# An existence result related to two-phase flows with dynamical capillary pressure

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# An existence result related to two-phase flows with dynamic capillary pressure

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**Abstract.** *We consider a nonlinear degenerate pseudo-parabolic equation arising in the modeling of immiscible two-phase flows within porous media when the dynamic capillary pressure $p_c = \pi(s) + \tau\partial_t s$ is a function of both the saturation $s$ and its time derivative $\partial_t s$. We show the existence of a weak solution to the problem using the compactness of a sequence of regularizations of the problem.*

## 1 The problem and main result

We consider the nonlinear degenerate pseudo-parabolic equation arising in the modeling of an incompressible immiscible two-phase flow within a porous medium:

$$\partial_t s + \nabla \cdot (\mathbf{F}(x,t;s) - H(s)\nabla p_c) = 0. \tag{1}$$

The capillary pressure $p_c$ includes dynamic effects [6], [3], i.e.

$$p_c = \pi(u) + \tau \partial_t s, \tag{2}$$

where $\tau > 0$. The function $H$ is supposed to be Lipschitz continuous, satisfying

$$H(0) = H(1) = 0, \quad H(u) > 0 \text{ if } u \in (0,1). \tag{3}$$

The convective term $\mathbf{F}(x,t,s)$, taking both the gravity and total flow rate into account, is supposed to satisfy

$$\text{for a.e. } (x,t), \quad \mathbf{F}(x,t;\cdot) \text{ is Lipschitz continuous}, \tag{4}$$

and, for all $t \geq 0$,

$$\text{for all } u \in [0,1], \quad \nabla \cdot \mathbf{F}(x,t;u) = 0. \tag{5}$$

The function $\pi$ is supposed to be increasing on $(0,1)$, and such that

$$H(\cdot)\pi'(\cdot) \in L^\infty(0,1), \text{ and } \sqrt{\pi'} \in L^1(0,1). \tag{6}$$

It is worth noticing that the assumptions (3)-(6)are satisfied in the commonly used models from oil-engineering.

We consider the flow in an open bounded subset of $\mathbb{R}^d$, with a Lipschitz boundary $\partial\Omega$ and on a time interval $[0,T]$. We denote by $Q = \Omega \times [0,T]$. We prescribe a boundary condition $s_D$ defined on $\partial\Omega \times [0,T]$ admitting an extension, still denoted by $s_D$, belonging to $C^1([0,T];H^1(\Omega))$, that is supposed to be essentially bounded far from 0 and 1: there exists $\eta \in (0,1)$ such that

$$\text{for a.e. } (x,t) \in Q, \quad s_D(x,t) \in [\eta, 1-\eta]. \tag{7}$$

We also prescribe an initial condition $s_0 \in H^1(\Omega)$, satisfying the *finite entropy condition*

$$\int_\Omega \Gamma(s_0(x))dx < \infty, \tag{8}$$

where $\Gamma$ is the convex function defined by

$$\Gamma(u) = \int_{s_D}^{u} \int_{s_D}^{v} \frac{1}{H(a)} da dv.$$

Note that the condition (8) in particular implies that $s_0 \in L^\infty(\Omega;[0,1])$. As a last tool to introduce, we denote by $\zeta$ a primitive form of $\sqrt{\pi'}$. We now give the definition of a weak solution to the problem.

**Definition** A function $s$ is said to be a weak solution to the problem if

1. $s \in L^\infty(Q;[0,1]) \cap H^1(Q)$, $(s - s_D) \in L^\infty((0,T); H_0^1(\Omega))$, $s_{|t=0} = s_0$,
2. $\zeta(s) \in L^2((0,T); H^1(\Omega))$ and $\sqrt{H(s)}\partial_t \nabla s \in L^2(Q)$,
3. for all $\phi \in C_c^\infty(Q)$,

$$\iint_Q \partial_t s \phi dx dt - \iint_Q \mathbf{F}(x,t;s) \cdot \nabla \phi dx dt + \iint_Q H(s) \left( \nabla \pi(s) + \tau \partial_t \nabla s \right) \cdot \nabla \phi dx dt = 0. \quad (9)$$

The main result of our contribution [1], mixed with some of those presented in [4], is the following:

**Theorem** Under assumptions (3)–(8), there exists a weak solution.

## 2 Key ideas of the proof

The proof of the above theorem relies on compactness properties of a sequence of approximate solutions $(s_n)_n$ obtained by a regularization of the problem, by approximating the function $H$ by the functions $H_n$, given by $H_n(u) = H(u) + \dfrac{1}{n}$. By a choice of convenient test functions (see [1] and [4]), uniform estimates with respect to $n$ are derived. In particular, one has

$$\|\partial_t s_n\|_{L^2(Q)} \le C, \qquad \|\nabla s\|_{L^\infty((0,T);L^2(\Omega))} \le C, \quad (10)$$

$$\|\sqrt{H(s_n)}\partial_t \nabla s_n\|_{L^2(Q)} \le C, \quad (11)$$

and

$$\|\Gamma_n(s_n)\|_{L^\infty((0,T);L^1(\Omega))} \le C, \quad (12)$$

where $\Gamma_n(u) = \int_{s_D}^u \int_{s_D}^v \frac{1}{H_n(a)} da dv$, and where $C$ denotes a generic finite quantity independent on $n$. Hence it follows from (10) that there exists $s \in H^1(Q)$ such that, up to a subsequence, $s_n$ tends to $s$ almost everywhere in $Q$ and weakly in $H^1(Q)$. From (12), we deduce that $s \in L^\infty(Q;[0,1])$. From (11), we can claim that there exists $\xi \in L^2(Q)$ such that, up to a new subsequence,

$$\sqrt{H(s_n)}\partial_t \nabla s_n \rightharpoonup \xi \quad \text{weakly in } L^2(Q).$$

The major difficulty consists in identifying $\xi$ as $\sqrt{H(s)}\partial_t \nabla s$. This has been performed in [1] by a tricky use of the *div-curl lemma* [5] and of Vitali convergence theorem [2].

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