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HOW TO EXTEND ROBERTS' LAW FOR ECCENTRICALLY DRIVEN, INVERTED SLIDER-CRANKS

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Abstract. The extension of Roberts' Law concerns inverted slider-cranks with an eccentricity and a general tracing-point attached to the moving plane of the slider. As both curve-cognates degenerate in this case, the infinite turning-joint has *first* been replaced by a *finite* far-away joint, but such that the coupler-plane also meets *two* accuracy-positions of the erstwhile slider.
One of the curve-cognates may then be adapted or modified until up to *six* accuracy-positions are met. They seem to be sufficient to attain an extraordinary good approximation of the entire branch of the original curve. Besides, even the transmission-angle attains a value about twice as large as the one obtained with the *auxiliary* four-bar representing the *first* approximation.
Cases, where the inverted slider-crank turns into a crank-and-rocker, into a double crank or even into a stretchable four-bar, are shown. The latter corresponds with one of Grashof's border cases for the inverted slider-crank. Roberts' Law then ensures that each inverted slider-crank, having a fully revolving crank, possesses *three* four-bar branch-cognates this way.

1. Introduction

Though Roberts' Law (ref. [1]) describes the general existence of *three* four-bar curve cognates, all able to produce the *same* four-bar coupler curve, its application seems to collapse when applied on a degenerated four-bar such as the inverted slider-crank.
However, a previous paper [4] showed a practical circumvention in the special case where the inverted slider-cranks produced *symmetrical* curves. Then, a very good approximation was obtained based on *stretch-rotation* and *symmetrization*. However, for the more general type of the (eccentric) inverted slider-crank, symmetrization isn't applicable and has to be replaced by another procedure, apparently leading to a similar concurrence between the curves as in the symmetrical case.
In the *eccentric* case, we are going to start from in this paper, also *non*-Grashof types occur. Then, there is no crank making a complete revolution, whereas the curve produced will be singular branched as then only *one* branch occurs as the gradual merging result of two different branches originally produced by Grashof types.
In the *symmetrical* case the replacement four-bar reproduced only *one* branch generated by the then centrically driven inverted slider-crank. Naturally, something alike is to be expected for the eccentric case. Thus, when an eccentrically driven inverted slider-crank is of the Grasfof-type, two branches appear, each possibly replaceable by *one* coupler-branch of a Grashof four-bar. For the *Non*-Grashof type only part of the curve may be reproduced. This part then corresponds with either a forward - or, otherwise, a backward stroke of the "input-rocker" between its extreme positions (Fig. 9).

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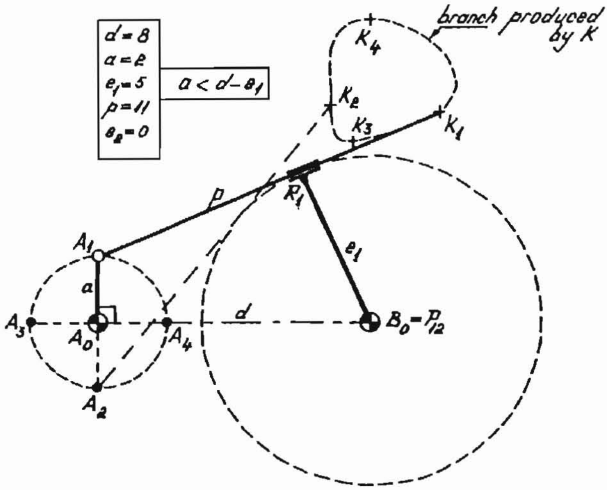

Fig. 1: Initial coupler branch produced by point K attached to the slider of an inverted slider-crank (4 accuracy positions chosen to design a 4-bar branch-cognate with)

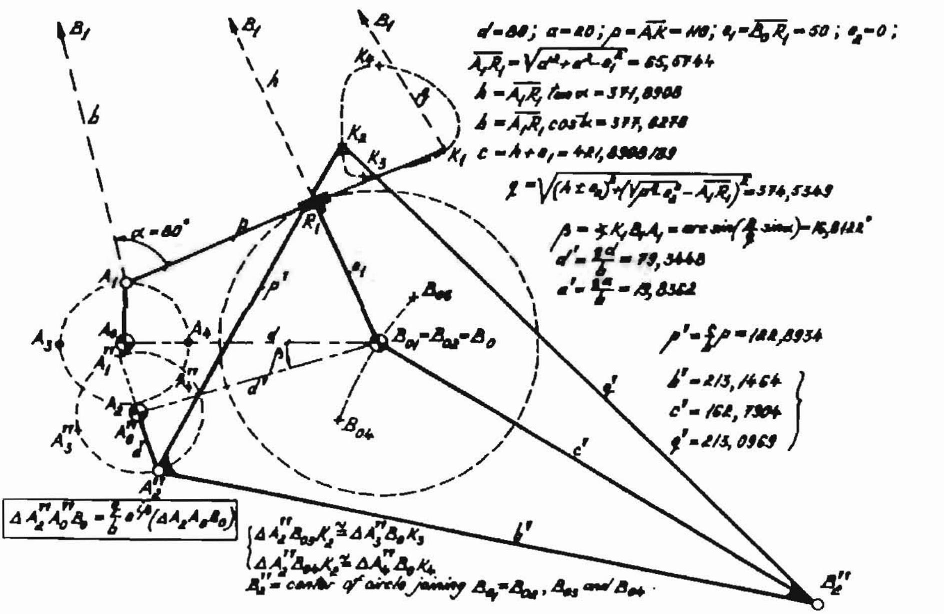

Fig. 2: Design of 4-bar branch cognate (nearly) producing the same coupler branch as the one produced by the eccentric inverted slider-crank (4 position synthesis with position reduction)

2. Investigated Types

It is possible to distinguish between *four* types of eccentric inverted slider cranks. Two of them are of the Grashof type, whereas two others are *Non*-Grashof. For the Grashof-type either $a + e_1 < d$ (Fig. 1), or $d + e_1 < a$ (Figrs. 3, 10).

For the *Non*-Grashof type, the crank-circle about A_0 has to intersect the eccentricity-circle (about B_0) at real points R and $\bar{R}$, leading to the very existence of $\Delta\ A_0RB_0$ meeting the three *Non*-Grashof conditions:

$a < d + e_1$

$d < e_1 + a$

$e_1 < a + d$

Of these, the last one has to be met anyway, otherwise no real mechanism is to be drawn with real tangents to the eccentricity-circle. Thus, as $e_1 < a + d$ represents an almost trivial condition, true also for Grashof linkages, only two cases are left for the *Non*-Grashof type. They are:

case I $\quad d - e_1 < a < d'$ (Fig. 9)
case II $\quad d < a < d + e_1$

Note that for the centric case, for which $e_1 = 0$, the Non-Grashof types disappear. Further, border-cases appear when either $a + e_1 = d$ (Fig. 6) or $d + e_1 = a$.

Of course, in order to be complete, all occurring cases are of interest when looking for the possible existence of four-bar branch cognates when Grashof linkages are at hand, or "half curve cognates" when *Non*-Grashof inverted slider cranks are presented (Fig. 9).

3. Design of the 4-bar branch-cognate

The actual design of the four-bar cognate replacing the inverted slider-crank occurs in *two* stages: (Figrs. 2, 4, 7, (9) and 10)
In the 1st stage a first approximation is adopted leading to an *auxiliary* four-bar, based on *two* specifically chosen accuracy positions of the coupler-point and on a simultaneous replacement of the slider.
In the 2nd stage, part of Roberts' Configuration is used to find another crank-circle, simultaneously giving the designer the opportunity to improve his first approximation of the branch through two other pairs of specifically chosen accuracy positions.
As a result, only *one* branch of the 4-bar coupler curve approximates the chosen branch, initially produced by the inverted slider-crank. For the other branch one finds another branch-cognate somewhat different from the first.
Accuracy positions, as will be proved, occur in pairs. That is to say, for each accuracy-position another exists leading to the same result. So, basically, the design uses only three accuracy positions, in reality being six.
The first pair is to be allocated for the auxiliary four-bar, whereas the remaining pairs are to be used for the final mechanism found in the 2nd stage.

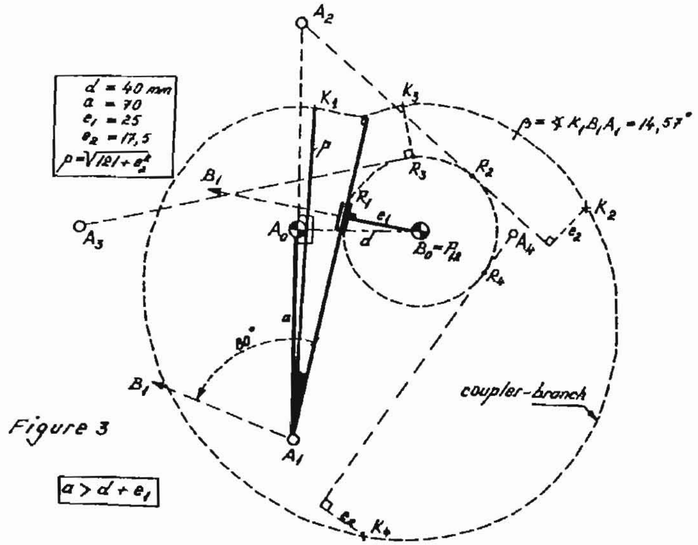

Fig. 3: Original coupler-branch produced by a coupler point K of an inverted slider-crank with eccentricity e_1 (4 accuracy positions are used to obtain the 4-bar branch cognate)

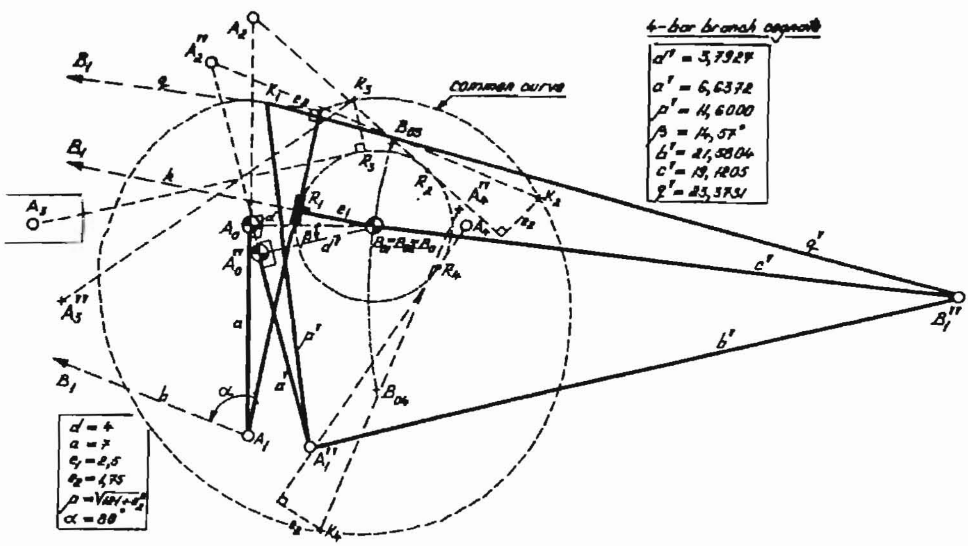

Fig. 4: The ecc. inverted slider-crank as well as the double crank (being the branch cognate) producing the same coupler branch

3.1 THE AUXILIARY FOUR-BAR

In order to simplify matters and to attain a unified method for all cases, the first pair of accuracy-positions are taken to correspond with the perpendicular positions of the input-crank with respect to the frame. (Figrs. **1, 3, 6, 10**). Then, P_{12}, the virtual *rotation center* of the positions, A_1K_1 and A_2K_2, coincides at the center B_0 of the eccentricity-circle. (The mid-normal of $B_1{}^{*}B_2{}^{*}$ always joins B_0.)
One further replaces the slider A_1R_1, touching the eccentricity-circle at R_1, by a basically required *large* "circle" with center B_1. Then, the last point, becoming a finite turning-joint, has to comply with the first two accuracy-positions, whence B_1 joins the normal of A_1K_1, simultaneously meeting point $P_{12} = B_0$.

In order to have a large length for $\quad c = \overline{B_1B_0} = h + e_1 \qquad (1)$

one chooses the angle $\alpha = \sphericalangle B_1A_1R_1 = 80^0$. (In case $\alpha = 90^0$, one reattains the inverted slider-crank.)
The dimensions of the auxiliary mechanism (A_0-$A_1K_1B_1$-B_0) having the same crank, the same coupler-point, and also the same center - B_0 - as the inverted slider-crank we started from, is then to be established from the initial dimensions: $\overline{A_0B_0} = d$; $\overline{A_1A_0} = a$;

$\overline{B_0R_1} = e_1$; $\overline{A_1K_1} = p$ and e_2, the latter being the shortest distance from K_1 to A_1R_1.

We so find that:

$$\overline{A_1R_1} = \sqrt{d^2 + a^2 - e_1^2} \qquad (2)$$

$$\overline{B_1R_1} = h = \overline{A_1R_1}\tan\alpha \qquad (3)$$

$$\overline{A_1B_1} = b = \overline{A_1R_1}\cos^{-1}\alpha \qquad (4)$$

$$\overline{B_1K_1} = q = \sqrt{(h \pm e_2)^2 + (\sqrt{p^2 - e_2^2} - \overline{A_1R_1})^2} \qquad (5)$$

$$\sphericalangle K_1B_1A_1 = \beta = \arcsin[\frac{p}{q}\sin(\alpha - \arcsin\frac{e_2}{p})] \qquad (6)$$

Clearly, the design of the auxiliary mechanism is the same for all Grashof-types having a fully revolving crank. Thus,

for $a \le d - e_1$ with $d > e_1$ (Figrs. **1, 6**)
for $a \ge d + e_1$ with $d > e_1$ (Fig. **3**)
and for $a \ge d + e_1$ with $d < e_1$ (Fig. **10**)

Of course, as $\alpha < 90^0$, the curve produced by the auxiliary mechanism will deviate a bit from the initial curve as produced by the eccentric inverted slider-crank. However, the deviation is to be corrected by application of the 2 nd stage leading to the final replacement mechanism.

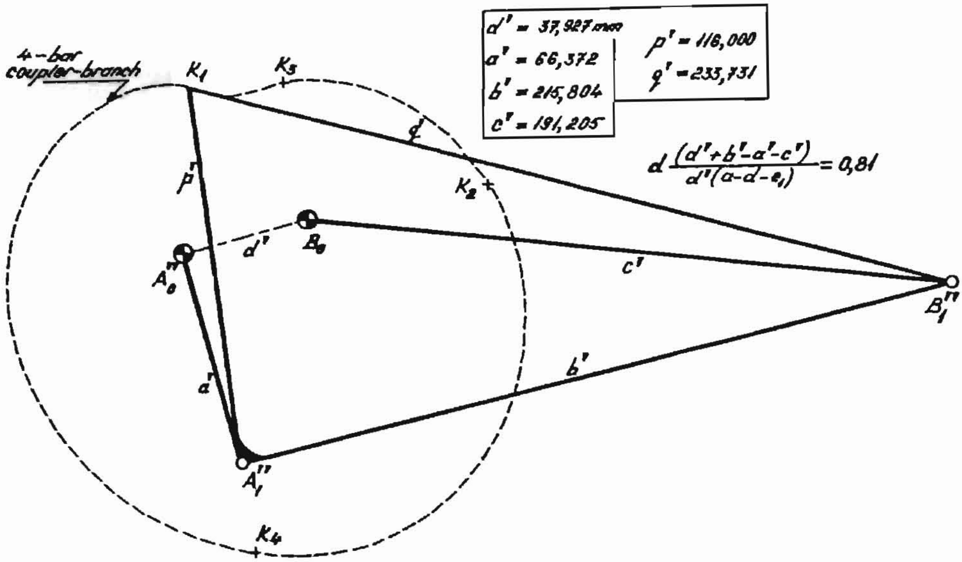

Fig. 5: The double crank as a four-bar branch cognate of an eccentrically driven inverted slider-crank (the 2 mechanisms produce about the same branch with nearly the same input velocity)

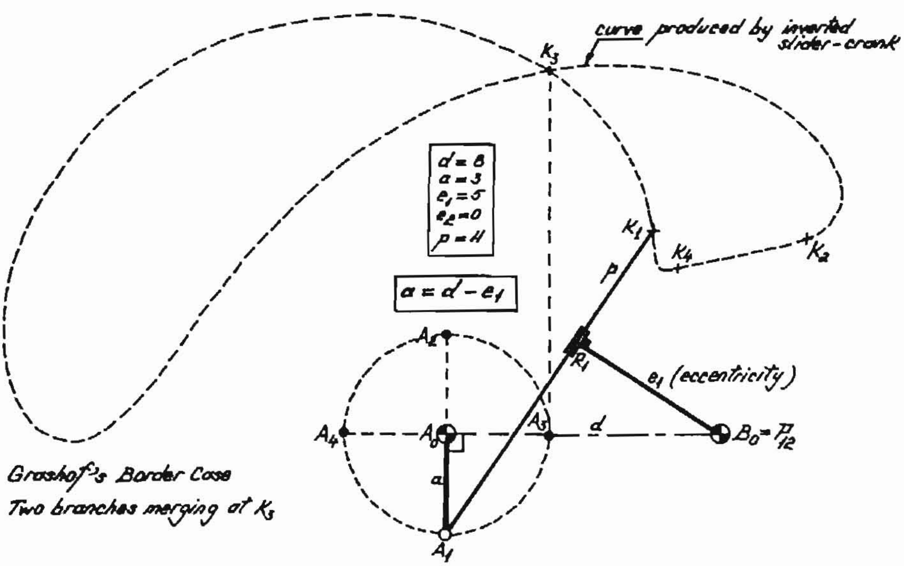

Fig. 6: Eccentrically driven, inverted slider crank (4 accuracy positions to be used for the design of a 4-bar branch cognate)

3.2 FINAL FOUR-BAR BRANCH COGNATE

Naturally, further enlargement of the angle α leads to lower values for the transmission angle $\mu_1 = \sphericalangle B_0B_1A_1$. (For instance, if $\alpha = 90^0$, then $\mu_1 = 0^0$)
As this is undesired with regard to force-transmission, other ways have to be found. Whence, we abide by $\alpha = 80^0$, but look at Roberts' Law and ditto Configuration instead. Though Roberts' Law may then be applied caused by the now finite location of point B_1, the transformation does not change the curve itself.
However, if we consider a particular curve-cognate of Roberts', namely the one having a crank rotating with the same speed as the angular velocity of the initial crank, it becomes feasible to change *three* of its cognate-dimensions, in order to meet *three* different accuracy positions. To carry this out in detail, the following procedure is adopted:

1. First, change the auxiliary four-bar into its Roberts'curve-cognate having a crank $A_0''A''$ rotating with the same speed as the initial crank A_0A,

2. Secondly, replace turning-joint $(B_1'')_{cognate}$ or $(B_2'')_{cognate}$ until B'' meets the three (or six) accuracy-positions being chosen.

In more detail, this would lead to the *stretch-rotation* of $\Delta\ A_1A_0B_0$ about B_0 with the complex multiplication-factor $\frac{q}{b}e^{i\beta}$, yielding

$$\Delta\ A_1''A_0''B_0 = \frac{q}{b}\ e^{i\beta}(\Delta\ A_1A_0B_0)\ . \tag{7}$$

For the 2nd accuracy-position, a similar equation holds:

$$\Delta\ A_2''A_0''B_0 = \frac{q}{b}e^{i\beta}(\Delta\ A_2A_0B_0) \tag{8}$$

Hence, we so obtain the simple formulas:

$$\overline{A_0''B_0} = d' = \frac{q}{b}d \tag{9}$$

$$\overline{A_1''A_0''} = a' = \frac{q}{b}a \tag{10}$$

Further, $\sphericalangle\ B_0A_0''A_1'' = 90^0 = -\sphericalangle\ B_0A_0''A_2''$ (11)

Whereas, similarly according to Roberts' Configuration, also

$$\overline{A_2''K_2} = \overline{A_1''K_1} = p' = \frac{c}{b}p \tag{12}$$

Naturally, the perpendicular bisectors of $\overline{A_1''A_2''}$ and of $\overline{K_1K_2}$ intersect at B_0. Thus,

$$\Delta\ A_2''B_0K_2 \equiv \Delta\ A_1''B_{02}K_1 \tag{13}$$

yielding $B_{01} = B_0 = B_{02}$. (14)

Then, with two other chosen accuracy positions from the initially given mechanism, say A_3K_3 and A_4K_4, it becomes quite possible to determine accurate corresponding positions,

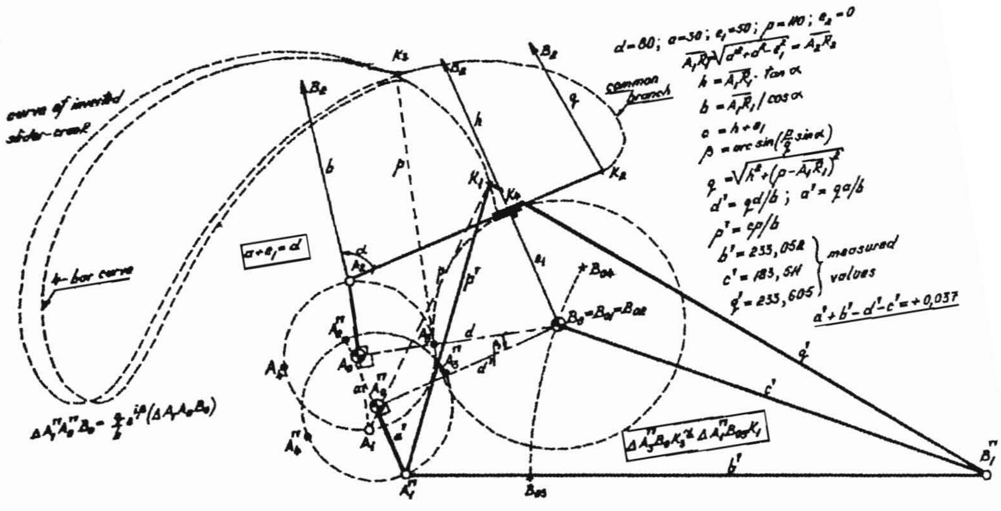

Fig. 7: Design of four-bar Branch-Cognate of inverted slider-crank
(Grashof's Border case $a + e_1 = d$ leading to $a' + b' > d' + c'$)

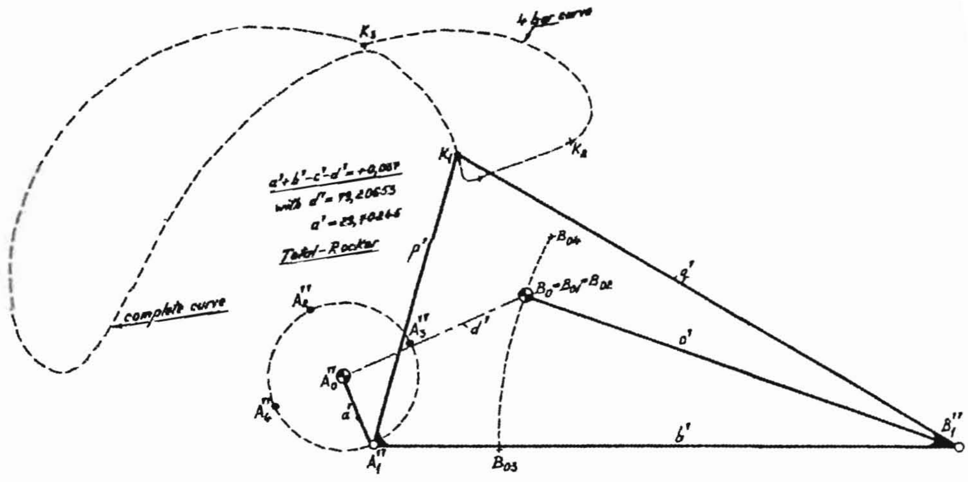

Fig. 8: Four-bar Branch Cognate of inverted slider-crank

A_3'' and A_4'', at the cognated crank-circle about A_0''. This may be realized by intersection of circles having radius p' respectively about K_3 and K_4, with the cognated crank-circle about A_0''

When, for instance, A_3 as well as A_4 join the initial fixed link A_0B_0, we find that both, A_3'' as well as A_4'', slightly deviates from the line containing the cognated fixed link, that is to say from $A_0''B_0$. Anyway, having determined their exact locations at the cognated crank-circle, we may establish the locations of the points, B_{03} and B_{04}. The way to find these points, will be based on the so-called *attachment-method*, hence, on the equations:

$$\Delta\ A_3''B_0K_3 = \Delta\ A_1''B_{03}K_1 \tag{15}$$

$$\Delta\ A_4''B_0K_4 = \Delta\ A_1''B_{04}K_1 \tag{16}$$

The circle joining the *three* points $B_{01} = B_{02}$, B_{03} and B_{04} then has the required turning-joint B_1'' for its center.

As a result, one obtains the new dimensions:

$$\overline{A_1''B_1''} = b' \quad , \quad \overline{B_0B_1''} = c' \quad \text{and} \quad \overline{K_1B_1''} = q' \tag{17}$$

In the particular case for which the initial branch is *symmetrical*, which occurs when the two eccentricities are zero, B_1'' coincides at the center of a circle joining K_1, B_0 and A_1''. In that case $b' = c' = q'$, meaning that the four-bar is of the λ-type (ref.[4]).

The accuracy-positions chosen, corresponded with the initial crank-positions dividing *360^0 into equal parts of 90^0*.

A *measure* for the accuracy of the cognate curve may be found by comparing the *Grashof-distances* of the two mechanisms. Then, the best approximation will be the one having its *Grashof-distance-ratio* much nearer the value 1 than with other approximations. For the mechanisms of Fig. 2, for instance, the auxiliary four-bar yields the Grashof-distance-ratio $\dfrac{a + c - d - b}{a - d + e_1} = 1.57$, whereas the final branch-cognate results into the ratio $\dfrac{d(a' + b' - c' - d')}{d'(a - d + e_1)} = 0.924$. Thus, the latter represents the better mechanism. (Note that the Grashof-distance-ratio equals 1 when two Roberts' 4-bar curve cognates are compared.)

For the auxiliary mechanism of Fig. 4, we obtain the Grashof-distance-ratio

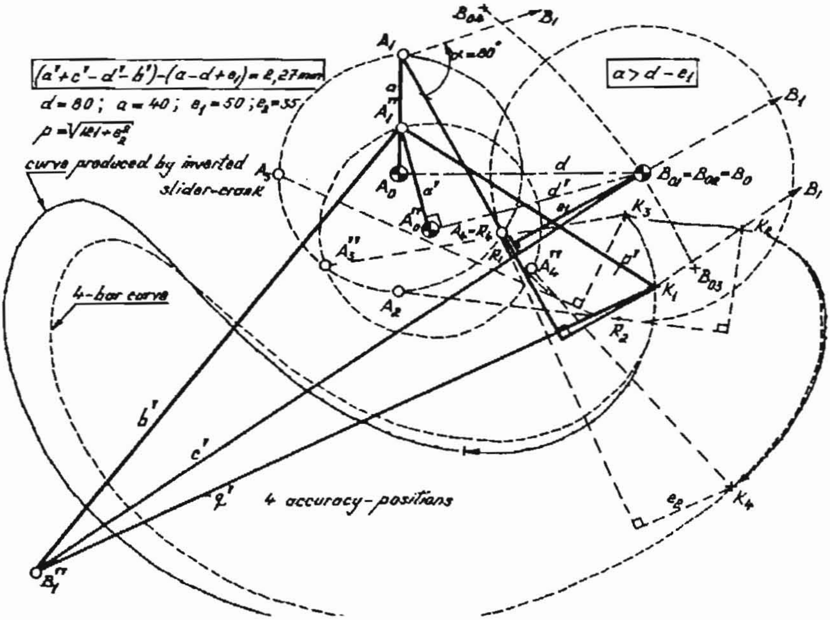

Fig. 9: Non-Grashof Curve-Cognate

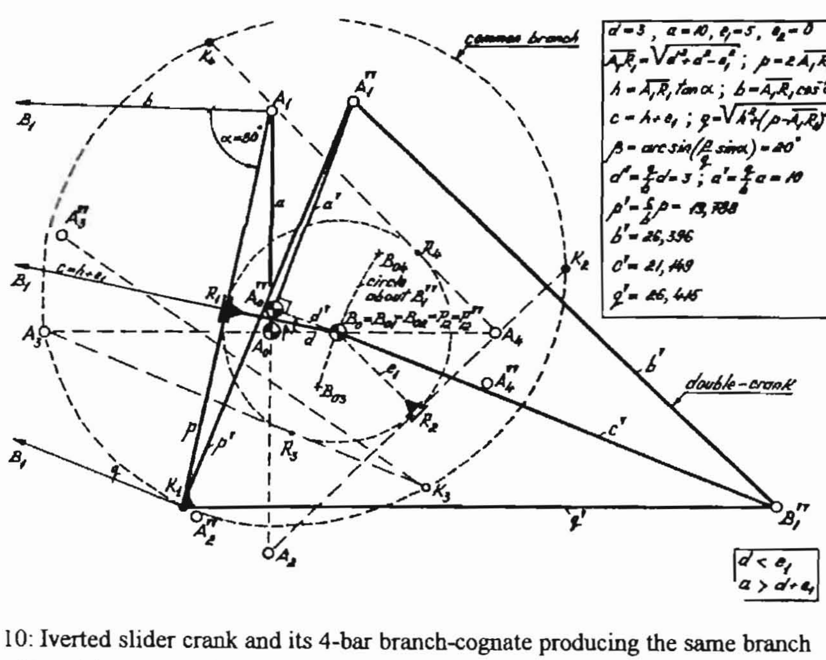

Fig. 10: Iverted slider crank and its 4-bar branch-cognate producing the same branch (4 position-synthesis using a point position reduction method: $B_{01} = B_{02}$)

$\frac{d + c - a - b}{a - d - e_1} = 2.34$, whereas the branch-cognate gives rise to the ratio

$\frac{d(d' + b' - a' - c')}{d'(a - d - e_1)} = 0.81.$

Even for Non-Grashof linkages, namely for those without revolving bars, the method remains applicable. Fig. 9, for instance, yields for the auxiliary mechanism the Grashof-distance-ratio $\frac{a + b - c - d}{a - d + e_1} = 1.65$, whereas the final curve-cognate gives the ratio

$\frac{d(a' + c' - d' - b')}{d'(a - d + e_1)} = 1.35.$

It is quite possible that an other distribution of our accuracy-positions, gives a better Grashof-ratio. Remarkable better results though, are not to be expected. For instance, if we choose point A_4 of Fig. 9 at the other intersection of the crank - and the eccentricity-circle, we obtain the ratio $\frac{d(a' + c' - d' - b')}{d'(a - d + e_1)} = 0.73.$ Now, its distance to 1 is only slightly less than with the 1.35-value belonging to the mechanism demonstrated in Fig. 9. In Grashof's border case, such as the particular one demonstrated in the Figrs. 6 and 7, we rather observe the *difference* between these Grashof-distances.
Then, for the auxiliary mechanism we precisely observe the value

$$(a + c - b - d) - (a + e_1 - d) = \sqrt{d^2 + a^2 - e_1^2}\ .\ (\sin\alpha - 1)\cos^{-1}\alpha =$$

$$= -40\sqrt{3}\ \tan\tfrac{1}{2}[(\pi/2) - \alpha] = -40\sqrt{3}\ \tan 5^0 = -6.0614\ .$$

As in this case each *Grashof-distance* should tend to zero, a considerable improvement is obtained with the final branch-cognate as demonstrated in Fig. 8. Then, namely

$$\frac{d}{d'}\ .\ (a' + b' - c' - d') - (a + e_1 - d) = +\ 0.037\ .$$

A random *circle* about B_0 intersecting the cognated crank-circle at possible accuracy-positions A''_m and A''_n , yields a virtual rotation-centre $P_{mn} = B_0$, giving $B_{0m} = B_{0n}$. Indeed, accuracy-positions only occur in pairs. Thus, each accuracy-position has to be counted twice, unless of course the cognated crank-positions already are at a circle about B_0, as will be the case for the accuracy-positions, A''_1 and A''_2 .
Generally, the transmission angle $\mu_1 = \sphericalangle A''_1B''_1B_0$ appears to be *about twice* as large as the one obtained with the auxiliary four-bar or with its Roberts' curve-cognate. Complete application of Roberts' Law *would* have lead to a point $(B''_1)_{cognate}$ as the fourth vertex of a linkage parallelogram $B_0B_1K_1(B''_1)_{cognate}$. However, such a point $(B''_1)_{cognate}$ would have been much farther away, giving about half the transmission-angle. All examples demonstrated show this same phenomenon. We conclude, that the 2 nd stage leads to a better approximation of the generated branch, simultaneously giving about twice the transmission angle in comparision to the one obtained in the 1 st stage.

Theoretically, the *circular* curve described by point B_0 of the Stephenson-2 six-bar $(A_0 - A - B^* - B_0 - A_0'' - A_1'' - K_1)$ with respect to the link $A_1''K_1$ will be a Stephenson-2 six-bar curve of order 16, (ref.[5]). Clearly, this curve approximates a circle very neatly, although in reality not more than 16 intersections exist. (We count 2 times 16, minus the (8 times 2) intersections of the circle at the two circular or isotropic points of the curve.)

4. Conclusions

Branches of curves produced by inverted slider-cranks having an eccentricity and/or an eccentrically located coupler point, are to be reproduced by even *three* different four-bar branch-cognates. In the two cases the curves are singular branched, the three four-bar curve-cognates reproduce only part of the curve, namely the part corresponding to either a forward - or, otherwise a backward stroke of the input-rocker. Better transmission angles may be attained at the cost of the accuracy of the reproduction. In most cases though a high accuracy is obtained with acceptable transmission angles. (The accuracy of the replacement-method has been measured with a newly introduced *Grashof-distance-ratio.*)

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