Tensors and second quantization

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

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by

J. de Graaf

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
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J. de GRAAF

Abstract

Starting from a pair of vector spaces $\{\mathbf{X},\, \mathcal{L}(\mathbf{X})\}$, with $\mathbf{X}$ an inner product space and $\mathcal{L}(\mathbf{X})$, the space of linear mappings $\mathbf{X} \to \mathbf{X}$, we construct a six-tuple

$$\{\, \mathbf{X},\, \mathcal{L}(\mathbf{X}),\, \mathbf{X}^\sharp,\, \mathcal{L}(\mathbf{X}^\sharp),\, \mathfrak{C},\, (\cdot)^\sharp \,\}.$$

Here $\mathbf{X}^\sharp$ is again an inner product space and $\mathcal{L}(\mathbf{X}^\sharp)$ the space of its linear mappings. It is required that $\mathbb{C} \subset \mathbf{X}^\sharp,\ \mathbf{X} \subset \mathbf{X}^\sharp$, as linear subspaces.
$\mathbb{1} \in \mathbb{C}$ denotes 1.

Further, $\mathfrak{C}$ denotes a *creation* map

$$\mathfrak{C}:\ \mathbf{X} \to \mathcal{L}(\mathbf{X}^\sharp)\ :\ \ \mathbf{g} \mapsto \mathfrak{C}[\mathbf{g}] = \widehat{\mathbf{g}}, \quad \text{with} \quad \widehat{\mathbf{g}}\mathbb{1} = \mathbf{g},$$

and $(\cdot)^\sharp$ denotes a *lifting* map

$$(\cdot)^\sharp\ :\ \mathcal{L}(\mathbf{X}) \to \mathcal{L}(\mathbf{X}^\sharp)\ :\ \ \mathrm{K} \mapsto \mathrm{K}^\sharp,$$

such that, whenever $t \mapsto \boldsymbol{\psi}_j(t)$ solves an evolution equation in $\mathbf{X}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}_j(t) = \mathrm{K}\boldsymbol{\psi}_j(t),\ j = 1, 2, 3, \ldots,$$

then any product of operator valued functions

$$t \mapsto \widehat{\boldsymbol{\psi}}_1(t)\widehat{\boldsymbol{\psi}}_2(t)\cdots\widehat{\boldsymbol{\psi}}_k(t) \in \mathcal{L}(\mathbf{X}^\sharp),\ k = 1, 2, \ldots,$$

solves the associated *commutator equation* in $\mathcal{L}(\mathbf{X}^\sharp)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\widehat{\boldsymbol{\psi}}_1(t)\widehat{\boldsymbol{\psi}}_2(t)\cdots\widehat{\boldsymbol{\psi}}_k(t)\right) = \left[\,\mathrm{K}^\sharp\ ,\ \widehat{\boldsymbol{\psi}}_1(t)\widehat{\boldsymbol{\psi}}_2(t)\cdots\widehat{\boldsymbol{\psi}}_k(t)\,\right].$$

Furthermore, $t \mapsto \widehat{(\boldsymbol{\psi})}_j(t)\mathbb{1} = \boldsymbol{\psi}_j(t) \in \mathbf{X} \subset \mathbf{X}^\sharp$.

We also note that $t \mapsto \widehat{\boldsymbol{\psi}}_1(t)\widehat{\boldsymbol{\psi}}_2(t)\cdots\widehat{\boldsymbol{\psi}}_k(t)\mathbb{1} \in \mathbf{X}^\sharp$ represents the state of k identical systems 'living apart together'. Cf. the free field 'formalism' in physics.

Such constructions can be realized in many different ways (section 2). However in Quantum Field Theory one requires additional relations between the *creation operator* $\mathfrak{C}$ and its adjoint $\mathfrak{A} = \mathfrak{C}^\star$, the *annihilation operator*. These are the so called *Canonical (Anti-)Commutation Relations*, (section 3). Here, unlike in books on theoretical physics, the combinatorial aspects of those

[1] This note is meant to be Appendix K in the lecture notes 'Tensorrekening en Differentiaalmeetkunde'.

restrictions are dealt with in full detail. Annihilation/Creation operators don't grow on trees! However, apart from the way of presentation, nothing new is claimed here.

This note is completely algebraic. For topological extensions of the maps $\mathfrak{C}$, $\mathfrak{A}$ to distribution spaces we refer to Part III in [EG], where a mathematical interpretation of Dirac's formalism has been presented.

1 Some preliminary tensor bookkeeping

Let $\mathbf{X}$ be a complex vector space endowed with an inner product $(\,\cdot\,,\,\cdot\,)$.
We follow the physicists good convention of *anti-linearity* in the first entry and *linearity* in the second entry.
Typically, $\mathbf{X}$ is a complex Hilbert Space or a dense linear subspace of a complex Hilbert Space. This dense linear subspace has to be chosen a common invariant 'C^∞*-domain*' or '*analyticity domain*' for the set of operators under consideration. Of course, all those technicalities can be overlooked if $\dim \mathbf{X} < \infty$.

Notation

- The k-fold tensor product $\overset{k}{\mathbf{X}} = \underbrace{\mathbf{X} \bigotimes \cdots \bigotimes \mathbf{X}}_{k \text{ times}}$, $\quad k = 1, 2, \ldots$. We define $\overset{0}{\mathbf{X}} = \mathbb{C}$.

- A polyvector in $\overset{k}{\mathbf{X}}$ is the special tensor $\overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k$, where $\overset{k}{\mathbf{u}}_j \in \mathbf{X}\,, 1 \leq j \leq k$.

  The linear span of polyvectors is dense in $\overset{k}{\mathbf{X}}$. The unit in $\overset{0}{\mathbf{X}}$ is denoted $\mathbb{1}$.
  Note that if a k-tensor happens to be representable as a polyvector, such representation is not unique. Also the splitting of an 'entangled' k-tensor into a sum of polyvectors is not unique. Happily, in the sequel there is no need to bother about this.

- The inner product $(\,\cdot\,,\,\cdot\,)_k$ on $\overset{k}{\mathbf{X}}$ is derived from the inner product on $\mathbf{X}$.
  For two polyvectors it is defined by the product

$$(\overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k\,,\, \overset{k}{\mathbf{v}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{v}}_k)_k = (\overset{k}{\mathbf{u}}_1\,,\, \overset{k}{\mathbf{v}}_1) \bullet \cdots \bullet (\overset{k}{\mathbf{u}}_k\,,\, \overset{k}{\mathbf{v}}_k), \tag{1.1}$$

  followed by sesqui-linear extension.
  If for the $\overset{k}{\mathbf{u}}_j$ we pick (with possible repetition) elements from an othonormal basis $\{\mathbf{e}_\ell\}_{\ell=\mathbf{0}}^\infty \subset \mathbf{X}$, then the set of all possible such choices provides an orthonormal basis for $\overset{k}{\mathbf{X}}$.

Sometimes it is usefull to put a positive, possibly k-dependent, constant in front of (1.1).

- By $\overset{\otimes}{\mathbf{X}}$ is denoted the set of 'finite' sums in the direct orthogonal sum $\bigoplus_{k=0}^{\infty} \overset{k}{\mathbf{X}}$.

  So, only orthogonal sums with a *finite* number of terms $\neq 0$ are considered.

  When needed $\mathsf{U} \in \overset{\otimes}{\mathbf{X}}$ is orthogonally split $\mathsf{U} = \overset{0}{\mathsf{U}} \oplus \overset{1}{\mathsf{U}} \oplus \overset{2}{\mathsf{U}} \oplus \ldots$, with $\overset{k}{\mathsf{U}} \in \overset{k}{\mathbf{X}}$.

  For the special elements $\mathbb{1} \oplus 0 \oplus 0 \oplus 0 \cdots \oplus 0 \cdots$ and $0 \oplus \mathbf{g} \oplus 0 \cdots \oplus 0 \cdots$, we keep to the notation $\mathbb{1}$ and $\mathbf{g}$, respectively.

- The inner product on $\overset{\otimes}{\mathbf{X}}$ is taken to be the standard 'direct sum inner product' derived from the $(\,\cdot\,,\,\cdot\,)_k$.

- The respective vector spaces of linear mappings on and between $\mathbf{X}$, $\overset{k}{\mathbf{X}}$ and $\overset{\otimes}{\mathbf{X}}$ are denoted by $\mathcal{L}(\mathbf{X})$, $\mathcal{L}(\overset{k}{\mathbf{X}}, \overset{k-1}{\mathbf{X}})$, $\mathcal{L}(\overset{\otimes}{\mathbf{X}})$, etc.

  The set of bijective mappings $\mathrm{R} : \mathcal{L}(\mathbf{X}) \to \mathcal{L}(\mathbf{X})$ for which there exists a bijective adjoint $\mathrm{R}^\star : \mathcal{L}(\mathbf{X}) \to \mathcal{L}(\mathbf{X})$ is denoted by $\mathcal{B}\mathrm{ij}(\mathbf{X})$.

- For $\mathrm{K} \in \mathcal{L}(\mathbf{X})$ we introduce $\mathrm{K}^{k\,\boxplus} : \overset{k}{\mathbf{X}} \to \overset{k}{\mathbf{X}}$ by $\mathrm{K}^{0\,\boxplus} = 0$, if $k = 0$, and

  $$\mathrm{K}^{k\,\boxplus}(\overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k) = (\mathrm{K}\,\overset{k}{\mathbf{u}}_1) \otimes \overset{k}{\mathbf{u}}_2 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k + \cdots + \overset{k}{\mathbf{u}}_1 \otimes \overset{k}{\mathbf{u}}_2 \otimes \cdots \otimes (\mathrm{K}\,\overset{k}{\mathbf{u}}_k)\,, \quad k = 1, 2, \ldots,$$

  followed by linear extension.

- For $\mathrm{K} \in \mathcal{L}(\mathbf{X})$ we introduce $\mathrm{K}^{\boxplus} : \overset{\otimes}{\mathbf{X}} \to \overset{\otimes}{\mathbf{X}}$ by

  $$\mathrm{K}^{\boxplus} = \mathrm{diag}\,[\,\mathrm{H}^{0\boxplus}, \mathrm{K}^{1\boxplus}, \mathrm{K}^{2\boxplus}, \ldots, \mathrm{K}^{k\boxplus}, \ldots]$$

- For $\mathrm{R} \in \mathcal{L}(\mathbf{X})$ we introduce $\mathrm{R}^{k\otimes} : \overset{k}{\mathbf{X}} \to \overset{k}{\mathbf{X}}$ by $\mathrm{R}^{0\otimes} = \mathrm{I}$, if $k = 0$, and

  $$\mathrm{R}^{k\otimes}(\overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k) = (\mathrm{R}\,\overset{k}{\mathbf{u}}_1) \otimes (\mathrm{R}\,\overset{k}{\mathbf{u}}_2) \otimes \cdots \otimes (\mathrm{R}\,\overset{k}{\mathbf{u}}_k)$$

  followed by linear extension.

- For $\mathrm{R} \in \mathcal{L}(\mathbf{X})$ we introduce $\mathrm{R}^{\otimes} : \overset{\otimes}{\mathbf{X}} \to \overset{\otimes}{\mathbf{X}}$ by

  $$\mathrm{R}^{\otimes} = \mathrm{diag}\,[\,\mathrm{R}^{0\otimes}, \mathrm{R}^{1\otimes}, \mathrm{R}^{2\otimes}, \ldots, \mathrm{R}^{k\otimes}, \ldots]$$

- Note that for $\mathrm{K} \in \mathcal{L}(\mathbf{X})$

  $$e^{t\mathrm{K}^{k\boxplus}} = \left(e^{t\mathrm{K}}\right)^{k\otimes}, \quad k = 0, 1, 2, \ldots, \qquad \text{and} \qquad e^{t\mathrm{K}^{\boxplus}} = \left(e^{t\mathrm{K}}\right)^{\otimes}.$$

2 Lifting Evolution Equations to a Tensor Algebra

PROBLEM I
Investigate the existence of a pair $\mathfrak{A}$, $\mathfrak{C}$ of linear mappings with the properties,

$$\mathfrak{A} : \mathbf{X} \to \mathcal{L}(\overset{\otimes}{\mathbf{X}}) \ : \ \mathbf{g} \mapsto \mathfrak{A}[\mathbf{g}], \quad \mathfrak{C} : \mathbf{X} \to \mathcal{L}(\overset{\otimes}{\mathbf{X}}) \ : \ \mathbf{h} \mapsto \mathfrak{C}[\mathbf{h}], \tag{2.1}$$

where $\mathfrak{A}$ depends anti-linearly on $\mathbf{g}$ and where $\mathfrak{C}$ depends linearly on $\mathbf{g}$, such that

$$\begin{array}{rcl} \forall\, \mathbf{g} \in \mathbf{X} & : & \mathfrak{A}[\mathbf{g}]\mathbb{1} = 0, \quad \mathfrak{C}[\mathbf{g}]\mathbb{1} = \mathbf{g}, \quad \mathfrak{A}[\mathbf{g}]^\star = \mathfrak{C}[\mathbf{g}], \\ \forall\, \mathrm{R} \in \mathcal{B}\mathrm{ij}(\mathbf{X}) \ \forall\, \mathbf{g} \in \mathbf{X} & : & \begin{cases} \mathfrak{A}[\mathrm{R}\mathbf{g}] = \mathrm{R}^{\otimes}\, \mathfrak{A}[\mathbf{g}]\, (\mathrm{R}^{-1})^{\otimes}, \\ \mathfrak{C}[\mathrm{R}\mathbf{g}] = (\mathrm{R}^{-\star})^{\otimes}\, \mathfrak{C}[\mathbf{g}]\, (\mathrm{R}^{\star})^{\otimes}. \end{cases} \end{array} \tag{2.2}$$

Such operators are named *annihilation* and *creation* operators, respectively.

Definition 2.1
For any fixed $\mathbf{g} \in \mathbf{X}$ and $k = 1, 2, \ldots$, the linear mappings $\mathfrak{c}_k[\mathbf{g}]$, $\mathfrak{a}_k[\mathbf{g}]$ are introduced. As a start they are defined on polyvectors and next linearly extended.

$$\begin{array}{lcl} \mathfrak{c}_k[\mathbf{g}] & : & \overset{k-1}{\mathbf{X}} \to \overset{k}{\mathbf{X}} \ : \ \mathfrak{c}_k[\mathbf{g}]\,(\overset{k-1}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k-1}{\mathbf{u}}_{k-1}) = \overset{k-1}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k-1}{\mathbf{u}}_{k-1} \otimes \mathbf{g} \\ \mathfrak{a}_k[\mathbf{g}] & : & \overset{k}{\mathbf{X}} \to \overset{k-1}{\mathbf{X}} \ : \ \mathfrak{a}_k[\mathbf{g}]\,(\overset{k}{\mathbf{v}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{v}}_k) = (\mathbf{g}\,,\, \overset{k}{\mathbf{v}}_k)\, \overset{k}{\mathbf{v}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{v}}_{k-1} \end{array} \tag{2.3}$$

Note that $\mathfrak{c}_1[\mathbf{g}]\mathbb{1} = \mathbf{g}$, $\mathfrak{c}_k[\mathbf{g}]\mathbf{h} = \mathbf{h} \otimes \mathbf{g}$, $\mathfrak{a}_1[\mathbf{g}]\mathbb{1} = 0$, $\mathfrak{a}_1[\mathbf{g}]\mathbf{f} = (\mathbf{g}\,,\,\mathbf{f})$.

Properties 2.2

(a) $\mathbf{g} \mapsto \mathfrak{c}_k[\mathbf{g}]$ is linear. $\mathbf{g} \mapsto \mathfrak{a}_k[\mathbf{g}]$ is anti-linear.

(b) $\mathfrak{c}_k[\mathbf{g}]^\star = \mathfrak{a}_k[\mathbf{g}]$.

(c) $\forall\, \mathrm{R} \in \mathcal{B}\mathrm{ij}(\mathbf{X}) \ : \ \mathfrak{c}_k[\mathrm{R}\mathbf{g}] = \mathrm{R}^{k\otimes}\, \mathfrak{c}_k[\mathbf{g}]\, (\mathrm{R}^{-1})^{(k-1)\otimes}$

(d) $\forall\, \mathrm{R} \in \mathcal{B}\mathrm{ij}(\mathbf{X}) \ : \ \mathfrak{a}_k[\mathrm{R}\mathbf{g}] = (\mathrm{R}^{-\star})^{(k-1)\otimes}\, \mathfrak{c}_k[\mathbf{g}]\, (\mathrm{R}^\star)^{k\otimes}$

Definition 2.3
For any $\mathsf{U} \in \overset{\otimes}{\mathbf{X}}$ split $\mathsf{U} = \overset{0}{\mathsf{U}} \oplus \overset{1}{\mathsf{U}} \oplus \overset{2}{\mathsf{U}} \oplus \ldots$, with $\overset{k}{\mathsf{U}} \in \overset{k}{\mathbf{X}}$.
Take a fixed sequence $k \mapsto \theta_k > 0$, $k = 1, 2, \ldots$ and define two shift-like operators

$$\begin{array}{l} \mathfrak{C} \ : \ \mathbf{g} \to \mathfrak{C}[\mathbf{g}] \in \mathcal{L}(\overset{\otimes}{\mathbf{X}}) : \quad \mathbf{g} \mapsto \left\{ \mathsf{U} \mapsto \mathfrak{C}[\mathbf{g}]\mathsf{U} = \bigoplus_{k=1}^{\infty} \theta_k \mathfrak{c}_k[\mathbf{g}] \overset{k-1}{\mathsf{U}} \right\} \\ \mathfrak{A} \ : \ \mathbf{g} \to \mathfrak{A}[\mathbf{g}] \in \mathcal{L}(\overset{\otimes}{\mathbf{X}}) : \quad \mathbf{g} \mapsto \left\{ \mathsf{V} \mapsto \mathfrak{A}[\mathbf{g}]\mathsf{V} = \bigoplus_{k=0}^{\infty} \theta_{k+1} \mathfrak{a}_{k+1}[\mathbf{g}] \overset{k+1}{\mathsf{V}} \right\} \end{array} \tag{2.4}$$

The 'matrix representations' of the annihilation/creation operators are co-diagonal matrices. Their entries are operators in $\mathcal{L}(\overset{k}{\mathbf{X}}, \overset{\ell}{\mathbf{X}})$, with $k, \ell = 0, 1, 2, \ldots$.

$$\mathfrak{A}[\mathbf{g}] = \begin{pmatrix} 0 & \theta_1 \mathfrak{a}_1[\mathbf{g}] & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \theta_2 \mathfrak{a}_2[\mathbf{g}] & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \theta_3 \mathfrak{a}_3[\mathbf{g}] & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \theta_k \mathfrak{a}_k[\mathbf{g}] & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \ddots & \cdots \end{pmatrix}$$

$$\mathfrak{C}[\mathbf{h}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \theta_1 \mathfrak{c}_1[\mathbf{h}] & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & \theta_2 \mathfrak{c}_2[\mathbf{h}] & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & \theta_k \mathfrak{c}_k[\mathbf{h}] & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \ddots & 0 & \cdots & \cdots \end{pmatrix}$$

Theorem 2.4
The operators $\mathfrak{A}$ and $\mathfrak{C}$ in the previous definition, solve PROBLEM I if $\theta_1 = 1$ is taken. In particular we have for all $\mathbf{g} \in \mathbf{X}$, $\mathrm{K} \in \mathcal{L}(\mathbf{X})$, $t \in \mathbb{R}$

$$\begin{aligned} \mathfrak{A}[e^{t\mathrm{K}}\mathbf{g}] &= e^{t\mathrm{K}^{\boxplus}} \, \mathfrak{A}[\mathbf{g}] \, e^{-t\mathrm{K}^{\boxplus}}, \\ \mathfrak{C}[e^{t\mathrm{K}}\mathbf{g}] &= e^{-t(\mathrm{K}^{\boxplus})^\star} \, \mathfrak{C}[\mathbf{g}] \, e^{t(\mathrm{K}^{\boxplus})^\star} = e^{-t(\mathrm{K}^\star)^{\boxplus}} \, \mathfrak{C}[\mathbf{g}] \, e^{t(\mathrm{K}^\star)^{\boxplus}}. \end{aligned} \tag{2.5}$$

The functions $t \mapsto \mathbf{G}(t) = \mathfrak{A}[e^{t\mathrm{K}}\mathbf{g}] \in \mathcal{L}(\overset{\otimes}{\mathbf{X}})$ and $t \mapsto \mathbf{G}^\star(t) = \mathfrak{C}[e^{t\mathrm{K}}\mathbf{g}] \in \mathcal{L}(\overset{\otimes}{\mathbf{X}})$ solve the commutator evolution equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G}(t) &= \left[\mathrm{K}^{\boxplus}, \, \mathbf{G}(t)\right], \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{G}^\star(t) &= -\left[(\mathrm{K}^{\boxplus})^\star, \, \mathbf{G}^\star(t)\right]. \end{aligned} \tag{2.6}$$

Application to Schrödinger-type evolution equations
If $t \mapsto \boldsymbol{\psi}(t) \in \mathbf{X}$ solves the Schrödinger-type evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\psi}(t) = -i\mathrm{H}\boldsymbol{\psi}(t), \tag{2.7}$$

with H self-adjoint, then the functions $t \mapsto \mathfrak{A}[\boldsymbol{\psi}(t)] \in \mathcal{L}(\overset{\otimes}{\mathbf{X}})$ and $t \mapsto \mathfrak{A}[\boldsymbol{\psi}(t)] \in \mathcal{L}(\overset{\otimes}{\mathbf{X}})$ both solve the Heisenberg-type (= commutator-type) evolution equation in $\mathcal{L}(\overset{\otimes}{\mathbf{X}})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{C}[\boldsymbol{\psi}(t)] = -i\Big[\mathrm{H}^{\boxplus}, \, \mathfrak{C}[\boldsymbol{\psi}(t)]\Big] = -i\big\{\mathrm{H}^{\boxplus} \, \mathfrak{C}[\boldsymbol{\psi}(t)] - \mathfrak{C}[\boldsymbol{\psi}(t)]\mathrm{H}^{\boxplus}\big\}. \tag{2.8}$$

Applying this to the element $\mathbb{1} \in \overset{\otimes}{\mathbf{X}}$, we get back the original solution $t \mapsto \psi(t)$.
Note the minus sign discrepancy between (2.7) and [H]p.20: (2.37).

In applications the following modification is important

Theorem 2.5
Let $\mathscr{P}_k : \overset{k}{\mathbf{X}} \to \overset{k}{\mathbf{X}}$, $k = 0, 1, 2, \ldots$, *all be orthogonal projections.*
Suppose that for $k = 0, 1, 2, \ldots$ *the operators* $\mathscr{P}_k$ *and* $\mathrm{K}^{k\boxplus}$ *commute:* $\mathscr{P}_k \mathrm{K}^{k\boxplus} = \mathrm{K}^{k\boxplus} \mathscr{P}_k$.
Define the projection $\mathscr{P} : \overset{\otimes}{\mathbf{X}} \to \overset{\otimes}{\mathbf{X}}$ *by* $\mathscr{P} = \mathrm{diag}[\, \mathscr{P}_0 \,, \mathscr{P}_1 \,, \mathscr{P}_2 \,, \ldots]$.
Then also the functions $t \mapsto \mathscr{P}\mathbf{G}(t)\mathscr{P} \in \mathcal{L}(\overset{\otimes}{\mathbf{X}})$ *and* $t \mapsto \mathscr{P}\mathbf{G}^\star(t)\mathscr{P} \in \mathcal{L}(\overset{\otimes}{\mathbf{X}})$ *solve the equations (2.6).*
The operators in those operator valued functions are in $\mathcal{L}\big(\mathscr{P}(\overset{\otimes}{\mathbf{X}})\big)$.

Proof Put the operator $\mathscr{P}$ on both sides of the operator equations (2.6). We get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{P}\mathbf{G}(t)\mathscr{P} &= \big[\mathscr{P}\mathrm{K}^{\boxplus}\mathscr{P} \,,\, \mathscr{P}\mathbf{G}(t)\mathscr{P}\big] \,, \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{P}\mathbf{G}(t)\mathscr{P})^\star &= -\big[(\mathscr{P}\mathrm{K}^{\boxplus}\mathscr{P})^\star \,,\, (\mathscr{P}\mathbf{G}(t)\mathscr{P})^\star\big] \,. \end{aligned} \tag{2.9}$$

■

Note that, by way of example, $\mathscr{P}_k \mathrm{K}^{k\boxplus} = \mathrm{K}^{k\boxplus} \mathscr{P}_k$ holds if $\mathrm{K}^{k\boxplus}$ is normal **and** $\mathscr{P}_k$ projects on (the closure of) an invariant subspace of it. In its turn 'normality' is guaranteed if K is self-adjoint, skew-adjoint or unitary. Etc.

3 Canonical (Anti-)Commutation Relations

Of extreme importance in Quantum Field Theory are applications of Theorem 2.5 with the projection $\mathscr{P}^+$ on the **symmetric tensors** and the projection $\mathscr{P}^-$ on the **anti-symmetric tensors**, combined with the special choice of the constants: $\theta_k = \sqrt{k}$, cf. Def. 2.3, in the Annihilation-Creation operator pair.
For the following inspiration has been drawn from the Appendix on Multilinear Algebra, section 12 in [D].

Theorem 3.1 *If the inner product on* $\overset{k}{\mathbf{X}}$ *is chosen to be (a positive scalar multiple of) the inner product induced by* $(\,\cdot\,,\,\cdot\,)$ *on* $\mathbf{X}$*, the linear extensions of the mappings*

$$\mathscr{P}_k^- : \overset{k}{\mathbf{X}} \to \overset{k}{\mathbf{X}} \;:\; \overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k \;\mapsto \mathscr{P}_k^-\big(\overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k\,\big) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}^k} (-)^\sigma \overset{k}{\mathbf{u}}_{\sigma(1)} \otimes \cdots \otimes \overset{k}{\mathbf{u}}_{\sigma(k)}\,,$$

$$\mathscr{P}_k^+ : \overset{k}{\mathbf{X}} \to \overset{k}{\mathbf{X}} \;:\; \overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k \;\mapsto \mathscr{P}_k^+\big(\overset{k}{\mathbf{u}}_1 \otimes \cdots \otimes \overset{k}{\mathbf{u}}_k\,\big) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}^k} \overset{k}{\mathbf{u}}_{\sigma(1)} \otimes \cdots \otimes \overset{k}{\mathbf{u}}_{\sigma(k)}\,,$$

are orthogonal projections on the anti-symmetric and symmetric tensors in $\overset{k}{\mathbf{X}}$*, respectively.*

Proof
• Consider the antisymmetric case. We will show that for any pair of polyvectors $\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_k\,, \mathbf{g}_1 \otimes \cdots \otimes \mathbf{g}_k \in \overset{k}{\mathbf{X}}$, with $\mathbf{f}_j\,, \mathbf{g}_\ell \in \mathbf{X}$, the tensors

$$\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_k - \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}^k} (-)^\sigma\, \mathbf{f}_{\sigma(1)} \otimes \cdots \otimes \mathbf{f}_{\sigma(k)}, \quad \text{and} \quad \frac{1}{k!} \sum_{\tau \in \mathfrak{S}^k} (-)^\tau\, \mathbf{g}_{\tau(1)} \otimes \cdots \otimes \mathbf{g}_{\tau(k)},$$

are orthogonal. This is checked by calculating the inner product

$$\frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}^k} (-)^\sigma \sum_{\tau \in \mathfrak{S}^k} (-)^\tau \big(\mathbf{f}_{\sigma(1)} \otimes \cdots \otimes \mathbf{f}_{\sigma(k)}\,,\, \mathbf{g}_{\tau(1)} \otimes \cdots \otimes \mathbf{g}_{\tau(k)}\big) =$$

$$= \frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}^k} (-)^\sigma \sum_{\tau \in \mathfrak{S}^k} (-)^\tau (\mathbf{f}_{\sigma(1)}\,, \mathbf{g}_{\tau(1)}) \cdot \cdots \cdot (\mathbf{f}_{\sigma(k)}\,, \mathbf{g}_{\tau(k)}) =$$

$$= \frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}^k} (-)^\sigma \sum_{\tau \in \mathfrak{S}^k} (-)^\tau (\mathbf{f}_1\,, \mathbf{g}_{\tau\sigma^{-1}(1)}) \cdot \cdots \cdot (\mathbf{f}_k\,, \mathbf{g}_{\tau\sigma^{-1}(k)}) =$$

$$= \frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}^k} \sum_{\tau \in \mathfrak{S}^k} (-)^{\tau\sigma^{-1}} (\mathbf{f}_1\,, \mathbf{g}_{\tau\sigma^{-1}(1)}) \cdot \cdots \cdot (\mathbf{f}_k\,, \mathbf{g}_{\tau\sigma^{-1}(k)}) =$$

$$= \frac{1}{(k!)^2} \sum_{\sigma \in \mathfrak{S}^k} \sum_{\tau \in \mathfrak{S}^k} (-)^\tau (\mathbf{f}_1\,, \mathbf{g}_{\tau(1)}) \cdot \cdots \cdot (\mathbf{f}_k\,, \mathbf{g}_{\tau(k)}) = \frac{1}{k!} \sum_{\tau \in \mathfrak{S}^k} (-)^\tau (\mathbf{f}_1\,, \mathbf{g}_{\tau(1)}) \cdot \cdots \cdot (\mathbf{f}_k\,, \mathbf{g}_{\tau(k)}).$$

For the 3rd line in this derivation we have rearranged the product of inner products in such a way that the $\mathbf{f}_{\sigma(\cdot)}$ appear in the natural order. The 'inner product partner' of $\mathbf{f}_\ell$ is easily found if we put, for a moment, $\sigma(j_\ell) = \ell$. Then $\tau(j_\ell) = \tau(\sigma^{-1}(\ell))$.
For the 5th line note that, for fixed σ the permutations $\tau\sigma^{-1}$ run through the whole of $\mathfrak{S}^k$.
• For the symmetric case just omit all factors of type $(-)^\sigma$ in the previous consideration. ■

Remarks

- In the antisymmetric case one usually denotes

$$\mathscr{P}_k^-(\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_k) = \mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_k,$$

which is named *multi-vector*. This notation is consistent wit the notation for the 'exterior product', because of

$$(\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_k) \wedge (\mathbf{f}_{k+1} \wedge \cdots \wedge \mathbf{f}_{k+\ell}) = \mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_k \wedge \mathbf{f}_{k+1} \wedge \cdots \wedge \mathbf{f}_{k+\ell}\,.$$

This follows from the definition

$$(\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_k) \wedge (\mathbf{f}_{k+1} \wedge \cdots \wedge \mathbf{f}_{k+\ell}) = \mathscr{P}_{k+\ell}^-\Big((\mathbf{f}_1 \wedge \cdots \wedge \mathbf{f}_k) \otimes (\mathbf{f}_{k+1} \wedge \cdots \wedge \mathbf{f}_{k+\ell})\Big)\,,$$

and the property

$$\mathscr{P}_{k+\ell}^-\Big(\mathscr{P}_k^-(\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_k) \otimes \mathscr{P}_\ell^-(\mathbf{f}_{k+1} \otimes \cdots \otimes \mathbf{f}_{k+\ell})\Big) = \mathscr{P}_{k+\ell}^-\Big(\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_k \otimes \mathbf{f}_{k+1} \otimes \cdots \otimes \mathbf{f}_{k+\ell}\Big). \tag{3.1}$$

In order to prove the latter we define for any $\rho \in \mathfrak{S}^k\,,\ \sigma \in \mathfrak{S}^\ell$
the elements $\rho', \sigma' \in \mathfrak{S}^{k+\ell}$ by

$$\rho' :\ (1, 2, \ldots, k+\ell) \mapsto (\rho(1), \ldots, \rho(k), k+1, \ldots, k+\ell),$$

$$\sigma' :\ (1, 2, \ldots, k+\ell) \mapsto (1, \ldots, k, k+\sigma(1), \ldots, k+\sigma(\ell)).$$

Note that ρ', σ' commute and $(-)^{\rho'} = (-)^\rho\,,\ (-)^{\sigma'} = (-)^\sigma$.

Rewrite the left hand side of the desired identity

$$\frac{1}{k!}\frac{1}{\ell!}\frac{1}{(k+\ell)!}\sum_{\rho\in\mathfrak{S}^k}\sum_{\sigma\in\mathfrak{S}^\ell}\sum_{\tau\in\mathfrak{S}^{k+\ell}}(-)^\rho(-)^\sigma(-)^\tau\mathbf{f}_{\tau(\rho(1))}\otimes\cdots\otimes\mathbf{f}_{\tau(\rho(k))}\otimes\mathbf{f}_{\tau(k+\sigma(1))}\otimes\cdots\otimes\mathbf{f}_{\tau(k+\sigma(\ell))} =$$

$$= \frac{1}{k!}\frac{1}{\ell!}\frac{1}{(k+\ell)!}\sum_{\rho'\in\mathfrak{S}^k}\sum_{\sigma'\in\mathfrak{S}^\ell}\sum_{\tau\in\mathfrak{S}^{k+\ell}}(-)^{\tau\rho'\sigma'}\mathbf{f}_{\tau\rho'\sigma'(1))}\otimes\cdots\otimes\mathbf{f}_{\tau\rho'\sigma'(k+\ell)},$$

and note that the inner sum does not depend on ρ', σ' because for any fixed ρ', σ' the permutations $\tau\rho'\sigma'$ cover the whole of $\mathfrak{S}^{k+\ell}$ precisely once, which leads to the right hand side of (3.2)

- In the symmetric case one usually denotes

$$\mathscr{P}_k^+(\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_k) = \mathbf{f}_1 \vee \cdots \vee \mathbf{f}_k = \mathbf{f}_1\mathbf{f}_2 \cdots \mathbf{f}_k.$$

  The projected space $\mathscr{P}_k^+(\overset{k}{\mathbf{X}})$ corresponds with the space of polynomials of degree k on $\mathbf{X}$, via linear combinations of

$$\mathbf{X} \ni \mathbf{x} \quad \mapsto \quad (\mathbf{f}_1\,,\,\mathbf{x})(\mathbf{f}_2\,,\,\mathbf{x}) \bullet \cdots \bullet (\mathbf{f}_k\,,\,\mathbf{x}) \in \mathbb{C}.$$

The next observation enables us to apply Theorem 2.5.

Theorem 3.2
For all $\mathrm{K} \in \mathcal{L}(\mathbf{X})$ *and* $k = 0, 1, 2, \ldots$, *the operators* $\mathrm{K}^{k\boxplus}$, $\mathscr{P}_k^-$, $\mathscr{P}_k^+ : \overset{k}{\mathbf{X}} \to \overset{k}{\mathbf{X}}$, *commute.*

Proof
- Consider the anti-symmetric case,

$$\mathrm{K}^{k\boxplus}\mathscr{P}_k^- = \mathrm{K}^{k\boxplus}\Big(\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}^k}(-)^\sigma \mathbf{f}_{\sigma(1)} \otimes \mathbf{f}_{\sigma(2)} \otimes \cdots \otimes \mathbf{f}_{\sigma(k)} =$$

$$\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}^k}(-)^\sigma \mathrm{K}\mathbf{f}_{\sigma(1)} \otimes \mathbf{f}_{\sigma(2)} \otimes \cdots \otimes \mathbf{f}_{\sigma(k)} + \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}^k}(-)^\sigma \mathbf{f}_{\sigma(1)} \otimes \mathrm{K}\mathbf{f}_{\sigma(2)} \otimes \cdots \otimes \mathbf{f}_{\sigma(k)} + \cdots$$

$$\cdots + \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}^k}(-)^\sigma \mathbf{f}_{\sigma(1)} \otimes \mathbf{f}_{\sigma(2)} \otimes \cdots \otimes \mathrm{K}\mathbf{f}_{\sigma(k)}. \tag{3.2}$$

From each of those sums we first select the polyvectors that are built from the vectors $\mathrm{K}\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3, \ldots$, $\mathbf{f}_k$.

$$\begin{aligned}
&\sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathrm{K}\mathbf{f}_1 \otimes \mathbf{f}_{1+\sigma'(1)} \otimes \cdots \otimes \mathbf{f}_{1+\sigma'(k-1)} \; + \\
&\quad - \sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathbf{f}_{1+\sigma'(1)} \otimes \mathrm{K}\mathbf{f}_1 \otimes \mathbf{f}_{1+\sigma'(2)} \otimes \cdots \otimes \mathbf{f}_{1+\sigma'(k-1)} \; + \\
&\qquad + \sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathbf{f}_{1+\sigma'(1)} \otimes \mathbf{f}_{1+\sigma'(2)} \otimes \mathrm{K}\mathbf{f}_1 \otimes \cdots \otimes \mathbf{f}_{1+\sigma'(k-1)} \; + \\
&\qquad\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad\qquad +(-1)^k \sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathbf{f}_{1+\sigma'(1)} \otimes \mathbf{f}_{1+\sigma'(2)} \otimes \cdots \otimes \mathbf{f}_{1+\sigma'(k-1)} \otimes \mathrm{K}\mathbf{f}_1 \; = \\
&\qquad\qquad\qquad = \mathscr{P}_k^-\big(\mathrm{K}\mathbf{f}_1 \otimes \mathbf{f}_2 \otimes \cdots \otimes \mathbf{f}_k\big).
\end{aligned}$$

Note that the correct signs are easily found by specializing to $\sigma' \in \mathfrak{S}^k$ equal to the identical permutation.
Next we fix j, $1 \le j \le k$ and select from the respective sums in (3.2) all polyvectors that are built from the vectors $\mathbf{f}_1, \ldots \mathbf{f}_{j-1}$, $\mathrm{K}\mathbf{f}_j$, $\mathbf{f}_{j+1}, \ldots, \mathbf{f}_k$. The $\dot{+}$ is employed for 'addition

modulo k' of the indices. We find

$$
\begin{aligned}
&(-1)^{(k-j+1)(j-1)}\Big\{\sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathrm{K}\mathbf{f}_j\otimes\mathbf{f}_{j\dotplus\sigma'(1)}\otimes\cdots\otimes\mathbf{f}_{j\dotplus\sigma'(k-1)}\ +\\
&\quad-\sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathbf{f}_{j\dotplus\sigma'(1)}\otimes\mathrm{K}\mathbf{f}_j\otimes\mathbf{f}_{j\dotplus\sigma'(2)}\otimes\cdots\otimes\mathbf{f}_{j\dotplus\sigma'(k-1)}\ +\\
&\qquad+\sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathbf{f}_{j\dotplus\sigma'(1)}\otimes\mathbf{f}_{j\dotplus\sigma'(2)}\otimes\mathrm{K}\mathbf{f}_j\otimes\mathbf{f}_{j\dotplus\sigma'(3)}\otimes\cdots\otimes\mathbf{f}_{j\dotplus\sigma'(k-1)}\ +\\
&\qquad\quad\cdot\ \cdot\\
&\qquad\qquad+(-1)^k\sum_{\sigma'\in\mathfrak{S}^{k-1}}(-)^{\sigma'}\mathbf{f}_{j\dotplus\sigma'(1)}\otimes\mathbf{f}_{j\dotplus\sigma'(2)}\otimes\cdots\otimes\mathbf{f}_{j\dotplus\sigma'(k-1)}\otimes\mathrm{K}\mathbf{f}_j\Big\}\ =\\
&=(-1)^{(k-j+1)(j-1)}\mathscr{P}_k^-\big(\mathrm{K}\mathbf{f}_j\otimes\mathbf{f}_{j+1}\otimes\cdots\otimes\mathbf{f}_k\otimes\mathbf{f}_1\otimes\cdots\mathbf{f}_{j-1}\big)=\\
&\qquad\qquad\qquad=\mathscr{P}_k^-\big(\mathbf{f}_1\otimes\cdots\otimes\mathbf{f}_{j-1}\otimes\mathrm{K}\mathbf{f}_j\otimes\mathbf{f}_{j+1}\cdots\otimes\mathbf{f}_k\big).
\end{aligned}
$$

Note that $(-1)^{(k-j+1)(j-1)}$ is the sign of the permutation in $\mathfrak{S}^k$ that puts the sequence $j,\ j+1,\ \ldots,\ k,\ 1,\ \ldots,\ j-1$ in the natural order.
• With the results just obtained, it now follows

$$
\begin{aligned}
&\mathscr{P}_k^-\mathrm{K}^{k\boxplus}\big(\mathbf{f}_1\otimes\cdots\otimes\mathbf{f}_k\big)=\\
&\quad=\mathscr{P}_k^-\big(\mathrm{K}\mathbf{f}_1\otimes\mathbf{f}_2\otimes\cdots\otimes\mathbf{f}_k+\mathbf{f}_1\otimes\mathrm{K}\mathbf{f}_2\otimes\cdots\otimes\mathbf{f}_k+\cdots+\mathbf{f}_1\otimes\mathbf{f}_2\otimes\cdots\otimes\mathrm{K}\mathbf{f}_k\big)=\\
&\qquad=\mathrm{K}^{k\boxplus}\mathscr{P}_k^-\big(\mathbf{f}_1\otimes\cdots\otimes\mathbf{f}_k\big).
\end{aligned}
$$

■

Theorem 3.3 $\Big($**Canonical (Anti-)Commutation Relations**$\Big)$

• **Boson Case:** *The Creation-Annihilation pair*

$$
\begin{aligned}
&\overset{+}{\mathfrak{C}}:\ \mathbf{g}\mapsto\overset{+}{\mathfrak{C}}[\mathbf{g}]\in\mathcal{L}\big((\mathscr{P}^+(\overset{\otimes}{\mathbf{X}}))\big):\quad\mathbf{g}\mapsto\Big\{\,\mathsf{U}\mapsto\overset{+}{\mathfrak{C}}[\mathbf{g}]\mathsf{U}=\textstyle\bigoplus_{k=1}^{\infty}\sqrt{k}\mathscr{P}_k^+\mathfrak{c}_k[\mathbf{g}]\overset{k-1}{\mathsf{U}}\in\overset{\otimes}{\mathbf{X}}\,\Big\}\\
&\overset{+}{\mathfrak{A}}:\ \mathbf{g}\mapsto\overset{+}{\mathfrak{A}}[\mathbf{g}]\in\mathcal{L}\big((\mathscr{P}^+(\overset{\otimes}{\mathbf{X}}))\big):\quad\mathbf{g}\mapsto\Big\{\,\mathsf{V}\mapsto\overset{+}{\mathfrak{A}}[\mathbf{g}]\mathsf{V}=\textstyle\bigoplus_{k=0}^{\infty}\sqrt{k+1}\mathscr{P}_{k+1}^+\mathfrak{a}_{k+1}[\mathbf{g}]\overset{k+1}{\mathsf{V}}\in\overset{\otimes}{\mathbf{X}}\,\Big\}
\end{aligned}
\tag{3.3}
$$

is a pair of adjoint operators that satisfies the Commutation Relations

$$
\forall\ \mathbf{f}\,,\,\mathbf{g}\,\in\mathbf{X}\quad\begin{cases}\overset{+}{\mathfrak{C}}[\mathbf{f}]\,\overset{+}{\mathfrak{C}}[\mathbf{g}]-\overset{+}{\mathfrak{C}}[\mathbf{g}]\,\overset{+}{\mathfrak{C}}[\mathbf{f}]=0,\qquad\overset{+}{\mathfrak{A}}[\mathbf{f}]\,\overset{+}{\mathfrak{A}}[\mathbf{g}]-\overset{+}{\mathfrak{A}}[\mathbf{g}]\,\overset{+}{\mathfrak{A}}[\mathbf{f}]=0,\\ \qquad\qquad\overset{+}{\mathfrak{C}}[\mathbf{f}]\,\overset{+}{\mathfrak{A}}[\mathbf{g}]-\overset{+}{\mathfrak{A}}[\mathbf{g}]\,\overset{+}{\mathfrak{C}}[\mathbf{f}]=(\mathbf{f}\,,\,\mathbf{g})\,\mathrm{I}.\end{cases}
\tag{3.4}
$$

• **Fermion Case:** *Suppose* $\dim\mathbf{X}=\infty$. *The Creation-Annihilation pair*

$$
\begin{aligned}
&\overset{-}{\mathfrak{C}}:\ \mathbf{g}\mapsto\overset{-}{\mathfrak{C}}[\mathbf{g}]\in\mathcal{L}\big((\mathscr{P}^-(\overset{\otimes}{\mathbf{X}}))\big):\quad\mathbf{g}\mapsto\Big\{\,\mathsf{U}\mapsto\overset{-}{\mathfrak{C}}[\mathbf{g}]\mathsf{U}=\textstyle\bigoplus_{k=1}^{\infty}\sqrt{k}\mathscr{P}_k^-\mathfrak{c}_k[\mathbf{g}]\overset{k-1}{\mathsf{U}}\in\overset{\otimes}{\mathbf{X}}\,\Big\}\\
&\overset{-}{\mathfrak{A}}:\ \mathbf{g}\mapsto\overset{-}{\mathfrak{A}}[\mathbf{g}]\in\mathcal{L}\big((\mathscr{P}^-(\overset{\otimes}{\mathbf{X}}))\big):\quad\mathbf{g}\mapsto\Big\{\,\mathsf{V}\mapsto\overset{+}{\mathfrak{A}}[\mathbf{g}]\mathsf{V}=\textstyle\bigoplus_{k=0}^{\infty}\sqrt{k+1}\mathscr{P}_{k+1}^-\mathfrak{a}_{k+1}[\mathbf{g}]\overset{k+1}{\mathsf{V}}\in\overset{\otimes}{\mathbf{X}}\,\Big\}
\end{aligned}
\tag{3.5}
$$

is a pair of adjoint operators that satisfies the Anti-Commutation Relations

$$\forall\ \mathbf{f}\,,\ \mathbf{g}\in\mathbf{X}\quad\begin{cases}\bar{\mathfrak{C}}[\mathbf{f}]\,\bar{\mathfrak{C}}[\mathbf{g}]+\bar{\mathfrak{C}}[\mathbf{g}]\,\bar{\mathfrak{C}}[\mathbf{f}]=0, \qquad \bar{\mathfrak{A}}[\mathbf{f}]\,\bar{\mathfrak{A}}[\mathbf{g}]+\bar{\mathfrak{A}}[\mathbf{g}]\,\bar{\mathfrak{A}}[\mathbf{f}]=0,\\ \bar{\mathfrak{C}}[\mathbf{f}]\,\bar{\mathfrak{A}}[\mathbf{g}]+\bar{\mathfrak{A}}[\mathbf{g}]\,\bar{\mathfrak{C}}[\mathbf{f}]=(\mathbf{f}\,,\,\mathbf{g})\,\mathrm{I}.\end{cases}\tag{3.6}$$

Proof

We only deal with the Fermion-case. The Boson-case is a lot easier because meticulous bookkeeping of minus signs is not needed. The proofs proceed by applying the (hoped for) identities on arbitrary polyvectors of arbitrary length. Doing so, the proof is reduced to operations with $\mathfrak{c}_k$ and $\mathfrak{a}_k$. In our calculations factors with a roof $\widehat{\quad}$ have to be skipped.

- With (3.1)

$$\mathscr{P}^-_{k+2}\mathfrak{c}_{k+2}[\mathbf{f}]\mathscr{P}^-_{k+1}\mathfrak{c}_{k+1}[\mathbf{g}]\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)=\mathscr{P}^-_{k+2}\mathfrak{c}_{k+2}[\mathbf{f}]\mathscr{P}^-_{k+1}\Big(\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)\otimes\mathbf{g}\Big)=$$

$$=\mathscr{P}^-_{k+2}\mathfrak{c}_{k+2}[\mathbf{f}]\mathscr{P}^-_{k+1}\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\mathbf{g}\big)=\mathscr{P}^-_{k+2}\Big(\mathscr{P}^-_{k+1}\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\mathbf{g}\big)\otimes\mathbf{f}\Big)=$$

$$=\mathscr{P}^-_{k+2}\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\mathbf{g}\otimes\mathbf{f}\big)=\overset{k}{\mathbf{u}}_1\wedge\cdots\wedge\overset{k}{\mathbf{u}}_k\wedge\mathbf{g}\wedge\mathbf{f}.$$

Adding to this a similar expression with $\mathbf{f}$ and $\mathbf{g}$ interchanged we get 0.

- If two operators commute, so do their adjoints. This could also straightforwardly be proved starting from the identity

$$\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)=$$

$$=\frac{1}{k(k-1)}\sum_{1\le\ell<m\le k}(-1)^{2k-\ell-m-1}\Big(\mathscr{P}^-_{k-2}\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_\ell}\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_m}\otimes\cdots\overset{k}{\mathbf{u}}_k\big)\Big)\otimes\Big(\overset{k}{\mathbf{u}}_\ell\otimes\overset{k}{\mathbf{u}}_m-\overset{k}{\mathbf{u}}_m\otimes\overset{k}{\mathbf{u}}_\ell\Big).$$

- Finally the 3rd identity in (3.6)

$$\textbf{I.}\qquad\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)=\frac{1}{k}\sum_{\ell=1}^{k}(-1)^{k-\ell}\Big(\mathscr{P}^-_{k-1}\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_\ell}\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)\Big)\otimes\overset{k}{\mathbf{u}}_\ell$$

$$\mathfrak{a}_k[\mathbf{g}]\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)=\frac{1}{k}\sum_{\ell=1}^{k}(-1)^{k-\ell}(\mathbf{g}\,,\,\overset{k}{\mathbf{u}}_\ell)\Big(\mathscr{P}^-_{k-1}\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_\ell}\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)\Big)$$

$$\mathfrak{c}_k[\mathbf{h}]\mathscr{P}^-_{k-1}\mathfrak{a}_k[\mathbf{g}]\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)=\frac{1}{k}\sum_{\ell=1}^{k}(-1)^{k-\ell}(\mathbf{g}\,,\,\overset{k}{\mathbf{u}}_\ell)\Big(\mathscr{P}^-_{k-1}\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_\ell}\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)\Big)\otimes\mathbf{h}$$

$$\mathscr{P}^-_k\mathfrak{c}_k[\mathbf{h}]\mathscr{P}^-_{k-1}\mathfrak{a}_k[\mathbf{g}]\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\big)=\frac{1}{k}\sum_{\ell=1}^{k}(-1)^{k-\ell}(\mathbf{g}\,,\,\overset{k}{\mathbf{u}}_\ell)\mathscr{P}^-_k\big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_\ell}\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\mathbf{h}\big)$$

II. Put $\mathbf{h} = \overset{k}{\mathbf{u}}_{k+1}$, wherever convenient.

$$\mathscr{P}^-_{k+1}\Big(\mathfrak{c}_{k+1}[\mathbf{h}]\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k)\Big) = \mathscr{P}^-_{k+1}\Big(\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k)\otimes\mathbf{h}\Big) = \mathscr{P}^-_{k+1}\Big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\mathbf{h}\Big).$$

$$\mathfrak{a}_{k+1}[\mathbf{g}]\mathscr{P}^-_{k+1}\Big(\mathfrak{c}_{k+1}[\mathbf{h}]\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k)\Big) = \mathfrak{a}_{k+1}[\mathbf{g}]\mathscr{P}^-_{k+1}\Big(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\overset{k}{\mathbf{u}}_{k+1}\Big) =$$

$$= \frac{1}{k+1}\sum_{\ell=1}^{k+1}(-1)^{k+1-\ell}(\mathbf{g}\,,\,\overset{k}{\mathbf{u}}_\ell)\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_\ell}\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\overset{k}{\mathbf{u}}_{k+1}) =$$

$$= \frac{1}{k+1}(\mathbf{g}\,,\,\mathbf{h})\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k) +$$

$$-\frac{1}{k+1}\sum_{\ell=1}^{k}(-1)^{k-\ell}(\mathbf{g}\,,\,\overset{k}{\mathbf{u}}_\ell)\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\widehat{\overset{k}{\mathbf{u}}_\ell}\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k\otimes\mathbf{h})$$

Note that this expression equals 0 if it happens that $k \geq \dim\mathbf{X}$.

III. Adding the results of I and II

$$k\mathscr{P}^-_k\mathfrak{c}_k[\mathbf{h}]\mathscr{P}^-_{k-1}\mathfrak{a}_k[\mathbf{g}]\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k) + (k+1)\mathscr{P}^-_{k+1}\mathfrak{a}_{k+1}[\mathbf{g}]\mathscr{P}^-_{k+1}\Big(\mathfrak{c}_{k+1}[\mathbf{h}]\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k)\Big) =$$

$$= (\mathbf{g}\,,\,\mathbf{h})\mathscr{P}^-_k(\overset{k}{\mathbf{u}}_1\otimes\cdots\otimes\overset{k}{\mathbf{u}}_k).$$

The factors k and $k+1$ fit in precisely with the choice $\theta_k = \sqrt{k}$. ■

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