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## An existence result related to two-phase flows with dynamic capillary pressure

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**Abstract.** *We consider a nonlinear degenerate pseudo-parabolic equation arising in the modeling of immiscible two-phase flows within porous media when the dynamic capillary pressure  $p_c = \pi(s) + \tau \partial_t s$  is a function of both the saturation  $s$  and its time derivative  $\partial_t s$ . We show the existence of a weak solution to the problem using the compactness of a sequence of regularizations of the problem.*

### 1 The problem and main result

We consider the nonlinear degenerate pseudo-parabolic equation arising in the modeling of an incompressible immiscible two-phase flow within a porous

medium:

$$\partial_t s + \nabla \cdot (\mathbf{F}(x, t; s) - H(s)\nabla p_c) = 0. \quad (1)$$

The capillary pressure  $p_c$  includes dynamic effects [6], [3], i.e.

$$p_c = \pi(u) + \tau \partial_t s, \quad (2)$$

where  $\tau > 0$ . The function  $H$  is supposed to be Lipschitz continuous, satisfying

$$H(0) = H(1) = 0, \quad H(u) > 0 \text{ if } u \in (0, 1). \quad (3)$$

The convective term  $\mathbf{F}(x, t, s)$ , taking both the gravity and total flow rate into account, is supposed to satisfy

$$\text{for a.e. } (x, t), \quad \mathbf{F}(x, t; \cdot) \text{ is Lipschitz continuous,} \quad (4)$$

and, for all  $t \geq 0$ ,

$$\text{for all } u \in [0, 1], \quad \nabla \cdot \mathbf{F}(x, t; u) = 0. \quad (5)$$

The function  $\pi$  is supposed to be increasing on  $(0, 1)$ , and such that

$$H(\cdot)\pi'(\cdot) \in L^\infty(0, 1), \text{ and } \sqrt{\pi'} \in L^1(0, 1). \quad (6)$$

It is worth noticing that the assumptions (3)-(6) are satisfied in the commonly used models from oil-engineering.

We consider the flow in an open bounded subset of  $\mathbb{R}^d$ , with a Lipschitz boundary  $\partial\Omega$  and on a time interval  $[0, T]$ . We denote by  $Q = \Omega \times [0, T]$ . We prescribe a boundary condition  $s_D$  defined on  $\partial\Omega \times [0, T]$  admitting an extension, still denoted by  $s_D$ , belonging to  $C^1([0, T]; H^1(\Omega))$ , that is supposed to be essentially bounded far from 0 and 1: there exists  $\eta \in (0, 1)$  such that

$$\text{for a.e. } (x, t) \in Q, \quad s_D(x, t) \in [\eta, 1 - \eta]. \quad (7)$$

We also prescribe an initial condition  $s_0 \in H^1(\Omega)$ , satisfying the *finite entropy condition*

$$\int_{\Omega} \Gamma(s_0(x)) dx < \infty, \quad (8)$$

where  $\Gamma$  is the convex function defined by

$$\Gamma(u) = \int_{s_D}^u \int_{s_D}^v \frac{1}{H(a)} da dv.$$

Note that the condition (8) in particular implies that  $s_0 \in L^\infty(\Omega; [0, 1])$ . As a last tool to introduce, we denote by  $\zeta$  a primitive form of  $\sqrt{\pi'}$ . We now give the definition of a weak solution to the problem.

**Definition** A function  $s$  is said to be a weak solution to the problem if

1.  $s \in L^\infty(Q; [0, 1]) \cap H^1(Q)$ ,  $(s - s_D) \in L^\infty((0, T); H_0^1(\Omega))$ ,  $s|_{t=0} = s_0$ ,
2.  $\zeta(s) \in L^2((0, T); H^1(\Omega))$  and  $\sqrt{H(s)}\partial_t \nabla s \in L^2(Q)$ ,
3. for all  $\phi \in C_c^\infty(Q)$ ,

$$\begin{aligned} \iint_Q \partial_t s \phi dx dt - \iint_Q \mathbf{F}(x, t; s) \cdot \nabla \phi dx dt \\ + \iint_Q H(s) (\nabla \pi(s) + \tau \partial_t \nabla s) \cdot \nabla \phi dx dt = 0. \end{aligned} \quad (9)$$

The main result of our contribution [1], mixed with some of those presented in [4], is the following:

**Theorem** Under assumptions (3)–(8), there exists a weak solution.

## 2 Key ideas of the proof

The proof of the above theorem relies on compactness properties of a sequence of approximate solutions  $(s_n)_n$  obtained by a regularization of the problem, by approximating the function  $H$  by the functions  $H_n$ , given by  $H_n(u) = H(u) + \frac{1}{n}$ . By a choice of convenient test functions (see [1] and [4]), uniform estimates with respect to  $n$  are derived. In particular, one has

$$\|\partial_t s_n\|_{L^2(Q)} \leq C, \quad \|\nabla s\|_{L^\infty((0, T); L^2(\Omega))} \leq C, \quad (10)$$

$$\|\sqrt{H(s_n)}\partial_t \nabla s_n\|_{L^2(Q)} \leq C, \quad (11)$$

and

$$\|\Gamma_n(s_n)\|_{L^\infty((0, T); L^1(\Omega))} \leq C, \quad (12)$$

where  $\Gamma_n(u) = \int_{s_D}^u \int_{s_D}^v \frac{1}{H_n(a)} da dv$ , and where  $C$  denotes a generic finite quantity independent on  $n$ . Hence it follows from (10) that there exists  $s \in H^1(Q)$  such that, up to a subsequence,  $s_n$  tends to  $s$  almost everywhere in  $Q$  and weakly in  $H^1(Q)$ . From (12), we deduce that  $s \in L^\infty(Q; [0, 1])$ . From (11), we can claim that there exists  $\xi \in L^2(Q)$  such that, up to a new subsequence,

$$\sqrt{H(s_n)}\partial_t \nabla s_n \rightharpoonup \xi \quad \text{weakly in } L^2(Q).$$

The major difficulty consists in identifying  $\xi$  as  $\sqrt{H(s)}\partial_t \nabla s$ . This has been performed in [1] by a tricky use of the *div-curl lemma* [5] and of Vitali convergence theorem [2].

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## REFERENCES

- [1] C. Cancès, C. Choquet, Y. Fan and I.S. Pop, *Existence of weak solutions to a degenerate pseudo-parabolic equation modeling two-phase flow in porous media*, CASA Report 10-75, Eindhoven University of Technology (2010).
- [2] G. B. Folland. *Real Analysis: Modern Techniques and Their Applications* (second ed.) Wiley-Interscience, 1999.
- [3] S.M. Hassanizadeh and W.G. Gray, *Thermodynamic basis of capillary pressure in porous media*, *Water Resour. Res.* **29** (1993), 3389–3405.
- [4] A. Mikelić, *A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure*, *J. Differential Equations* **248** (2010), 1561–1577.
- [5] F. Murat, *Compacité par compensation*, *Ann. Sc. Norm. Sup. Pisa* **5** (1978), 489–507.
- [6] D. Pavone, *Macroscopic equations derived from space averaging for immiscible two-Phase flow in porous media*, *Oil & Gas Science and Technology - Rev. IFP* **44**(1989), No. 1, 29–41.