Solution to Problem 63-14: A resistance problem

Thus

$$2^{2a+2b+c} \approx 3^{a+b+c},$$

or

$$\log_2 3 \approx \frac{2a+2b+c}{a+b+c}.$$

The most likely periods for a finite cycle, which are here given by the sum $a + b + c$, are therefore the *denominators* in the best rational approximations of $\log_2 3$. Since the continued fraction

$$\log_2 3 = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\cdots$$

has the convergents

$$\frac{1}{1}, \quad \frac{2}{1}, \quad \frac{3}{2}, \quad \frac{8}{5}, \quad \frac{19}{12}, \quad \frac{65}{41}, \cdots,$$

we do find denominators of 1, 2, 5, and 12.

However, it has not been proven that these denominators constitute the only allowable periods. Nor has a finite cycle of period 41 been discovered. Nor have any other finite cycles been so far discovered.

Problem 63-14, *A Resistance Problem*, by RON L. GRAHAM (Bell Telephone Laboratories).

A regular n-gon is given such that each vertex is connected to the center and to its two neighboring (nearest) vertices by means of unit resistors. Determine the equivalent resistance R_n between two adjacent vertices.

Solution. By C. J. BOUWKAMP (Philips Research Laboratories, and Technological University, Eindhoven, Netherlands).

Let A and B be two neighboring vertices; let an electric current I enter the "wheel" at A, and let it leave at B, as due to an applied voltage V across AB. Assuming $n > 2$ and $0 \leqq k < n$, let currents i_{2k} flow from the center C to the successive vertices along the circumference, and let currents i_{2k+1} flow in the circumferential resistors. The current from C to A is i_0, that from C to B is $i_{2n-2} = -i_0$, and that from B to A is $i_{2n-1} = -2i_0 = V$. The input current at A is $I = i_1 - 3i_0$, hence

$$R_n = \frac{-2i_0}{i_1 - 3i_0}. \tag{1}$$

Now, by applying alternately Kirchhoff's mesh-voltage and node-current laws, we easily obtain the recurrence relation

$$i_k = i_{k-1} + i_{k-2}, \tag{2}$$

holding for $1 < k < 2n - 1$.

Let $x = \frac{1}{2}(1 + \sqrt{5})$ and $y = -1/x$ denote the two zeros of the characteristic polynomial $z^2 - z - 1$ of (2). Then, with $u_k = (x^k - y^k)/(x - y)$, $i_k (k > 1)$. can be linearly expressed in terms of i_0 and i_1, as follows:

$$i_k = u_{k-1}i_0 + u_k i_1 .$$

Here $\{u_k\}$, $k = 1, 2, \cdots$, is the well-known sequence of Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots .$$

Since

$$-i_0 = i_{2n-2} = u_{2n-3}i_0 + u_{2n-2}i_1 ,$$

we find

$$\frac{-i_1}{i_0} = \frac{1 + u_{2n-3}}{u_{2n-2}},$$

and substitution of this in (1) gives, after some transformation,

$$R_n = \frac{2u_{2n-2}}{1 + u_{2n-2} + u_{2n}}, \tag{3}$$

which solves the problem in question.

However, (3) can be simplified considerably if we distinguish between even and odd values of n. Let $v_k = x^k + y^k$; then $\{v_k\}$, $k = 1, 2, \cdots$, is the sequence of Lucas numbers:

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \cdots .$$

As is well-known, the Fibonacci and Lucas sequences constitute two linearly independent solutions of the difference equation (2). There exist many relations between these numbers; for example,

$$u_{n-1} + u_{n+1} = v_n , \qquad v_{n-1} + v_{n+1} = 5u_n , \tag{4}$$

which can easily be proved by induction.

Now, if n is odd we have $i_{n-1} = 0$ in virtue of symmetry; this leads to

$$\frac{-i_1}{i_0} = \frac{u_{n-2}}{u_{n-1}} = \frac{v_{n-3} + v_{n-1}}{v_{n-2} + v_n}.$$

If this is inserted in (1) we get after a few manipulations the set of formulas:

$$\begin{aligned} R_n &= \frac{2u_{n-1}}{u_{n-1} + u_{n+1}} = \frac{2u_{n-1}}{v_n} = \frac{2}{5}\left(1 + \frac{v_{n-2}}{v_n}\right), \\ \frac{1}{R_n} &= \frac{1}{2}\left(1 + \frac{u_{n+1}}{u_{n-1}}\right), \qquad n \text{ odd}. \end{aligned} \tag{5}$$

On the other hand, if n is even we have $i_n = -i_{n-2}$, again by symmetry, and thus

$$\frac{-i_1}{i_0} = \frac{u_{n-3} + u_{n-1}}{u_{n-2} + u_n} = \frac{v_{n-2}}{v_{n-1}}.$$

Therefore, the analog of (5) is obtained from (5) by interchange of u and v, except for an extra factor of 5 in the middle:

$$(6)\qquad \begin{aligned} R_n &= \frac{2v_{n-1}}{v_{n-1} + v_{n+1}} = \frac{2v_{n-1}}{5u_n} = \frac{2}{5}\left(1 + \frac{u_{n-2}}{u_n}\right), \\ \frac{1}{R_n} &= \frac{1}{2}\left(1 + \frac{v_{n+1}}{v_{n-1}}\right), \end{aligned} \qquad n \text{ even.}$$

There is yet another way to evaluate R_n which deserves mention. Let n be an odd number; take (6) for $n + 1$ and (5) for n; then

$$\frac{5}{2} R_{n+1} - 1 = \frac{u_{n-1}}{u_{n+1}} = \left(\frac{2}{R_n} - 1\right)^{-1}.$$

If n is even, these relations hold if u is replaced by v. Therefore, if n is arbitrary we have the recurrence relation

$$(7)\qquad R_{n+1} = \frac{4}{5(2 - R_n)},$$

which with $R_3 = \frac{1}{2}$ allows the numerical evaluation of R_n independently of the Fibonacci and Lucas numbers.

The behavior of R_n for large values of n is determined by

$$R_\infty = \lim_{n\to\infty} R_n = 1 - 5^{-1/2} = 0.5527\ 8640\ 4\cdots.$$

The limit is attained monotonically from below. For $n \geqq 20$, R_n equals R_∞ up to 8 decimals; see Table 1.

Generalization of the problem. In what follows we determine the equivalent resistance between any two vertices of the wheel.

First, let r_n denote the equivalent resistance between the vertex A and the center C. Then

$$(8)\qquad r_n = 1 - R_n.$$

To prove (8), let S_n denote the equivalent resistance between A and B if the unit resistor between A and B is deleted. Then, by the parallel-connection theorem, $R_n^{-1} = 1 + S_n^{-1}$. Similarly, we have $r_n^{-1} = 1 + s_n^{-1}$, where s_n is the equivalent resistance between A and C if the unit resistor between A and C is deleted. Now, the wheel is not only highly symmetric but it is also a self-dual network; this implies $s_n S_n = 1$ which, with the two identities above, is (8).

Secondly, let $R_{n,m}$ denote the equivalent resistance between A and some vertex D along the circumference of the wheel such that there are m unit resistors between A and D, $0 < m < n$. Obviously, $R_{n,1} = R_n$, and in this case the input current is $i_1 - 3i_0$, while the applied voltage is $-2i_0$. Now, let this very current enter the wheel at A, let it leave at B, let it again enter at B, let it leave at the next vertex, and so on, until it leaves the wheel at D. The new set of currents is obtained by simply adding the partial currents of the $m - 1$ steps (at each step the subscripts of the currents increase by 2). For example, the current from C to

TABLE 1

Numerical values

n	u_n	v_n	R_n exact	R_n approximate
1	1	1		
2	1	3		
3	2	4	1/2	.5000 0000
4	3	7	8/15	.5333 3333
5	5	11	6/11	.5454 5455
6	8	18	11/20	.5500 0000
7	13	29	16/29	.5517 2414
8	21	47	58/105	.5523 8095
9	34	76	21/38	.5526 3158
10	55	123	152/275	.5527 2727
11	89	199	110/199	.5527 6382
12	144	322	199/360	.5527 7778
13	233	521	288/521	.5527 8311
14	377	843	1042/1885	.5527 8515
15	610	1364	377/682	.5527 8592
16	987	2207	2728/4935	.5527 8622
17	1597	3571	1974/3571	.5527 8633
18	2584	5778	3571/6460	.5527 8638
19	4181	9349	5168/9349	.5527 8639
20	6765	15127	18698/33825	.5527 8640

$R_\infty = .5527\ 8640\ 4 \cdots$

A becomes

$$\dot{i}_0' = \dot{i}_0 + \dot{i}_2 + \cdots + \dot{i}_{2m-2} .$$

If we know $\dot{i}_0'$, we can calculate $R_{n,m}$ because

$$R_{n,m} = \frac{-2\dot{i}_0'}{\dot{i}_1 - 3\dot{i}_0} . \tag{9}$$

Evidently, $\dot{i}_0'$ can be linearly expressed in $\dot{i}_0$ and $\dot{i}_1$, as follows:

$$\dot{i}_0' = \dot{i}_0 + \sum_{k=1}^{m-1} \dot{i}_{2k} = \dot{i}_0 \sum_{k=0}^{m-1} u_{2k-1} + \dot{i}_1 \sum_{k=1}^{m-1} u_{2k}$$

$$= (1 + u_{2m-2})\dot{i}_0 + (-1 + u_{2m-1})\dot{i}_1 .$$

If this is substituted in (9) we get, after some transformation,

$$R_{n,m} = 2\,\frac{1 + u_{2n-1} - u_{2m-1} + u_{2n-2}\,u_{2m-2} - u_{2m-1}\,u_{2n-3}}{1 + u_{2n-2} + u_{2n}} .$$

To eliminate the product terms in the numerator, we replace them by their explicit expressions in terms of x and y, so as to obtain

$$u_{2n-2}u_{2m-2} - u_{2m-1}u_{2n-3} = -u_{2n-2m-1} ,$$

and therefore

$$(10) \qquad R_{n,m} = R_{n,n-m} = 2\,\frac{1 + u_{2n-1} - u_{2m-1} - u_{2n-2m-1}}{1 + u_{2n-2} + u_{2n}}.$$

This formula holds for $n \geqq 3$ and $0 \leqq m \leqq n$. In fact, the numerator vanishes if m is either n or 0, as it should. From (10) may be derived

$$(11) \qquad R_{n,m} = (u_{2m-1} + u_{2m+1} - 2)R_n + 2(1 - u_{2m-1}),$$

which is useful for numerical calculations if a list of R_n (see Table 1) is available. In particular, (11) gives

$$R_{n,2} = 5R_n - 2, \qquad R_{n,3} = 8(2R_n - 1),$$
$$R_{n,4} = 3(15R_n - 8), \qquad R_{n,5} = 11(11R_n - 6).$$

Again, the general expression (10) can be much simplified if we distinguish between the four possibilities as to the parity of n and of m. Without going into details of proof, we state the final result:

$$(12) \qquad R_{n,m} = \begin{cases} 2u_m u_{n-m}/u_n, & n \text{ even, } m \text{ even,} \\ 2v_m v_{n-m}/5u_n, & n \text{ even, } m \text{ odd,} \\ 2u_m v_{n-m}/v_n, & n \text{ odd, } m \text{ even,} \\ 2v_m u_{n-m}/v_n, & n \text{ odd, } m \text{ odd.} \end{cases}$$

Equivalently, we have

$$(13) \qquad R_{n,m} = \frac{2}{\sqrt{5}}\,\frac{(p^m - 1)(p^{n-m} - 1)}{p^n - 2}, \qquad p = x^2 = \frac{1}{2}(3 + \sqrt{5}).$$

Equation (13) was independently obtained by N. G. de Bruijn. In fact, it was his formula (13) that led us to the establishment of (12) given (10).

Also solved by S. D. Bedrosian (University of Pennsylvania), John W. Cell (North Carolina State University), William D. Fryer (Cornell Aeronautical Laboratory), Charles A. Halijak (Kansas State University), Philip G. Kirmser (Kansas State University), George E. Radke (Philadelphia Electric Company), Sidney Spital (California State Polytechnic College) and the proposer.

Problem 63-15, *On a Periodic Solution of a Differential Equation*, by G. W. Veltkamp (Technological University, Eindhoven).

a) Consider the differential equation

$$(1) \qquad \frac{dy}{dt} + f(y) = p(t),$$

where

(i) p is continuous and periodic with period 1,
(ii) f is continuously differentiable for all y,
(iii) $f(y_2) > f(y_1)$ whenever $y_2 > y_1$,