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Abstract In this paper, we present a new gauge technique for the Newton Raphson method to solve the periodic steady state (PSS) analysis of free-running oscillators in the time domain. To find the frequency a new equation is added to the system of equations. Our equation combines a generalized eigenvector with the time derivative of the solution. It is dynamically updated within each Newton-Raphson iteration. The method is applied to an analytic benchmark problem and to an LC oscillator. It provides better convergence properties than when using the popular phase-shift condition. It also does not need additional information about the solution. The method can also easily be implemented within the Harmonic Balance framework.

1 Introduction

Designing an oscillator requires a Periodic-Steady State (PSS) analysis. The PSS solution can be found by long time integration, starting from perturbing the DC-solution. It is needed for phase noise analysis [2]. Time integration is robust (it al-

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ways works: the DC-solution is an unstable PSS solution), but the convergence can be very slow. Therefore dedicated solution methods have been presented based in time-domain, frequency-domain or by hybrid circuit-state representations [1, 7, 11]. In these methods the period T or the frequency f is an additional unknown. To make the solution unique an additional equation, like a phase-shift condition, is added [3, 5, 8]. The overall system of equations is solved by a Newton-Raphson method that needs initial estimates for the solution as well as for T (or f).
This paper presents a Newton-Raphson based method with a dynamic additional condition to find the PSS solution of a free-running oscillator in the time domain. Here generalized eigenvectors of the linearized circuit equations and the time derivative at each time step provide a new robust gauge equation for the Newton-Raphson equations. The method is applied to an analytic benchmark problem and to an LC oscillator. The efficiency of the method is verified through numerical experiments. It provides better convergence properties than when using the popular phase-shift condition. It also does not need additional information about the solution.

2 The autonomous oscillator problem

The PSS problem for autonomous circuits on one overall period T is defined as a system of Differential-Algebraic Equations (DAEs) in the following form,

$$\frac{\mathrm{d}\mathbf{q}(\mathbf{x})}{\mathrm{d}t} + \mathbf{j}(\mathbf{x}) = \mathbf{0} \in \mathbb{R}^n, \tag{1}$$

$$\mathbf{x}(0) = \mathbf{x}(T), \tag{2}$$

where $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$ and T are unknown; $\mathbf{q}$ and $\mathbf{j}$ are known functions of $\mathbf{x}$. In the above autonomous circuit, there is a non-trivial PSS solution in the absence of sources. Here the period T (or the frequency $f = 1/T$) is unknown and is determined by the system. By transforming the simulation time interval $[0, T]$ to the standard interval $[0, 1]$, f enters the above equations as a parameter

$$f\frac{\mathrm{d}\mathbf{q}(\mathbf{x})}{\mathrm{d}t} + \mathbf{j}(\mathbf{x}) = 0. \tag{3}$$

Taking f as extra unknown, we need an extra equation to complete the system. Usually one requires the additional constraint condition

$$\mathbf{c}^T\mathbf{x}(t_c) - c = 0, \tag{4}$$

to provide a non-zero value for some vector $\mathbf{c}$ which makes the phase-shift unique. For instance, one provides the value of a particular coordinate of $\mathbf{x}$ at some time t_c.

3 Newton procedure

We discretize $[0,1]$ using equidistant time points $t_i = i\Delta t$ for $i = 0,\ldots,N$ with $N\Delta t = 1$. Thus, $t_0 = 0$, $t_N = 1$. Let $\mathbf{x}_i$ approximate $\mathbf{x}(t_i)$ and $\mathbf{X} = \begin{bmatrix}\mathbf{x}_0 \cdots \mathbf{x}_{N-1}\end{bmatrix}^T$. We discretize (3) by applying Simpson's Rule on the (overlapping) sub-intervals $[t_{i-1}, t_{i+1}]$, for $i = 1,\ldots,N$, yielding

$$\mathbf{F}_i(\mathbf{X}, f) = f\,\frac{\mathbf{q}(\mathbf{x}_{i+1}) - \mathbf{q}(\mathbf{x}_{i-1})}{2\Delta t} + \frac{\mathbf{j}(\mathbf{x}_{i-1}) + 4\,\mathbf{j}(\mathbf{x}_i) + \mathbf{j}(\mathbf{x}_{i+1})}{6}, \quad i = 1,\ldots,N. \tag{5}$$

For $i = N-1$ and $i = N$ we apply the periodicity constraint $\mathbf{x}_N = \mathbf{x}_0$ and $\mathbf{x}_{N+1} = \mathbf{x}_1$. Let $t_c = t_{k'}$ for some k' and redefine $\mathbf{c}$ to apply to $\mathbf{X}$. We write $\mathbf{q}_i = \mathbf{q}(\mathbf{x}_i)$ and similarly for $\mathbf{j}_i$. The Newton-Raphson method to solve the discrete systems becomes

$$\mathbf{M}^k \begin{bmatrix} \mathbf{X}^{k+1} - \mathbf{X}^k \\ f^{k+1} - f^k \end{bmatrix} = - \begin{bmatrix} \mathbf{F}(\mathbf{X}^k, f^k) \\ \mathbf{c}^T\mathbf{X}^k - c \end{bmatrix}, \tag{6}$$

in which $\mathbf{X}^k = \begin{bmatrix}\mathbf{x}_0^k \cdots \mathbf{x}_{N-1}^k\end{bmatrix}^T$ and

$$\mathbf{F}(\mathbf{X}, f) = \begin{bmatrix} f\,\frac{\mathbf{q}_2 - \mathbf{q}_0}{2\Delta t} + \frac{\mathbf{j}_0 + 4\mathbf{j}_1 + \mathbf{j}_2}{6} \\ \vdots \\ f\,\frac{\mathbf{q}_0 - \mathbf{q}_{N-2}}{2\Delta t} + \frac{\mathbf{j}_{N-2} + 4\mathbf{j}_{N-1} + \mathbf{j}_0}{6} \\ f\,\frac{\mathbf{q}_1 - \mathbf{q}_{N-1}}{2\Delta t} + \frac{\mathbf{j}_{N-1} + 4\mathbf{j}_N + \mathbf{j}_1}{6} \end{bmatrix}, \quad \mathbf{M}^k = \begin{bmatrix} \mathbf{A}^k & \mathbf{b}^k \\ \mathbf{c}^T & \delta \end{bmatrix}. \tag{7}$$

Here

$$\mathbf{A}^k = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{X}^k, f^k} = f^k \cdot \mathbf{C}^k + \mathbf{G}^k, \quad \mathbf{b}^k = \left.\frac{\partial \mathbf{F}}{\partial f}\right|_{\mathbf{X}^k, f^k}, \tag{8}$$

for suitable matrices $\mathbf{C}$ and $\mathbf{G}$, that are composed by the local Jacobians $\mathbf{C}_i = \left.\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_i}$ and $\mathbf{G}_i = \left.\frac{\partial \mathbf{j}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_i}$ and the discretization step size,

$$\mathbf{C} = \tfrac{1}{2\Delta t}\begin{bmatrix} -\mathbf{C}_0 & \mathbf{0} & \mathbf{C}_2 & & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{C}_1 & \mathbf{0} & \mathbf{C}_3 & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \mathbf{C}_0 & & \cdots & \mathbf{0} & -\mathbf{C}_{N-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_1 & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{C}_{N-1} \end{bmatrix}, \quad \mathbf{G} = \tfrac{1}{6}\begin{bmatrix} \mathbf{G}_0 & 4\mathbf{G}_1 & \mathbf{G}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_1 & 4\mathbf{G}_2 & \mathbf{G}_3 & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \mathbf{G}_0 & & \cdots & \mathbf{0} & \mathbf{G}_{N-2} & 4\mathbf{G}_{N-1} \\ 4\mathbf{G}_0 & \mathbf{G}_1 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{G}_{N-1} \end{bmatrix}. \tag{9}$$

Usually $\delta = 0$ in (7). The matrix $\mathbf{A}$ becomes badly conditioned when the Newton iterands converge. This is due to the fact that the time derivative of the PSS solution solves the linearized homogeneous circuit equations when linearized at the PSS solution. Hence when the discretization is exact this time derivative of the ultimate PSS is in the kernel of $\mathbf{A}$. Due to this conditioning problem the vectors $\mathbf{b}$ and $\mathbf{c}$ and (scalar) value δ are really needed to make the matrix $\mathbf{M}$ non-singular (otherwise one could use a Schur complement approach). $\mathbf{b}$ must have non-trivial components

in Ker($\mathbf{A}$) and in Ker($\mathbf{A}^T$), both. A similar statement holds for $\mathbf{c}$. Hence Ker($\mathbf{A}$) $\not\perp$ Ker($\mathbf{A}^T$).

4 Bordered matrices

Theorem 1. *Let* $\mathbf{A}^+$ *be the Moore-Penrose inverse of* $\mathbf{A}$ *[6]. Define* $\mathbf{g}$, $\mathbf{h}$, $\mathbf{u}$, $\mathbf{v}$, α *by*

$$\begin{aligned}
&\mathbf{g} = \mathbf{A}^+\mathbf{b}, \quad \mathbf{h} &&= \mathbf{c}^*\mathbf{A}^+ \quad \text{(least squares approximations)},\\
&\mathbf{u} = (\mathbf{I}-\mathbf{A}\mathbf{A}^+)\mathbf{b}, \quad \mathbf{v} &&= \mathbf{c}^*(\mathbf{I}-\mathbf{A}^+\mathbf{A}) \quad \text{(projection errors)},\\
&\alpha = \delta - \mathbf{c}^*\mathbf{A}^+\mathbf{b}.
\end{aligned}$$

Then $\mathbf{g}$, $\mathbf{h}$, $\mathbf{u}$ *and* $\mathbf{v}$ *satisfy*

$$\begin{aligned}
\mathbf{A}^+\mathbf{u} = 0, \quad \mathbf{v}\mathbf{A}^+ &= 0,\\
\mathbf{u}^+\mathbf{A} = 0, \quad \mathbf{A}\mathbf{v}^+ &= 0, \quad [(\mathbf{u}^+)^T \in \mathrm{Ker}(\mathbf{A}^T),\ \ \mathbf{v}^+ \in \mathrm{Ker}(\mathbf{A})],\\
\mathbf{h}\mathbf{A}+\mathbf{v} = \mathbf{c}^*, \quad \mathbf{A}\mathbf{g}+\mathbf{u} &= \mathbf{b},\\
\mathbf{v}\mathbf{g} = 0, \quad \mathbf{h}\mathbf{u} &= 0,\\
\mathbf{h}\mathbf{A}\mathbf{A}^+ = \mathbf{h}, \quad \mathbf{A}^+\mathbf{A}\mathbf{g} &= \mathbf{g}, \quad \mathbf{h}\mathbf{A}\mathbf{g} = \delta - \alpha.
\end{aligned}$$

We are now able to derive more detailed expressions for the generalized inverse of a bordered matrix. See also [2,3] and [4,9] for cases where $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

Theorem 2. *Let*

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^* & \delta \end{bmatrix}, \quad \tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{A} & \mathbf{u} \\ \mathbf{v} & \alpha \end{bmatrix}. \tag{10}$$

Assume $\mathbf{u} \neq 0$ *and* $\mathbf{v} \neq 0$*, then*

$$\mathbf{M}^+ = \begin{bmatrix} \mathbf{A}^+ - \mathbf{g}\mathbf{u}^+ - \mathbf{v}^+\mathbf{h} - \delta\,\mathbf{v}^+\mathbf{u}^+ & \mathbf{v}^+ \\ \mathbf{u}^+ & 0 \end{bmatrix}, \quad \tilde{\mathbf{M}}^+ = \begin{bmatrix} \mathbf{A}^+ - \alpha\,\mathbf{v}^+\mathbf{u}^+ & \mathbf{v}^+ \\ \mathbf{u}^+ & 0 \end{bmatrix}. \tag{11}$$

The expression for $\tilde{\mathbf{M}}^+$ follows by checking the Moore-Penrose conditions [6]. For $\mathbf{M}^+$ we note that, when $\delta = \alpha + \mathbf{c}^*\mathbf{g}$,

$$\mathbf{M} = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{h} & 1 \end{pmatrix} \tilde{\mathbf{M}} \begin{pmatrix} \mathbf{I} & \mathbf{g} \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\mathbf{M}^+ = \begin{pmatrix} \mathbf{I} & -\mathbf{g} \\ 0 & 1 \end{pmatrix} \tilde{\mathbf{M}}^+ \begin{pmatrix} \mathbf{I} & 0 \\ -\mathbf{h} & 1 \end{pmatrix}.$$

Let Ker($\mathbf{A}$) $= \langle \mathbf{a} \rangle$, Ker($\mathbf{A}^T$) $= \langle \mathbf{a}_T \rangle$ ($\mathbf{a}$ and $\mathbf{a}_T$ unit vectors) and let $\mathbf{b} \in \langle \mathbf{a}_T \rangle$ and $\mathbf{c} \in \langle \mathbf{a} \rangle$. Then the most simple expressions appear because $\mathbf{g} = \mathbf{0}$, $\mathbf{h} = \mathbf{0}$, $\mathbf{u} = \mathbf{b}$, $\mathbf{v} = \mathbf{c}^*$. Furthermore, there also is robustness in the sense that if we have other

choices then the bordered matrix may still be non-singular. Note that the lower right entries in $\mathbf{M}^+$ and $\tilde{\mathbf{M}}^+$ are zero (which may not happen for $\mathbf{M}$ or $\tilde{\mathbf{M}}$).
For the bordered matrix $\mathbf{M}^k$ in (7) the choice of $\mathbf{b}^k$ comes from the partial differentiation with respect to the chosen additional unknown f. The choice of $\mathbf{c}$ depends on the "gauge" equation that we add to the system. The matrix $\mathbf{A}$ is a matrix pencil, hence a choice for a generalized (kernel) eigenvector is best here. As equation we prefer the bi-orthogonality equation. This prevents all problems with determining the location of the oscillation and the range of values of the PSS solution.

5 Using generalized eigenvalue methods

A proper dynamic expression within the loops for the vector c can increase the convergence rate of the Newton method. Generalized eigenvalue methods for matrix pencils are good candidates for obtaining a dynamic vector $\mathbf{c}$ to make $\mathbf{M}$ non-singular. Applying these methods in each Newton iteration gives the eigentriples $(\mathbf{v}, \mathbf{w}, \lambda)$ such that $[\lambda f \mathbf{C} + \mathbf{G}]\mathbf{v} = \mathbf{0}$ and $\mathbf{w}^T[\lambda f \mathbf{C} + \mathbf{G}] = \mathbf{0}$. Generalized eigenvalue methods are provided by the DPA (Dominant Pole Algorithm) and RQI (Raleigh Quotient Iteration) [10]. Here a combination of these methods (SARQI) is used to obtain a good accuracy and convergence rate.
The $\mathbf{v}$ and $\mathbf{w}$ have a bi-orthogonality relation with the matrix $\mathbf{C}$, $\mathbf{w}^T\mathbf{C}\mathbf{v} = 1$. In Section 3 we observed that in the limit when the Newton approximations are close to the exact solution, the right-hand side eigenvector $\mathbf{v}$ for the λ closest to 1 is close to $\mathrm{d}\mathbf{X}/\mathrm{d}t$ (up to a normalisation factor). Hence by approximating the bi-orthogonality relation by

$$\mathbf{w}^T \cdot \mathbf{C} \cdot \left.\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}\right|_{\mathbf{X}=\mathbf{X}^k} - 1 = 0. \tag{12}$$

we obtain a good choice for a dynamic gauge equation within each iteration of the Newton method. To write (12) even in the form $\mathbf{c}^T\mathbf{X} - c = 0$, we express $\mathrm{d}\mathbf{X}/\mathrm{d}t$ into $\mathbf{X}$. Spectral differentiation [12] provides $\mathrm{d}\mathbf{X}/\mathrm{d}t = \mathbf{D} \cdot \mathbf{X}$ with good accuracy using some matrix $\mathbf{D}$. This results in a choice $\mathbf{c}^T = \mathbf{w}^T \cdot \mathbf{C} \cdot \mathbf{D}$ and $c = 1$.
We observe that we always can compare $\mathbf{v}$ with $\mathrm{d}\mathbf{X}/\mathrm{d}t$ for convergence. We may even consider $\mathbf{c}^T = \mathbf{v}^T$. Additionally we can compare λf_{old} with f. We finally note that spectral differentiation easily fits Harmonic Balance implementations.

6 Analytic benchmark oscillator

As an example, consider the analytic benchmark problem [7],

$$\begin{aligned} \tfrac{\mathrm{d}y}{\mathrm{d}t} &= z + \varepsilon\left(1 - \sqrt{y^2+z^2}\right) y, \\ \tfrac{\mathrm{d}z}{\mathrm{d}t} &= -y + \varepsilon\left(1 - \sqrt{y^2+z^2}\right) z. \end{aligned} \tag{13}$$

The fact that we can tune convergence speed with ε makes this particular problem a suitable benchmark problem. For all ε the exact PSS solution of this problem is $y(t) = \sin(t - t_c)$, $z(t) = \cos(t - t_c)$, where t_c is some constant phase shift. The period $T = 2\pi$. By defining $r^2 = y^2 + z^2$, the system of equations (13) can be written in the form of (1),

$$\mathbf{x} = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{j} = \begin{bmatrix} -\varepsilon(1-r)y - z \\ -\varepsilon(1-r)z + y \end{bmatrix}.$$

Starting with initial conditions $T_0 = 2.2\pi$, $y_0(t) = 1.5\sin(t + \pi/4)$, $z_0(t) = \cos(t)$, and $N = 101$ (100 actual time grid points), the PSS solution is obtained using the old phase-shift condition method and the new eigenvector condition method. Figure 1 shows the initial guess and the PSS solution of $y(t)$ for both methods when $\varepsilon = 0.1$. For both methods we determine the maximum of the normalized correction of the

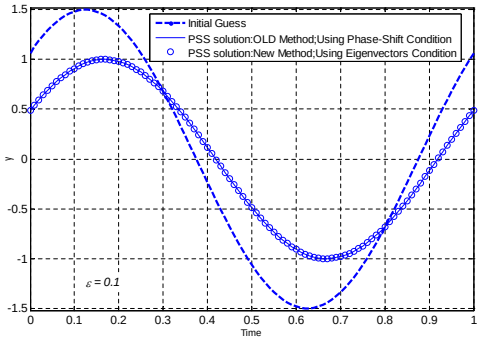

Fig. 1 Initial guess and *PSS* solution of $y(t)$ for different methods when $\varepsilon = 0.1$.

solution and the normalized frequency correction

$$\Delta \mathbf{X}^k\Big|_{\text{Normalized}} = ||\mathbf{X}^{k+1} - \mathbf{X}^k||_\infty / ||\mathbf{X}^k||_\infty, \quad \Delta f^k\Big|_{\text{Normalized}} = |f^{k+1} - f^k| / |f^k|$$

during each k-th Newton-Raphson iteration; the results are presented in Fig. 2. The better convergence behaviour of the new method is clearly observed. Although the simulation time and memory usage of the old method with a good phase-shift condition are smaller than that of the new method, the former method does not converge without enough information about $\mathbf{x}$ (see the curves with a $\times$ mark). Because of the observed robustness on the non-singularity of $\mathbf{M}^k$ (Section 4), one may stop the dynamic update of the gauge equation when the process starts converging.

7 LC oscillator

For many applications the free running oscillator can be modeled as an *LC* tank with a nonlinear resistor that is governed by the following differential equations for the

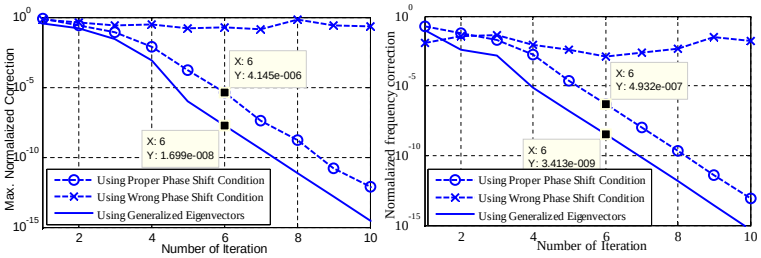

Fig. 2 Maximum of the normalized correction and normalized frequency correction for each iteration when $\varepsilon = 0.1$ for different methods.

unknowns v as the nodal voltage and i as the inductor current.

$$\begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R} & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} S\tanh(\frac{G_n v(t)}{S}) \\ 0 \end{bmatrix} = \mathbf{0} \tag{14}$$

$$v(0) = v_0, \quad i(0) = i_0. \tag{15}$$

where C, L and R are the capacitance, inductance and resistance, respectively. The voltage controlled nonlinear resistor is defined by the S and G_n parameters. For example, consider an oscillator designed for a frequency of 6 GHz with $L = 0.53\ nH$, $C = 1.33\ pF$, $R = 250\ \Omega$, $S = 1/R$, and $G_n = -1.1/R$. Starting with initial conditions $T_0 = 2.2\pi$, $v_0(t) = \sin(t)$, $i_0(t) = 0.2\sin(t)$, and $N = 101$ (100 actual grid points), the PSS solutions are obtained using the old phase-shift condition method and with the new eigenvector gauge method. The comparisons of the methods using the maximum of the normalized correction and the normalized frequency correction with respect to the iteration number k are presented in Fig. 3 showing similar improvement as in the previous example.

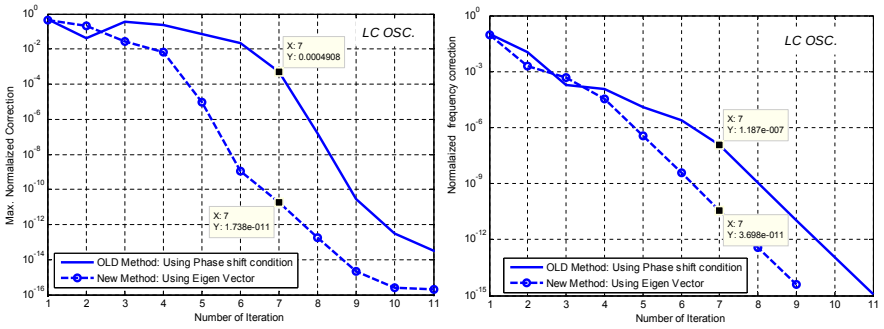

Fig. 3 Maximum of the normalized correction and normalized frequency correction for each iteration for different methods.

8 Conclusion

A new time-domain technique for the Newton-Raphson simulation of a free-running oscillator was presented. The generalized eigenvectors for the eigenvalue closest to 1 and the time derivative of the solution provide a robust gauge equation that is dynamically updated within each Newton-Raphson iteration. It was verified that the new method has better convergence properties compared to the popular phase-shift condition method and does not need additional information about the solution. The gauge equation also easily fits a Harmonic Balance environment.

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