$ (see section 2.2). Using the relations it follows that $\Xi = \pm H_2$. These points correspond to zero dimensional symplectic leaves on the final orbit space that correspond to the cases where the reduced phase space reduces to an isolated point. Such a point can bee traced back through the different stages of the reduction to see that its corresponding fibre in the original phase space is topological equivalent to $\mathbb{T}^2$. A similar argument holds for $Fix_{\mathbb{CP}^3}(G_{L_1})$. The points on the edges correspond to the four normal modes found on $\mathbb{CP}^3$ in $Fix_{\mathbb{CP}^3}(G_{\Xi,L_1})$, and thus the corresponding fibre in the original phase space is topological equivalent to $\mathbb{S}^1$. **q.e.d.**

These theorems describe all relative equilibria corresponding to tori of dimension one and two. Relative equilibria corresponding to three dimensional tori will correspond to critical points of the Hamiltonian system on the regular parts of the reduced phase spaces for the $\mathbb{T}^3$ reduction, i.e. stationary points of the reduced system on the parts of the reduced phase spaces that are symplectic leaves of maximal dimension. These are the points that correspond to solutions of equation (36) under the conditions given by (37). That is, the points where the reduced Hamiltonian is tangent to the reduced phase space. When such a point coincides with a singular point of the reduced phase space one will obtain one of the lower dimensional tori found above. However, such a torus might then be fibred with still lower dimensional tori. An example is found in section 2.2 where the circles of stationary points correspond to $\mathbb{T}^2$ fibred with $\mathbb{S}^1$. These are special cases of the relative equilibria described in Theorem 5.4.

6 Generalized Zeeman model, the case $\lambda = 0$

Recall that, taking $\lambda = 0$, our model on the second reduced phase space with $\xi = 0$ is equivalent to the normalized perturbed Keplerian system modeling the hydrogen atom subject to the Zeeman potential. Allowing all values of ξ we call the resulting model the generalized Zeeman model. Using the general result given in section 4 the relative equilibria are given by the admissible roots of the polynomial (36), i.e. those roots that lie on the reduced phase space. For this case the polynomial (36) reduces to the following expression

$$p(K) = 4\ (Kn - l\xi)\left(-5\,nK^5 + l\xi\,K^4 + C_0K^3 + C_1K^2 + C_2K + C_3\right)\ , \tag{38}$$

where

$$\begin{aligned}
C_0(n,\xi,l) &= 10\,n\left(l^2 + n^2 + \xi^2\right),\\
C_1(n,\xi,l) &= -2\,l\xi\left(\xi^2 + 29\,n^2 + l^2\right),\\
C_2(n,\xi,l) &= -n\left(5\,l^4 - 26\,n^2\xi^2 - 26\,n^2l^2 - 18\,\xi^2l^2 + 5\,n^4 + 5\,\xi^4\right),\\
C_3 &= l\xi\ \left(l^4 - 15\,n^4 + \xi^4 - 2\,n^2l^2 - 2\,\xi^2l^2 - 2\,n^2\xi^2\right).
\end{aligned}$$

In particular the discriminant locus D for (38) describes where in the parameter space the number of solutions and thus the number of relative equilibria changes. The discriminant of (38) is (omitting the multiplicative constant)

$$D(n,\xi,l) = n^2\xi^2l^2\,(l-\xi)^2(l+\xi)^2(\xi-n)^2(l-n)^2(\xi+n)^2(l+n)^2D_1(n,\xi,l)D_2(n,\xi,l)D_3(n,\xi,l)\ ,$$

with

$$\begin{aligned}
D_1(n,\xi,l) &= \left(\xi^2l^2 - \xi^2n^2 - l^2n^2 + 5n^4\right)^2\ ,\\
D_2(n,\xi,l) &= \left(9\,n^2l^2 - 45\,n^4 - \xi^2l^2 + 9\,n^2\xi^2\right)^2\ ,\\
D_3 &= \xi^8l^8 + 15625\,\xi^8n^8 - 87500\,\xi^6n^{10} + 82000\,n^8\xi^2l^6 - 2992\,n^4\xi^6l^6\\
&\quad + 76\,n^2\xi^6l^8 - 15480\,n^6\xi^4l^6 - 76\,n^2\xi^8l^6 - 17500\,n^6\xi^2l^8 - 185052\,\xi^4n^8l^4\\
&\quad + 1950\,\xi^8n^4l^4 + 77400\,\xi^2n^{10}l^4 - 74800\,n^{12}\xi^2l^2 + 15480\,\xi^6n^6l^4 + 87500\,l^6n^{10}\\
&\quad 77400\,\xi^4n^{10}l^2 - 17500\,\xi^8n^6l^2 + 82000\,\xi^6n^8l^2 + 1950\,n^4\xi^4l^8 + 15625\,l^8n^8\\
&\quad + 48750\,\xi^4n^{12} - 9500\,\xi^2n^{14} + 625\,n^{16} + 48750\,l^4n^{12} - 9500\,l^2n^{14}
\end{aligned}$$

The discriminant is zero if $n = 0$ in which case the first reduced phase space reduces to a point and we find the origin as a stationary point. When $\xi = n$ or $l = n$ the third reduced phase space is a point corresponding to a single relative equilibrium. When $\xi = 0$ or $l = 0$ the discriminant has a double zero corresponding to a double zero of the equation. However in this case the Hamiltonian as well as the reduced phase space are symmetric with respect to the reflection $K \to -K$, and, although we find a double

admissible solution of (38), the double solution corresponds to two relative equilibria on different energy levels. Furthermore the discriminant is zero if $\xi = l$ or $\xi = -l$. Crossing these lines the discriminant does not change sign and consequently there is no change in the number of solutions, perhaps possibly at these lines. However at these lines the number of admissible solutions does not change. At these lines one of the relative equilibria correspond to a relative equilibrium on a Hamiltonian level surface caused by the fact that this level surface passes through the singular point of a singular reduced phase space. Note that $|\xi| \leqslant n$ and $|l| \leqslant n$. Consequently D_1 and D_2 are strictly positive. Thus all bifurcations will take place along the set given by $D_3(n, \xi, l) = 0$ which is drawn in figure (4) and turns out to be a square with cusp-like vertices.

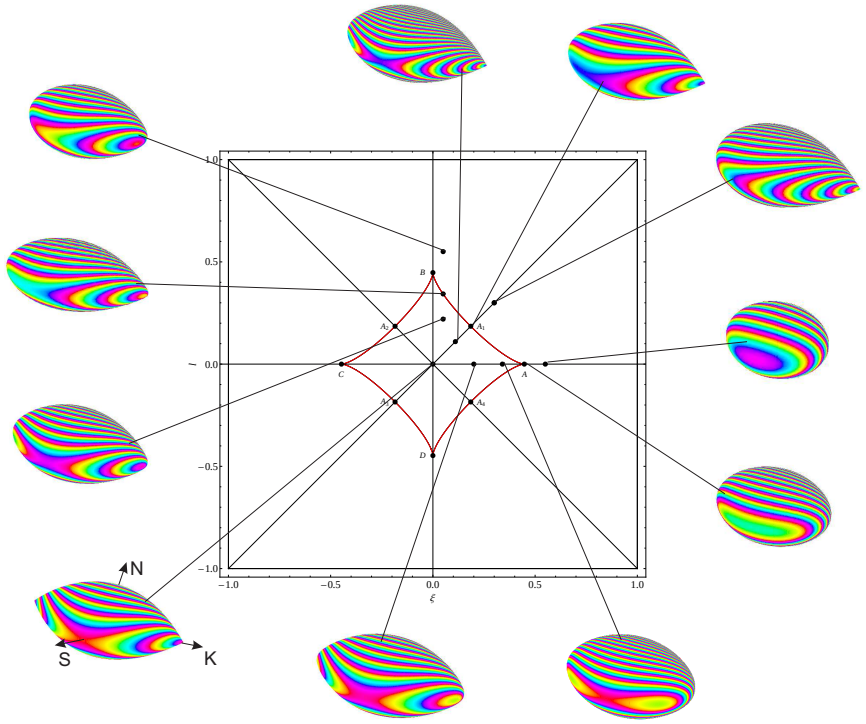

Figure 4: Curve of bifurcation $D_3(n, \xi, l) = 0$ for $n = 1$ together with the reduced phase spaces

In figure (4) we also find the different intersections of reduced phase spaces and Hamiltonian level surfaces which are illustrated by painting the reduced phase space, a method introduced in [4]. Furthermore in figure (6) it is illustrated how the saddle-center points move along the reduced phase spaces passing through a pitchfork point. By carefully studying the intersections of reduced phase spaces and Hamiltonian level surfaces we obtain the following result.

Theorem 6.1 *The bifurcation surface $D_3(n, \xi, l) = 0$ defines a region around the On axis in the space of parameters. Inside the region there are four relative equilibria, three stable and one unstable. Outside there are two stable relative equilibria. Fixing a value of n, saddle-center bifurcations take place when we cross $D_3 = 0$, except for the cuspidal points A, B, C and D, i.e. the points $(n, 0, \pm\frac{n}{\sqrt{5}})$ and $(n, \pm\frac{n}{\sqrt{5}}, 0)$, where pitchfork bifurcation occur when we move along the lines $\xi = 0$ or $l = 0$.*

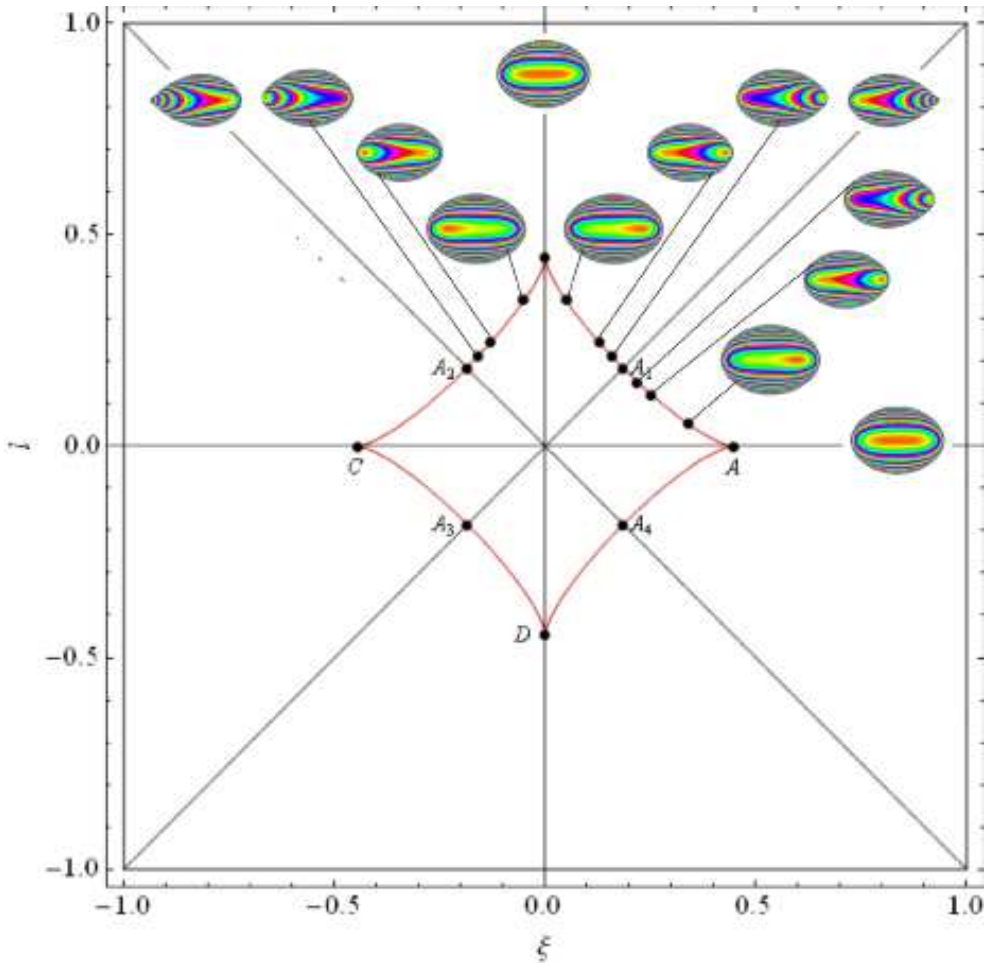

Figure 5: Shifting of the saddle-centre bifurcation point

In this case we may also draw the singularity of the energy-moment map (H_2, Ξ, L_1, H) for a fixed value n of H_2, which is given in figure (6). The lower part of this singularity is the most interesting part and is given in figure (7). The equations for this surface are obtained in the following way. Set the Hamiltonian (32) equal to h and solve for N. Substitute this N into te relation (30) defining the third reduced phase space, and put $S = 0$. One obtains an equation in K with parameters h, ξ and l if one sets $\lambda = 0$ and takes n fixed. The discriminant locus of this equation then describes the singularity of the energy-momentum mapping in ξ, l, h-space. Note that one has to take into account that the relation (30) is subjected to the inequalities defining the reduced phase space. The h-axis is the vertical axis. One recovers the curve $D_3(n, \xi, l) = 0$ in figure (7). Inside this curve there are four h-values corresponding to a relative equilibrium, outside there are two.

Remark 6.2 The section $\xi = 0$ of the singularity of the energy-moment map is the same as the one presented for several normalized perturbed Keplerian systems, including the Zeeman problem, in [5],[30].

Remark 6.3 Note that the stability of the stationary points on the two-dimensional

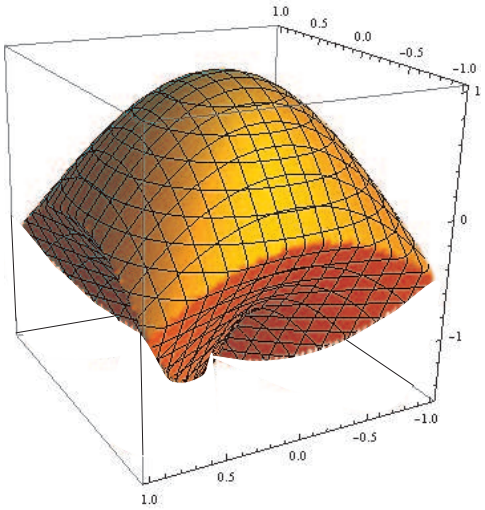

Figure 6: Singularity of the energy-moment map

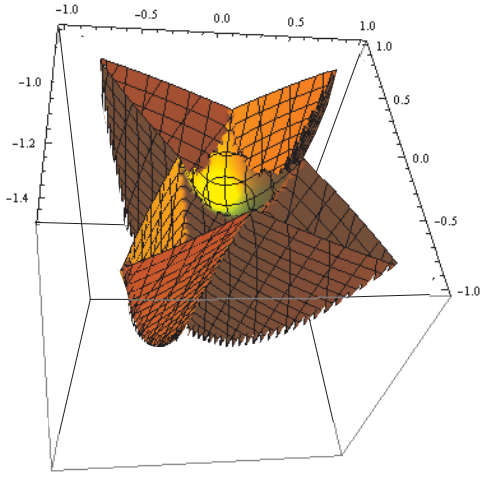

Figure 7: Detail of the singularity of the energy-moment map

reduced phase space follows directly from the geometry. The stability of the corresponding solutions in original phase space can now be studied using the notion of G_μ-stability as introduced in [27].

7 Hamiltonian Hopf Bifurcations in the case $\xi = l$

Consider the relative equilibrium $\mathbf{z}_e = (n, 0, 0, \xi, 0, 0)$ on the second reduced phase space. This equilibrium, after the third reduction has been implemented, corresponds to a cone-like singular point of that orbit space. We are interested in studying the possible existence of Hamiltonian Hopf bifurcations, degenerate or not.

The matrix of the tangent flow of the second reduced vector field (24)at z_e takes the following form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\xi n\,(2+\lambda) & 0 & 0 & \Gamma \\ 0 & 2\xi n\,(2+\lambda) & 0 & 0 & -\Gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2(\lambda-1)\,(\xi^2-5n^2) & 0 & 0 & -4\xi n(\lambda-1) \\ 0 & 2(\lambda-1)\,(\xi^2-5n^2) & 0 & 0 & 4\xi n(\lambda-1) & 0 \end{pmatrix} \tag{39}$$

where $\Gamma = 2(4\lambda-1)n^2 + 2(1-\lambda)\xi^2$. The associated characteristic polynomial is

$$p(X) = X^2(X^4 + aX^2 + b) \tag{40}$$

where

$$a = (8\,\xi^4 - 52\,n^2\xi^2 + 160\,n^4)\,\lambda^2 + (-200\,n^4 - 16\,\xi^4 + 104\,n^2\xi^2)\,\lambda - 16\,n^2\xi^2 + 8\,\xi^4 + 40\,n^4$$
$$b = 16\,(\lambda-1)^2\,(20\,n^4\lambda - 5\,n^4 + \xi^4\lambda - \xi^4 - 11\,n^2\xi^2\lambda + 2\,n^2\xi^2)^2$$

(41)

Solutions of $p(X) = 0$ are 0 (double) and

$$X = \pm\sqrt{-(9n^2\xi^2\lambda^2 + \Delta) \pm n\xi\lambda i\sqrt{|\Delta|}} \tag{42}$$

where

$$\Delta = (4\,\xi^4 - 35\,n^2\xi^2 + 80\,n^4)\,\lambda^2 + (-8\,\xi^4 + 52\,n^2\xi^2 - 100\,n^4)\,\lambda + 4\,\xi^4 - 8\,n^2\xi^2 + 20\,n^4. \tag{43}$$

The curve $\Delta = 0$ in the parametric plane (ξ, λ) has the graph given in figure (8).

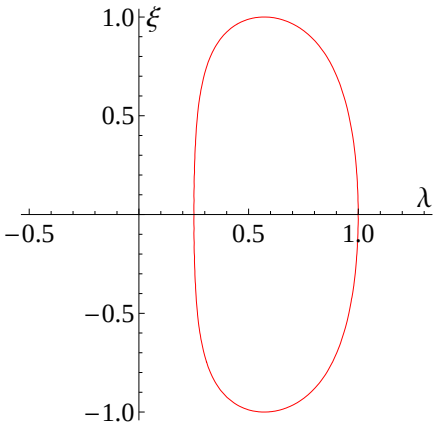

Figure 8: The curve $\Delta = 0$ for $n = 1$

This curve will play a key role in the analysis of the Hopf bifurcation as we will see later on.

Note that when $\Delta = 0$ we have a double pair of purely imaginary values $\pm 3n\xi\lambda i$. Moreover, when $\Delta < 0$ we have two pairs or complex eigenvalues and for $\Delta > 0$ we have two pair of imaginary eigenvalues. Finally, when $\xi = 0$ and $\lambda = 1/4$ or 1, we see that the linear system is nilpotent with a zero eigenvalue of multiplicity four.

A nonlinear normal mode of a Hamiltonian system is a periodic solution near equilibrium with period close to that of a periodic trajectory of the linearized vector field. We will consider the normal modes associated to the rectilinear trajectories through the origin.

We will use the geometric criterium [21], [23], [10] in order to determine the presence of non degenerate Hamiltonian Hopf bifurcations. For a (standard) Hamiltonian Hopf bifurcation to take place one needs in fact three transversality conditions to hold true, which are described below in geometric terms as conditions T.1-3.. We assume that for $\lambda = \lambda_0$ the Hamiltonian level surface L_{λ_0} given by $H_{\lambda_0}((S, K, N) = H_{\lambda_0}((0, n, 0)$is tangent to the reduced phase space at $(S, K, N) = (0, n, 0)$, the cone-like singularity of the reduced phase space.

(T1) The coefficient of the semisimple part of the linearized system at the equilibrium and for the bifurcation value of the parameter has to be nonzero.(This condition can be relaxed if the S^1–symmetry generated by S is externally imposed), i.e. the the bifurcation of eigenvalues should be be from two pairs of purely imaginary eigenvalues through a double pair into the complex plane.

(T2) L^λ enters the cone as λ passes through λ_0 with a nonzero rate of change of the angle between L^ν and the reduced phase space.

(T3) L^{λ_0} has second order contact with the reduced phase space in $(0, n, 0)$.

Theorem 7.1 (Hopf-Bifurcation for $K = n$) *We consider the 4D system given by the Van der Waals Hamiltonian, where $\Xi = \xi$ and $L_1 = l$ are first integrals of he system. For the corresponding normalized system truncated after terms of order six we have that:*

1. *For a fixed value n , of the oscillator energy, the geometric locus in the parameter plane (ξ, λ) where there exists a nondegenerate supercritical Hopf bifurcation is given by*

$$\Delta = (80n^4 - 35n^2\xi^2 + 4\xi^4)\,\lambda^2 - 4\,(25n^4 - 13n^2\xi^2 + 2\xi^4)\,\lambda + 4\,(5n^4 - 2n^2\xi^2 + \xi^4) = 0$$

   with $\xi \neq 0$.

2. *For $\xi = 0$ we have two degenerate Hamiltonian Hopf bifurcations for the parameter values $\lambda = 1/4$ and $\lambda = 1$.*

3. *When $\xi = n$ the problem is degenerate for $\lambda = 4/7$.*

Proof : Let us consider the function

$$g_\lambda(K) = \frac{3n}{2}\left(\frac{3\lambda}{2} - 1\right)K^2 + \xi^2(1-\lambda)K \pm (n-K)\sqrt{(K+n)^2 - 4\xi^2} \tag{44}$$

describing the difference between the reduced Hamiltonian and the lower and upper arc of the reduced phase space in the $S = 0$ plane. Imposing that $K = n$ be a root of g_λ, we obtain the corresponding values of the energy. They are given by

$$h = -\frac{3\,(2 - 3\,\lambda)n^2 + 4\,(\lambda - 1)\,\xi^2}{4}\,n.$$

Moreover, imposing that $K = n$ is a critical value of g_λ we obtain

$$\frac{3n^2}{2}\,(3\lambda - 2) + \xi^2(1-\lambda) \mp \frac{n\,(4+\lambda)}{2}\sqrt{n^2 - \xi^2} = 0. \tag{45}$$

After some computations we obtain that (45) is equivalent to $\Delta = 0$. In other words, we have verified the condition $H1$. in the curve $\Delta = 0$. In order to satisfy the transversality condition $H2$ we ought to have

$$\frac{dg'_\lambda(n)}{d\lambda} = \frac{9n^2}{2} - \xi^2 \pm \frac{n\sqrt{n^2 - \xi^2}}{2} \neq 0, \tag{46}$$

but this is the case. Indeed, for a fixed n it is easy to see that there does not exist ξ verifying the equation. Finally with respect to condition $H3$ it results that $-6\sqrt{n^2-\xi^2}+9\sqrt{n^2-\xi^2}\lambda = 3\sqrt{-(\xi-n)(\xi+n)}(-2+3\lambda)$

$$\begin{aligned} g_{\lambda}^{''}(n) &= \frac{n\left(n(\lambda-4)+3\sqrt{n^2-\xi^2}(3\lambda-2)\right)}{2\sqrt{n^2-\xi^2}} \neq 0 \\ g_{\lambda}^{''}(n) &= \frac{n\left(n(\lambda-4)+3\sqrt{n^2-\xi^2}(2-3\lambda)\right)}{2\sqrt{n^2-\xi^2}} \neq 0 \end{aligned} \tag{47}$$

Plotting $g_{\lambda}^{''}(n)$ and $\Delta = 0$ it is clear that only for $\xi = 0$ and $\lambda = 1/4$ or 1 this condition is not satisfied. Moreover there exists another degenerate situation when $\xi = n$, because the thrice reduce space then collapses to a point. In that case we have $\lambda = 4/7$ (see fig. (9). **q.e.d.**

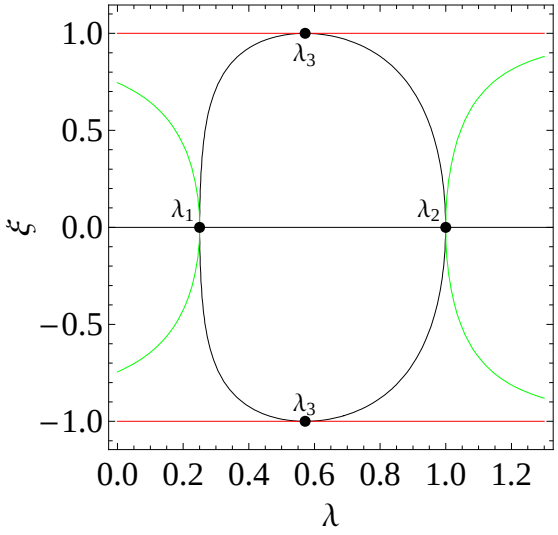

Figure 9: Graphical proof of the condition $H3$. The green lines are $g_{\lambda}^{''}(n)$ according to the sign of g_{λ}. The parameter values $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = 1$ correspond to degenerate Hamiltonian Hopf Bifurcation for $n = 1$. The parameter value $\lambda_3 = \frac{4}{7}$ is a critical case for $n = \pm 1$.

Remark 7.2 At a Hamiltonian Hopf bifurcation the equilibrium changes from stable to unstable. Inside the curve $\Delta = 0$ in figure (8) is stable, outside this curve the equilibrium is unstable.

When $\xi = l = 0$ the reduced phase space has an additional cone-like singularity at $K = -n$, which is a candidate for the presence of a Hamiltonian Hopf bifurcation. Therefore we have to study the system at the equilibrium $\mathbf{z}_e = (-n, 0, 0, 0, 0, 0)$ of the second reduced phase space.

The matrix of the tangent flow of the second reduced vector field (24)at z_e takes the following form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,n^2(4\lambda-1) \\ 0 & 0 & 0 & 0 & -2\,n^2(4\lambda-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10n^2(\lambda-1) & 0 & 0 & 0 \\ 0 & 10n^2(\lambda-1) & 0 & 0 & 0 & 0 \end{pmatrix} \tag{48}$$

The associated characteristic polynomial is

$$p(X) = X^2\left(X^2 + 20n^4\,(4\lambda-1)\,(\lambda-1)\right)^2 \tag{49}$$

This polynomial has a zero root of multiplicity six for $\lambda = 1$ or $\lambda = 1/4$. The system is nilpotent of multiplicity six. For $\lambda > 1$ or $\lambda < 1/4$ this polynomial has a pair of imaginary roots of multiplicity 2. For $1/4 < \lambda < 1$ has a pair of real roots of multiplicity 2.

Theorem 7.3 (Hopf-Bifurcation for $K = -n$ and $\xi = l = 0$) *Under the conditions of the previous theorem we have a Hamiltonian Hopf bifurcation only for $\lambda = 1,\ 1/4$, these are degenerate Hamiltonian Hopf bifurcations for the system under consideration.*

Proof : In a similar way to the previous proof, we consider the function

$$g_\lambda(K) = \frac{3n}{2}\left(\frac{3\lambda}{2} - 1\right)K^2 \pm n(4-\lambda)(n-K)^2 \tag{50}$$

Imposing that $K = -n$ is a critical value of g_λ we obtain $-5n^2(\lambda-1)$ for positive sign and $-n^2(4\lambda-1)$ for minus sign. Only for $\lambda = 1,\ 1/4$ the previous expressions are zero. On the other and

$$\frac{dg_\lambda^{'}(n)}{d\lambda} = \begin{cases} -5n^2 \text{ for } + \\ -4n^2 \text{ for } - \end{cases}$$

and the condition $H2$ is verified. Finally, the condition $H3$ is not verified for the values $\lambda = 1,\ 1/4$ since

$$g_\lambda^{''}(n) = \begin{cases} 5n(\lambda-1) \text{ for } + \\ n(4\lambda-1) \text{ for } - \end{cases}$$

q.e.d.

Remark 7.4 For $\xi = -l$ and $K = -n$ one may prove similar results as in the theorems above. Also along $\xi = -l$ one finds Hamiltonian Hopf bifurcations.

8 Van der Waals problem: $\Xi = 0$. Relative equilibria and bifurcations

As is said before, the reduced space for $\Xi = 0$ relates to normalized 3-DOF perturbed Keplerian systems. More on perturbed Keplerian systems can be found in Cushman's survey [5] on this issue. The Van der Waals model is not among the examples in [5]. However, from the generic study done about the first order double reduced Hamiltonian, one notices that the generalized Van der Waals family for $\Xi = 0$ falls in the class of systems studied in [5]. However in [5] dependence on a parameter was not considered. It will turn out that there are special values of the physical parameter λ that will give rise to degeneracy of the normalized flow in the sense that on the third reduced phase space there will be nonisolated relative equilibria. In [12] and [14, 15] this degeneracy was related to integrability in the 3D case. Part of the results in this section can also be found in the preliminary report in [8].

The dynamics related to the generalized Van der Waals potential corresponds to the case $\Xi = 0$, with the integral $L_1 = l$ as the axial symmetry. In the case the reduced space is defined by

$$[\,(n+l)^2 - K^2\,][\,(n-l)^2 - K^2\,] = 4N^2 + 4S^2. \tag{51}$$

which is diffeomorphic to an S^2 if $l \neq 0$, $l < |n|$ as we have shown in Section 4. If $l > 0$, the domain of the reduced phase space is $-|n-l| \leq K \leq |n-l|$. When $l < 0$, it will be $-|n+l| \leq K \leq |n+l|$. When $l = 0$ the reduced phase space is not a manifold but has two singular points, and we will study this case separately. The Hamiltonian function reduces to

$$\overline{\mathcal{H}}_{L_1}(K, N) = \frac{3n}{4}(3\lambda - 2)\, K^2 + \frac{n}{2}(4 - \lambda)\, N. \tag{52}$$

the level surfaces of which are, in general, parabolic cylinders. The nature of these level surfaces changes with λ. $\overline{\mathcal{H}}_{L_1}{-}1(h)$ is a plane $N = 3h/(5n)$ when $\lambda = 2/3$; for values $\lambda < 2/3$ the parabolic cylinder has a maximum. When $\lambda > 2/3$ it has a minimum. There is a special case related to the value $\lambda = 4$. For this case the Hamiltonian surfaces are

$$\mathcal{H} = \frac{15n}{2}\, K^2$$

i.e. parallel planes to the fundamental plane $0SN$ which intersect the reduced space in one circle if $K = 0$, and a pair of circles if $K \neq 0$. The corresponding system of differential equations given by the Poisson structure is now

$$\frac{dK}{dt} = 0, \qquad \frac{dN}{dt} = 60nK\,S, \qquad \frac{dS}{dt} = -60nKN. \tag{53}$$

Thus the flow is a pure rotation, except for the circle $N^2 + S^2 = (n^2 - l^2)^2/4$ in the plane $K = 0$, which is a circle of stationary points. There are also two isolated equilibria at the points $(0, \pm|n \pm l|, 0)$. The two limitting cases are $n = |l|$, which correspond to the reduced spaces shrunk to a point, and $l = 0$ which is, in the context of perturbed Keplerian systems

referred to as the polar case. The relative equilibria correspond to rectilinear solutions in configuration space. Because of the presence of nonisolated stationary points on the third reduced phase space we are in a degenerate case, which might be influenced by adding higher order terms to our normal form.

Next consider the generic case $\lambda \neq 4$ and determine the tangencies between the reduced space given by equation (51) and the quadratic function

$$N = \frac{3(3\lambda - 2)}{2(4 - \lambda)} K^2 + \frac{2h}{n(4 - \lambda)}.$$

in the plane $S = 0$. The sign of the coefficient of K^2 as a function of λ, will be negative, positive and negative in $\lambda \in (0, 2/3)$, $(2/3, 4)$ and $(4, \infty)$ respectively. We immediately obtain the first derivative from equation (51)

$$\frac{dN}{dK} = \frac{[K^2 - (n^2 + l^2)]\, K}{2N},$$

and likewise from the Hamiltonian

$$\frac{dN}{dK} = 3\, \frac{(3\lambda - 2)}{4 - \lambda}\, K.$$

Equaling both expressions

$$\frac{[K^2 - (n^2 + l^2)]\, K}{2N} = 3\, \frac{(3\lambda - 2)}{4 - \lambda}\, K$$

we find that for $K = 0$ we will have a tangent contact, corresponding to the lower and higher point of the reduced space $(0,\, 0,\, \pm\frac{1}{2}(n^2 - l^2))$, whichever value of the integral and physical parameter. Then, assuming $K \neq 0$, we search for the other values related to tangent contact in

$$6\, \frac{(3\lambda - 2)}{4 - \lambda}\, N = K^2 - (n^2 + l^2).$$

Replacing the expression of N, $(\pm)$ we obtain

$$9 \left(\frac{3\lambda - 2}{4 - \lambda} \right)^2 [(n + l)^2 - K^2\,][(n - l)^2 - K^2\,] = [K^2 - (n^2 + l^2)]^2,$$

which is a biquadratic equation in K, whose roots we have to discuss. Let

$$\mathcal{P}(K) = C_4\, K^4 + C_2\, K^2 + C_0 = 0$$

with

$$C_4 = 5(4\lambda - 1)(\lambda - 1),$$

$$C_2 = -10(4\lambda - 1)(\lambda - 1)(l^2 + n^2),$$

$$C_0 = [5(\lambda - 1)l^2 - (4\lambda - 1)n^2][(4\lambda - 1)l^2 - 5(\lambda - 1)n^2]$$

Then, if $\lambda = 1$ and $l = 0$, the polynomial is zero. Also, if $\lambda = 1/4$ and $l = 0$, the polynomial is zero. Excluding these two values of the external parameter, we have $K = 0$ as double root when $C_0 = 0$. And this leads to the stationary points given by

$$l = n\sqrt{\frac{4\lambda - 1}{5(\lambda - 1)}},\ \lambda \in [0, 1/4]; \quad l = n\sqrt{\frac{5(\lambda - 1)}{4\lambda - 1}},\ \lambda \in [1, 4]; \quad l = n\sqrt{\frac{4\lambda - 1}{5(\lambda - 1)}},\ \lambda \in [4, \infty]$$

which correspond to pitchfork bifurcations. This recovers results from [12] (see also fig. (10))
For $\lambda \in (0, 1/4)$ and $\lambda \in (1, 4)$, the two stable equilibria emanating from the bifurcation point are

$$K = \pm\sqrt{l^2 + n^2 - \frac{3(2 - 3\lambda)}{\sqrt{5(1 - 5\lambda + 4\lambda^2)}}},$$

and for $\lambda > 4$

$$K = \pm\sqrt{l^2 + n^2 + \frac{3(2 - 3\lambda)}{\sqrt{5(1 - 5\lambda + 4\lambda^2)}}},$$

and the value of N is obtained replacing K in equation (51).

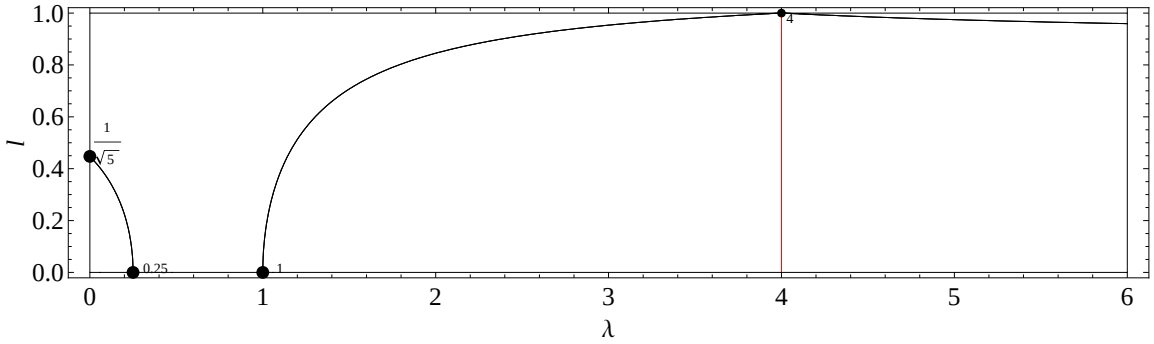

Figure 10: Relative equilibria for the Van der Waals family [12]

Note that when $\lambda \in (0, 1/4)$, and l approaching 0 the stable points will move towards the the singular points $(0, \pm n, 0)$ of the third reduced phase space for $\xi = 0, l = 0$. In the case $\lambda \in (1, 4)$ and $\lambda \in (4, \infty)$, the two equilibria bifurcate at $\lambda = 4$. In [12] these orbits are related to the associated 3D perturbed Keplerian system using Delaunay variables. The pitchfork bifurcation lines are related to circular orbits at critical inclinations and equatorial orbits at criticaleccenricities. The stable stationary points bifurcating correspond to circular equatorial orbits.

Finally we will consider in some detail the 'polar case´,that is, $\xi = l = 0$ (see also [13]). In this case the section of the reduced phase space with the plane $S = 0$ as well as the section of the reduced Hamiltonian with the plane $S = 0$. We get

$$N = \pm\frac{1}{2}(n^2 - K^2), \qquad N = \frac{3}{2}\,\frac{3\lambda - 2}{4 - \lambda}K^2 + \frac{2\,h}{n(4 - \lambda)}.$$

The two parabola´s coincide for $\lambda = \frac{1}{4}$ and $\lambda = 1$ giving rise to a whole parabola of stationary solutions. For $\lambda = \frac{1}{4}$ along the topside of the reduced phase space and $\lambda = 1$ along the bottom side of the reduced phase space. Again we have a degenerate situation. These degenerate situations are of interest because the claim is that for these particular values of the parameter the original system will be integrable, also in the 4D case. Note that these values also correspond to Hamiltonian Hopf bifurcations (see section 7). Because of the degeneracy a higher order analysis of the normal form will be needed to show the presence of a Hamiltonian Hopf bifurcation in these cases. However, if these degenerate cases correspond to integrable systems the degeneracy might be of infinite co-dimension (compare [20, 22]).

Another interesting feature of the polar case is that for $\dfrac{1}{4} < \lambda < 1$ the flow has a heteroclinic trajectory for $h = -3(3\lambda-2)n^3/4$, that passes through the points $(0, \pm n, 0)$, which may be parametrized as follows

$$N = \frac{3}{2}\frac{3\lambda-2}{4-\lambda}(K^2-n^2), \qquad S = \pm\frac{3}{2}\sqrt{\frac{5(4\lambda-1)(1-\lambda)}{4-\lambda}}(K^2-n^2), \qquad K = K$$

These heteroclinic solutions bifurcate at $\lambda = \frac{1}{4}$ and $\lambda = 1$ through the parabola of stationary points into a stationary point with two homoclinic orbits surrounding the singular points of the reduced phase space (see fig. (11)) Note that at $\lambda = 4$ these

Figure 11: Three snapshots of the separatrix. Left $\lambda = 1/5$, center $\lambda = 1/2$ and right $\lambda = 2$

homoclinic orbits bifurcate through a circle of stationary points by coinciding with each other.

For the sake of completeness we include the explicit expressions of the separatrices:

• Homoclinic for $0 < \lambda < 1/4$

$$K = n\tanh\omega_1 t, \quad N = \frac{3(2-3\lambda)n^2}{2(\lambda-4)}\mathrm{sech}^2\omega_1 t, \quad S = \frac{\sqrt{5(1-\lambda)(4\lambda-1)}n^2}{(\lambda-4)}\mathrm{sech}^2\omega_1 t,$$

where $\omega_1(\lambda) = 2n^2\sqrt{5(1-\lambda)(4\lambda-1)}$.

• Heteroclinic for $1/4 < \lambda < 1$

$$K = \pm\frac{n\sqrt{4-\lambda}}{\sqrt{4\lambda-1}}\mathrm{sech}\omega_2 t,\ N = \frac{n^2}{2} + \frac{3(2-3\lambda)n^2}{2(4\lambda-1)}\,\mathrm{sech}^2\omega_2 t,\ S = \pm\frac{n^2\sqrt{5(\lambda-1)}}{\sqrt{4\lambda-1}}\frac{\mathrm{senh}\,\omega_2 t}{\cosh^2\omega_2 t}$$

where $\omega_2(\lambda) = 2n^2\sqrt{5(4-\lambda)(\lambda-1)}$. Note that for $\lambda = 2/3$ the heteroclinic is a plane curve.

• Homoclinic for $1 < \lambda < 4$

$$K = \frac{\pm n(\lambda-4)\operatorname{sech}\omega_3 t}{\sqrt{5}\sqrt{(\lambda-5)\lambda+4}},\ N = -\frac{n^2}{2} + \frac{3(3\lambda-2)n^2}{10(\lambda-1)}\operatorname{sech}^2\omega_3 t,\ S = \frac{\pm n^2\sqrt{(4\lambda-1)}}{\sqrt{5(\lambda-1)}}\frac{\operatorname{senh}\omega_3 t}{\cosh^2\omega_3 t}$$

Finally in fig. (12) and fig. (13) the orbits on the reduced phase space for the polar case are illustrated by painting the reduced energy.

Acknowledgements

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Authors appear in alphabetical order.

$\lambda = 0$: **'polar Zeeman'**

$\lambda = 1/10$

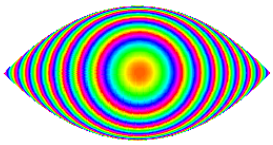
$\lambda = 1/4$ **integrable case**: south pole view: **infinite equilibria**

$\lambda = 0.26$

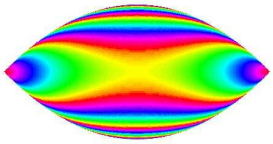
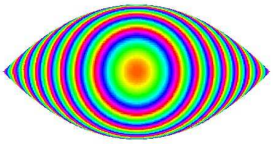
$\lambda = 2/3$ **space sliced by horizontal planes**

$\lambda = 9/10$

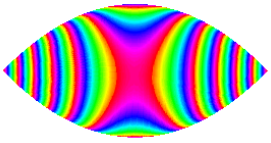
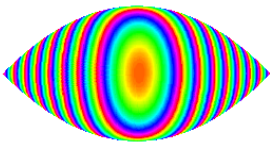
Figure 12: Case $\Xi = L_1 = 0$: 'north pole' (left) and 'south pole' (right) views of the flow on the thrice reduced space for several values of the physical parameter λ. The integrable case $\lambda = 1/4$ is related to a Hopf bifurcation connecting the singular points

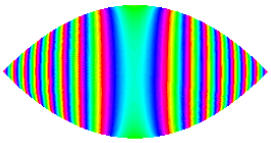
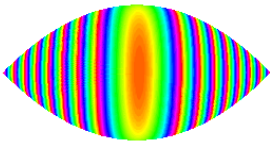
$\lambda = 1$, **integrable case**. north pole view: Infinite equilibria

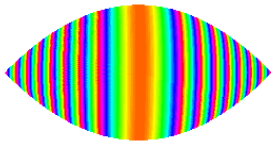
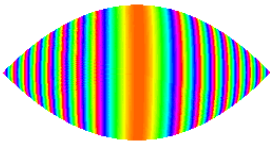
$\lambda = 1.1$

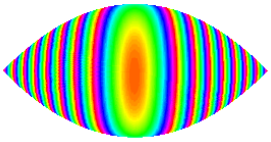
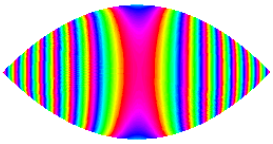
$\lambda = 2$ **Van der Waals**

$\lambda = 3.5$

$\lambda = 4$, **integrable case**. Oyster bifurcation. Infinite equilibria in the plane $K = 0$

$\lambda = 6$: stable equilibrium is at the north; the unstable a the south.

Figure 13: Case $\Xi = L_1 = 0$: 'north pole' (left) and 'south pole' (right) views of the flow on the thrice reduced space for several values of the physical parameter λ. From $\lambda = 1$ to $\lambda = 6$ we find two integrable cases. When $\lambda = 1$ the bifurcation associated to it involves the singular points: it is a degenerate Hopf bifurcation. When $\lambda = 4$ we identify another bifurcation called oyster-bifurcations by some authors.

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