Nonperiodic inspections to guarantee a prescribed level of reliability

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Nonperiodic Inspections to Guarantee a Prescribed Level of Reliability

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Abstract: A cost-optimal nonperiodic inspection policy is derived for complex multicomponent systems. The model takes into consideration the degradation of all the components in the system with the use of a Bessel process with drift. The inspection times are determined by a deterministic function and depend on the system's performance measure. The nonperiodic policy is developed by evaluating the expected lifetime costs and the optimal policy by an optimal choice of inspection function. The model thus gives a guaranteed level of reliability throughout the life of the project.

Keywords and Phrases: Wiener process, Bessel process, regenerative process

9.1 Introduction

The aim of the chapter is to derive a cost-optimal inspection and maintenance policy for a multicomponent system whose state of deterioration is modelled with the use of a Markov stochastic process. Each component in the system undergoes a deterioration described by a Wiener process. The proposed model takes into account the different deterioration processes by considering a multivariate state description $\mathbf{W}_t$. The performance measure R_t of the system is a functional on the underlying process and is not monotone. Decisions are made by setting a critical level for the process. Because it is nonmonotone the performance measure can cross the critical level in both directions but will eventually grow without limit. Our decisions are thus based on the probability that the performance measure never returns below the critical level. By choosing the critical level appropriately we thus guarantee a minimum level of reliability.

9.2 The Model

9.2.1 The considered processes

A system S consisting of N components (or subsystems) is considered. It is assumed that each component experiences its own way of deteriorating through time and that the N deteriorations are independent; that is, the deterioration of any component has no influence on the deterioration of the $N-1$ remaining components. The proposed model takes into account the different N deterioration processes as follows. Each component undergoes a deterioration described by a Wiener process. The components are labelled C_i, $i \in \{1,\dots,N\}$ and the corresponding Wiener processes are $W_t^{(i)}$, $i \in \{1,\dots,N\}$, where

$$W_t^{(i)} = \mu_i t + \sigma B_t^{(i)}, \qquad \forall i \in \{1,\dots,N\}. \tag{9.1}$$

The above Wiener processes have different drift terms (the μ_is) but for simplicity the volatility terms (σ) are assumed identical and each component is assumed to be new at time $t=0:\ W_0^{(i)}=0$. The independence is modelled by considering N independent Brownian motions $B_t^{(i)}$s. The next step consists in considering the following N-dimensional Wiener process:

$$\begin{aligned}\mathbf{W}_t &= \left(W_t^{(1)}, W_t^{(2)}, \dots, W_t^{(N)}\right)\\ &= \underline{\mu} t + \sigma \mathbf{B}_t \\ \mathbf{W}_0 &= \underline{0}\end{aligned} \tag{9.2}$$

with

$$\underline{\mu} = \begin{pmatrix}\mu_1\\ \vdots \\ \mu_N\end{pmatrix}, \qquad \mathbf{B}_t = \begin{pmatrix}B_t^{(1)}\\ \vdots \\ B_t^{(N)}\end{pmatrix}. \tag{9.3}$$

Decisions are based on a summary measure of performance which corresponds to a functional on the underlying process $A(W_t)$, as in Newby and Barker (2006). In this study the functional used to describe the system's performance measure is the Euclidean norm R_t,

$$\begin{aligned}R_t &= \|\mathbf{W}_t\|_2\\ &= \sqrt{\sum_{i=1}^{N}(W_t^{(i)})^2}\end{aligned}. \tag{9.4}$$

R_t is the radial part of a drifting Brownian motion starting at the origin; it therefore corresponds to a Bessel process with drift $Bes_0(\nu, \mu)$ starting at the origin with index ν and drift μ [Rogers and Pitman (1980)], where:

$$\nu = \frac{1}{2}N - 1 \quad \text{and} \quad \mu = \sqrt{\sum_{i=1}^{N} \mu_i^2}. \tag{9.5}$$

Remark 9.2.1 The radial part of a Brownian motion with drift starting from any other point $R_0 \neq 0$ does not correspond to a Bessel process with drift $Bes_X(\nu, \mu)$ [Rogers and Pitman (1980)].

9.2.2 Maintenance actions and nonperiodic inspections

The model proposed in this chapter aims at giving an optimal maintenance and inspection policy. The efficiency of the policy entirely depends on the inspection times and the type of maintenance on the system.

Maintenance actions are determined by comparing the observed system state R_t with a critical level ξ. However, rather than considering the first hitting time at this threshold, decisions are based on the last exit time from this critical level. For a general process X_t the last exit time is

$$H_\xi^x = \sup_{t \in \mathbb{R}^+} \{X_t \leq \xi | X_0 = x\}.$$

In a monotone process both the first hitting time and last exit times are stopping times and the distributions of these times are relatively straightforward to obtain. The Bessel process R_t describing the performance measure is nonmonotone so that the last exit time is not a stopping time but the probability $\mathbb{P}[H_\xi^0 \leq t]$ is known.

Decision rules for maintenance are made with the help of a maintenance function. In our particular case, the process chosen is the Euclidean norm of an n-dimensional Wiener process which corresponds to a Bessel process only when the process starts from the initial state 0. Hence it is a necessity to always consider the process starting from state 0. This rules out the usual repair model, that describes the effect of maintenance on the system by determining a new starting point for the process. The problem is tackled by considering changes in the value of the critical threshold ξ, rather than a new starting point for the process, and hence affects the time taken to traverse the distance to the critical threshold. After a repair the system is described by the same process starting from zero but with the critical threshold reduced to the distance between the repaired state and the original threshold. We introduce a repair function which models the amount by which the threshold is lowered after undertaking a repair on the system. The function introduced is denoted by d and if $\{\tau_1, \tau_2, \ldots\}$ refer to the inspection times, d may be defined as

$$d : \mathbb{R}^+ \to \mathbb{R}^+ \qquad R_{\tau_i} \mapsto d\,(R_{\tau_i})\,. \tag{9.6}$$

It is a function of the performance measure of the system at inspection times. The choice for d is made among the set of bijective functions. The bijective property for d is required when the derived cost functions are numerically evaluated with an appropriate choice of quadrature points. The idea is that rather than considering R_t starting from a new initial state after the maintenance action with the same threshold value ξ, we reset the value R_{τ_i} to 0 and consider a lower threshold $\xi\,' = \xi - d\,(R_{\tau_i})$. This may also be regarded as a shift of the x-axis of amount $d\,(R_{\tau_i})$ upwards. As far as the decision problem is concerned, the Markov property of the process is exploited and allows a copy of the original process to be considered:

$$\mathbb{P}[R_t < \xi \,|\, R_0 = x] = \mathbb{P}[R'_t < \xi - x \,|\, R'_0 = 0] \tag{9.7}$$

with

$$\begin{aligned} R_t &= \|\mathbf{W}_t\|_2 \\ R'_t &= \|\mathbf{W}_{\tau_i^+ + t} - \mathbf{W}_{\tau_i^+}\|_2 \end{aligned}\,. \tag{9.8}$$

Recall that $\mathbf{W}_t$ is the n-dimensional process describing the state of the system. The process observed to be in state $\mathbf{W}_{\tau_i}$ is repaired instantaneously and restarts in state $\mathbf{W}_{\tau_i^+}$ where $\|\mathbf{W}_{\tau_i^+}\|_2 = x$: the repair undertaken on the system can therefore be interpreted as a componentwise repair. R'_t is an equivalent process with the same probability structure and starting at the origin. In the more usual notation

$$\mathbb{P}^x[R_t < \xi] = \mathbb{P}^0[R'_t < \xi - x] \tag{9.9}$$

with the superscript indicating the starting point.

The proposed model considers a nonperiodic inspection policy, the reason for this being that it is a more general approach and often results in policies with lower costs, particularly in cases where high costs of lost production are taken into consideration. Rather than considering a dynamic programming problem as did Newby and Dagg (2004), the optimization problem is simplified by using an inspection scheduling function m as introduced in Grall *et al.* (2002). The scheduling function is a decreasing function of $d\,(R_{\tau_i})$, the amount by which the threshold is decreased, and determines the amount of time until the next inspection time

$$\begin{aligned} m : \quad & \mathbb{R}^+ && \to [m_{\min}, m_{\max}] \\ & d\,(R_{\tau_i}) && \mapsto m\,[d\,(R_{\tau_i})]\,. \end{aligned} \tag{9.10}$$

With τ_i $(i \in \mathbb{N})$ denoting the times at which the system is inspected and R_{τ_i} its performance, the next inspection time τ_{i+1} is deduced using the relation

$$\tau_{i+1} = \tau_i + m\,[d\,(R_{\tau_i})]\,. \tag{9.11}$$

Consequently, it is the state of the performance measure that determines the next inspection time. The choice for m is made among the set of decreasing functions

$$\forall\, i, j \in \mathbb{N} :\ d\left(R_{\tau_i}\right) \leq d\left(R_{\tau_j}\right) \Leftrightarrow m\left[d\left(R_{\tau_i}\right)\right] \geq m\left[d\left(R_{\tau_j}\right)\right]. \tag{9.12}$$

This allows us to model the fact that the worse the performance of the system is (and hence the lower the value for the new critical threshold after repair is) the more frequently it needs to be inspected. We note that the great advantage with this approach is that it preserves continuity within the model. The approach here is to optimize the total expected cost with respect to the inspection scheduling function. The inspection functions form a two-parameter family and these two parameters, a and b, are allowed to vary to locate the optimum values. The function can be thus written $m\left[\,.\,|\,a,\, b\right]$ leading to a total expected cost function $v_{\xi}\left(a, b\right)$ which is optimized with respect to a and b. The two parameters are defined in the following way,

$$\begin{aligned} m\left[0\,|\,a,\, b\right] &= a, \\ m\left[R_t\,|\,a,\, b\right] &= \alpha, \qquad \text{if } R_t \geq b, \end{aligned} \tag{9.13}$$

for some fixed chosen value $\alpha \in [0, a]$. From the above, we may deduce that $m_{min} = \alpha$ and $m_{max} = a$. These parameters have physical interpretations:

(i) Parameter a corresponds to the amount of time elapsed before the first inspection (i.e., when the system is new)

(ii) Parameter b controls changes in frequency of inspections.

As the choice of inspection scheduling functions is made among the set of decreasing functions, one may deduce

$$\forall\, i \in \mathbb{N}, \qquad \tau_{i+1} - \tau_i \leq a.$$

(That is, the amount of time between any two consecutive inspections will not exceed a.) Moreover, the parameter b sets a lower bound for the process R_t below which the system's performance is assumed to be insufficient; this therefore justifies a periodic inspection of the system of period α.

To ensure tractability of the optimization and of the effects of the chosen function on the optimal cost, choices for m are confined within the set of polynomials of order less than or equal to 2. We note, however, that the proposed models are not restricted to this choice of inspection scheduling functions and can be extended to any other type of function. Particular attention is paid to the convexity or concavity property of m; this allows different rates of inspections as time passes to be considered. The following three expressions for m are investigated,

$$m_1\left[x|\,a,\,b\right] = \max\left\{1, a - \frac{a-1}{b}x\right\} \tag{9.14}$$

$$m_2\left[x|\,a,\,b\right] = \begin{cases} \dfrac{(x-b)^2}{b^2}(a-1)+1, & 0 \leqslant x \leqslant b \\ 1, & x > b \end{cases} \tag{9.15}$$

$$m_3\left[x|\,a,\,b\right] = \begin{cases} -\left(\dfrac{\sqrt{a-1}}{b}x\right)^2 + a, & 0 \leqslant x \leqslant b \\ 1, & x > b \end{cases} \tag{9.16}$$

with $a > 1$ in all cases. Note that if $a = 1$ the policy becomes a periodic inspection policy with period $\tau = a = 1$ and in the case where $a < 1$ the policy inspects less frequently for a more deteriorated system.

Remark 9.2.2 In the rest of the chapter, the notations $m\left(x\right)$ and $v_{\xi-x}$ are used rather than $m\left(x|a,b\right)$ and $v_{\xi-x}\left(a,b\right)$, for clarity.

The function m_1 resembles the inspection scheduling function considered in the numerical example section of Grall *et al.* (2002) and constitutes a reference for our numerical results. Note that whereas the time until the next inspection decreases rather quickly when dealing with m_2, m_3 allows greater time between the inspections when the state of the system is still small. The function m_2 might be thought appropriate for a system experiencing early failures (infant mortality), whereas m_3 is more appropriate for a system that is unlikely to fail in its early age.

9.2.3 Features of the model

Model assumptions

1. Without loss of generality, it is assumed that the system's initial performance is maximum (i.e., $R_0 = 0$) with initial critical threshold ξ.

2. Inspections are nonperiodic, perfect (in the sense that they reveal the true state of the system), and they are instantaneous.

3. Maintenance actions are instantaneous.

4. The system's performance is only known at inspection times, however, the moment at which the performance does not meet the prescribed criteria is immediately known (self-announcing): the system is then instantaneously replaced by a new one with cost C_f.

5. Each inspection incurs a fixed cost C_i.

6. Each maintenance action on the system incurs a cost determined by a function C_r. It is a function of the performance of the system at inspection time.

Settings for the model

1. The state space in which the process R_t evolves is partitioned by the critical threshold ξ as follows.

$$\mathbb{R}^+ = [0, \xi) \cup [\xi, +\infty) \,. \tag{9.17}$$

   Because the process R_t is nonmonotone, the first time at which the process hits the threshold ξ is not considered as the time at which the system fails. Instead, we use the transience and positivity properties of the process to define the system as unsafe when it has escaped from the interval $[0, \xi)$. This time is the last exit time $H_\xi^0 = \sup_{t \in \mathbb{R}^+} \{R_t \leq \xi | R_0 = 0\}$.

2. The system is inspected at inspection times $\{\tau_1, \tau_2, \ldots\}$. The time between inspections τ_{i-1} and τ_i is T_i, $i \in \mathbb{N}$ and is determined by using an inspection scheduling function m, described in Section 9.2.2. The sequence of inspection times $(\tau_i)_{i \in Z^+}$ is strictly increasing and satisfies:

$$\begin{aligned} \tau_0 &= 0 \\ \tau_i &= \sum_{k=1}^{i} T_k \\ T_i &= \tau_i - \tau_{i-1}, \qquad i \geq 1. \end{aligned} \tag{9.18}$$

   At inspection time τ_i, the corresponding system's state is R_{τ_i} and appropriate maintenance action (repair or do nothing) is undertaken. Let τ_i^* denote the times at which the system is replaced: at such times the process $(R_t)_{t \geq 0}$ is reset to zero. These times are regeneration times and allow us to derive an expression for the total expected cost of inspection and maintenance.

3. At inspection time $t = \tau$ (prior to any maintenance action), the system's performance is R_τ.

4. Given that the system's initial performance is maximum (i.e., $R_0 = 0$), decisions on the level of maintenance (replacement or imperfect maintenance) are made on the basis of the indicator function $\mathbf{1}_{\{H_\xi^0 > \tau\}}$. By this it is meant that decisions on whether to replace the system are taken on the basis of the process having definitively escaped from the interval $[0, \xi)$.

5. Deterministic maintenance at inspection time is modelled with the use of the following maintenance function,

$$d(x) = \begin{cases} x, & x < \frac{\xi}{K} \\ kx, & x \geq \frac{\xi}{K} \end{cases} \tag{9.19}$$

with corresponding cost function

$$C_r(x) = \begin{cases} 0, & x < \frac{\xi}{K} \\ 100, & x \geq \frac{\xi}{K} \end{cases} \tag{9.20}$$

with constants $k \in (0,1]$ and $K \in (1,+\infty)$. The constant k determines the amount of repair undertaken on the system; K is arbitrarily chosen and sets the region of repairs for the system.

9.3 Expected Total Cost

In this section we propose an expression for the expected total cost of inspections and maintenance. The Markov property of the Bessel process allows the total cost to be expressed via a recursive approach: a conditioning argument on the threshold value is considered. The notation $V_{\xi-x}$ is used to denote the total cost of maintenance, where $\xi - x$ refers to the threshold value. The maintenance decisions are made using the exit time from the region of acceptable performance. The time $H^0_{\xi-x}$ can never be known by observation because observing any up-crossing of the threshold reveals a potential exit time but there remains the possibility of a further down-crossing and up-crossing in the future. This is the meaning of the fact that $H^0_{\xi-x}$ is not a stopping time. In a nonprobabilistic context, the process $H^0_{\xi-x}$ is described by a noncausal model. The difficulty is readily resolved because the probability that the last exit time occurs before the next inspection is known. In the light of these observations the decision rules are formulated as follows.

- $\mathbf{1}_{\{H^0_{\xi-x} > m(x)\}} = 1$: performance of the system (evaluated with respect to the last time the process hits the critical threshold) meets the prescribed criteria until the next scheduled inspection. Upon inspection, the system's performance is $R_{m(x)}$. The system is inspected, and a cost of inspection C_i is considered. The maintenance brings the system state of degradation back to a lower level $d\left(R_{m(x)}\right)$ with cost $C_r\left(R_{m(x)}\right)$. Future costs enter by looking at the process starting from the origin and with the new critical threshold set up equal to $\xi - d\left(R_{m(x)}\right)$. The system is then next inspected after $m\left[d\left(R_{m(x)}\right)\right]$ units of time.

- $\mathbf{1}_{\{H^0_{\xi-x}>m(x)\}} = 0$: the performance fails to meet the prescribed criteria between two inspections. The system is replaced with cost C_f and the process restarts from the origin. Future costs are then taken into consideration by looking at the process starting from the origin and with the new critical threshold set up equal to ξ.

9.3.1 Expression of the expected total cost

We first give the expression for the total cost and then take the expectation. This is done by considering the above different scenarios

$$\begin{aligned} V_{\xi-x} &= \left(C_i + V_{\xi-d\left(R_{m(x)}\right)} + C_r\left(R_{m(x)}\right)\right)\mathbf{1}_{\{performance\ acceptable\}} \\ &\quad + \left(C_f + V_\xi\right)\mathbf{1}_{\{performance\ not\ acceptable\}} \\ &= \left(C_i + V_{\xi-d\left(R_{m(x)}\right)} + C_r\left(R_{m(x)}\right)\right)\mathbf{1}_{\{H^0_{\xi-x}>m(x)\}} \\ &\quad + \left(C_f + V_\xi\right)\mathbf{1}_{\{H^0_{\xi-x}\le(x)\}}. \end{aligned} \tag{9.21}$$

Taking the expectation leads to:

$$\begin{aligned} v_{\xi-x} &= \mathbb{E}[V_{\xi-x}] \\ &= \mathbb{E}\left[\left(C_f + V_\xi\right)\mathbf{1}_{\{H^0_{\xi-x}\le m(x)\}}\right] \\ &\quad + \mathbb{E}\left[\left(C_i + V_{\xi-d\left(R_{m(x)}\right)} + C_r\left(R_{m(x)}\right)\right)\mathbf{1}_{\{H^0_{\xi-x}>m(x)\}}\right] \\ &= \left(C_f + v_\xi\right)\mathbb{E}\left[\mathbf{1}_{\{H^0_{\xi-x}\le m(x)\}}\right] \\ &\quad + \mathbb{E}\left[\left(C_i + V_{\xi-d\left(R_{m(x)}\right)} + C_r\left(R_{m(x)}\right)\right)\mathbf{1}_{\{H^0_{\xi-x}>m(x)\}}\right] \\ &= \left(C_f + v_\xi\right)\mathbb{P}\left[H^0_{\xi-x}\le m(x)\right] \\ &\quad + \int_0^{+\infty}\left(C_i + C_r(y) + v_{\xi-d(y)}\right)\mathbb{P}\left[H^0_{\xi-x}>m(x)\right]f^0_{m(x)}(y)\,dy \\ &= \left(C_f + v_\xi\right)\mathbb{P}\left[H^0_{\xi-x}\le m(x)\right] \\ &\quad + \int_0^{+\infty}\left(C_i + C_r(y) + v_{\xi-d(y)}\right)\mathbb{P}\left[H^0_{\xi-x}>m(x)\right]f^0_{m(x)}(y)\,dy. \end{aligned} \tag{9.22}$$

Using the density of the last hitting time h^0_ξ and the transition density f^0_t of the process R_t

$$\begin{aligned} v_{\xi-x} &= \left(C_f + v_\xi\right)\int_0^{m(x)} h^0_{\xi-x}(t)\,dt \\ &\quad + \int_0^{+\infty}\left(C_i + C_r(y) + v_{\xi-d(y)}\right)\left(1 - \int_0^{m(x)} h^0_{\xi-x}(t)\,dt\right)f^0_{m(x)}(y)\,dy \end{aligned}$$

$$
\begin{aligned}
&= C_i \left(1 - \int_0^{m(x)} h^0_{\xi-x}(t)\, dt\right) + (C_f + v_\xi) \int_0^{m(x)} h^0_{\xi-x}(t)\, dt \\
&\quad + \left(1 - \int_0^{m(x)} h^0_{\xi-x}(t)\, dt\right) \int_0^{+\infty} C_r(y)\, f^0_{m(x)}(y)\, dy \\
&\quad + \int_0^{+\infty} v_{\xi-d(y)} \left(1 - \int_0^{m(x)} h^0_{\xi-x}(t)\, dt\right) f^0_{m(x)}(y)\, dy. \qquad (9.23)
\end{aligned}
$$

In (9.22) the expected value $\mathbb{E}\left[V_{\xi-d\left(R_{m(x)}\right)}\mathbf{1}_{\{H^0_{\xi-x}>m(x)\}}\right]$ is required. The expected value is derived by using the conditional independence of $H^0_{\xi-x}$ and R_τ. The independence allows the factorization of the integrals as shown in the appendix.

Rearranging (9.23) above gives

$$
v_{\xi-x} = Q(x) + \lambda(x)\, v_\xi + \int_0^{d^{-1}(\xi)} v_{\xi-d(y)} K\{x, y\}\, dy, \qquad (9.24)
$$

where

$$
\begin{aligned}
\lambda(x) &= \int_0^{m(x)} h^0_{\xi-x}(t)\, dt \\
Q(x) &= (1 - \lambda(x)) \left(C_i + \int_0^{+\infty} C_r(y)\, f^0_{m(x)}(y)\, dy\right) + C_f \lambda(x) \qquad (9.25) \\
K\{x, y\} &= \left(1 - \int_0^{m(x)} h^0_{\xi-x}(t)\, dt\right) f^0_{m(x)}(y).
\end{aligned}
$$

Note that now the limit in the integral in (9.24) is finite. The justification for this change of limit is that the expected cost $v_{\xi-x}$ is assumed to be zero when the critical threshold is negative. Indeed, a negative threshold in the model would either mean that the system never reaches a critical state or that it is always in a failed state; hence no maintenance action needs to be considered, setting the expected cost of maintenance to zero.

9.3.2 Obtaining the solutions

The equation (9.24) is solved numerically: an approximation to the continuous problem is constructed by discretizing the integrals giving a set of linear matrix equations. The discrete problem is solved using the methods described in Press *et al.* (1992). First, note that at $t = 0$ the system is new. Under this condition, we rewrite Equation (9.24) as follows.

$$
v_{\xi-x} = Q(x) + \lambda(x)\, v_{\xi-x} + \int_0^{d^{-1}(\xi)} v_{\xi-d(y)} K\{x, y\}\, dy. \qquad (9.26)
$$

Yielding to the following Fredholm equation,

$$\{1-\lambda(x)\}v_{\xi-x} = Q(x) + \int_0^{d^{-1}(\xi)} v_{\xi-d(y)} K\{x,y\}\,dy. \tag{9.27}$$

Rewriting (9.24) as (9.27) does not affect the solution to the equation and will allow the required solution to be obtained by a homotopy argument based on ξ. Indeed both Equations (9.24) and (9.27) are identical when $x = 0$; we therefore solve Equation (9.27) and get the solution for $x = 0$. The Nystrom routine with the N-point Gauss–Legendre rule at the points y_j, $j \in \{1,\ldots,N\}$ is applied to (9.27); we get

$$\{1-\lambda(x)\}v_{\xi-x} = Q(x) + \sum_{j=1}^{N} v_{\xi-d(y_j)} K\{x,y_j\}\,w_j. \tag{9.28}$$

We then evaluate the above at the following appropriate points $x_i = d(y_i)$ and obtain:

$$\{1-\lambda(x_i)\}v_{\xi-x_i} = Q(x_i) + \sum_{j=1}^{N} v_{\xi-d(y_j)} K\{x_i,y_j\}\,w_j, \tag{9.29}$$

which, because $v_{\xi-x_i}$ and $v_{\xi-d(y_i)}$ are evaluated at the same points, can be rewritten in the following matrix form,

$$(\mathbf{D}-\mathbf{K})\,\mathbf{v} = \mathbf{Q}, \tag{9.30}$$

where:

$$\begin{aligned}\mathbf{v}_i &= v_{\xi-x_i}\\ \mathbf{D}_{i,j} &= (1-\lambda(x_i))\,\mathbf{1}_{\{i=j\}}\\ \mathbf{K}_{i,j} &= K\{x_i,y_j\}\,w_j\\ \mathbf{Q}_i &= Q(x_i)\,.\end{aligned} \tag{9.31}$$

Having obtained the solution at the quadrature points by solving inversion of the matrix $\mathbf{D}-\mathbf{K}$, we get the solution at any other quadrature point x by simply using Equation (9.28) as an interpolatory formula.

Remark 9.3.1 $K\{x,y\}$ in (9.25) is the product of a density function by a survival function hence it is bounded by the maximum of the density which, by the Fredholm alternative, ensures that the equation in (9.30) has a solution (i.e., $\mathbf{D}-\mathbf{K}$ is invertible).

Because we are interested in a system which is new at time $t = 0$, we just choose the quadrature point $x_i = 0$, which justifies that rewriting (9.24) as (9.27) does not affect the solution to the equation.

9.4 Numerical Results and Comments

This section presents results from numerical experiments. The values of the parameters for the process used to model the degradation of the system and the different costs used were chosen arbitrarily to show some important features of the inspection policy. The initial value for the critical threshold is $\xi = 5$, the Bessel process considered is $Bes_0\,(0.5, 1)$, and the values for the cost of inspection and the cost of failure are $C_i = 50$ and $C_f = 200$.

The corresponding costs of repair are chosen to be dependent on the state of the system found at inspection as follows.

$$C_r\,(y) = \begin{cases} 0, & y < \frac{\xi}{2} \\ 100, & y \geq \frac{\xi}{2}. \end{cases} \tag{9.32}$$

The purpose of the present model is to find an optimal inspection policy for the expected total cost of inspection and maintenance of the system. Three different types of inspection policies are considered with the use of the three inspection scheduling functions m_1, m_2, and m_3 defined in Section 9.2.2. The expected total costs are minimized with respect to the two parameters a and b.

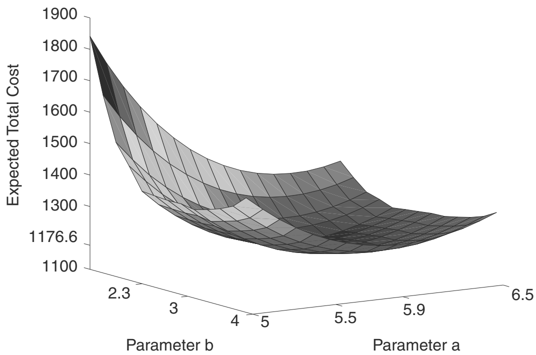
The numerical results for the case of small maintenance on the system ($k = 0.9$) are shown in Figure 9.1. In the case of a large amount of maintenance ($k = 0.1$), the numerical results are shown in Figure 9.2. The optimal values a_i^*, b_i^*, and v_i^* ($i = \{1, 2, 3\}$) for a, b, and v_ξ, respectively, in the different scenarios, are summarized in Table 9.1.

We first note that the surfaces obtained clearly show the presence of an optimal policy for each inspection function considered. In the case $k = 0.1$ with inspection function m_2, the optimal inspection policy seems to strongly depend on parameter a only, which is the first time of inspection of the system. The choice for b does not seem to be of much importance.

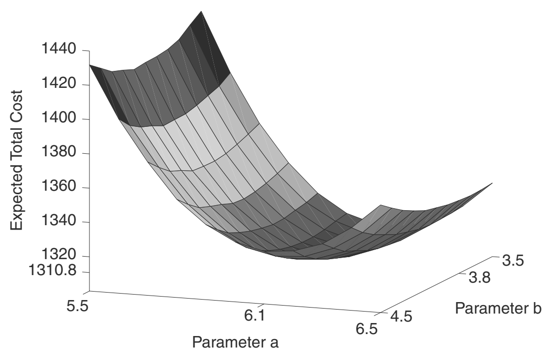
Even if the optimal inspection policy gives a value b^* which is less than ξ, we note that the choice $b > 5\,(\equiv \xi)$ is not meaningless: indeed the value R_{τ_i} of the process at inspection time τ_i may be greater then ξ: it is the last hitting time of ξ by the process that defines the process as unsafe.

From Table 9.1, we note that the optimal costs are smaller for $k = 0.1$ than for $k = 0.9$. This makes sense, because in both cases the same values for the costs were considered: the case $k = 0.1$ corresponding to more repair, the system will tend to deteriorate slower and therefore will require less maintenance resulting in a smaller total cost. In both cases $k = 0.9$ and $k = 0.1$, we note that the value for v^* increases with the convexity of the inspection function: $v_3^* < v_1^* < v_2^*$.

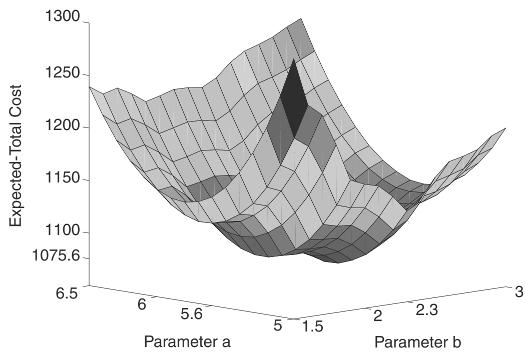
Plots of the optimal inspection functions in Figure 9.3 show that the smallest value for a is a_3, corresponding to the first inspection time for a new system



(a) Inspection function: m_1

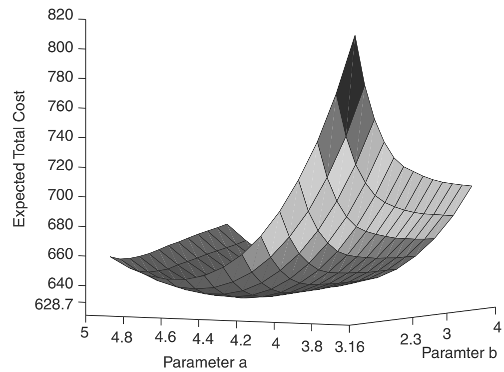


(b) Inspection function: m_2

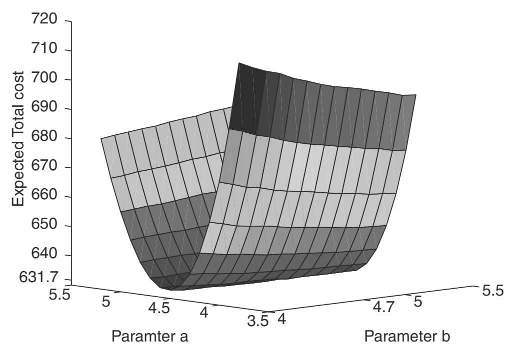


(c) Inspection function: m_3

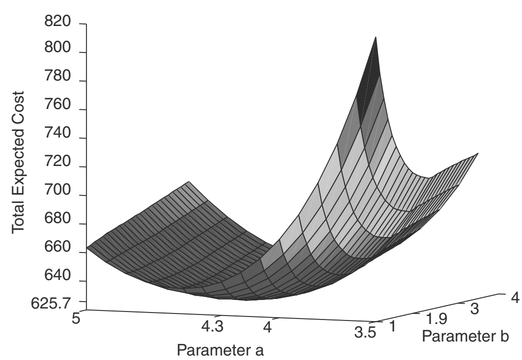
Figure 9.1. Surface representations of the expected total costs with different inspection scheduling functions, $k = 0.9$.



(a) Inspection function: m_1



(b) Inspection function: m_2



(c) Inspection function: m_3

Figure 9.2. Surface representations of the expected total costs with different inspection scheduling functions, $k = 0.1$.

Table 9.1. Optimal values of the parameters a and b for the three inspection scheduling functions

Inspection Policies		$k = 0.9$	$k = 0.1$
	a_1^*	5.9	4.5
m_1	b_1^*	2.3	2.3
	v_1^*	1176.6	628.73
	a_2^*	6.1	4.5
m_2	b_2^*	3.8	4.7
	v_2^*	1310.8	631.71
	a_3^*	5.6	4.3
m_3	b_3^*	2.3	1.9
	v_3^*	1089.3	625.67

when inspection function m_3 is used. However, when the value of the process reaches some value (rather close to 0), the function m_3 crosses m_1 and m_2 to lie above them. It then crosses m_2 a second time to return below it. We may deduce that for this process an optimal policy is first to allow a long time between the inspections, then to change strategy drastically to a small interval or an almost periodic inspection policy of period 1. This change of inspection decision within the same policy m_3 happens earlier when $k = 0.1$.

9.5 Conclusion

The proposed model provides optimal nonperiodic inspection policies for a complex multicomponent system whose state is described by a multivariate Wiener process. Decisions are made on the basis of the state of a performance measure defined by the Euclidean norm of the multivariate process and the last exit time from an interval rather than the first hitting time. The models are optimized in the sense that they result in a minimum expected maintenance cost, whose expression uses a conditioning argument on the critical threshold's value. The nonperiodicity of the inspection times is modelled with the use of an inspection scheduling function, introduced in Grall *et al.* (2002), which determines the next time to inspect the system based on the value of the performance measure at inspection time. The numerical results obtained show the presence of a cost-optimal inspection policy in each of the six cases, where different inspection functions and different amounts of repair are considered. Attention is paid to

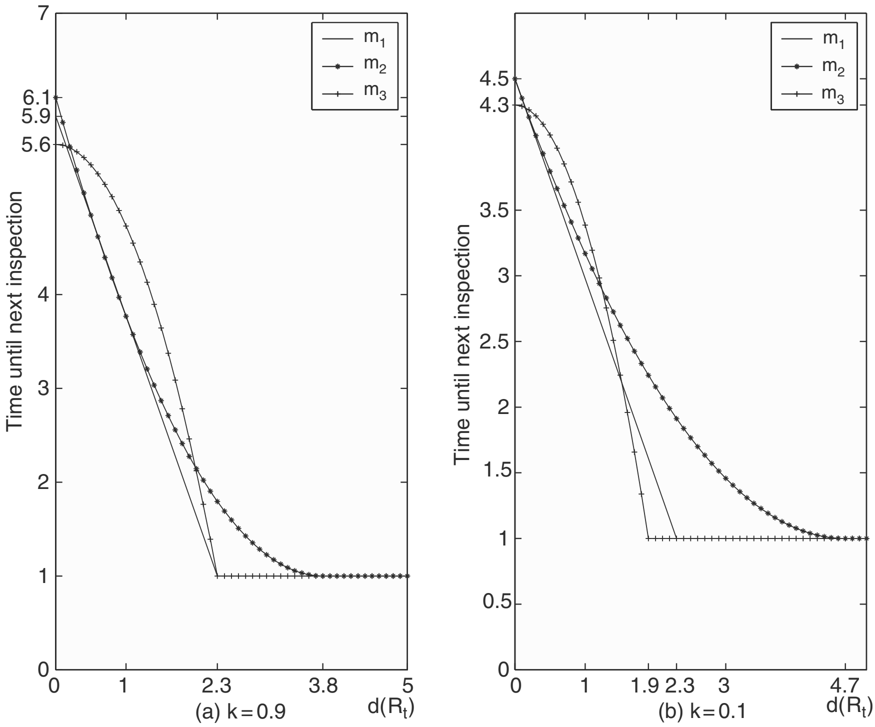


Figure 9.3. Optimal inspection scheduling functions.

the influence of the convexity of the inspection function on the optimal expected total cost: the value for the optimal cost v^* increases with the convexity of the inspection function.

Appendix

Let $f_{R_{m(x)},H^0_{\xi-x}}$ be the joint probability density function of the process at time $m(x)$ and the last exit time from the interval $[0,\xi-x)$. We may deduce:

$$
\begin{aligned}
&E\left[V_{\xi-d\left(R_{m(x)}\right)}\times\mathbf{1}_{\{H^0_{\xi-x}>m(x)\}}\right]\\
&\quad=\int_0^{+\infty}\int_0^{+\infty}v_{\xi-d(y)}\times\mathbf{1}_{\{t>m(x)\}}f_{R_{m(x)},H^0_{\xi-x}}(y,t)\,dydt\\
&\quad=\int_0^{+\infty}\int_0^{+\infty}v_{\xi-d(y)}\times\mathbf{1}_{\{t>m(x)\}}f_{R_{m(x)}|H^0_{\xi-x}=t}(y)\,h^0_{\xi-x}(t)\,dydt\\
&\quad=\int_{m(x)}^{+\infty}\int_0^{+\infty}v_{\xi-d(y)}f_{R_{m(x)}|H^0_{\xi-x}>m(x)}(y)\,h^0_{\xi-x}(t)\,dydt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{m(x)}^{+\infty} h_{\xi-x}^{0}(t) \int_{0}^{+\infty} v_{\xi-d(y)} f_{R_{m(x)}|H_{\xi-x}^{0}>m(x)}(y)\, dydt \\
&= \int_{m(x)}^{+\infty} h_{\xi-x}^{0}(t)\, dt \int_{0}^{+\infty} v_{\xi-d(y)} f_{R_{m(x)}}(y)\, dy \\
&= \left(1 - \int_{0}^{m(x)} h_{\xi-x}^{0}(t)\, dt\right) \int_{0}^{+\infty} v_{\xi-d(y)} f_{R_{m(x)}}(y)\, dy \\
&= \left(1 - \int_{0}^{m(x)} h_{\xi-x}^{0}(t)\, dt\right) \int_{0}^{+\infty} v_{\xi-d(y)} f_{m(x)}^{0}(y)\, dy.
\end{aligned}
$$

The conditional independence allows the replacement of $f_{R_{m(x)}|H_{\xi-x}^{0}>m(x)}$ by $f_{R_{m(x)}}$: as $H_{\xi-x}^{0} > m(x)$, the process may still be in the region $[0, \xi - x)$ and hence the region of integration remains $[0, +\infty)$.

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