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HARMONIC MEASURES VERSUS QUASICONFORMAL MEASURES FOR HYPERBOLIC GROUPS

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ABSTRACT. We establish a dimension formula for the harmonic measure of a finitely supported and symmetric random walk on a hyperbolic group. We also characterize random walks for which this dimension is maximal. Our approach is based on the Green metric, a metric which provides a geometric point of view on random walks and, in particular, which allows us to interpret harmonic measures as quasiconformal measures on the boundary of the group.

1. INTRODUCTION

It is a leading thread in hyperbolic geometry to try to understand properties of hyperbolic spaces by studying their large-scale behaviour. This principle is applied through the introduction of a canonical compactification which characterises the space itself. For instance a hyperbolic group Γ in the sense of Gromov admits a natural boundary at infinity $\partial\Gamma$: it is a topologically well-defined compact set on which Γ acts by homeomorphisms. Together, the pair consisting of the boundary $\partial\Gamma$ with the action of Γ characterises the hyperbolicity of the group. Topological properties of $\partial\Gamma$ also encode the algebraic structure of the group. For instance one proves that Γ is virtually free if and only if $\partial\Gamma$ is a Cantor set (see [41] and also [11] for other results in this vein). Moreover, the boundary is endowed with a canonical quasiconformal structure which determines the quasi-isometry class of the group (see [26] and the references therein for details).

Characterising special subclasses of hyperbolic groups such as cocompact Kleinian groups often requires the construction of special metrics and measures on the boundary which carry some geometrical information. For example, M. Bonk and B. Kleiner proved that a group admits a cocompact Kleinian action on the hyperbolic space $\mathbb{H}^n$, $n \geq 3$, if and only if its boundary has topological dimension $n - 1$ and carries an Ahlfors-regular metric of dimension $n - 1$ [8].

There are two main constructions of measures on the boundary of a hyperbolic group: quasiconformal measures and harmonic measures. Let us recall these constructions.

Given a cocompact properly discontinuous action of Γ by isometries on a pointed proper geodesic metric space (X, w, d), the Patterson-Sullivan procedure consists in taking weak limits of

$$\frac{1}{\sum_{\gamma \in \Gamma} e^{-sd(w,\gamma(w))}} \sum_{\gamma \in \Gamma} e^{-sd(w,\gamma(w))} \delta_{\gamma(w)}$$

as s decreases to the logarithmic volume growth

$$v \stackrel{\text{def.}}{=} \limsup_{R \to \infty} \frac{1}{R} \log |B(w, R) \cap \Gamma(w)| \,.$$

Patterson-Sullivan measures are quasiconformal measures and Hausdorff measures of ∂X when endowed with a visual metric.

Given a probability measure μ on Γ, the random walk $(Z_n)_n$ starting from the neutral element e associated with μ is defined by

$$Z_0 = e \,;\; Z_{n+1} = Z_n \cdot X_{n+1} \,,$$

where (X_n) is a sequence of independent and identically distributed random variables of law μ. Under some mild assumptions on μ, the walk $(Z_n)_n$ almost surely converges to a point $Z_\infty \in \partial \Gamma$. The law of Z_∞ is by definition the harmonic measure ν.

The purpose of this work is to investigate the interplay between those two classes of measures and take advantage of this interplay to derive information on the geometry of harmonic measures. We show that, for a general hyperbolic group, the Hausdorff dimension of the harmonic measure can be explicitly computed and satisfies a 'dimension-entropy-rate of escape' formula. We also characterise those harmonic measures of maximal dimension.

The usual tool for this kind of result is to replace the action of the group by a linear-in-time action of a dynamical system and then to apply the thermodynamic formalism to it: for free groups and Fuchsian groups, a Markov-map F_Γ has been introduced on the boundary which is orbit-equivalent to Γ [12, 33]. For discrete subgroups of isometries of a Cartan-Hadamard manifold, one may work with the geodesic flow [31, 32, 21, 23].

Our approach is different for both these methods seem difficult to implement for general hyperbolic groups. On the one hand, it is not obvious how to associate a Markov map with a general hyperbolic group, even using the automatic structure of the group. On the other hand, the construction of the geodesic flow for general hyperbolic spaces is delicate and its mixing properties do not seem strong enough to apply the thermodynamic formalism. Instead, we will combine geometric arguments with the special features of random walks to derive our results. As such, we believe our approach to be more elementary.

1.1. **Geometric setting.** Given a hyperbolic group Γ, we let $\mathcal{D}(\Gamma)$ denote the collection of hyperbolic left-invariant metrics on Γ and quasi-isometric to a word metric induced by a finite generating set of Γ. In general these metrics do not come from proper geodesic metric spaces as we will see (cf. Theorem 1.1 for instance). In the sequel, we will distinguish the group as a space and as acting on a space: we keep the notation Γ for the group, and we denote by X the group as a metric space endowed with a metric $d \in \mathcal{D}(\Gamma)$. We may equivalently write $(X, d) \in \mathcal{D}(\Gamma)$. We will often require a base point which we will denote by $w \in X$.

This setting enables us to capture in particular the following two situations.

- Assume that Γ admits a cocompact properly discontinuous action by isometries on a proper geodesic space (Y, d). Pick $w \in Y$ such that $\gamma \in \Gamma \mapsto \gamma(w)$ is a bijection, and consider $X = \Gamma(w)$ with the restriction of d.
- We may choose $(X, d) = (\Gamma, d_G)$ where d_G is the Green metric associated with a random walk (see Theorem 1.1).

Let μ be a symmetric probability measure the support of which generates Γ. Even if the support of μ may be infinite, we will require some compatibility with the geometry of the quasi-isometry class of $\mathcal{D}(\Gamma)$. Thus, we will often assume one of the following two assumptions. Given a metric $(X, d) \in \mathcal{D}(\Gamma)$, we say that the random walk has *finite first moment* if

$$\sum_{\gamma \in \Gamma} d(w, \gamma(w)) \mu(\gamma) < \infty \, .$$

We say that the random walk has an *exponential moment* if there exists $\lambda > 0$ such that

$$\sum_{\gamma \in \Gamma} e^{\lambda d(w, \gamma(w))} \mu(\gamma) < \infty \, .$$

Note that both these conditions only depend on the quasi-isometry class of the metric.

1.2. **The Green metric.** The analogy between both families of measures – quasiconformal and harmonic – has already been pointed out in the literature. Our first task is to make this empirical fact a theorem i.e., we prove that harmonic measures are indeed quasiconformal measures for a well-chosen metric: given a symmetric law μ on Γ such that its support generates Γ, let $F(x, y)$ be the probability that the random walk started at x ever hits y. Up to a constant factor, $F(x, y)$ coincides with the Green function

$$G(x, y) \overset{\text{def.}}{=} \sum_{n=0}^{\infty} \mathbb{P}^x[Z_n = y] = \sum_{n=0}^{\infty} \mu^n(x^{-1}y) \, ,$$

where $\mathbb{P}^x$ denotes the probability law of the random walk (Z_n) with $Z_0 = x$ (if $Z_0 = e$, the neutral element of Γ, we will simply write $\mathbb{P}^e = \mathbb{P}$), and where, for each $n \geq 1$, μ^n is the law of Z_n i.e., the nth convolution power of the measure μ.

We define the *Green metric* between x and y in Γ by

$$d_G(x, y) \overset{\text{def.}}{=} - \log F(x, y) \, .$$

This metric was first introduced by S. Blachère and S. Brofferio in [6] and further studied in [7]. It is well-defined as soon as the walk is transient i.e., eventually leaves any finite set. This is the case as soon as Γ is a non-elementary hyperbolic group.

Non-elementary hyperbolic groups are non-amenable and for such groups and finitely supported laws μ, it was proved in [6] that the Green and word metrics are quasi-isometric. Nevertheless it does not follow from this simple fact that d_G is hyperbolic, see the discussion below, § 1.7.

We first prove the following:

Theorem 1.1. *Let Γ be a non-elementary hyperbolic group, μ a symmetric probability measure on Γ the support of which generates Γ.*

(i) *Assume that μ has an exponential moment, then $d_G \in \mathcal{D}(\Gamma)$ if and only if for any r there exists a constant $C(r)$ such that*

(1) $$F(x, y) \leq C(r) F(x, v) F(v, y)$$

whenever x, y and v are points in a locally finite Cayley graph of Γ and v is at distance at most r from a geodesic segment between x and y.

(ii) *If $d_G \in \mathcal{D}(\Gamma)$ then the harmonic measure is Ahlfors regular of dimension $1/\varepsilon$, when $\partial \Gamma$ is endowed with a visual metric d_ε^G of parameter $\varepsilon > 0$ induced by d_G.*

Visual metrics are defined in the next section.

A. Ancona proved that (1) holds for finitely supported laws μ. Condition (1) has also been coined by V. Kaimanovich as the key ingredient in proving that the Martin boundary coincides with the geometric (hyperbolic) boundary [23, Thm 3.1] (See also § 1.5 and § 3.2 for a further discussion on the relationships between the Green metric and the Martin boundary).

Theorem 1.1 in particular yields

Corollary 1.2. *Let Γ be a non-elementary hyperbolic group, μ a finitely supported symmetric probability measure on Γ the support of which generates Γ. Then its associated Green metric d_G is a left-invariant hyperbolic metric on Γ quasi-isometric to Γ such that the harmonic measure is Ahlfors regular of dimension $1/\varepsilon$, when $\partial\Gamma$ is endowed with a visual metric d_ε^G of parameter $\varepsilon > 0$ induced by d_G.*

Our second source of examples of random walks satisfying (1) will come from Brownian motions on Riemannian manifolds of negative curvature. The corresponding law μ will then have infinite support (see § 1.6 and § 6).

1.3. **Dimension of the harmonic measure at infinity.** Let $(X, d) \in \mathcal{D}(\Gamma)$. We fix a base point $w \in X$ and consider the random walk on X started at w i.e., the sequence of X-valued random variables $(Z_n(w))$ defined by the action of Γ on X. There are (at least) two natural asymptotic quantities one can consider: the *asymptotic entropy*

$$h \overset{\text{def.}}{=} \lim_n \frac{-\sum_{\gamma \in \Gamma} \mu^n(\gamma) \log \mu^n(\gamma)}{n} = \lim_n \frac{-\sum_{x \in \Gamma(w)} \mathbb{P}[Z_n(w) = x] \log \mathbb{P}[Z_n(w) = x]}{n}$$

which measures the way the law of $Z_n(w)$ is spread in different directions, and the *rate of escape* or *drift*

$$\ell \overset{\text{def.}}{=} \lim_n \frac{d(w, Z_n(w))}{n},$$

which estimates how far $Z_n(w)$ is from its initial point w. (The above limits for h and ℓ are almost sure and in L^1 and they are finite as soon as the law has a finite first moment.)

We obtain the following.

Theorem 1.3. *Let Γ be a non-elementary hyperbolic group, $(X, d) \in \mathcal{D}(\Gamma)$, d_ε be a visual metric of ∂X, and let $B_\varepsilon(a, r)$ be the ball of center $a \in \partial X$ and radius r for the distance d_ε. Let ν be the harmonic measure of a random walk (Z_n) whose increments are given by a symmetric law μ with finite first moment such that $d_G \in \mathcal{D}(\Gamma)$.*

The pointwise Hausdorff dimension $\lim_{r \to 0} \frac{\log \nu(B_\varepsilon(a, r))}{\log r}$ exists for ν-almost every $a \in \partial X$, and is independent from the choice of a. More precisely, for ν-almost every $a \in \partial X$,

$$\lim_{r \to 0} \frac{\log \nu(B_\varepsilon(a, r))}{\log r} = \frac{\ell_G}{\varepsilon \ell}$$

where $\ell > 0$ denotes the rate of escape of the walk with respect to d and $\ell_G \overset{\text{def.}}{=} \lim_n \frac{d_G(w, Z_n(w))}{n}$ the rate of escape with respect to d_G.

We recall that the dimension of a measure is the infimum Hausdorff dimension of sets of positive measure. In [7], it was shown that $\ell_G = h$ the asymptotic entropy of the walk. From Theorem 1.3, we deduce that

Corollary 1.4. *Under the assumptions of Theorem 1.3,*

$$\dim \nu = \frac{h}{\varepsilon \ell}$$

where h denotes the asymptotic entropy of the walk and ℓ its rate of escape with respect to d.

This dimension formula already appears in the work of F. Ledrappier for random walks on free groups [33]. See also V. Kaimanovich, [24]. For general hyperbolic groups, V. Leprince established the inequality $\dim \nu \le h/(\varepsilon \ell)$ and made constructions of harmonic measures with arbitrarily small dimension [29]. More recently, V. Leprince established that $h/\varepsilon \ell$ is also the box dimension of the harmonic measure under the sole assumption that the random walk has a finite first moment [30]. Note however that the notion of box dimension is too weak to ensure the existence of the pointwise Hausdorff dimension almost everywhere.

This formula is also closely related to the dimension formula proved for ergodic invariant measures with positive entropy in the context of geometric dynamical systems: the drift corresponds to a Lyapunov exponent [45].

1.4. **Characterisation of harmonic measures with maximal dimension.** Given a random walk on a finitely generated group Γ endowed with a left-invariant metric d, the so-called *fundamental inequality* between the asymptotic entropy h, the drift ℓ and the logarithmic growth rate v of the action of Γ reads

$$h \le \ell v \,.$$

It holds as soon as all these objects are well-defined (cf. [7]). Corollary 1.4 provides a geometric interpretation of this inequality in terms of the harmonic measure: indeed, since v/ε is the dimension of $(\partial X, d_\varepsilon)$, see [13], it is clearly larger than the dimension of ν.

A. Vershik suggested the study of the case of equality (see [16, 43]). For any hyperbolic group, Theorem 1.1 implies that the equality $h = \ell v$ holds for the Green metric and Theorem 1.5 below shows that the equality for some $d \in \mathcal{D}(\Gamma)$ implies d is almost proportional to d_G. In particular, given a metric in $\mathcal{D}(\Gamma)$, all the harmonic measures for which the (fundamental) equality holds belong to the same class of quasiconformal measures.

In the sequel, two measures will be called *equivalent* if they share the same sets of zero measure.

Theorem 1.5. *Let Γ be a non-elementary hyperbolic group and $(X, d) \in \mathcal{D}(\Gamma)$; let d_ε be a visual metric of ∂X, and ν the harmonic measure given by a symmetric law μ with an exponential moment, the support of which generates Γ. We further assume that $(X, d_G) \in \mathcal{D}(\Gamma)$. We denote by ρ a quasiconformal measure on $(\partial X, d_\varepsilon)$. The following propositions are equivalent.*

   (i) *We have the equality $h = \ell v$.*
   (ii) *The measures ρ and ν are equivalent.*
   (iii) *The measures ρ and ν are equivalent and the density is almost surely bounded and bounded away from 0.*
   (iv) *The map $(\Gamma, d_G) \xrightarrow{Id} (X, vd)$ is a $(1, C)$-quasi-isometry.*
   (v) *The measure ν is a quasiconformal measure of $(\partial X, d_\varepsilon)$.*

This theorem is the counterpart of a result of F. Ledrappier for Brownian motions on universal covers of compact Riemannian manifolds of negative sectional curvature [31], see also

§ 1.6. Similar results have been established for the free group with free generators, see [33]. The case of equality $h = \ell v$ has also been studied for particular sets of generators of free products of finite groups [36]. For universal covers of finite graphs, see [34].

Theorem 1.5 enables us to compare random walks and decide when their harmonic measures are equivalent.

Corollary 1.6. *Let Γ be a non-elementary hyperbolic group with two finitely supported symmetric probability measures μ and $\widehat{\mu}$ where both supports generate Γ. We consider the random walks (Z_n) and $(\widehat{Z}_n)$. Let us denote their Green functions by G and $\widehat{G}$ respectively, the asymptotic entropies by h and $\widehat{h}$, and the harmonic measures seen from the neutral element e by ν and $\widehat{\nu}$. The following propositions are equivalent.*

(i) *We have the equality*
$$\widehat{h} = \lim \frac{-1}{n} \log G(e, \widehat{Z}_n)$$
in L^1 and almost surely.

(ii) *We have the equality*
$$h = \lim \frac{-1}{n} \log \widehat{G}(e, Z_n)$$
in L^1 and almost surely.

(iii) *The measures ν and $\widehat{\nu}$ are equivalent.*

(iv) *There is a constant C such that*
$$\frac{1}{C} \le \frac{G(x,y)}{\widehat{G}(x,y)} \le C \,.$$

1.5. **The Green metric and the Martin compactification.** Given a probability measure μ on a countable group Γ, one defines the Martin kernel
$$K(x,y) = K_y(x) \overset{\text{def.}}{=} \frac{G(x,y)}{G(e,y)} \,.$$

By definition, the *Martin compactification* $\Gamma \cup \partial_M \Gamma$ is the smallest compactification of Γ endowed with the discrete topology such that the Martin kernel continuously extends to $\Gamma \times (\Gamma \cup \partial_M \Gamma)$. Then $\partial_M \Gamma$ is called the *Martin boundary*.

A general theme is to identify the Martin boundary with a geometric boundary of the group. It was observed in [7] that the Martin compactification coincides with the Busemann compactification of (Γ, d_G). We go one step further by showing that the Green metric provides a common framework for the identification of the Martin boundary with the boundary at infinity of a hyperbolic space (cf. [1, 3, 23]).

Theorem 1.7. *Let Γ be a countable group, μ a symmetric probability measure the support of which generates Γ. We assume that the corresponding random walk is transient. If the Green metric is hyperbolic, then the Martin boundary consists only of minimal points and it is homeomorphic to the hyperbolic boundary of (Γ, d_G).*

In particular, if Γ is a non-elementary hyperbolic group and if $d_G \in \mathcal{D}(\Gamma)$, then $\partial_M \Gamma$ is homeomorphic to $\partial \Gamma$.

One easily deduces from Corollary 1.2:

Corollary 1.8. (A. Ancona) *Let Γ be a non-elementary hyperbolic group, μ a finitely supported probability measure the support of which generates Γ. Then the Martin boundary is homeomorphic to the hyperbolic boundary of Γ.*

In §6.3, we provide examples of hyperbolic groups with random walks for which the Green metric is hyperbolic, but not in the quasi-isometry class of the group, and also examples of non-hyperbolic groups for which the Green metric is nonetheless hyperbolic. These examples are constructed by discretising Brownian motions on Riemannian manifolds (see below).

1.6. **Brownian motion revisited.** Let M be the universal covering of a compact Riemannian manifold of negative curvature with deck transformation group Γ i.e., the action of Γ is isometric, cocompact and properly discontinuous. The Brownian motion (ξ_t) on M is the diffusion process generated by the Laplace-Beltrami operator. It is known that the Brownian motion trajectory almost surely converges to some limit point $\xi_\infty \in \partial M$. The law of ξ_∞ is the harmonic measure of the Brownian motion. The notions of rate of escape and asymptotic entropy also make perfect sense in this setting.

Refining a method of T. Lyons and D. Sullivan [35], W. Ballmann and F. Ledrappier construct in [3] a random walk on Γ which mirrors the trajectories of the Brownian motion and to which we may apply our previous results. This enables us to recover the following results.

Theorem 1.9. *Let M be the universal covering of a compact Riemannian manifold of negative curvature with logarithmic volume growth v. Let d_ε be a visual distance on ∂M. Then*

$$\dim \nu = \frac{h_M}{\varepsilon \ell_M}$$

where h_M and ℓ_M denote the asymptotic entropy and the drift of the Brownian motion respectively. Furthermore, $h_M = \ell_M v$ if and only if ν is equivalent to the Hausdorff measure of dimension v/ε on $(\partial M, d_\varepsilon)$.

The first result is folklore and explicitly stated by V. Kaimanovich in the introduction of [21], but we know of no published proof. The second statement is due to F. Ledrappier [31]. Note that more is known: the equality $h_M = \ell_M v$ is equivalent to the equality of ν with the canonical *conformal* measure on $(\partial M, d_\varepsilon)$, and this is possible only if M is a rank 1 symmetric space [32, 5].

1.7. **Quasiruled hyperbolic spaces.** As previously mentioned, S. Blachère and S. Brofferio proved that, for finitely supported laws, the Green metric d_G is quasi-isometric to the word metric. But since d_G is defined only on a countable set, it is unlikely to be the restriction of a proper geodesic metric (which would have guaranteed the hyperbolicity of (Γ, d_G)). Therefore, the proof of Theorem 1.1 requires the understanding of which metric spaces among the quasi-isometry class of a given geodesic hyperbolic space are also hyperbolic. For this, we coin the notion of a quasiruler: a τ-*quasiruler* is a quasigeodesic $g : \mathbb{R} \to X$ such that, for any $s < t < u$,

$$d(g(s), g(t)) + d(g(t), g(u)) - d(g(s), g(u)) \leq 2\tau.$$

A metric space will be *quasiruled* if constants (λ, c, τ) exist so that the space is (λ, c) - quasigeodesic and if every (λ, c)-quasigeodesic is a τ-quasiruler. We refer to the Appendix for details on the definitions and properties of quasigeodesics and quasiruled spaces. We prove the following characterisation of hyperbolicity, interesting in its own right.

Theorem 1.10. *Let X be a geodesic hyperbolic metric space, and $\varphi : X \to Y$ a quasi-isometry, where Y is a metric space. Then Y is hyperbolic if and only if it is quasiruled.*

Theorem 1.10 will be used to prove that the hyperbolicity of d_G is equivalent to condition (1) in Theorem 1.1. We complete this discussion by exhibiting for any hyperbolic group, a non-hyperbolic left-invariant metric in its quasi-isometry class (cf. Proposition A.11).

1.8. **Outline of the paper.** In Section 2, we recall the main facts on hyperbolic groups which will be used in the paper. In Section 3, we recall the construction of random walks, discuss some of their properties and introduce the Green metric. We also prove Theorem 1.7 and Theorem 1.1. We then draw some consequences on the harmonic measure and the random walk. The following Section 4 deals with the proof of Theorem 1.3. In Section 5, we deal with Theorem 1.5 and its corollary. Finally, Theorem 1.9 is proved in Section 6. The appendices are devoted to quasiruled spaces. We prove Theorem 1.10 in Appendix A, and we show that quasiruled spaces retain most properties of geodesic hyperbolic spaces: in Appendix B, we show that the approximation of finite configurations by trees still hold, and we explain why M. Coornaert's theorem on quasiconformal measures remains valid in this setting.

1.9. **Notation.** A distance in a metric space will be denoted either by $d(\cdot, \cdot)$ or $|\cdot - \cdot|$. If a and b are positive, $a \lesssim b$ means that there is a universal positive constant u such that $a \leq ub$. We will write $a \asymp b$ when both $a \lesssim b$ and $b \lesssim a$ hold. Throughout the article, dependance of a constant on structural parameters of the space will not be notified unless needed. Sometimes, it will be convenient to use Landau's notation $O(\cdot)$.

2. Hyperbolicity in metric spaces

Let (X, d) be a metric space. It is said to be *proper* if closed balls of finite radius are compact. A *geodesic curve (resp. ray, segment)* is a curve isometric to $\mathbb{R}$ (resp. $\mathbb{R}_+$, a compact interval of $\mathbb{R}$). The space X is said to be *geodesic* if every pair of points can be joined by a geodesic segment.

Given three points $x, y, w \in X$, one defines the Gromov inner product as follows:

$$(x|y)_w \overset{\text{def.}}{=} (1/2)\{|x - w| + |y - w| - |x - y|\} .$$

Definition. A metric space (X, d) is *δ-hyperbolic* ($\delta \geq 0$) if, for any $w, x, y, z \in X$, the following ultrametric type inequality holds

$$(y|z)_w \geq \min\{(x|y)_w, (x|z)_w\} - \delta .$$

We shall write $(\cdot|\cdot)_w = (\cdot|\cdot)$ when the choice of w is clear from the context.

Hyperbolicity is a large-scale property of the space. To capture this information, one defines the notion of quasi-isometry.

Definition. Let X, Y be two metric spaces and $\lambda \geq 1$, $c \geq 0$ two constants. A map $f : X \to Y$ is a *(λ, c)-quasi-isometric embedding* if, for any $x, x' \in X$, we have

$$\frac{1}{\lambda}|x - x'| - c \leq |f(x) - f(x')| \leq \lambda|x - x'| + c .$$

The map f is a *(λ, c)-quasi-isometry* if, in addition, there exist a quasi-isometric embedding $g : Y \to X$ and a constant C such that $|g \circ f(x) - x| \leq C$ for any $x \in X$. Equivalently,

f is a quasi-isometry if it is a quasi-isometric embedding such that Y is contained in a C-neighborhood of $f(X)$. We then say that f is C-*cobounded*.

In the sequel, we will always choose the constants so that that a (λ, c)-quasi-isometry is c-cobounded.

Definition. A *quasigeodesic curve (resp. ray, segment)* is the image of $\mathbb{R}$ (resp. $\mathbb{R}_+$, a compact interval of $\mathbb{R}$) by a quasi-isometric embedding.

In a geodesic hyperbolic metric space (X, d), quasigeodesics always shadow genuine geodesics i.e., given a (λ, c)-quasigeodesic q, there is a geodesic g such that $d_H(g, q) \leq K$, where d_H denotes the Hausdorff distance, and K only depends on δ, λ and c [17, Th. 5.6].

Compactification. Let X be a proper hyperbolic space, and $w \in X$ a base point. A sequence (x_n) tends to infinity if, by definition, $(x_n|x_m) \to \infty$ as $m, n \to \infty$. The *visual* or *hyperbolic* boundary ∂X of X is the set of sequences which tend to infinity modulo the equivalence relation defined by: $(x_n) \sim (y_n)$ if $(x_n|y_n) \to \infty$. One may also extend the Gromov inner product to points at infinity in such a way that the inequality

$$(y|z) \geq \min\{(x|y), (x|z)\} - \delta,$$

now holds for any points $w, x, y, z \in X \cup \partial X$.

For each $\varepsilon > 0$ small enough, there exists a so-called *visual metric* d_ε on ∂X i.e which satisfies for any $a, b \in \partial X$: $d_\varepsilon(a, b) \asymp e^{-\varepsilon(a|b)}$.

We shall use the notation $B_\varepsilon(a, r)$ to denote the ball in the space $(\partial X, d_\varepsilon)$ with center a and radius r.

We refer to [17] for the details (chap. 6 and 7).

Busemann functions. Let us assume that (X, d) is a hyperbolic space. Let $a \in \partial X$, $x, y \in X$. The function

$$\beta_a(x, y) \stackrel{\text{def.}}{=} \sup\left\{\limsup_{n\to\infty}[d(x, a_n) - d(y, a_n)]\right\},$$

where the supremum is taken over all sequences $(a_n)_n$ in X which tends to a, is called the *Busemann function* at the point a.

Shadows. Let $R > 0$ and $x \in X$. The *shadow* $\mho(x, R)$ is the set of points $a \in \partial X$ such that $(a|x)_w \geq d(w, x) - R$.

Approximating finitely many points by a tree (cf. Theorem B.1) yields:

Proposition 2.1. *Let (X, d) be a hyperbolic space. For any $\tau \geq 0$, there exist positive constants C, R_0 such that for any $R > R_0$, $a \in \partial X$ and $x \in X$ such that $(w|a)_x \leq \tau$,*

$$B_\varepsilon\left(a, \frac{1}{C}e^{R\varepsilon}e^{-\varepsilon|w-x|}\right) \subset \mho(x, R) \subset B_\varepsilon\left(a, Ce^{R\varepsilon}e^{-\varepsilon|w-x|}\right).$$

Shadows will enable us to control measures on the boundary of a hyperbolic group, see the lemma of the shadow in the next paragraph.

2.1. **Hyperbolic groups.** Let X be a hyperbolic proper metric space and Γ a subgroup of isometries which acts properly discontinuously on X i.e., for any compact sets K and L, the number of group elements $\gamma \in \Gamma$ such that $\gamma(K) \cap L \neq \emptyset$ is finite. For any point $x \in X$, its orbit $\Gamma(x)$ accumulates only on the boundary ∂X, and its set of accumulation points turns out to be independent of the choice of x; by definition, $\overline{\Gamma(x)} \cap \partial X$ is the *limit set* $\Lambda(\Gamma)$ of Γ.

An action of a group Γ on a metric space is said to be *geometric* if

(1) each element acts by isometry;
(2) the action is properly discontinuous;
(3) the action is cocompact.

For example, if Γ is a finitely generated group, S is a finite symmetric set of generators, one may consider the Cayley graph X associated with S: the set of vertices are the elements of the group, and pairs $(\gamma, \gamma') \in \Gamma \times \Gamma$ define an edge if $\gamma^{-1}\gamma' \in S$. Endowing X with the metric which makes each edge isometric to the segment $[0, 1]$ defines the *word metric associated with S*. It turns X into a geodesic proper metric space on which Γ acts geometrically by left-translation.

We recall Švarc-Milnor's lemma which provides a sort of converse statement, see [17]:

Lemma 2.2. *Let X be a geodesic proper metric space, and Γ a group which acts geometrically on X. Then Γ is finitely generated and X is quasi-isometric to any locally finite Cayley graph of Γ.*

A group Γ is *hyperbolic* if it acts geometrically on a geodesic proper hyperbolic metric space (e.g. a locally finite Cayley graph). In this case, one has $\Lambda(\Gamma) = \partial X$. Then Švarc-Milnor's lemma above implies that Γ is finitely generated.

We will say that a metric space (X, d) is *quasi-isometric* to the group Γ if it is quasi-isometric to a locally finite Cayley graph of Γ.

Let Γ be a hyperbolic group geometrically acting on (X, d). The action of Γ extends to the boundary. Busemann functions, visual metrics and the action of Γ are related by the following property: for any $a \in \partial X$ and any $\gamma \in \Gamma$, there exists a neighborhood V of a such that, for any $b, c \in V$,

$$d_\varepsilon(\gamma(b), \gamma(c)) \asymp L_\gamma(a) d_\varepsilon(b, c)$$

where $L_\gamma(a) \overset{\text{def.}}{=} e^{\varepsilon\beta_a(w, \gamma^{-1}(w))}$. Moreover, Γ also acts on measures on ∂X through the rule $\gamma^*\rho(A) \overset{\text{def.}}{=} \rho(\gamma A)$.

A hyperbolic group is said to be *elementary* if it is finite or quasi-isometric to $\mathbb{Z}$. We will only be dealing with non-elementary hyperbolic groups.

2.2. **Quasiconformal measures.** We now assume that Γ is a hyperbolic group and $(X, d) \in \mathcal{D}(\Gamma)$ (recall the definition in Section 1.1).

The next theorem summarizes the main properties of quasiconformal measures on the boundary of X. It was proved by M. Coornaert in [13] in the context of geodesic spaces. We state here a more general version to cover the case $d \in \mathcal{D}(\Gamma)$. We justify the validity of this generalisation at the end of the appendix. We refer to Section 4 for the definitions of the Hausdorff measure and dimension.

Theorem 2.3. *Let Γ be a non-elementary hyperbolic group and $(X, d) \in \mathcal{D}(\Gamma)$. For any small enough $\varepsilon > 0$, then $0 < \dim_H (\partial X, d_\varepsilon) < \infty$ and*

$$v \stackrel{def.}{=} \limsup \frac{1}{R} \log |\{\Gamma(w) \cap B(w, R)\}| = \varepsilon \cdot \dim_H (\partial X, d_\varepsilon) .$$

Let ρ be the Hausdorff measure on ∂X of dimension $\alpha \stackrel{def.}{=} v/\varepsilon$;

(i) *ρ is Ahlfors-regular of dimension α i.e., for any $a \in \partial X$, for any $r \in (0, \mathrm{diam}\partial X)$, $\rho(B_\varepsilon(a, r)) \asymp r^\alpha$. In particular, $0 < \rho(\partial X) < \infty$.*

(ii) *ρ is a Γ-quasiconformal measure i.e., for any $\gamma \in \Gamma$, $\rho \ll \gamma^* \rho \ll \rho$ and*

$$\frac{d\gamma^* \rho}{d\rho} \asymp (L_\gamma)^\alpha \rho \ a.e.. $$

(iii) *The action of Γ is ergodic for ρ i.e., for any Γ-invariant Borelian B of ∂X,*

$$\rho(B) = 0 \ \text{ or } \ \rho(\partial X \backslash B) = 0 .$$

Moreover, if ρ' is another Γ-quasiconformal measure, then $\rho \ll \rho' \ll \rho$ and $\dfrac{d\rho}{d\rho'} \asymp 1$ a.e. and

$$|\{\Gamma(w) \cap B(w, R)\}| \asymp e^{vR} .$$

The class of measures thus defined on ∂X is called the *Patterson-Sullivan* class. It does not depend on the choice of the parameter ε but it does depend on the metric d.

The study of quasiconformal measures yields the following key estimate [13]:

Lemma 2.4. (Lemma of the shadow) *Under the assumptions of Theorem 2.3, there exists R_0, such that if $R > R_0$, then, for any $x \in X$,*

$$\rho(\mho(x, R)) \asymp e^{-vd(w, x)}$$

where the implicit constants do not depend on x.

3. Random walks and Green metric for hyperbolic groups

Let Γ be a hyperbolic group, and let us consider the set $\mathcal{D}(\Gamma)$ of left-invariant hyperbolic metrics on Γ which are quasi-isometric to Γ. We fix such a metric $(X, d) \in \mathcal{D}(\Gamma)$ with a base point $w \in X$, and we consider a symmetric probability measure μ on Γ with finite first moment i.e.

$$\sum_{\gamma \in \Gamma} \mu(\gamma) d(w, \gamma(w)) < \infty .$$

The random walk $(Z_n)_n$ starting from the neutral element e associated with μ is defined by the recursion relations:

$$Z_0 = e \, ; \, Z_{n+1} = Z_n \cdot X_{n+1} ,$$

where (X_n) is a sequence of independent and identically distributed random variables of law μ. Thus, for each n, Z_n is a random variable taking its values in Γ. We use the notation $Z_n(w)$ for the image of the base point $w \in X$ by Z_n. The *rate of escape*, or *drift* of the random walk $Z_n(w)$ is the number ℓ defined as

$$\ell \stackrel{def.}{=} \lim_n \frac{d(w, Z_n(w))}{n} ,$$

where the limit exists almost surely and in L^1 by the sub-additive ergodic Theorem (J. Kingman [28], Y. Derriennic [14]).

If Γ is elementary, then its boundary is either empty or finite. In either case, there is no interest in looking at properties at the boundary. We will assume from now on that Γ is non-elementary. In particular, Γ is non-amenable so not only is the random walk always transient, ℓ is also positive (cf. [25, §7.3]).

There are different ways to prove that almost any trajectory of the random walk has a limit point $Z_\infty(w) \in \partial X$. We recall below a theorem by V. Kaimanovich (cf. Theorem 7.3 in [25] and §7.4 therein) since it contains some information on the way $(Z_n(w))$ actually tends to $Z_\infty(w)$ that will be used later.

Theorem 3.1. (V. Kaimanovich). *Let Γ be a non-elementary hyperbolic group and $(X, d) \in \mathcal{D}(\Gamma)$, and let us consider a symmetric probability measure μ with finite first moment the support of which generates Γ. Then $(Z_n(w))$ almost surely converges to a point $Z_\infty(w)$ on the boundary.*

For any $a \in \partial X$, we choose a quasigeodesic $[w, a)$ from w to a in a measurable way.

For any n, there is a measurable map π_n from ∂X to X such that $\pi_n(a) \in [w, a)$, and, for almost any trajectory of the random walk,

$$(2) \qquad \lim_{n \to \infty} \frac{|Z_n(w) - \pi_n(Z_\infty(w))|}{n} = 0 \,.$$

The actual result was proved for geodesic metrics d. Once proved in a locally finite Cayley graph, one may then use a quasi-isometry to get the statement in this generality.

The estimate (2) will be improved in Corollary 3.9 under the condition that d_G belongs to $\mathcal{D}(\Gamma)$.

The *harmonic measure* ν is then the law of $Z_\infty(w)$ i.e., it is the probability measure on ∂X such that $\nu(A)$ is the probability that $Z_\infty(w)$ belongs to the set A. More generally, we let ν_γ be the harmonic measure for the random walk started at the point $\gamma(w)$, $\gamma \in \Gamma$ i.e. the law of $\gamma(Z_\infty(w))$. Comparing with the action of Γ on ∂X, we see that $\gamma^* \nu = \nu_{\gamma^{-1}}$.

3.1. **The Green metric.** Let Γ be a countable group and μ a symmetric law the support of which generates Γ.

For $x, y \in \Gamma$, we define $F(x, y)$ as the probability that a random walk starting from x hits y in finite time i.e., the probability there is some n such that $xZ_n = y$. S. Blachère and S. Brofferio [6] have defined the Green metric by

$$d_G(x, y) \stackrel{\text{def.}}{=} -\log F(x, y) \,.$$

The Markov property implies that F and the Green function G satisfy

$$G(x, y) = F(x, y)G(y, y) \,.$$

Since $G(y, y) = G(e, e)$, we then get that

$$F(x, y) = \frac{G(x, y)}{G(e, e)}$$

i.e. F and G only differ by a multiplicative contant and

$$d_G(x, y) = \log G(e, e) - \log G(x, y) \,.$$

This function d_G is known to be a left-invariant metric on Γ (see [6, 7] for details).

We end this short introduction to the Green metric with the following folklore property.

Lemma 3.2. *Let μ be a symmetric probability measure on Γ which defines a transient random walk. Then (Γ, d_G) is a proper metric space i.e., balls of finite radius are finite.*

PROOF. It is enough to prove that $G(e, x)$ tends to 0 as x leaves any finite set.

Fix $n \geq 1$; by definition of convolution and by the Cauchy-Schwarz inequality,

$$\mu^{2n}(x) = \sum_{y \in \Gamma} \mu^n(y)\mu^n(y^{-1}x) \leq \sqrt{\sum_{y \in \Gamma} \mu^n(y)^2} \sqrt{\sum_{y \in \Gamma} \mu^n(y^{-1}x)^2} \,.$$

Since we are summing over the same set, it follows that

$$\sum_{y \in \Gamma} \mu^n(y)^2 = \sum_{y \in \Gamma} \mu^n(y^{-1}x)^2$$

and the symmetry of μ implies that

$$\sum_{y \in \Gamma} \mu^n(y)^2 = \sum_{y \in \Gamma} \mu^n(y)\mu^n(y^{-1}) = \mu^{2n}(e) \,.$$

Therefore, $\mu^{2n}(x) \leq \mu^{2n}(e)$. Similarly,

$$\mu^{2n+1}(x) = \sum_{y \in \Gamma} \mu(y)\mu^{2n}(y^{-1}x) \leq \sum_{y \in \Gamma} \mu(y)\mu^{2n}(e) \leq \mu^{2n}(e) \,.$$

Since the walk is transient, it follows that $G(e, e)$ is finite, so, given $\varepsilon > 0$, there is some $k \geq 1$ such that

$$\sum_{n \geq k} \mu^{2n}(e) \leq \sum_{n \geq 2k} \mu^n(e) \leq \varepsilon \,.$$

On the other hand, since μ^n is a probability measure for all n, there is some finite subset K of Γ such that, for all $n \in \{0, \ldots, 2k-1\}$, $\mu^n(K) \geq 1 - \varepsilon/(2k)$. Therefore, if $x \notin K$, then

$$G(e, x) = \sum_{0 \leq n < 2k} \mu^n(x) + \sum_{n \geq 2k} \mu^n(x) \leq \sum_{0 \leq n < 2k} \mu^n(\Gamma \setminus K) + 2\sum_{n \geq k} \mu^{2n}(e) \leq \varepsilon + 2\varepsilon \,.$$

The lemma follows. ■

3.2. **The Martin boundary.** Let Γ be a countable group and μ be a symmetric probability measure on Γ. We assume that the support of μ generates Γ and that the corresponding random walk is transient.

A non-negative function h on Γ is μ-*harmonic* (harmonic for short) if, for all $x \in \Gamma$,

$$h(x) = \sum_{y \in \Gamma} h(y)\mu(x^{-1}y) \,.$$

A positive harmonic function h is *minimal* if any other positive harmonic function v smaller than h is proportional to h.

The *Martin kernel* is defined for all $(x, y) \in \Gamma \times \Gamma$ by

$$K(x, y) \stackrel{\text{def.}}{=} \frac{G(x, y)}{G(e, y)} = \frac{F(x, y)}{F(e, y)} \,.$$

We endow Γ with the discrete topology. Let us briefly recall the construction of the Martin boundary $\partial_M \Gamma$: let $\Psi : \Gamma \to C(\Gamma)$ be defined by $y \longmapsto K_y = K(\cdot, y)$. Here $C(\Gamma)$ is the space of real-valued functions defined on Γ endowed with the topology of pointwise convergence. It turns out that Ψ is injective and thus we may identify Γ with its image. The closure of $\Psi(\Gamma)$ is compact in $C(\Gamma)$ and, by definition, $\partial_M \Gamma = \overline{\Psi(\Gamma)} \setminus \Psi(\Gamma)$ is the Martin boundary. In the compact space $\Gamma \cup \partial_M \Gamma$, for any initial point x, the random walk $Z_n(x)$ almost surely converges to some random variable $Z_\infty(x) \in \partial_M \Gamma$ (see for instance E. Dynkin [15], A. Ancona [1] or W. Woess [44]).

To every point $\xi \in \partial_M \Gamma$ corresponds a positive harmonic function K_ξ. Every minimal function arises in this way: if h is minimal, then there are a constant $c > 0$ and $\xi \in \partial_M \Gamma$ such that $h = cK_\xi$. We denote by $\partial_m \Gamma$ the subset of $\partial_M \Gamma$ consisting of (normalised) minimal positive harmonic functions.

Choquet's integral representation implies that, for any positive harmonic function h, there is a unique probability measure κ^h on $\partial_m \Gamma$ such that

$$h = \int K_\xi d\kappa^h(\xi) \,.$$

We will also use L. Naïm's kernel Θ on $\Gamma \times \Gamma$ defined by

$$\Theta(x, y) \stackrel{\text{def.}}{=} \frac{G(x, y)}{G(e, x)G(e, y)} = \frac{K_y(x)}{G(e, x)} \,.$$

As the Martin kernel, Naïm's kernel admits a continuous extension to $\Gamma \times (\Gamma \cup \partial_M \Gamma)$. In terms of the Green metric, one gets

$$(3) \qquad\qquad \log \Theta(x, y) = 2(x|y)_e^G - \log G(e, e) \,,$$

where $(x|y)_e^G$ denotes the Gromov product with respect to the Green metric. See [38] for properties of this kernel.

We shall from now on assume that the Green metric d_G is hyperbolic. Then it has a visual boundary that we denote by $\partial_G \Gamma$. We may also compute the Busemann function in the metric d_G, say β_a^G. Sending y to some point $a \in \partial_G \Gamma$ in the equation $d_G(e, y) - d_G(x, y) = \log K(x, y)$, we get that $\beta_a^G(e, x) = \log K_a(x)$.

We now start preparing the proof of Theorem 1.7 in the next lemma and proposition. We define an equivalence relation $\sim_M$ on $\partial_M \Gamma$: say that $\xi \sim_M \zeta$ if there exists a constant $C \geq 1$ such that

$$\frac{1}{C} \leq \frac{K_\xi}{K_\zeta} \leq C \,.$$

Given $\xi \in \partial_M \Gamma$, we denote by $M(\xi)$ the class of ξ.

We first derive some properties of this equivalence relation:

Lemma 3.3.   (i) *There exists a constant $E \geq 1$ such that for all sequences (x_n) and (y_n) in Γ converging to ξ and ζ in $\partial_M \Gamma$ respectively and such that $\Theta(x_n, y_n)$ tends to infinity, then*

$$\frac{1}{E} \leq \frac{K_\xi}{K_\zeta} \leq E \,;$$

in particular, $\xi \sim_M \zeta$.

   (ii) *For any $\xi \in \partial_M \Gamma$, there is some $\zeta \in M(\xi)$ and a sequence (y_n) in Γ which tends to some point $a \in \partial_G \Gamma$ in the sense of Gromov, to $\zeta \in \partial_M \Gamma$ in the sense of Martin and such that $\Theta(y_n, \xi)$ tends to infinity.*

(iii) *Let $\xi, \zeta \in \partial_M \Gamma$. If $\zeta \notin M(\xi)$, then there is a neighborhood $V(\zeta)$ of ζ in Γ and a constant M such that*

$$K_\xi(x) \leq MG(e, x)$$

for any $x \in V(\zeta)$.

PROOF.

(i) Fix $z \in \Gamma$ and n large enough so that $(x_n|y_n)_e^G \gg d_G(e, z)$; we consider the approximate tree T associated with $F = \{e, z, x_n, y_n\}$ and the $(1, C)$-quasi-isometry $\varphi : (F, d_G) \to (T, d_T)$ (cf. Theorem B.1).

On the tree T, we have

$$|d_T(\varphi(e), \varphi(x_n)) - d_T(\varphi(z), \varphi(x_n))| = |d_T(\varphi(e), \varphi(y_n)) - d_T(\varphi(z), \varphi(y_n))|,$$

so that

$$|(d_G(e, x_n) - d_G(z, x_n)) - (d_G(e, y_n) - d_G(z, y_n))| \leq 2C.$$

In terms of the Martin kernel,

$$|\log K_{x_n}(z) - \log K_{y_n}(z)| \leq 2C.$$

Letting n go to infinity yields the result.

(ii) Let (y_n) be a sequence such that

$$\lim K_\xi(y_n) = \sup K_\xi.$$

Since K_ξ is harmonic, the maximum principle implies that (y_n) leaves any compact set. But the walk is symmetric and transient so Lemma 3.2 implies that $G(e, y_n)$ tends to 0.

Furthermore, for n large enough, $K_\xi(y_n) \geq K_\xi(e) = 1$, so that

$$\Theta(y_n, \xi) \geq \frac{1}{G(e, y_n)} \to \infty.$$

Let (x_n) be a sequence in Γ which tends to ξ. For any n, there is some m such that

$$|K_\xi(y_n) - K_{x_m}(y_n)| \leq G(e, y_n).$$

It follows that

$$\Theta(y_n, x_m) \geq \Theta(y_n, \xi) - \frac{|K_\xi(y_n) - K_{x_m}(y_n)|}{G(e, y_n)} \geq \Theta(y_n, \xi) - 1.$$

Therefore, applying part (i) of the lemma, we see that any limit point of (y_n) in $\partial_M \Gamma$ belongs to $M(\xi)$.

Moreover, for any such limit point $\zeta \in \partial_M \Gamma$, we get that

$$\Theta(y_n, \zeta) \geq \frac{1}{E} \Theta(y_n, \xi).$$

Applying the same argument as above, we see that, for any $M > 0$, there is some n and m_n such that, if $m \geq m_n$ then

$$\Theta(y_n, y_m) \geq M - 1.$$

From (3) we conclude, using a diagonal procedure, that there exist a subsequence (n_k) such that (y_{n_k}) tends to infinity in the Gromov topology.

(iii) Since $\zeta \notin M(\xi)$, there is a neighborhood $V(\zeta)$ and a constant M such that $\Theta(x, \xi) \leq M$ for all $x \in V(\zeta)$. Otherwise, we would find $y_n \to \zeta$ with $\Theta(y_n, \xi)$ going to infinity, and the argument above would imply $\zeta \in M(\xi)$. Therefore,

$$K_\xi(x) \leq MG(e, x).$$

■

Proposition 3.4. *Every Martin point is minimal.*

PROOF. We observe that if K_ξ is minimal, then $M(\xi) = \{\xi\}$. Indeed, if $\zeta \in M(\xi)$, then

$$K_\xi \geq K_\xi - \frac{1}{C}K_\zeta \geq 0$$

for some constant $C \geq 1$. The minimality of K_ξ implies that K_ξ and K_ζ are proportional and, since their value at e is 1, $K_\xi = K_\zeta$ i.e., $\xi = \zeta$.

Let $\xi \in \partial_M\Gamma$. There is unique probability measure κ^ξ on $\partial_m\Gamma$ such that

$$K_\xi = \int K_\zeta d\kappa^\xi(\zeta).$$

By Fatou-Doob-Naïm Theorem, for κ^ξ-almost every ζ, the ratio $G(e, x)/K_\xi(x)$ tends to 0 as x tends to ζ in the fine topology [1, Thm. II.1.8]. From Lemma 3.3 (iii), it follows that κ^ξ is supported by $M(\xi)$. In particular, $M(\xi)$ contains a minimal point. ■

PROOF OF THEOREM 1.7. Since every Martin point is minimal, Lemma 3.3, (ii), implies that for every $\xi \in \partial_M\Gamma$, there is some sequence (x_n) in Γ which tends to ξ in the Martin topology and to some point a in the hyperbolic boundary as well.

Let us prove that the point a does not depend on the sequence. If (y_n) is another sequence tending to ξ, then

$$\limsup_{n,m\to\infty} \Theta(x_n, y_m) = \infty$$

because $\Theta(\xi, x_n)$ tends to infinity. Therefore, there is a subsequence of (y_n) which tends to a in the Gromov topology. Since we have only one accumulation point, it follows that a is well-defined. This defines a map $\phi : \partial_M\Gamma \to \partial_G\Gamma$.

Now, if (x_n) tends to a in the Gromov topology, then it has only one accumulation point in the Martin boundary as well by Lemma 3.3, (i). So the map ϕ is injective. The surjectivity follows from the compactness of $\partial_M\Gamma$.

To conclude the proof, it is enough to prove the continuity of ϕ since $\partial_M\Gamma$ is compact. Let $M > 0$ and $\xi \in \partial_M\Gamma$ be given. We consider a sequence (x_n) which tends to ξ as in Lemma 3.3. Let C be the constant given by Theorem B.1 for 4 points. We pick n large enough so that $(x_n|\phi(\xi))_e^G \geq M + 2C + \log 2$. Let

$$A = \min\{K_\xi(x), x \in B_G(e, d_G(x_n, e))\}.$$

Let $\zeta \in \partial_M\Gamma$ such that $|K_\xi - K_\zeta| \leq (A/2)$ on $B_G(e, d_G(x_n, e))$. It follows that

$$1/2 \leq \frac{K_\zeta}{K_\xi} \leq 3/2.$$

Approximating $\{e, x_n, \phi(\xi), \phi(\zeta)\}$ by a tree, we conclude that $(\phi(\xi)|\phi(\zeta))_e^G \geq M$, proving the continuity of ϕ. ■

3.3. **Hyperbolicity of the Green metric.** We start with a characterisation of the hyperbolicity of the Green metric in the quasi-isometry class of the group.

Proposition 3.5. *Let Γ be a non-elementary hyperbolic group and μ a symmetric probability measure with Green function G. We fix a finite generating set S and consider the associated word metric d_w. The Green metric d_G is quasi-isometric to d_w and hyperbolic if and only if the following two conditions are satisfied.*

(ED) *There are positive constants C_1 and c_1 such that, for all $\gamma \in \Gamma$,*

$$G(e, \gamma) \leq C_1 e^{-c_1 d_w(e, \gamma)}$$

(QR) *For any $r \geq 0$, there exists a positive constant $C(r)$ such that*

$$G(e, \gamma) \leq C(r) G(e, \gamma') G(\gamma', \gamma)$$

whenever $\gamma, \gamma' \in \Gamma$ and γ' is at distance at most r from a d_w-geodesic segment between e and γ.

Remark. Even though hyperbolicity is an invariant property under quasi-isometries between geodesic metric spaces, this is not the case when we do not assume the spaces to be geodesic (see the appendix).

PROOF. We first assume that $d_G \in \mathcal{D}(\Gamma)$. The quasi-isometry property implies that condition (ED) holds. The second condition (QR) follows from Theorem A.1.

Indeed, since d_G is hyperbolic and quasi-isometric to a word distance, then (Γ, d_G) is quasiruled. This is sufficient to ensure that condition (QR) holds for $r = 0$. The general case $r \geq 0$ follows: let y be the closest point to γ' on a geodesic between e and γ and note that

$$d_G(e, \gamma') + d_G(\gamma', \gamma) \leq d_G(e, y) + d_G(y, \gamma) + 2d_G(y, \gamma') \leq \log C(0) + d_G(e, \gamma) + 2d_G(y, \gamma').$$

Thus one may choose $C(r) = C(0) \exp(2c)$ where $c = \sup d_G(y, \gamma')$ for all pair y, γ' at distance less than r. This last sup is finite because d_G is quasi-isometric to a word metric.

For the converse, we assume that both conditions (ED) and (QR) hold and let $C = \max\{d_G(e, s), \ s \in S\}$. For any $\gamma \in \Gamma$, we consider a d_w-geodesic $\{\gamma_j\}$ joining e to γ. It follows that

$$d_G(e, \gamma) \leq \sum_j d_G(\gamma_j, \gamma_{j+1}) \leq C d_w(e, \gamma).$$

From (ED), we obtain

$$d_G(e, \gamma) \geq c_1 d_w(e, \gamma) - \log C_1.$$

Since both metrics are left-invariant, it follows that d_w and d_G are quasi-isometric.

Condition (QR) implies that d_w-geodesics are not only quasigeodesics for d_G, but also quasirulers, cf. Appendix A. Indeed, since the two functions F and G only differ by a multiplicative factor, condition (QR) implies that there is a constant τ such that, for any d_w-geodesic segment $[\gamma_1, \gamma_2]$ and any $\gamma \in [\gamma_1, \gamma_2]$, we have

$$d_G(\gamma_1, \gamma) + d_G(\gamma, \gamma_2) \leq 2\tau + d_G(\gamma_1, \gamma_2).$$

Theorem A.1, (iii) implies (i), implies that (Γ, d_G) is a hyperbolic space. ∎

To prove the first statement of Theorem 1.1, it is now enough to establish the following lemma.

Lemma 3.6. *Let Γ be a non-elementary hyperbolic group, and μ a symmetric probability measure with finite exponential moment. Then condition (ED) holds.*

When μ is finitely supported, the lemma was proved by S. Blachère and S. Brofferio using the Carne-Varopoulos estimate [6].

PROOF. Let us fix a word metric d_w induced by a finite generating set S, so that $d_w \in \mathcal{D}(\Gamma)$.

Since Γ is non-amenable, Kesten's criterion implies that there are positive constants C and a such that

$$(4) \qquad \forall \gamma \in \Gamma, \quad \mu^n(\gamma) \leq \mu^n(e) \leq C e^{-an}.$$

For a proof, see [44, Cor. 12.5].

We assume that $\mathbb{E}[\exp \lambda d_w(e, Z_1)] = E < \infty$ for a given $\lambda > 0$. For any $b > 0$, it follows from the exponential Tchebychev inequality that

$$\mathbb{P}\left[\sup_{1 \leq k \leq n} d_w(e, Z_k) \geq nb\right] \leq e^{-\lambda bn} \mathbb{E}\left[\exp\left(\lambda \sup_{1 \leq k \leq n} d_w(e, Z_k)\right)\right].$$

But then, for $k \leq n$,

$$d_w(e, Z_k) \leq \sum_{1 \leq j \leq n-1} d_w(Z_j, Z_{j+1}) = \sum_{1 \leq j \leq n-1} d_w(e, Z_j^{-1} Z_{j+1}).$$

The increments $(Z_j^{-1} Z_{j+1})$ are independent random variables and all follow the same law as Z_1. Therefore

$$(5) \qquad \mathbb{P}\left[\sup_{1 \leq k \leq n} d_w(e, Z_k) \geq nb\right] \leq e^{-\lambda bn} E^n = e^{(-\lambda b + \log E)n}.$$

We choose b large enough so that $c \overset{\text{def.}}{=} -\lambda b + \log E < 0$.

We have

$$G(e, \gamma) = \sum_n \mu^n(\gamma) = \sum_{1 \leq k \leq |\gamma|/b} \mu^k(\gamma) + \sum_{k > |\gamma|/b} \mu^k(\gamma),$$

where we have set $|\gamma| = d_w(e, \gamma)$. The estimates (5) and (4) respectively imply that

$$
\begin{aligned}
\sum_{1 \leq k \leq |\gamma|/b} \mu^k(\gamma) &\leq \frac{|\gamma|}{b} \sup_{1 \leq k \leq |\gamma|/b} \mu^k(\gamma) \leq \frac{|\gamma|}{b} \mathbb{P}[\exists k \leq |\gamma|/b \text{ s.t. } Z_k = \gamma] \\
&\leq \frac{|\gamma|}{b} \mathbb{P}\left[\sup_{1 \leq k \leq |\gamma|/b} d_w(e, Z_k) \geq |\gamma|\right] \lesssim |\gamma| e^{-c|\gamma|}
\end{aligned}
$$

and

$$\sum_{k > |\gamma|/b} \mu^k(\gamma) \lesssim e^{-(a/b)|\gamma|}.$$

Therefore, (ED) holds.   ∎

When Γ is hyperbolic and μ has finite support, A. Ancona [1] proved that the Martin boundary is homeomorphic to the visual boundary ∂X. The key point in his proof is the following estimate (see [44, Thm. 27.12] and Theorem 1.7).

Theorem 3.7. (A. Ancona) *Let Γ be a non-elementary hyperbolic group, X a locally finite Cayley graph endowed with a geodesic metric d so that Γ acts canonically by isometries, and let μ be a finitely supported symmetric probability measure the support of which generates Γ. For any $r \geq 0$, there is a constant $C(r) \geq 1$ such that*

$$F(x,v)F(v,y) \leq F(x,y) \leq C(r)F(x,v)F(v,y)$$

whenever $x, y \in X$ and v is at distance at most r from a geodesic segment between x and y.

This implies together with Lemma 3.6 that when μ is finitely supported, both conditions (ED) and (QR) hold. Therefore, Proposition 3.5 implies that $d_G \in \mathcal{D}(\Gamma)$. We have just established the first statement of Corollary 1.2.

3.4. **Martin kernel vs Busemann function: end of the proof of Theorem 1.1.** We assume that $X = \Gamma$ equipped with the Green metric d_G belongs to $\mathcal{D}(\Gamma)$ throughout this paragraph.

Notation. When we consider notions with respect to d_G, we will add the exponent G to distinguish them from the same notions in the initial metric d. Thus Busemann functions for d_G will be written β_a^G. The visual metric on ∂X seen from w for the original metric will be denote by d_ε, and by d_ε^G for the one coming from d_G. Balls at infinity will be denoted by B_ε and B_ε^G.

Let us recall that the Martin kernel is defined by

$$K(x,y) = \frac{F(x,y)}{F(w,y)} = \exp\left\{d_G(w,y) - d_G(x,y)\right\}.$$

By definition of the Martin boundary $\partial_M X$, the kernel $K(x,y)$ continuously extends to a μ-harmonic positive function $K_a(\cdot)$ when y tends to a point $a \in \partial_M X$. We recall that, by Theorem 1.7, we may - and will - identify $\partial_M X$ with the visual boundary ∂X.

As we already mentioned Γ acts on $\partial_M X$, so on its harmonic measure and we have $\gamma^* \nu = \nu_{\gamma^{-1}}$. Besides, see e.g. G. Hunt [19] or W. Woess [44, Th. 24.10] for what follows, ν and ν_γ are absolutely continuous and their Radon-Nikodym derivatives satisfy

$$\frac{d\nu_\gamma}{d\nu}(a) = K_a(\gamma(w)).$$

We already computed the Busemann function in the metric d_G in part 3.2: $\beta_a^G(w,x) = \log K_a(x)$. Thus we have proved that

$$\frac{d\gamma^* \nu}{d\nu}(a) = \exp \beta_a^G(w, \gamma^{-1}w).$$

It follows at once that ν is a quasiconformal measure on $(\partial X, d_\varepsilon^G)$ of dimension $1/\varepsilon$. Actually, ν is even a *conformal measure* since we have a genuine equality above. Therefore ν belongs to the Patterson-Sullivan class associated with the metric d_G. According to Theorem 2.3, it is in particular comparable to the Hausdorff measure for the corresponding visual metric. This ends both the proofs of Theorem 1.1 and of Corollary 1.2.

We note that, comparing the statements in Theorem 1.1 (ii) and Theorem 2.3, we recover the equality $v_G = 1$ already noticed in [6] for random walks on non-amenable groups. See also [7].

3.5. **Consequences.** We now draw consequences of the hyperbolicity of the Green metric.

We refer to the appendices for properties of quasiruled spaces.

3.5.1. *Deviation inequalities.* We study the lateral deviation of the position of the random walk with respect to the quasiruler $[w, Z_\infty(w))$ where, for any $x \in X$ and $a \in \partial X$, we chose an arbitrary quasiruler $[x, a)$ from x to a in a measurable way.

Proposition 3.8. *Assume that Γ is a non-elementary hyperbolic group, $(X, d) \in \mathcal{D}(\Gamma)$, and μ is a symmetric law so that the associated Green metric belongs to $\mathcal{D}(\Gamma)$. The following holds*

(i) *There is a positive constant b so that, for any $D \geq 0$ and $n \geq 0$,*
$$\mathbb{P}[d(Z_n(w), [w, Z_\infty(w))) \geq D] \lesssim e^{-bD}.$$

(ii) *There is a constant τ_0 such that for any positive integers m, n, k,*
$$\mathbb{E}[(Z_m(w)|Z_{m+n+k}(w))_{Z_{m+n}(w)}] \leq \tau_0.$$

PROOF.

Proof of (i). Observe that

$$\begin{aligned}
\mathbb{P}[d(Z_n(w), [w, Z_\infty(w))) \geq D] &= \sum_{z \in X} \mathbb{P}[d(Z_n(w), [w, Z_\infty(w))) \geq D, \, Z_n(w) = z] \\
&= \sum_{z \in X} \mathbb{P}[d(z, [w, Z_n^{-1}Z_\infty(z))) \geq D, \, Z_n(w) = z] \\
&= \sum_{z \in X} \mathbb{P}[d(z, [w, Z_n^{-1}Z_\infty(z))) \geq D]\mathbb{P}[Z_n(w) = z] \\
&= \sum_{z \in X} \mathbb{P}[d(z, [w, Z_\infty(z))) \geq D]\mathbb{P}[Z_n(w) = z]
\end{aligned}$$

The second equality holds because $\gamma w = z$ implies that $\gamma^{-1}Z_\infty(z) = Z_\infty(w)$. The third equality comes from the independence of $Z_n = X_1 X_2 \cdots X_n$ and $Z_n^{-1}Z_\infty = X_{n+1}X_{n+2}\cdots$. The last equality uses the fact that $Z_n^{-1}Z_\infty$ and Z_∞ have the same law.

On the event $\{d(z, [w, Z_\infty(z))) \geq D\}$, we have in particular $d(w, z) \geq D$ and we can pick $x \in [w, z]$ such that $d(z, x) = D + O(1)$. Then, because the triangle $(w, z, Z_\infty(z))$ is thin and since $d(z, [w, Z_\infty(z))) \geq D$, we must have $Z_\infty(z) \in \mho_z(x, R)$. As usual R is a constant that does not depend on z, D or $Z_\infty(z)$. We now apply the lemma of the shadow Lemma 2.4 to the Green metric to deduce that

$$\mathbb{P}[d(z, [w, Z_\infty(z))) \geq D] \leq \mathbb{P}^z[Z_\infty(z) \in \mho_z(x, R)] = \nu_z(\mho_z(x, R)) \lesssim e^{-d_G(z,x)}.$$

Finally, using the quasi-isometry between d and d_G, it follows that

$$\mathbb{P}[d(Z_n(w), [w, Z_\infty(w))) \geq D] \lesssim e^{-bD}.$$

Proof of (ii). Using the independence of the increments of the walk, one may first assume that $m = 0$.

Let us choose $Y_n(w) \in [w, Z_\infty(w))$ such that $d(w, Y_n(w))$ is as close from $(Z_n(w)|Z_\infty(w))$ as possible. Since the space (X, d) is quasiruled, it follows that $d(w, Y_n(w)) = (Z_n(w)|Z_\infty(w)) + O(1)$.

(We only use Landau's notation $O(1)$ for estimates that are uniform with respect to the trajectory of (Z_n). Thus the line just above means that there exists a <u>deterministic</u> constant C such that

$$|d(w, Y_n(w)) - (Z_n(w)|Z_\infty(w))| \leq C.$$

The same convention applies to the rest of the proof.)

Let us define

$$A_0 = \{d(w, Y_n(w)) \leq d(w, Y_{n+k}(w))\}$$

and, for $j \geq 1$,

$$A_j = \{j - 1 < d(w, Y_n(w)) - d(w, Y_{n+k}(w)) \leq j\}.$$

Approximating $\{w, Z_n(w), Z_{n+k}(w), Z_\infty(w)\}$ by a tree, it follows that, on the event A_0,

$$(w|Z_{n+k}(w))_{Z_n(w)} \leq d(Z_n(w), [w, Z_\infty(w))) + O(1)$$

and that, on the event A_j,

$$(w|Z_{n+k}(w))_{Z_n(w)} \leq d(Z_n(w), [w, Z_\infty(w))) + j + O(1).$$

Therefore

$$\mathbb{E}[(w|Z_{n+k}(w))_{Z_n(w)}] \leq \mathbb{E}[d(Z_n(w), [w, Z_\infty(w)))] + \sum_{j \geq 1} j\mathbb{P}(A_j) + O(1).$$

If $d(w, Y_n(w)) - d(w, Y_{n+k}(w)) \geq j$ then $d(Z_{n+k}(w), [Z_n(w), Z_\infty(w))) \geq j$ so that

$$\mathbb{P}(A_{j+1}) \leq \mathbb{P}[d(Z_{n+k}(w), [Z_n(w), Z_\infty(w))) \geq j].$$

Using (i) for the random walk starting at $Z_n(w)$, we get

$$\sum_{j \geq 1} j\mathbb{P}(A_j) \lesssim 1.$$

On the other hand,

$$\mathbb{E}[d(Z_n(w), [w, Z_\infty(w)))] = \int_0^\infty \mathbb{P}[d(Z_n(w), [w, Z_\infty(w))) \geq D]\, dD \lesssim \int_0^\infty e^{-bD}\, dD = 1/b.$$

The proposition follows. ∎

We now improve the estimate (2) in Theorem 3.1 when $d_G \in \mathcal{D}(\Gamma)$.

Corollary 3.9. *Let Γ be a non-elementary hyperbolic group, $(X, d) \in \mathcal{D}(\Gamma)$ and μ a symmetric law such that $d_G \in \mathcal{D}(\Gamma)$, then we have*

(6)
$$\limsup \frac{d(Z_n(w), [w, Z_\infty(w)))}{\log n} < \infty \quad \mathbb{P} \ a.s.$$

PROOF. It follows from Proposition 3.8 that we may find a constant $\kappa > 0$ so that

$$\mathbb{P}[d(Z_n(w), [w, Z_\infty(w))) \geq \kappa \log n] \leq \frac{1}{n^2}.$$

Therefore, the Borel-Cantelli lemma implies that

$$\limsup \frac{d(Z_n(w), [w, Z_\infty(w)))}{\log n} < \infty \quad \mathbb{P} \ a.s.$$

and the corollary follows. ∎

3.5.2. *Escape of the random walk from balls.* We assume here that μ is a symmetric and finitely supported probability measure on a non-elementary hyperbolic group Γ and that the support of μ generates Γ. We want to compare the harmonic measure with the uniform measure on the spheres for the Green metric. We define the (exterior) sphere of the ball $B_G(w, R)$ by

$$\partial B_G(w, R) \overset{\text{def.}}{=} \{x \in X \ : \ x \notin B_G(w, R) \text{ and } \exists \gamma \in Supp(\mu) \text{ s.t. } \gamma^{-1}(x) \in B_G(w, R)\}.$$

The harmonic measure ν_R on $\partial B_G(w, R)$ is the law of the first point visited outside $B_G(w, R)$.

As the volume of the sphere $\partial B_G(w, R)$ equals e^R up to a multiplicative constant (see [6]), we need to compare $\nu_R(\cdot)$ with e^{-R}. In other words, we have to bound the ratio between the measure $\nu_R(\cdot)$ and the hitting probability $F(w, \cdot)$. Observe that, in principle, there could be points on the sphere that are visited by the walk a long time after it left the ball. We shall see that this scenario can only take place on a finite scale.

In the following we only consider quasigeodesics for (X, d) and (X, d_G) that are geodesics for a given word metric $d_w \in \mathcal{D}(\Gamma)$.

Proposition 3.10. *There exist positive constants $C_1 < 1$ and C_2 such that for any positive real R, the harmonic measure ν_R on the sphere $\partial B_G(w, R)$ satisfies*

$$\forall x \in \partial B_G(w, R), \ \exists y \in B_G(x, C_1) \cap \partial B_G(w, R) \ \ s.t. \ C_2 e^{-R} \leq \nu_R(y) \leq e^{-R}.$$

PROOF. The upper bound (valid for any $x \in \partial B_G(w, R)$) obviously follows from the definition of the Green metric: if $y \notin B_G(w, R)$, then

$$\nu_R(y) \leq F(w, y) = \exp(-d_G(w, y)) \leq e^{-R}.$$

For the lower bound, we consider a quasigeodesic from w to x and denote by y the first point of $\partial B_G(w, R)$ along that path. Since μ has finite support, $d_G(w, x)$ and $d_G(w, y)$ only differ by an additive constant. The quasiruler property then implies that y is at a bounded distance from x.

Let $\mathcal{E} = \mathcal{E}(R)$ denote the set of points $z \in \partial B_G(w, R)$ such that there is a quasigeodesic reaching z from w entirely contained in $B_G(w, R)$ (except for the last step toward z).

Let $z \in \mathcal{E}$. Since y and z belong to $\partial B_G(w, R)$, then $d_G(w, z)$ and $d_G(w, y)$ only differ by an additive constant and we have

$$(7) \qquad d_G(y, z) \geq d_G(y, z) + (d_G(w, y) - d_G(w, z) - C) = 2(w|z)_y - C$$

Let k_0 be an integer and define

$$\mathcal{E}_0 \overset{\text{def.}}{=} \{z \in \mathcal{E} \ : \ (w|z)_y \leq k_0\}$$

and for all integer $k \geq k_0$,

$$\mathcal{E}_k \overset{\text{def.}}{=} \{z \in \mathcal{E} \ : \ k < (w|z)_y \leq k + 1\}.$$

We denote by τ_R the first hitting time of $\partial B_G(w, R)$ by the random walk and by τ_y the first hitting time of y. Then

$$F(w, y) = \mathbb{P}[\tau_y < \infty, Z_{\tau_R}(w) \in \mathcal{E}_0] + \sum_{k=k_0}^{\infty} \sum_{z \in \mathcal{E}_k} \mathbb{P}[\tau_y < \infty, Z_{\tau_R}(w) = z]$$

At this point, we need to use the Strong Markov property to say that once we know that $Z_{\tau_R}(w) = z$ and $z \neq y$, the hitting time of y must occur after τ_R. Then, the finiteness of τ_y

depends only on the position z disregarding the behavior of the random walk up to time τ_R. Namely,

$$\mathbb{P}[\tau_y < \infty, \, Z_{\tau_R}(w) = z] = \mathbb{P}^z[\tau_y < \infty]\mathbb{P}[Z_{\tau_R}(w) = z]\,.$$

Using (7), the definition of $(\mathcal{E}_k)$ and the inequality $\mathbb{P}[Z_{\tau_R}(w) = z] \leq \mathbb{P}[\tau_z < \infty] \leq e^{-R}$, we get that

$$(8) \qquad\qquad F(w,y) \leq \mathbb{P}[Z_{\tau_R}(w) \in \mathcal{E}_0] + C \sum_{k=k_0}^{\infty} e^{-2k}e^{-R}\#\mathcal{E}_k\,.$$

We need an upper bound on $\#\mathcal{E}_k$. Take $z \in \mathcal{E}_k$, and let y_{R-k} be the point at distance $R - k$ from w along the quasigeodesic $[w, y]$.

As the triangle (w, z, y) is thin, the center of the associated approximate tree is at a bounded distance from the point y_{R-k}. Then, since for any z in $\mathcal{E}_k$, $(w|y)_z - k$ is bounded by a constant, the set $\mathcal{E}_k$ is therefore included in the ball $B_G(y_{R-k}, k+C)$ for some constant C. Thus $\#\mathcal{E}_k \lesssim e^k$ and

$$(9) \qquad\qquad C \sum_{k=k_0}^{\infty} e^{-2k}e^{-R}\#\mathcal{E}_k \leq C(k_0)e^{-R}$$

with $C(k_0)$ tending to 0 when k_0 tends to infinity.

As μ is finitely supported, $\partial B_G(w, R)$ is at a bounded distance from $B_G(w, R)$. So $y \in B_G(w, R + C(\mu))$ and $F(w, y) \geq e^{-C(\mu)}e^{-R}$. Now choose k_0 so that $C(k_0) < (1/2)e^{-C(\mu)}$ and take $R > k_0$. Then (8) and (9) give

$$(10) \qquad\qquad \mathbb{P}[Z_{\tau_R}(w) \in \mathcal{E}_0] \geq \frac{1}{2}e^{-C(\mu)}e^{-R}\,.$$

We conclude that $\nu_R(\mathcal{E}_0) \gtrsim e^{-R}$. Take $y' \in \mathcal{E}_0$ so that $(w|y')_y \leq k_0$. By the definition of the set $\mathcal{E}$ and by the thinness of the triangle (w, y, y'), there exists a path joining y and y' within $B_G(w, R)$ of length at most $c(k_0)$, a constant depending only on k_0 and δ. Therefore, there exists a constant $c'(k_0, \mu)$ such that

$$\nu_R(y) \geq \nu_R(y')c'(k_0, \mu)\,.$$

Finally, as $\#\mathcal{E}_0$ is bounded above by a constant, (10) gives

$$\nu_R(y) \gtrsim \sum_{y' \in \mathcal{E}_0} \nu_R(y') = \nu_R(\mathcal{E}_0) \gtrsim \varepsilon^{-R}\,.$$

■

Remark. Proposition 3.10 says that the harmonic measure on spheres is well spread out and that the harmonic measure of a bounded domain of the sphere of radius R if e^{-R} up to a multiplicative constant. Approximating the balls of ∂X by shadows, we get that ν is Alfhors-regular of dimension $1/\varepsilon$, hence quasiconformal. Therefore, we get an alternative proof of the second statement of Theorem 1.1 when μ has finite support.

3.5.3. *The doubling condition for the harmonic measure.* Let us recall that a measure m is said to be *doubling* if there exists a constant $C > 0$ such that, for any ball B of radius at most the diameter of the space then $m(2B) \leq Cm(B)$.

Proposition 3.11. *Let Γ be a non-elementary hyperbolic group, $(X, d) \in \mathcal{D}(\Gamma)$ and let μ be a symmetric law such that $d_G \in \mathcal{D}(\Gamma)$. The harmonic measure is doubling with respect to the visual measure d_ε on ∂X.*

PROOF. The modern formulation of Efremovich and Tichonirova's theorem (cf. Theorem 6.5 in [9] and references therein) states that quasi-isometries between hyperbolic proper geodesic spaces $\Phi : X \to Y$ extend as quasisymmetric maps $\phi : \partial X \to \partial Y$ between their visual boundaries i.e., there is an increasing homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|\phi(a) - \phi(b)| \leq \eta(t)|\phi(a) - \phi(c)|$$

whenever $|a - b| \leq t|a - c|$.

Since $d_G \in \mathcal{D}(\Gamma)$, the spaces involved are visual. Thus, the statement remains true since we may still approximate properly the space by trees, cf. Appendix B.

Since (X, d) and (X, d_G) are quasi-isometric, the boundaries are thus quasisymmetric with respect to d_ε and d_ε^G. Furthermore, ν is doubling with respect to d_ε^G since it is Ahlfors-regular, and this property is preserved under quasisymmetry. ∎

Basic properties on quasisymmetric maps include [18]. More information on boundaries of hyperbolic groups, and the relationships between hyperbolic geometry and conformal geometry can be found in [10, 26].

4. DIMENSION OF THE HARMONIC MEASURE ON THE BOUNDARY OF A HYPERBOLIC METRIC SPACE

Theorem 1.3 will follow from Proposition 4.1 and Proposition 4.2.

We recall the definition of the rates of escape ℓ and ℓ_G of the random walk with respect to d or d_G respectively.

$$\ell \overset{\text{def.}}{=} \lim_n \frac{d(w, Z_n(w))}{n} \text{ and } \ell_G \overset{\text{def.}}{=} \lim_n \frac{d_G(w, Z_n(w))}{n} \,.$$

We will first prove

Proposition 4.1. *Let Γ be a non-elementary hyperbolic group and let $(X, d) \in \mathcal{D}(\Gamma)$. Let μ be a symmetric probability measure on Γ the support of which generates Γ such that $d_G \in \mathcal{D}(\Gamma)$ and with finite first moment*

$$\sum_{\gamma \in \Gamma} d_G(w, \gamma(w))\mu(\gamma) < \infty \,.$$

Let ν be the harmonic measure seen from w on ∂X.

For ν-a.e. $a \in \partial X$,

$$\lim_{r \to 0} \frac{\log \nu(B_\varepsilon(a, r))}{\log r} = \frac{\ell_G}{\varepsilon \ell} \,,$$

where B_ε denotes the ball on ∂X for the visual metric d_ε.

Remark. Recall from [7] that μ having finite first moment with respect to the Green metric is a consequence of μ having finite entropy.

PROOF. It is convenient to introduce an auxiliary word metric d_w which is of course geodesic. We may then consider the visual quasiruling structure $\mathcal{G}$ induced by the d_w-geodesics for both metrics d and d_G via the identity map, cf. the appendix.

We combine Propositions 2.1 and B.5 to get that, for a fixed but large enough R, for any $a \in \partial X$ and $x \in [w, a) \subset \mathcal{G}$

$$B_\varepsilon(a, (1/C)e^{-\varepsilon d(w,x)}) \subset \mho_{\mathcal{G}}(x, R) \subset B_\varepsilon(a, Ce^{-\varepsilon d(w,x)})$$

and
$$B_\varepsilon^G(a, (1/C)e^{-\varepsilon d_G(w,x)}) \subset \mho_\mathcal{G}(x, R) \subset B_\varepsilon^G(a, Ce^{-\varepsilon d_G(w,x)})$$
for some positive constant C. We recall that the shadows $\mho_\mathcal{G}(x, R)$ are defined using geodesics for the word metric d_w.

The doubling property of ν with respect to the visual metric d_ε implies that

(11)
$$\nu(B_\varepsilon(a, Ce^{-\varepsilon d(w,x)})) \asymp \nu(\mho_\mathcal{G}(x, R))$$

for any $x \in [w, a)$.

Let $\eta > 0$; by definition of the drift, there is a set of full measure with respect to the law of the trajectories of the random walk, in which for any sequence $(Z_n(w))$ and for n large enough, we have $|d(w, Z_n(w)) - \ell n| \leq \eta n$ and $|d_G(w, Z_n(w)) - \ell_G n| \leq \eta n$.

From Theorem 3.1 applied to the metrics d and d_G, we get that, for n large enough, $d(Z_n(w), \pi_n(Z_\infty(w))) \leq \eta n$ and $d_G(Z_n(w), \pi_n(Z_\infty(w))) \leq \eta n$.

We conclude that

(12)
$$\begin{cases} |d(w, \pi_n(Z_\infty(w))) - \ell n| \leq 2\eta n \\ |d_G(w, \pi_n(Z_\infty(w))) - \ell_G n| \leq 2\eta n \end{cases}$$

Set
$$r_n = e^{-\varepsilon d(w, \pi_n(Z_\infty(w)))}.$$
Therefore, using (11) with $a = Z_\infty(w)$ and $x = \pi_n(Z_\infty(w))$, we get
$$\nu(B_\varepsilon(Z_\infty(w), r_n)) \asymp \nu(\mho_\mathcal{G}(\pi_n(Z_\infty(w)), R)) \asymp e^{-d_G(w, \pi_n(Z_\infty(w)))}$$
where the right-hand part comes from the fact that ν is a quasiconformal measure of dimension $1/\varepsilon$ for the Green visual metric and the lemma of the shadow (Lemma 2.4). Hence we deduce from (12) that, if n is large enough, then

(13)
$$\left| \frac{\log \nu(B_\varepsilon(Z_\infty(w), r_n))}{\log r_n} - \frac{\ell_G}{\varepsilon \ell} \right| \lesssim \eta.$$

Since the measure ν is doubling (Proposition 3.11), ν is also α-homogeneous for some $\alpha > 0$, (cf. [18, Chap. 13]) i.e., there is a constant $C > 0$ such that, if $0 < r < R < \text{diam}\partial X$ and $a \in \partial X$, then
$$\frac{\nu(B_\varepsilon(a, R))}{\nu(B_\varepsilon(a, r))} \leq C \left(\frac{R}{r} \right)^\alpha.$$
From
$$\left| \log \frac{e^{-\varepsilon n \ell}}{r_n} \right| \leq 2n\varepsilon\eta$$
it follows that
$$\left| \log \frac{\nu(B_\varepsilon(Z_\infty(w), e^{-\varepsilon n \ell}))}{\nu(B_\varepsilon(Z_\infty(w), r_n))} \right| \leq 2n\alpha\varepsilon\eta + O(1).$$
Therefore
$$\limsup_n \left| \frac{\log \nu(B_\varepsilon(Z_\infty(w), e^{-\varepsilon n \ell}))}{\log e^{-\varepsilon n \ell}} - \frac{\log \nu(B_\varepsilon(Z_\infty(w), r_n))}{\log r_n} \right| \lesssim \eta.$$

Since $\eta > 0$ is arbitrary, it follows from (13) that
$$\lim_{r \to 0} \frac{\log \nu(B_\varepsilon(Z_\infty(w), r))}{\log r} = \lim_{n \to \infty} \frac{\log \nu(B_\varepsilon(Z_\infty(w), e^{-\varepsilon n \ell}))}{\log e^{-\varepsilon n \ell}} = \lim_{n \to \infty} \frac{\log \nu(B_\varepsilon(Z_\infty(w), r_n))}{\log r_n} = \frac{\ell_G}{\varepsilon \ell}.$$

In other words, for ν almost every $a \in \partial X$,

$$\lim_{r \to 0} \frac{\log \nu(B_\varepsilon(a, r))}{\log r} = \frac{\ell_G}{\varepsilon \ell} \, .$$

■

It remains to prove that ν has dimension $\ell_G / \varepsilon \ell$. This is standard.

Hausdorff measures. Let $s, t \geq 0$, we set

$$\mathcal{H}_s^t(X) \stackrel{\text{def.}}{=} \inf \left\{ \sum r_i^s, \ B_i = B(x_i, r_i), \ X \subset (\cup B_i), r_i \leq t \right\} ,$$

where we consider covers by balls.

The s-dimensional measure is then

$$\mathcal{H}_s(X) \stackrel{\text{def.}}{=} \lim_{t \to 0} \mathcal{H}_s^t(X) = \sup_{t > 0} \mathcal{H}_s^t(X) \, .$$

The *Hausdorff dimension* $\dim_H X$ of X is the number $s \in [0, \infty]$ such that, for $s' < s$, $\mathcal{H}_{s'}(X) = \infty$ holds and for all $s' > s$, $\mathcal{H}_{s'}(X) = 0$.

The *Hausdorff dimension* $\dim \nu$ of a measure ν is the infimum of the Hausdorff dimensions over all sets of full measure.

Replacing covers by balls by covers by any kind of sets in the definition of $\mathcal{H}_s^t(X)$ and replacing radii by diameters would not change the value of $\dim \nu$.

For more properties, one can consult [37].

Proposition 4.2. *Let X be a proper metric space and ν a Borel regular probability measure on X. If, for ν-almost every $x \in X$,*

$$\lim_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} = \alpha$$

then $\dim \nu = \alpha$.

We recall the proof for the convenience of the reader. We will use the following covering lemma.

Lemma 4.3. *Let X be a proper metric space and $\mathcal{B}$ a family of balls in X with uniformly bounded radii. Then there is a subfamily $\mathcal{B}' \subset \mathcal{B}$ of pairwise disjoint balls such that*

$$\cup_\mathcal{B} B \subset \cup_{\mathcal{B}'} (5B) \, .$$

For a proof of the lemma, see Theorem 2.1 in [37].

PROOF OF PROP. 4.2. Let $s > \alpha$, and choose $\eta > 0$ small enough so that $\beta := s - \alpha - \eta > 0$. For ν-almost every x, a radius $r_x > 0$ exists so that

$$\left| \frac{\log \nu(B(x, r))}{\log r} - \alpha \right| \leq \eta \, ,$$

for $r \in (0, r_x]$.

Let us denote by $Y = \{ x \in X \ : \ r_x < \infty \}$, which is of full measure. Let us fix $t \in (0, 1)$. For any $x \in Y$, we choose $\rho_x = \min\{r_x, t\}$. We apply Lemma 4.3 to $\{B(x, \rho_x)\}$ and obtain a

subfamily $\mathcal{B}_t$. It follows that Y is covered by $5\mathcal{B}_t$ and

$$
\mathcal{H}_s^{5t}(Y) \ \leq \sum_{\mathcal{B}_t}(5\rho_x)^s \qquad\qquad \leq 5^s t^\beta \sum_{\mathcal{B}_t}\rho_x^{\alpha+\eta}
$$

$$
\lesssim t^\beta \sum_{\mathcal{B}_t}\nu(B(x,\rho_x)) \ \lesssim t^\beta \nu\left(\cup_{\mathcal{B}_t}B(x,\rho_x)\right)
$$

$$
\lesssim t^\beta
$$

which tends to 0 with t. Therefore $\mathcal{H}_s(Y) = 0$ and so $\dim_H Y \leq s$ for all $s > \alpha$. Whence $\dim \nu \leq \alpha$.

Conversely, let Y be a set of full measure. There is a subset $Z \subset Y$ such that $\nu(Z) \geq 1/2$ and such that the convergence of $\log \nu(B(x,r))/\log r$ to α is uniform on Z (Egorov theorem). Fix $s < \alpha$ and let us consider $\eta > 0$ small enough so that $\gamma = \alpha - \eta - s > 0$. There exists $0 < r_0 \leq 5$ such that, for any $r \in (0, r_0)$ and any $x \in Z$,

$$
\left|\frac{\log \nu(B(x,r))}{\log r} - \alpha\right| \leq \eta \, .
$$

Let $\mathcal{B}$ be a cover of Z by balls of radius ρ_x smaller than $t \leq r_0/5$. Pick a subfamily $\mathcal{B}_t = \{B(x, \rho_x)\}$ using Lemma 4.3. Then $5\mathcal{B}_t$ covers Z and

$$
1/2 \leq \sum_{\mathcal{B}_t}\nu(5B) \leq 5^{\alpha-\eta}\sum_{\mathcal{B}_t}\rho_x^{\alpha-\eta} \lesssim t^\gamma \sum_{\mathcal{B}_t}\rho_x^s \, .
$$

This proves that $\mathcal{H}_s^t(Z) \gtrsim t^{-\gamma}$ so that $\dim_H Y \geq \dim_H Z \geq \alpha$. ∎

5. Harmonic measure of maximal dimension

This section is devoted to the proof of Theorem 1.5 and its corollary.

5.1. **The fundamental equality.** We assume that $d \in \mathcal{D}(\Gamma)$, μ is a probability measure with exponential moment such that $d_G \in \mathcal{D}(\Gamma)$. Thus there exists $\lambda > 0$ such that

$$
E \overset{\text{def.}}{=} \mathbb{E}\left[e^{\lambda d(w, Z_1(w))}\right] < \infty \, .
$$

The main issue in the proof of Theorem 1.5 is the following implication which we prove first:

Proposition 5.1. *Under the hypotheses of Theorem 1.5, if $h = \ell v$, then ρ and ν are equivalent.*

Let R be the constant coming from the lemma of the shadow (Lemma 2.4) and write $\mho(x)$ for $\mho(x, R)$.

Let us now define

$$
\varphi_n = \frac{\rho(\mho(Z_n(w)))}{\nu(\mho(Z_n(w)))} \quad \text{and} \quad \phi_n = \log \varphi_n \, .
$$

Since μ^n is the law of Z_n, observe that, if $\beta \in (0, 1]$, then

$$
\mathbb{E}[\varphi_n^\beta] = \sum_{\gamma \in \Gamma}\mu^n(\gamma)\left(\frac{\rho(\mho(\gamma(w)))}{\nu(\mho(\gamma(w)))}\right)^\beta \text{ and } \mathbb{E}[\phi_n] = \sum_{\gamma \in \Gamma}\mu^n(\gamma)\log\left(\frac{\rho(\mho(\gamma(w)))}{\nu(\mho(\gamma(w)))}\right) \, .
$$

We start with two lemmata.

Lemma 5.2. *There are finite constants $C_1 \geq 1$ and $\beta \in (0,1]$ such that, for all $N \geq 1$,*

$$\frac{1}{N} \sum_{1 \leq n \leq N} \mathbb{E}[\varphi_n^\beta] \leq C_1.$$

When μ is finitely supported, one can choose $\beta = 1$ in the lemma.

PROOF. Let $N \geq 1$ and $1 \leq n \leq N$ be chosen. We will first prove that there are some κ and β independent from N and n such that

$$(14) \qquad R_\kappa \overset{\text{def.}}{=} \sum_{\gamma,\ d(w,\gamma(w)) \geq \kappa N} \left(\frac{\rho(\mho(\gamma(w)))}{\nu(\mho(\gamma(w)))} \right)^\beta \mu^n(\gamma) \lesssim 1.$$

We have already seen that the logarithmic volume growth rate for the Green metric is 1. Then, from the lemma of the shadow (Lemma 2.4) applied to both metrics, we get

$$(15) \qquad \nu(\mho(\gamma(w))) \asymp e^{-d_G(w,\gamma(w))} = F(w,\gamma(w)) \asymp G(w,\gamma(w)) = \sum_k \mu^k(\gamma)$$

and

$$(16) \qquad \rho(\mho(\gamma(w))) \asymp e^{-v d(w,\gamma(w))}.$$

On the other hand, since d_G is quasi-isometric to d, it follows that there is a constant $c > 0$ such that

$$\frac{\rho(\mho(\gamma(w)))}{\nu(\mho(\gamma(w)))} \lesssim e^{cd(w,\gamma(w))}.$$

Hence

$$R_\kappa \lesssim \sum_{k \geq \kappa N} e^{c\beta k} \sum_{k \leq d(w,\gamma(w)) < k+1} \mu^n(\gamma).$$

But μ^n is the distribution of Z_n so that

$$\sum_{k \leq d(w,\gamma(w)) < k+1} \mu^n(\gamma) \leq \mathbb{P}[d(w, Z_n(w)) \geq k].$$

From the exponential Tchebychev inequality, one obtains

$$(17) \qquad R_\kappa \lesssim \sum_{k \geq \kappa N} e^{(c\beta - \lambda)k} \mathbb{E}\left[e^{\lambda d(w, Z_n(w))} \right]$$

Now,

$$d(w, Z_n(w)) \leq \sum_{0 \leq j < N} d(Z_j(w), Z_{j+1}(w)) = \sum_{0 \leq j < N} d(w, Z_j^{-1} Z_{j+1}(w))$$

since Γ acts by isometries. Thus, the independance of the increments of the walk implies

$$\mathbb{E}\left[e^{\lambda d(w, Z_n(w))} \right] \leq E^N.$$

If we take $\beta \overset{\text{def.}}{=} \min\{\lambda/2c, 1\}$ then (17) becomes

$$R_\kappa \lesssim \sum_{k \geq \kappa N} e^{(-\lambda/2)k} E^N \lesssim e^{-(\lambda/2)\kappa N} E^N.$$

The estimate (14) is obtained by choosing $\kappa = 2 \log E / \lambda$.

We now prove that

$$
(18) \qquad P_N \stackrel{\text{def.}}{=} \frac{1}{N} \sum_{1 \leq n \leq N} \sum_{\gamma \in \Gamma : d(w, \gamma(w)) \leq \kappa N} \left(\frac{\rho(\mho(\gamma(w)))}{\nu(\mho(\gamma(w)))} \right)^{\beta} \mu^n(\gamma) \lesssim 1 \,.
$$

Both (14) and (18) implies the lemma.

Note that since $\beta \leq 1$, it follows that $\varphi_n^{\beta} \leq \max\{1, \varphi_n\} \leq 1 + \varphi_n$.

Hence:

$$
P_N \quad \lesssim 1 + \frac{1}{N} \sum_{n=1}^{N} \sum_{\gamma \in \Gamma : d(w, \gamma(w)) \leq \kappa N} \frac{\rho(\mho(\gamma(w)))}{\nu(\mho(\gamma(w)))} \mu^n(\gamma)
$$

$$
\lesssim 1 + \frac{1}{N} \sum_{\gamma \in \Gamma : d(w, \gamma(w)) \leq \kappa N} \frac{\displaystyle\sum_{n=1}^{N} \mu^n(\gamma)}{\nu(\mho(\gamma(w)))} \rho(\mho(\gamma(w))).
$$

But (15) implies that

$$
\frac{\displaystyle\sum_{n=1}^{N} \mu^n(\gamma)}{\nu(\mho(\gamma(w)))} \lesssim 1
$$

so that

$$
(19) \qquad P_N \lesssim 1 + \frac{1}{N} \sum_{d(w,x) \leq \kappa N} \rho(\mho(x)) \,.
$$

Since $\rho(\mho(x)) \asymp e^{-vd(w,x)}$ by (16) and since there are approximately e^{vk} elements in the d-ball of radius k (Theorem 2.3), we have

$$
\sum_{d(w,x) \leq \kappa N} \rho(\mho(x)) \asymp \sum_{1 \leq n \leq \kappa N} e^{vn} e^{-vn} \,,
$$

and

$$
\sum_{d(w,x) \leq \kappa N} \rho(\mho(x)) \lesssim N \,.
$$

Therefore, the estimate (18) follows from (19). $\blacksquare$

Lemma 5.3. *There is a finite constant $C_2 \geq 0$ such that the sequence $(\mathbb{E}(\phi_n) + C_2)_{n \geq 1}$ is subadditive and $(1/n)\phi_n$ tends to $h - \ell v$ a.s. and in expectation.*

PROOF. By the lemma of the shadow (Lemma 2.4),

$$
\frac{1}{n}\phi_n = \frac{1}{n}d_G(w, Z_n(w)) - \frac{1}{n}vd(w, Z_n(w)) + O(1/n)
$$

so, from Kingman ergodic theorem it follows that $(1/n)\phi_n$ converges almost surely and in expectation towards

$$
\ell_G - \ell v = h - \ell v \,,
$$

since $h = \ell_G$, see [7].

Let $m, n \geq 1$. It also follows from the lemma of the shadow and the triangle inequality for d_G that

$$
\mathbb{E}[\phi_{m+n}] - (\mathbb{E}[\phi_m] + \mathbb{E}[\phi_n]) \leq v\mathbb{E}[d(w, Z_m(w) + d(Z_m(w), Z_{m+n}(w)) - d(w, Z_{m+n}(w))] + O(1) \,.
$$

So Proposition 3.8 implies the existence of some constant C_2 such that

$$\mathbb{E}[\phi_{m+n}] - (\mathbb{E}[\phi_m] + \mathbb{E}[\phi_n]) \leq C_2 \,.$$

This gives the desired subadditivity. ∎

PROOF OF PROPOSITION 5.1. We shall prove that if ρ and ν are not equivalent, then $h < \ell v$.

Assuming that ρ and ν are not equivalent, the ergodicity of both measures implies that φ_n tends to 0 $\mathbb{P}$-a.s.

Choose $\eta \in (0, e^{-1}]$.

By Egorov theorem, there exist two measurable sets A and $B = A^c$ such that $\mathbb{P}[A] \leq \eta$ and $(\varphi_n|_B)_n$ converges uniformly to 0.

For any $n \geq 1$,

$$\mathbb{E}[\phi_n] = \int_A \phi_n d\mathbb{P} + \int_B \phi_n d\mathbb{P} \,.$$

Since $(\varphi_n|_B)_n$ uniformly converges to 0, there exists n_0 such that for n larger than n_0, $\phi_n|_B \leq \log \eta$ and therefore

$$\int_B \phi_n d\mathbb{P} \leq \mathbb{P}[B] \log \eta \leq (1 - \eta) \log \eta \,.$$

Choose β and C_1 as in Lemma 5.2. Jensen inequality yields

$$\int_A \phi_n d\mathbb{P} \leq \frac{\mathbb{P}[A]}{\beta} \log \int_A \varphi_n^\beta \frac{d\mathbb{P}}{\mathbb{P}[A]} \leq \frac{\eta}{\beta} \log(1/\eta) + \frac{\eta}{\beta} \log^+ \mathbb{E}[\varphi_n^\beta] \,,$$

where we have used $\eta \leq 1/e$.

But Lemma 5.2 implies that $\liminf \mathbb{E}[\varphi_n^\beta] < 2C_1$. So that there exists $p \geq n_0$ such that $\mathbb{E}[\varphi_p^\beta] \leq 2C_1$.

Hence,

$$\mathbb{E}[\phi_p] \leq (1 - \eta) \log \eta + \frac{\eta}{\beta} \log(1/\eta) + \frac{\eta}{\beta} \log(2C_1) \,.$$

When η tends to 0, the right-hand side tends to $-\infty$. Therefore, if we fix η small enough, there exists p such that

$$\mathbb{E}[\phi_p] + C_2 \leq -1 \,,$$

where C_2 is the constant appearing in Lemma 5.3.

Lemma 5.3 now implies that

$$\frac{1}{k}(\mathbb{E}[\phi_{kp}] + C_2) \leq \mathbb{E}[\phi_p] + C_2 \leq -1$$

for $k \geq 1$. As $(1/pk)\mathbb{E}[\phi_{pk}]$ tends to $(h - \ell v)$, letting k go to infinity, one obtains

$$(h - \ell v) \leq \frac{-1}{p} < 0 \,.$$

∎

Remark. In view of the proof of Proposition 5.1, one might wonder whether it is always true that a doubling measure of maximal dimension in an Ahlfors-regular space, as ν is, has to be equivalent to the Hausdorff measure of the same dimension. This property turns out

to be false in general. We are grateful to P. Mattila for pointing out to us its invalidity and to Y. Heurteaux for providing an explicit example of a doubling measure of dimension 1 in the unit interval $[0, 1]$ which is singular to the Lebesgue measure. We briefly describe his construction.

Let us consider a sequence of integers $(T_n)_n$ tending to infinity and satisfying

(1) $T_{2n} - T_{2n-1}$ is equivalent to T_{2n};
(2) $T_{2n+1} - T_{2n}$ is negligible in front of T_{2n};
(3) $T_{2n+1} - T_{2n}$ tends to infinity.

We then fix a weight $p \in (0, 1/2)$, define the sequence $(p_k)_k$ as follows:

- If $T_{2n-1} < k \le T_{2n}$, then $p_k = 1/2$;
- If $T_{2n} < k \le T_{2n+1}$, then $p_k = p$.

This means that the value of p_k is either p or $1/2$, and that, asymptotically, the mean of $(p_k)_{1 \le k \le n}$ tends towards $1/2$: $\lim_N (1/N) \sum_{k=1}^{N} p_k = 1/2$.

Let us code the dyadic intervals of $[0, 1]$ of the nth generation as $I_{a_1 \ldots a_n}$, with $a_1, \ldots, a_n = 0$ or 1. We define the measure m by setting

$$\frac{m(I_{a_1 \cdots a_{n+1}})}{m(I_{a_1 \cdots a_n})} = p_n \quad \text{if} \quad a_{n+1} = a_n$$

and

$$\frac{m(I_{a_1 \cdots a_{n+1}})}{m(I_{a_1 \cdots a_n})} = 1 - p_n \quad \text{if} \quad a_{n+1} \ne a_n\,.$$

The measure m we have just defined has dimension 1 (any set of dimension less than 1 is m-negligible), is doubling (see [4] for similar constructions), but it is singular with respect to the Lebesgue measure since $2^n m(I_{a_1 \ldots a_n})$ tends to 0 a.e.

5.2. **Equivalent measures.** We let Γ be a non-elementary hyperbolic group, $(X, d) \in \mathcal{D}(\Gamma)$, and μ a probability measure on Γ so that $d_G \in \mathcal{D}(\Gamma)$. This section is devoted to proving

Proposition 5.4. *If ρ and ν are equivalent then their density is almost surely bounded i.e., there is a constant $C \ge 1$ such that for any Borel set $A \subset \partial X$,*

$$\frac{1}{C}\nu(A) \le \rho(A) \le C\nu(A)\,.$$

We will work with the space $\partial^2 X$ of distinct points $(a, b) \in \partial X \times \partial X$, $a \ne b$, which is reminiscent to the geodesic flow of a negatively curved manifold. The group Γ acts on $\partial^2 X$ by the diagonal action $\gamma \cdot (a, b) = (\gamma(a), \gamma(b))$, $\gamma \in \Gamma$.

We define the following two σ-finite measures on $\partial^2 X$:

$$d\tilde{\rho}(a, b) = \frac{d\rho(a) \otimes d\rho(b)}{\exp 2v(a|b)} \quad \text{and} \quad d\tilde{\nu}(a, b) = \frac{d\nu(a) \otimes d\nu(b)}{\exp 2(a|b)^G}\,,$$

where we define

$$(a|b)^G \stackrel{\text{def.}}{=} \liminf_{(a_n),(b_n) \to a,b} (a_n|b_n)^G\,.$$

We recall that since ν is a conformal measure, $\tilde{\nu}$ is invariant, and it is furthermore ergodic [23, Thm 3.3]. On the other hand, ρ being just a quasiconformal measure, it follows that $\tilde{\rho}$ is just quasi-invariant, cf. [13]. This implies the existence of a constant $C \geq 1$ such that, for any Borel set $A \subset \partial^2 X$,

$$\frac{1}{C}\tilde{\rho}(A) \leq \tilde{\rho}(\gamma(A)) \leq C\tilde{\rho}(A) \,.$$

PROOF OF PROPOSITION 5.4. By assumption, there is a positive ν-integrable function J such that $d\rho = J d\nu$. Therefore, $d\tilde{\rho} = \tilde{J} d\tilde{\nu}$ holds with

$$\tilde{J}(a,b) = J(a)J(b)\frac{\exp 2(a|b)^G}{\exp 2v(a|b)} \,.$$

We shall first prove that $\tilde{J}$ is essentially constant (and non-zero). There is a constant $C > 1$ such that the set

$$A \overset{\text{def.}}{=} \{(1/C) \leq \tilde{J} \leq C\}$$

has positive $\tilde{\nu}$-measure. Since $\tilde{\nu}$ is ergodic, for $\tilde{\nu}$-almost every $(a,b) \in \partial^2 X$, there exists $\gamma \in \Gamma$ such that $\gamma(a,b) \in A$. It follows from the invariance of $\tilde{\nu}$ and the quasi-invariance of $\tilde{\rho}$ that

$$\tilde{J}(a,b) \asymp \tilde{J}(\gamma(a),\gamma(b)) \,.$$

This proves the claim.

Therefore, for $\tilde{\nu}$-almost every (a,b),

$$J(a)J(b) \asymp \frac{\exp 2v(a|b)}{\exp 2(a|b)^G} \,.$$

Let us assume that $\log J$ is unbounded in a neighborhood U of a point $a \in \partial X$. We may find a point $b \in \partial X$ with $J(b)$ finite and non-zero, and far enough from U so that

$$\frac{\exp 2v(c|b)}{\exp 2(c|b)^G} \asymp 1$$

for any $c \in U$. This proves that $\log J$ had to be bounded in U: a contradiction. ∎

5.3. **Geometric characterisation of the fundamental inequality.** We may now turn to the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. We first prove that (i), (ii) and (iii) are equivalent. Then we prove that (iii) implies (iv), (iv) implies (v) which implies (iii).

- From Proposition 5.1, we deduce that (i) implies (ii). Proposition 5.4 says that (ii) implies (iii). Furthermore, if ν and ρ are equivalent, then they have the same Hausdorff dimension. So, from Corollary 1.4 and Theorem 2.3, we get that

$$\frac{h}{\ell\varepsilon} = \dim \nu = \dim \rho = \frac{v}{\varepsilon} \,,$$

  and thus $h = \ell v$.
- To prove that (iii) implies (iv), we apply the lemma of the shadow (Lemma 2.4): it follows that, for any $\gamma \in \Gamma$,

$$e^{-vd(w,\gamma(w))} \asymp \rho(\mho(\gamma(w))) \asymp \nu(\mho(\gamma(w))) \asymp e^{-d_G(w,\gamma(w))}$$

  whence the existence of a constant C such that

$$|vd(w,\gamma(w)) - d_G(w,\gamma(w))| \leq C \,.$$

Since Γ acts transitively by isometries for both metrics, it follows that (X, vd) and (X, d_G) are $(1, C)$-quasi-isometric.

- Assuming (iv), it follows that Busemann functions coincide up to the multiplicative factor v. Therefore, the Radon-Nikodym derivative of $\gamma^*\nu$ with respect to ν at a point $a \in \partial X$ is proportional to $\exp(-v\beta_a(w, \gamma^{-1}(w)))$ a.e. Therefore, ν is a quasiconformal measure for $(\partial X, d_\varepsilon)$. This is (v).

- For the last implication, (v) implies (iii), one can use the uniqueness statement in Theorem 2.3 to get that ρ and ν are equivalent and have bounded density. This proves (iii). ∎

5.4. Simultaneous random walks. We now turn to the proof of Corollary 1.6.

PROOF OF COROLLARY 1.6. Let us consider the Green metric d_G associated with μ and denote by $\widehat{\ell}$ the drift of $(\widehat{Z}_n)$ in the metric space (Γ, d_G). Theorem 1.1 implies that $d_G \in \mathcal{D}(\Gamma)$.

Assumption (i) translates into $\widehat{h} = \widehat{\ell}$. Since $v_G = 1$, this means that $\widehat{\nu}$ has maximal dimension in the boundary of (Γ, d_G) endowed with a visual metric. Therefore Theorem 1.5 implies the equivalence between (i) and (iii).

Exchanging the roles of μ and $\widehat{\mu}$ gives the equivalence between (ii) and (iii).

If $\widehat{d}_G$ denotes the Green metric for $\widehat{\mu}$, then (iv) means that d_G and $\widehat{d}_G$ are $(1, C)$-quasi-isometric, which is equivalent to (iii) by Theorem 1.5. ∎

6. DISCRETISATION OF BROWNIAN MOTION

We let M be the universal covering of a Riemannian manifold N of pinched negative curvature and finite volume with deck transformation group Γ i.e., $M/\Gamma = N$. We let d denote the distance defined by the Riemannian structure on M. Note that when N is compact, Γ acts geometrically on M, and since it has negative curvature, it follows that Γ is hyperbolic and that M is quasi-isometric to Γ by Švarc-Milnor's lemma (Lemma 2.2).

We consider the diffusion process (ξ_t) generated by the Laplace-Beltrami operator Δ on M. That is, we let p_t be the fundamental solution of the heat equation $\partial_t = \Delta$. Then there is a probability measure $\mathbb{P}^y$ on the family Ξ^y of continuous curves $\xi : \mathbb{R}_+ \to M$ with $\xi_0 = y$ such that, for any Borel sets $A_1, A_2, ..., A_n$, and any times $t_1 < t_2 < \ldots < t_n$,

$$\mathbb{P}^y(\xi_{t_1} \in A_1, \ldots, \xi_{t_n} \in A_n)$$
$$= \int_{A_1} \int_{A_2} \ldots \int_{A_n} p_{t_1}(y, x_1) p_{t_2-t_1}(x_1, x_2) \ldots p_{t_n-t_{n-1}}(x_{n-1}, x_n) dx_1 \ldots dx_n.$$

If μ is a positive measure on M, we write $\mathbb{P}^\mu = \int_M \mathbb{P}^y \mu(dy)$, and this defines a measure on the set of Brownian paths Ξ.

As for random walks, the following limit exists almost surely and in L^1 and we call it the drift of the Brownian motion:

$$\ell_M \overset{\text{def.}}{=} \lim \frac{d(\xi_0, \xi_t)}{t};$$

it is also known that $\ell_M > 0$ and that (ξ_t) almost surely converges to a point ξ_∞ in ∂M [40, 39]. The distribution of ξ_∞ is the harmonic measure. Furthermore, V. Kaimanovich has

defined an asymptotic entropy h_M which shares the same properties as for random walks [20]: for any $y \in M$,

$$h_M \stackrel{\text{def.}}{=} \lim \frac{-1}{t} \int p_t(y,x) \log p_t(y,x) dx \,.$$

He also proved that the fundamental inequality $h_M \leq \ell_M v$ remains valid in this setting, where v denotes the logarithmic volume growth rate of M.

6.1. **The discretised motion.** W. Ballmann and F. Ledrappier have refined a method of T. Lyons and D. Sullivan [35], further studied by A. Ancona [1], V. Kaimanovich [22], and by A. Karlsson and F. Ledrappier [27] which replaces the Brownian motion by a random walk on Γ [3]. The construction goes as follows in our specific case.

Let $\pi : M \to N$ be the universal covering and let us fix a base point $w \in M$. Fix $\varepsilon > 0$ smaller than the injectivity radius of N at $\pi(w)$, and consider $\overline{V} = B(\pi(w), \varepsilon)$ in N; for D large enough, the set $\overline{F} = \{G_{\overline{V}}(\pi(w), \cdot) \geq D\}$ is compact in $\overline{V}$, where $G_{\overline{V}}$ denotes the Green function of the Brownian motion killed outside $\overline{V}$. There exists a so-called Harnack constant $C < \infty$ such that, for any positive harmonic function h on $\overline{V}$ and any points $a, b \in \overline{F}$, $h(a)/h(b) \leq C$ holds.

Let $V = \pi^{-1}(\overline{V})$, $F = \pi^{-1}(\overline{F})$, $V^x = B(x, \varepsilon)$ and $F_x = F \cap V^x$ for $x \in X \stackrel{\text{def.}}{=} \Gamma(w)$. If $y \in F_x$, we set $\chi(y) = x$. (Note that χ is well defined thanks to the choice of ε.)

Let ξ_t be a sample path of the Brownian motion. We define inductively the following Markov stopping times $(R_n)_{n \geq 1}$ and $(S_n)_{n \geq 0}$ as follows.

Set $S_0 = 0$ if $\xi_0 \notin X$, and $S_0 = \min\{t \geq 0, \ \xi_t \notin V^{\xi_0}\}$. Then, for $n \geq 1$, let

$$\begin{cases} R_n = \min\{t \geq S_{n-1}, \ \xi_t \in F\} \\ S_n = \min\{t \geq R_n, \ \xi_t \notin V^{X_n}\} \end{cases}$$

with $X_n = \chi(\xi_{R_n})$.

Let us also define recursively for $k \geq 0$ on $\Xi \times [0,1]^{\mathbb{N}}$,

$$\begin{cases} N_0(\xi, \alpha) = 0 \\ \\ N_k(\xi, \alpha) = \min\{n > N_{k-1}(\xi, \alpha), \ \alpha_n < \kappa_n(\xi)\} \end{cases}$$

where

$$\kappa_n(\xi) = \frac{1}{C} \frac{d\varepsilon_{X_n}^V}{d\varepsilon_{\xi_{R_n}}^V}(\xi_{S_n}) \,,$$

and, for $z \in F$, ε_z^V denotes the distribution of ξ_{S_1} for sample paths ξ_t starting at z. We also set

$$T_k = S_{N_k} \,.$$

For y in M, we let $\widetilde{\mathbb{P}}^y$ denote the product measure of $\mathbb{P}^y \times \lambda^{\mathbb{N}}$, where λ is the Lebesgue measure on $[0,1]$. We then define on X, the law

$$\mu_y(x) = \widetilde{\mathbb{P}}^y[X_{N_1} = x] \,.$$

The following properties are known to hold [35, 22, 3, 27].

Theorem 6.1. *Let us define $\mu(\gamma) = \mu_w(\gamma(w))$, and $Z_k(w) = X_{N_k}$ with $Z_0(w) = w$.*

(i) *The random sequence $(Z_n(w))$ is the random walk generated by μ: for any $x_1 = \gamma_1(w), \ldots, x_n = \gamma_n(w) \in X$,*

$$\widetilde{\mathbb{P}}^w(Z_1 = x_1, \ldots, Z_n = x_n) = \mu(\gamma_1)\mu(\gamma_1^{-1}\gamma_2)\ldots\mu(\gamma_{n-1}^{-1}\gamma_n).$$

(ii) *The measure μ is symmetric with full support but has a finite first moment with respect to d.*

(iii) *The Green function G_μ of the random walk is proportional to the Green function G_M of M.*

(iv) *There exists a positive constant T such that the following limit exists almost surely and in L^1:*

$$\lim \frac{S_{N_k}}{k} = T.$$

(v) *Almost surely and in L^1,*

$$\lim \frac{d(\xi_{kT}, Z_k(w))}{k} = 0.$$

(vi) *The harmonic measures for the Brownian motion and the random walk coincide.*

We are able to prove the following:

Theorem 6.2. *Under the notation and assumptions from above, let d_G denote the Green metric associated with μ. If N is compact, then $d_G \in \mathcal{D}(\Gamma)$ and*

$$\dim \nu = \frac{h_M}{\varepsilon \ell_M}$$

where h_M and ℓ_M denote the entropy and the drift of the Brownian motion respectively.

PROOF. The acronyms (ED) and (QR) below refer to Proposition 3.5. Since M has pinched negative curvature, it follows that $G_M(x,y) \lesssim e^{-cd(x,y)}$ holds for some constant $c > 0$, see [2, (2.4) p. 434]. By part (iii) of Theorem 6.1, G_μ and G_M are proportional. Therefore G_μ also satisfies $G_\mu(x,y) \lesssim e^{-cd(x,y)}$ and (ED) is proved. Furthermore, A. Ancona's Theorem 3.7 also holds for the Brownian motion, see [1], showing that (QR) holds as well. Both these properties imply that $(X, d_G) \in \mathcal{D}(\Gamma)$ by Proposition 3.5.

The identity $h_\mu = h_M \cdot T$ was proved by V. Kaimanovich [20, 22]. Furthermore, from Theorem 6.1 (4), it follows that almost surely,

$$\ell_\mu = \lim \frac{d(w, Z_k(w))}{k} = \lim \frac{d(w, \xi_{kT})}{k} = \ell_M \cdot T.$$

Thus, Corollary 1.4 implies that

$$\dim \nu = \frac{h_M}{\varepsilon \ell_M}.$$

∎

The computation of the drift can also be found in [27].

6.2. **Exponential moment for the discretised motion.** In [1], A. Ancona wrote in a remark that the random walk defined above has a finite exponential moment when N is compact. Since this fact is crucial to us, we provide here a detailed proof. This will enable us to apply Theorem 1.5 and conclude the proof of Theorem 1.9.

Theorem 6.3. *If N is compact, then the random walk (Z_n) defined in Theorem 6.1 has a finite exponential moment.*

The proof requires intermediate estimates on the Brownian motion. The main step is an estimate on the position of ξ_{S_1}:

Proposition 6.4. *There are positive constants C_1 and c_1 such that, for any $r \geq 1$,*

$$\sup_{y \in M} \mathbb{P}^y[d(\xi_0, \xi_{S_1}) \geq r] \leq C_1 e^{-c_1 r} \,.$$

Proposition 6.4 follows from the following lemma.

Lemma 6.5. *We write $\xi_t^* = \sup_{0 \leq s \leq t} d(\xi_0, \xi_s)$. There are constants $m > 0$, $c_2 > 0$ and $C_2 > 0$ such that*

$$\sup_{y \in M} \mathbb{P}^y[\xi_t^* \geq mt] \leq C_2 e^{-c_2 t} \,.$$

PROOF. We first prove that all the exponential moments of ξ_1^* are finite. Our proof relies on the following upper Gaussian estimate valid as soon as the curvature is bounded (see e.g. [40, §6] for a proof): for any $y \in M$ and any $t \geq 2$,

$$\mathbb{P}^y[\xi_1^* \geq t] \leq \exp\left(-ct^2\right) \,,$$

for some constant c that does not depend on y nor on t.

Hence, if $\lambda > 0$ then

$$\mathbb{E}^y\left[e^{\lambda \xi_1^*}\right] = 1 + \int_{u > 0} e^u \mathbb{P}^y[\xi_1^* \geq (u/\lambda)] du$$

$$\leq 1 + \int_{u > 0} e^{u - c\frac{u^2}{\lambda^2}} du < \infty \,.$$

Let $y \in M$ and $m > 0$. It follows from the exponential Tchebychev inequality that

$$\mathbb{P}^y[\xi_t^* \geq mt] \leq e^{-\lambda mt} \mathbb{E}^y\left[e^{\lambda \xi_t^*}\right] \,.$$

We remark that, for $n \geq 1$ and $t \in (n-1, n]$,

$$\xi_t^* \leq \sum_{0 \leq k < n} \sup_{k \leq s \leq k+1} d(\xi_k, \xi_s) \,.$$

It follows from the Markov property that, for all $y \in M$,

$$\mathbb{E}^y\left[e^{\lambda \xi_t^*}\right] \leq \left(\sup_{z \in M} \mathbb{E}^z\left[e^{\lambda \xi_1^*}\right]\right)^n \,.$$

Therefore

$$\mathbb{P}^y[\xi_t^* \geq mt] \lesssim e^{-\lambda mt} \left(\sup_{z \in M} \mathbb{E}^z\left[e^{\lambda \xi_1^*}\right]\right)^t \,.$$

So, if m is chosen large enough, we will find $c_2 > 0$ so that

$$\mathbb{P}^y[\xi_t^* \geq mt] \lesssim e^{-c_2 t} \,.$$

∎

PROOF OF PROPOSITION 6.4. The compactness of N easily implies the following upper bound on the first hitting time S_1 using the orthogonal decomposition of $L^2(N)$ (see [39, (5.2)]): there are positive constants C_3 and c_3 such that, for any $y \in M$,

$$(20) \qquad \mathbb{P}^y[S_1 \geq k] \leq C_3 e^{-c_3 k}$$

Let us consider $\kappa > 0$ that will be fixed later.

$$\mathbb{P}^y[d(y, \xi_{S_1}) \geq r] \leq \mathbb{P}^y[d(y, \xi_{S_1}) \geq r; \ S_1 \leq \kappa] + \mathbb{P}^y[d(y, \xi_{S_1}) \geq r; \ S_1 \geq \kappa].$$

From (20), it follows that

$$\mathbb{P}^y[d(y, \xi_{S_1}) \geq r] \lesssim \mathbb{P}^y[\xi_\kappa^* \geq r] + e^{-c_3\kappa}.$$

Choosing $\kappa = r/m$, Lemma 6.5 implies that

$$\mathbb{P}^y[d(y, \xi_{S_1}) \geq r] \lesssim e^{-\frac{c_2}{m}r} + e^{-\frac{c_3}{m}r},$$

and the proposition follows. ∎

PROOF OF THEOREM 6.3. Let $r \geq 1$ and $k \geq 1$, and $\lambda > 0$ that will be fixed later.

The exponential Tchebychev inequality yields

$$\mathbb{P}^y[d(\xi_0, \xi_{S_k}) \geq r] \leq e^{-\lambda r} \mathbb{E}^y\left[e^{\lambda d(\xi_0, \xi_{S_k})}\right].$$

But

$$d(\xi_0, \xi_{S_k}) \leq \sum_{0 \leq j < k} d(\xi_{S_j}, \xi_{S_{j+1}})$$

so the strong Markov property implies that

(21) $$\mathbb{P}^y[d(\xi_0, \xi_{S_k}) \geq r] \leq e^{-\lambda r} \left(\sup_{z \in M} \mathbb{E}^z\left[e^{\lambda d(\xi_0, \xi_{S_1})}\right]\right)^k.$$

Using Proposition 6.4 and its notation, we get that for any $z \in M$,

$$\mathbb{E}^z\left[e^{\lambda d(\xi_0, \xi_{S_1})}\right] = 1 + \int_{u>0} e^u \mathbb{P}^y[d(\xi_0, \xi_{S_1}) \geq (u/\lambda)] du$$

$$\leq 1 + C_1 \int_{u>0} e^u e^{-c_1 u/\lambda} du.$$

We choose $\lambda < c_1$; there exists a positive constant C_4 such that

$$\sup_{z \in M} \mathbb{E}^z\left[e^{\lambda d(\xi_0, \xi_{S_1})}\right] \leq \frac{1 + C_4\lambda}{1 - (\lambda/c_1)}.$$

Plugging this last inequality in (21) yields

(22) $$\mathbb{P}^y[d(\xi_0, \xi_{S_k}) \geq r] \lesssim e^{-\lambda r + kc_4\lambda}$$

for some constant $c_4 > 0$.

We note that, for any $x \in X$, any $z \in F_x$ and $u \in \partial V^x$,

$$\frac{d\varepsilon_x^V}{d\varepsilon_z^V}(u) \geq (1/C)$$

where C is the Harnack constant. Observe that this estimate is uniform with respect to $u \in \partial V^x$ and $z \in F_x$. Therefore,

$$\begin{aligned}
\widetilde{\mathbb{P}}^y[T_1 \geq S_k \mid \xi] &= \widetilde{\mathbb{P}}^y \left[\cap_{n=1}^{k-1} \{ \kappa_n(\xi) < \alpha_n \} \mid \xi \right] \\
&\leq \widetilde{\mathbb{P}}^y \left[\cap_{n=1}^{k-1} \{ (1/C^2) < \alpha_n \} \mid \xi \right] = \widetilde{\mathbb{P}}^y \left[\cap_{n=1}^{k-1} \{ (1/C^2) < \alpha_n \} \right] \\
&= \prod_{n=1}^{k-1} \widetilde{\mathbb{P}}^y[(1/C^2) < \alpha_n] \lesssim (1 - (1/C^2))^k .
\end{aligned}$$

(23)

In (23), we used the notation $\widetilde{\mathbb{P}}^y[. \mid \xi]$ to denote the conditional probability given the Brownian path ξ. Note that S_k, being a function of ξ, does not depend on the sequence α. We used this fact for the second equality above; see also [27] for a different argument leading to the same conclusion.

From (22) and (23), it then follows that

$$\begin{aligned}
\widetilde{\mathbb{P}}^y[d(y, \xi_{T_1}) \geq r] &= \sum_{k \geq 1} \widetilde{\mathbb{P}}^y[d(y, \xi_{S_k}) \geq r; \; S_k = T_1] \\
&= \sum_{k \geq 1} \widetilde{\mathbb{E}}^y[\widetilde{\mathbb{P}}^y[S_k = T_1 \mid \xi]; \; d(y, \xi_{S_k}) \geq r] \\
&\lesssim \sum_{k \geq 1} (1 - (1/C^2))^k \widetilde{\mathbb{P}}^y[d(y, \xi_{S_k}) \geq r] \\
&\lesssim e^{-\lambda r} \sum_{k \geq 1} (1 - (1/C^2))^k e^{\lambda c_4 k} .
\end{aligned}$$

Thus, there is some $\lambda_0 > 0$ so that if we choose $\lambda \in (0, \lambda_0]$ then this last series is convergent and we find

$$\widetilde{\mathbb{P}}^y[d(y, \xi_{T_1}) \geq r] \lesssim e^{-\lambda r} .$$

Consequently, noting that $d(Z_1(w), \xi_{T_1}) \leq \varepsilon$ and choosing $\lambda = \lambda_0$,

$$\mathbb{E}\left[e^{(\lambda_0/2)d(y, Z_1(w))} \right] \lesssim 1 + \int_{u>0} e^u \widetilde{\mathbb{P}}[d(y, \xi_{T_1}) > 2u/\lambda_0] du \lesssim 1 + \int_{u>0} e^{-u} du < \infty . \qquad \blacksquare$$

6.3. **Examples.** Let us fix $n \geq 2$ and consider the hyperbolic space $\mathbb{H}^n$ of constant sectional curvature -1. The explicit form of the Green function on this space shows easily that, given $w, x, y, z \in \mathbb{H}^n$ which are at distance $c > 0$ apart from one another, one has

(24) $$\Theta(x, y) \gtrsim \min\{\Theta(x, z), \Theta(z, y)\}$$

where Θ is Naïm's kernel, and the implicit constant depends only on c. Let N be a finite volume hyperbolic manifold with deck transformation group Γ acting on $\mathbb{H}^n$. The estimate (24) shows that the Green metric d_G on Γ associated with the discretised Brownian motion on $\mathbb{H}^n$ is hyperbolic. Moreover, the estimate (ED) holds as well, so that the Green metric d_G is quasi-isometric to the restriction of the hyperbolic metric to the orbit $\Gamma(o)$ of a base point $o \in \mathbb{H}^n$. Since N has finite volume, the limit set of Γ is the whole sphere at infinity, and it coincides with the visual boundary of (Γ, d_G). Therefore, Theorem 1.7 implies that the Martin boundary coincides with $\partial \mathbb{H}^n$, homeomorphic to $\mathbb{S}^{n-1}$. We omit the details.

We apply this construction in two special cases.

If we consider for N a punctured 2-torus with a complete hyperbolic metric of finite volume (as in [3]), we obtain an example of a random walk on the free group for which the Green metric is hyperbolic but its boundary $\mathbb{S}^1$ does not coincide with the boundary of the group (which is a Cantor set). Therefore, d_G does not belong to the quasi-isometry class of the free group.

If we consider now for N a complete hyperbolic 3-manifold of finite volume with a rank 2 cusp, then its fundamental group is not hyperbolic since it contains a subgroup isomorphic to $\mathbb{Z}^2$, but the Green metric is hyperbolic nonetheless.

Appendix A. Quasiruled hyperbolic spaces

For geodesic spaces, hyperbolicity admits many characterisations based on geodesic triangles (cf. Prop. 2.21 from [17]). Most of them still hold when the space X is just a length space (see eg. [42]). For instance, a geodesic hyperbolic space satisfies Rips condition, namely, a constant δ exists such that any edge of a geodesic triangle is at distance at most δ from the two other edges.

It is known that if X and Y are two quasi-isometric geodesic spaces, then X is hyperbolic if and only if Y is (Theorem 5.12 in [17]). This statement is known to be false in general if we do not assume both spaces to be geodesic (Example 5.12 from [17], and Proposition A.11 below).

Since quasi-isometries do not preserve small-scales of metric spaces, in particular geodesics, it is therefore important to find other coarse characterisations of hyperbolicity. Such a characterisation is the purpose of this appendix. We propose a setting which enables us to go through the whole theory of quasiconformal measures as if the underlying space was geodesic.

Definition. A *quasigeodesic curve (resp. ray, segment)* is the image of $\mathbb{R}$ (resp. $\mathbb{R}_+$, a compact interval of $\mathbb{R}$) by a quasi-isometric embedding. A space is said to be *quasigeodesic* if there are constants λ, c such that any pair of points can be connected by a (λ, c)-quasigeodesic.

The image of a geodesic space by a quasi-isometry is thus quasigeodesic. But as it was mentioned earlier, hyperbolicity need not be preserved.

Definition. A τ-*quasiruler* is a quasigeodesic $g : \mathbb{R} \to X$ (resp. quasisegment $g : I \to X$, quasiray $g : \mathbb{R}_+ \to X$) such that, for any $s < t < u$,

$$(g(s)|g(u))_{g(t)} \leq \tau.$$

Let X be a metric space. Let $\lambda \geq 1$ and $\tau, c > 0$ be constants. A *quasiruling structure* $\mathcal{G}$ is a set of τ-quasiruled (λ, c)-quasigeodesics such any pair of points of X can be joined by an element of $\mathcal{G}$.

A metric space will be *quasiruled* if constants (λ, c, τ) exist so that the space is (λ, c)-quasigeodesic and if every (λ, c)-quasigeodesic is a τ-quasiruler i.e., the set of quasigeodesics defines a quasiruling structure. The data of a quasiruled space are thus the constants (λ, c) for the quasigeodesics and the constant τ given by the quasiruler property of the (λ, c)-quasigeodesics.

A quasi-isometric embedding $f : X \to Y$ between a geodesic metric space X into a metric space Y is τ-*ruling* if the image of any geodesic segment is a τ-quasiruler. Then the images of geodesics of X define a quasiruling structure $\mathcal{G}$ of Y. In this situation, we will say that $\mathcal{G}$ is induced by X.

Theorem A.1. *Let X be a geodesic hyperbolic metric space, and $\varphi : X \to Y$ a quasi-isometry, where Y is a metric space. The following statements are equivalent:*

  (i) *Y is hyperbolic;*
  (ii) *Y is quasiruled;*
  (iii) *φ is ruling.*

Moreover if Y is a hyperbolic quasiruled space, then Y is isometric to a quasiconvex subset of a geodesic hyperbolic metric space Z.

Furthermore, if Γ acts geometrically on Y, then Γ is a quasiconvex group acting on Z.

Theorem 1.10 is a consequence from Theorem A.1.

We refer to [17] for any undefined notion used in the sequel.

A.1. **Straightening of configurations.** Let $I = [a, b] \subset \mathbb{R}$ be a closed connected subset. We assume throughout this section that constants (λ, c, τ) are fixed.

Lemma A.2. *Let $g : I \to X$ be a quasiruler. There is a $(1, c_1)$-quasi-isometry*

$$f : g(I) \to [0, |g(b) - g(a)|] ,$$

for some c_1 which depends only on the data (λ, c and τ).

PROOF. For any $x \in g(I)$, let $f(x) = \min\{|x - g(a)|, |g(b) - g(a)|\}$. Thus

(25) $$||x - g(a)| - f(x)| \leq 2\tau.$$

Let $x, y \in g(I)$ with $x = g(s)$ and $y = g(t)$, and let us assume that $s < t$.

  • We apply (25) repeatedly. On the one hand,

$$|f(x) - f(y)| \leq ||x - g(a)| - |y - g(a)|| + 4\tau \leq |x - y| + 4\tau .$$

  On the other hand, since $s < t$, it follows that

$$|x - g(a)| + |x - y| \leq |y - g(a)| + 2\tau$$

  so that

$$|f(x) - f(y)| \geq |x - y| - 8\tau.$$

  Hence f is a $(1, 8\tau)$-quasi-isometric embedding.
      Note that the constants above are not sharp (a case by case treatment would divide most of them by 2).

  • If $|a - b| \leq 2$, then $|f(g(a)) - f(g(b))| = |g(a) - g(b)| \leq 2\lambda + c$ and f is cobounded. Otherwise, $|a - b| > 2$. Let $s_j = a + j$ for $j \in \mathbb{N} \cap [0, |b - a|]$. It follows that

$$|f(g(s_j)) - f(g(s_{j+1}))| \leq \lambda|s_j - s_{j+1}| + c + 4\tau \leq \lambda + c + 4\tau.$$

  The set $\{f(g(s_j))\}_j$ is a chain in $[0, |g(b) - g(a)|]$ which joins 0 to $|f(g(a)) - f(g(b))| = |g(b) - g(a)|$; since two consecutive points of $\{f(g(s_j))\}_j$ are at most $\lambda + c + 4\tau$ apart, it follows that its $(\lambda + c + 4\tau)$-neighborhood covers $[0, |g(a) - g(b)|]$, hence f is a quasi-isometry. ∎

Remark. If f_a denotes the map as above and $f_b : g(I) \to [0, |g(b) - g(a)|]$ the map such that $f_b(g(b)) = 0$, then $||f_a(x) + f_b(x) - |g(a) - g(b)|| \leq 2\tau$ holds.

Definition. Given three points $\{x, y, z\}$, there is a tripod T and an isometric embedding $f : \{x, y, z\} \to T$ such that the images are the endpoints of T. We let $\bar{c}$ denote the center of T.

A quasitriangle Δ is given by three points x, y, z together with three quasirulers joining them. We will denote the edges by $[x, y]$, $[x, z]$ and $[y, z]$. Such a quasitriangle is δ-thin if any segment is in the δ-neighborhood of the two others.

Lemma A.3. *Let Δ be a δ-thin quasitriangle with vertices $\{x, y, z\}$. There is a $(1, c_2)$-quasi-isometry*

$$f_\Delta : \Delta \to T\,,$$

where T is the tripod associated with $\{x, y, z\}$ and c_2 depends only on the data (δ, λ, c, τ).

PROOF. Let us define f_Δ using Lemma A.2 on each edge. This map is clearly cobounded.

Let $u, v \in \Delta$. Since Δ is thin, one may find two points $u', v' \in \Delta$ on the same edge such that $|u - u'| \leq \delta$ and $|v - v'| \leq \delta$, so that

$$||u - v| - |u' - v'|| \leq 2\delta.$$

If u and u' belong to the same edge, then

$$|f_\Delta(u) - f_\Delta(u')| \leq |u - u'| + c_1 \leq \delta + c_1\,.$$

Otherwise, let x be the common vertex of the edges containing u and u', then it follows from (25) that

$$|f_x(u) - f_x(u')| \leq |u - u'| + 4\tau \leq \delta + 4\tau$$

and similarly for v and v'. Thus

$$|f_\Delta(u) - f_\Delta(u')|, |f_\Delta(v) - f_\Delta(v')| \leq c'\,,$$

where c' depends only on the data.

It follows that

$$||f_\Delta(u) - f_\Delta(v)| - |f_\Delta(u') - f_\Delta(v')|| \leq 2c'.$$

But since u' and v' belong to the same edge, Lemma A.2 implies that

$$||f_\Delta(u') - f_\Delta(v')| - |u' - v'|| \leq c_1\,,$$

so

$$||f_\Delta(u) - f_\Delta(v)| - |u' - v'|| \leq 2c' + c_1$$

and finally

$$||f_\Delta(u) - f_\Delta(v)| - |u - v|| \leq (2c' + c_1 + 2\delta).$$

∎

In the situation of Lemma A.3 we have

$$|(f_\Delta(x)|f_\Delta(y))_{f_\Delta(z)} - (x|y)_z| \leq C\,,$$

for some universal constant $C > 0$; thus, we may find points $c_x \in [y, z]$, $c_y \in [x, z]$ and $c_z \in [y, x]$ such that

$$|f_\Delta(c_x) - \bar{c}|, |f_\Delta(c_y) - \bar{c}|, |f_\Delta(c_z) - \bar{c}| \leq c_3,$$

and

$$\mathrm{diam}\{c_x, c_y, c_z\} \leq c_3,$$

where c_3 depends only on the data.

Proposition A.4. *Let X be a metric space endowed with a quasiruling structure $\mathcal{G}$ such that all quasitriangles are δ-thin. Then X is hyperbolic quantitatively: the constant of hyperbolicity only depends on $(\delta, \lambda, c, \tau)$.*

PROOF. Let us fix $w, x, y, z \in X$. Let us consider the following triangles: $A = \{w, x, z\}$ and $B = \{w, x, y\}$. Let us denote by T_A, T_B and $\bar{c}_A$, $\bar{c}_B$ the associated tripod and center respectively, and let us define $Q = T_A \cup T_B$ where both copies $f_A([w, x])$ and $f_B([w, x])$ of $[w, x]$ have been identified. This metric space Q is topologically an "$\times$", and so is of course 0-hyperbolic.

Let us define $f : A \cup B \to Q$ by sending A under f_A and B under f_B.

The restriction of f to A and to B is a $(1, c_2)$-quasi-isometry by Lemma A.3.

It follows that
$$|f(y) - f(z)| = |f(y) - \bar{c}_B| + |\bar{c}_B - \bar{c}_A| + |\bar{c}_A - f(z)|.$$
One may find $c_A, c_B \in [w, x]$ such that $|f(c_A) - \bar{c}_A| \leq c_3$ and $|f(c_B) - \bar{c}_B| \leq c_3$. Lemma A.3 implies that $|f(y) - f(c_B)| = |y - c_B|$ and $|f(c_A) - f(z)| = |c_A - z|$ up to an additive constant. Therefore, $|f(y) - \bar{c}_B| = |y - c_B|$ and $|\bar{c}_A - f(z)| = |c_A - z|$ up to an additive constant too. By Lemma A.2, $|\bar{c}_B - \bar{c}_A| = |c_B - c_A|$ up to an additive constant, whence the existence of some constant $c_4 > 0$ such that
$$|f(y) - f(z)| \geq |y - c_B| + |c_B - c_A| + |c_A - z| - c_4 \geq |y - z| - c_4.$$

Hence $(f(y)|f(z))_{f(w)} \leq (y|z)_w + c_4$. It follows from the hyperbolicity of Q that
$$(y|z)_w \geq \min\{(f(x)|f(z))_{f(w)}, (f(y)|f(x))_{f(w)}\} - c_4$$
and since the restrictions of f to A and B are $(1, c_2)$-quasi-isometries,
$$\min\{(f(x)|f(z))_{f(w)}, (f(y)|f(x))_{f(w)}\} - c_4 \geq \min\{(x|z)_w, (y|x)_w\} - c_5$$
for some constant c_5. We have just established that for any w, x, y, z,
$$(y|z)_w \geq \min\{(x|z)_w, (y|x)_w\} - c_5.$$

∎

A.2. **Embeddings of hyperbolic spaces.** We recall a theorem of M. Bonk and O. Schramm (Theorem 4.1 in [9]):

Theorem A.5. *Any δ-hyperbolic space X can be isometrically embedded into a complete geodesic δ-hyperbolic space Y.*

We will show that if Γ acts isometrically on X, then so is the case on Y. To prove this we need to review the construction of the set Y.

The first lemma, which we recall, is the basic step in the construction.

Lemma A.6. *Let X be δ-hyperbolic metric space, and let $a \neq b$ be in X. If, for every x, $(|a - b|/2, |a - b|/2) \neq (|a - x|, |b - x|)$, then there is a δ-hyperbolic space $X[a, b] = X \cup \{m\}$ such that $(|a - b|/2, |a - b|/2) = (|a - m|, |b - m|)$. Furthermore, for any $x \in X$,*
$$|x - m| = \frac{|a - b|}{2} + \sup_{w \in X}(|x - w| - \max\{|a - w|, |b - w|\}).$$

We call m the middle point of $\{a, b\}$.

Lemma A.7. *A δ-hyperbolic metric space X embeds isometrically into a δ-hyperbolic space X^* such that, for any $(a, b) \in X$, there exists a middle point $m = m(a, b) \in X^*$.*

PROOF. They apply a transfinite induction: let $\phi : \omega \to X \times X$ be an ordinal of $X \times X$. Define inductively $X(\alpha)$ as follows. Set $X(0) = X$. If $\alpha = \beta + 1 \leq \omega + 1$, then define $X(\alpha) = X(\beta)[\phi(\alpha)]$. Clearly, $X(\alpha)$ is δ-hyperbolic. If α is a limit ordinal, set

$$X(\alpha) = (\cup_{\beta < \alpha} X(\beta)) [\phi(\alpha)].$$

Here too, $X(\alpha)$ is δ-hyperbolic since δ-hyperbolicity is preserved under increasing unions. The space $X^* = X(\omega + 1)$ fulfills the requirements. ∎

For $\alpha \leq \omega + 1$, let us define $m(\alpha) = m(\phi(\alpha))$ the middle of $\phi(\alpha) = (a(\alpha), b(\alpha))$, and let $D(\alpha) = |a - b|$. If $x^* \in X$, set $\alpha(x^*) = 0$; otherwise, let $P(x^*)$ be the set of ordinals α such that $x^* \in X(\alpha)$. Let us define $\alpha(x^*)$ as the minimum of $P(x^*)$; it follows that $x^* = m(\alpha)$. We let $D(x^*) = D(\alpha)$. We also write $\phi(\alpha) = (a(x^*), b(x^*))$.

Lemma A.8. *Let $\alpha < \beta$, then*

$$|m(\alpha) - m(\beta)| = \frac{D(\beta)}{2} + \sup_{w \in X(\alpha)} \{|w - m(\alpha)| - \max\{|w - a(\beta)|, |w - b(\beta)|\}\}.$$

PROOF. Let

$$Z = \left\{ \gamma \in \omega, \ |m(\alpha) - m(\gamma)| = \frac{D(\gamma)}{2} + \sup_{w \in X(\alpha)} \{|w - m(\alpha)| - \max\{|w - a(\gamma)|, |w - b(\gamma)|\}\} \right\}.$$

The set Z contains $\{\gamma \leq \alpha + 1\}$ by definition. Let us assume that $\{\gamma < \beta\} \subset Z$ for some $\beta > \alpha$. Pick $\gamma \in Z$, so that $\alpha < \gamma < \beta$. Given $\varepsilon > 0$, there is some $w \in X(\alpha)$ so that

$$|m(\alpha) - m(\gamma)| \leq \frac{D(\gamma)}{2} + |w - m(\alpha)| - \max\{|w - a(\gamma)|, |w - b(\gamma)|\} + \varepsilon.$$

Since $w \in X(\alpha)$ is fixed,

$$|m(\gamma) - a(\beta)| \geq \frac{D(\gamma)}{2} + |w - a(\beta)| - \max\{|w - a(\gamma)|, |w - b(\gamma)|\}.$$

A similar statement holds for $b(\beta)$ instead of $a(\beta)$. Therefore

$$\max\{|m(\gamma) - a(\beta)|, |m(\gamma) - b(\beta)|\} \geq$$
$$\frac{D(\gamma)}{2} + \max\{|w - a(\beta)|, |w - b(\beta)|\} - \max\{|w - a(\gamma)|, |w - b(\gamma)|\},$$

and

$$|m(\alpha) - m(\gamma)| - \max\{|m(\gamma) - a(\beta)|, |m(\gamma) - b(\beta)|\} \leq$$
$$|m(\alpha) - w| - \max\{|w - a(\beta)|, |w - b(\beta)|\} + \varepsilon.$$

It follows that, for each $\alpha < \gamma < \beta$, there is some $w \in X(\alpha)$ such that the supremum in the definition of $|m(\alpha) - m(\beta)|$ is attained within $X(\alpha)$. Hence $\beta \in Z$, so $Z = X^*$ by induction. ∎

Lemma A.9. *Let $0 < \alpha < \beta$. Then $|m(\alpha) - m(\beta)|$ can be computed as*

$$\frac{D(\alpha)}{2} + \frac{D(\beta)}{2} + \sup_{w, w' \in X} \{|w - w'| - (\max\{|w - a(\alpha)|, |w - b(\alpha)|\} + \max\{|w' - a(\beta)|, |w' - b(\beta)|\})\}.$$

PROOF. We endow $\omega \times \omega$ with the lexicographical order, and we consider $\omega' = \{(\alpha, \beta), \alpha < \beta\}$. We assume by transfinite induction that the lemma is true for any $(\alpha, \beta) < (\widehat{\alpha}, \widehat{\beta})$. By Lemma A.8, there is some $\widehat{w} \in X(\widehat{\alpha})$ such that

$$|m(\widehat{\alpha}) - m(\widehat{\beta})| \leq \frac{D(\widehat{\beta})}{2} + |\widehat{w} - m(\widehat{\alpha})| - \max\{|\widehat{w} - a(\widehat{\beta})|, |\widehat{w} - b(\widehat{\beta})|\} + \varepsilon.$$

It follows from the induction assumption that there are points $w', w \in X$ such that

$$|m(\widehat{\alpha}) - \widehat{w}| - \varepsilon \leq \frac{D(\widehat{\alpha})}{2} + \frac{D(\widehat{w})}{2} + |w - w'| - (\max\{|w - a(\widehat{\alpha})|, |w - b(\widehat{\alpha})|\}$$
$$+ \max\{|w' - a(\widehat{w})|, |w' - b(\widehat{w})|\}).$$

But

$$\max\{|\widehat{w} - a(\widehat{\beta})|, |\widehat{w} - b(\widehat{\beta})|\} \geq \frac{D(\widehat{w})}{2} + \max\{|w' - a(\widehat{\beta})|, |w' - b(\widehat{\beta})|\}$$
$$- \max\{|w' - a(\widehat{w})|, |w' - b(\widehat{w})|\},$$

so

$$|m(\widehat{\alpha}) - m(\widehat{\beta})| - 2\varepsilon \leq \frac{D(\widehat{\alpha})}{2} + \frac{D(\widehat{\beta})}{2} + |w - w'| - (\max\{|w - a(\widehat{\alpha})|, |w - b(\widehat{\alpha})|\}$$
$$+ \max\{|w' - a(\widehat{\beta})|, |w' - b(\widehat{\beta})|\}).$$

This establishes the lemma. ∎

Corollary A.10. *If Γ acts on X by isometry, then it acts also on X^* by isometry.*

PROOF. If $x^* \in X^* \setminus X$ and $g \in \Gamma$, we let $g(x^*) = m(g(a(x^*)), g(b(x^*))))$. The fact that $g : X^* \to X^*$ acts by isometry follows from Lemma A.9 since the distance between two points relies only on points inside X. ∎

The construction now goes as follows. Define $X_0 = X$, and $X_{n+1} = X_n^*$, for $n \geq 0$. The space $X' = \cup_{n \in \mathbb{N}} X_n$ is a metric δ-hyperbolic space such that any pair of points admits a midpoint in X'. Note that if Γ acts on X by isometry, then it also acts by isometry on X'.

To obtain a complete geodesic space, M. Bonk and O. Schramm use again a transfinite induction. Let ω_0 be the first uncountable ordinal. They define a metric space $Z(\alpha)$ for each ordinal $\alpha < \omega_0$ such that $Z(\alpha) \supset Z(\beta)$ if $\alpha > \beta$. We set $Z(0)$ as the completion of X'. More generally, if $\alpha = \beta + 1$, define $Z(\alpha)$ as the completion of $Z(\beta)'$. For limit ordinals α, we define $Z(\alpha)$ as the completion of $\cup_{\beta < \alpha} Z(\beta)'$. It follows that for each $\alpha < \omega_0$, the metric space $Z(\alpha)$ is complete, δ-hyperbolic, and admits an isometric action of Γ if X did.

The construction is completed by letting $Y = \cup_{\alpha < \omega_0} Z(\alpha)$. As above, an action of a group Γ by isometry on X extends canonically as an action by isometry on Y.

A.3. **Quasiruled spaces and hyperbolicity.** We prove Theorem A.1 in four steps.

A.3.1. Let us assume that Y is a quasigeodesic δ-hyperbolic space. It follows from Theorem A.5 that there are a δ-hyperbolic geodesic metric space $\widehat{Y}$ and an isometric embedding $\iota : Y \to \widehat{Y}$. Thus, for any quasigeodesic segment $g : [a, b] \to Y$, $\iota(g)$ shadows a genuine geodesic $\widehat{g} = [\iota(g(a)), \iota(g(b))]$ from $\widehat{Y}$ at distance $H = H(\lambda, c, \delta)$. In other words, for any $t \in [a, b]$, there is a point $\widehat{y}_t \in \widehat{g}$ such that $|\iota(g(t)) - \widehat{y}_t| \leq H$. It follows that

$$(g(a)|g(b))_{g(t)} \leq (\iota(g(a))|\iota(g(b)))_{\widehat{y}_t} + H = H$$

since $\iota(g(a))$, $\iota(g(b))$ and $\widehat{y}_t$ belong to a geodesic segment.

Therefore, Y is quasiruled.

A.3.2.   If Y is quasiruled, then φ is ruling since the image under φ is a quasigeodesic, hence a quasiruler by definition.

A.3.3.   Let us now assume that X is a geodesic hyperbolic space and $\varphi : X \to Y$ is a quasi-isometry into a metric space Y. It follows that Y is quasigeodesic and that the edge of the image of any geodesic triangle is at a bounded distance from the two other edges i.e., triangles are δ-thin. If φ is ruling, then Proposition A.4 applies, and proves that Y is hyperbolic.

A.3.4.   The statement concerning group actions follows from above and the previous section.

A.4. **Non-hyperbolic invariant metric on a hyperbolic group.** In [17], the authors provide an example of a non-hyperbolic metric space quasi-isometric to $\mathbb{R}$. One could wonder if, in the case of groups, the invariance of hyperbolicity holds for quasi-isometric and invariant metrics. In this section, we disprove this statement.

Proposition A.11. *For any hyperbolic group, a left-invariant metric quasi-isometric to a word metric exists which is not hyperbolic.*

We are grateful to C. Pittet and I. Mineyev for having pointed out to us the metric d in the following proof as a possible candidate.

PROOF. Let Γ be a hyperbolic group and let $|.|$ denote a word metric. We define the metric

$$d(x,y) = |x-y| + \log(1 + |x-y|) \, .$$

Clearly, $|x-y| \leq d(x,y) \leq 2|x-y|$ holds and d is left-invariant by Γ.

Let us prove that (Γ, d) is not quasiruled, hence not hyperbolic by Theorem A.1.

Let g be a geodesic for $|.|$ which we identify with $\mathbb{Z}$. Since (Γ, d) is bi-Lipschitz to $(\Gamma, | \cdot |)$, it is a $(2,0)$-quasigeodesic for d. But

$$d(0,n) + d(n,2n) - d(0,2n) = \log(1+n)^2/(1+2n)$$

asymptotically behaves as $\log n$. Therefore g is not quasiruled.   ∎

APPENDIX B. APPROXIMATE TREES AND SHADOWS

Approximate trees is an important tool to understand hyperbolicity in geodesic spaces. Here, we adapt their existence to the setting of hyperbolic quasiruled metric spaces following E. Ghys and P. de la Harpe (Theorem 2.12 in [17]).

Theorem B.1. *Let (X, w) be a δ-hyperbolic metric space and let $k \geq 0$.*

(i) *If $|X| \leq 2^k + 2$, then there is a finite metric pointed tree T and a map $\phi : X \to T$ such that:*
   → *$\forall x \in X$, $|\phi(x) - \phi(w)| = |x - w|$,*
   → *$\forall x, y \in X$, $|x - y| - 2k\delta \leq |\phi(x) - \phi(y)| \leq |x - y|$.*

(ii) *If there are τ-quasiruled rays $(X_i, w_i)_{1 \le i \le n}$ with $n \le 2^k$ such that $X = \cup X_i$, then there is a pointed $\mathbb{R}$-tree T and a map $\phi : X \to T$ such that*
$\to \ \forall x \in X, \ |\phi(x) - \phi(w)| = |x - w|$,
$\to \ \forall x, y \in X, \ |x - y| - 2(k + 1)\delta - 4c - 2\tau \le |\phi(x) - \phi(y)| \le |x - y|$, *where* $c = \max\{|w - w_i|\}$.

We repeat the arguments in [17]. The proofs of the first two lemmata can be found in [17], and the last one is the quasiruled version of [17, Lem. 2.14]. In the three lemmata, X is assumed to be δ-hyperbolic. Furthermore, we will omit the subscript w for the inner product and write $(\cdot|\cdot) = (\cdot|\cdot)_w$.

Lemma B.2. *We define*

$\to \ (x|y)' = \sup \min\{(x_{i-1}|x_i), \ 2 \le i \le L\}$, *where the supremum is taken over all finite chains* $x_1, \ldots, x_L$ *with* $x_1 = x$ *and* $x_L = y$,
$\to \ |x - y|' = |x - w| + |y - w| - 2(x|y)'$,
$\to \ x \sim y$ *if* $|x - y|' = 0$.

Then $\sim$ is an equivalence relation and $|\cdot|'$ is a distance on $X/\sim$ which makes it a 0-hyperbolic space. Moreover, for any $x \in X$, $|x - w|' = |x - w|$ holds, and for any $x, y \in X$, $|x - y|' \le |x - y|$.

Lemma B.3. *If $|X| \le 2^k + 2$ then for any chain $x_1, \ldots, x_L \in X$,*

$$(x_1|x_L) \ge \min_{2 \le j \le L} \{(x_{j-1}|x_j)\} - k\delta,$$

holds.

Lemma B.4. *Let $X = \cup_{i=1}^n X_i$ where (X_i, w_i) are τ-quasiruled rays. If $n \le 2^k$ then, for any chain $x_1, \ldots, x_L \in X$,*

$$(x_1|x_L) \ge \min_{2 \le j \le L} \{(x_{j-1}|x_j)\} - (k+1)\delta - 2c - \tau.$$

PROOF. First, $(x|y)_w \le \min\{|x - w|, |y - w|\}$ holds for any $x, y \in X$, and if $x, y \in X_i$ then $|(x|y)_{w_i} - \min\{|x - w_i|, |y - w_i|\}| \le \tau$, and $|x - w_i| \ge |x - w| - |w - w_i| \ge |x - w| - c$. Similarly, $|y - w_i| \ge |y - w| - c$. Thus, $(x|y)_{w_i} \ge \min\{|x - w|, |y - w|\} - c - \tau$ and

$$(x|y)_w \ge (x|y)_{w_i} - c \ge \min\{|x - w|, |y - w|\} - 2c - \tau \ge \min\{(x|x')_w, (y|y')_w\} - 2c - \tau$$

for all $x', y' \in X$.

Let $x_1, \ldots, x_L \in X$ be a chain. We will write $X(x_j)$ to denote the quasiruled ray X_i which contains x_j. Either, for all $j \ge 2$, $x_j \notin X(x_1)$, or there is a maximal index $j > 1$ such that $x_j \in X(x_1)$. Hence, it follows from above that $(x_1|x_j) \ge \min_{2 \le i \le j}\{(x_{i-1}|x_i)\} - 2c - \tau$. In this case, let us consider $x_1, x_j, x_{j+1}, \ldots, x_L$.

We inductively extract a chain (x_i') of length at most $2n \le 2^{k+1}$ which contains x_1 and x_L and such that at most two elements belong to a common X_i, and in this case, they have successive indices. It follows from Lemma B.3 and from above that

$$(x_1|x_L) \ge \min\{(x_{i-1}'|x_i')\} - (k+1)\delta \ge \min\{(x_{i-1}|x_i)\} - (k+1)\delta - 2c - \tau.$$

$\blacksquare$

PROOF OF THEOREM B.1. The theorem follows as soon as we have found a quasi-isometric embedding $\phi : X \to T$ with T 0-hyperbolic.

Lemma B.2 implies that $X/\sim$ is 0-hyperbolic and that $\phi : X \to X/\sim$ satisfies $|\phi(x) - \phi(w)|' = |x - w|$ and $|\phi(x) - \phi(y)|' \le |x - y|$.

For case (i), Lemme B.3 shows that $(x|y) \geq (x|y)' - k\delta$ i.e.,

$$|\phi(x) - \phi(y)|' \geq |x - y| - 2k\delta.$$

For case (ii), Lemme B.4 shows that $(x|y) \geq (x|y)' - (k+1)\delta - 2c - \tau$ i.e.,

$$|\phi(x) - \phi(y)|' \geq |x - y| - 2(k+1)\delta - 4c - 2\tau.$$

■

Visual quasiruling structures. Let (X, d, w) be a hyperbolic space endowed with a quasiruling structure $\mathcal{G}$. We say that $\mathcal{G}$ is *visual* if any pair of points in $X \cup \partial X$ can be joined by a τ-quasiruled (λ, c)-quasigeodesic. If X is a proper space, then any quasiruling structure can be completed into a visual quasiruling structure. Also, if Y is a hyperbolic geodesic proper metric space and $\varphi : Y \to X$ is ruling, then the induced quasiruling structure is also visual. This fact can in particular be applied when Y is a locally finite Cayley graph of a non-elementary hyperbolic group Γ, $(X, d) \in \mathcal{D}(\Gamma)$ and φ is the identity map. Thus one endows (X, d) with a visual quasiruling structure.

Shadows. Let (X, d, w) be a hyperbolic quasiruled space endowed with a visual quasiruling structure $\mathcal{G}$. We already defined the shadow $\mho(x, R)$ in Section 2 as the set of points $a \in \partial X$ such that $(a|x)_w \geq d(w, x) - R$. An alternative definition is: let $\mho_{\mathcal{G}}(x, R)$ be the set of points $a \in \partial X$ such that there is a quasiruler $[w, a) \in \mathcal{G}$ which intersects

$$B(x, R) = \{y \in X \ : \ d(x, y) < R\}.$$

The following holds by applying Theorem B.1, since $\mathcal{G}$ is visual.

Proposition B.5. *Let X be a hyperbolic space endowed with a visual quasiruling structure $\mathcal{G}$. There exist positive constants C, R_0 such that for any $R > R_0$, $a \in \partial X$ and $x \in [w, a) \in \mathcal{G}$,*

$$\mho_{\mathcal{G}}(x, R - C) \subset \mho(x, R) \subset \mho_{\mathcal{G}}(x, R + C).$$

The whole theory of quasiconformal measures for hyperbolic groups acting on geodesic spaces in [13] is based on the existence of approximate trees. Therefore, the same proof as in [13] leads to Theorem 2.3 and Lemma 2.4. Since quasiconformal measures are Ahlfors-regular, the lemma of the shadow also holds for shadows defined by visual quasiruling structures.

Note that, in a hyperbolic space endowed with a visual quasiruling structure, Theorem B.1 implies that the definition of Busemann functions we gave in Section 2 is equivalent to the classical one given below:

Busemann functions. Let us assume that (X, w) is a pointed hyperbolic quasiruled space. Let $a \in \partial X$, $x, y \in X$ and $h : \mathbb{R}_+ \to X$ a quasiruled ray such that $h(0) = y$ and $\lim_\infty h = a$. We define $\beta_a(x, h) \stackrel{\text{def.}}{=} \limsup(|x - h(t)| - |y - h(t)|)$ and

$$\beta_a(x, y) \stackrel{\text{def.}}{=} \sup\{\beta_a(x, h), \text{ with } h \text{ as above}\}.$$

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