A class of degenerate pseudo-parabolic equations : existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 10-44
July 2010

A class of degenerate pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization

by

Y. Fan, I.S. Pop

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

A class of degenerate pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization

Y. Fan, I.S. Pop

July 15, 2010

Abstract

In this paper, we investigate a class of degenerate pseudo-parabolic equations. Such equations model two-phase flow in porous media where dynamic effects are included in the capillary pressure. The existence and uniqueness of a weak solution are proved, and error estimates for an Euler implicit time discretization are obtained.

1 Introduction

In this paper, we focus on the following pseudo-parabolic equation:

$$u_t + \nabla \cdot \vec{F}(u) = \nabla \cdot (H(u)\nabla u) + \tau \Delta u_t, \tag{1.1}$$

where $H : \mathbb{R} \longrightarrow [0, +\infty)$ is smooth and bounded. Note that in particular H may become 0 for some value of u, we call this situation degenerate. A two-phase porous media flow taking into account dynamic effects in the phase pressure difference is proposed in [12],

$$u_t + \nabla \cdot \vec{F}(u) = \nabla \cdot (H(u)\nabla p), \tag{1.2}$$

with $p = p_c(u) + \tau\partial_t u$. Clearly, (1.1) is a simpler version of (1.2), as the degeneracy is encountered only in the second order term. Here we study the existence and uniqueness of weak solutions to (1.1), complemented with initial and boundary conditions. We do so by applying a discretization in time, for which we also give error estimates.

Pseudo-parabolic equations arise in many real life applications such as radiation with time delay [17], degenerate double-diffusion models [3], heat conduction models [23] and models for lightning propagation [2], etc. Existence and uniqueness of weak solutions to nonlinear pseudo-parabolic equations are proved in [20], while existence of weak solutions for some degenerate cases is studied in [18], [19]. A nonlinear parabolic-Sobolev equation is studied in [25]. In [21], homogenization of a pseudoparabolic system is considered. Travelling wave solutions and their relation to non-standard shock solutions to hyperbolic conservation laws are investigated in [4], [7] for linear higher order terms. This analysis is pursued in [6] for degenerate situations. Numerical schemes for dynamic capillary effects in heterogeneous porous media are given in [13] and a numerical scheme for the pore-scale simulation of crystal dissolution and precipitation in porous media is studied in

[8]. The case of discontinuous initial data is analyzed in [5]. Superconvergence of a finite element approximation to similar equation is investigated in [1] and time-stepping Galerkin methods are analyzed in [10] and [11], where two difference-approximation schemes are considered. In [22], Fourier spectral methods for pesudo-parabolic equations are analyzed.

The analysis below is carried out under the following assumptions:

- (A1) Ω is an open, bounded and connected domain in $\mathbb{R}^d$, with Lipschitz continuous boundary. With $T > 0$ given we denote $Q = \Omega \times (0, T]$.
- (A2) τ is a given strictly positive number.
- (A3) The vector valued $\vec{F}$ satisfies $\vec{F} = \vec{v} f(u)$, where $\vec{v} \in \mathbb{R}^d$ is a fixed vector. The functions f and H are $C^{1,1}$ satisfying $0 \leq f \leq 1$, $0 \leq H \leq M$ for some $M > 0$. We denote L an upper bound for the Lipschitz constants of f, H, f', H'.

Remark 1.1 *We take $\vec{v} \in \mathbb{R}^d$ for the ease of presentation. However, the results below can be extended to more general cases, such as $\vec{v}$ is a divergence free vector field, or $\vec{F}$ is a given $C^{1,1}$ vector valued function.*

In this paper, we use standard notations. In particular, $L^2(\Omega)$ stands for the square Lebesgue integrable functions on Ω , $W^{1,2}(\Omega)$ requests the same also for the derivatives of first order. $W_0^{1,2}(\Omega)$ is a subset of $W^{1,2}(\Omega)$ whose elements have zero boundary values. Furthermore, $W^{-1,2}(\Omega)$ is the dual space of $W_0^{1,2}(\Omega)$.
The initial and boundary conditions of (1.1) are given as follows:

$$u(\cdot, 0) = u^0, \quad \text{and} \quad u|_{\partial\Omega} = 0, \tag{1.3}$$

where $u^0 \in W_0^{1,2}(\Omega)$. We seek a weak solution to the following

Problem P Find $u \in W^{1,2}(0, T; W_0^{1,2}(\Omega))$ such that

$$\begin{aligned} &\int_0^T \int_\Omega u_t \phi dx dt - \int_0^T \int_\Omega \vec{F}(u) \nabla \phi dx dt \\ &+ \int_0^T \int_\Omega H(u) \nabla u \nabla \phi dx dt + \tau \int_0^T \int_\Omega \nabla u_t \nabla \phi dx dt = 0, \end{aligned} \tag{1.4}$$

for any $\phi \in L^2(0, T; W_0^{1,2}(\Omega))$.

This paper is organized as follows: Section 2 provides the existence of weak solutions to **Problem P**. The uniqueness of the weak solution is proved in Section 3. In Section 4, some error estimates for an Euler implicit time discretization scheme are obtained, and in Section 5, an iterative approach for solving the time discretization nonlinear problems is discussed and some numerical computations are given to verify the theoretical results. In the last section, some conclusions are given.

2 Existence

We show the existence of a weak solution to **Problem P** by the method of Rothe (see [14]), based on the Euler implicit time stepping. Before defining the time discretization we mention the following elementary inequality, which will be used several times later:

$$ab \leq \frac{1}{2\delta}a^2 + \frac{\delta}{2}b^2, \quad \text{for any} \quad a, b \in \mathbb{R} \quad \text{and} \quad \delta > 0. \tag{2.1}$$

2.1 Time discretization

With $N \in \mathbb{N}$, let $\Delta t = T/N$ and consider the following:
Problem P^{n+1} Given $u^n \in W_0^{1,2}(\Omega), n \in \{0, 1, 2, ..., N-1\}$, find $u^{n+1} \in W_0^{1,2}(\Omega)$ such that

$$\begin{aligned}(u^{n+1} - u^n, \phi) + \Delta t(\nabla \cdot \vec{F}(u^{n+1}), \phi) + \Delta t(H(u^{n+1})\nabla u^{n+1}, \nabla\phi) +\\ \tau(\nabla(u^{n+1} - u^n), \nabla\phi) = 0,\end{aligned} \tag{2.2}$$

for any $\phi \in W_0^{1,2}(\Omega)$, here $(\cdot, \cdot)$ means L^2 inner product. Note that this is the weak formulation of

$$\frac{u^{n+1} - u^n}{\Delta t} + \nabla \cdot \vec{F}(u^{n+1}) = \nabla(H(u^{n+1})\nabla u^{n+1}) + \tau\Delta\frac{(u^{n+1} - u^n)}{\Delta t}. \tag{2.3}$$

We have the following:

Lemma 2.1 **Problem P^{n+1}** *has a solution.*

Proof . Note that u^{n+1} can be identified formally with the solution of the following equation:

$$-\nabla \cdot ((\Delta t H(X) + \tau)\nabla X) + \Delta t\nabla \cdot \vec{F}(X) + X - u^n + \tau\Delta u^n = 0. \tag{2.4}$$

If $u^n \in C_0^{2,1}(\Omega)$, Theorem 8.2 from Chapter 4 in [16] provides the existence of $u^{n+1} = X \in C_0^{2,1}(\Omega)$ solving (2.4).

If $u^n \in W_0^{1,2}(\Omega)$, there exists a sequence $\{u_k^n\}_{k\in\mathbb{N}} \subseteq C_0^{2,1}(\Omega)$ converging to u^n in $W^{1,2}(\Omega)$. Solving (2.4) gives the sequence $\{X_k\}_{k\in\mathbb{N}} \subseteq C_0^{2,1}(\Omega)$ with u_k^n instead of u^n. Consider the weak form of (2.4):

$$\begin{aligned}\Delta t(H(X_k)\nabla X_k, \nabla\phi) + \tau(\nabla X_k, \nabla\phi) - \Delta t(\vec{F}(X_k), \nabla\phi)\\ +(X_k, \phi) = (u_k^n, \phi) + \tau(\nabla u_k^n, \nabla\phi),\end{aligned} \tag{2.5}$$

for any $\phi \in W_0^{1,2}(\Omega)$.

Taking $\phi = X_k \in W_0^{1,2}(\Omega)$ with $\vec{\mathcal{F}}(X_k) = \int_0^{X_k} \vec{F}(v)dv$ gives

$$(\vec{F}(X_k), \nabla X_k) = \int_\Omega \vec{F}(X_k)\nabla X_k dx = \int_{\partial\Omega} \vec{\nu} \cdot \vec{\mathcal{F}}(0)dx = 0, \tag{2.6}$$

where $\vec{\nu}$ is the outer normal vector to $\partial\Omega$. By (2.1),

$$\tau \int_\Omega |\nabla X_k|^2 dx + \int_\Omega |X_k|^2 dx \le \tau \int_\Omega |\nabla u_k^n|^2 dx + \int_\Omega |u_k^n|^2 dx \le C, \tag{2.7}$$

where C is a positive constant.

By the construction of $\{u_k^n\}$, we have

$$(u_k^n, \phi) \to (u^n, \phi), \tag{2.8}$$

$$(\nabla u_k^n, \nabla\phi) \to (\nabla u^n, \nabla\phi). \tag{2.9}$$

for any $\phi \in W_0^{1,2}(\Omega)$.
Further, since $\{X_k\}_{k\in\mathbb{N}}$ and $\{\nabla X_k\}_{k\in\mathbb{N}}$ are uniformly bounded in $L^2(\Omega)$, there exists a subsequence (still denoted as X_k) weakly converging to some X in $W_0^{1,2}(\Omega)$. Clearly,

$$(X_k, \phi) \to (X, \phi), \tag{2.10}$$

$$(\nabla X_k, \nabla\phi) \to (\nabla X, \nabla\phi), \tag{2.11}$$

$$(\vec{F}(X_k), \nabla\phi) \to (\vec{F}(X), \nabla\phi), \tag{2.12}$$

for any $\phi \in W_0^{1,2}(\Omega)$.

Define

$$\mathcal{H}(y) := \int_0^y H(v)dv. \tag{2.13}$$

Since $X_k \to X$ strongly in $L^2(\Omega)$ and according to (A3), we know that $\mathcal{H}(X_k) \to \mathcal{H}(X)$ strongly in $L^2(\Omega)$. Further, $\mathcal{H}(X_k)$ is uniformly bounded in $W_0^{1,2}(\Omega)$. Therefore

$$(\nabla\mathcal{H}(X_\varepsilon), \nabla\phi) \to (\nabla\mathcal{H}(X), \nabla\phi). \tag{2.14}$$

Then from (2.8), (2.9), (2.10), (2.11), (2.12) and (2.14), X is a solution to **Problem** $\mathbf{P}^{n+1}$.
□

Lemma 2.2 *The solution of* ***Problem*** $\boldsymbol{P}^{n+1}$ *is unique, at least if* Δt *is small enough.*

Proof . Assume we have two solutions X and Y. Define

$$\mathcal{G}(y) = \int_0^y (H(v) + \frac{\tau}{\Delta t})dv, \tag{2.15}$$

and subtract the equation for Y from the equation for X, taking $\phi = \mathcal{G}(X) - \mathcal{G}(Y)$ in the result gives
(2.16)
$$\Delta t||\nabla(\mathcal{G}(X)-\mathcal{G}(Y))||^2_{L^2(\Omega)} - \Delta t(\vec{F}(X)-\vec{F}(Y), \nabla(\mathcal{G}(X)-\mathcal{G}(Y))) + (X-Y, \mathcal{G}(X)-\mathcal{G}(Y)) = 0.$$

Therefore,

(2.17)

$$\Delta t||\nabla(\mathcal{G}(X)-\mathcal{G}(Y))||^2_{L^2(\Omega)}+\frac{\tau}{\Delta t}||X-Y||^2_{L^2(\Omega)} \leq \frac{\Delta t}{2}(L||X-Y||^2_{L^2(\Omega)}+||\nabla(\mathcal{G}(X)-\mathcal{G}(Y))||^2_{L^2(\Omega)}).$$

If $\Delta t^2 < \frac{2\tau}{L}$,

$$||X - Y||_{L^2(\Omega)} = 0, \tag{2.18}$$

implying the uniqueness.

2.2 A priori estimates

Having established the existence for the time discretization problems, we proceed with investigating **Problem P**. To this end, we obtain some a priori estimates.

Lemma 2.3 *For any $n \in \{0, 1, 2, ..., N-1\}$, we have:*

$$||u^{n+1}||^2_{L^2(\Omega)} + \tau||\nabla u^{n+1}||^2_{L^2(\Omega)} \leq C, \tag{2.19}$$

$$||u^{n+1} - u^n||^2_{L^2(\Omega)} + \tau||\nabla(u^{n+1} - u^n)||^2_{L^2(\Omega)} \leq C(\Delta t)^2, \tag{2.20}$$

here C denotes a positive constant.

Proof . **1**. Taking $\phi = u^{n+1}$ in (2.2) gives

(2.21)

$$||u^{n+1}||^2_{L^2(\Omega)}+\tau||\nabla u^{n+1}||^2_{L^2(\Omega)}+\Delta t\int_\Omega H(u^{n+1})|\nabla u^{n+1}|^2dx = (u^n, u^{n+1})+\tau(\nabla u^n, \nabla u^{n+1}).$$

Since u^{n+1} vanishes on $\partial\Omega$, with $\mathcal{F}(u^{n+1}) = \int_0^{u^{n+1}} \vec{F}(v)dv$ we have

$$(\nabla \cdot \vec{F}(u^{n+1}), u^{n+1}) = -\int_\Omega \vec{F}(u^{n+1})\nabla u^{n+1}dx = \int_{\partial\Omega} \vec{\nu} \cdot \vec{\mathcal{F}}(0)dx = 0,$$

together with (2.1) yields

$$||u^{n+1}||^2_{L^2(\Omega)} + \tau||\nabla u^{n+1}||^2_{L^2(\Omega)} \leq ||u^n||^2_{L^2(\Omega)} + \tau||\nabla u^n||^2_{L^2(\Omega)}. \tag{2.22}$$

Since $u^0 \in W^{1,2}_0(\Omega)$, this implies

$$||u^n||^2_{L^2(\Omega)} + \tau||\nabla u^n||^2_{L^2(\Omega)} \leq C. \tag{2.23}$$

2. Taking $\phi = u^{n+1} - u^n$ in (2.2) gives

$$\begin{aligned} &||u^{n+1} - u^n||^2_{L^2(\Omega)} - \Delta t(\vec{F}(u^{n+1}), \nabla \cdot (u^{n+1} - u^n)) \\ &+\Delta t(H(u^{n+1})\nabla u^{n+1}, \nabla(u^{n+1} - u^n)) + \tau||\nabla(u^{n+1} - u^n)||^2_{L^2(\Omega)} = 0, \end{aligned} \tag{2.24}$$

Using (2.1) and the boundedness of $\vec{F}$, we have

$$||u^{n+1} - u^n||^2_{L^2(\Omega)} + \frac{\tau}{2}||\nabla(u^{n+1} - u^n)||^2_{L^2(\Omega)} \leq C(\Delta t)^2. \quad \square \tag{2.25}$$

Remark 2.1 *From the proof of (2.19), if we define* $||u^n||^2 = ||u^n||^2_{L^2(\Omega)} + \tau||\nabla u^n||^2_{L^2(\Omega)}$, *then* $||u^n||$ *decreases as* n *increases. Further, from (2.20) one immediately obtains*

$$\sum_{k=1}^{N} ||u^k - u^{k-1}||^2_{L^2(\Omega)} \leq C\Delta t, \tag{2.26}$$

$$\sum_{k=1}^{N} ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)} \leq C\Delta t. \tag{2.27}$$

2.3 Existence

To show the existence of a solution to **Problem P**, we start by defining

$$U_N(t) = u^{k-1} + \frac{t - t^{k-1}}{\Delta t}(u^k - u^{k-1}), \quad \text{and} \quad \overline{U}_N(t) = u^k, \tag{2.28}$$

for $t^{k-1} = (k-1)\Delta t \leq t < t^k = k\Delta t, k = 1, 2...N$. We have the following result:

Theorem 2.1 **Problem P** *has a solution.*

Proof . According to the a priori estimates in Lemma 2.3,

$$\begin{aligned} \int_0^T ||U_N(t)||^2_{L^2(\Omega)}dt &= \sum_{k=1}^{N}\int_{t^{k-1}}^{t^k} ||u^{k-1} + \frac{t - t^{k-1}}{\Delta t}(u^k - u^{k-1})||^2_{L^2(\Omega)}dt \\ &\leq 2\sum_{k=1}^{N}\int_{t^{k-1}}^{t^k} (||u^{k-1}||^2_{L^2(\Omega)} + ||u^k - u^{k-1}||^2_{L^2(\Omega)})dt \\ &\leq C. \end{aligned} \tag{2.29}$$

Similarly,

$$\int_0^T ||\nabla U_N(t)||^2_{L^2(\Omega)}dt \leq C, \tag{2.30}$$

$$\int_0^T ||\partial_t U_N||^2_{L^2(\Omega)}dt = \frac{1}{\Delta t}\sum_{k=1}^{N} ||u^k - u^{k-1}||^2_{L^2(\Omega)} \leq C, \tag{2.31}$$

and

$$\begin{aligned} \int_0^T ||\partial_t \nabla U_N||^2_{L^2(\Omega)}dt &= \sum_{k=1}^{N}\int_{t^{k-1}}^{t^k} ||\frac{1}{\Delta t}\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)}dt \\ &= \frac{1}{\Delta t}\sum_{k=1}^{N} ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)} \leq C. \end{aligned} \tag{2.32}$$

Therefore $\{U_N\}_{N\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(0, T; W_0^{1,2}(\Omega))$, so it has a subsequence (still denoted as $\{U_N\}$) that converges weakly to some $U \in W^{1,2}(0, T; W_0^{1,2}(\Omega))$. Therefore

U_N converges strongly to U in $L^2(Q)$.

We now exploit a general principle that relates the piecewise linear and the piecewise constant interpolation (see e.g. [15] for a proof of the corresponding lemma): if one interpolation converges strongly in $L^2(Q)$, then the other interpolation also converges strongly in $L^2(Q)$. From the convergence of U_N, we conclude that $\overline{U}_N$ also converges strongly in $L^2(Q)$. With $\mathcal{H}$ defined in (2.13), the boundedness of H implies that $\mathcal{H}(\overline{U}_N)$ is uniformly bounded in $L^2(0,T;W_0^{1,2}(\Omega))$. As in the proof of Lemma 2.1, one gets

$$\nabla\mathcal{H}(\overline{U}_N) \rightharpoonup \nabla\mathcal{H}(U). \tag{2.33}$$

From (2.2), we know

$$\begin{aligned}\int_0^T\int_\Omega \partial_t U_N(t)\phi dxdt &- \int_0^T\int_\Omega \vec{F}(\overline{U}_N(t))\nabla\phi dxdt \\ &+ \int_0^T\int_\Omega \nabla\mathcal{H}(\overline{U}_N(t))\nabla\phi dxdt + \tau\int_0^T\int_\Omega \partial_t\nabla U_N(t)\nabla\phi dxdt = 0,\end{aligned} \tag{2.34}$$

for any $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$,

Using the weak convergence of U_N and $\mathcal{H}(\overline{U}_N)$, we consider a sequence $\Delta t \to 0$ and pass to the limit in (2.34). This shows that U is a solution to **Problem P**. □

Remark 2.2 *As will be proved in the following section, the solution of* **Problem P** *is unique. Therefore the convergence holds along any* $\Delta t \searrow 0$.

3 Uniqueness

Here we show that the solution to **Problem P** is unique. To do so, we use the following result (see e.g. Chapter 6 in [9]):

Proposition 3.1 *Let* $g \in L^2(\Omega)$. *The equation*

$$-\Delta G = g \quad in \quad \Omega, \tag{3.1}$$

with boundary condition $G|_{\partial\Omega} = 0$ *has a unique weak solution* $G \in W_0^{1,2}(\Omega)$, *satisfying*

$$||\nabla G||_{W^{1,2}(\Omega)} = ||g||_{W^{-1,2}(\Omega)} \leq C||g||_{L^2(\Omega)}. \tag{3.2}$$

We use this for proving the uniqueness result:

Theorem 3.1 *The solution of* **Problem P** *is unique.*

Proof . Assume u and v are two solutions, we have $(u-v)(\cdot,0) = 0$ and for any $\tilde{t} > 0$,

$$\begin{aligned}\int_0^{\tilde{t}}\int_\Omega (u-v)_t\phi dxdt &- \int_0^{\tilde{t}}\int_\Omega (\vec{F}(u) - \vec{F}(v))\nabla\phi dxdt \\ &+ \int_0^{\tilde{t}}\int_\Omega \nabla(\mathcal{H}(u) - \mathcal{H}(v))\nabla\phi dxdt + \tau\int_0^{\tilde{t}}\int_\Omega \nabla(u-v)_t\nabla\phi dxdt = 0,\end{aligned} \tag{3.3}$$

for any $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$.

Taking $g = u - v$ in Proposition 3.1, there exists a $G_{u-v} \in W_0^{1,2}(\Omega)$ such that

$$(\nabla G_{u-v}, \nabla \psi) = (u - v, \psi), \tag{3.4}$$

for any $\psi \in W_0^{1,2}(\Omega)$, satisfying

$$||G_{u-v}||_{W^{1,2}(\Omega)} \leq C||u - v||_{L^2(\Omega)}. \tag{3.5}$$

Note that through u and v, G_{u-v} also depends on t. First, by (3.4) for any $\tilde{t} > 0$

$$\begin{aligned} &\int_0^{\tilde{t}} \int_\Omega (u-v)_t G_{u-v} dxdt \\ = &\int_\Omega (u-v) G_{u-v}|_0^{\tilde{t}} dx - \int_\Omega \int_0^{\tilde{t}} (u-v)\partial_t G_{u-v} dtdx \\ = &\int_\Omega |\nabla G_{u-v}|^2|_0^{\tilde{t}} dx - \int_0^{\tilde{t}} \int_\Omega \nabla G_{u-v} \nabla \partial_t G_{u-v} dtdx \\ = &\frac{1}{2} \int_\Omega |\nabla G_{u-v}(\cdot, \tilde{t})|^2 dx, \end{aligned} \tag{3.6}$$

as $G_{u-v}(\cdot, 0) = 0$. Further, by (A3)

$$\int_0^{\tilde{t}} \int_\Omega (\vec{F}(u) - \vec{F}(v)) \nabla G_{u-v} dxdt \leq C \int_0^{\tilde{t}} \int_\Omega |u-v||\nabla G_{u-v}| dxdt \leq C \int_0^{\tilde{t}} \int_\Omega |u-v|^2 dxdt. \tag{3.7}$$

Next the monotonicity of $\mathcal{H}$ implies

$$\int_0^{\tilde{t}} \int_\Omega \nabla(\mathcal{H}(u) - \mathcal{H}(v)) \nabla G_{u-v} dxdt = \int_0^{\tilde{t}} \int_\Omega (\mathcal{H}(u) - \mathcal{H}(v))(u-v) dxdt \geq 0, \tag{3.8}$$

Finally,

$$\begin{aligned} &\tau \int_0^{\tilde{t}} \int_\Omega \partial_t \nabla(u-v) \nabla G_{u-v} dxdt \\ = &\tau \int_0^{\tilde{t}} \int_\Omega \partial_t (u-v)(u-v) dxdt \\ = &\frac{\tau}{2} \int_\Omega (u-v)(\cdot, \tilde{t})^2 dx. \end{aligned} \tag{3.9}$$

Therefore taking $\phi = G_{u-v}$ in (3.3) gives

$$\frac{1}{2}||\nabla G_{u-v}(\cdot, \tilde{t})||^2_{L^2(\Omega)} + \frac{\tau}{2}||(u-v)(\cdot, \tilde{t})||^2_{L^2(\Omega)} \leq C \int_0^{\tilde{t}} \int_\Omega |u-v|^2 dxdt. \tag{3.10}$$

By Gronwall's inequality, $||(u-v)(\cdot, \tilde{t})||_{L^2(\Omega)} = 0$. Since $\tilde{t}$ is arbitrary, this gives uniqueness.

4 Error estimates

From the above we see that the approximating sequence U_N converges strongly to U in $L^2(Q)$. In this section we will estimate the error $U_N - U$. Recalling (3.3), we have

$$\int_0^T \int_\Omega \partial_t U_N(t)\phi dxdt - \int_0^T \int_\Omega \vec{F}(\overline{U}_N(t))\nabla\phi dxdt + \int_0^T \int_\Omega \nabla\mathcal{H}(\overline{U}_N(t))\nabla\phi dxdt + \tau \int_0^T \int_\Omega \partial_t \nabla U_N(t)\nabla\phi dxdt = 0. \tag{4.1}$$

Denote

$$e_u(t) = u(t) - U_N(t), \quad \text{and} \quad e_H(t) = \mathcal{H}(u(t)) - \mathcal{H}(U_N(t)). \tag{4.2}$$

Obviously, $e_u, e_H \in W_0^{1,2}(\Omega)$ and $e_u(\cdot, 0) = e_{\mathcal{H}}(\cdot, 0) = 0$.

Theorem 4.1 *The following estimate holds:*

$$||e_u||_{L^\infty(0,T;L^2(\Omega))} \leq C\Delta t. \tag{4.3}$$

Proof . Subtracting (4.1) from (1.4) gives

$$\int_0^{\tilde{t}} \int_\Omega \partial_t e_u \phi dxdt - \int_0^{\tilde{t}} \int_\Omega (\vec{F}(u(t)) - \vec{F}(\overline{U}_N(t)))\nabla\phi dxdt + \int_0^{\tilde{t}} \int_\Omega \nabla(\mathcal{H}(u(t)) - \mathcal{H}(\overline{U}_N(t)))\nabla\phi dxdt + \tau \int_0^{\tilde{t}} \int_\Omega \partial_t \nabla e_u \nabla\phi dxdt = 0. \tag{4.4}$$

Taking $g = e_u$ in Proposition 3.1 provides a $G_{e_u} \in W_0^{1,2}(\Omega)$ satisfying

$$(\nabla G_{e_u}, \nabla\psi) = (e_u, \psi), \tag{4.5}$$

for any $\psi \in W_0^{1,2}(\Omega)$, and

$$||G_{e_u}||_{W^{1,2}(\Omega)} \leq C||e_u||_{L^2(\Omega)}. \tag{4.6}$$

We will use G_{e_u} as test function in (4.4). As in Section 3 we have for any $\tilde{t} > 0$

$$\int_0^{\tilde{t}} \int_\Omega \partial_t e_u G_{e_u} dxdt = \frac{1}{2}\int_\Omega (\nabla G_{e_u}(\cdot, \tilde{t}))^2 dx = \frac{1}{2}||e_u(\tilde{t})||^2_{W^{-1,2}}. \tag{4.7}$$

Further,

$$\begin{aligned} &\int_0^{\tilde{t}} \int_\Omega (\vec{F}(u(t)) - \vec{F}(\overline{U}_N(t)))\nabla G_{e_u} dxdt \\ \leq\ & C_1 \int_0^{\tilde{t}} ||e_u||^2_{L^2(\Omega)} dxdt + \int_0^{\tilde{t}} \int_\Omega (\vec{F}(U_N(t)) - \vec{F}(\overline{U}_N(t)))\nabla G_{e_u} dxdt \\ \leq\ & C_1 \int_0^{\tilde{t}} ||e_u||^2_{L^2(\Omega)} dxdt + C_2 \int_0^{\tilde{t}} ||U_N - \overline{U}_N||_{L^2(\Omega)} ||\nabla G_{e_u}||_{L^2(\Omega)} dt \end{aligned} \tag{4.8}$$

Since $U_N - \overline{U}_N = \frac{t_k - t}{\Delta t}(u^k - u^{k-1})$, for $t \in (t^{k-1}, t^k)$. By (2.26), we get $||U_N - \overline{U}_N||_{L^2(\Omega)} \leq C\Delta t$, therefore

$$\int_0^{\tilde{t}} \int_\Omega (\vec{F}(u(t)) - \vec{F}(\overline{U}_N(t)))\nabla G_{e_u} dxdt \leq (C_1 + \frac{1}{2}) \int_0^{\tilde{t}} ||e_u||^2_{L^2(\Omega)} dxdt + C_3(\Delta t)^2 \tag{4.9}$$

Similarly,

$$\begin{aligned} & \int_0^{\tilde{t}} \int_\Omega \nabla(\mathcal{H}(u(t)) - \mathcal{H}(\overline{U}_N(t)))\nabla G_{e_u} dxdt \\ = & \int_0^{\tilde{t}} \int_\Omega \nabla e_H \nabla G_{e_u} dxdt + \int_0^{\tilde{t}} \int_\Omega \nabla(\mathcal{H}(U_N(t)) - \mathcal{H}(\overline{U}_N(t)))\nabla G_{e_u} dxdt \\ = & \int_0^{\tilde{t}} \int_\Omega e_u e_H dxdt + \int_0^{\tilde{t}} \int_\Omega \nabla(\mathcal{H}(U_N(t)) - \mathcal{H}(\overline{U}_N(t)))\nabla G_{e_u} dxdt \\ \geq & \int_0^{\tilde{t}} \int_\Omega \nabla(\mathcal{H}(U_N(t)) - \mathcal{H}(\overline{U}_N(t)))\nabla G_{e_u} dxdt \\ = & \sum_{k=1}^{N} \int_{t^{k-1}}^{t^k} \int_\Omega \Big(\mathcal{H}(U_N(t)) - \mathcal{H}(\overline{U}_N(t))\Big) e_u dtdx \\ \geq & -C \int_\Omega \sum_{k=1}^{N} \int_{t^{k-1}}^{t^k} (\frac{1}{4}|u^k - u^{k-1}|^2 + |e_u|^2) dtdx \\ \geq & -C(\Delta t)^2 - \frac{1}{2} \int_\Omega \int_0^{\tilde{t}} |e_u|^2 dtdx, \end{aligned} \tag{4.10}$$

and

$$\tau \int_0^{\tilde{t}} \int_\Omega \partial_t \nabla e_u \nabla G_{e_u} dxdt = \tau \int_\Omega \partial_t e_u e_u dxdt = \frac{\tau}{2} \int_\Omega e_u(\cdot, \tilde{t})^2 dx. \tag{4.11}$$

Using above, taking $\psi = G_{e_u}$ in (4.4) gives

$$\frac{1}{2} \int_\Omega (\nabla G_{e_u}(\cdot, \tilde{t}))^2 dx + \frac{\tau}{2} \int_\Omega e_u(\cdot, \tilde{t})^2 dx \leq C_1(\Delta t)^2 + C_2 \int_0^{\tilde{t}} \int_\Omega |e_u|^2 dxdt. \tag{4.12}$$

Using Gronwall's inequality, we obtain the estimate

$$||e_u||_{L^\infty(0,T;L^2(\Omega))} \leq C\Delta t. \tag{4.13}$$

□

Remark 4.1 *From (4.13), since H is Lipschitz continuous, we immediately obtain*

$$||e_H(\cdot, t)||_{L^\infty(0,T;L^2(\Omega))} \leq C\Delta t. \tag{4.14}$$

5 Numerical example

In this section, we give a numerical example to verify the theoretical findings. We solve the following equation in $Q = (0, 1) \times (0, 1]$

$$\frac{\partial u}{\partial t} = \frac{1}{6}\frac{\partial}{\partial x}([u]_+\frac{\partial u}{\partial x}) + \frac{1}{6}\frac{\partial^3 u}{\partial^2 x \partial t} - \frac{1}{2(1+t)^2}, \tag{5.1}$$

with initial and boundary conditions

$$u(x, 0) = x(1 - x), \quad u(0, t) = u(1, t) = 0. \tag{5.2}$$

Here

$$[u]_+ = \begin{cases} u & \text{if} \quad u > 0, \\ 0 & \text{if} \quad u \le 0. \end{cases} \tag{5.3}$$

therefore the equation becomes degenerate whenever $u \le 0$. For the equation (5.1), the exact solution is

$$u(x, t) = \frac{x(1 - x)}{1 + t}. \tag{5.4}$$

In the following, we use this solution to test the numerical scheme.

5.1 Numerical scheme

Before giving the numerical results, we present an iterative scheme to solve the time discretization problems. To do so, taking $\Delta t = 1/N (N \in \mathbb{N})$ and denoting $f(t) = \frac{1}{2(1+t)^2}$, formally we get

$$\frac{u^n - u^{n-1}}{\Delta t} = \frac{1}{6}\partial_x([u^n]_+\partial_x u^n) + \frac{1}{6}\partial_{xx}(\frac{u^n - u^{n-1}}{\Delta t}) - f(t^n). \tag{5.5}$$

Define the Kirchhoff transform

$$v = \beta(u) := \frac{1}{6}\int_0^u (\Delta t[s]_+ + 1)ds = \begin{cases} \frac{\Delta t}{12}u^2 + \frac{1}{6}u, & \text{if} \quad u > 0 \\ \frac{1}{6}u, & \text{if} \quad u \le 0, \end{cases} \tag{5.6}$$

instead of solving (5.5), we seek $v^n = \beta(u^n)$ such that

$$\beta^{-1}(v^n) - \partial_{xx}v^n = u^{n-1} - \frac{1}{6}\partial_{xx}u^{n-1} - \Delta t f(t^n). \tag{5.7}$$

with $v^n = 0$ at $x = 0$ and $x = 1$. To solve (5.7), we use the following iteration method inspired from [26], pp. 90-100 (also see e.g. [8], [24]):

$$6v^{n,i} - \partial_{xx}v^{n,i} = 6v^{n,i-1} - \beta^{-1}(v^{n,i-1}) + \alpha(u^{n-1}, t^n), \tag{5.8}$$

where $i = 1, 2...$ and

$$\alpha(u^{n-1}, t^n) = u^{n-1} - \frac{1}{6}\partial_{xx} u^{n-1} - \Delta t f(t^n). \tag{5.9}$$

This iteration requires a starting point $v^{n,0}$. As will be proved below, the iteration is convergent for any $v^{n,0}$. However, for the practical reasons, we choose $v^{n,0} = v^{n-1} = \beta(u^{n-1})$.

Lemma 5.1 *The iteration method (5.8) is convergent in the $W^{1,2}(0,1)$ norm.*

Proof . We write (5.9) in weak form, find $v^{n,i} \in W_0^{1,2}(0,1)$ such that

$$(6v^{n,i}, \phi) + (\partial_x v^{n,i}, \partial_x \phi) = (6v^{n,i-1} - \beta^{-1}(v^{n,i-1}), \phi) + (\alpha(u^{n-1}, t^n), \phi). \tag{5.10}$$

for any $\phi \in W_0^{1,2}(0,1)$. Similarly,

$$(6v^{n,i-1}, \phi) + (\partial_x v^{n,i-1}, \partial_x \phi) = (6v^{n,i-2} - \beta^{-1}(v^{n,i-2}), \phi) + (\alpha(u^{n-1}, t^n), \phi). \tag{5.11}$$

Subtracting (5.10) from (5.11),

$$\begin{aligned} &6(v^{n,i} - v^{n,i-1}, \phi) + (\partial_x(v^{n,i} - v^{n,i-1}), \partial_x \phi) \\ &= 6(v^{n,i-1} - v^{n,i-2}, \phi) - (\beta^{-1}(v^{n,i-1}) - \beta^{-1}(v^{n,i-2}), \phi). \end{aligned} \tag{5.12}$$

Taking $\phi = v^{n,i} - v^{n,i-1}$ gives,

$$\begin{aligned} &6||v^{n,i} - v^{n,i-1}||^2_{L^2(\Omega)} + ||\partial_x(v^{n,i} - v^{n,i-1})||^2_{L^2(\Omega)} \leq \\ &||v^{n,i} - v^{n,i-1}||_{L^2(\Omega)} \cdot ||6(v^{n,i-1} - v^{n,i-2}) - (\beta^{-1}(v^{n,i-1}) - \beta^{-1}(v^{n,i-2}))||_{L^2(\Omega)}. \end{aligned} \tag{5.13}$$

From the definition of β, we have

$$\beta'(u) = \begin{cases} \frac{1}{6}(u\Delta t + 1), & \text{if} \quad u \geq 0 \\ \frac{1}{6}, & \text{otherwise.} \end{cases} \tag{5.14}$$

Therefore

$$(\beta^{-1})'(v) = \frac{1}{\beta'(u)} \in (0, 6]. \tag{5.15}$$

From (5.13), we obtain

$$6||v^{n,i} - v^{n,i-1}||^2_{L^2(\Omega)} + ||\partial_x(v^{n,i} - v^{n,i-1})||^2_{L^2(\Omega)} \leq 6||v^{n,i} - v^{n,i-1}||^2_{L^2(\Omega)} \cdot ||v^{n,i-1} - v^{n,i-2}||^2_{L^2(\Omega)}.$$

Using Poincaré inequality, $||u||_{L^2(0,1)} \leq ||\partial_x u||_{L^2(0,1)}$ for any $u \in W_0^{1,2}(0,1)$. Therefore

$$\begin{aligned} &||v^{n,i} - v^{n,i-1}||^2_{L^2(\Omega)} + \frac{1}{6}||\partial_x(v^{n,i} - v^{n,i-1})||^2_{L^2(\Omega)} \\ &\leq \frac{1}{2}(||v^{n,i} - v^{n,i-1}||^2_{L^2(\Omega)} + ||(v^{n,i-1} - v^{n,i-2})||^2_{L^2(\Omega)}) \\ &\leq \frac{1}{2}||v^{n,i} - v^{n,i-1}||^2_{L^2(\Omega)} + \frac{3}{8}||(v^{n,i-1} - v^{n,i-2})||^2_{L^2(\Omega)} + \frac{1}{8}||\partial_x(v^{n,i-1} - v^{n,i-2})||^2_{L^2(\Omega)} \end{aligned} \tag{5.16}$$

Define $||v^{n,i}||^2 = ||v^{n,i} - v^{n,i-1}||^2_{L^2(\Omega)} + \frac{1}{3}||\partial_x(v^{n,i} - v^{n,i-1})||^2_{L^2(\Omega)}$ (equivalent to the $W^{1,2}$ norm), we obtain

$$||v^{n,i}||^2 \leq \frac{3}{4}||v^{n,i-1}||^2, \tag{5.17}$$

using Banach fixed point theorem, we obtain the convergence of the iteration method (5.9).

5.2 Numerical results

We compute the numerical solution u^N of (5.1) and estimate the error $e_u = u - u_N$, with u the exact solution of (5.1). For simplicity, we only compute e_u at $t = 1$. To this aim, finite difference scheme on uniform mesh with $dx = 10^{-5}$ is coupled with different time stepping $dt = 10^{-1}, 10^{-2}, 10^{-3}$ and 10^{-4}. To solve the nonlinear problem at any two steps, we perform 3 to 4 iterations. This is sufficient to achieve $||v^{n,i} - v^{n,i-1}||_{L^2(\Omega)} \leq 10^{-5}$. The numerical results are presented in Table 1. As follows from Theorem 4.1, the error satisfies

$$||e_u(\cdot, 1)||_{L^2(\Omega)} \leq C\Delta t. \tag{5.18}$$

This is confirmed by the Table 1. In particular, we estimate C to 0.066.

dt	$\|\|e_u(\cdot,1)\|\|_{L^2(\Omega))}$	$ratio(\|\|e_u\|\|/dt)$
10^{-1}	6.1997×10^{-3}	6.1997×10^{-2}
10^{-2}	6.447×10^{-4}	6.447×10^{-2}
10^{-3}	6.4632×10^{-5}	6.4632×10^{-2}
10^{-4}	6.5842×10^{-6}	6.5842×10^{-2}

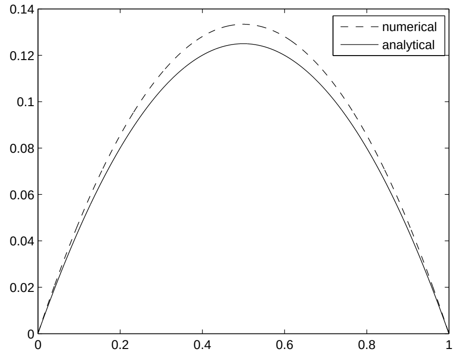
Table 1: Errors $e_u(\cdot, 1)$ for different dt

Figure 1 also displays numerical solutions for $u(\cdot, 1)$ at $t = 1$, compared to the exact solution for $dt = 10^{-1}$ and $dt = 10^{-2}$. For $dt = 10^{-3}$ and $dt = 10^{-4}$, one could not distinguish between the numerical solution and the analytical one.

6 Conclusion

In this paper, a class of degenerate pseudo-parabolic equations is investigated. This involves a vanishing nonlinear factor in the second order differential operator. We employ the Rothe method for proving the existence of a solution, and use a Green function approach for the uniqueness. Further, we estimate the error between the exact and the time discrete solution. Finally, these theoretical estimates are confirmed by a numerical example.

Acknowledgement We thank Prof. Andro Mikelić, Dr. Clément Cancès for their useful suggestions and discussions. Part of the work of Y.Fan was supported by the International Research Training Group NUPUS funded by the German Research Foundation DFG (GRK 1398) and The Netherlands Organisation for Scientific Research NWO (DN 81-754).



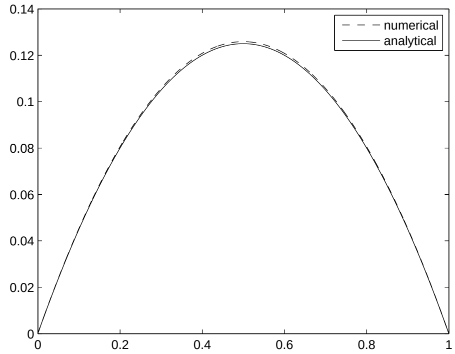


Figure 1: Numerical solution and exact solution for $dx = 10^{-5}, dt = 10^{-1}$(left) and $dt = 10^{-2}$(right)

References

[1] D.N. Arnold, Jr.J. Douglas and V. Thomée, *Superconvergence of a Finite Element Approximation to the Solution of a Sobolev Equation in a Single Space Variable*, Math. Comp. **36** (1981), 53–63.

[2] B.C. Aslan, W.W. Hager and S. Moskow, *A generalized eigenproblem for the Laplacian which arises in lightning*, J. Math. Anal. Appl. **341** (2008), 1028–1041.

[3] G.I. Barenblatt, I.P. Zheltov and I.N. Kochina, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks (strata)*, J. Appl. Math. Mech. **24** (1960), 1286–1303.

[4] C. Cuesta, J. Hulshof, *A model problem for groundwater flow with dynamic capillary pressure: stability of travelling waves*, Nonlinear Anal. **52** (2003), 1199–1218.

[5] C.M. Cuesta, I.S. Pop, *Numerical schemes for a pseudo-parabolic Burgers equation: Discontinuous data and long-time behaviour*, J. Comput. Appl. Math. **224** (2009), 269–283.

[6] C.J. van Duijn, Y. Fan, L.A. Peletier and I.S. Pop, *Travelling wave solutions for degenerate pseudo-parabolic equation modelling two-phase flow in porous media*, CASA Report **10-01**, TU Eindhoven, 2010.

[7] C.J. van Duijn, L.A. Peletier and I.S. Pop, *A new class of entropy solutions of the Buckley-Leverett equation*, SIAM J. Math. Anal. **39** (2007), 507–536.

[8] V.M. Devigne, I.S. Pop, C.J. van Duijn, T. Clopeau, *A numerical scheme for the pore-scale simulation of crystal dissolution and precipitation in porous media*, SIAM J. Numer. Anal. **46** (2008), 895–919.

[9] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, **19**. American Mathematical Society, Providence, RI, 2010.

[10] R.E. Ewing, *Time-stepping Galerkin methods for nonlinear Sobolev partial differential equations*, SIAM J. Numer. Anal. **15** (1978), 1125–1150.

[11] W.H. Ford and T.W. Ting, *Unifrom error estimates for difference approximations to nonlinear pseudo-parabolic partial differential equations*, SIAM J. Numer. Anal. **11** (1974), 155–169.

[12] S.M. Hassanizadeh and W.G. Gray, *Thermodynamic basis of capillary pressure in porous media*, Water Resour. Res. **29** (1993), 3389–3405.

[13] R. Helmig, A. Weiss, B. Wohlmuth, *Dynamic capillary effects in heterogeneous porous media*, Comput. Geosci. **11** (2007), 261–274.

[14] J. Kačur, *Method of Rothe in evolution equations*, Teubner Texts Math. **80**, Teubner Verlagsgesellschaft, Leipzig, 1985.

[15] M. Lenzinger and B. Schweizer, *Two-phase flow equations with outflow boundary conditions in the hydrophobic-hydrophilic case*, Nonlinear Anal. **73** (2010), 840–853.

[16] O.A. Ladyzhenskaya, N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York-London 1968.

[17] E. Milne, *The diffusion of imprisoned radiation through a gas*, J. London Math. Soc. **1** (1926), 40–51.

[18] A. Mikelić, H. Bruining, *Analysis of model equations for stress-enhanced diffusion in coal layers. I. Existence of a weak solution*, SIAM J. Math. Anal. **40** (2008), 1671–1691.

[19] A. Mikelić, *A global existence result for the equations describing unsatured flow in porous media with dynamic capillary pressure*, J. Differential Equations **248** (2010), 1561–1577.

[20] M. Ptashnyk, *Nonlinear pseudoparabolic equations as sigular limit of reaction-diffusion equations*, Appl. Anal. **85** (2006), 1285–1299.

[21] M. Peszyńska, R. Showalter, S. Yi, *Homogenization of a pseudoparabolic system*, Appl. Anal. **88** (2009), 1265–1282.

[22] A. Quarteroni, *Fourier spectral methods for pseudo-parabolic equations*, SIAM J. Numer. Anal. **24** (1987), 323–335.

[23] L.I. Rubenstein, *On the problem of the process of propagation of heat in heterogeneous media*, IZV. Akad. Nauk SSSR, Ser. Geogr. **1** (1948).

[24] F.A. Radu, I.S. Pop, P. Knabner, *Mixed finite elements for the Richards' equation: linearization procedure*, J. Comput. Appl. Math. **168**(2004), 365–373.

[25] R.E. Showalter, *A nonlinear parabolic-Sobolev equation*, J. Math. Anal. Appl. **50** (1975), 183–190.

[26] J.Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, 1994.

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
10-40	V. Chalupecký T. Fatima A. Muntean	Multiscale sulfate attack on sewer pipes: Numerical study of a fast micro-macro mass transfer limit	July '10
10-41	E.J.W. ter Maten J. Rommes	Predicting 'parasitic ffects' in large-scale circuits	July '10
10-42	M.E. Rudnaya R.M.M. Mattheij J.M.L. Maubach	Derivative-based image quality measure	July '10
10-43	K. Kumar T.L. van Noorden I.S. Pop	Effective dispersion equations for reactive flows involving free boundaries at the micro-scale	July '10
10-44	Y. Fan I.S. Pop	A class of degenerate pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization	July '10

Ontwerp: de Tantes,
Tobias Baanders, CWI