# Notes on solving Maxwell equations, part 2, Green's function for stratified media

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# Notes on solving Maxwell equations Part 2: Green's function for stratified media

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## 1 Introduction

In the previous report (part 1), the problem and its governing equations are described and is discarded in this report. The finite element method in part 1, or any other method for that matter, determines the fields in and close to the scatterer (near-field) that is used to construct the fields in the far-field. The goal of part 2 is to find far-field expressions formulated as total fields or the Radar Cross Section (RCS) of the scattered fields. The far-field is calculated from the scatterer problem in the contrast formulation. The scatterer then acts as a radiating object with a known source $J$. Using Green's function theory, the far-field solution is just the convolution of that source with the fundamental solution $G$ to the Maxwell equation.

Without loss of generality, the expressions are formulated in total fields $E$ and $H$. Again, the time convention for the time-harmonic term $\exp(-i\omega t)$ is used, but in contrary to part 1, the quantities are in full dimensions, following closely the notation used by Chew, Balanis and others.

## 2 Green's function

The electric field dyadic Green's function $\bar{G}^E$ in a homogeneous medium is the starting point. It consists of the fundamental solutions to Helmholtz equation, which can be written in a Fourier expansion of plane waves. This expansion allows embedding in a multilayer medium. Finally, the vector potential approach is used to derive the "potential" Green's function $\bar{G}^A$. The latter is a two-step derivation, where the electric and magnetic fields are functions of the vector potentials and involves less evaluation in case of far-field calculations.

## 2.1 Homogeneous medium

Recall the Maxwell equation in homogeneous medium (part 1)

$$\nabla \times \nabla \times E - k^2 E = i\omega\mu J.$$

This is written as three Helmholtz equations (Cartesian coordinates)

$$\nabla^2 E(r) + k^2 E(r) = -i\omega\mu \left[ \bar{I} + \frac{\nabla\nabla}{k^2} \right] J(r), \tag{1}$$

which solution is the convolution

$$E(r) = i\omega\mu \int_\Omega g(r' - r) \left[ \bar{I} + \frac{\nabla'\nabla'}{k^2} \right] J(r') \mathrm{d}r'$$

or (Chew, page 27)

$$E(r) = i\omega\mu \int_\Omega J(r') \left[ \bar{I} + \frac{\nabla'\nabla'}{k^2} \right] g(r' - r) \mathrm{d}r'.$$

with $g(r)$ the fundamental solution to the Helmholtz equation (see next subsection). Alternatively, this can be written as

$$E(r) = i\omega\mu \int_\Omega J(r') \bar{G}^E(r', r) \mathrm{d}r',$$

with the dyadic Green's function $\bar{G}^E(r, r')$ (second rank tensor)

$$\bar{G}^E(r', r) = \left[ \bar{I} + \frac{\nabla'\nabla'}{k^2} \right] g(r' - r),$$

that satisfies the following equation (Chew, page 31)

$$\nabla \times \nabla \times \bar{G}^E(r, r') - k^2 \bar{G}^E(r, r') = \bar{I}\delta(r - r').$$

It can be shown that (Chew, page 28)

$$(\bar{G}^E(r', r))^T = \bar{G}^E(r, r') = \left[ \bar{I} + \frac{\nabla\nabla}{k^2} \right] g(r - r'),$$

So, the convolution of this dyadic Green's function and the source finally is

$$E(r) = i\omega\mu \int_{\Omega} \bar{G}^{E}(r, r')\, J(r')\mathrm{d}r'.$$

This integral is defined properly provided that $r \notin \Omega$ as $\nabla\nabla$ is of order $1/|r-r|^3$ when $r \to r'$ and should de redefined in that case (Chew, page 28).

## 2.2 Helmholtz equation

The fundamental solution to the scalar wave equation or Helmholtz equation

$$(\nabla^2 + k^2)g(r, r') = -\delta(r - r')$$

is

$$g(r, r') = g(r', r) = g(r - r') = \frac{\exp(ik|r - r'|)}{4\pi|r - r'|}.$$

Suppose that the Fourier transform of the solution exists

$$g(r, r') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \tilde{g}(k_x, k_y, k_z) \exp(ik_x(x-x')+ik_y(y-y')+ik_z(z-z'))\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z.$$

The Fourier transform of the Helmholtz equation

$$\int_{-\infty}^{\infty} (k^2-k_x^2-k_y^2-k_z^2)\tilde{g}(k_x, k_y, k_z) \exp(ik_x(x-x')+ik_y(y-y')+ik_z(z-z'))\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z = \int_{-\infty}^{\infty} \exp(ik_x(x - x') + ik_y(y - y') + ik_z(z - z'))\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z$$

implies the Fourier components

$$\tilde{g}(k_x, k_y, k_z) = \frac{1}{k^2 - k_x^2 - k_y^2 - k_z^2}.$$

Eliminating $k_z$, with use of the poles $k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2}$ and Jordan's Lemma (allowing a small loss) results in the Weyl identity (Chew, page 65)

$$\frac{\exp(ik|r-r'|)}{|r-r'|} = \frac{i}{2\pi}\int\limits_{-\infty}^{\infty} \frac{1}{k_z}\exp(ik_x(x-x') + ik_y(y-y') + ik_z|z-z'|)\,\mathrm{d}k_x\mathrm{d}k_y, \tag{2}$$

where $k_z^2 = k^2 - k_x^2 - k_y^2$, $\Im\mathrm{m}(k_z) > 0$ and $\Re\mathrm{e}(k_z) > 0$, for all $k_x$ and $k_y$ in the integration.

The commonly used spectral decomposition of point source solutions can be formulated as Fourier integrals.

## 2.3 Fourier integrals

For short notation we introduce the (shifted) 2D Fourier transforms (cf. Michalski 1990)

$$\tilde{f} = \mathcal{F}\left(f(x-x', y-y')\right) = \int\limits_{-\infty}^{\infty} f(x-x', y-y')\exp(-ik_x(x-x')-ik_y(y-y'))\mathrm{d}x\mathrm{d}y.$$

$$f = \mathcal{F}^{-1}\left(\tilde{f}(k_x, k_y)\right) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} \tilde{f}(k_x, k_y)\exp(ik_x(x-x') + ik_y(y-y'))\mathrm{d}k_x\mathrm{d}k_y,$$

which can be used to switch easily from different representations. By introducing cylindrical coordinates,

$$x - x' = \rho\cos(\phi),\ y - y' = \rho\sin(\phi),$$

$$k_x = k_\rho\cos(\alpha),\ k_y = k_\rho\sin(\alpha),$$

$$\rho = \sqrt{(x-x')^2 + (y-y')^2},\ \phi = \arctan\left(\frac{y-y'}{x-x'}\right),$$

and $k_\rho^2 = k_x^2 + k_y^2$, we can express various inverse Fourier integrals that arise in Sommerfeld-type integrals

$$S_n\left[\tilde{G}(k_\rho)\right](k_\rho\rho) = \frac{1}{2\pi}\int\limits_0^{\infty} \tilde{G}(k_\rho)J_n(k_\rho\rho)k_\rho^{n+1}\mathrm{d}k_\rho, \tag{3}$$

where $J_n$ is the Bessel's function of order $n$. Explicitly, the inverse Fourier transforms are

$$\mathcal{F}^{-1}(\tilde{G}) = G = S_0\left[\tilde{G}\right],$$

$$\mathcal{F}^{-1}(ik_x\tilde{G}) = \frac{\partial}{\partial x}G = -\cos(\phi)S_1\left[\tilde{G}\right],$$

$$\mathcal{F}^{-1}(ik_y\tilde{G}) = \frac{\partial}{\partial y}G = -\sin(\phi)S_1\left[\tilde{G}\right],$$

$$\mathcal{F}^{-1}(k_x^2\tilde{G}) = -\frac{\partial^2}{\partial x^2}G = -\frac{1}{2}\left(\cos(2\phi)S_2\left[\tilde{G}\right] - S_0\left[k_\rho^2\tilde{G}\right]\right),$$

$$\mathcal{F}^{-1}(k_y^2\tilde{G}) = -\frac{\partial^2}{\partial y^2}G = \frac{1}{2}\left(\cos(2\phi)S_2\left[\tilde{G}\right] + S_0\left[k_\rho^2\tilde{G}\right]\right),$$

$$\mathcal{F}^{-1}(k_xk_y\tilde{G}) = -\frac{\partial^2}{\partial x\partial y}G = -\frac{1}{2}\sin(2\phi)S_2\left[\tilde{G}\right].$$

The Fourier component of the Sommerfeld identity are easily obtained

$$\mathcal{F}\left(\frac{\exp(ik|r-r'|)}{4\pi|r-r'|}\right) = \frac{i}{2}\frac{\exp(ik_z|z-z'|)}{k_z}, \tag{4}$$

where the Sommerfeld identity is given by

$$\begin{aligned} 4\pi g(r,r') = \frac{\exp(ik|r-r'|)}{|r-r'|} &= i\int_0^\infty \frac{k_\rho}{k_z}J_0(k_\rho\rho)\exp(ik_z|z-z'|)\,\mathrm{d}k_\rho \\ &= \frac{i}{2}\int_{-\infty}^\infty \frac{k_\rho}{k_z}H_0^{(1)}(k_\rho\rho)\exp(ik_z|z-z'|)\,\mathrm{d}k_\rho. \end{aligned} \tag{5}$$

The physical interpretation is that a spherical wave is expanded as an integral summation of conical (or cylindrical) waves multiplied by a plane wave in $z$-direction (Chew, page 66).

For postprocessing purposes, one may use the standard (non-shifted) Fourier transform $\mathcal{H}$ of $f$ is $\hat{f}$ (hat). It is given by a well-known of the Fourier transform

$$\hat{f}(k_x, k_y; x', y') = \mathcal{H}\left(f(x - x', y - y')\right) = \int_{-\infty}^{\infty} f(x{-}x', y{-}y') \exp(-ik_x x{-}ik_y y) \mathrm{d}x\mathrm{d}y$$

$$= \int_{-\infty}^{\infty} f(x - x', y - y') \exp(-ik_x(x - x') - ik_y(y - y')) \exp(-ik_x x' - ik_y y') \mathrm{d}x\mathrm{d}y$$

$$= \tilde{f}(k_x, k_y; x', y') \exp(-ik_x x' - ik_y y')$$

The spectral representation is used to derive the Green's function in terms of Herzian dipoles where we derive the fields given point sources.

## 2.4 Hertzian dipoles

In 3D, a general source is composed of the three (linear independent) vector sources: the Hertzian dipoles.

Suppose a current source is defined as $J(r) = \hat{\alpha} I \ell \delta(r - r')$, radiating from $r = r'$, $I\ell$ is constant and $\hat{\alpha}$ is the direction of the dipole). The electric and magnetic fields result from the convolution with the Green's function (note that $J(r) = \hat{\alpha} I \ell$ in $r = r'$ and $J(r) = 0$ elsewhere)

$$E(r) = i\omega\mu \left( \bar{I} + \frac{\nabla\nabla}{k^2} \right) \cdot \hat{\alpha} I \ell \frac{\exp(ik|r - r'|)}{4\pi|r - r'|}$$

$$H(r) = \nabla \times \hat{\alpha} I \ell \frac{\exp(ik|r - r'|)}{4\pi|r - r'|}.$$

If there is a spectral representation $\tilde{E}_z(k_\rho, r)$ of $E_z$, the other components can be derived using the Maxwell equations (see Chew, page 76)

$$E(r) = \int_{-\infty}^{\infty} \tilde{E}(k_\rho, r) \mathrm{d}k_\rho, \; H(r) = \int_{-\infty}^{\infty} \tilde{H}(k_\rho, r) \mathrm{d}k_\rho,$$

where the $x$- and $y$-components for $\tilde{E}$ and $\tilde{H}$ are given by

$$\begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \end{pmatrix} = \frac{1}{k_\rho^2} \left[ i \begin{pmatrix} k_x \\ k_y \end{pmatrix} \frac{\partial}{\partial z} \tilde{E}_z + \begin{pmatrix} -k_y \\ k_x \end{pmatrix} \omega\mu \tilde{H}_z \right] \tag{6}$$

$$\begin{pmatrix} \tilde{H}_x \\ \tilde{H}_y \end{pmatrix} = \frac{1}{k_\rho^2}\left[ i \begin{pmatrix} k_x \\ k_y \end{pmatrix} \frac{\partial}{\partial z} \tilde{H}_z - \begin{pmatrix} -k_y \\ k_x \end{pmatrix} \omega\epsilon \tilde{E}_z \right], \tag{7}$$

where $\tilde{E}_z$ and $\tilde{H}_z$ are the spectral components of $E_z$ and $H_z$, respectively.

The next sections list the explicit relations for the field due to the electric dipoles (Chew, page 71); similar results are obtained by duality (Chew, page 74). Recall that the $z$-components determines the TE and TM polarization.

### 2.4.1 Vertical electric dipole (VED)

The VED $\hat{\alpha} = \hat{z}$ gives the components (Chew, page 71), with

$$E_z = \frac{iI\ell}{4\pi\omega\epsilon}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\frac{\exp(ik|r-r'|)}{4\pi|r-r'|},\ H_z = 0. \tag{8}$$

Assuming that the dipole is of unit strength ($I\ell = 1$) and using the Sommerfeld identity (5), the Fourier components of the $z$-components of the electric and magnetic fields are (a function of $k_\rho$)

$$\omega\epsilon\tilde{E}_z = -\frac{1}{2}\frac{k_\rho^2}{k_z}\exp(ik_z|z-z'|),\ \tilde{H}_z = 0.$$

By using (7), the other components are

$$\omega\epsilon\tilde{E}_x = \pm\frac{1}{2}k_x\exp(ik_z|z-z'|),\ \omega\epsilon\tilde{E}_y = \pm\frac{1}{2}k_y\exp(ik_z|z-z'|)$$

$$\tilde{H}_x = \frac{1}{2}\frac{k_y}{k_z}\exp(ik_z|z-z'|),\ \tilde{H}_y = -\frac{1}{2}\frac{k_x}{k_z}\exp(ik_z|z-z'|).$$

### 2.4.2 Horizontal electric dipole (HED)

The HED $\hat{\alpha} = \hat{x}$ is given by the components

$$E_z = \frac{iI\ell}{\omega\epsilon}\frac{\partial^2}{\partial z\partial x}\frac{\exp(ik|r-r'|)}{4\pi|r-r'|},\ H_z = -I\ell\frac{\partial}{\partial y}\frac{\exp(ik|r-r'|)}{4\pi|r-r'|}. \tag{9}$$

Assuming that $I\ell = 1$ and using the Sommerfeld identity (5), the Fourier components are

$$\omega\epsilon\tilde{E}_x = -\frac{1}{2}(k^2 - k_x^2)\frac{1}{k_z}\exp(ik_z|z-z'|),\ \omega\epsilon\tilde{E}_y = \frac{1}{2}k_xk_y\frac{1}{k_z}\exp(ik_z|z-z'|)$$

$$\omega\epsilon\tilde{E}_z = \pm\frac{1}{2}k_x \exp(ik_z|z-z'|)$$

$$\tilde{H}_x = 0,\ \tilde{H}_y = \mp\frac{1}{2}\exp(ik_z|z-z'|)$$

$$\tilde{H}_z = \frac{1}{2}\frac{k_y}{k_z}\exp(ik_z|z-z'|).$$

For the HED in $y$-direction, the $z$-components are

$$E_z = \frac{iI\ell}{\omega\epsilon}\frac{\partial^2}{\partial z\partial y}\frac{\exp(ik|r-r'|)}{4\pi|r-r'|},\ H_z = I\ell\frac{\partial}{\partial x}\frac{\exp(ik|r-r'|)}{4\pi|r-r'|}$$

with the Fourier components

$$\omega\epsilon\tilde{E}_x = \frac{1}{2}k_x k_y\frac{1}{k_z}\exp(ik_z|z-z'|),\ \omega\epsilon\tilde{E}_y = -\frac{1}{2}(k^2-k_y^2)\frac{1}{k_z}\exp(ik_z|z-z'|)$$

$$\omega\epsilon\tilde{E}_z = \pm\frac{1}{2}k_y \exp(ik_z|z-z'|),$$

$$\tilde{H}_x = \mp\frac{1}{2}\exp(ik_z|z-z'|),\ \tilde{H}_y = 0$$

$$\tilde{H}_z = \frac{1}{2}\frac{k_x}{k_z}\exp(ik_z|z-z'|).$$

The $\tilde{E}$ components form the elements of the electric Green's function $\tilde{\bar{G}}^E$. The $\tilde{H}$ is here the magnetic field due to the electric dipoles.

### 2.4.3 Summary

The spectral components of the dyadic Green's function $\bar{G}^E$ for the electric field in an homogeneous medium is simply the 2D Fourier transform of the dyad $\bar{G}^E(r,r') = \left[\bar{I} + \nabla\nabla/k^2\right] g(r-r')$, explicitly

$$\tilde{\bar{G}}^E(k_x,k_y;z,z') = \frac{1}{k^2}\begin{bmatrix} k^2-k_x^2 & -k_x k_y & ik_x\frac{\partial}{\partial z} \\ k_x k_y & k^2-k_y^2 & ik_y\frac{\partial}{\partial z} \\ ik_x\frac{\partial}{\partial z} & ik_y\frac{\partial}{\partial z} & k_\rho^2 \end{bmatrix}\tilde{g}(k_x,k_y;z,z').$$

Given the relation $i\omega\mu\tilde{H} = \tilde{\nabla}\times\tilde{E}$, it can readily be seen that the columns of

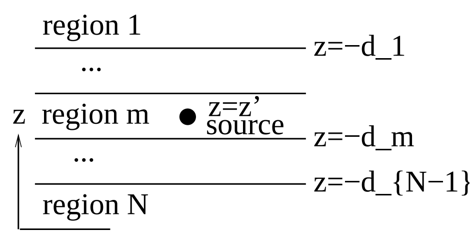


Figure 1: stratified medium

$$\tilde{\nabla} \times \tilde{\bar{G}}^E = \tilde{\nabla} \times \bar{I}\tilde{g} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & ik_y \\ \frac{\partial}{\partial z} & 0 & -ik_x \\ -ik_y & ik_x & 0 \end{bmatrix} \tilde{g}(k_x, k_y; z, z')$$

are the magnetic fields due to the two HEDs and the VED, respectively.

When dipoles are placed in a stratified medium, reflection from the interfaces should be incorporated. The general solution is a linear sum of planar waves, which propagation through stratified media can be calculated explicitly.

## 2.5 Stratified Media

The construction of this (dyadic) Green's function is done via the homogeneous Green's function for homogeneous media and its convolution. The Sommerfeld (5) or Weyl identities (2) expand the solution in planar waves travelling in $z$-direction. The $z$-variation of the solution in free space in $z'$ is given by

$$F(z, z') = \exp(ik_z|z - z'|).$$

Suppose the stratified medium consists of $N$ layers and a (dipole) source is located in medium $m$ (see Figure 1), the solution is constructed inside this medium and the solution outside this medium is constructed recursively.

### 2.5.1 Planar waves

The simplest case of a stratified medium is the half-plane, which consists of two layers or $N = 2$.

An incoming TE planar wave in region 1 is reflected by the layer in region 2 and is written as

$$E^m_{1y}(z) = e_0 \left[\exp(-ik_{1z}z) + R_{12} \exp(2ik_{1z}d_1 + ik_z z)\right].$$

At $z = -d_1$ the field matches to the transmitted wave in region 2

$$E_{2y}^m(z) = e_0 T_{12} \exp(-ik_{2z}z).$$

Using the boundary conditions for the electric and magnetic fields, the Fresnel coefficients are calculated

$$R_{12} = \frac{\mu_2 k_{1z} - \mu_1 k_{2z}}{\mu_2 k_{1z} + \mu_1 k_{2z}},$$

$$T_{12} = \frac{2\mu_2 k_{1z}}{\mu_2 k_{1z} + \mu_1 k_{2z}}.$$

This procedure can be generalized for multiple layers.

### 2.5.2 Generalized reflection coefficient

The $\tilde{R}_{ij}$ are the generalized reflection coefficients in the stratified media. For both TE and TM these are derived from a recursive relation (Chew, pages 52+53)

$$\tilde{R}_{i,i+1} = \frac{R_{i,i+1} + \tilde{R}_{i+1,i+2} \exp(2ik_{i+1,z}(d_{i+1} - d_i))}{1 + R_{i,i+1}\tilde{R}_{i+1,i+2} \exp(2ik_{i+1,z}(d_{i+1} - d_i))},\ \tilde{R}_{N,N+1} = 0,$$

and

$$\tilde{R}_{i,i-1} = \frac{R_{i,i-1} + \tilde{R}_{i-1,i-2} \exp(2ik_{i-1,z}(d_i - d_{i-1}))}{1 + R_{i,i-1}\tilde{R}_{i-1,i-2} \exp(2ik_{i-1,z}(d_i - d_{i-1}))},\ \tilde{R}_{10} = 0,$$

with the Fresnel reflection and transmission coefficients for layers $i$ and $i+1$ as if they would be in half-space (Chew, page 49)

$$R_{i,i+1}^{TE} = \frac{\mu_{i+1} k_{iz} - \mu_i k_{i+1,z}}{\mu_{i+1} k_{iz} + \mu_i k_{i+1,z}},$$

$$R_{i,i+1}^{TM} = \frac{\epsilon_{i+1} k_{iz} - \epsilon_i k_{i+1,z}}{\epsilon_{i+1} k_{iz} + \epsilon_i k_{i+1,z}}.$$

Furthermore, $R_{ji} = -R_{ij}$ and the transmission coefficient is $T_{ij} = 1 + R_{ij}$ and $k_{iz}/\mu_i(1 - R_{ij}) = k_{jz}/\mu_j T_{ij}$.

### 2.5.3 Inside the source region

If the source is embedded in same medium $m$, this wave is accompanied by the reflection from the layer beneath and transmission wave from above. For TE or TM modes, we replace $F$ by

$$F(z,z') = \exp(ik_{mz}|z-z'|) + A_m \exp(-ik_{mz}z) + C_m \exp(ik_{mz}z), \tag{10}$$

where $k_{mz}$ is the $z$-component of the wave vector in region $m$. The two extra terms are reflections at the boundaries $z = -d_{m-1}$ and $z = -d_m$ and constraints can be found for $A_m$ and $C_m$. For $z > z'$ at $z = -d_{m-1}$,

$$A_m \exp(ik_{mz}d_{m-1}) = \tilde{R}_{m,m-1} \left[\exp(ik_{mz}|d_{m-1}+z'|) + C_m \exp(-ik_{mz}d_{m-1})\right]$$

and for $z < z'$ at $z = -d_m$,

$$C_m \exp(-ik_{mz}d_m) = \tilde{R}_{m,m+1} \left[\exp(ik_{mz}|d_m+z'|) + A_m \exp(ik_{mz}d_m)\right],$$

where $\tilde{R}_{ij}$ are the generalized reflection coefficients. Solving $A_m$ and $D_m$ from the two relations gives

$$\begin{aligned} A_m = {} & \exp(-ik_{mz}d_{m-1})\tilde{R}_{m,m-1} \\ & \left[\exp(-ik_{mz}(d_{m-1}+z')) + \tilde{R}_{m,m+1} \exp(ik_{mz}(2d_m - d_{m-1} + z'))\right] \tilde{M}_m \end{aligned}$$

$$\begin{aligned} C_m = {} & \exp(ik_{mz}d_m)\tilde{R}_{m,m+1} \\ & \left[-\exp(ik_{mz}(d_m+z')) + \tilde{R}_{m,m-1} \exp(-ik_{mz}(2d_{m-1} - d_m + z'))\right] \tilde{M}_m, \end{aligned}$$

with

$$\tilde{M}_m = \left[1 - \tilde{R}_{m,m+1}\tilde{R}_{m,m-1} \exp(2ik_{mz}(d_m - d_{m-1})\right]^{-1}.$$

Using this $F$, the contributions by the reflection and transmission waves to the dipoles can be derived.

### 2.5.4 Vertical electric dipole (VED)

If the reflected waves from the top and bottom layers are included and the components of the electric fields are

$$\omega\epsilon\tilde{E}_x = \frac{1}{2}k_x\left[\pm\exp(ik_z|z-z'|) + A_v^e\exp(ik_z z) - B_v^e\exp(-ik_z z)\right]$$

$$\omega\epsilon\tilde{E}_y = \frac{1}{2}k_y\left[\pm\exp(ik_z|z-z'|) + A_v^e\exp(ik_z z) - B_v^e\exp(-ik_z z)\right]$$

$$\omega\epsilon\tilde{E}_z = -\frac{1}{2}\frac{k_\rho^2}{k_z}\left[\exp(ik_z|z-z'|) + A_v^e\exp(ik_z z) + B_v^e\exp(-ik_z z)\right],$$

and the magnetic field components are

$$\tilde{H}_x = -\frac{1}{2}\frac{k_y}{k_z}\left[\exp(ik_z|z-z'|) + A_v^e\exp(ik_z z) + B_v^e\exp(-ik_z z)\right]$$

$$\tilde{H}_y = \frac{1}{2}\frac{k_x}{k_z}\left[\exp(ik_z|z-z'|) + A_v^e\exp(ik_z z) + B_v^e\exp(-ik_z z)\right] \tag{11}$$

$$\tilde{H}_z = 0$$

### 2.5.5 Horizontal electric dipole (HED)

In the stratified medium, the reflected waves from the top and bottom layers are included and the electric field components are

$$\omega\epsilon\tilde{E}_x = -\frac{1}{2}\left[(k^2-k_x^2)\frac{1}{k_z}\exp(ik_z|z-z'|) + \frac{1}{k_\rho^2}\left(-k_x^2k_z^2B_h^e + k_y^2k^2A_h^e\right)\frac{1}{k_z}\exp(-ik_z z) + \frac{1}{k_\rho^2}\left(k_x^2k_z^2D_h^e + k_y^2k^2C_h^e\right)\frac{1}{k_z}\exp(ik_z z)\right]$$

$$\omega\epsilon\tilde{E}_y = \frac{1}{2}k_xk_y\left[\frac{1}{k_z}\exp(ik_z|z-z'|) + \frac{1}{k_\rho^2}\left(k_z^2B_h^e + k^2A_h^e\right)\frac{1}{k_z}\exp(-ik_z z) + \frac{1}{k_\rho^2}\left(-k_z^2D_h^e + k^2C_h^e\right)\frac{1}{k_z}\exp(ik_z z)\right]$$

$$\omega\epsilon\tilde{E}_z = \frac{1}{2}k_x \left[\pm\exp(ik_z|z-z'|) + B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right]$$

and the magnetic field components

$$\tilde{H}_x = \frac{1}{2}\frac{k_x k_y}{k_\rho^2}\left[(A_h^e + B_h^e)\exp(-ik_z z) + (D_h^e - C_h^e)\exp(ik_z z)\right]$$

$$\tilde{H}_y = \mp\frac{1}{2}\exp(ik_z|z-z'|) + \frac{1}{2}\frac{1}{k_\rho^2}\Big[(k_y^2 A_h^e - k_x^2 B_h^e)\exp(-ik_z z) + (-k_y^2 D_h^e - k_x^2 C_h^e)\exp(ik_z z)\Big] \quad (12)$$

$$\tilde{H}_z = \frac{1}{2}\frac{k_y}{k_z}\left[\exp(ik_z|z-z'|) + A_h^e\exp(-ik_z z) + C_h^e\exp(ik_z z)\right]$$

Note that the reflected up-going wave $D_h^e\exp(ik_z z)$ will have a minus sign due to the negative sign of the primary down going field $\exp(ik_z|z-z'|)$ in $\omega\epsilon\tilde{E}_z$.

For the HED in $y$-direction, change the role of $x$ and $y$ gives the $z$-components

$$\omega\epsilon\tilde{E}_z = \frac{1}{2}k_y \left[\pm\exp(ik_z|z-z'|) + B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right]$$

$$\tilde{H}_z = -\frac{1}{2}\frac{k_x}{k_z}\left[\exp(ik_z|z-z'|) + A_h^e\exp(-ik_z z) + C_h^e\exp(ik_z z)\right].$$

Note the sign change in the magnetic field component $H_z = I\ell\frac{\partial}{\partial x}g(r,r')$.

### 2.5.6 Coefficients

The coefficients $A, B, C$ and $D$ represent TM and TE waves as given in the table

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $A_h^e$ | $A_h^m$ | $A_v^e$ | $A_v^m$ | $B_h^e$ | $B_h^m$ | $B_v^e$ | $B_v^m$ |
| TE | TM | TM | TE | TM | TE | TM | TE |
| $C_h^e$ | $C_h^m$ | $C_v^e$ | $C_v^m$ | $D_h^e$ | $D_h^m$ | $D_v^e$ | $D_v^m$ |
| TE | TM | TE | TM | TM | TE | TM | TE |

and are functions of the source location $z'$. In the above notation we have (the superscripts $e$ and $m$ are omitted, $-d_m < z' < -d_{m-1}$)

$$B_v = A_h = \exp(-ik_{mz}d_{m-1})\tilde{R}_{m,m-1}\left[\exp(-ik_{mz}(d_{m-1}+z')) \right. \\ \left. +\tilde{R}_{m,m+1}\exp(ik_{mz}(2d_m - d_{m-1}+z'))\right]\tilde{M}_m \tag{13}$$

$$A_v = C_h = \exp(ik_{mz}d_m)\tilde{R}_{m,m+1}\left[\exp(ik_{mz}(d_m+z')) \right. \\ \left. +\tilde{R}_{m,m-1}\exp(-ik_{mz}(2d_{m-1} - d_m+z'))\right]\tilde{M}_m \tag{14}$$

$$D_v = B_h = \exp(-ik_{mz}d_{m-1})\tilde{R}_{m,m-1}\left[\exp(-ik_{mz}(d_{m-1}+z')) \right. \\ \left. -\tilde{R}_{m,m+1}\exp(ik_{mz}(2d_m - d_{m-1}+z'))\right]\tilde{M}_m \tag{15}$$

$$C_v = D_h = \exp(ik_{mz}d_m)\tilde{R}_{m,m+1}\left[-\exp(ik_{mz}(d_m+z')) \right. \\ \left. +\tilde{R}_{m,m-1}\exp(-ik_{mz}(2d_{m-1} - d_m+z'))\right]\tilde{M}_m. \tag{16}$$

Note the minus signs appearing in the square brackets of coefficients $D_v = B_h$ and $C_v = D_h$. This is due to the minus sign of the down going wave from the HED (and HMD) solutions. Also useful are the relations between the coefficients

$$\frac{\partial}{\partial z'}A_h = -ik_{mz}B_h$$

$$\frac{\partial}{\partial z'}C_h = -ik_{mz}D_h$$

$$\frac{\partial}{\partial z'}B_h = -ik_{mz}A_h$$

$$\frac{\partial}{\partial z'}D_h = -ik_{mz}C_h,$$

$$A_v^e = C_h^m,\ B_v^e = A_h^m,\ D_v^e = B_h^m,\ C_v^e = D_h^m.$$

For $m = 1$, we may also write

$$D_h^e - C_h^e = D_h^m - C_h^m,$$

and for $m = N$,

$$A_h^e + B_h^e = A_h^m + B_h^m.$$

### 2.5.7 Summary

These components build up the electric field dyadic Green's function $\tilde{\bar{G}}^E$ for problem defined on a stratified medium. The next subsection uses the Fourier modes of dipole solutions derived previously in the vector potential approach.

## 2.6 Vector potential approach

In a source free medium, the magnetic flux $B$ and electric flux $D$ are soleniodal, i.e.

$$\nabla \cdot B = 0, \ \nabla \cdot D = 0.$$

Then the arbitrary vectors $A$ and $F$, the potential vectors, exist such that

$$B_A = \mu H_A = \nabla \times A,$$

$$D_F = \epsilon E_F = -\nabla \times F.$$

The definition of the curl above together with the divergence of $A$ and $F$, uniquely define the vector potentials. We choose the Lorentz condition or gauge to define the divergence (Balanis, page 257). Substituting this into the Maxwell equations, the total electric $E = E_A + E_F$ and magnetic $H = H_A + H_F$ fields yield (Balanis, page 260)

$$E = i\omega \left[ \bar{I} + \frac{\nabla\nabla}{k^2} \right] A - \frac{i\omega}{\epsilon} \nabla \times F \tag{17}$$

$$H = i\omega \left[ \bar{I} + \frac{\nabla\nabla}{k^2} \right] F + \frac{i\omega}{\mu} \nabla \times A. \tag{18}$$

So, a field is a result of both vector potentials; the scatterer problem gives the vector potentials in terms of Green's functions and sources and is described in the next sections.

### 2.6.1 Green's function

The Green's function is not unique and one of the commonly used is the "traditional" form for the vector potentials $\bar{G}^A$ and $\bar{G}^F$ (Michalski 1990) and are given by

$$\bar{G}^{A,F} = \begin{bmatrix} G_{xx} & 0 & 0 \\ 0 & G_{xx} & 0 \\ G_{zx} & G_{zy} & G_{zz} \end{bmatrix}. \tag{19}$$

For the $x$- and $y$-horizontal dipoles only two components per column (direction) need to be specified (Sommerfeld, page 257). Using this form, the following integrals define vector potentials for volume sources, with $J$ the electric field density and $M$ the (non-physical) magnetic field density, are defined (Balanis, page 276)

$$A = \int_V \bar{G}^A(r, r')\, J(r')\mathrm{d}V' \tag{20}$$

$$F = \int_V \bar{G}^F(r, r')\, M(r')\mathrm{d}V'. \tag{21}$$

For a radiating surface $S$ (in free space) with linear densities $J_S$ and $M_S$

$$A = \oint_S \bar{G}^A(r, r')\, J_S(r')\mathrm{d}S' \tag{22}$$

$$F = \oint_S \bar{G}^F(r, r')\, M_S(r')\mathrm{d}S'. \tag{23}$$

For magnetic and electric currents $I_e$ and $I_m$ these reduce to line integrals over $C$

$$A = \int_C \bar{G}^A(r, r')\, I_e(r')\mathrm{d}l'$$

$$F = \int_C \bar{G}^F(r, r')\, I_m(r')\mathrm{d}l'.$$

The direct relation between the electric/magnetic Green's function from the previous section and vector potential Green's functions are

$$\mu\, \bar{G}^E(r, r') = \left[\bar{I} + \frac{\nabla\nabla}{k^2}\right] \bar{G}^A(r, r'). \tag{24}$$

$$\epsilon\, \bar{G}^H(r, r') = \left[\bar{I} + \frac{\nabla\nabla}{k^2}\right] \bar{G}^F(r, r'). \tag{25}$$

In the next section, the elements in the vector potential Green's function are derived for an observer at $z$ and source at $z'$ situated in the same layer. The solution outside the source layer is determined in the Appendix.

### 2.6.2 Observer inside source layer

By using VED, HED, VMD and HMD dipoles the contributing elements to dyadic $\bar{G}^{A,F}$ can be constructed.

Use the $TE_z$- and $TM_z$-components of the fields generated by the dipoles and derive the components as explained in the previous section. Then look for a vector potential $A$ in spectral domain such that

$$\nabla \times A = \mu H_A$$

or in spectral form

$$\begin{bmatrix} ik_y\tilde{A}_z - \frac{\partial}{\partial z}\tilde{A}_y \\ -\left(ik_x\tilde{A}_z - \frac{\partial}{\partial z}\tilde{A}_x\right) \\ ik_x\tilde{A}_y - ik_y\tilde{A}_x \end{bmatrix} = \begin{bmatrix} \mu\tilde{H}_x \\ \mu\tilde{H}_y \\ \mu\tilde{H}_z \end{bmatrix}.$$

The vector potential $A$ only needs two components to be specified. The result will be the first column in the dyadic $\bar{G}^A$.

For a HED in $x$-direction, the spectral components of the magnetic field is given by (12). The vector potential components for a HED become, with the choice $\tilde{A}_y = 0$,

$$ik_y\tilde{A}_z = \mu\tilde{H}_x, \; -ik_y\tilde{A}_x = \mu\tilde{H}_z$$

or

$$\tilde{A}_x = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + A_h^e\exp(-ik_z z) + C_h^e\exp(ik_z z)\right],$$

$$\tilde{A}_y = 0,$$

$$\tilde{A}_z = \frac{\mu}{2i}\frac{k_x}{k_\rho^2}\left[(A_h^e + B_h^e)\exp(-ik_z z) + (D_h^e - C_h^e)\exp(ik_z z)\right].$$

The HED in $y$-direction one chooses $\tilde{A}_x = 0$, which implies

$$ik_x\tilde{A}_y = \mu\tilde{H}_z, \; -ik_x\tilde{A}_z = \mu\tilde{H}_y$$

or

$$\tilde{A}_x = 0,$$

$$\tilde{A}_y = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + A_h^e \exp(-ik_z z) + C_h^e \exp(ik_z z)\right],$$

$$\tilde{A}_z = \frac{\mu}{2i}\frac{k_y}{k_\rho^2}\left[(A_h^e + B_h^e)\exp(-ik_z z) + (D_h^e - C_h^e)\exp(ik_z z)\right],$$

which is the same as switching role $x$ and $y$ in the derivation for the HED in $x$-direction.

For a VED the magnetic components are given by (11). Thus, the choice $\tilde{A}_y = 0$ in the potential vector results in

$$\tilde{A}_x = 0,\ -ik_z\tilde{A}_z = \mu\tilde{H}_y,\ ik_y\tilde{A}_z = \mu\tilde{H}_x$$

or

$$\tilde{A}_x = 0,$$

$$\tilde{A}_y = 0,$$

$$\tilde{A}_z = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + A_v^e \exp(ik_z z) + B_v^e \exp(-ik_z z)\right].$$

All components of the vector potential Green's functions $\tilde{G}^{A,F}$ can be obtained (cf. Dural & Aksun 1995)

$$\tilde{G}_{xx}^A = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + A_h^e \exp(-ik_z z) + C_h^e \exp(ik_z z)\right]$$

$$\tilde{G}_{zx}^A = \frac{\mu}{2ik_z}\left[\frac{k_x k_z}{k_\rho^2}(A_h^e + B_h^e)\exp(-ik_z z) + \frac{k_x k_z}{k_\rho^2}(D_h^e - C_h^e)\exp(ik_z z)\right]$$

$$\tilde{G}_{zz}^A = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + A_v^e \exp(ik_z z) + B_v^e \exp(-ik_z z)\right]$$

$$\tilde{G}_{xx}^F = -\frac{\epsilon}{2ik_z}\left[\exp(ik_z|z-z'|) + A_h^m \exp(-ik_z z) + C_h^m \exp(ik_z z)\right]$$

$$\tilde{G}_{zx}^F = \frac{\epsilon}{2ik_z}\left[\frac{k_x k_z}{k_\rho^2}(A_h^m + B_h^m)\exp(-ik_z z) + \frac{k_x k_z}{k_\rho^2}(D_h^m - C_h^m)\exp(ik_z z)\right]$$

$$\tilde{G}^F_{zz} = -\frac{\epsilon}{2ik_z}\left[\exp(ik_z|z-z'|) + A^m_v \exp(ik_z z) + B^m_v \exp(-ik_z z)\right].$$

The elements from the $y$-oriented HED and HMD follow directly by rotation in the $x, y$-plane

$$\tilde{G}^{A,F}_{yy} = \tilde{G}^{A,F}_{xx}, \quad \tilde{G}^{A,F}_{zy}/k_y = \tilde{G}^{A,F}_{zx}/k_x.$$

The spatial Green's function are obtained using the inverse Fourier transforms

$$G^A_{xx} = \mathcal{F}^{-1}(\tilde{G}) = S_0[\tilde{G}] = S_0[\tilde{G}^A_{xx}] = \frac{1}{2\pi}\int_0^\infty \tilde{G}^A_{xx} J_0(k_\rho \rho) k_\rho \mathrm{d}k_\rho$$

$$= -\frac{\mu}{4\pi i}\int_0^\infty \left[\exp(ik_z|z-z'|) + A^e_h \exp(-ik_z z) + C^e_h \exp(ik_z z)\right]\frac{k_\rho}{k_z} J_0(k_\rho \rho)\mathrm{d}k_\rho$$

$$= \mu\frac{\exp(ik|r-r'|)}{4\pi|r-r'|} - \frac{\mu}{4\pi i}\int_0^\infty \left[A^e_h \exp(-ik_z z) + C^e_h \exp(ik_z z)\right]\frac{k_\rho}{k_z} J_0(k_\rho \rho)\mathrm{d}k_\rho$$

$$G^A_{zx} = \mathcal{F}^{-1}(ik_x\tilde{G}) = -\cos(\phi) S_1[\tilde{G}] = -\cos(\phi) S_1\left[\frac{\tilde{G}^A_{zx}}{ik_x}\right] = -\cos(\phi)\frac{1}{2\pi}\int_0^\infty \frac{\tilde{G}^A_{zx}}{ik_x} J_1(k_\rho \rho) k^2_\rho \mathrm{d}k_\rho$$

$$= \frac{\mu}{4\pi}\cos(\phi)\int_0^\infty \left[(A^e_h + B^e_h)\exp(-ik_z z) + (D^e_h - C^e_h)\exp(ik_z z)\right] J_1(k_\rho \rho)\mathrm{d}k_\rho,$$

$$G^A_{zz} = \mathcal{F}^{-1}(\tilde{G}) = S_0[\tilde{G}] = S_0[\tilde{G}^A_{zz}] = \frac{1}{2\pi}\int_0^\infty \tilde{G}^A_{zz} J_0(k_\rho \rho) k_\rho \mathrm{d}k_\rho$$

$$= -\frac{\mu}{4\pi i}\int_0^\infty \left[\exp(ik_z|z-z'|) + A^e_v \exp(ik_z z) + B^e_v \exp(-ik_z z)\right]\frac{k_\rho}{k_z} J_0(k_\rho \rho)\mathrm{d}k_\rho$$

$$= \mu\frac{\exp(ik|r-r'|)}{4\pi|r-r'|} - \frac{\mu}{4\pi i}\int_0^\infty \left[A^e_v \exp(ik_z z) + B^e_v \exp(-ik_z z)\right]\frac{k_\rho}{k_z} J_0(k_\rho \rho)\mathrm{d}k_\rho.$$

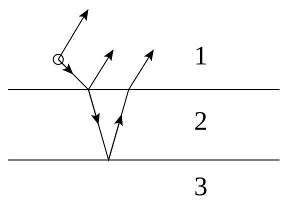


Figure 2: Waves reflection

### 2.6.3 Example

In the example problem $N = 3$, where sources are in region 1 and 2. The interfaces are located at $z = -d_1$ and $z = -d_2$.

First, a source at $z = z'$ in region 1 ($m = 1$ and $n = 1$). The generalized reflection coefficient is

$$\tilde{R}_{12} = \frac{R_{12} + R_{23}\exp(2ik_{2z}(d_2 - d_1))}{1 + R_{12}R_{23}\exp(2ik_{2z}(d_2 - d_1))}$$

with $k_{2z} = \sqrt{k_2^2 - k_\rho^2}$ (and $\tilde{M}_1 = 1$). The spectral Green's functions become

$$\tilde{G}^A_{xx} = -\frac{\mu_1}{2ik_{1z}}\left[\exp(ik_{1z}|z - z'|) + C^e_h \exp(ik_{1z}z)\right]$$

$$\tilde{G}^A_{zx} = \frac{\mu_1}{2ik_{1z}}\left[\frac{k_x k_{1z}}{k_\rho^2}(D^e_h - C^e_h)\exp(ik_{1z}z)\right]$$

$$\tilde{G}^A_{zz} = -\frac{\mu_1}{2ik_{1z}}\left[\exp(ik_{1z}|z - z'|) + A^e_v \exp(ik_{1z}z)\right]$$

$$\tilde{G}^F_{xx} = -\frac{\epsilon_1}{2ik_{1z}}\left[\exp(ik_{1z}|z - z'|) + C^m_h \exp(ik_{1z}z)\right]$$

$$\tilde{G}^F_{zx} = \frac{\epsilon_1}{2ik_{1z}}\left[\frac{k_x k_{1z}}{k_\rho^2}(D^m_h - C^m_h)\exp(ik_{1z}z)\right]$$

$$\tilde{G}^F_{zz} = -\frac{\epsilon_1}{2ik_{1z}}\left[\exp(ik_{1z}|z - z'|) + A^m_v \exp(ik_{1z}z)\right],$$

with coefficients (either TE or TM)

$$A_v = C_h = \tilde{R}_{12}\exp(ik_{1z}(2d_1 + z'))$$

$$D_h = -\tilde{R}_{12} \exp(ik_{1z}(2d_1 + z')).$$

To obtain the closed-form spatial Green's functions by doing the Fourier inverses (3). The construction of the solution for observers outside the source layer is described in the Appendix.

Now, we have relations for the fields as function of the sources applied in the whole domain. We use these to calculate the fields in the desired subdomain.

#### 2.6.4 Summary

The electric Green's function is obtained using the dipole solution directly (sections 2.5.4 and 2.5.5) or by applying (24) to the vector potential Green's function

$$\mu\tilde{\tilde{G}}^E = \frac{1}{k^2} \begin{bmatrix} k^2 - k_x^2 & -k_x k_y & ik_x \frac{\partial}{\partial z} \\ -k_x k_y & k^2 - k_x^2 & ik_y \frac{\partial}{\partial z} \\ ik_x \frac{\partial}{\partial z} & ik_y \frac{\partial}{\partial z} & k^2 - k_z^2 \end{bmatrix} \tilde{\tilde{G}}^A$$

and similarly $\epsilon\tilde{\tilde{G}}^H = 1/k^2[\cdots]\tilde{\tilde{G}}^F$ by duality.

# 3 Far-field reconstruction

The ingredient for the calculation of RCS for a scatterer is a far-field description of the electric field $E$.

The far-field is constructed using Green's functions for a given electric and magnetic currents on the scatterer. The idea is to evaluate or approximate the convolutions to obtain the far-field quantities as function of the currents in the scatterer.

## 3.1 Radar Cross Section

The bi-static RCS is defined in terms of the time-averaged Poynting vector

$$RCS(\theta, \phi) \equiv \lim_{r \to \infty} \left[ 4\pi r^2 \frac{P_{scat}}{P_{inc}} \right], \tag{26}$$

where $P_i = \frac{1}{2}\Re e(E \times H^*)$ is the power density. A property of solutions for point sources in homogeneous media is the $\exp(ikr)/r$ behavior in the far field. Furthermore, it can be shown that $P_i = |E_\theta^i|^2 + |E_\phi^i|^2$ for the far-field, where the radial components $E_r$ and $H_r$ vanish (Balanis, page 280).

For 2D problems, the bistatic scattering width is defined as

$$SW(\theta) = \lim_{r\to\infty}\left[2\pi r\frac{|E^s|^2}{|E^i|^2}\right].$$

The observation points far from the scatterer, $r \geq 2D^2/\lambda$, defines the far-field (Fraunhofer) region, where $D$ is the maximal dimension of the scatterer and $\lambda$ is the wavelength . In this region the maximum phase error is $\pi/8 \sim 22.5^\circ$ (Balanis, page 286). The limit is the minimum distance for the RCS to be valid. Another limit defines the near-field $r \leq 0.62\sqrt{D^3/\lambda^3}$ (Fresnel).

For $r \to \infty$ in a homogeneous medium, the fields are orthogonal to the radial direction (plane waves and $\nabla\nabla \sim 1/r^2$); we may write in terms of vector potentials $A$ and $F$ (Balanis, page 285).

# 4 Numerical evaluation

All representations involve (multiple) integrals over infinite (frequency) domains of rapid oscillating functions containing singularities. Also the sheer number of integrals, for each location requires a new evaluation, makes the evaluation of the solution costly. Straightforward numerical integration requires many sample points or do not even converge due to singularities.

The next subsections describe techniques to deal with the problems. The path in the Sommerfeld intergrals can be extended to the complex plane to avoid singularities. Some singularities can be subtracted analytically. Finally, the integrand can be approximated by exponentials, which integral is derived analytical.

## 4.1 Sommerfeld integration path (SIP)

Sommerfeld integrals can be evaluated using Cauchy theorem. The integral

$$I = \int_0^\infty f(x)\mathrm{d}x$$

is an integration of complex function $f$ along the real $x$-axis. Suppose the function $f$ has a singularity at $x = x_0$. A new contour $C$ is parametrized by $s(t)$, with $s(0) = 0$ and $s(\infty) = \infty$ can be defined and

$$I = \int_C f(z)\mathrm{d}z = \int_0^\infty f(s(t))\frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t.$$

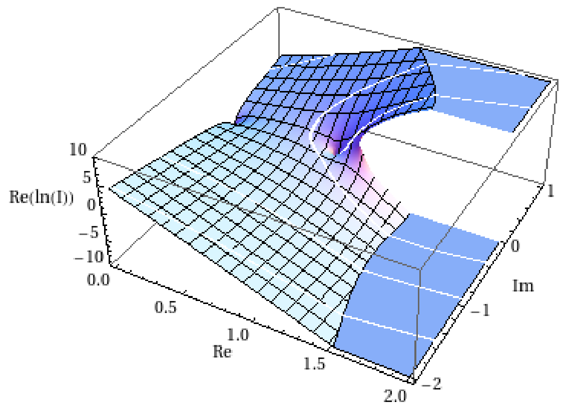


Figure 3: Real part of ln of a typical integrand.

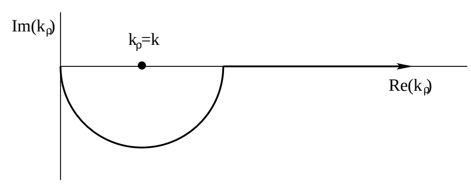


Figure 4: FSIP, dot represents a pole.

For semi-infinite Sommerfeld integrals, a pole is at $k_\rho = k$. In this context, the contour $s(t)$ is the so-called Sommerfeld integration path (SIP) or folded SIP (FSIP).

## 4.2 Singularity subtraction

For sources and observation point are close, the integrand in the Sommerfeld integrals can be highly oscillatory for large frequencies and singular in the limit. However, we have (Sommerfeld)

$$\int_0^\infty \frac{\exp(ik_z\alpha)}{k_z} J_0(k_\rho\rho)\mathrm{d}k_\rho = \frac{\exp(ikr)}{r},\ r = \sqrt{\rho^2+\alpha^2}$$

$$\int_0^\infty \exp(k_\rho\alpha) J_1(k_\rho\rho)\mathrm{d}k_\rho = \frac{1}{\rho}\left(1+\frac{\alpha}{\sqrt{\rho^2+\alpha^2}}\right).$$

Take the limit of the integrand for $k_\rho \to \infty$ and use the above identities to add and subtract the singularity from the integrand.

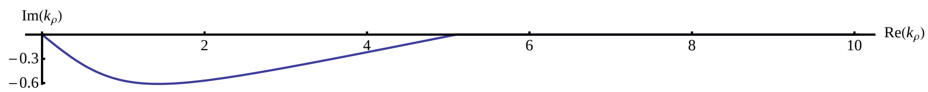


Figure 5: Deformed SIP for GPOF method

### 4.3 GPOF

The general pencil-of-function method (GPOF) can be used to approximate the integrand in the Sommerfeld integrals. The idea is to approximate the Fourier components, for given $z$ and $z'$, as function of $k_\rho$ in a sum of exponentials, i.e

$$\tilde{G}(k_\rho(t))|_{z,z'} = -\frac{\mu}{2ik_z}\sum_{i=1}^{M} b_i \exp(s_i t)$$

where $t$ is the running variable in a linear parametrization of $k_z(t)/k = \alpha + \beta t$ and by using $k_\rho(t) = \sqrt{k^2 - k_z^2(t)}$.

Then we can write as a function of $k_z(k_\rho) = \sqrt{k^2 - k_\rho^2}$

$$\tilde{G}(k_\rho)|_{z,z'} = -\frac{\mu}{2ik_z}\sum_{i=1}^{M} b_i \exp\left(s_i \frac{k_z(k_\rho)/k - \alpha}{\beta}\right) = \\ -\frac{\mu}{2ik_z}\sum_{i=1}^{M} b_i \exp\left(-s_i\frac{\alpha}{\beta}\right)\exp\left(ik_z(k_\rho)\frac{s_i}{i\beta}\right).$$

Use the Sommerfeld identity here to get a closed-form 3D Greens' function

$$G(r,r')|_{z,z'} = \mathcal{F}^{-1}(\tilde{G}) = \mu\sum_{i=1}^{M} b_i \exp\left(-s_i\frac{\alpha}{\beta}\right)\frac{\exp(ikr_i)}{4\pi r_i}$$

with $r_i = \sqrt{(\rho-\rho')^2 - (s_i/\beta)^2}$ (which are complex functions).

The residues $b_i$ and the poles $s_i$ are obtained using GPOF method (Cheng & Yang, 2005).

The approximation can be improved by taking two piecewise linear parametrizations of the SIP in the complex $k_z$ plane. One starting from $k$ going to the positive imaginary axis and the other is on the positive imaginary axis. This is the two-level DCIM (Aksun 1996, Aksun 2005).

The $G^A_{zx}$ and $G^A_{zy}$ elements are approximated in the same way. We have $\mathcal{F}^{-1}(\tilde{G}^A_{zx}/ik_x) = \int G^A_{zx}\mathrm{d}x$, when using $G^{\mathrm{prim}}(r) = \exp(ikr)/(4\pi r)$, we get

$$G^A_{zx} = \frac{\partial}{\partial x}\left(\mathcal{F}^{-1}\left(\frac{\tilde{G}_{zx}}{ik_x}\right)\right) = \mu \sum_{i=1}^{M} b_i \exp\left(-s_i \frac{\alpha}{\beta}\right) \frac{\mathrm{d}G^{\mathrm{prim}}}{\mathrm{d}r} \frac{\partial r_i}{\partial x},$$

with

$$\frac{\mathrm{d}G^{\mathrm{prim}}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\exp(ikr)}{4\pi r}\right) = (ikr-1)\frac{\exp(ikr)}{4\pi r^2}.$$

For second-order derivatives (e.g. in $G^E_{xx}$ and $G^E_{yy}$)

$$\frac{\partial^2 G^{\mathrm{prim}}}{\partial x^2} = \frac{\mathrm{d}^2 G^{\mathrm{prim}}}{\mathrm{d}r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\mathrm{d}G^{\mathrm{prim}}}{\mathrm{d}r}\frac{\partial^2 r}{\partial x^2}$$

and mixed derivatives (e.g. in $G^E_{xy}$)

$$\frac{\partial^2 G^{\mathrm{prim}}}{\partial x \partial y} = \frac{\mathrm{d}^2 G^{\mathrm{prim}}}{\mathrm{d}r^2}\frac{\partial r}{\partial x}\frac{\partial r}{\partial y} + \frac{\mathrm{d}G^{\mathrm{prim}}}{\mathrm{d}r}\frac{\partial^2 r}{\partial x \partial y}$$

both using

$$\frac{\mathrm{d}^2 G^{\mathrm{prim}}}{\mathrm{d}r^2} = \left((ikr-1)^2+1\right)\frac{\exp(ikr)}{4\pi r^3}.$$

The partial derivatives of the radius $r = r(x, y)$ are

$$\begin{aligned} \tfrac{\partial r}{\partial x} &= \tfrac{x-x'}{r} \\ \tfrac{\partial^2 r}{\partial x^2} &= \tfrac{1}{r} - \tfrac{(x-x')^2}{r^3} \\ \tfrac{\partial^2 r}{\partial x \partial y} &= -(x-x')(y-y')\tfrac{1}{r^3}. \end{aligned}$$

The derivatives $\partial/\partial x(\mathcal{F}^{-1}(...))$ can be done in a straightforward fashion (either symbolically or numerically).

## 4.4 Summary

Assuming that the current density $J(r')$ in a domain $\Omega'$ is given, the solution to the Maxwell equations can be constructed by the following steps in the vector potential approach (no need for the equivalence principle). The main ingredient is the Green's function $\bar{G}^A$.

#### 4.4.1 Green's function

1. Each column in the (dyadic) Green's function (19) is derived from the solutions for 2 HEDs and a VED, using equations (9) and (8).
2. These solutions are expanded in planar waves in $z$-direction using Fourier transformations (3), resulting in Sommerfeld type integrals.
3. The stratified medium is included by adding the reflected waves from the adjacent layers as in (10), which are either TE or TM waves.
4. The vector potential $A$ for the electric field is the integral (20).
5. The field $E$ outside that domain is evaluated by relations (17) and (18).

The resulting expression for the total field can be evaluated numerically.

#### 4.4.2 Numerical evaluation

1. Define observer points in which the $\bar{I} + \nabla\nabla/k^2$ operator can be approximated by finite difference (central difference and bi-linear interpolation).
2. Do volume integration of the potential vector (20) involving the product $\bar{G}^A(r, r') \cdot J(r')$ for the components $(\bar{G}^A \cdot J)_x$, $(\bar{G}^A \cdot J)_y$ and $(\bar{G}^A \cdot J)_z$ for each observer point (adaptive rules like Romberg or Filon integration; Gauss-Chebychev or Clenshaw-Curtis integration rules).
3. Evaluate the Sommerfeld integrals contained in $\bar{G}^A$ using a SIP as depicted in Figure 4 (integration rules for slowly decaying, highly oscillating integrands) or create closed-form approximations using GPOF method.

The evaluation of $\nabla\nabla$-operator can be avoided by using the Green's function for the electric field $\bar{G}^E$. This involves more Sommerfeld integrals to be evaluated.

# 5 2D problem

The relations for the full 3D Maxwell equations reduce when the solution is (semi)constant in one direction (say $y$-direction). The derivations in the next section follow the same recipe as in the 3D case.

## 5.1 Homogeneous medium

The Green's function is a dyadic function and has the same functionality as in 3D. However, the fundamental solution $g(r, r')$ is the solution of the 2D Helmholtz equation.

## 5.2 Helmholtz equation

The fundamental solution to the 2D Helmholtz equation,

$$(\nabla^2 + k^2)g(r, r') = -\delta(x - x')\delta(z - z')$$

is

$$g(r, r') = \frac{i}{4} H_0^{(1)}(k|r - r'|).$$

Suppose that the Fourier transform $\tilde{g}$ of the solution exists

$$g(r, r') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(k_x, k_z) \exp(ik_x(x - x') + ik_z(z - z'))\mathrm{d}k_x \mathrm{d}k_z.$$

The Fourier transform of the 2D Helmholtz equation

$$\int_{-\infty}^{\infty} (k^2 - k_x^2 - k_z^2)\tilde{g}(k_x, k_z) \exp(ik_x(x - x') + ik_z(z - z'))\mathrm{d}k_x \mathrm{d}k_z = \int_{-\infty}^{\infty} \exp(ik_x(x - x') + ik_z(z - z'))\mathrm{d}k_x \mathrm{d}k_z$$

implies the Fourier components

$$\tilde{g}(k_x, k_z) = \frac{1}{k^2 - k_x^2 - k_z^2}$$

eliminating $k_z$ with poles $k_z = \pm\sqrt{k^2 - k_x^2}$ and using Jordan's Lemma (allowing a small loss) gives

$$g(r, r') = \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{\exp(ik_x(x - x') + ik_z|z - z'|)}{k_z} \mathrm{d}k_x =$$

$$\frac{i}{2\pi} \int_{0}^{\infty} \frac{\exp(ik_z|z - z'|)}{k_z} \cos(k_x(x - x'))\mathrm{d}k_x$$

with $k_z = \sqrt{k^2 - k_x^2}$. Here we recognize the $F(z, z') = \exp(ik_z|z - z'|)$ term, which allows the use of equation (10) and the other following derivations.

## 5.3 Fourier integrals

The inverse Fourier transform

$$\mathcal{F}^{-1}(\tilde{G}) = \frac{1}{\pi}\int_0^\infty \tilde{G}(k_x)\cos(k_x(x-x'))\mathrm{d}k_x,$$

$$\mathcal{F}^{-1}(ik_x\tilde{G}) = \frac{\partial}{\partial x}G = -\frac{1}{\pi}\int_0^\infty \tilde{G}(k_x)\sin(k_x(x-x'))k_x\mathrm{d}k_x,$$

$$\mathcal{F}^{-1}(k_x^2\tilde{G}) = -\frac{\partial^2}{\partial x^2}G = \frac{1}{\pi}\int_0^\infty \tilde{G}(k_x)\cos(k_x(x-x'))k_x^2\mathrm{d}k_x.$$

The Fourier transform of the fundamental solution is (cf. Weyl identity)

$$\mathcal{F}\left(\frac{i}{4}H_0^{(1)}(k|r-r'|)\right) = \frac{i}{2}\frac{\exp(ik_z|z-z'|)}{k_z}. \tag{27}$$

The next subsections discusses the oblique incidence case (TE/TM) and the conical case (general).

## 5.4 Hertzian dipoles

The 2D solution satisfies the 3D Maxwell equations. So, a dipole at $r = r'$ (is actually a line source at $x = x'$ and $z = z'$) gives

$$E(r) = i\omega\mu\left(\bar{I} + \frac{\nabla\nabla}{k^2}\right)\hat{\alpha}I\ell\frac{i}{4}H_0^{(1)}(k|r-r'|)$$

$$H(r) = \nabla\times\hat{\alpha}I\ell\frac{i}{4}H_0^{(1)}(k|r-r'|).$$

We use the dipoles to derive the columns of the dyadic function. Suppose $I\ell = 1$, then the HED $\hat{\alpha} = \hat{x}$ (e.g. $J(r) = \hat{x}I\ell\delta(r-r')$) gives the TM and TE components

$$E_z = \frac{i}{\omega\epsilon}\frac{\partial^2}{\partial z\partial x}\frac{i}{4}H_0^{(1)}(k|r-r'|),\ H_z = 0,$$

the HED $\hat{\alpha} = \hat{y}$ gives the TM and TE components

$$E_z = 0,\ H_z = \frac{\partial}{\partial x}\frac{i}{4}H_0^{(1)}(k|r - r'|),$$

and the VED $\hat{\alpha} = \hat{z}$ gives the TM and TE components

$$E_z = \frac{i}{\omega\epsilon}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\frac{i}{4}H_0^{(1)}(k|r - r'|),\ H_z = 0.$$

From these relations, the spectral $z$-components are easily extracted. The other components are derived from (6) and (7), with $k_\rho = k_x$, or

$$\tilde{E}_x = \frac{i}{k_x}\frac{\partial}{\partial z}\tilde{E}_z,\ \tilde{E}_y = \frac{1}{k_x}\omega\mu\tilde{H}_z$$

$$\tilde{H}_x = \frac{i}{k_x}\frac{\partial}{\partial z}\tilde{H}_z,\ \tilde{H}_y = -\frac{1}{k_x}\omega\epsilon\tilde{E}_z$$

For the $\hat{\alpha} = \hat{x}$, the components are (TM)

$$\omega\epsilon\tilde{E}_x = -\frac{1}{2}k_z \exp(ik_z|z - z'|), \tilde{E}_y = 0,$$

$$\omega\epsilon\tilde{E}_z = \pm\frac{1}{2}k_x \exp(ik_z|z - z'|)$$

$$\tilde{H}_x = 0,\ \tilde{H}_y = \mp\frac{1}{2}\exp(ik_z|z - z'|),\ \tilde{H}_z = 0$$

for $\hat{\alpha} = \hat{y}$, (TE)

$$\tilde{E}_x = 0, \omega\epsilon\tilde{E}_y = -\frac{1}{2}\frac{k^2}{k_z}\exp(ik_z|z - z'|), \tilde{E}_z = 0,$$

$$\tilde{H}_x = \pm\frac{1}{2}\exp(ik_z|z - z'|),\ \tilde{H}_y = 0,$$

$$\tilde{H}_z = -\frac{1}{2}\frac{k_x}{k_z}\exp(ik_z|z - z'|)$$

and for $\hat{\alpha} = \hat{z}$, (TM)

$$\omega\epsilon\tilde{E}_x = \pm\frac{1}{2}k_x \exp(ik_z|z - z'|), \tilde{E}_y = 0,$$

$$\omega\epsilon\tilde{E}_z = -\frac{1}{2}\frac{k_x^2}{k_z}\exp(ik_z|z-z'|)$$

$$\tilde{H}_x = 0,\ \tilde{H}_y = \frac{1}{2}\frac{k_x}{k_z}\exp(ik_z|z-z'|), \tilde{H}_z = 0.$$

These are columns of the electric and magnetic dyadic Green's function.

## 5.5 Vector potential

The 2D Green's function $\bar{G}^A$ for stratified media will be (call it the traditional form)

$$\bar{G}^A = \begin{bmatrix} G^A_{xx} & 0 & 0 \\ 0 & G^A_{yy} & 0 \\ 0 & 0 & G^A_{zz} \end{bmatrix}.$$

For the construction of the vector potential we have to satisfy

$$\mu H_A = \nabla \times A$$

or the spectral representation

$$\begin{bmatrix} -\frac{\partial}{\partial z}\tilde{A}_y \\ -\left(ik_x\tilde{A}_z - \frac{\partial}{\partial z}\tilde{A}_x\right) \\ ik_x\tilde{A}_y \end{bmatrix} = \begin{bmatrix} \mu\tilde{H}_x \\ \mu\tilde{H}_y \\ \mu\tilde{H}_z \end{bmatrix}.$$

For HED $\hat{\alpha} = \hat{x}$, from $\tilde{H}_x = \tilde{H}_z = 0$ directly follows $\tilde{A}_y = 0$ and with the choice $\tilde{A}_z = 0$ we have

$$\tilde{A}_x = -\frac{\mu}{2ik_z}\exp(ik_z|z-z'|),\ \tilde{A}_y = \tilde{A}_z = 0.$$

HED $\hat{\alpha} = \hat{y}$, from $\tilde{H}_y = 0$ with the choice $\tilde{A}_z = 0$ follows

$$\tilde{A}_x = 0,\ \tilde{A}_y = -\frac{\mu}{2ik_z}\exp(ik_z|z-z'|),\ \tilde{A}_z = 0.$$

VED $\hat{\alpha} = \hat{z}$, from $\tilde{H}_x = 0$ (and $\tilde{H}_z = 0$) that $\tilde{A}_y = 0$ with the choice $\tilde{A}_x = 0$

$$\tilde{A}_x = \tilde{A}_y = 0,\ \tilde{A}_z = -\frac{\mu}{2ik_z}\exp(ik_z|z-z'|).$$

## 5.6 Stratified media

Adding the stratified media to the expressions do not add extra elements to the dyad $\tilde{G}^A$.

For HED $\hat{\alpha} = \hat{x}$ the previous results are replaced by (TM)

$$\omega\epsilon\tilde{E}_x = -\frac{1}{2}k_z\left[\exp(ik_z|z-z'|) - B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right]$$

$$\tilde{E}_y = 0$$

$$\omega\epsilon\tilde{E}_z = \frac{1}{2}k_x\left[\pm\exp(ik_z|z-z'|) + B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right],$$

$$\tilde{H}_x = 0$$

$$\tilde{H}_y = -\frac{1}{2}\left[\pm\exp(ik_z|z-z'|) + B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right]$$

$$\tilde{H}_z = 0$$

thus with $\frac{\partial}{\partial z}\tilde{A}_x = -\mu\frac{1}{k_x}\omega\epsilon\tilde{E}_z$ we get

$$\tilde{A}_x = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) - B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right].$$

The $\tilde{A}_x$ is the first element of the first column in the dyad (TM)

$$\tilde{G}^A_{xx} = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right].$$

The other components are derived in a similar way (TE and TM, respectively)

$$\tilde{G}^A_{yy} = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + A_h^e\exp(-ik_z z) + C_h^e\exp(ik_z z)\right],$$

$$\tilde{G}^A_{zz} = -\frac{\mu}{2ik_z}\left[\exp(ik_z|z-z'|) + A_v^e\exp(ik_z z) + B_v^e\exp(-ik_z z)\right].$$

Applying the inverse Fourier transform finally gives

$$G_{xx}^A = \mathcal{F}^{-1}(\tilde{G}_{xx}^A) = \frac{1}{\pi}\int\limits_0^\infty \tilde{G}_{xx}^A(k_x)\cos(k_x(x-x'))\mathrm{d}k_x =$$

$$-\frac{\mu}{2\pi i}\int\limits_0^\infty \left[\exp(ik_z|z-z'|) + B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right]\frac{1}{k_z}\cos(k_x(x-x'))\mathrm{d}k_x =$$

$$\mu\frac{i}{4}H_0^{(1)}(k|r-r'|) - \frac{\mu}{2\pi i}\int\limits_0^\infty \left[B_h^e\exp(-ik_z z) + D_h^e\exp(ik_z z)\right]\frac{1}{k_z}\cos(k_x(x-x'))\mathrm{d}k_x,$$

$$G_{yy}^A = \mathcal{F}^{-1}(\tilde{G}_{yy}^A) = \frac{1}{\pi}\int\limits_0^\infty \tilde{G}_{yy}^A(k_x)\cos(k_x(x-x'))\mathrm{d}k_x =$$

$$-\frac{\mu}{2\pi i}\int\limits_0^\infty \left[\exp(ik_z|z-z'|) + A_h^e\exp(-ik_z z) + C_h^e\exp(ik_z z)\right]\frac{1}{k_z}\cos(k_x(x-x'))\mathrm{d}k_x =$$

$$\mu\frac{i}{4}H_0^{(1)}(k|r-r'|) - \frac{\mu}{2\pi i}\int\limits_0^\infty \left[A_h^e\exp(-ik_z z) + C_h^e\exp(ik_z z)\right]\frac{1}{k_z}\cos(k_x(x-x'))\mathrm{d}k_x,$$

$$G_{zz}^A = \mathcal{F}^{-1}(\tilde{G}_{zz}^A) = \frac{1}{\pi}\int\limits_0^\infty \tilde{G}_{zz}^A(k_x)\cos(k_x(x-x'))\mathrm{d}k_x =$$

$$-\frac{\mu}{2\pi i}\int\limits_0^\infty \left[\exp(ik_z|z-z'|) + A_v^e\exp(ik_z z) + B_v^e\exp(-ik_z z)\right]\frac{1}{k_z}\cos(k_x(x-x'))\mathrm{d}k_x =$$

$$\mu\frac{i}{4}H_0^{(1)}(k|r-r'|) - \frac{\mu}{2\pi i}\int\limits_0^\infty \left[A_v^e\exp(ik_z z) + B_v^e\exp(-ik_z z)\right]\frac{1}{k_z}\cos(k_x(x-x'))\mathrm{d}k_x.$$

The coefficients $B_h^e$, $D_h^e$, $A_h^e$, $C_h^e$, $A_v^e$ and $C_v^e$ are solved at the boundaries of the source region and are given by relations (13) to (16).

In a similar way the total fields for a multi-layer medium are derived. For $\hat{\alpha} = \hat{y}$, we get (TE)

$$\tilde{E}_x = 0$$

$$\omega\epsilon\tilde{E}_y = -\frac{1}{2}\frac{k^2}{k_z}\left[\exp(ik_z|z-z'|) + A_h^e\exp(-ik_z z) + C_h^e\exp(ik_z z)\right]$$

$$\tilde{E}_z = 0,$$

$$\tilde{H}_x = \frac{1}{2}\left[\pm \exp(ik_z|z-z'|) - A_h^e \exp(-ik_z z) + C_h^e \exp(ik_z z)\right]$$

$$\tilde{H}_y = 0$$

$$\tilde{H}_z = -\frac{1}{2}\frac{k_x}{k_z}\left[\exp(ik_z|z-z'|) + A_h^e \exp(-ik_z z) + C_h^e \exp(ik_z z)\right]$$

For $\hat{\alpha} = \hat{z}$, (TM)

$$\omega\epsilon\tilde{E}_x = \frac{1}{2}k_x\left[\pm \exp(ik_z|z-z'|) + A_v^e \exp(ik_z z) - B_v^e \exp(-ik_z z)\right]$$

$$\tilde{E}_y = 0$$

$$\omega\epsilon\tilde{E}_z = -\frac{1}{2}\frac{k_x^2}{k_z}\left[\exp(ik_z|z-z'|) + A_v^e \exp(ik_z z) + B_v^e \exp(-ik_z z)\right],$$

$$\tilde{H}_x = 0$$

$$\tilde{H}_y = \frac{1}{2}\frac{k_x}{k_z}\left[\exp(ik_z|z-z'|) + A_v^e \exp(ik_z z) + B_v^e \exp(-ik_z z)\right]$$

$$\tilde{H}_z = 0.$$

The columns in dyadic Green's function $\tilde{\bar{G}}^E$ are the corresponding dipole solutions $\tilde{E}$. The non-zeros in this dyadic for the electric field are

$$\tilde{G}_{xx}^E\left(= \frac{i}{k_x}\frac{\partial}{\partial z}\tilde{G}_{zx}^E\right) = -\frac{1}{k^2}\frac{1}{2i}k_z\left[\exp(ik_z|z-z'|) - B_h^e \exp(-ik_z z) + D_h^e \exp(ik_z z)\right]$$

$$\tilde{G}_{zx}^E = \frac{1}{k^2}\frac{1}{2i}k_x\left[\pm \exp(ik_z|z-z'|) + B_h^e \exp(-ik_z z) + D_h^e \exp(ik_z z)\right]$$

$$\tilde{G}_{yy}^E = -\frac{1}{2i}\frac{1}{k_z}\left[\exp(ik_z|z-z'|) + A_h^e \exp(-ik_z z) + C_h^e \exp(ik_z z)\right]$$

$$\tilde{G}^E_{xz}\left(=\frac{i}{k_x}\frac{\partial}{\partial z}\tilde{G}^E_{zz}\right)=\frac{1}{k^2}\frac{1}{2i}k_x\left[\pm\exp(ik_z|z-z'|)+A^e_v\exp(ik_z z)-B^e_v\exp(-ik_z z)\right]$$

$$\tilde{G}^E_{zz}=-\frac{1}{k^2}\frac{1}{2i}\frac{k_x^2}{k_z}\left[\exp(ik_z|z-z'|)+A^e_v\exp(ik_z z)+B^e_v\exp(-ik_z z)\right].$$

Note again that $G^A \sim \mu G^E \sim E/(i\omega\mu)$ or $\mu G^E \sim \omega\epsilon E/ik^2$. This is also obtained by applying (24) to the vector potential Green's function, giving the expression

$$\mu\tilde{\bar{G}}^E=\frac{1}{k^2}\left[\begin{array}{ccc} k^2-k_x^2 & 0 & ik_x\frac{\partial}{\partial z} \\ 0 & k^2 & 0 \\ ik_x\frac{\partial}{\partial z} & 0 & k^2-k_z^2 \end{array}\right]\tilde{\bar{G}}^A.$$

Here you can see that for the $\mathrm{TM}_z$ case, only the $G_{yy}$ component is needed.

### 5.7 Conical case

In the conical case, the solution has the same $y$-dependency as the incoming field, namely $g(y)=\exp(ik_y y)$. We need a reformulation of the Maxwell equation to get the 2D equation for this case. Start with (1) and solve this with the separation of variables $E(x,y,z)=\hat{E}(x,z)g(y)$ and $J(x,y,z)=\hat{J}(x,z)g(y)$. The $\nabla$-operator reduces to $\hat{\nabla}=[\partial_x+ik_y+\partial_z]^T$. So

$$\nabla^2 E(r)+k^2E(r)=-i\omega\mu\left[\bar{I}+\frac{\nabla\nabla}{k^2}\right]J(r)$$

becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\hat{E}(x,z)+\hat{k}^2\hat{E}(x,z)=-i\omega\mu\left[\bar{I}+\frac{\hat{\nabla}\hat{\nabla}}{k^2}\right]\hat{J}(x,z),$$

with $\hat{k}^2=k^2-k_y^2$ and

$$\hat{\nabla}\hat{\nabla}=\left[\begin{array}{ccc} \partial_x^2 & ik_y\partial_x & \partial_{xz}^2 \\ ik_y\partial_x & -k_y^2 & ik_y\partial_z \\ \partial_{zx}^2 & ik_y\partial_z & \partial_z^2 \end{array}\right].$$

This again leads to the solution using 2D Green's function (free space)

$$\hat{\tilde{G}}^E(r,r') = \left[\bar{I} + \frac{\hat{\nabla}\hat{\nabla}}{k^2}\right]\hat{g}(r,r'),$$

with $\hat{g}(r,r')$ is the fundamental solution to the 2D Helmholtz equation

$$\hat{g}(r,r') = \frac{i}{4}H_0^{(1)}\left(\hat{k}^2|r-r'|\right) = \frac{i}{2\pi}\int_0^\infty \frac{\exp(ik_z|z-z'|)}{k_z}\cos(k_x(x-x'))\mathrm{d}k_x$$

and $k_z = \sqrt{\hat{k}^2 - k_x^2}$. The stratified media is added as done before and the relation between the field and vector potential Green's function is given by

$$\mu\hat{\tilde{G}}^E = \frac{1}{k^2}\begin{bmatrix} k^2-k_x^2 & -k_yk_x & ik_x\frac{\partial}{\partial z} \\ -k_yk_x & k^2-k_y^2 & ik_y\frac{\partial}{\partial z} \\ ik_x\frac{\partial}{\partial z} & ik_y\frac{\partial}{\partial z} & k^2-k_z^2 \end{bmatrix}\hat{\tilde{G}}^A. \tag{28}$$

Note that $k_y$ is a constant fixed by the incident wave. Following the definition in part 1, set $k_y = k_0\sin(\theta)\sin(\phi)$.

## 5.8 GPOF

The 2D Green's functions are easily obtained by substituting the term $G^{\text{prim}}(r) = (i/4)H_0^{(1)}(kr)$ in which the radius reduces to $\rho - \rho' = x - x'$. For the derivatives use

$$\frac{\mathrm{d}G^{\text{prim}}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{i}{4}H_0^{(1)}(kr)\right) = -\frac{ik}{4}H_1^{(1)}(kr)$$

and

$$\frac{\mathrm{d}^2G^{\text{prim}}}{\mathrm{d}r^2} = -\frac{ik^2}{8}\left(H_0^{(1)}(kr) - H_2^{(1)}(kr)\right).$$

For the conical case we apply (28) to obtain the field Green's function. In this case 7 function need to be approximated, i.e. $\hat{\tilde{G}}_{xx}^A$, $\hat{\tilde{G}}_{yy}^A$, $\hat{\tilde{G}}_{zz}^A$, $\frac{\partial}{\partial z}\hat{\tilde{G}}_{xx}^A$, $\frac{\partial}{\partial z}\hat{\tilde{G}}_{yy}^A$, $\frac{\partial}{\partial z}\hat{\tilde{G}}_{zz}^A$ and $\frac{\partial^2}{\partial z^2}\hat{\tilde{G}}_{zz}^A$, or if the primary term is subtracted

$$\hat{\tilde{G}}_{xx}^A - \hat{\tilde{G}}^{\text{prim}} = -\frac{\mu}{2i}\frac{1}{k_z}\left[B_h^e\exp(-ik_zz) + D_h^e\exp(ik_zz)\right]$$

$$\tilde{\hat{G}}_{yy}^{A} - \tilde{\hat{G}}^{\text{prim}} = -\frac{\mu}{2i}\frac{1}{k_z}\left[A_h^e \exp(-ik_z z) + B_h^e \exp(ik_z z)\right]$$

$$\tilde{\hat{G}}_{zz}^{A} - \tilde{\hat{G}}^{\text{prim}} = -\frac{\mu}{2i}\frac{1}{k_z}\left[A_v^e \exp(ik_z z) + B_v^e \exp(-ik_z z)\right]$$

$$\frac{\partial}{\partial z}\left(\tilde{\hat{G}}_{xx}^{A} - \tilde{\hat{G}}^{\text{prim}}\right) = -\frac{\mu}{2}\left[-B_h^e \exp(-ik_z z) + D_h^e \exp(ik_z z)\right]$$

$$\frac{\partial}{\partial z}\left(\tilde{\hat{G}}_{yy}^{A} - \tilde{\hat{G}}^{\text{prim}}\right) = -\frac{\mu}{2}\left[-A_h^e \exp(-ik_z z) + B_h^e \exp(ik_z z)\right]$$

$$\frac{\partial}{\partial z}\left(\tilde{\hat{G}}_{zz}^{A} - \tilde{\hat{G}}^{\text{prim}}\right) = -\frac{\mu}{2}\left[A_v^e \exp(ik_z z) - B_v^e \exp(-ik_z z)\right]$$

$$\frac{\partial^2}{\partial z^2}\left(\tilde{\hat{G}}_{zz}^{A} - \tilde{\hat{G}}^{\text{prim}}\right) = \frac{\mu}{2i}k_z\left[A_v^e \exp(ik_z z) + B_v^e \exp(-ik_z z)\right].$$

The GPOF sum can directly be transformed to the closed-form Green's functions using identity (27). The $x$-derivatives are applied analytically to these functions using the derivatives of the primary Green's functions as described above, with $k = \hat{k}$.

## 5.9 Far-field

For the 2D case you can find the expressions in the book of Allen Taflove (Taflove 2000), where the Hankel functions are approximated.

In order to obtain the leading order far-field spectral field, we can use a stationary-phase approximation. This gives a relation of the spectral field at infinity in a fixed direction from the origin. Suppose the electric field at $z \geq 0$ component can be written as

$$E(x,z) = \int_{-\infty}^{+\infty} \hat{E}(k_x;0)\exp(ik_x x + ik_z z)\mathrm{d}k_x.$$

We are interested in the far-field value of this field at $r_\infty$ in the direction

$$s = \left(\frac{x}{r}, \frac{z}{r}\right),$$

where $r = \sqrt{x^2+z^2}$ is the distance from $r_\infty$ to the origin. So, we may write

$$E_\infty(s_x, s_z) = \int\limits_{|k_x|\le k} \hat{E}(k_x;0)\exp\left(ikr\left(\frac{k_x}{k}s_x + \frac{k_z}{k}s_z\right)\right)\mathrm{d}k_x.$$

if the evanescent waves decay at infinity (homogeneous waves). Changing variable $p = k_x/k$ we then have

$$E_\infty(s_x, s_z) = \int\limits_{|p|\le 1} k\hat{E}(kp;0)\exp\left(ikr\left(ps_x + m(p)s_z\right)\right)\mathrm{d}p,$$

with $m(p) = \sqrt{1-p^2}$. Now, we have the far-field $E_\infty$ written in the desired form. Given a direction $s = (s_x, s_z)$ (remember that $s_z = \sqrt{1-s_x^2}$), approximate the function

$$F(\kappa; s_x) = \int\limits_{-1}^{1} a(p)\exp\left(i\kappa g(p;s_x)\right)\mathrm{d}p,$$

with $F(\kappa; s_x) = E_\infty(s_x, s_z)$, $\kappa = kr$, $a(p) = k\hat{E}(kp;0)$ and $g(p; s_x) = ps_x + ms_z$, using the stationary-phase method as $\kappa \to \infty$. It can be shown that the critical points of the second kind (end points of the integration path) are of order $1/\kappa$. Let we continue with the critical points of the first kind, thus the points $p_1$ where $g'(p_1) = 0$. We have

$$g'(p) = s_x + m's_z = s_x - \frac{p}{m}s_z$$

$$g''(p) = -s_z\left(\frac{m - pm'}{m^2}\right) = -\frac{s_z}{m}\left(1 + \frac{p^2}{m^2}\right).$$

The (first and only) critical point is at $p_1 = s_x$ and concequently $m_1 = s_z$. So,

$$g''(p_1) = -\left[1 + \left(\frac{s_x}{s_z}\right)^2\right] = -\frac{1}{s_z^2} < 0$$

and we have the approximation, as $\kappa \to \infty$

$$F^{(1)}(\kappa; s_x) \sim \sqrt{\frac{-2\pi}{\kappa g''(p_1)}}a(p_1)\exp(i\kappa g(p_1))\exp\left(-\frac{i\pi}{4}\right).$$

The electric field is then given by

$$E_\infty(s_x, s_z) \sim \sqrt{s_z^2 \frac{2\pi}{kr}} k\hat{E}(ks_x; 0) \exp(ikr) \exp\left(-\frac{i\pi}{4}\right)$$

where the far-field approximation of the Hankel function $H_0^{(1)}(kr)$ can be recognized, or

$$E_\infty(s_x, s_z) \sim ks_z\pi\hat{E}(ks_x; 0)H_0^{(1)}(kr).$$

For a given direction $s$, the mode $k_x = ks_x$ ($k_z = ks_z$) is the only contribution for $r \to \infty$ and is given by

$$\hat{E}(k_x; 0) = \frac{E_\infty(k_x/k, k_z/k)}{\pi k_z H_0^{(1)}(kr)}.$$

Plugging this back into the Fourier transformation we started with gives

$$E(x, z) = \frac{1}{\pi H_0^{(1)}(kr)} \int_{|k_x| \leq k} E_\infty(k_x/k, k_z/k) \frac{1}{k_z} \exp(ik_x x + ik_z z) \mathrm{d}k_x.$$

For a analytical solution to a problem, the far-field $E_\infty$ should be derived. For instance, the general solution to a conducting or dielectric cylinder

$$E(x, z) = \sum_{n=-\infty}^{n=\infty} A_n H_n^{(1)}(kr) \exp(in\phi)$$

using the far-field aproximation for the Hankel functions, we arrive at

$$\hat{E}(k_x; 0) = \frac{\sum A_n \sqrt{\frac{2}{\pi kr}} \exp(i(kr - \pi/4)) \exp(-in\pi/2) \exp(in\phi)}{\pi k_z \sqrt{\frac{2}{\pi kr}} \exp(i(kr - \pi/4))}$$

This simplifies to

$$\hat{E}(k_x; 0) = \frac{1}{\pi k_z} \sum_{n=-\infty}^{n=\infty} A_n \exp(in(\pi/2 - \phi)).$$

Higher-order terms can be obtained using the Method of Steepest Descent (Chew, page 82).

# 6 1D problem

For the sake of competeness, the one-dimensional Helmholtz equation has the trivial solution

$$g(z, z') = \frac{i}{2} \frac{\exp(k|z - z'|)}{k}.$$

Using this in the derivation of the vector potential, the (free space) Green's function reduces to

$$\mu \bar{G}^E(z, z') = \bar{G}^A(z, z') = \mu \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} g(z, z'),$$

which convolution gives $\text{TEM}_z$ waves ($H_z = E_z = 0$).

Inclusion of multiple layers follows the same way as in 2D and 3D. Note that $k_z \equiv k$ is the only wave number present in a 1D problem. Basically, $g(z, z')$ is a single Fourier mode.

Also for the 1D problem the oblique and conical case the Green's functions can be derived in the same way.

# 7 Integral equation

The Green's function convolutions give rise to integral equations (integro-differential equations) for the electric fields (EFIE and MPIE). The equations can be solved using for instance the Method of Moments (MoM).

## 7.1 Equivalence principle

Huygens' principle or surface equivalence theorem uses the Gauss' divergence theorem to replace the volume integral with a integral over surface $S$ and $S_{\text{inf}}$.

Assuming that the solution and Green's functions vanish at $S_{\text{inf}} \to \infty$, we may write (Chew, page 32)

$$E(r') = -\oint_S n \times E(r) \cdot \nabla' \times \bar{G}^E(r, r') - i\omega\mu(n \times H(r)) \cdot \bar{G}^E(r, r')\text{d}S.$$

It can be shown (Chew, page 32) that this is equivalent to

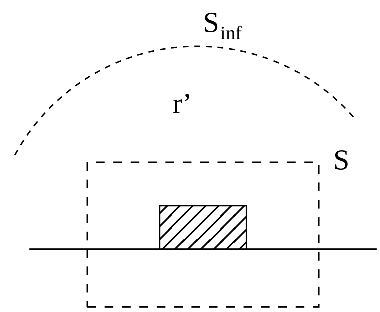


Figure 6: Equivalence principle.

$$E(r') = -\nabla' \times \oint_S \bar{G}^H(r', r) \cdot n \times E(r) \mathrm{d}S + i\omega\mu \oint_S \bar{G}^E(r', r) \cdot n \times H(r) \mathrm{d}S,$$

which is similar to the potentional approach (22) and (23), with $J_S = n \times H$ and $M_S = -n \times E$ (use $\nabla \times \nabla\nabla/k^2 = 0$, relations (24) and (25)). From the uniqueness theorem (Chew, page 32), the solution can be derived either from given $J_S$ or $M_S$, provided that the Green's functions satisfy the boundary conditions $\nabla' \times \bar{G}^H = 0$ or $\bar{G}^E = 0$ on the surface $S$, respectively. For an unbounded homogeneous medium $\bar{G}^H = \bar{G}^E$.

### 7.2 EFIE

The Green's function can be defined via the contrast field $E^{\mathrm{c}} = E - E^{\mathrm{m}}$, where $E^{\mathrm{m}}$ is the known background solution. This field satifies the Maxwell equation

$$\nabla \times \nabla \times E^{\mathrm{c}} - k^2 E^{\mathrm{c}} = k^2 E^{\mathrm{m}} - \nabla \times \nabla \times E^{\mathrm{m}}$$

or

$$\nabla \times \nabla \times E^{\mathrm{c}} - k_{\mathrm{m}}^2 E^{\mathrm{c}} = (k^2 - k_{\mathrm{m}}^2)(E^{\mathrm{m}} + E^{\mathrm{c}}),$$

where $k_m$ is the wave number in the stratified media (without the scatterer). Using the Green's function $\bar{G}^E$ for multi-layer media we obtain the following electric field integral equation (EFIE) for $E^{\mathrm{c}}$ (use $i\omega\mu J = (k^2 - k_{\mathrm{m}}^2)(E^{\mathrm{m}} + E^{\mathrm{c}})$)

$$E^{\mathrm{c}}(r) = \int_V \bar{G}^E(r, r')(k_{\mathrm{res}}^2 - k_0^2)(E^{\mathrm{m}}(r') + E^{\mathrm{c}}(r'))dr',$$

with $V$ is the scattering object area, $k_{\mathrm{res}}$ and $k_0$ constant material properties in the scatterer ($k^2 - k_m^2$ vanishes outside this area). Rearranging terms gives

$$\frac{E^{\mathrm{c}}(r)}{k_{\mathrm{res}}^2 - k_0^2} - \int_V \bar{G}^E(r,r')E^{\mathrm{c}}(r')\mathrm{d}r' = \int_V \bar{G}^E(r,r')E^{\mathrm{m}}(r')\mathrm{d}r'.$$

The 2D TE$_z$ half plane problem, with $E^{\mathrm{m}} = \hat{y}E_y^{\mathrm{m}}$, simplifies the EFIE considerably. In this case, $\bar{G}^E = \bar{G}^A/\mu$ resulting in

$$\frac{E_y^{\mathrm{c}}(r)}{k_{\mathrm{res}}^2 - k_0^2} - \frac{1}{\mu}\int_V G_{yy}^A(r,r')E_y^{\mathrm{c}}(r')\mathrm{d}r' = \frac{1}{\mu}\int_V G_{yy}^A(r,r')E_y^{\mathrm{m}}(r')\mathrm{d}r',$$

Take $E_y^{\mathrm{c}}(r) = \sum_{n=1}^N a_n g_n(r)$ and $N$ test functions $w_m(r) = \delta(r - r_m)$, with $r_m$ the locations in the center of segment $S_m$. Note that this method of moments is now similar to the point-matching method. The basis functions are piecewise constant $g_n(r) = 1$ for $r$ in segment $S_n$ and zero outside. $E_y^{\mathrm{m}}(r)$ is the background plane wave solution for the problem without scatterer (normal incident)

$$E_y^{\mathrm{m}}(r) = \exp(-ik_{1z}z) + R_{12}\exp(2ik_{1z}d_1 + ik_{1z}z)$$

which is evaluated in all $r_m$ and approximated by $E_y^{\mathrm{m}}(r) = \sum_{m=1}^N E_y^{\mathrm{m}}(r_m)g_m(r)$. The EFIE can be written as a linear system $[Z_{mn}][I_n] = [V_m]$

$$Z_{mn} = \frac{1}{k_{\mathrm{res}}^2 - k_0^2} - A_{mn},\ V_m = A\,E_y^{\mathrm{m}}(r_m),\ I_n = a_n,$$

with $A_{mn} = \frac{1}{\mu}\int_{S_n} G_{yy}^A(r_m,r')dr'$.

## 7.3 MPIE

The Mixed Potential Integral Equation (MPIE) is a reformulation of the EFIE and is given by

$$E(r) = i\omega\left[\int_V \bar{G}^A(r,r')\,J(r')\mathrm{d}r + \frac{1}{k^2}\nabla\phi^A(r)\right],$$

with

$$\phi^A(r) = \int_V \left[\nabla\cdot\bar{G}^A(r,r')\right]J(r')\mathrm{d}r$$

The equation of continuity states that $\nabla \cdot J = i\omega q$. By assuming that there exist a scalar function $K_\phi^A(r,r')$ and a vector function $P^A(r,r')$, such that

$$\nabla \cdot \bar{G}^A(r,r') = \nabla' K_\phi^A(r,r') + P^A(r,r'),$$

then (provided that the scatterer does not intersect with the interfaces, Michalski 1990)

$$\begin{aligned}\phi^A(r) &= \int_V K_\phi^A(r,r')\,\nabla' \cdot J(r')\mathrm{d}r' + \int_V P^A(r,r') \cdot J(r')\mathrm{d}r' \\ &= i\omega \int_V K_\phi^A(r,r')\,q(r')\mathrm{d}r' + \int_V P^A(r,r') \cdot J(r')\mathrm{d}r.\end{aligned}$$

Add $\nabla P^A$ to $\bar{G}^A$ to obtain an alternative dyadic Green's function $\bar{K}^A = \bar{G}^A + \nabla P^A$. Together with the scalar potential kernel $K_\phi^A$, the solution can be written as the mixed field integral equation

$$E(r) = i\omega \left[ \int_V \bar{K}^A(r,r')\,J(r')\mathrm{d}r + \frac{i\omega}{k^2} \nabla \int_V K_\phi^A(r,r')\,q(r')\mathrm{d}r' \right].$$

The choice $P_x^A = P_y^A = 0$ result in the "alternative" Green's function and has all the continuity properties in $z$ and $z'$ at the interfaces (Michalski 1990).

# 8 Appendix

In the appendix, various problems are presented which have known analytical solutions. Furthermore, the Green's function for observers that are not in the same layer as the sources is given. At the end, some useful identities are given.

## 8.1 Problems

The following problems can be solved analytically and are used to validate the methods.

### 8.1.1 Conducting cylinder

The perfectly conducting cylinder is oriented in the direction of the $z$-axis (Balanis, page 603 converted to $\exp(-i\omega t)$ convention). Define the total electric field as $E^{\mathrm{t}} = E^{\mathrm{i}} + E^{\mathrm{s}}$, with

$$E^{\mathrm{i}} = \hat{z}E_z^{\mathrm{i}} = \hat{z}E_0 \exp(ikx) = \hat{z}E_0 \sum_{n=-\infty}^{\infty} i^n J_n(k\rho) \exp(in\phi).$$

The boundary condition for PEC, $n \times E^{\mathrm{t}} = 0$ implies $E_z^{\mathrm{t}} = 0$, and the outgoing waves are combination of Hankel functions $H_n^{(1)}$

$$E_z^{\mathrm{s}} = \sum_{n=-\infty}^{\infty} c_n H_n^{(1)}(k\rho).$$

Then, for all $n$ at the boundary $\rho = a$,

$$E_0 i^n J_n(ka) \exp(in\phi) + c_n H_n^{(1)}(ka) = 0.$$

Solving coefficients $c_n$

$$c_n = -i^n E_0 \frac{J_n(ka)}{H_n^{(1)}(ka)} \exp(in\phi)$$

The total field is

$$E_z^{\mathrm{s}}(\phi, \rho) = -E_0 \sum_{n=-\infty}^{\infty} \frac{i^n J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(k\rho) \exp(in\phi)$$

The induced current is (Balanis, page 605)

$$J_S(\phi) = \hat{z} \frac{2E_0}{\pi a \omega \mu} \sum_{n=-\infty}^{\infty} \frac{i^n \exp(in\phi)}{H_n^{(1)}(ka)}$$

The vector potential $A$ for this problem is defined as

$$A = \hat{z}\mu \int_0^{2\pi} \frac{i}{4} H_0^{(1)}(k|r - r'|) J_S(\phi') a \, \mathrm{d}\phi'.$$

Note that from the vector potential approach $E^{\mathrm{s}} = i\omega(I + \nabla\nabla/k^2)A = i\omega A$ as in this problem the vector $A$ only has a $z$-component.

### 8.1.2 Dielectric cylinder

The material inside the cylinder with radius $a$ has refraction index $n_2$ and outside the cylinder $n_1$, with the corresponding wave numbers $k_i = 2\pi n_i/\lambda$. The the scattered field is given by ($\exp(i\omega t)$-convention)

$$E_y^{\mathrm{s}}(\rho, \phi) = \sum_{n=-\infty}^{\infty} A_n H_n^{(2)}(k_1\rho)\exp(in\phi)$$

$$A_n = -B_n^{\mathrm{ext}}/C_n$$

$$B_n^{\mathrm{ext}} = i^n \exp\left[-in\left(\frac{\pi}{2}+\theta\right)\right]$$

$$C_n = \frac{H_n^{(2)}(k_1a)J_n'(k_2a) - H_n^{(2)}{}'(k_1a)J_n(k_2a)}{J_n(k_1a)J_n'(k_2a) - J_n'(k_1a)J_n(k_2a)}.$$

## 8.2 Green's function for the stratified medium

The case that the observer is outside the source region is presented here.

### 8.2.1 Outside source region

The solution has amplitudes that are related to the amplitudes on the boundary of the source region (Chew, page 79).

If $z' \in R_m$ and $z \in R_n$, $n < m$ (above the source), the up going field can be written as

$$F_+(z, z') = A_n^+\left[\exp(ik_{nz}z) + \tilde{R}_{n,n-1}\exp(-2ik_{nz}d_{n-1} - ik_{nz}z)\right]$$

Factor $A_n^+$ is related to the amplitude at the upper boundary of the source region $z = -d_{m-1}$

$$A_m^+ = \exp(-ik_{mz}(z' + d_{m-1})) + C_m\exp(-ik_{mz}d_{m-1}) + A_m\exp(ik_{mz}d_{m-1})$$

or

$$A_m^+ = \left[\exp(-ik_{mz}z') + \exp(ik_{mz}(z' + 2d_m))\tilde{R}_{m,m+1}\right]\exp(-ik_{mz}d_{m-1})\tilde{M}_m$$

in a recursive way as

$$A_i^+ \exp(ik_{iz}d_i) = A_{i+1}^+ \exp(ik_{i+1,z}d_i) S_{i+1,i}^+ \tag{29}$$

$$S_{i+1,i}^+ = \frac{T_{i+1,i}}{1 - R_{i,i+1}\tilde{R}_{i,i-1}\exp(2ik_{iz}(d_i - d_{i-1}))}.$$

Similarly, if $z' \in R_m$ and $z \in R_n$, $n > m$ (beneath source), we have the down going wave

$$F_-(z,z') = A_n^- \left[\exp(-ik_{nz}z) + \tilde{R}_{n,n+1}\exp(2ik_{nz}d_n + ik_{nz}z)\right].$$

Factor $A_n^-$ is related to the amplitude at the lower source region boundary $z = -d_m$

$$A_m^- = \exp(ik_{mz}(z' + d_m)) + A_m \exp(ik_{mz}d_m) + C_m \exp(2ik_{mz}d_m)$$

or

$$A_m^- = \left[\exp(ik_{mz}z') + \exp(-ik_{mz}(z' + 2d_{m-1}))\tilde{R}_{m,m-1}\right]\exp(ik_{mz}d_m)\tilde{M}_m$$

in a recursive way as

$$A_i^- \exp(ik_{iz}d_i) = A_{i-1}^- \exp(ik_{i-1,z}d_{i-1}) S_{i-1,i}^- \tag{30}$$

$$S_{i-1,i}^- = \frac{T_{i-1,i}}{1 - R_{i,i-1}\tilde{R}_{i,i+1}\exp(2ik_{iz}(d_i - d_{i-1}))}.$$

### 8.2.2 Observer outside source layer

The fields outside the source region $m$ are related to the amplitude $A_m^+$ and $A_m^-$ of outgoing fields at the boundary of the source region. At $z = -d_{m-1}$ and $z = -d_m$, the amplitudes of the $x$-component of the waves from a HED/HMD dipole

$$A_m^+ = \exp(-ik_{mz}(z' + d_{m-1})) + C_h \exp(-ik_{mz}d_{m-1}) + A_h \exp(ik_{mz}d_{m-1})$$

$$A_m^- = \exp(ik_{mz}(z' + d_m)) + C_h \exp(-ik_{mz}d_m) + A_h \exp(ik_{mz}d_m),$$

the $z$-component from a HED or HMD dipole

$$A_m^+ = \frac{k_x k_{mz}}{k_\rho^2}(A_h + B_h)\exp(ik_{mz}d_{m-1}) + \frac{k_x k_{mz}}{k_\rho^2}(D_h - C_h)\exp(-ik_{mz}d_{m-1})$$

$$A_m^- = \frac{k_x k_{mz}}{k_\rho^2}(A_h + B_h)\exp(ik_{mz}d_m) + \frac{k_x k_{mz}}{k_\rho^2}(D_h - C_h)\exp(-ik_{mz}d_m)$$

and the $z$-component from a VED or VMD dipole

$$A_m^+ = \exp(-ik_{mz}(z' + d_{m-1})) + A_v \exp(-ik_{mz}d_{m-1}) + B_v \exp(ik_{mz}d_{m-1})$$

$$A_m^- = \exp(ik_{mz}(z' + d_m)) + A_v \exp(-ik_{mz}d_m) + B_v \exp(ik_{mz}d_m).$$

The corresponding elements of Green's function are $(n < m)$

$$\tilde{G}_{xx}^A = \tilde{G}_{zz}^A = -\frac{\mu_n}{2ik_{mz}} A_n^+ \left[\exp(ik_{nz}z) + \tilde{R}_{n,n-1}\exp(-2ik_{nz}d_{n-1} - ik_{nz}z)\right],$$

$$\tilde{G}_{zx}^A = \frac{\mu_n}{2ik_{mz}} A_n^+ \left[\exp(ik_{nz}z) + \tilde{R}_{n,n-1}\exp(-2ik_{nz}d_{n-1} - ik_{nz}z)\right],$$

$$\tilde{G}_{xx}^F = \tilde{G}_{zz}^F = -\frac{\epsilon_n}{2ik_{mz}} A_n^+ \left[\exp(ik_{nz}z) + \tilde{R}_{n,n-1}\exp(-2ik_{nz}d_{n-1} - ik_{nz}z)\right],$$

$$\tilde{G}_{zx}^F = \frac{\epsilon_n}{2ik_{mz}} A_n^+ \left[\exp(ik_{nz}z) + \tilde{R}_{n,n-1}\exp(-2ik_{nz}d_{n-1} - ik_{nz}z)\right],$$

and $(n > m)$

$$\tilde{G}_{xx}^A = \tilde{G}_{zz}^A = -\frac{\mu_n}{2ik_{mz}} A_n^- \left[\exp(-ik_{nz}z) + \tilde{R}_{n,n+1}\exp(2ik_{nz}d_n + ik_{nz}z)\right],$$

$$\tilde{G}_{zx}^A = \frac{\mu_n}{2ik_{mz}} A_n^- \left[\exp(-ik_{nz}z) + \tilde{R}_{n,n+1}\exp(2ik_{nz}d_n + ik_{nz}z)\right],$$

$$\tilde{G}_{xx}^F = \tilde{G}_{zz}^F = -\frac{\epsilon_n}{2ik_{mz}} A_n^- \left[\exp(-ik_{nz}z) + \tilde{R}_{n,n+1}\exp(2ik_{nz}d_n + ik_{nz}z)\right],$$

$$\tilde{G}_{zx}^F = \frac{\epsilon_n}{2ik_{mz}} A_n^- \left[\exp(-ik_{nz}z) + \tilde{R}_{n,n+1}\exp(2ik_{nz}d_n + ik_{nz}z)\right],$$

where $\tilde{R}$ corresponds to either $\mathrm{TE}_z$ or $\mathrm{TM}_z$ waves (note that component $\tilde{G}_{zx}$ has both) and amplitudes $A_n^+$ and $A_n^-$ satisfy the recursive relations (29) and (30) , respectively.

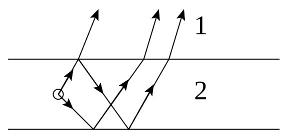


Figure 7: Wave reflections.

### 8.2.3 Example

A source in region 2 ($m = 2$ and $n = 1$). The fields in region 1 can be written as

$$\tilde{G}^A_{xx} = \tilde{G}^A_{zz} = -\frac{\mu_1}{2ik_{1z}} A_1^+ \exp(ik_{1z}z)$$

$$\tilde{G}^A_{zx} = \frac{\mu_1}{2ik_{1z}} \left[\frac{k_x k_{1z}}{k_\rho^2} A_1^+ \exp(ik_{1z}z)\right]$$

$$\tilde{G}^F_{xx} = \tilde{G}^F_{zz} = -\frac{\epsilon_1}{2ik_{1z}} A_1^+ \exp(ik_{1z}z)$$

$$\tilde{G}^F_{zx} = \frac{\epsilon_1}{2ik_{1z}} \left[\frac{k_x k_{1z}}{k_\rho^2} A_1^+ \exp(ik_{1z}z)\right]$$

and the amplitudes $A_1^+$ of the up-going wave in region 1

$$A_1^+ = \exp(-ik_{1z}d_1) A_2^+ \exp(ik_{2z}d_1) T_{21}$$

is written in terms of the amplitudes $A_2^+$ at the boundary $z = -d_1$ of region 2 for the six types of waves (3 x TE$_z$ + 3 x TM$_z$)

$$A_2^+ = [\exp(-ik_{2z}z') + C_h] \exp(-ik_{2z}d_1)$$

$$A_2^+ = (D_h - C_h) \exp(-ik_{2z}d_1)$$

$$A_2^+ = [\exp(-ik_{2z}z') + A_v] \exp(-ik_{2z}d_1),$$

with coefficients for the up going waves given by

$$A_v = C_h = R_{23} [\exp(ik_{2z}(2d_1 + z')) + R_{21} \exp(-ik_{2z}(2(d_1 - d_2) + z'))] \tilde{M}_2$$

$$D_h = R_{23}\left[-\exp(ik_{2z}(2d_2+z')) + R_{21}\exp(-ik_{2z}(2(d_1-d_2)+z'))\right]\tilde{M}_2.$$

$$\tilde{M}_2 = \left[1 - R_{23}R_{21}\exp(2ik_{2z}(d_2-d_1))\right]^{-1}.$$

Again, coefficients $A$, $C$, and $D$ are either $\mathrm{TE}_z$ or $\mathrm{TM}_z$.

## 8.3 Useful identities

Beside the identities mentioned in Part 1, more of those particularly for the Green's function theory.

### 8.3.1 Spherical coordinates

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arccos(z/r) \\ \phi &= \arctan(y/x) \\ x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\cos\phi \\ z &= r\cos\theta \end{aligned}$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} E_r \\ E_\theta \\ E_\phi \end{bmatrix}$$

with the inverse

$$\begin{bmatrix} E_r \\ E_\theta \\ E_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

### 8.3.2 Cylindrical coordinates

In cylindrical coordinates $(r, \phi, z)$ and $E = (E_r, E_\phi, E_z)^T$

$$\begin{bmatrix} E_r \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

The curl operator is

$$\nabla \times E = \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\phi}E_z - \frac{\partial}{\partial z}E_\phi \\ -(\frac{\partial}{\partial r}E_z - \frac{\partial}{\partial z}E_r) \\ \frac{1}{r}(\frac{\partial}{\partial r}(rE_\phi - \frac{\partial}{\partial\phi}E_r) \end{bmatrix}.$$

### 8.3.3 Hankel and Bessel's functions

$J_n$ are the Bessel's functions of the first kind and $Y_n$ the Bessel's functions of the second kind. Valid relation (you may interchange $J$ and $Y$)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^m J_m(x)\right) = x^m J_{m-1}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-m} J_m(x)\right) = -x^{-m} J_{m+1}(x)$$

$$J_{-m} = (-1)^m J_m$$

$$\frac{\mathrm{d}}{\mathrm{d}x} J_m(x) = \frac{1}{2}\left(J_{m-1}(x) - J_{m+1}(x)\right) = \frac{m J_m(z)}{z} - J_{m+1}(z)$$

$$J_m(x) = \frac{x}{2m}\left(J_{m-1}(x) + J_{m+1}(x)\right).$$

Hankel functions are defined as $H_n^{(1)}(z) = J_n(z) + iY_n(z)$ and $H_n^{(2)}(z) = J_n(z) - iY_n(z)$. So,

$$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}.$$

$$H_0^{(1)}(-z) = -H_0^{(2)}(z)$$

$$\frac{\mathrm{d}}{\mathrm{d}z} H_n^{(1,2)}(z) = \frac{1}{2}\left(H_{n-1}^{(1,2)}(x) - H_{n+1}^{(1,2)}(x)\right) = \frac{n H_n^{(1,2)}(z)}{z} - H_{n+1}^{(1,2)}(z)$$

Addition theorem (Balanis, page 597)

$$H_0^{(1,2)}(\beta|\rho - \rho'|) = \sum_{n=-\infty}^{\infty} J_n(\beta\rho') H_n^{(1,2)}(\beta\rho)\exp(in(\phi - \phi')),\ \rho \geq \rho'$$

$$\exp(\pm i\beta x) = \exp(\pm i\beta\rho\cos(\phi)) = \sum_{n=-\infty}^{\infty} i^{\pm n} J_n(\beta\rho)\exp(in\phi)$$

### 8.3.4 Limits

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),\ x \to \infty$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),\ x \to \infty$$

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} \exp\left(i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right),\ x \to \infty$$

$$H_n^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} \exp\left(-i\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right),\ x \to \infty$$

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