Rational representations and controller synthesis of L2 behaviors

Citation for published version (APA):
Mutsaers, M. E. C., & Weiland, S. (2012). Rational representations and controller synthesis of L2 behaviors. *Automatica*, *48*(1), 1-14. https://doi.org/10.1016/j.automatica.2011.09.009

DOI:
10.1016/j.automatica.2011.09.009

Document status and date:
Published: 01/01/2012

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

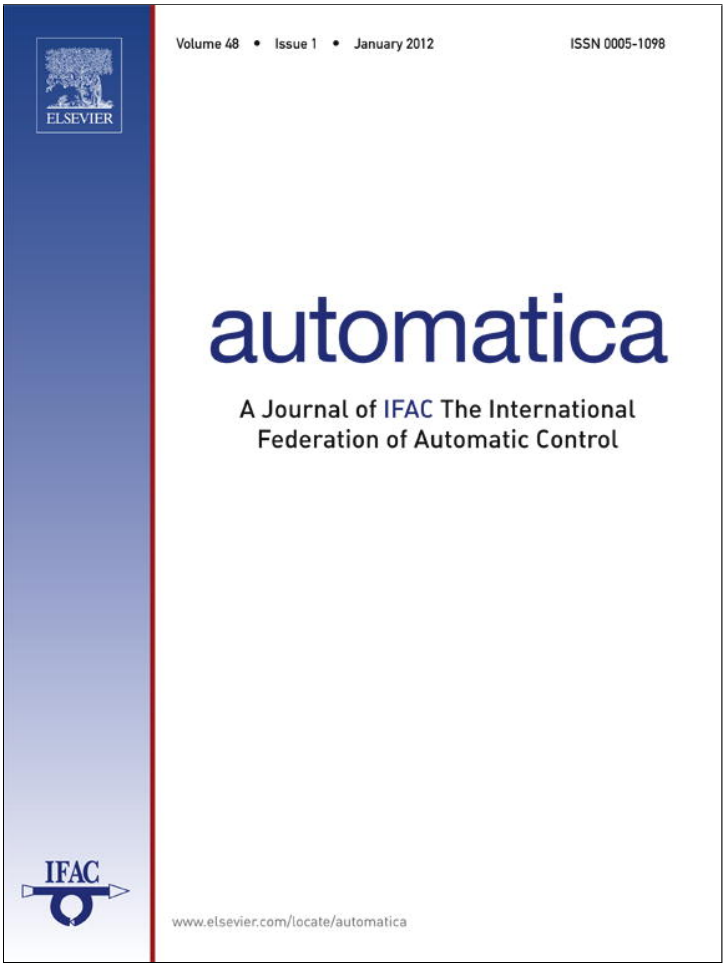
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Download date: 17. Nov. 2023



Volume 48 • Issue 1 • January 2012 ISSN 0005-1098

automatica

A Journal of IFAC The International Federation of Automatic Control

www.elsevier.com/locate/automatica

Automatica 48 (2012) 1–14

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Rational representations and controller synthesis of $\mathscr{L}_2$ behaviors[☆]

Mark Mutsaers[1], Siep Weiland

Eindhoven University of Technology, Department of Electrical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

ARTICLE INFO

Article history:
Received 26 February 2010
Received in revised form
5 June 2011
Accepted 9 June 2011
Available online 11 October 2011

Keywords:
Behavioral systems
$\mathscr{L}_2$ systems
Representation theory
Elimination
Controller synthesis

ABSTRACT

This paper considers linear dynamical systems restricted to square integrable trajectories. Following the behavioral formalism, a number of relevant classes of linear and shift-invariant $\mathscr{L}_2$ systems are defined. It is shown that rational functions, analytic in specific half-spaces of the complex plane, prove most useful for representing such systems. For various classes of $\mathscr{L}_2$ systems, this paper provides a complete characterization of system equivalence in terms of rational kernel representations of $\mathscr{L}_2$ systems. In addition, a complete solution is given for the problem when selected (non-manifest) variables of an $\mathscr{L}_2$ system can be completely eliminated from their behavior. This elimination theorem has considerable independent interest in general modeling problems. It is shown that the elimination result is key in the solution of the problem for realizing an arbitrary $\mathscr{L}_2$ system as the interconnection of a given $\mathscr{L}_2$ system and a to-be-synthesized $\mathscr{L}_2$ system. In the context of control, this problem amounts to characterizing the existence and parameterization of all controllers that, after interconnection with a given plant, constitute a desired controlled system.

1. Introduction

For years, the behavioral theory of dynamical systems has been advocated as a natural vantage point from which to address general questions on modeling, identification, model equivalence and control. Within this theory, quite some research effort has been devoted to studying equations of the form

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0, \tag{1}$$

which represents a system of differential equations in the signal w and where $P(\xi)$ is a polynomial in the indeterminate ξ, with real matrix-valued coefficients. Here, (1) is a compact notation for the general class of systems that can be represented by any finite number of linear, ordinary, constant coefficient differential equations in, say, $\mathtt{w}$ variables that evolve over time. The interest in models of this type stems from the fact that many first-principle modeling exercises naturally lead to systems of ordinary differential equations with real coefficients. Eq. (1) is called a kernel representation of a system and its associated behavior is the set of sufficiently often differentiable functions $w : \mathbb{T} \to \mathbb{R}^{\mathtt{w}}$ (in $\mathtt{w}$ variables and defined on some time set $\mathbb{T} \subset \mathbb{R}$) that satisfy (1). If differentiation in (1) is not understood in a generalized sense of distributions, then there is a technical difficulty about the function space in which solutions w of (1) are assumed to reside. Since many relevant linear, shift-invariant function spaces are dense in the space $\mathcal{C}^\infty$ of infinitely differentiable functions, the restriction to this signal space resolves this complication and is the reason for interpreting the solution set of (1) in this sense.

It is the purpose of this paper to investigate model classes in which solutions of (1) belong to the Lebesgue space of square integrable functions on the time set $\mathbb{T} = \mathbb{R}_+$, $\mathbb{T} = \mathbb{R}_-$ or $\mathbb{T} = \mathbb{R}$. The reason for investigating these model classes lies in the importance of square integrable trajectories in many control questions where performance and stability requirements are specified in terms of square integrable trajectories only. In addition, the study of solutions of (1) restricted to specific Hilbert spaces leads to important questions on system representation and system equivalence.

Although this work is inspired by the study of L_2 systems defined on different time sets, we heavily exploit the fact that the space of square Lebesgue integrable functions on $\mathbb{T} \subseteq \mathbb{R}$ is isomorphic to complex valued Hilbert or Hardy spaces via the (unilateral or bilateral) Laplace transform. Hilbert spaces of complex valued functions $w : \mathbb{C} \to \mathbb{C}^{\mathtt{w}}$ that are square integrable on the imaginary axis (possibly with different domains of analyticity) are closed under multiplication with rational functions $P(s)$ (also with different domains of analyticity). This observation

---

[☆] This work was supported by the Dutch Technologiestichting STW under project number EMR.7851. The material in this paper was partially presented at the 47th IEEE Conference on Decision and Control (CDC 2008), December 9–11, 2008, Cancun, Mexico, and the joint 48th IEEE conference on Decision and Control and 28th Chinese Control Conference (CCDC 2009), December 16–18, 2009, Shanghai, PR China. This paper was recommended for publication in revised form by Associate Editor Maria Elena Valcher under the direction of Editor Roberto Tempo.

E-mail addresses: M.E.C.Mutsaers@tue.nl (M. Mutsaers), S.Weiland@tue.nl (S. Weiland).

[1] Tel.: +31 40 2473579; fax: +31 40 2434582.

0005-1098/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2011.09.009

naturally leads one to investigate representations of the form

$$P(s)w(s) = 0, \tag{2}$$

where $w(s)$ is the Laplace transform of a solution of (1) and where $P(s)$ is a *real rational* function (i.e., every entry of P is a quotient of polynomials with real coefficients) in $s \in \mathbb{C}$. Clearly, solutions of (1) with compact support satisfy (2) on taking Laplace transforms. Here, the system associated with (2) with P being real rational will be the collection of all $w \in \mathcal{L}_2$ that satisfy (2). This functional analytic interpretation of (2) proves useful for solving questions on synthesis, representation, normalization, elimination and interconnection of $\mathcal{L}_2$ systems. These questions will be addressed in this paper.

Models inferred from first principles generally lead to higher order differential equations and one may therefore argue that rational kernel operators of the form (2) are less interesting from a general modeling point of view. This is true. However, the functional analytic tools for rational model representations allow for possibilities such as scaling, normalization, projection and approximation that cannot be paralleled by polynomial methods. It is for this reason that a thorough understanding of system representations by rational operators prove a useful alternative to (polynomial) differential operators. Earlier investigations in, e.g., Trentelman (2010), Trentelman, Yoe, and Praagman (2007) and Willems and Yamamoto (2007) have studied interpretations of (1) with rational functions P. In these papers, solutions of (1) with rational P are defined by all infinitely often differentiable functions w that satisfy the polynomial differential equation $N(\mathrm{d}/\mathrm{d}t)w = 0$, where N is a (or any) factor in the left-coprime factorization $P = D^{-1}N$ of P over the ring of polynomials. In this paper we take a different point of view. First, we do not consider $\mathcal{C}^\infty$ signals with time as the independent variable, but rather work with the Hardy spaces $\mathcal{H}_2^+, \mathcal{H}_2^-$ or the Hilbert space $\mathcal{L}_2$ as signal spaces of interest. Second, we exploit the inner product structure on the signal space to infer a rich theory on rational representations of dynamical systems. This paper extends a number of results that were obtained in Weiland and Stoorvogel (1997) for a class of discrete ℓ_2 systems to continuous time systems.

The outline of the paper is as follows. First, questions of system equivalence, elimination and synthesis that will be discussed in this paper are introduced in Section 2. Section 3 introduces notation. Section 4 deals with rational representations of $\mathcal{L}_2$ systems. Three classes of $\mathcal{L}_2$ systems are introduced and we present for each model class complete results on system equivalence and for the elimination of latent variables. The roles of $\mathcal{L}_2$ behaviors in interconnected systems and specific controller synthesis problems are discussed in Section 5. Conclusions are presented in Section 6. All proofs are collected in the Appendix.

2. Problem formulation

Following the behavioral formalism, a dynamical system (Belur, 2003; Willems, 1989, 2007) is a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B}), \tag{3}$$

where $\mathbb{T} \subseteq \mathbb{R}$ or $\mathbb{T} \subseteq \mathbb{C}$ is the time or frequency axis, $\mathbb{W}$ is the signal space, which will be a w-dimensional vector space throughout, and $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the behavior, that is defined in more explicit terms in Section 4.

We consider $\mathcal{L}_2$ behaviors, which are closed, shift-invariant subspaces of $\mathcal{L}_2$. This means that, contrary to the usual behavioral models, ours does not consider function classes with time as the independent variable but uses frequency, i.e., $\mathbb{T} = \mathbb{C}$. More particularly, we distinguish between closed, invariant subspaces of $\mathcal{L}_2$ that contain the open right complex half-plane, the open left complex half-plane or the imaginary axis in their domain of analyticity. This leads to three distinct classes of $\mathcal{L}_2$ systems, each of them allowing system representations as the kernel (null space) of a rational operator. For each of these classes, three main questions will be addressed:

1. *System equivalence*. In the context of this paper, the question of system equivalence means for finding conditions under which two rational operators represent the same behavior. We provide a complete answer to this question for each of the three system classes.
2. *Elimination of variables and equations*. The elimination problem amounts to finding conditions under which a distinguished auxiliary variable can be completely eliminated from the defining equations of a system. Specifically, we consider systems $\Sigma = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathcal{B}_{\text{full}})$ whose behaviors are closed, invariant subsets of $\mathcal{L}_2$ and described by the kernel of a rational operator. Variables of such systems consist of pairs (w, ℓ) with w a manifest variable that is of interest to the user, and ℓ a latent variable that is used as an auxiliary variable for describing the model. Every latent variable system induces a system whose behavior $\mathcal{B} = \{w \mid \exists \ell \text{ such that } (w, \ell) \in \mathcal{B}_{\text{full}}\}$ is the projection of the latent variable behavior on its manifest variable. We will be interested in finding necessary and sufficient conditions under which the induced manifest behavior again admits a rational kernel representation. We address this question for each of the three model classes of $\mathcal{L}_2$ systems. For locally integrable or infinitely differentiable solutions of polynomial differential operators, a complete answer to this question has been given in Polderman (1997) and Polderman and Willems (1998).
3. *Synthesis of controlled systems*. Controlled systems $\Sigma_K = (\mathbb{T}, \mathbb{W}_K, \mathcal{K})$ are obtained either by full or by partial interconnections of systems $\Sigma_P = (\mathbb{T}, \mathbb{W}_P, \mathcal{P})$ and $\Sigma_C = (\mathbb{T}, \mathbb{W}_C, \mathcal{C})$ that are referred to as plants and controllers, respectively. Fig. 1(a)–(b) illustrate the main idea. The plant Σ_P and controller Σ_C share a distinguished variable c, called the *interconnection variable* that is constrained by the joint laws of Σ_P and Σ_C. For full interconnections, all variables are shared, which means that the interconnection is simply the intersection $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$. Partial interconnections are more general as the interconnection variable c is not necessarily manifest. For a given plant Σ_P and a desired controlled system Σ_K, the *controller synthesis problem* amounts to synthesizing, if it exists, a controller Σ_C that after interconnection with Σ_P results in the desired controlled system Σ_K. In Section 5, this problem is addressed for both full and partial interconnections. Existence and non-uniqueness of controllers are characterized, and we aim to parameterize all controllers that establish a desired controlled system after (full or partial) interconnection. As mentioned in the introduction, earlier research for cases with infinitely smooth behaviors has been carried out for this problem in Trentelman et al. (2007) and Willems (1997, 2007).

3. Notation

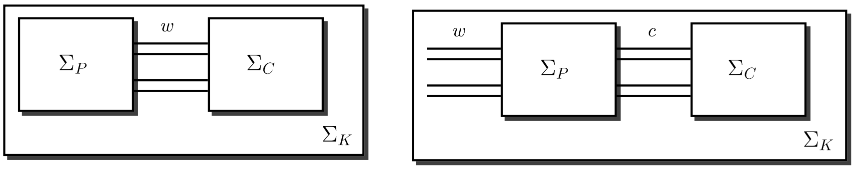
Hardy spaces are denoted by $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$, where $p = 1, 2, \ldots, \infty$, and defined by

$$\mathcal{H}_p^+ := \{f : \mathbb{C}_+ \to \mathbb{C}^{\mathrm{w}} \mid \|f\|_{\mathcal{H}_p^+} < \infty, f \text{ is analytic}\}, \quad \text{and}$$

$$\mathcal{H}_p^- := \{f : \mathbb{C}_- \to \mathbb{C}^{\mathrm{w}} \mid \|f\|_{\mathcal{H}_p^-} < \infty, f \text{ is analytic}\},$$

where $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$, $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$ and $s = \sigma + j\omega$. So, functions in $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$ are analytic in $\mathbb{C}_+$ and in $\mathbb{C}_-$, respectively, and their norm is defined as

$$\|f\|_{\mathcal{H}_p^+} = \begin{cases} \sup\limits_{\sigma>0} \left(\dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} |f(\sigma + j\omega)|^p \mathrm{d}\omega \right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \sup\limits_{\sigma>0} \sup\limits_{\omega\in\mathbb{R}} |f(\sigma + j\omega)|, & p = \infty, \end{cases}$$



(a) Full interconnection.

(b) Partial interconnection.

Fig. 1. Interconnection problems.

with $\|f\|_{\mathcal{H}_p^-}$ similarly defined for functions in $\mathcal{H}_p^-$. Here, $|f| := \sqrt{\sum_{i=1}^{w} |f_i|^2}$ denotes the Euclidean norm. It is well known that the tangential limits $\sigma \downarrow 0$ in the above expressions exist and belong to $\mathcal{L}_2$, which makes $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ closed subspaces of $\mathcal{L}_2$; cf. Francis (1987). Since $\mathcal{L}_2 = \mathcal{H}_2^- \oplus \mathcal{H}_2^+$, elements $w \in \mathcal{L}_2$ can be uniquely decomposed as $w = w_+ + w_-$ with $w_+ = \Pi_+ w \in \mathcal{H}_2^+$ and $w_- := \Pi_- w \in \mathcal{H}_2^-$. Here, $\Pi_+ : \mathcal{L}_2 \to \mathcal{H}_2^+$ and $\Pi_- : \mathcal{L}_2 \to \mathcal{H}_2^-$ denote the canonical projections from $\mathcal{L}_2$ to $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$, respectively.

The norm of a complex valued matrix $F \in \mathbb{C}^{p \times q}$ is defined as $\|F\| := \sqrt{\lambda_{\max}(F^*F)}$, which is the largest singular value of F. A complex valued matrix $F(j\omega)$ belongs to the Lebesgue space $\mathcal{L}_\infty$ if its norm $\|F(j\omega)\|$ is essentially bounded for all frequencies $\omega \in \mathbb{R}$. The corresponding norm is defined as $\|F\|_{\mathcal{L}_\infty} := \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \|F(j\omega)\|$. Functions $F : \mathbb{C}_+ \to \mathbb{C}^{p \times q}$, analytic on $\mathbb{C}_+$, belong to the space $\mathcal{H}_\infty^+$ if the norm $\|F\|_{\mathcal{H}_\infty^+} := \sup_{\sigma > 0} \sup_{\omega \in \mathbb{R}} \|F(\sigma + j\omega)\| < \infty$. The space $\mathcal{H}_\infty^-$ is defined similarly.

The prefixes $\mathcal{R}$ and $\mathcal{U}$ denote rational matrices and units in the Hardy spaces $\mathcal{H}_\infty^+$ and $\mathcal{H}_\infty^-$, as, e.g., $\mathcal{RH}_\infty^- := \{F \in \mathcal{H}_\infty^- \mid F \text{ is rational}\}$ and $\mathcal{UH}_\infty^- := \{U \in \mathcal{RH}_\infty^- \mid U^{-1} \in \mathcal{RH}_\infty^-\}$. Note that units are necessarily square rational matrices. Elements of $\mathcal{RH}_\infty^+$ and $\mathcal{RH}_\infty^-$ will be referred to as stable and anti-stable functions, respectively. See Francis (1987) and Vidyasagar (1985) for more details about Hardy spaces.

The ring $\mathcal{RH}_\infty^+$ admits an extension that consists of stable rational functions with possible poles at infinity:

$$\mathcal{RH}_{\infty,*}^+ := \left\{ f \mid \exists k \geq 0, \exists \alpha < 0 \text{ s.t. } \frac{1}{(s-\alpha)^k} f(s) \in \mathcal{RH}_\infty^+ \right\}. \quad (4)$$

Matrix-valued functions in $\mathcal{RH}_{\infty,*}^+$ are understood as matrices whose elements satisfy the right-hand side of (4) with $f : \mathbb{C}_+ \to \mathbb{C}$. Similarly, we define the extension $\mathcal{RH}_{\infty,*}^-$ (resp., $\mathcal{RL}_{\infty,*}$) as the space of complex valued functions f for which there exist $k \geq 0$ and $\alpha > 0$ such that $\frac{1}{(s-\alpha)^k} f(s) \in \mathcal{RH}_\infty^-$ (resp., $\frac{1}{(s-\alpha)^k} f(s) \in \mathcal{RL}_\infty$ for some $k \geq 0$ and $\alpha \neq 0$). These extended spaces are characterized as follows.

Lemma 3.1.

$$\begin{aligned} &\mathcal{RH}_{\infty,*}^+ = \mathcal{RH}_\infty^+ + \mathbb{R}[s], \qquad \mathcal{RH}_{\infty,*}^- = \mathcal{RH}_\infty^- + \mathbb{R}[s], \\ &\mathcal{RL}_{\infty,*} = \mathcal{RL}_\infty + \mathbb{R}[s], \end{aligned} \quad (5)$$

where $\mathbb{R}[s]$ denotes the class of polynomials with real matrix-valued coefficients.

The proof can be found in the Appendix. The space of units in $\mathcal{RH}_{\infty,*}^+$ ($\mathcal{RH}_{\infty,*}^-$, $\mathcal{RL}_{\infty,*}$) is denoted by $\mathcal{UH}_{\infty,*}^+$ and consists of all $U \in \mathcal{RH}_{\infty,*}^+$ ($U \in \mathcal{RH}_{\infty,*}^-$, $U \in \mathcal{RL}_{\infty,*}$) such that $U^{-1} \in \mathcal{RH}_{\infty,*}^+$ ($U^{-1} \in \mathcal{RH}_{\infty,*}^-$, $U^{-1} \in \mathcal{RL}_{\infty,*}$).

Every $P \in \mathcal{RH}_\infty^-$ (or $\tilde{P} \in \mathcal{RH}_\infty^+$) defines the usual multiplication of a Laurent operator in the frequency domain as $(Pw)(s) = P(s)w(s)$. Specifically:

Lemma 3.2. *Let $P \in \mathcal{RH}_\infty^-$ and $\tilde{P} \in \mathcal{RH}_\infty^+$ define multiplicative operators $(Pw)(s) = P(s)w(s)$ and $(\tilde{P}w)(s) = \tilde{P}(s)w(s)$, with possible domains $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$. Then*

$$\begin{aligned} &P : \mathcal{L}_2 \to \mathcal{L}_2, \qquad P : \mathcal{H}_2^+ \to \mathcal{L}_2, \qquad P : \mathcal{H}_2^- \to \mathcal{H}_2^-, \\ &\tilde{P} : \mathcal{L}_2 \to \mathcal{L}_2, \qquad \tilde{P} : \mathcal{H}_2^+ \to \mathcal{H}_2^+, \qquad \tilde{P} : \mathcal{H}_2^- \to \mathcal{L}_2. \end{aligned}$$

The kernel (or null space) of a rational multiplication operator P defined on $\mathcal{L}_2$, $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$ is denoted by $\ker P$, $\ker_+ P$ and $\ker_- P$, respectively. Thus, $\ker_+ P = \{w \in \mathcal{H}_2^+ \mid Pw = 0\}$.

Let $P \in \mathcal{RH}_\infty^-$ and consider the corresponding multiplication operators as in Lemma 3.2. P is called $\mathcal{L}_2$, $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$ *inner* if $\|Pw\|_2 = \|w\|_2$ for all $w \in \mathcal{L}_2$, $w \in \mathcal{H}_2^+$ or $w \in \mathcal{H}_2^-$, respectively. We call P co-inner if its Hermitian transpose is inner. A matrix $P \in \mathcal{RH}_\infty^-$ (or $P \in \mathcal{RH}_{\infty,*}^-$) is called *outer* if for every $\lambda \in \mathbb{C}_-$, $P(\lambda)$ has full row rank. If P is outer, then P has a right inverse which is analytic in $\mathbb{C}_-$. It is easily seen that all elements in $\mathcal{UH}_\infty^-$ and $\mathcal{UH}_{\infty,*}^-$ are outer. Outer functions are necessarily square or wide while inner functions are square or tall. Similar definitions apply to $\mathcal{RH}_\infty^+$. For further properties of inner and outer functions, we refer the reader to Francis (1987), Kailath (1980) and Vidyasagar (1985).

The τ-shift operator $\hat{\sigma}_\tau$ on a signal $\hat{w} : \mathbb{R} \to \mathbb{R}$ is defined as

$$(\hat{\sigma}_\tau \hat{w})(t) = \hat{w}(t - \tau).$$

We call $\hat{\sigma}_\tau$ a right (left) shift whenever $\tau > 0$ ($\tau < 0$). Let $\mathcal{L}$, $\mathcal{L}_+$, $\mathcal{L}_-$ denote the usual bilateral and unilateral Laplace transforms defined on square integrable functions on $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$, respectively. We will be interested in operators $\sigma_\tau : \mathcal{L}_2 \to \mathcal{L}_2$, $\sigma_\tau^+ : \mathcal{H}_2^+ \to \mathcal{H}_2^+$ and $\sigma_\tau^- : \mathcal{H}_2^- \to \mathcal{H}_2^-$, with $\tau \in \mathbb{R}$, that commute with the Laplace transform according to $\mathcal{L}\hat{\sigma}_\tau = \sigma_\tau \mathcal{L}$, $\mathcal{L}_+\hat{\sigma}_\tau = \sigma_\tau^+ \mathcal{L}_+$ and $\mathcal{L}_-\hat{\sigma}_\tau = \sigma_\tau^- \mathcal{L}_-$. These operators are defined by setting

$$(\sigma_\tau w)(s) = \mathrm{e}^{-s\tau} w(s),$$

$$(\sigma_\tau^+ w)(s) = \begin{cases} \mathrm{e}^{-s\tau} w(s), & [\tau > 0] \\ \mathrm{e}^{-s\tau} \left(w(s) - \displaystyle\int_0^{-\tau} \hat{w}(t) \mathrm{e}^{-st} \mathrm{d}t \right), & [\tau < 0] \end{cases}$$

$$(\sigma_\tau^- w)(s) = \begin{cases} \mathrm{e}^{-s\tau} \left(w(s) - \displaystyle\int_{-\tau}^{0} \hat{w}(t) \mathrm{e}^{-st} \mathrm{d}t \right), & [\tau > 0] \\ \mathrm{e}^{-s\tau} w(s). & [\tau < 0]. \end{cases}$$

Here, $\hat{w} := \mathcal{L}_+^{-1} w$ for $w \in \mathcal{H}_2^+$ and $\hat{w} := \mathcal{L}_-^{-1} w$ for $w \in \mathcal{H}_2^-$. Obviously, σ_0 is the identity map. Note that $\sigma_\tau : \mathcal{L}_2 \to \mathcal{L}_2$ defines an isometry (for all $\tau \in \mathbb{R}$) and that $\sigma_\tau^+ : \mathcal{H}_2^+ \to \mathcal{H}_2^+$ and $\sigma_\tau^- : \mathcal{H}_2^- \to \mathcal{H}_2^-$ define isometries only if $\tau \geq 0$ and $\tau \leq 0$, respectively. We will drop the superscript $+$ and $-$ in σ_τ^+, σ_τ^- whenever the domain of the operators is clear from the context.

Definition 3.3. A subset $\mathcal{B}$ of $\mathcal{L}_2$ (or $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$) is said to be *left invariant* if $\sigma_\tau \mathcal{B} \subseteq \mathcal{B}$ for all $\tau < 0$. It is said to be *right invariant* if $\sigma_\tau \mathcal{B} \subseteq \mathcal{B}$ for all $\tau > 0$.

4. Equivalence and elimination for rational representations

In this section, behaviors of dynamical systems are defined as closed subspaces of $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ represented by the null spaces of rational operators (Mutsaers & Weiland, 2008). Behavioral inclusion, equivalence and elimination of variables will be discussed in terms of rational operators. The results will be compared with earlier research on infinitely smooth behaviors represented by rational differential operators (Trentelman, 2010; Willems & Yamamoto, 2006, 2008). Throughout this section, we will use the variables w and ℓ, which are elements of $\mathcal{L}_2$, $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$.

4.1. Anti-stable rational operators

Let $P \in \mathcal{RH}_\infty^-$ be a rational operator with $\mathtt{w}$ columns. We associate three dynamical systems with P by setting

$$\begin{aligned} \Sigma &:= (\mathbb{C}, \mathbb{C}^{\mathtt{w}}, \mathcal{B}), \\ \Sigma_+ &:= (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathcal{B}_+), \\ \Sigma_- &:= (\mathbb{C}_-, \mathbb{C}^{\mathtt{w}}, \mathcal{B}_-), \end{aligned} \tag{6a}$$

where

$$\begin{aligned} \mathcal{B} &:= \{w \in \mathcal{L}_2 | Pw = 0\} = \ker P, \\ \mathcal{B}_+ &:= \{w \in \mathcal{H}_2^+ | Pw \in \mathcal{H}_2^-\} = \ker_+ \Pi_+ P, \\ \mathcal{B}_- &:= \{w \in \mathcal{H}_2^- | Pw = 0\} = \ker_- P. \end{aligned} \tag{6b}$$

Here Π_+ denotes the canonical projection $\Pi_+ : \mathcal{L}_2 \to \mathcal{H}_2^+$. The subsets $\mathcal{B} \subset \mathcal{L}_2, \mathcal{B}_+ \subset \mathcal{H}_2^+, \mathcal{B}_- \subset \mathcal{H}_2^-$ define behaviors of dynamical systems Σ, Σ_+, Σ_- (respectively) in the frequency domain, i.e. as subsets of complex valued functions. We refer to P as a *rational kernel representation* of these systems. The corresponding time domain models of (6a) are inferred via the inverse Laplace transform according to $\hat{\Sigma} := (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathcal{L}^{-1}\mathcal{B})$, $\hat{\Sigma}_+ := (\mathbb{R}_+, \mathbb{R}^{\mathtt{w}}, \mathcal{L}_+^{-1}\mathcal{B}_+)$ and $\hat{\Sigma}_- := (\mathbb{R}_-, \mathbb{R}^{\mathtt{w}}, \mathcal{L}_-^{-1}\mathcal{B}_-)$.

Lemma 4.1. *For $P \in \mathcal{RH}_\infty^-$ the behaviors $\mathcal{B}$, $\mathcal{B}_+$ and $\mathcal{B}_-$ in* (6b) *are closed, left invariant subspaces of $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$, respectively.*

The proof of this lemma can be found in the Appendix. Systems of the form (6) will generally be referred to as left invariant $\mathcal{L}_2$ systems.

Definition 4.2. The classes of all linear and left invariant systems in $\mathcal{L}_2, \mathcal{H}_2^+$ and $\mathcal{H}_2^-$ that admit representations by anti-stable rational operators as in (6) are denoted by $\mathbb{L}, \mathbb{L}_+$ and $\mathbb{L}_-$, respectively.

We call a rational kernel representation P *minimal* if any other rational kernel representation of the system has at least as many rows as P. A rational kernel representation is minimal if and only if P has full row rank. For a dynamical system Σ in the class $\mathbb{L}$, the *output cardinality* of its behavior $\mathcal{B}$ is defined as $\mathtt{p}(\mathcal{B}) = \operatorname{rowrank}(P)$, where $P \in \mathcal{RH}_\infty^-$ represents $\mathcal{B}$ as in (6b). The output cardinality therefore reflects the number of independent restrictions that are imposed on the system. It is easily shown that $\mathtt{p}(\mathcal{B})$ is, in fact, independent of the representation P and that $\mathtt{p}(\mathcal{B})$ can be interpreted as the dimension of the output variable in one (or any) input–output representation of Σ. Similarly, the *input cardinality* of $\mathcal{B}$ is the number $\mathtt{m}(\mathcal{B}) = \mathtt{w} - \mathtt{p}(\mathcal{B})$, which represents the degree of under-determination of the restrictions that the system imposes on its $\mathtt{w}$ variables. For systems in the model classes $\mathbb{L}_+$ and $\mathbb{L}_-$ the input and output cardinality are defined in a similar manner.

A complete characterization of inclusions and equivalence of systems in the model classes $\mathbb{L}$, $\mathbb{L}_+$ and $\mathbb{L}_-$ is given in the following result.

Theorem 4.3 (*Inclusion and Equivalence*)**.** *Let two systems in the class $\mathbb{L}$ (or $\mathbb{L}_+$ or $\mathbb{L}_-$) with behaviors $\mathcal{B}_1$, $\mathcal{B}_2$ (or $\mathcal{B}_{1,+}$, $\mathcal{B}_{2,+}$ or $\mathcal{B}_{1,-}$, $\mathcal{B}_{2,-}$) be represented by full rank $P, Q \in \mathcal{RH}_\infty^-$, respectively, as in* (6)*. We then have:*

1. *inclusions of behaviors:*
    i. $\mathcal{B}_2 \subset \mathcal{B}_1 \iff \exists F \in \mathcal{RL}_{\infty,*}$ *s.t.* $P = FQ$,
    ii. $\mathcal{B}_{2,+} \subset \mathcal{B}_{1,+} \iff \exists F \in \mathcal{RH}_{\infty,*}^-$ *s.t.* $P = FQ$,
    iii. $\mathcal{B}_{2,-} \subset \mathcal{B}_{1,-} \iff \exists F \in \mathcal{RL}_{\infty,*}$ *s.t.* $P = FQ$;
2. *equivalence of behaviors:*
    i. $\mathcal{B}_1 = \mathcal{B}_2 \iff \exists U \in \mathcal{UL}_{\infty,*}$ *s.t.* $P = UQ$,
    ii. $\mathcal{B}_{1,+} = \mathcal{B}_{2,+} \iff \exists U \in \mathcal{UH}_{\infty,*}^-$ *s.t.* $P = UQ$,
    iii. $\mathcal{B}_{1,-} = \mathcal{B}_{2,-} \iff \exists U \in \mathcal{UL}_{\infty,*}$ *s.t.* $P = UQ$;
3. *if, in addition, Q is co-inner, then the statements in item 1 are equivalent to the existence of $F \in \mathcal{RL}_\infty, F \in \mathcal{RH}_\infty^-$ and $F \in \mathcal{RL}_\infty$, in* i–iii *respectively, such that $P = FQ$; if also P is co-inner, then the statements in item 2 are equivalent to the existence of $U \in \mathcal{UL}_\infty, U \in \mathcal{UH}_\infty^-$ and $U \in \mathcal{UL}_\infty$, in* i–iii *respectively, such that $P = UQ$.*

Example 4.4. Let $P(s) = \frac{s+1}{s-1}$ and $Q(s) = \frac{1}{s-2}$. Then $P, Q \in \mathcal{RH}_\infty^-$ and $P = FQ$ with $F(s) = \frac{(s+1)(s-2)}{s-1}$. Since F is analytic in $\mathbb{C}_-$ and $\frac{1}{s-\alpha}F(s) \in \mathcal{RH}_\infty^-$ for any $\alpha > 0$, it follows that $F \in \mathcal{RH}_{\infty,*}^-$. Statement 1ii of Theorem 4.3 thus promises that $\mathcal{B}_{2,+} \subset \mathcal{B}_{1,+}$ where $\mathcal{B}_{1,+} := \ker_+ \Pi_+ P$ and $\mathcal{B}_{2,+} := \ker_+ \Pi_+ Q$. Indeed, $\mathcal{B}_{2,+} = \{0\}$ and $\mathcal{B}_{1,+} = \left\{\frac{c}{s+1} \mid c \in \mathbb{C}\right\} \subset \mathcal{H}_2^+$. Since $\mathcal{B}_{2,+}$ is also represented by the (inner and) co-inner function $Q(s) = 1$, the same conclusion follows from statement 3 of Theorem 4.3 as $P = FQ$ with $F(s) = \frac{s+1}{s-1}$, which belongs to $\mathcal{RH}_\infty^-$.

Example 4.5. Let $P(s) = \begin{bmatrix} 1 & -T(s) \end{bmatrix}$ with $T(s) = \frac{s+1}{s-1}$. Then P defines a system in the model class $\mathbb{L}$ whose behavior $\mathcal{B}_1 = \ker P$ is the $\mathcal{L}_2$ *graph* associated with the transfer function T, i.e., $\mathcal{B}_1 = \{w = (y, u) \in \mathcal{L}_2 \mid y = Tu\}$. If $T = D^{-1}N$ is a normalized left-coprime factorization of T over $\mathcal{RH}_\infty^-$ then $P = UQ$ with $Q = \begin{bmatrix} D & -N \end{bmatrix}$ and $U = D^{-1}$. Since $U \in \mathcal{UL}_{\infty,*}$, statement 2i of Theorem 4.3 claims that $\mathcal{B}_1 = \mathcal{B}_2$ with $\mathcal{B}_2 = \ker Q$. Since $QQ^* = I$, it follows that every system in $\mathbb{L}$ admits a co-inner kernel representation.

Remark 4.6. Theorem 4.3 substantially differs from the equivalence results in Gottimukkala, Fiaz, and Trentelman (2011), Trentelman (2010) and Willems and Yamamoto (2007, 2008) where $\mathcal{C}^\infty$ behaviors are defined as kernels of rational differential operators P. In Gottimukkala et al. (2011), it is shown that the controllable parts of the $\mathcal{C}^\infty$ kernels of rational operators P and Q coincide if and only if there exists a *unitary* matrix $U \in \mathcal{UL}_{\infty,*}$ such that $P = UQ$.

Remark 4.7. The explicit construction of the operators F and U in Theorem 4.3 is an application of the Beurling–Lax theorem (Rosenblum & Rovnyak, 1997). We refer the reader to the proof of Theorem 4.3 for details.

Next, we consider latent variable systems for the three model classes $\mathbb{L}, \mathbb{L}_+$ and $\mathbb{L}_-$. Let $\Sigma_\ell = (\mathbb{C}, \mathbb{C}^{\mathtt{w}} \times \mathbb{C}^\ell, \mathcal{B}_{\text{full}}) \in \mathbb{L}$ be a system in which variables are decomposed into a manifest variable w and a latent variable ℓ. Let $\Sigma_{\ell,+} \in \mathbb{L}_+$ and $\Sigma_{\ell,-} \in \mathbb{L}_-$ denote latent variable systems with behaviors $\mathcal{B}_{\text{full},+}$ and $\mathcal{B}_{\text{full},-}$ with a similar variable decomposition. This means that there exists $P = [P_1 \quad P_2] \in \mathcal{RH}_\infty^-$ such that

$$\begin{aligned}
\mathcal{B}_{\text{full}} &:= \left\{ \operatorname{col}(w,\ell) \in \mathcal{L}_2 \mid P\begin{bmatrix} w \\ \ell \end{bmatrix} = 0 \right\} = \ker P, \\
\mathcal{B}_{\text{full},+} &:= \left\{ \operatorname{col}(w,\ell) \in \mathcal{H}_2^+ \mid P\begin{bmatrix} w \\ \ell \end{bmatrix} \in \mathcal{H}_2^- \right\} = \ker_+ \Pi_+ P, \\
\mathcal{B}_{\text{full},-} &:= \left\{ \operatorname{col}(w,\ell) \in \mathcal{H}_2^- \mid P\begin{bmatrix} w \\ \ell \end{bmatrix} = 0 \right\} = \ker_- P,
\end{aligned} \tag{7}$$

where P is decomposed according to the variables (w, ℓ). Associate with (7) the manifest behaviors

$$\begin{aligned}
\mathcal{B}_{\text{manifest}} &:= \{ w \in \mathcal{L}_2 \mid \exists \ell \in \mathcal{L}_2 \text{ s.t. } \operatorname{col}(w,\ell) \in \mathcal{B}_{\text{full}} \}, \\
\mathcal{B}_{\text{manifest},+} &:= \{ w \in \mathcal{H}_2^+ \mid \exists \ell \in \mathcal{H}_2^+ \text{ s.t. } \operatorname{col}(w,\ell) \in \mathcal{B}_{\text{full},+} \}, \\
\mathcal{B}_{\text{manifest},-} &:= \{ w \in \mathcal{H}_2^- \mid \exists \ell \in \mathcal{H}_2^- \text{ s.t. } \operatorname{col}(w,\ell) \in \mathcal{B}_{\text{full},-} \}.
\end{aligned}$$

That is, the manifest behaviors consist of the projection of the full behaviors on the manifest variable w. From a general modeling point of view, the modeler is interested in the manifest behavior only, but the representation of this system is typically implicitly described by means of auxiliary or latent variables. We therefore address the question of when the manifest behaviors define systems in $\mathbb{L}$, $\mathbb{L}_+$ and $\mathbb{L}_-$, respectively, and whether one can find explicit representations for the manifest system. This is formalized as follows.

Definition 4.8. The full behaviors in (7) are said to be ℓ-eliminable if there exists a $P' \in \mathcal{RH}_\infty^-$ such that

$$\begin{aligned}
\mathcal{B}_{\text{manifest}} &= \{ w \in \mathcal{L}_2 \mid P'w = 0 \} = \ker P' \quad \text{or} \\
\mathcal{B}_{\text{manifest},+} &= \{ w \in \mathcal{H}_2^+ \mid P'w \in \mathcal{H}_2^- \} = \ker_+ \Pi_+ P' \quad \text{or} \\
\mathcal{B}_{\text{manifest},-} &= \{ w \in \mathcal{H}_2^- \mid P'w = 0 \} = \ker_- P'.
\end{aligned}$$

Thus, in an ℓ-eliminable system, one can find a kernel representation for its induced manifest behavior. The following elimination theorem is the main result of this section.

Theorem 4.9 (*Elimination*). *Let $P = [P_1 \quad P_2] \in \mathcal{RH}_\infty^-$ be full row rank and define the full system behaviors as in (7) and consider the equation*

$$Q = P_1 + P_2 X. \tag{8}$$

We have, with respect to (8), that

$$\begin{aligned}
\mathcal{B}_{\text{full}} \text{ is } \ell\text{-eliminable} &\iff \exists X \in \mathcal{RL}_\infty \text{ s.t. } Q \in \mathcal{RH}_\infty^- \\
&\qquad \text{and } \operatorname{rowrank}(Q) = \\
&\qquad \mathtt{p}(\mathcal{B}_{\text{full}}) - \operatorname{rowrank}(P_2), \\
\mathcal{B}_{\text{full},+} \text{ is } \ell\text{-eliminable} &\iff \exists X \in \mathcal{RH}_\infty^+ \text{ s.t. } Q \in \mathcal{RH}_\infty^- \\
&\qquad \text{and } \operatorname{rowrank}(Q) = \\
&\qquad \mathtt{p}(\mathcal{B}_{\text{full},+}) - \operatorname{rowrank}(P_2), \\
\mathcal{B}_{\text{full},-} \text{ is } \ell\text{-eliminable} &\iff \exists X \in \mathcal{RH}_\infty^- \text{ s.t. } Q \in \mathcal{RH}_\infty^- \\
&\qquad \text{and } \operatorname{rowrank}(Q) = \\
&\qquad \mathtt{p}(\mathcal{B}_{\text{full},-}) - \operatorname{rowrank}(P_2).
\end{aligned}$$

Moreover, in each of these cases, the corresponding manifest behavior of Definition 4.8 is represented by the rational operator $P' = Q$.

The elimination problem has been investigated earlier. For polynomial representations of $\mathcal{C}^\infty$ systems, it has been shown in Polderman and Willems (1998) that elimination of latent variables is always possible. The same result has been obtained for discrete time systems. The elimination problem for $\mathcal{C}^\infty$ solutions of rational differential operators has been mentioned in Willems and Yamamoto (2008); however no concrete solution was presented in that paper. Theorem 4.9 shows that in the context of the Hardy and Lebesgue spaces, that we introduced here, elimination of latent variables from systems in the model classes $\mathbb{L}$, $\mathbb{L}_+$ and $\mathbb{L}_-$ is only possible under the stated conditions. For results of eliminability in terms of conditions from geometric control theory, we refer the reader to Mutsaers and Weiland (2010, 2011).

Example 4.10. Consider the latent variable system with behavior given by

$$\mathcal{B}_\ell = \left\{ (w,\ell) \in \mathcal{H}_2^+ \,\middle|\, \begin{bmatrix} 2\dfrac{(s-2)(s-3)}{(s-7)(s-8)} & \dfrac{s-\alpha}{s-7} \\ \dfrac{s+4}{s-8} & 0 \end{bmatrix} \begin{bmatrix} w \\ \ell \end{bmatrix} \in \mathcal{H}_2^- \right\}.$$

Here, α is a non-zero real constant. By Theorem 4.9 this system is ℓ-eliminable if there exists $X \in \mathcal{RH}_\infty^+$ such that Q in (8) belongs to $\mathcal{RH}_\infty^-$ and satisfies the proper rank conditions. This implies that

$$2\frac{(s-2)(s-3)}{(s-7)(s-8)} + \frac{s-\alpha}{s-7}X(s) \in \mathcal{RH}_\infty^-, \tag{9}$$

and the rank condition implies that

$$2\frac{(s-2)(s-3)}{(s-7)(s-8)} + \frac{s-\alpha}{s-7}X(s) = U(s)\frac{s+4}{s-8},$$

for some $U \in \mathcal{UH}_\infty^-$. Since the poles in the right part of this equation are always in $\mathbb{C}_+$, the left part should also satisfy this. However, the poles of X are in $\mathbb{C}_-$. Hence, $\alpha < 0$ is a necessary condition for ℓ-eliminability. It follows that this system is ℓ-eliminable if and only if $\alpha < 0$. Indeed, with $X(s) = -\frac{s-3}{s-\alpha}$ and $U(s) = \frac{s-3}{s-7} \in \mathcal{UH}_\infty^-$ we obtain that

$$\frac{s-3}{s-7}\left(2\frac{s-2}{s-8} - 1\right) = \frac{s-3}{s-7}\left(\frac{s+4}{s-8}\right),$$

which fulfills the rank condition. Moreover, also (9) holds with $X \in \mathcal{RH}_\infty^+$ if and only if $\alpha < 0$.

4.2. Stable rational operators

So far, we have considered anti-stable rational operators for defining $\mathcal{L}_2$ systems. This subsection defines model classes of $\mathcal{L}_2$ systems through *stable* rational operators. The material in this subsection is analogous to that in the previous subsection and will therefore be stated without further discussion or proof. Let $\tilde{P} \in \mathcal{RH}_\infty^+$ and consider the following three dynamical systems:

$$\begin{aligned}
\Sigma &:= (\mathbb{C}, \mathbb{C}^{\mathtt{w}}, \mathcal{B}), \\
\Sigma_+ &:= (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathcal{B}_+), \\
\Sigma_- &:= (\mathbb{C}_-, \mathbb{C}^{\mathtt{w}}, \mathcal{B}_-),
\end{aligned} \tag{10a}$$

where

$$\begin{aligned}
\mathcal{B} &:= \{ w \in \mathcal{L}_2 | \tilde{P}w = 0 \} = \ker \tilde{P}, \\
\mathcal{B}_+ &:= \{ w \in \mathcal{H}_2^+ | \tilde{P}w = 0 \} = \ker_+ \tilde{P}, \\
\mathcal{B}_- &:= \{ w \in \mathcal{H}_2^- | \tilde{P}w \in \mathcal{H}_2^+ \} = \ker_- \Pi_- \tilde{P}.
\end{aligned} \tag{10b}$$

Here, Π_- is the canonical projection from $\mathcal{L}_2$ onto $\mathcal{H}_2^-$.

Lemma 4.11. *For $\tilde{P} \in \mathcal{RH}_\infty^+$, the behaviors $\mathcal{B}$, $\mathcal{B}_+$, and $\mathcal{B}_-$ in (10b) are closed, right invariant subspaces of $\mathcal{L}_2$, $\mathcal{H}_2^+$, and $\mathcal{H}_2^-$, respectively.*

Hence, kernels of anti-stable rational operators define left invariant subspaces, and kernels of stable rational operators are right invariant.

Definition 4.12. The classes of all linear and right invariant systems in $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ that admit representations by stable rational operators as in (10) are denoted by $\mathbb{M}$, $\mathbb{M}_+$ and $\mathbb{M}_-$, respectively.

Theorem 4.13 (*Inclusion and Equivalence*)**.** *Let two systems in the class $\mathbb{M}$ (or $\mathbb{M}_+, \mathbb{M}_+$) with behaviors $\mathscr{B}_1, \mathscr{B}_2$ (or $\mathscr{B}_{1,+}, \mathscr{B}_{2,+}$ or $\mathscr{B}_{1,-}, \mathscr{B}_{2,-}$) be represented by full rank $\tilde P, \tilde Q \in \mathscr{RH}_\infty^+$, respectively, as in (10). We then have:*

1. *inclusions of behaviors:*
   - i. $\mathscr{B}_2 \subset \mathscr{B}_1 \iff \exists \tilde F \in \mathscr{RL}_{\infty,*}$ *s.t.* $\tilde P = \tilde F \tilde Q$,
   - ii. $\mathscr{B}_{2,+} \subset \mathscr{B}_{1,+} \iff \exists \tilde F \in \mathscr{RL}_{\infty,*}$ *s.t.* $\tilde P = \tilde F \tilde Q$,
   - iii. $\mathscr{B}_{2,-} \subset \mathscr{B}_{1,-} \iff \exists \tilde F \in \mathscr{RH}_{\infty,*}^+$ *s.t.* $\tilde P = \tilde F \tilde Q$;
2. *equivalence of behaviors:*
   - i. $\mathscr{B}_1 = \mathscr{B}_2 \iff \exists \tilde U \in \mathscr{UL}_{\infty,*}$ *s.t.* $\tilde P = \tilde U \tilde Q$,
   - ii. $\mathscr{B}_{1,+} = \mathscr{B}_{2,+} \iff \exists \tilde U \in \mathscr{UL}_{\infty,*}$ *s.t.* $\tilde P = \tilde U \tilde Q$,
   - iii. $\mathscr{B}_{1,-} = \mathscr{B}_{2,-} \iff \exists \tilde U \in \mathscr{UH}_{\infty,*}^+$ *s.t.* $\tilde P = \tilde U \tilde Q$;
3. *if in addition $\tilde Q$ is co-inner, then the statements in item 1 are equivalent to the existence of $\tilde F \in \mathscr{RL}_\infty$, $\tilde F \in \mathscr{RL}_\infty$ and $\tilde F \in \mathscr{RH}_\infty^+$, in* i–iii *respectively, such that $\tilde P = \tilde F \tilde Q$; if also $\tilde P$ is co-inner, then the statements in item 2 are equivalent with the existence of $\tilde U \in \mathscr{UL}_\infty$, $\tilde U \in \mathscr{UL}_\infty$ and $\tilde U \in \mathscr{UH}_\infty^+$, in* i–iii *respectively, such that $\tilde P = \tilde U \tilde Q$.*

Next, consider the elimination problem for latent variable systems in the model classes $\mathbb{M}$, $\mathbb{M}_+$ and $\mathbb{M}_-$. Let $\tilde P = [\tilde P_1 \quad \tilde P_2] \in \mathscr{RH}_\infty^+$ be decomposed according to the partition of the variable $\mathrm{col}(w, \ell)$ and consider $\mathscr{B}_{\mathrm{full}} = \ker \tilde P$, $\mathscr{B}_{\mathrm{full},+} = \ker_+ \tilde P$ and $\mathscr{B}_{\mathrm{full},-} = \ker_- \Pi_- \tilde P$, as defined in a similar manner as in (7). In the following result we provide necessary and sufficient conditions for the complete elimination of the variable ℓ and an explicit representation of the corresponding manifest behaviors as kernels of stable rational operators:

Theorem 4.14 (*Elimination*)**.** *Let $\tilde P = [\tilde P_1 \quad \tilde P_2] \in \mathscr{RH}_\infty^+$ be full row rank and define full system behaviors as in (10) and consider the equation*

$$\tilde Q = \tilde P_1 + \tilde P_2 \tilde X. \tag{11}$$

We have, with respect to (11), that

$$\begin{aligned}
\mathscr{B}_{\mathrm{full}} \text{ is } \ell\text{-eliminable} \iff{}& \exists \tilde X \in \mathscr{RL}_\infty \text{ s.t. } \tilde Q \in \mathscr{RH}_\infty^+ \\
&\text{and } \mathrm{rowrank}(\tilde Q) = \\
&\mathrm{p}(\mathscr{B}_{\mathrm{full}}) - \mathrm{rowrank}(\tilde P_2), \\
\mathscr{B}_{\mathrm{full},+} \text{ is } \ell\text{-eliminable} \iff{}& \exists \tilde X \in \mathscr{RH}_\infty^+ \text{ s.t. } \tilde Q \in \mathscr{RH}_\infty^+ \\
&\text{and } \mathrm{rowrank}(\tilde Q) = \\
&\mathrm{p}(\mathscr{B}_{\mathrm{full},+}) - \mathrm{rowrank}(\tilde P_2), \\
\mathscr{B}_{\mathrm{full},-} \text{ is } \ell\text{-eliminable} \iff{}& \exists \tilde X \in \mathscr{RH}_\infty^- \text{ s.t. } \tilde Q \in \mathscr{RH}_\infty^+ \\
&\text{and } \mathrm{rowrank}(\tilde Q) = \\
&\mathrm{p}(\mathscr{B}_{\mathrm{full},-}) - \mathrm{rowrank}(\tilde P_2).
\end{aligned}$$

Moreover, in each of these cases, the corresponding manifest behavior is represented as the kernel of the stable rational operator $\tilde Q$.

The proofs of Theorems 4.13 and 4.14 are similar to the proofs of Theorems 4.3 and 4.9 and are not included in this paper.

5. Controller synthesis

This section answers the third question posed in Section 2, namely the controller synthesis problem. Given are two systems Σ_P and Σ_K, both represented by means of rational kernel representations. We address the question of how to synthesize a third system Σ_C, belonging to the same model class as Σ_P and Σ_K, such that the interconnection of Σ_P and Σ_C coincides with Σ_K. Because this question is of evident interest in control, we will refer to Σ_P as the *plant*, to Σ_C as the *controller* and to Σ_K as the *controlled system*. The problem then amounts to synthesizing a controller for a given plant that yields a given controlled system after interconnecting plant and controller. Here, we distinguish between full and partial interconnections as explained in Section 2. For the latter case, we will illustrate the results obtained by giving an example.

In this section, we focus on the system class $\mathbb{L}_+$. However, all results extend to the system classes $\mathbb{L}$, $\mathbb{L}_-$ and $\mathbb{M}_{(\pm)}$ without additional technical problems. For simplicity of notation, throughout this section we omit the subscript $_+$ in the definitions of systems (Σ) and their corresponding behaviors.

5.1. The full interconnection problem

For systems in the class $\mathbb{L}_+$, the synthesis problem by *full interconnection* is formalized as follows.

Problem 5.1. Let two systems $\Sigma_P = (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathscr{P}) \in \mathbb{L}_+$ and $\Sigma_K = (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathscr{K}) \in \mathbb{L}_+$ be given.

- i. Verify whether there exists $\Sigma_C = (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathscr{C}) \in \mathbb{L}_+$ such that $\mathscr{P} \cap \mathscr{C} = \mathscr{K}$. Any such system is said to *implement $\mathscr{K}$ for $\mathscr{P}$* by *full interconnection* through w.
- ii. If such a controller exists, find a representation $C_0 \in \mathscr{RH}_\infty^-$ for the system Σ_C, in the sense that its behavior $\mathscr{C} = \ker_+ \Pi_+ C_0$ implements $\mathscr{K}$ for $\mathscr{P}$.
- iii. Characterize the set $\mathbb{C}_{\mathrm{par}}$ of all $C \in \mathscr{RH}_\infty^-$ for which the behavior $\mathscr{C} = \ker_+ \Pi_+ C$ implements $\mathscr{K}$ for $\mathscr{P}$.

The synthesis algorithm that will be derived in this section is inspired by the polynomial analog that has been treated in Polderman and Willems (1998) and Trentelman et al. (2007). Specifically, we provide an explicit algorithm that leads to the set of all rational representations of behaviors $\mathscr{C}$ that implement $\mathscr{K}$ for $\mathscr{P}$. The main result is stated as follows.

Theorem 5.2. *Let the systems $\Sigma_P = (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathscr{P}) \in \mathbb{L}_+$ and $\Sigma_K = (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathscr{K}) \in \mathbb{L}_+$ be represented by the rational operators $P, K \in \mathscr{RH}_\infty^-$, respectively.*

- i. *There exists a controller $\Sigma_C = (\mathbb{C}_+, \mathbb{C}^{\mathtt{w}}, \mathscr{C}) \in \mathbb{L}_+$ that implements $\mathscr{K}$ for $\mathscr{P}$ by full interconnection if and only if there exists an outer function $X \in \mathscr{RH}_{\infty,*}^-$ such that $P = XK$.*
- ii. *The set $\mathbb{C}_{\mathrm{par}}$ of all possible kernel representations of controllers that implement $\mathscr{K}$ for $\mathscr{P}$ by full interconnection is given as the output of Algorithm 5.4.*

By Theorem 4.3, the condition in item i of Theorem 5.2 implies that $\mathscr{K} \subset \mathscr{P}$. Hence, the inclusion $\mathscr{K} \subset \mathscr{P}$ is a necessary condition for the existence of a controller that implements $\mathscr{K}$ for $\mathscr{P}$ by full interconnection. This condition is, however, not sufficient. This is unlike the situation for $\mathscr{C}^\infty$ behaviors discussed in Polderman and Willems (1998) and Trentelman et al. (2007) where the inclusion $\mathscr{K} \subset \mathscr{P}$ is a necessary and sufficient condition for guaranteeing the existence of a $(\mathscr{C}^\infty)$ controller that implements $\mathscr{K}$ for $\mathscr{P}$. The fact that we consider systems in $\mathbb{L}_+$ over the function space $\mathscr{H}_2^+$ therefore makes an important difference in synthesis questions when compared to $\mathscr{C}^\infty$ systems. We illustrate this in the following example.

Example 5.3. Given is the plant behavior $\mathscr{P} = \ker_+ \Pi_+ P$ with $P = \begin{bmatrix} \frac{(s-\alpha)(s+2)}{(s-3)(s-4)} & \frac{(s-\alpha)(s+5)}{(s-1)(s-2)} \end{bmatrix} \in \mathscr{RH}_\infty^-$, with α a non-zero real constant. The desired controlled behavior $\mathscr{K} = \ker_+ \Pi_+ K$ is represented by $K = \mathrm{diag}\left(\frac{s+2}{s-2}, \frac{s+5}{s-1}\right) \in \mathscr{RH}_\infty^-$. By Theorem 5.2, there exists a controller that implements $\mathscr{K}$ for $\mathscr{P}$ if and only if there exists an outer $X \in \mathscr{RH}_{\infty,*}^-$ such that $P = XK$. Such an X exists and is given by $X = \left[\frac{s-\alpha}{(s-2)(s-3)(s-4)} \; \frac{s-\alpha}{s-2}\right] \in \mathcal{RH}_\infty^-$, which is outer if and only if $\alpha > 0$. For $\alpha < 0$, we do not fulfill the condition of Theorem 5.2. In that case $\mathcal{K} \subset \mathcal{P}$ and the transient $w(s) = \frac{1}{s-\alpha} w_0 \in \mathcal{H}_2^+$, with $w_0 \in \mathbb{C}^2$ an arbitrary vector, belongs to $\mathcal{P}$ but not to $\mathcal{K}$. Now note that for any controller $\mathcal{C} = \ker_+ \Pi_+ C$, with $C = [C_1 \; C_2] \in \mathcal{RH}_\infty^-$, we have that $\det(\begin{bmatrix} P \\ C \end{bmatrix}) = C_2(s)\frac{(s-\alpha)(s+2)}{(s-3)(s-4)} - C_1(s)\frac{(s-\alpha)(s+5)}{(s-1)(s-2)}$. This implies that $w(s)$ belongs to the full interconnection of $\mathcal{P}$ and $\mathcal{C}$. Conclude that for $\alpha < 0$, we have that $\mathcal{K} \subset \mathcal{P}$ but $\mathcal{K}$ cannot be implemented for $\mathcal{P}$.

The algorithm

The following algorithm yields an explicit construction of all controllers Σ_C that solve Problem 5.1 for the class $\mathbb{L}_+$ of $\mathcal{L}_2$ systems.

Algorithm 5.4. Let $P, K \in \mathcal{RH}_\infty^-$ define the behaviors $\mathcal{P}$ and $\mathcal{K}$ corresponding to the systems $\Sigma_P \in \mathbb{L}_+$ and $\Sigma_K \in \mathbb{L}_+$, respectively.

Aim: Find all $C \in \mathcal{RH}_\infty^-$ that define systems $\Sigma_C \in \mathbb{L}_+$ with behavior $\mathcal{C} = \ker_+ \Pi_+ C$ such that $\mathcal{C}$ implements $\mathcal{K}$ for $\mathcal{P}$ in the sense that $\mathcal{P} \cap \mathcal{C} = \mathcal{K}$ by full interconnection.

Step 1: Find an outer rational function $X \in \mathcal{RH}_{\infty,*}^-$ such that $P = XK$. If no such X exists, the algorithm ends and no controller exists that implements $\mathcal{K}$ for $\mathcal{P}$. In this case, set $\mathbb{C}_{\text{par}} = 0$.

Step 2: Determine a unitary function $U \in \mathcal{UH}_{\infty,*}^-$ which brings X into the form: $\overline{X} = XU = [X_1 \; 0]$, where $X_1 \in \mathcal{UH}_{\infty,*}^-$.

Step 3: Define $W := [0 \; I]U^{-1} \in \mathcal{RH}_{\infty,*}^-$, where the dimension of the identity matrix equals the number of zero-columns in $\overline{X}$.

Step 4: Set $C_0 := WK \in \mathcal{RH}_{\infty,*}^-$. Define $\alpha > 0$ and $k \geq 0$ such that $C := \frac{1}{(s-\alpha)^k} C_0 \in \mathcal{RH}_\infty^-$. Then, let $\mathcal{C} = \ker_+ \Pi_+ C$ be such that $\Sigma_C := (\mathbb{C}_+, \mathbb{C}^{\mathrm{w}}, \mathcal{C}) \in \mathbb{L}_+$ implements $\mathcal{K}$ for $\mathcal{P}$.

Step 5: Set

$$\mathbb{C}_{\text{par}} = \left\{ \frac{1}{(s-\alpha)^k}(Q_1 P + Q_2 WK) \in \mathcal{RH}_\infty^- \mid \right.$$

$$\left. Q_1 \in \mathcal{RH}_{\infty,*}^-, Q_2 \in \mathcal{UH}_{\infty,*}^-, \alpha > 0, k \geq 0 \right\}. \tag{12}$$

Output: $\mathbb{C}_{\text{par}}$ is a parameterization of all controllers Σ_C that implement $\mathcal{K}$ for $\mathcal{P}$ by ranging over all kernel representations $\mathcal{C} = \ker_+ \Pi_+ C$ with $C \in \mathbb{C}_{\text{par}}$.

This explicit construction results in full plant–controller interconnections with the property that $\mathrm{p}(\mathcal{P}) + \mathrm{p}(\mathcal{C}) = \mathrm{p}(\mathcal{K})$. In the terminology used in Polderman and Willems (1998) and Trentelman et al. (2007), these are referred to as *regular interconnections* and they realize the idea that controllers do not duplicate laws that are already present in the plant to establish the controlled system.

5.2. *The partial interconnection problem*

In this subsection we consider the more general synthesis problem with partial interconnections of dynamical systems $\Sigma_P = (\mathbb{C}_+, \mathbb{C}^{\mathrm{w}} \times \mathbb{C}^{\mathrm{c}}, \mathcal{P}_{\text{full}})$ and $\Sigma_K = (\mathbb{C}_+, \mathbb{C}^{\mathrm{w}}, \mathcal{K})$ in the model class $\mathbb{L}_+$, represented by the rational operators $P, K \in \mathcal{RH}_\infty^-$, respectively. Here, Σ_P is a latent variable system as introduced in Section 4, so $P = [P_1 \; P_2]$ is decomposed according to the manifest and latent variables w and c of dimensions w and c, respectively.

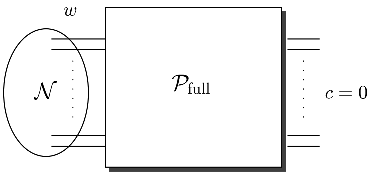


Fig. 2. The hidden behavior $\mathcal{N}$.

Problem 5.5. Let two linear left invariant systems $\Sigma_P = (\mathbb{C}_+, \mathbb{C}^{\mathrm{w}} \times \mathbb{C}^{\mathrm{c}}, \mathcal{P}_{\text{full}}) \in \mathbb{L}_+$ and $\Sigma_K = (\mathbb{C}_+, \mathbb{C}^{\mathrm{w}}, \mathcal{K}) \in \mathbb{L}_+$ be given.

i. Verify whether there exists a linear left invariant system $\Sigma_C = (\mathbb{C}_+, \mathbb{C}^{\mathrm{c}}, \mathcal{C}) \in \mathbb{L}_+$ such that

   $$\mathcal{K} = \{w \in \mathcal{H}_2^+ \mid \exists c \in \mathcal{H}_2^+ \text{ s.t. } (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}.$$

   Any such system is said to *implement* $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$ by *partial* interconnection.

ii. If such a controller exists, find a representation $C \in \mathcal{RH}_\infty^-$ for the system Σ_C in the sense that its behavior $\mathcal{C} = \ker_+ \Pi_+ C$ implements $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$.

To solve this problem, we associate with the system Σ_P a set $\mathcal{N}$ that we refer to as the *hidden behavior*. For the model class $\mathbb{L}_+$ it is defined as

$$\begin{aligned} \mathcal{N} &:= \{w \in \mathcal{H}_2^+ \mid \mathrm{col}(w, 0) \in \mathcal{P}_{\text{full}}\} \\ &= \{w \in \mathcal{H}_2^+ \mid P_1 w \in \mathcal{H}_2^-\} = \ker_+ \Pi_+ P_1, \end{aligned}$$

according to the decomposition made between manifest and latent variables. The hidden behavior is illustrated in Fig. 2 and is named hidden since it is not possible to estimate trajectories in $\mathcal{N}$ by observing the latent variable c only. Problem 5.5 can be solved under suitable conditions as is shown in the following theorem. This result is inspired by the controller implementation theorem introduced in Willems and Trentelman (2002).

Theorem 5.6. *Let the systems $\Sigma_P = (\mathbb{C}_+, \mathbb{C}^{\mathrm{w}} \times \mathbb{C}^{\mathrm{c}}, \mathcal{P}_{\text{full}}) \in \mathbb{L}_+$ and $\Sigma_K = (\mathbb{C}_+, \mathbb{C}^{\mathrm{w}}, \mathcal{K}) \in \mathbb{L}_+$ be represented by $P, K \in \mathcal{RH}_\infty^-$, respectively. Let $P = [P_1 \; P_2]$ be decomposed according to w and c. Suppose that $\mathcal{P}_{\text{full}}$ is c-eliminable. Then $\mathcal{N} = \ker_+ \Pi_+ P_1$ and, by Theorem 4.9, there exists $P_{\text{man}} \in \mathcal{RH}_\infty^-$ such that $\mathcal{P}_{\text{manifest}} = \ker_+ \Pi_+ P_{\text{man}}$. Moreover, there exists a controller $\Sigma_C = (\mathbb{C}_+, \mathbb{C}^{\mathrm{c}}, \mathcal{C}) \in \mathbb{L}_+$ that implements $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$ if and only if there exist outer functions $X, Y \in \mathcal{RH}_{\infty,*}^-$ such that*

$$P_{\text{man}} = XK \quad \text{and} \quad K = YP_1. \tag{13}$$

The proof of this theorem is also constructive and is given in the Appendix. The conditions in (13) imply that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}_{\text{manifest}}$, which are necessary and sufficient conditions for the existence of a $\mathcal{C}^\infty$ controller Σ_C that implements $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$ as discussed in Polderman and Willems (1998) and Trentelman et al. (2007). However, these conditions are not sufficient for the partial interconnection problem for systems in the model class $\mathbb{L}_+$, as in the full interconnection case.

The algorithm

An explicit construction of a controller $\Sigma_C \in \mathbb{L}_+$ that implements $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$ by partial interconnection is given by the following algorithm.

Algorithm 5.7. Let $P, K \in \mathcal{RH}_\infty^-$ define the behaviors $\mathcal{P}_{\text{full}}$ and $\mathcal{K}$ corresponding to the systems $\Sigma_P \in \mathbb{L}_+$ and $\Sigma_K \in \mathbb{L}_+$, respectively.

Assumption: $\mathcal{P}_{\text{full}}$ is c-eliminable.

Aim: Find $C \in \mathcal{RH}_\infty^-$ that defines the behavior $\mathcal{C}$ of system $\Sigma_C \in \mathbb{L}_+$ as

$$\mathcal{C} = \{c \in \mathcal{H}_2^+ \mid Cc \in \mathcal{H}_2^-\} = \ker_+ \Pi_+ C,$$

such that $\mathcal{C}$ implements $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$ by partial interconnection through c.

Step 1: Use Theorem 4.9 to obtain $P_{\text{man}} \in \mathcal{RH}_\infty^-$ such that

$$\mathcal{P}_{\text{manifest}} = \{w \in \mathcal{H}_2^+ \mid P_{\text{man}} w \in \mathcal{H}_2^-\} = \ker_+ \Pi_+ P_{\text{man}}.$$

Step 2: Find an outer rational function $X \in \mathcal{RH}_{\infty,*}^-$ such that $K = XP_1$. If no such X exists, the algorithm stops and no controller can be found.

Step 3: Find an outer rational $Y \in \mathcal{RH}_{\infty,*}^-$ such that $P_{\text{man}} = YK$. If no such Y exists, the algorithm stops here.

Step 4: Determine a unitary function $U \in \mathcal{UH}_{\infty,*}^-$ which brings Y into the form: $\overline{Y} = YU = [Y_1\ 0]$, with $Y_1 \in \mathcal{UH}_{\infty,*}^-$.

Step 5: Define $W := [0\ I]U^{-1} \in \mathcal{RH}_{\infty,*}^-$, where the dimension of the identity matrix equals the number of zero-columns in $\overline{Y}$.

Step 6: The controller Σ_C with behavior $\mathcal{C} = \ker_+ \Pi_+ C$ is given by

$$C = \frac{1}{(s-\alpha)^k} WXP_2,$$

where $\alpha > 0$ and $k \geq 0$ are such that $C \in \mathcal{RH}_\infty^-$.

5.3. *The example*

To illustrate the algorithm for controller synthesis by partial interconnection, consider the following input–state–output system:

$$\Sigma_P : \begin{cases} \dot{x} = Ax + B_1 d + B_2 u, \\ z = C_1 x + D_{11} d + D_{12} u, \\ y = C_2 x + D_{21} d + D_{22} u, \end{cases} \tag{14}$$

with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}, \qquad D_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad D_{21} = D_{22} = 0.$$

In this example, $w := \operatorname{col}(z, d)$ is the manifest variable and $c := \operatorname{col}(y, u)$ denotes the variable that is available for (partial) interconnection with a controller. The controlled system Σ_K is defined by the state space equations

$$\Sigma_K : \begin{cases} \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -14 \end{bmatrix} x + \begin{bmatrix} -1.4615 \\ -1.4545 \\ -4.3597 \end{bmatrix} d, \\ z = \begin{bmatrix} 1 & 0 & -2.8583 \\ 0 & 2 & 3.0027 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d \end{cases}$$

which were obtained by substitution of the static output feedback law $u = -3y$ in (14). The $\mathcal{L}_2$ behaviors of the plant and the controlled system are viewed as elements in the model class $\mathbb{L}_+$ and represented by anti-stable rational operators in $\mathcal{RH}_\infty^-$. In this case, $\mathcal{P}_{\text{full}} = \ker_+ \Pi_+ P$ and $\mathcal{K} = \ker_+ \Pi_+ K$, where $P(s) = \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix} \in \mathcal{RH}_\infty^-$ is decomposed accordingly with $\operatorname{col}(w, c)$, with

$$P_1(s) = \begin{bmatrix} -\dfrac{s+1}{s-5} & 0 & \dfrac{2}{s-5} \\ 0 & -\dfrac{s+3}{s-3} & \dfrac{s+5}{s-3} \\ 0 & 0 & -\dfrac{3}{s-1} \end{bmatrix} \quad \text{and}$$

$$P_2(s) = \begin{bmatrix} 0 & \dfrac{s-4}{s-5} \\ 0 & -\dfrac{2s+16}{s-3} \\ -\dfrac{s+5}{s-1} & \dfrac{3}{s-1} \end{bmatrix},$$

and $K(s)$ given in Box I Given P and K, we apply Algorithm 5.7 to find a controller that implements $\mathcal{K}$ for $\mathcal{P}$ by partial interconnection.

Step 1: To obtain a representation of the manifest behavior $\mathcal{P}_{\text{manifest}}$, we first eliminate the latent variable c in the full plant behavior. For this, we start by creating zero-rows in P_2, as discussed in the proof of Theorem 4.9, by pre-multiplying P with U defined by

$$U(s) = \begin{bmatrix} \dfrac{2(s+6)}{s-3} & \dfrac{s-4}{s-5} & 0 \\ \dfrac{s-2}{s-3} & 0 & 0 \\ 0 & 0 & \dfrac{s-2}{s-3} \end{bmatrix} \quad \text{with}$$

$$U(s)^{-1} = \begin{bmatrix} 0 & \dfrac{s-3}{s-2} & 0 \\ \dfrac{s-5}{s-4} & -\dfrac{2(s+6)(s-5)}{(s-2)(s-4)} & 0 \\ 0 & 0 & \dfrac{s-3}{s-2} \end{bmatrix}.$$

Since U and U^{-1} belong to $\mathcal{RH}_\infty^-$, we infer that $U \in \mathcal{UH}_\infty^-$ and we have that $U \in \mathcal{UH}_{\infty,*}^-$. This results in

$$\begin{aligned} U(s)P_1(s) &= \begin{bmatrix} -\dfrac{2(s+1)(s+6)}{(s-3)(s-5)} & -\dfrac{(s+3)(s-4)}{(s-3)(s-5)} & \dfrac{(s+1)(s+4)}{(s-3)(s-5)} \\ -\dfrac{(s+1)(s-2)}{(s-3)(s-5)} & 0 & \dfrac{2(s-2)}{(s-3)(s-5)} \\ 0 & 0 & -\dfrac{3(s-2)}{(s-1)(s-3)} \end{bmatrix} \\ &:= \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix}, \end{aligned}$$

$$\text{and} \quad U(s)P_2(s) = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{(s-2)(s-4)}{(s-3)(s-5)} \\ -\dfrac{(s+5)(s-2)}{(s-1)(s-3)} & \dfrac{3(s-2)}{(s-1)(s-3)} \end{bmatrix} := \begin{bmatrix} P_{21} \\ P_{22} \end{bmatrix}.$$

It is now easily seen that the conditions for eliminability of c in Theorem 4.9 are satisfied since there exists an $X \in \mathcal{RH}_\infty^+$ such that $P_{12} + P_{22}X \in \mathcal{RH}_\infty^-$ and that $\operatorname{rowrank}(P_{11}) = \operatorname{p}(\mathcal{P}_{\text{full}}) - \operatorname{rowrank}(P_2)$ (note that this operator X differs from the one used in Step 2). Hence, by the elimination theorem, $\mathcal{P}_{\text{manifest}} = \ker_+ \Pi_+ P_{\text{man}}$ with

$$P_{\text{man}}(s) = \begin{bmatrix} -\dfrac{2(s+1)(s+6)}{(s-3)(s-5)} & -\dfrac{(s+3)(s-4)}{(s-3)(s-5)} & \dfrac{(s+1)(s+4)}{(s-3)(s-5)} \end{bmatrix} \in \mathcal{RH}_\infty^-.$$

$$K(s) = \begin{bmatrix} -\dfrac{(s+1)(s+14.04)(s-2.923)}{(s-1)(s-3)(s-14)} & -\dfrac{0.1791(s+3)(s+9.248)}{(s-1)(s-3)(s-14)} & \dfrac{11.1791(s-0.5558)(s-3.052)}{(s-1)(s-3)(s-14)} \\ \dfrac{22.8059(s+1)(s-2.364)}{(s-1)(s-3)(s-14)} & -\dfrac{(s+3)(s-0.8513)(s-14.27)}{(s-1)(s-3)(s-14)} & \dfrac{(s-0.7962)(s-3.34)(s-23.99)}{(s-1)(s-3)(s-14)} \end{bmatrix} \in \mathcal{RH}_\infty^-$$

Box I.

$$X(s) = \begin{bmatrix} \dfrac{(s+14.04)(s-2.923)(s-5)}{(s-1)(s-3)(s-14)} & \dfrac{0.1791(s+9.248)}{(s-1)(s-14)} & -\dfrac{3(s-1.95)(s-5.286)}{(s-3)(s-14)} \\ -\dfrac{22.8059(s-2.364)(s-5)}{(s-1)(s-3)(s-14)} & \dfrac{(s-0.8513)(s-14.27)}{(s-1)(s-14)} & \dfrac{6(s-1.231)(s-10.49)}{(s-3)(s-14)} \end{bmatrix} \in \mathcal{RH}_\infty^-$$

Box II.

Step 2: We need to verify the existence of an outer function $X \in \mathcal{RH}_{\infty,*}^-$ such that $K = XP_1$. The rational operator given in Box II fulfills this requirement because $\mathcal{RH}_\infty^- \subset \mathcal{RH}_{\infty,*}^-$.

Step 3: We need to verify the existence of an outer function $Y \in \mathcal{RH}_{\infty,*}^-$ such that $P_{\text{man}} = YK$. The rational operator

$$Y(s) = \begin{bmatrix} \dfrac{2(s-10.49)(s-1.231)}{(s-3)(s-5)} & \dfrac{(s-1.95)(s-5.286)}{(s-3)(s-5)} \end{bmatrix} \in \mathcal{RH}_\infty^-$$

fulfills this requirement.

Step 4: We need to post-multiply Y with a unitary operator U such that $YU = [Y_1\ 0]$, with Y_1 a unit. The matrix function

$$U(s) = \begin{bmatrix} \dfrac{s-1}{s-5} & -\dfrac{s-3}{s-2} \\ 0 & \dfrac{2(s-1.231)(s-3)(s-10.49)}{(s-1.95)(s-2)(s-5.286)} \end{bmatrix},$$

with inverse $U(s)^{-1}$

$$= \begin{bmatrix} \dfrac{s-5}{s-1} & \dfrac{0.5(s-1.95)(s-5)(s-5.286)}{(s-1)(s-1.231)(s-10.49)} \\ 0 & \dfrac{0.5(s-1.95)(s-2)(s-5.286)}{(s-1.231)(s-3)(s-10.49)} \end{bmatrix},$$

does indeed belong to $\mathcal{UH}_{\infty,*}^-$. Moreover,

$$Y_1(s) = \frac{2(s-1)(s-1.231)(s-10.49)}{(s-3)(s-5)^2} \in \mathcal{UH}_{\infty,*}^-$$

yields that Y_1 is a unitary function. This meets the conditions on U.

Step 5: The function $W := [0\ I]U^{-1}$ reads

$$W(s) = \begin{bmatrix} 0 & \dfrac{2(s-1.231)(s-3)(s-10.49)}{(s-1.95)(s-2)(s-5.286)} \end{bmatrix}.$$

Step 6: The controller Σ_C with behavior $\mathcal{C} = \ker_+ \Pi_+ C$ is given by the equation in Box III. There, $v(s) = \frac{(s-1.231)^2(s-10.49)^2}{(s-1.95)(s-2)(s-5.286)(s-14)} \in \mathcal{UH}_\infty^-$. By Theorem 4.3, $\mathcal{C} = \ker_+ \Pi_+ C_0$, with the equivalent kernel representation $C_0(s) = \begin{bmatrix} -\dfrac{12(s+5)}{s-1} & -\dfrac{4(s+5)}{s-1} \end{bmatrix} \in \mathcal{RH}_\infty^-$.
Note that this controller does indeed implement $\mathcal{K}$ for $\mathcal{P}$, since substitution of the law $u = 3y$ yields

$$-\frac{12(s+5)}{s-1}y - \frac{4(s+5)}{s-1}u = -\frac{12(s+5)}{s-1}y - \frac{4(s+5)}{s-1}3y = 0.$$

6. Conclusions

In this paper, systems are viewed as collections of functions that are square integrable on the imaginary axis. More specifically, we distinguish three classes of closed, left invariant systems that can be represented as kernels of rational operators in the class $\mathcal{RH}_\infty^+$ of stable rational functions, and three classes of closed right invariant systems that can be modeled as the null spaces of operators in $\mathcal{RH}_\infty^-$, the class of anti-stable rational functions. This defines six model classes of $\mathcal{L}_2$ systems. For each of these model classes we addressed the question of system equivalence. Necessary and sufficient conditions on rational functions have been derived that guarantee the equivalence of systems. We have presented necessary and sufficient conditions for the complete elimination of latent variables from an $\mathcal{L}_2$ latent variable system. More specifically, we presented conditions under which the induced manifest behavior of a latent variable system, represented as the kernel of a rational operator, can again be represented as the kernel of a rational operator. The results presented on equivalence and elimination of $\mathcal{L}_2$ systems that are represented by rational operators substantially differ from results on the elimination and equivalence of infinitely smooth solutions systems that are represented by polynomial differential equations.

We have applied the results to solve the controller synthesis problem in an analogous approach, as described in Trentelman et al. (2007). Explicit algorithms have been presented that synthesize a controller $\mathcal{C}$ that after interconnection with an $\mathcal{L}_2$ plant $\mathcal{P}$ gives a desired controlled behavior $\mathcal{K}$. In fact, we characterized all controllers (as $\mathcal{L}_2$ systems) that after interconnection with a given plant result in the desired controlled behavior. Two possible interconnection structures, namely full and partial interconnections, are distinguished for this controller synthesis problem.

Appendix. Proofs

We start this section with a lemma that proves useful in various proofs.

Lemma A.1. *Let $P \in \mathcal{RH}_\infty^-$, $k \geq 0$ and $\alpha > 0$. Then,*

$$\{w \in \mathcal{H}_2^+ \mid Pw \in \mathcal{H}_2^-\} = \left\{w \in \mathcal{H}_2^+ \mid \frac{1}{(s-\alpha)^k}Pw \in \mathcal{H}_2^-\right\}.$$

Moreover, let $z \in \mathcal{L}_2$. Then $\frac{1}{(s-\alpha)^k}z \in \mathcal{H}_2^-$ if and only if $z \in \mathcal{H}_2^-$.

Proof. For the first claim, we first verify the inclusion ($\subseteq$). Let $w \in \mathcal{H}_2^+$ be such that $z := Pw \in \mathcal{H}_2^-$. Since $k \geq 0$ and $\alpha > 0$, we have that $\frac{1}{(s-\alpha)^k} \in \mathcal{RH}_\infty^-$. Hence, by Lemma 3.2,

$$\begin{aligned} C = WXP_2 &= \left[-\frac{12(s+5)(s-1.231)^2(s-10.49)^2}{(s-1)(s-1.95)(s-2)(s-5.286)(s-14)} \quad -\frac{4(s+5)(s-1.231)^2(s-10.49)^2}{(s-1)(s-1.95)(s-2)(s-5.286)(s-14)} \right] \\ &= v(s) \left[-\frac{12(s+5)}{s-1} \quad -\frac{4(s+5)}{s-1} \right] \end{aligned}$$

Box III.

$\hat{z} := \frac{1}{(s-\alpha)^k} z \in \mathcal{H}_2^-$, which yields that $\frac{1}{(s-\alpha)^k} Pw \in \mathcal{H}_2^-$. To verify (⊇), take $w \in \mathcal{H}_2^+$ such that $\hat{z} := \frac{1}{(s-\alpha)^k} Pw \in \mathcal{H}_2^-$. By Lemma 3.2, we have that $z := Pw \in \mathcal{L}_2$. Decompose z as $z = z_- + z_+$ with $z_- = \Pi_- z \in \mathcal{H}_2^-$ and $z_+ = \Pi_+ z \in \mathcal{H}_2^+$. Substitution in the expression for $\hat{z} = \frac{1}{(s-\alpha)^k} z$ shows that

$$\frac{1}{(s-\alpha)^k} z_+ = \hat{z} - \frac{1}{(s-\alpha)^k} z_- \in \mathcal{H}_2^-. \tag{15}$$

We claim that z_+ is analytic in $\mathbb{C}$. To show this, first note that z_+ is analytic in $\mathbb{C}^+$, since $z_+ \in \mathcal{H}_2^+$. Also, z_+ is analytic in $\mathbb{C}^0$, since $z_+ = z - z_- \in \mathcal{L}_2$. Now, suppose that z_+ is not analytic at a point $s_0 \in \mathbb{C}^-$. Then, $\lim_{s \to s_0} z_+(s) = \infty$ and so there exists $m > 0$ such that $z_+(s) = \frac{1}{(s-s_0)^m} z_+'(s)$ with $z_+'(s)$ analytic in s_0. Then, for $k \geq 0$ and $\alpha > 0$,

$$\begin{aligned} \lim_{s \to s_0} \frac{1}{(s-\alpha)^k} z_+(s) &= \lim_{s \to s_0} \frac{1}{(s-\alpha)^k} \frac{1}{(s-s_0)^m} z_+'(s) \\ &= \frac{1}{(s_0-\alpha)^k} z_+'(s_0) \lim_{s \to s_0} \frac{1}{(s-s_0)^m} = \infty, \end{aligned}$$

which shows that $\frac{1}{(s-\alpha)^k} z_+(s)$ is not analytic in $s_0 \in \mathbb{C}^-$. This contradicts (15). Conclude that z_+ is analytic in $\mathbb{C}$. Since z_+ is bounded ($z_+ \in \mathcal{H}_2^+$) and analytic in $\mathbb{C}$, application of Liouville's boundedness theorem proves that z_+ is a constant function. Since $z_+ \in \mathcal{H}_2^+$, it follows that $z_+ = 0$. Consequently, $Pw = z = z_- + z_+ = z_- \in \mathcal{H}_2^-$, which proves (⊇). This completes the proof.

The second claim is immediate from the (⊇)-part of this proof. ☐

Proof of Lemma 3.1. To prove that $\mathcal{RH}_{\infty,*}^+ = \mathcal{RH}_\infty^+ + \mathbb{R}[s]$, we first show that $\mathcal{RH}_{\infty,*}^+ \supseteq \mathcal{RH}_\infty^+ + \mathbb{R}[s]$. Take arbitrary $f_1 \in \mathcal{RH}_\infty^+$ and $f_2 \in \mathbb{R}[s]$. Let $k \geq \text{degree}(\det f_2)$ and $\alpha < 0$. Then $\frac{1}{(s-\alpha)^k} \in \mathcal{RH}_\infty^+$ and

$$\frac{1}{(s-\alpha)^k}(f_1 + f_2) = \underbrace{\frac{1}{(s-\alpha)^k} f_1}_{\in \mathcal{RH}_\infty^+} + \frac{1}{(s-\alpha)^k} f_2 \in \mathcal{RH}_\infty^+$$

which, by (5), shows that $(f_1 + f_2) \in \mathcal{RH}_{\infty,*}^+$.

To verify the converse inclusion, let $f \in \mathcal{RH}_{\infty,*}^+$. Following (5), f is a rational function that is analytic in $\mathbb{C}^+$ with possible poles at infinity. Let $f = N(s)D(s)^{-1}$, with $N, D \in \mathbb{R}[s]$, be a right-coprime polynomial factorization of f. By the analyticity of f, $\det(D(\lambda)) \neq 0$, $\forall \lambda \in \mathbb{C}^+$. Moreover, there exist polynomials $Q, R \in \mathbb{R}[s]$ such that $N(s) = Q(s)D(s) + R(s)$ and $R(s)D(s)^{-1}$ is strictly proper (Vidyasagar, 1985). Hence, $f = N(s)D(s)^{-1} = Q(s) + R(s)D(s)^{-1}$ is a sum of a polynomial and a strictly proper rational function with poles in $\mathbb{C}^-$, i.e., $f = f_1 + f_2$ with $f_1 \in \mathcal{RH}_\infty^+, f_2 \in \mathbb{R}[s]$. This completes the proof. ☐

Proof of Lemma 4.1. To prove linearity, let $w_1, w_2 \in \mathcal{B}_+$. For $\lambda_1, \lambda_2 \in \mathbb{R}$, we have to verify whether $w := \lambda_1 w_1 + \lambda_2 w_2 \in \mathcal{B}_+$. This is indeed the case because $Pw = \lambda_1 P w_1 + \lambda_2 P w_2 \in \mathcal{H}_2^-$. To prove left invariance of $\mathcal{B}_+$, we need to show that for all $\tau \leq 0$ and $w \in \mathcal{B}_+$, $\sigma_\tau w \in \mathcal{B}_+$ holds. For all $\tau \leq 0$ we have that

$$P(s)(\sigma_\tau w)(s) = \mathrm{e}^{-s\tau} P(s) w(s) - P(s) \mathrm{e}^{-s\tau} \int_0^{-\tau} \hat{w}(t) \mathrm{e}^{-st} \mathrm{d}t.$$

Since $w \in \mathcal{B}_+$ we have $P(s)w(s) \in \mathcal{H}_2^-$ and therefore also $\mathrm{e}^{-s\tau} P(s) w(s) \in \mathcal{H}_2^-$ for $\tau \leq 0$. Moreover, with a change of variables $u := t + \tau$, we infer

$$\begin{aligned} \mathrm{e}^{-s\tau} \int_0^{-\tau} \hat{w}(t) \mathrm{e}^{-st} \mathrm{d}t &= \int_\tau^0 \hat{w}(u-\tau) \mathrm{e}^{-su} \mathrm{d}u \\ &= \int_{-\infty}^0 \hat{w}(u-\tau) \mathrm{e}^{-su} \mathrm{d}u \in \mathcal{H}_2^-, \end{aligned}$$

as $\hat{w}(\bullet - \tau) \in L_2^-$, for $\tau \leq 0$. Hence, $P(s)\mathrm{e}^{-s\tau} \int_0^{-\tau} \hat{w}(t) \mathrm{e}^{-st} \mathrm{d}t \in \mathcal{H}_2^-$. Consequently, $P(s)(\sigma_\tau w)(s) \in \mathcal{H}_2^-$ for $\tau \leq 0$. The proofs for $\mathcal{B}$ and $\mathcal{B}_-$ are similar and are omitted in this paper. ☐

Proof of Theorem 4.3. *Inclusions of behaviors*:

We prove the three statements on inclusions of behaviors through the following items:

- ($\mathcal{B}_2 \subset \mathcal{B}_1 \Longleftarrow \exists F \in \mathcal{RL}_{\infty,*}$ such that $P = FQ$):
  Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be represented by $P, Q \in \mathcal{RH}_\infty^-$. Suppose that $P = FQ$ with $F \in \mathcal{RL}_{\infty,*}$. Let $w \in \mathcal{B}_2$. Then $v := Qw = 0$ and we infer that $Pw = FQw = Fv = 0$. Therefore $w \in \mathcal{B}_1$. Since $w \in \mathcal{B}_2$ is arbitrary, we conclude that $\mathcal{B}_2 \subset \mathcal{B}_1$.
- ($\mathcal{B}_{2,+} \subset \mathcal{B}_{1,+} \Longleftarrow \exists F \in \mathcal{RH}_{\infty,*}^-$ such that $P = FQ$):
  Let $\mathcal{B}_{1,+}$ and $\mathcal{B}_{2,+}$ be represented by $P, Q \in \mathcal{RH}_\infty^-$ as in (6b). Suppose that $P = FQ$ with $F \in \mathcal{RH}_{\infty,*}^-$. Take $w \in \mathcal{B}_{2,+}$ and define $v := Qw$. Then, by definition of $\mathcal{B}_{2,+}$, we have that $v \in \mathcal{H}_2^-$. We infer that $z := Pw = FQw = Fv$, where we observe that $z \in \mathcal{L}_2$ since $P : \mathcal{H}_2^+ \to \mathcal{L}_2$. From (5) it follows that $\exists k \geq 0$ and $\exists \alpha > 0$ such that $\frac{1}{(s-\alpha)^k} F(s) \in \mathcal{RH}_\infty^-$. Hence $f := \frac{1}{(s-\alpha)^k} Fv = \frac{1}{(s-\alpha)^k} z \in \mathcal{H}_2^-$. Apply Lemma A.1 to infer that $z \in \mathcal{H}_2^-$. Hence, $z = Pw \in \mathcal{H}_2^-$, which shows that $w \in \mathcal{B}_{1,+}$. Since $w \in \mathcal{B}_{2,+}$ was arbitrary, we infer $\mathcal{B}_{2,+} \subset \mathcal{B}_{1,+}$.
- ($\mathcal{B}_{2,-} \subset \mathcal{B}_{1,-} \Longleftarrow \exists F \in \mathcal{RL}_{\infty,*}$ such that $P = FQ$):
  This proof is omitted, since it is similar to the proof of the first item.

To prove the converse implications, recall that any full row rank $P \in \mathcal{RH}_\infty^-$ admits an outer/co-inner factorization (Francis, 1987):

$$P = P_o P_{ci},$$

where $P_o \in \mathcal{RH}_\infty^-$ (square) is outer and $P_{ci} \in \mathcal{RH}_\infty^-$ (square or wide) is co-inner. Thus, P_{ci}^* is inner and $P_{ci}P_{ci}^* = I$. Since P_o is outer, its inverse P_o^{-1} exists and is analytic in $\mathbb{C}^-$ (Francis, 1987). Therefore, we have that $P_o^{-1} \in \mathcal{RH}_{\infty,*}^-$.

- ($\mathcal{B}_2 \subset \mathcal{B}_1 \Longrightarrow \exists F \in \mathcal{RL}_{\infty,*}$ such that $P = FQ$):
  Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are represented by $P, Q \in \mathcal{RH}_\infty^-$, respectively. Then

$$\begin{aligned} \mathcal{B}_2 &= \{w \in \mathcal{L}_2 \mid Qw = 0\} = \{w \in \mathcal{L}_2 \mid Q_o Q_{ci} w = 0\} \\ &= \{w \in \mathcal{L}_2 \mid Q_{ci} w = 0\} \\ &= \{w \in \mathcal{L}_2 \mid \langle Q_{ci} w, v \rangle_{\mathcal{L}_2} = 0,\ \forall v \in \mathcal{L}_2\} \\ &= \{w \in \mathcal{L}_2 \mid \langle w, Q_{ci}^* v \rangle_{\mathcal{L}_2} = 0,\ \forall v \in \mathcal{L}_2\} = (Q_{ci}^* \mathcal{L}_2)^\perp. \end{aligned}$$

Similarly, without using the factorization, we obtain that $\mathscr{B}_1 = (P^*\mathscr{L}_2)^\perp$.
If $\mathscr{B}_2 \subset \mathscr{B}_1$ then also $\mathscr{B}_1^\perp \subset \mathscr{B}_2^\perp$, and so

$$((P^*\mathscr{L}_2)^\perp)^\perp \subset ((Q_{ci}^*\mathscr{L}_2)^\perp)^\perp.$$

Equivalently, with over-bars denoting closures,

$$\overline{P^*\mathscr{L}_2} \subset \overline{Q_{ci}^*\mathscr{L}_2}. \tag{16}$$

The Beurling–Lax theorem (see the proof of Theorem 12.6 in Fuhrmann (1981, Chapter 2)) states that, if $\mathscr{M} = q\mathscr{H}$ for some inner function q and Hilbert space $\mathscr{H}$, then $\mathscr{M}$ is a closed invariant subspace of $\mathscr{H}$. Applying this to (16) gives

$$P^*\mathscr{L}_2 \subset \overline{P^*\mathscr{L}_2} \subset \overline{Q_{ci}^*\mathscr{L}_2} = Q_{ci}^*\mathscr{L}_2. \tag{17}$$

Now, we use a more general result for bounded operators A and B in Hilbert spaces (Theorem 7.1 in Fuhrmann (1981)), which states that $\operatorname{im} A \subset \operatorname{im} B$ if and only if $A = BC$ for some bounded operator C.
More explicitly, as in the proof of Theorem 7.1, define $B_0 := Q_{ci}^*|_{(\ker Q_{ci}^*)^\perp}$. Then B_0 is an injective mapping from $(\ker Q_{ci}^*)^\perp \to Q_{ci}^*\mathscr{L}_2$. Moreover, B_0^{-1} exists as a closed operator mapping $Q_{ci}^*\mathscr{L}_2$ into $(\ker Q_{ci}^*)^\perp$. Since $\operatorname{im} P^* \subset \operatorname{im} Q_{ci}^*$, the operator $C := B_0^{-1}P^*$ is a closed mapping from $\mathscr{L}_2$ to $(\ker Q_{ci}^*)^\perp$ and belongs to $\mathscr{RL}_\infty$. Now,

$$Q_{ci}^*C = Q_{ci}^*B_0^{-1}P^* = B_0B_0^{-1}P^* = P^*.$$

Consequently, by taking adjoints it follows that $P = C^*Q_{ci}$. Let $F := C^*Q_o^{-1}$. Since $C^* \in \mathscr{RL}_\infty$ and $Q_o^{-1} \in \mathscr{RH}_{\infty,*}^-$, we have that $F \in \mathscr{RL}_{\infty,*}$. Moreover, $FQ = C^*Q_o^{-1}Q = C^*Q_o^{-1}Q_oQ_{ci} = P$, which completes the proof.

- ($\mathscr{B}_{2,+} \subset \mathscr{B}_{1,+} \Longrightarrow \exists F \in \mathscr{RH}_{\infty,*}^-$ such that $P = FQ$):
This proof goes in a similar manner to the one in the previous item. However, we will make use of Lemma A.1 and claim that there exist $k \geq 0$ and $\alpha > 0$ such that

$$\begin{aligned}
\mathscr{B}_{2,+} &= \{w \in \mathscr{H}_2^+ \mid Qw =: z \in \mathscr{H}_2^-\}\\
&= \left\{w \in \mathscr{H}_2^+ \mid \frac{1}{(s-\alpha)^k}Qw = \frac{1}{(s-\alpha)^k}z \in \mathscr{H}_2^-\right\}\\
&= \left\{w \in \mathscr{H}_2^+ \mid \frac{1}{(s-\alpha)^k}Q_oQ_{ci}w = \frac{1}{(s-\alpha)^k}z \in \mathscr{H}_2^-\right\}\\
&= \left\{w \in \mathscr{H}_2^+ \mid \frac{1}{(s-\alpha)^k}Q_{ci}w \right.\\
&\qquad \left. = \frac{1}{(s-\alpha)^k}Q_o^{-1}z =: \hat{z} \in \mathscr{H}_2^-\right\}.
\end{aligned}$$

Indeed, since $Q_o^{-1} \in \mathscr{RH}_{\infty,*}$, the definition in (5) implies that there exist $k \geq 0$ and $\alpha > 0$ such that $\frac{1}{(s-\alpha)^k}Q_o^{-1} \in \mathscr{RH}_\infty^-$. For this choice of k and α it follows that $\hat{z} := \frac{1}{(s-\alpha)^k}Q_o^{-1}z \in \mathscr{H}_2^-$. Using this, and applying Lemma A.1 again, we obtain

$$\begin{aligned}
\mathscr{B}_{2,+} &= \left\{w \in \mathscr{H}_2^+ \mid \frac{1}{(s-\alpha)^k}Q_{ci}w = \hat{z} \in \mathscr{H}_2^-\right\}\\
&= \{w \in \mathscr{H}_2^+ \mid Q_{ci}w \in \mathscr{H}_2^-\}\\
&= \{w \in \mathscr{H}_2^+ \mid \langle w, Q_{ci}^*v\rangle_{\mathscr{H}_2^+} = 0,\ \forall v \in \mathscr{H}_2^+\}\\
&= (Q_{ci}^*\mathscr{H}_2^+)^\perp,
\end{aligned}$$

which represents $\mathscr{B}_{2,+}$ as the orthogonal complement of the image of an inner rational operator. This implies that the closure in (17) also vanishes in this case. Again applying Theorem 7.1 of Fuhrmann (1981), it follows that the bounded operator $C{:}\mathscr{H}_2^+ \to \mathscr{H}_2^+$ defined in the previous item belongs to $\mathscr{RH}_\infty^+$. This implies $F = C^*Q_o^{-1} \in \mathscr{RH}_{\infty,*}^-$ and satisfies $FQ = P$ as in the previous item.

- ($\mathscr{B}_{2,-} \subset \mathscr{B}_{1,-} \Longrightarrow \exists F \in \mathscr{RL}_{\infty,*}$ such that $P = FQ$):
This proof is omitted here, since it is similar to the proof of the last two implications. Here we will obtain that $C \in \mathscr{RH}_\infty^-$, resulting in $F \in \mathscr{RL}_{\infty,*}$.

Equality of behaviors:

We only show the proof for the equivalence $\mathscr{B}_{1,+} = \mathscr{B}_{2,+}$, which will be used in Section 5. With this proof, one can easily verify the other two equivalence conditions.

Let $\mathscr{B}_{1,+}$ and $\mathscr{B}_{2,+}$ be represented by full row rank operators $P, Q \in \mathscr{RH}_\infty^-$. Using the previous inclusion relations, we have that $\mathscr{B}_{1,+} = \mathscr{B}_{2,+}$ if and only if there exist $F_1 \in \mathscr{RH}_{\infty,*}^-$ and $F_2 \in \mathscr{RH}_{\infty,*}^-$ such that $P = F_1Q$ and $Q = F_2P$. A direct substitution then gives that $P = F_1F_2Q$ and $Q = F_2F_1P$. If P and Q have full row rank, it follows that $F_1 = F_2^{-1}$ which shows that both F_1 and F_2 belong to $\mathscr{UH}_{\infty,*}^-$. This completes the proof.

Using co-inner operators Q and P:

One can observe in the proof of the inclusions that when Q is co-inner, no outer/co-inner factorization has to be applied. In this case, we can verify whether $\operatorname{im} P^* \subset \operatorname{im} Q^*$ directly (since the closure of $\overline{Q^*\mathscr{L}_2} = Q^*\mathscr{L}_2$), and we obtain $F := C \in \mathscr{RL}_\infty$ as a bounded operator. For the case where also P is co-inner, equivalence of $\mathscr{B}_1 = \mathscr{B}_2$ holds when there exist $F_1, F_2 \in \mathscr{RL}_\infty$. Since we have shown that $F_1 = F_2^{-1}$, we know that $F_1, F_2 \in \mathscr{UL}_\infty$. Similar results can be obtained for the $\mathscr{H}_2^+$ and $\mathscr{H}_2^-$ behaviors. □

Proof of Theorem 4.9. We only show the second equivalence for systems in $\mathbb{L}_+$ as the proofs in the other cases are similar. To show this, let $U \in \mathscr{UH}_{\infty,*}^-$ be such that

$$UP_2 = \begin{bmatrix} P_{12} \\ 0 \end{bmatrix}$$

where P_{12} has full row rank. Define the decomposition

$$\tilde{P} := U[P_1\ P_2] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix}. \tag{18}$$

Then, by Theorem 4.3,

$$\begin{aligned}
\mathscr{B}_{\text{full},+} = \{(w, \ell) \in \mathscr{H}_2^+ \mid\ & P_{11}w + P_{12}\ell \in \mathscr{H}_2^- \\
& \text{and } P_{21}w \in \mathscr{H}_2^-\}.
\end{aligned} \tag{19}$$

It follows that $\mathscr{B}_{\text{full},+} = \mathscr{B}_{\text{full},+}^1 \cap \mathscr{B}_{\text{full},+}^2$, where

$$\begin{aligned}
\mathscr{B}_{\text{full},+}^1 &= \{(w, \ell) \in \mathscr{H}_2^+ \mid P_{11}w + P_{12}\ell \in \mathscr{H}_2^-\},\\
\mathscr{B}_{\text{full},+}^2 &= \{(w, \ell) \in \mathscr{H}_2^+ \mid P_{21}w \in \mathscr{H}_2^-\}.
\end{aligned}$$

Let $\mathscr{B}_{\text{manifest},+}^1$ be the manifest behavior associated with $\mathscr{B}_{\text{full},+}^1$ and let $\mathscr{B}_{\text{manifest},+}^2$ denote the manifest behavior associated with $\mathscr{B}_{\text{full},+}^2$.

($\Rightarrow$): Suppose that the system is ℓ-eliminable. First consider $\mathscr{B}_{\text{full},+}^1$. We first prove that $\mathscr{B}_{\text{manifest},+}^1 = \mathscr{H}_2^+$. To see this, let $p_1 = \mathrm{p}(\ker_+ \Pi_+[P_{11}\ P_{12}])$ be the output cardinality of $\mathscr{B}_{\text{full},+}^1$, and denote by $m_1 = \mathrm{m}(\ker_+ \Pi_+[P_{11}\ P_{12}]) = \dim(w) + \dim(\ell) - p_1$ the input cardinality of $\mathscr{B}_{\text{full},+}^1$. Since both P_{12} and $[P_{11}\ P_{12}]$ have full row rank, it follows that $p_1 = \operatorname{rowrank}(P_{12})$. This implies that the variables (w, ℓ) in $\mathscr{B}_{\text{full},+}^1$ admit a partitioning as

$$\begin{bmatrix} w \\ \ell \end{bmatrix} = \begin{bmatrix} w \\ \ell' \\ \ell'' \end{bmatrix},$$

where $u = \operatorname{col}(w, \ell')$ is an input variable (i.e., an unconstrained variable in $\mathscr{H}_2^+$) and $y = \ell''$ is an output variable. In particular, it follows that $w \in \mathcal{H}_2^+$ is unconstrained in $\mathcal{B}^1_{\mathrm{full},+}$ and therefore $\mathcal{B}^1_{\mathrm{manifest},+} = \mathcal{H}_2^+$.

Second, we construct the mapping X in (8). Define, for any $w \in \mathcal{H}_2^+$, the set of latent functions that are compatible with w as $\mathcal{L}(w) := \{\ell \in \mathcal{H}_2^+ \mid (w, \ell) \in \mathcal{B}^1_{\mathrm{full},+}\}$. Clearly, $\mathcal{L}(w)$ is non-empty and it is easily seen that $\mathcal{L}(w)$ is an affine set for any $w \in \mathcal{H}_2^+$. Indeed, if $\ell_1, \ell_2 \in \mathcal{L}(w)$ and $\alpha \in \mathbb{R}$ then $(w, \ell_i) \in \mathcal{B}^1_{\mathrm{full},+}$ for $i = 1, 2$ and, by linearity of $\mathcal{B}_{\mathrm{full}}$, also $\alpha(w, \ell_1) + (1-\alpha)(w, \ell_2) = (w, \alpha\ell_1 + (1-\alpha)\ell_2) \in \mathcal{B}^1_{\mathrm{full},+}$. This shows that $\alpha\ell_1 + (1-\alpha)\ell_2 \in \mathcal{L}(w)$. Any affine set can be written as

$$\mathcal{L}(w) = \mathcal{L}_0 + X(w), \tag{20}$$

where $\mathcal{L}_0 \subseteq \mathcal{H}_2^+$ and $X : \mathcal{H}_2^+ \to \mathcal{H}_2^+$ is linear. Here, $\mathcal{L}_0$ does not depend on w and it follows that $\mathcal{L}_0 = \mathcal{L}(0)$. This implies that $\mathcal{L}_0 = \ker_+ \varPi_+ P_{12}$. Without loss of generality, define $X : \mathcal{H}_2^+ \to \mathcal{H}_2^+$ in such a manner that (20) holds where $X(w)$ is orthogonal to $\mathcal{L}_0$, i.e., $\langle X(w), \mathcal{L}_0 \rangle = 0$. Suppose this is the case. We then claim that X is *unique*, *linear* and *shift invariant*. Linearity has already been shown.

- Uniqueness follows from the observation that whenever X_1 and X_2 satisfy $\langle X_1(w), \mathcal{L}_0 \rangle = 0$ and $\langle X_2(w), \mathcal{L}_0 \rangle = 0$ for all $w \in \mathcal{B}_{\mathrm{manifest}}$ then $\langle X_1(w) - X_2(w), \mathcal{L}_0 \rangle = 0$. On the other hand, (20) implies that $X_1(w) - X_2(w) \in \mathcal{L}_0$. But then $X_1(w) = X_2(w)$ for all $w \in \mathcal{B}_{\mathrm{manifest}}$.
- Shift invariance follows in a similar manner. Let $\ell \in \mathcal{L}(w)$, $\tau \le 0$. Then $\ell = \ell' + X(w)$ with $\ell' \in \mathcal{L}_0$ and consequently, $\sigma_\tau \ell = \sigma_\tau \ell' + \sigma_\tau X(w)$. Since $\mathcal{B}_{\mathrm{full}}$ is left invariant we infer that $(\sigma_\tau w, \sigma_\tau \ell) \in \mathcal{B}_{\mathrm{full}}$ and therefore $\sigma_\tau \ell \in \mathcal{L}(\sigma_\tau w) = \mathcal{L}_0 + X(\sigma_\tau w)$. It follows that $\sigma_\tau \ell = \sigma_\tau \ell' + X(\sigma_\tau w)$ and, using the uniqueness of X, we have that X commutes with σ_τ for any $\tau \le 0$.

Since $X : \mathcal{H}_2^+ \to \mathcal{H}_2^+$ is linear and shift invariant, it admits a representation as a multiplicative operator $[X(w)](s) = X(s)w(s)$ where $X \in \mathcal{H}_\infty^+$ is uniquely defined. See Theorem 1.3 in Weiss (1991). It follows that, for any $w \in \mathcal{H}_2^+$, the latent variable $\ell := Xw$ is compatible with w in the sense that $(w, Xw) \in \mathcal{B}^1_{\mathrm{full},+}$. In particular, $R_1 := P_{11} + P_{12}X$ satisfies

$$R_1 \mathcal{H}_2^+ = (P_{11} + P_{12}X)\mathcal{H}_2^+ \subseteq \mathcal{H}_2^-,$$

which proves that $R_1 = 0$. Since $X \in \mathcal{H}_\infty^+$ and $X\mathcal{H}_2^+$ is orthogonal to $\mathcal{L}_0 = \ker_+ \varPi_+ P_{12}$, it follows that $X = P_{12}^* Y$ for some $Y \in \mathcal{H}_\infty^+$. To prove that Y is rational, consider the Hankel operator $\varGamma_Y : \mathcal{H}_2^- \to \mathcal{H}_2^+$ defined as $\varGamma_Y = \varPi_+ Y$. Because $R_1 = 0$, $\operatorname{rank}(\varGamma_Y) = \dim(\varPi_+ (P_{12}P_{12}^*)^{-1} P_{11} \mathcal{H}_2^-)$ which is finite because $(P_{12}P_{12}^*)^{-1}P_{11}$ is rational. By Kronecker's theorem (Theorem 3.11 in Partington (1988)), Y will be rational. Hence, $Y \in \mathcal{RH}_\infty^+$ and it follows that $X = P_{12}^* Y \in \mathcal{RH}_\infty^+$.

Third, note that the manifest behavior $\mathcal{B}_{\mathrm{manifest},+} = \mathcal{B}^1_{\mathrm{manifest},+} \cap \mathcal{B}^2_{\mathrm{manifest},+}$. Since $\mathcal{B}^1_{\mathrm{manifest},+} = \mathcal{H}_2^+$, we infer that $\mathcal{B}_{\mathrm{manifest},+} = \mathcal{B}^1_{\mathrm{manifest},+} = \ker_+ \varPi_+ P_{21}$.

Finally, we prove that $Q \in \mathcal{RH}_\infty^-$ satisfies the rank conditions in Theorem 4.9. Since

$$R := \tilde{P} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

with $R_1 = 0$ and $R_2 = P_{21}$, it is immediate that $R \in \mathcal{RH}_\infty^-$. Moreover this satisfies $\operatorname{rowrank}(R) = \operatorname{rowrank}(R_2) = \operatorname{rowrank}(\tilde{P}) - \operatorname{rowrank}(P_{12})$. In (18), we have $\tilde{P} = UP$, and hence $P = U^{-1}\tilde{P} \in \mathcal{RH}_\infty^-$, which implies that $Q = U^{-1}R \in \mathcal{RH}_\infty^-$. This also does not change the rank conditions; hence $\operatorname{rowrank}(Q) = \mathtt{p}(\mathcal{B}_{\mathrm{full},+}) - \operatorname{rowrank}(P_2)$, which completes the proof.

($\Leftarrow$): Suppose there exists $X \in \mathcal{RH}_\infty^+$ such that $Q \in \mathcal{RH}_\infty^-$ and that the given row rank condition is fulfilled. We will show that the manifest behavior is given by $\mathcal{B}_{\mathrm{manifest},+} = \ker_+ \varPi_+ P_{21}$.

Take any $w \in \mathcal{B}_{\mathrm{manifest},+}$. Let ℓ be such that $(w, \ell) \in \mathcal{B}_{\mathrm{full},+}$, which implies using (19) that $P_{11}w + P_{12}\ell \in \mathcal{H}_2^-$ and $P_{21}w \in \mathcal{H}_2^-$, so $w \in \ker_+ \varPi_+ P_{21}$. Therefore, $\mathcal{B}_{\mathrm{manifest},+} \subset \ker_+ \varPi_+ P_{21}$. To prove the converse, we have to show that $\mathcal{B}_{\mathrm{manifest},+} \supset \ker_+ \varPi_+ P_{21}$. Take $w \in \ker_+ \varPi_+ P_{21}$ and define $\ell := Xw$, with the given $X \in \mathcal{RH}_\infty^+$. We then claim that $(w, \ell) \in \mathcal{B}_{\mathrm{full},+}$. Indeed, $l = Xw \in \mathcal{H}_2^+$ and

$$\begin{aligned} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix} \begin{bmatrix} w \\ \ell \end{bmatrix} &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} w \\ &= \begin{bmatrix} P_{11} + P_{12}X \\ P_{21} \end{bmatrix} w = Qw. \end{aligned} \tag{21}$$

We need to show that $Qw \in \mathcal{H}_2^-$. Since the row rank of Q equals $\mathtt{p}(\mathcal{B}_{\mathrm{full},+}) - \operatorname{rowrank}(P_2) = \operatorname{rowrank}(P_{21})$, there exists a $U \in \mathcal{UH}_\infty^-$ such that

$$UQ = U \begin{bmatrix} P_{11} + P_{12}X \\ P_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ P_{21} \end{bmatrix}.$$

Multiplication with elements in $\mathcal{UH}_\infty^-$ does not change the behavior by Theorem 4.3, so from (21) we obtain

$$UQw = U \begin{bmatrix} P_{11} + P_{12}X \\ P_{21} \end{bmatrix} w = \begin{bmatrix} 0 \\ P_{21} \end{bmatrix} w \in \mathcal{H}_2^-,$$

and hence $Qw \in \mathcal{H}_2^-$. Therefore we have $\mathcal{B}_{\mathrm{manifest},+} \supset \ker_+ \varPi_+ P_{21}$ and we have shown that $\mathcal{B}_{\mathrm{manifest},+} = \ker_+ \varPi_+ P_{21}$, which concludes the proof. □

Proof of Theorem 5.2. i. ($\Rightarrow$): Suppose $\Sigma_C \in \mathbb{L}_+$ implements $\mathcal{K}$ for $\mathcal{P}$. Hence there exists a $C \in \mathcal{RH}_\infty^-$ such that $\mathcal{C} = \ker_+ \varPi_+ C$. Then

$$\begin{aligned} \mathcal{P} \cap \mathcal{C} &= \ker_+(\varPi_+ P) \cap \ker_+(\varPi_+ C) \\ &= \ker_+ \left(\varPi_+ \begin{bmatrix} P \\ C \end{bmatrix} \right) = \ker_+ \varPi_+ K = \mathcal{K}. \end{aligned}$$

We can choose C such that $\begin{bmatrix} P \\ C \end{bmatrix}$ has full row rank. Then by applying Theorem 4.3, we obtain that there exists a $U \in \mathcal{UH}_{\infty,*}^-$ such that $\begin{bmatrix} P \\ C \end{bmatrix} = UK$. Let $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ be partitioned according to $\begin{bmatrix} P \\ C \end{bmatrix}$. Consequently, $P = U_1 K$ with $U_1 \in \mathcal{RH}_{\infty,*}^-$. Since U is a unitary function, U_1 is outer. Set $X = U_1$ to infer the implication. ($\Leftarrow$): Let an outer $X \in \mathcal{RH}_{\infty,*}^-$ be such that $P = XK$. Since X is outer, there exists a $U \in \mathcal{UH}_{\infty,*}^-$ such that $\overline{X} := XU = [X_1\ 0]$ where $X_1 \in \mathcal{UH}_{\infty,*}^-$. Define $\overline{W} := [0\ I]$ and consider $\overline{\varLambda} := \begin{bmatrix} \overline{X} \\ \overline{W} \end{bmatrix}$. Obviously, $\overline{\varLambda} \in \mathcal{UH}_{\infty,*}^-$. Define $\varLambda := \overline{\varLambda} U^{-1}$ and $W := \overline{W} U^{-1}$. Since $\overline{\varLambda}$ and U are elements in $\mathcal{UH}_{\infty,*}^-$, also $\varLambda \in \mathcal{UH}_{\infty,*}^-$. By Theorem 4.3,

$$\begin{aligned} \mathcal{K} &= \ker_+ \varPi_+ K = \ker_+ \varPi_+ \varLambda K \\ &= \ker_+ \varPi_+ \begin{bmatrix} X \\ W \end{bmatrix} K = \ker_+ \varPi_+ \begin{bmatrix} P \\ C_0 \end{bmatrix}, \end{aligned}$$

where we defined $C_0 := WK$. Note that $C_0 \in \mathcal{RH}_{\infty,*}^-$. Using the definition of $\mathcal{RH}_{\infty,*}^-$, we know that $\exists\alpha > 0$ and $\exists k \ge 0$ such that $C := \frac{1}{(s-\alpha)^k} C_0 \in \mathcal{RH}_\infty^-$. Applying Lemma A.1 results in $\{w \in \mathcal{H}_2^+ \mid C_0 w \in \mathcal{H}_2^-\} = \{w \in \mathcal{H}_2^+ \mid Cw \in \mathcal{H}_2^-\}$. The proof is then completed by setting $\mathcal{C} = \ker_+ \varPi_+ C$ which implements $\mathcal{K}$ for $\mathcal{P}$ by full interconnection.

ii. Observe that $U := \begin{bmatrix} I & 0 \\ Q_1 & Q_2 \end{bmatrix}$ belongs to $\mathcal{UH}_{\infty,*}^-$ for all $Q_1 \in \mathcal{RH}_{\infty,*}^-$ and $Q_2 \in \mathcal{UH}_{\infty,*}^-$. Then, using Theorem 4.3, we have

$$\mathcal{K} = \ker_+ \left(\varPi_+ \begin{bmatrix} P \\ C \end{bmatrix} \right) = \ker_+ \left(\varPi_+ \begin{bmatrix} I & 0 \\ Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} P \\ C \end{bmatrix} \right)$$

$$= \ker_+ \left(\Pi_+ \begin{bmatrix} P \\ Q_1 P + Q_2 W K \end{bmatrix}\right) := \ker \Pi_+ \begin{bmatrix} P \\ \tilde{C} \end{bmatrix},$$

where $Q_1 \in \mathcal{RH}^-_{\infty,*}$ and $Q_2 \in \mathcal{UH}^-_{\infty,*}$. We then have that $\ker_+ \Pi_+ C = \ker_+ \Pi_+ \tilde{C}$, with $\tilde{C} \in \mathcal{RH}^-_{\infty,*}$. Since Q_1 and Q_2 parameterize all possible unitary operators in $\mathcal{UH}^-_{\infty,*}$ with the structure of U, all possible functions $\tilde{C} \in \mathcal{RH}^-_{\infty,*}$ can be parameterized by $Q_1P + Q_2WK$. Using the definition of $\mathcal{RH}^-_{\infty,*}$ and Lemma A.1, $\exists\alpha > 0$ and $\exists k \geq 0$ such that $\frac{1}{(s-\alpha)^k}\tilde{C} \in \mathcal{RH}^-_\infty$ and that $\ker_+ \Pi_+ C = \ker_+ \Pi_+ \frac{1}{(s-\alpha)^k}\tilde{C}$. Hence, the set of controllers is parameterized by $\mathbb{C}_{\text{par}}$ as in (12). □

Proof of Theorem 5.6. (⇒): Suppose $\Sigma_C \in \mathbb{L}_+$ implements the desired behavior $\mathcal{K}$ for $\mathcal{P}_{\text{full}}$. This means that

$$\mathcal{K} = \{w \mid \exists c \text{ for which } (w,c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}.$$

In particular, any $w \in \mathcal{K}$ belongs to $\mathcal{P}_{\text{manifest}}$. Hence, $\mathcal{K} \subset \mathcal{P}_{\text{manifest}}$. By Theorem 4.3, there exists $X \in \mathcal{RH}^-_{\infty,*}$ such that $P_{\text{man}} = XK$. We need to verify whether X is outer. The full controlled behavior is given by

$$\mathcal{K}_{\text{full}} = \{(w,c) \in \mathcal{H}_2^+ \mid (w,c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}$$

$$= \left\{(w,c) \in \mathcal{H}_2^+ \mid \begin{bmatrix} P_1 & P_2 \\ 0 & C \end{bmatrix}\begin{bmatrix} w \\ c \end{bmatrix} \in \mathcal{H}_2^-\right\}. \qquad (22)$$

From Definition 4.8, it follows that $\mathcal{K}_{\text{full}}$ is c-eliminable since we have $\mathcal{K} = \{w \in \mathcal{H}_2^+ \mid Kw \in \mathcal{H}_2^-\} = \ker_+ \Pi_+ K$, with $K \in \mathcal{RH}^-_\infty$. By Theorem 4.9, there exists $X_k \in \mathcal{RH}^+_\infty$ such that $\mathcal{K} = \{w \in \mathcal{H}_2^+ \mid Qw \in \mathcal{H}_2^-\}$ with

$$Q := \begin{bmatrix} P_1 & P_2 \\ 0 & C \end{bmatrix}\begin{bmatrix} I \\ X_k \end{bmatrix} = \begin{bmatrix} P_1 + P_2X_k \\ CX_k \end{bmatrix} \in \mathcal{RH}^-_\infty.$$

It is assumed that $\mathcal{P}_{\text{full}}$ is also c-eliminable. Hence, there exists $X_p \in \mathcal{RH}^+_\infty$ such that $\mathcal{P}_{\text{manifest}} = \ker_+ \Pi_+ P_{\text{man}}$ with

$$P_{\text{man}} = \begin{bmatrix} P_1 & P_2 \end{bmatrix}\begin{bmatrix} I \\ X_p \end{bmatrix} = P_1 + P_2X_p \in \mathcal{RH}^-_\infty. \qquad (23)$$

As shown, $\mathcal{K} \subset \mathcal{P}_{\text{manifest}}$, so for any $w \in \mathcal{K}$ we can also use X_k for the elimination of c in $\mathcal{P}_{\text{full}}$ in (23) (with the restriction that $w \in \mathcal{K}$). Thus, for all $w \in \mathcal{K}$ we have that $(w, X_kw) \in \mathcal{P}_{\text{full}}$. Hence, there exists one mapping $\tilde{X} : w \to c$ that eliminates c in $\mathcal{P}_{\text{full}}$ as well as in $\mathcal{K}_{\text{full}}$ through

$$\tilde{X}w := \begin{cases} X_kw, & \forall w \in \mathcal{K}, \\ X_pw, & \forall w \in \mathcal{K}^\perp \cap \mathcal{P}, \end{cases}$$

$$\text{so } P_{\text{man}} = P_1 + P_2\tilde{X} \quad \text{and} \quad Q = \begin{bmatrix} P_1 + P_2\tilde{X} \\ C\tilde{X} \end{bmatrix} := \begin{bmatrix} P_{\text{man}} \\ C_{\text{man}} \end{bmatrix},$$

where P_{man} can be chosen to have full row rank, and redundant rows in C_{man} can be eliminated such that Q has full row rank. For all $w \in \mathcal{K}$ we have that $Qw \in \mathcal{H}_2^-$ as well as $Kw \in \mathcal{H}_2^-$. By Theorem 4.3, $\exists U \in \mathcal{UH}^-_{\infty,*}$ such that $Q = UK$, where we use the decomposition $U = [X^\top \ Y^\top]^\top$. Therefore, $P_{\text{man}} = XK$ with $X \in \mathcal{RH}^-_{\infty,*}$ outer, since $U \in \mathcal{UH}^-_{\infty,*}$.
We also need to show that $K = YP_1$ with $Y \in \mathcal{RH}^-_{\infty,*}$ outer. By linearity of the controller, 0 lies in $\mathcal{C}$, so

$$\mathcal{K}_0 := \{w \mid (w,0) \in \mathcal{P}_{\text{full}} \text{ and } 0 \in \mathcal{C}\} \subset \mathcal{K}.$$

Now observe that $\mathcal{K}_0 = \ker_+ \Pi_+ P_1 = \mathcal{N}$. Hence $\mathcal{N} \subset \mathcal{K}$, which implies that there exists $Y \in \mathcal{RH}^-_{\infty,*}$ such that $K = YP_1$. To verify the outer property, we introduce $\mathcal{N}_{\text{full}}$ as

$$\mathcal{N}_{\text{full}} = \left\{(w,c) \in \mathcal{H}_2^+ \mid \begin{bmatrix} P_1 & P_2 \\ 0 & C \\ 0 & C^\perp \end{bmatrix}\begin{bmatrix} w \\ c \end{bmatrix} \in \mathcal{H}_2^-\right\}, \qquad (24)$$

where we define $C^\perp \in \mathcal{RH}^-_\infty$ such that $\begin{bmatrix} C \\ C^\perp \end{bmatrix}$ has full rank. When $Cc \in \mathcal{H}_2^-$ and also $C^\perp c \in \mathcal{H}_2^-$, we do indeed have that $c = 0$, which should be the case for the hidden behavior. Since there exists a rational representation for $\mathcal{N}$, we know that we can eliminate c in (24) and so by Theorem 4.9 $\exists X_n \in \mathcal{RH}^-_\infty$ such that $\mathcal{N} = \ker_+ \Pi_+ Q'$ with

$$Q' := \begin{bmatrix} P_1 & P_2 \\ 0 & C \\ 0 & C^\perp \end{bmatrix}\begin{bmatrix} I \\ X_n \end{bmatrix} = \begin{bmatrix} P_1 + P_2X_n \\ CX_n \\ C^\perp X_n \end{bmatrix} \in \mathcal{RH}^-_\infty.$$

As shown, $\mathcal{N} \subset \mathcal{K}$; hence we can also use X_n to eliminate the variable c in (22) for all $w \in \mathcal{N}$. Extension for $w \in \mathcal{N}^\perp \cap \mathcal{K}$ yields the mapping $\tilde{X}'$, that can eliminate c in $\mathcal{N}_{\text{full}}$ as well as in $\mathcal{K}_{\text{full}}$, which is given by

$$\tilde{X}'w := \begin{cases} X_nw, & \forall w \in \mathcal{N}, \\ X_kw, & \forall w \in \mathcal{N}^\perp \cap \mathcal{K}, \end{cases}$$

$$\text{so } Q' = \begin{bmatrix} P_1 + P_2\tilde{X}' \\ C\tilde{X}' \\ C^\perp\tilde{X}' \end{bmatrix} = \begin{bmatrix} K \\ C^\perp\tilde{X}' \end{bmatrix},$$

where again K is chosen to have full row rank, and redundant rows in $C^\perp\tilde{X}'$ are removed to make Q' full row rank. For all $w \in \mathcal{N}$, we then have $Q'w \in \mathcal{H}_2^-$ and $P_1w \in \mathcal{H}_2^-$, so using Theorem 4.3, $\exists U' \in \mathcal{UH}^-_{\infty,*}$ such that $Q' = UP_1$. Decomposing U in $[Y^\top \ Z^\top]^\top$, we have $K = YP_1$ where Y is outer. This completes the proof.
(⇐): Let $X, Y \in \mathcal{RH}^-_{\infty,*}$ be outer functions such that $K = XP_1$ and $P_{\text{man}} = YK$. Since Y is outer, there exists a unitary function $U \in \mathcal{UH}^-_{\infty,*}$ such that $\overline{Y} := YU = [Y_1 \ 0]$ where $Y_1 \in \mathcal{UH}^-_{\infty,*}$. As in the proof of Theorem 5.2, we define $\overline{W} := [0 \ I]$ and consider $\overline{\Lambda} := \begin{bmatrix} \overline{Y} \\ \overline{W} \end{bmatrix}$. Obviously, $\overline{\Lambda} \in \mathcal{UH}^-_{\infty,*}$. Define $\Lambda := \overline{\Lambda}U^{-1}$ and $W := \overline{W}U^{-1}$. Since $\overline{\Lambda}$ and U are unitary operators, also $\Lambda \in \mathcal{UH}^-_{\infty,*}$. Using Theorem 4.3, we have

$$\mathcal{K} = \ker_+ \Pi_+ K = \ker_+ \Pi_+ \Lambda K$$

$$= \ker_+ \Pi_+ \begin{bmatrix} Y \\ W \end{bmatrix} K = \ker_+ \Pi_+ \begin{bmatrix} P_{\text{man}} \\ \overline{C} \end{bmatrix},$$

where we defined $\overline{C} := WK = WXP_1$ (using the condition $K = XP_1$). Note that $WX \in \mathcal{RH}^-_{\infty,*}$ and hence $\overline{C} \in \mathcal{RH}^-_{\infty,*}$. This operator $\overline{C}$ represents the behavior of Σ_C, however restricting the variable w instead of the variable c. This can be denoted by $\mathcal{C}_w = \{w \in \mathcal{H}_2^+ \mid \overline{C}w \in \mathcal{H}_2^-\}$. From the definition of $\mathcal{RH}^-_{\infty,*}$, there $\exists\alpha > 0$ and $\exists k \geq 0$ such that $\tilde{W} := \frac{1}{(s-\alpha)^k}WX \in \mathcal{RH}^-_\infty$. Then, given this α and k, we apply Lemma A.1 such that

$$\mathcal{C}_w = \ker_+ \Pi_+ \overline{C} = \ker_+ \Pi_+ WXP_1$$

$$= \ker_+ \Pi_+ \frac{1}{(s-\alpha)^k} WXP_1 = \ker_+ \Pi_+ \tilde{W}P_1.$$

Because $\operatorname{col}(w,c) \in \mathcal{P}_{\text{full}}$, we have $P_1w + P_2c \in \mathcal{H}_2^-$, hence $P_1w = -P_2c + v$ with a possible non-zero $v \in \mathcal{H}_2^-$. This results for all $w \in \mathcal{H}_2^+$ in

$$\overline{C}w \in \mathcal{H}_2^- \Rightarrow 0 = \Pi_+\overline{C}w = \Pi_+\tilde{W}P_1w$$

$$= \Pi_+(-\tilde{W}P_2c + \tilde{W}v) = -\Pi_+\tilde{W}P_2c,$$

because $\tilde{W}v \in \mathcal{H}_2^-$. Therefore, the behavior of the controller is given by

$$\mathcal{C} = \{c \in \mathcal{H}_2^+ \mid -\tilde{W}P_2c \in \mathcal{H}_2^-\}$$

$$= \ker_+ \Pi_+ \left(-\frac{1}{(s-\alpha)^k} WXP_2\right) := \ker_+ \Pi_+ C,$$

where $\alpha > 0$ and $k \geq 0$ such that $\tilde{W} \in \mathcal{RH}_\infty^-$. This implies that $C \in \mathcal{RH}_\infty^-$, which completes the proof. □

References

Belur, M. N. (2003). Control in a behavioral context. *Ph.D. thesis*. Rijksuniversiteit Groningen, The Netherlands.

Francis, B. A. (1987). *Lecture notes in control and information sciences, A course in H_∞ control theory*. Springer-Verlag.

Fuhrmann, P. A. (1981). *Linear systems and operators in Hilbert space*. McGraw-Hill.

Gottimukkala, S. V., Fiaz, S., & Trentelman, H. L. (2011). Equivalence of rational representations of behaviors. *Systems & Control Letters*, *60*, 119–127.

Kailath, T. (1980). *Linear systems*. Englewood Cliffs, NJ: Prentice-Hall.

Mutsaers, M., & Weiland, S. (2008). Controller synthesis for $\mathcal{L}_2$ behaviors using rational kernel representations. In *Proceedings of the 47th IEEE conference on decision and control* (pp. 5134–5139).

Mutsaers, M., & Weiland, S. (2010). On the elimination of latent variables in $\mathcal{L}_2$ behaviors. In *Proceedings of the 49th IEEE conference on decision and control* (pp. 7742–7747).

Mutsaers, M., & Weiland, S. (2011). Elimination of latent variables in behavioral systems: a geometric approach. In *Proceedings of the 18th IFAC world congress* (pp. 10117–10122).

Partington, J. R. (1988). *An introduction to Hankel operators*. Cambridge University Press.

Polderman, J. W. (1997). Proper elimination of latent variables. *Systems & Control Letters*, *32*(5), 261–269.

Polderman, J. W., & Willems, J. C. (1998). *Introduction to mathematical systems theory: a behavioral approach*. Springer-Verlag.

Rosenblum, M., & Rovnyak, J. (1997). *Hardy classes and operator theory*. Dover Publications.

Trentelman, H. L. (2010). *Perspectives in mathematical system theory, control, and signal processing* (pp. 239–249). Springer-Verlag (Chapter on behavioral equivalence of rational representations).

Trentelman, H. L., Yoe, R. Z., & Praagman, C. (2007). Polynomial embedding algorithms for controllers in a behavioral framework. *IEEE Transactions on Automatic Control*, *52*(11), 2182–2188.

Vidyasagar, M. (1985). *Control system synthesis: a factorization approach*. Cambridge, MA: MIT press.

Weiland, S., & Stoorvogel, A. (1997). Rational representations of behaviors: interconnectability and stabilizability. *Mathematics of Control, Signals, and Systems (MCSS)*, *10*(2), 125–164.

Weiss, G. (1991). Representation of shift-invariant operators on L^2 by H^∞ transfer functions: an elementary proof, a generalization to L^p, and a counterexample for L^∞. *Mathematics of Control, Signals, and Systems (MCSS)*, *4*, 193–203.

Willems, J. C. (1989). Models for dynamics. *Dynamics Reported*, *2*, 171–269.

Willems, J. C. (1997). On interconnections, control, and feedback. *IEEE Transactions on Automatic Control*, *42*(3), 326–339.

Willems, J. C. (2007). The behavioral approach to open and interconnected systems. *IEEE Control Systems Magazine*, *27*(6), 46–99.

Willems, J. C., & Trentelman, H. L. (2002). Synthesis of dissipative systems using quadratic differential forms: part I. *IEEE Transactions on Automatic Control*, *47*(1), 53–69.

Willems, J. C., & Yamamoto, Y. (2006). Behaviors defined by rational functions. In *Proc. 45th IEEE conference on decision and control* (pp. 550–552).

Willems, J. C., & Yamamoto, Y. (2007). Behaviors defined by rational functions. *Linear Algebra and its Applications*, *425*, 226–241.

Willems, J. C., & Yamamoto, Y. (2008). *Lecture notes in control and information sciences*: *Vol. 371*. *Recent advances in learning and control* (pp. 263–277). Berlin, Heidelberg: Springer (Chapter behaviors described by rational symbols and the parametrization of the stabilizing controllers).

Mark Mutsaers (born in Rijen, The Netherlands, 1983) received his M.Sc. degree in Control Engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands, in 2008. He is currently working towards the degree of Ph.D. in the Control Systems Group of the Department of Electrical Engineering at the same institution. His research interests include model reduction of large-scale dynamical systems, general systems theory and model predictive control.

Siep Weiland is Professor at the Control Systems Group, Department of Electrical Engineering, Eindhoven University of Technology. He received both his M.Sc. (1986) and Ph.D. degrees in Mathematics from the University of Groningen in the Netherlands. He was a postdoctoral research associate at the Department of Electrical Engineering and Computer Engineering, Rice University, Houston, USA, from 1991 to 1992. Since 1992 he has been affiliated to Eindhoven University of Technology. His research interests are the general theory of systems and control, robust control, model approximation, modeling and control of spatial–temporal systems, identification, and model predictive control. He was Associate Editor of the IEEE Transactions on Automatic Control from 1995 to 1999, of the European Journal of Control from 1999 to 2003, of the International Journal of Robust and Nonlinear Control from 2001 to 2004 and Associate Editor for Automatica from 2003 until 2006.