Hele-Shaw and Stokes flow with a source or sink : stability of spherical solutions

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Hele-Shaw and Stokes flow with a source or sink: Stability of spherical solutions

PROEFSCHRIFT

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Erwin Vondenhoff

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prof.dr. M.A. Peletier

Copromotor:
dr. G. Prokert

dedicated to the memory of my father

Contents

Chapter 1

Introduction

1.1 Moving boundary problems

The melting of ice, growth of tumours, winning of oil, and production of glasses are processes that have something in common. In all four cases the shape of a clump of matter (ice, tissue, or fluid) evolves over time. This evolution is due to a driving mechanism, namely temperature difference, nutrition in the cells, pressure variation, or surface tension. These processes are modeled as moving boundary problems. The crucial feature is that the boundary of the moving domain is part of the solution and has to be found. Besides the shape of the domain we often have to calculate physical quantities such as velocity, pressure, or temperature inside the domain. These quantities depend on the motion of the domain and vice versa. For example, the temperature of melting ice in water depends on the changing geometry. On the other hand, the evolution of the phase boundary is influenced by the temperature difference between water and ice.

The area of applications of moving boundary problems is very wide. From the examples above it may appear that we need to restrict ourselves to moving objects that are three-dimensional clumps of matter. This is not necessary. To give an example from population dynamics, consider a geographical region in which two competing species A and B live. One can divide this region into subregions, one in which species A lives, one with species B, and maybe a third subregion where the species coexist. The evolution of these regions can be understood from a model including quantities such as population density. Also in epidemiology a moving boundary between geographical domains can be used to study the spreading of a virus or a disease.

The subjects that we have mentioned up to now are all related to physics or biology. There are however many applications in other areas. An example from financial mathematics is pricing of American options [78]. A put option is a contract that gives the right to the holder to sell a risky asset like a stock within a specified period at a price that is fixed in advance. It is a natural question to ask what a fair price is for such an option. Many models for option pricing are based on the famous Black-Scholes partial differential equation [6] and include a so-called exercise boundary. The stock should be sold when its value reaches this boundary. The exercise boundary and the optimal option price, need to be determined simultaneously.

Roughly speaking, in applied mathematics there are three ways to approach moving

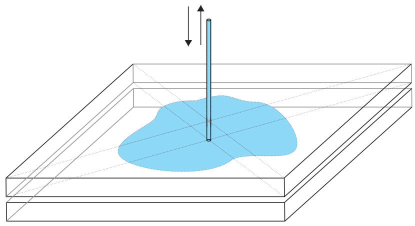


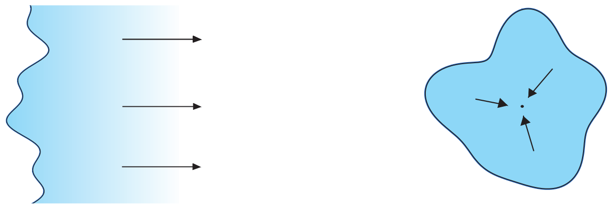
Figure 1.1: *Sketch of horizontal Hele-Shaw cell with injection or suction through a thin tube*

boundary problems. First of all, there is the numerical approach. In practise it is often inevitable to use numerical simulations to approximate solutions. A difficulty is that the geometries may become very complicated. Boundaries may become very irregular or even fractal. Another issue is that in some situations solutions suddenly vanish. We will discuss an example in which a solution breaks down in Section 1.2. The second approach is constructing exact analytical solutions. In Section 1.2 we discuss some cases in which this is possible. Apart from numerics and finding exact solutions, one can use analytical methods to prove qualitative properties like existence of solutions with certain regularity on some time interval. Besides existence there is the question of uniqueness. It is often believed that a model from the "real world" has precisely one solution. This is not always the case since reality is often simplified. Therefore, existence and uniqueness theorems are not only interesting for purely theoretical purposes but they also tell us whether a model is acceptable after simplifications or not. An example in which a mathematical model, that seems a reasonable description of reality, has no solutions, is a creeping flow past a cylinder. This example of non-existence is known as the Stokes paradox [52]. Besides existence and uniqueness a third condition for well-posedness is that a slight modification in the initial conditions must lead to small changes in the outcome.

In this thesis these three issues will be discussed for two important classes of moving boundary problems, namely Hele-Shaw flows and Stokes flows. In particular, it will be shown that certain solutions are asymptotically stable. This means that a small perturbation of these solutions decays during the evolution. The rate of this decay will be calculated. Several types of boundary conditions, that model different physical situations, will be considered.

1.2 Hele-Shaw flow

In 1898 the Hele-Shaw model was introduced to describe a liquid flow in a so-called Hele-Shaw cell [37]. This cell consists of two flat transparent parallel plates that are separated by a very small distance (see Figure 1.1). In the space between the plates a liquid layer is confined. In horizontal cells, gravity effects can be neglected. After averaging over the interstice, the liquid layer can be regarded as a two-dimensional bounded domain. It moves in the presence of one or more driving mechanisms. Dimensionless pressure p and velocity v are functions depending on two space variables x_1 and x_2 and time t. They are related via Darcy's law, named after a 19th century French hydraulic



(a) (b)

Figure 1.2: *(a) An unbounded liquid region with uniform suction at infinity. (b) A bounded liquid region with suction at a single point.*

engineer,

$$v = -\nabla p. \tag{1.1}$$

Furthermore, the fluid is assumed to be incompressible. Therefore

$$\operatorname{div} v = 0. \tag{1.2}$$

As a consequence, pressure is harmonic. Equations (1.1) and (1.2) are assumed to hold inside the moving liquid domain $t \mapsto \Omega(t)$.

In so-called classical Hele-Shaw flow p is zero on $\Gamma(t) := \partial\Omega(t)$. This models continuity of pressure over $\Gamma(t)$, assuming that pressure is zero outside the domain. We will discuss other boundary conditions later in this section.

On $\Gamma(t)$ we impose a kinematic boundary condition that says that the normal velocity v_n of the boundary is equal to $v \cdot n$ where n is the normal vector field. This is based on the assumption that the boundary moves with the particles, such that for any time t the set of particles at $\Gamma(t)$ is exactly the same. Completed with driving mechanisms we have defined a Hele-Shaw moving boundary problem.

Let us briefly discuss some applications of Hele-Shaw flow besides two-dimensional liquid flow in a Hele-Shaw cell. First of all, the three-dimensional version of the model is used to describe flows in porous media, like groundwater flow. Furthermore, the relatively simple model plays a paradigmatic role for understanding more complicated problems. Variations of the model are used to describe the growth of tumours that have the structure of a porous medium [7]. A problem that is related to the one-phase Hele-Shaw problem as well is the Muskat problem [57] in which two immiscible fluids are in contact with another and form an interface, on which continuity of pressure is assumed. Another related problem is the Stefan problem, that models melting of ice. Hele-Shaw flow can be regarded as the limit of this problem for small specific heat, see for instance [18].

An overview of the history of the Hele-Shaw problem is given in [41] and an overview of articles on Hele-Shaw flow until 1998 is given in [40]. To complete the specification of the classical Hele-Shaw problem we assume a driving mechanism. Let us consider the following two configurations:

- For unbounded domains, one can have uniform injection or suction at infinity as

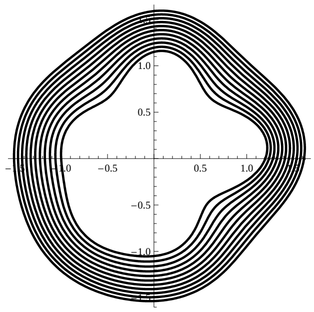


Figure 1.3: *An approximation of a solution to the classical Hele-Shaw moving boundary problem with injection. The lines denote boundaries of the domain for several values of time. As time tends to infinity, the domain "converges" to an expanding ball. The picture is generated from the linearised model (see Chapter 2).*

in Figure 1.2(a). This is incorporated in the model by assuming that

$$v \sim (-\mu, 0)^T, \quad x_1 \to +\infty. \tag{1.3}$$

For $\mu < 0$ the fluid retreats and for $\mu > 0$ the fluid expands.

- For bounded domains, one can have injection or suction at the origin (see Figure 1.2(b)). This is modeled by replacing (1.2) by

$$\operatorname{div} v = \mu\delta, \tag{1.4}$$

where δ is the Dirac-delta distribution. The area of the moving domain increases if $\mu > 0$ and shrinks for $\mu < 0$ with rate $|\mu|$.

Many other configurations have been considered. An example is a flow outside bubbles in a parallel-sided channel with a uniform translational motion ([55], [69]). Fluids outside bubbles with radial suction of fluid at infinity or injection of air in the origin have been studied by Howison and Gustafsson ([35], [42], [43], [44]).

In this thesis we focus on the situation in Figure 1.2(b) with a bounded domain and one source or sink such that (1.4) holds. The domain $\Omega(0)$ is assumed to be a small perturbation of a ball. It is clear that if the initial domain is exactly a ball, then the moving domain will be an expanding/shrinking ball. One of our goals is to answer the question of stability for this solution. We consider an initial domain that is a perturbed ball and investigate whether the moving domain "converges" to an expanding/shrinking ball. Moreover, we investigate the decay rate of perturbations. Figure 1.3 shows how due to injection a nearly spherical domain gradually takes more and more the shape of an expanding ball.

Many results in this thesis are based on linearisation of a parabolic equation that describes the domain evolution. Let us briefly discuss how linearisation methods have been used in the stability analysis for a travelling wave solution in the case of an unbounded domain with uniform injection or suction at infinity in two dimensions as in Figure 1.2(a) (see also Howison [45]). Identifying $\mathbb{R}^2$ and $\mathbb{C}$, the moving boundary $\Gamma(t)$

consisting of points with real part $-\mu t$ is a travelling wave solution that is initially (at $t = 0$) located at the imaginary axis. Let us consider a small perturbation of the initial boundary:

$$\Gamma(0) = \{\epsilon \sin(|k|y) + iy : y \in \mathbb{R}\}, \tag{1.5}$$

with ϵ small and $k \in \mathbb{Z}$. A stability analysis by Saffman and Taylor [66] shows that the solution to the linearised problem with homogeneous boundary conditions is given by

$$\Gamma(t) = \{-\mu t + \epsilon e^{-\mu|k|t} \sin(|k|y) + iy : y \in \mathbb{R}\}.$$

The travelling wave solution with an advancing boundary ($\mu > 0$) is linearly stable because all Fourier modes decay when t becomes large. On the other hand, the travelling wave solution for the receding boundary ($\mu < 0$) is linearly unstable. For this suction problem we discuss two regularisation methods that play an important role in this thesis. The homogeneous Dirichlet boundary condition for p is replaced by two types of other boundary conditions. We discuss what implications these conditions have for the linear analysis in the two-dimensional case.

- **Surface tension**: At the moving boundary we have the relation

  $$p = -\gamma\kappa, \tag{1.6}$$

  where κ is the mean curvature of the boundary of the liquid domain, taken negative for convex domains, and γ is a positive constant called the surface tension coefficient. In the context of the Stefan problem this equation is known as the Gibbs-Thomson relation. In Hele-Shaw models it is used to describe the influence of surface tension forces. If the initial boundary is the perturbed version (1.5) of the travelling wave solution, then the moving boundary will be

  $$\Gamma(t) = \{-\mu t + \epsilon e^{(-\gamma|k|^3 - \mu|k|)t} \sin(|k|y) + iy : y \in \mathbb{R}\},$$

  for the linearised problem, see also [45]. All Fourier modes decay if $\gamma > -\mu$.

- **Kinetic undercooling regularisation**: At the moving boundary we assume

  $$p + \beta \frac{\partial p}{\partial n} = 0,$$

  where n is the normal in outward direction and $\beta > 0$ is called the kinetic undercooling coefficient. The name kinetic undercooling originates from the Stefan problem, in which it models certain thermodynamic effects on the interface between ice and water. In a Hele-Shaw setting Romero [65] proposed to relate the term $\beta\frac{\partial p}{\partial n}$ to the second principal curvature of the liquid domain in the Hele-Shaw cell. This is the curvature of the thin meniscus of the liquid in the narrow gap between the two plates. The linear behaviour is

  $$\Gamma(t) = \{-\mu t + \epsilon e^{-\mu \frac{|k|}{1+\beta|k|} t} \sin(|k|y) + iy : y \in \mathbb{R}\}$$

  (see [45]). We see that for the linearised suction problem, Fourier modes grow at most exponentially with a factor $e^{\frac{|\mu|}{\beta}t}$ (that is independent of k).

Now we briefly discuss a method to construct exact solutions for the two-dimensional configuration with a bounded domain and a source or a sink (also discussed in [45]). On the boundary we assume $p = 0$.

Since p is harmonic, the two-dimensional problem can be treated as a problem in the complex plane in which p is the real part of a complex analytic function [31], [59]. By the Riemann mapping theorem there are time-dependent conformal mappings $f = f(\zeta, t)$ from the unit disk to the domain. These conformal mappings satisfy the Polubarinova-Galin evolution equation

$$\text{Re}\left(\zeta \frac{\partial f}{\partial \zeta} \overline{\frac{\partial f}{\partial t}}\right) = -\mu, \quad \text{for } |\zeta| = 1. \tag{1.7}$$

In [31], [45], [59], and [64] polynomial solutions to (1.7) with time dependent coefficients are discussed. Let us discuss the simplest non-trivial example given by

$$f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2,$$

with

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \frac{\mu}{a_1^2 - 4a_2^2} \begin{pmatrix} a_1 \\ -2a_2 \end{pmatrix}. \tag{1.8}$$

Figure 1.4 shows a corresponding domain evolution for the case of suction with $|a_2(0)| < \frac{1}{2}|a_1(0)|$. Note that a blow-up via a $\frac{3}{2}$-power cusp in the boundary occurs in finite time. At the time that the cusp is formed we have $|a_2| = \frac{1}{2}|a_1|$. If $|a_2| > \frac{1}{2}|a_1|$, then $f(\cdot, t)$ would no longer be injective. This results into self-intersecting boundaries and self-overlapping domains which is unacceptable. More generally, there are conditions on the coefficients of the polynomial solutions to (1.7). Huntingford [48] discusses these for Hele-Shaw flow and polynomials of degree 3. For more theory on injectivity of polynomials on the unit disk, see [16].

For a receding fluid in an unbounded domain finite-time blow-up results via cusp development have been found in [46].

We conclude this section by discussing some existence results for solutions to the Hele-Shaw problem.

The classical injection problem has been reduced to a variational inequality ([5], [17], [34]). Weak solutions have been introduced and existence results for all time have been proved. Moreover, a monotonicity result has been derived in [34]. The concept of a weak solution is more flexible than that of a classical solution since regularity and connectivity of domains are no issues.

In this thesis we are concerned with classical solutions. We want to obtain asymptotic stability results for the spherical solutions in the strongest possible norm. From the monotonicity for weak solutions we only obtain convergence in the C^0-topology (see Section 2.1 for explanation). Moreover, the monotonicity result only holds for the zero surface tension problem. An important restriction on our evolving domains is that they must be small star-shaped perturbations of balls. A bounded domain is said to be star-shaped (with respect to the origin) if each ray starting in the origin intersects the boundary of the domain at at most one point. Vasiliev and Markina [73] proved that

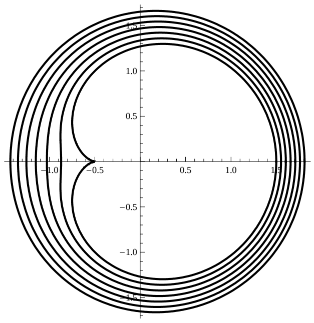


Figure 1.4: *Cusp development for the Hele-Shaw suction problem. The largest domain is the initial domain. For several values of time the moving domain is plotted. In finite time a cusp occurs.*

star-shapedness is preserved on some time interval for the 2D problem with injection and small surface tension. In [36] infinite lifetime of solutions is proved for the version of this problem without surface tension, assuming star-shapedness and analyticity of the initial boundary. Existence of classical short-time solutions for more general initial domains for a closely related problem has been proved by Escher and Simonett [21], [22]. Prokert [61] proved a global existence result in time for classical solutions for the case without a source or sink and $\gamma > 0$, assuming nearly spherical initial domains. Moreover, he proved that perturbations of a ball decay exponentially fast.

1.3 Stokes flow

Besides Hele-Shaw flow, Stokes flow with surface tension will be considered in this thesis. For Stokes flow we have

$$-\Delta v + \nabla p = 0 \tag{1.9}$$

and on the boundary

$$(\nabla v + \nabla v^T - pI)n = \gamma \kappa n \tag{1.10}$$

is assumed. Here I stands for the identity matrix. Again the fluid is assumed to be incompressible. The equations can be derived from the Navier-Stokes equations if one omits inertial terms.

Stokes flow appears in many moving boundary problems. As an example, the process of viscous sintering in glass industry can be modeled by means of a Stokes flow [51]. To study the growth of tumours, Stokes models are sometimes more apt than Hele-Shaw models. Although many tumours have a porous medium structure, there are tumours (e.g. breast cancers) that are more naturally modeled as a fluid ([25], [26], and [27]). More applications where Stokes flows are involved are given in [68].

Although for Stokes flow the components of v are not harmonic, in two dimensions it is still possible to represent v by means of two analytic functions since they are biharmonic [58]. Many authors used this representation to apply methods from complex analysis, see for instance [60]. Cummings, Howison, and King [12] found a set of conserved quantities for two-dimensional Stokes flow without surface tension. Furthermore, exact solutions can be constructed by means of conformal mappings from the unit disc. In [47] the particular case

$$f(\zeta, t) = a(t)\left(\zeta - \frac{b(t)}{n}\zeta^n\right),$$

with $b(t) < 1$ is studied and in [11] general cubic polynomials are treated, both with and without surface tension.

In [33] short-time existence results have been proved for the problem without injection or suction. In the same work, the authors show global existence for the case of an initial domain that is a small perturbation of a ball. In [19] joint spacial and temporal analyticity of the moving boundary has been proved. For the problem with injection or suction short-time existence and smoothness results have been proved in [60].

1.4 Objectives and main results

Again we denote a domain evolution by $t \mapsto \Omega(t)$ and $t \mapsto \Gamma(t)$ is its moving boundary.

Consider the situation with a source or a sink located at the origin, such that (1.4) holds. Suppose that the initial domain $\Omega(0)$ is the N-dimensional unit ball $\mathbb{B}^N := \{x \in \mathbb{R}^N : |x| < 1\}$. It is clear that for both injection and suction the evolving domain $\Omega(t)$ will be a ball for all t with volume equal to

$$\mathfrak{V}(t) = \mu t + \mathfrak{V}(0) = \mu t + \frac{\sigma_N}{N}, \tag{1.11}$$

where σ_N is the area of the unit sphere $\mathbb{S}^{N-1}$. This follows from

$$\frac{d\mathfrak{V}(t)}{dt} = \frac{d}{dt}\int_{\Omega(t)} dx = \int_{\Gamma(t)} v \cdot n d\sigma = \int_{\Omega(t)} \operatorname{div} v dx = \mu,$$

which is a consequence of (1.4) and the fact that the volume of $\mathbb{B}^N$ is equal to the quotient of the area of $\mathbb{S}^{N-1}$ and N. The radius of the evolving ball is therefore equal to

$$\alpha(t) = \sqrt[N]{\frac{\mu N t}{\sigma_N} + 1}. \tag{1.12}$$

The expanding ball in the presence of a source ($\mu > 0$) has infinite lifetime, whereas the shrinking ball ($\mu < 0$) vanishes at $t = \frac{\sigma_N}{|\mu| N}$. Throughout the thesis we refer to this expanding/shrinking ball as the trivial (spherical) solution or the trivial domain evolution.

One of our goals is to answer the stability question for this spherical solution in the context of Hele-Shaw and Stokes flow. By stability we mean that a domain that is initially a small perturbation of $\mathbb{B}^N$ gradually takes more and more the shape of the expanding/shrinking ball $\alpha(t)\mathbb{B}^N$. For the case of injection we intuitively expect that this is true, as Figure 1.3 suggests. We will prove this for Hele-Shaw flow with the three boundary conditions that we discussed. For Stokes flow we restrict ourselves to condition (1.10).

For the suction problem we expect that this is in general not the case. As we have seen in Section 1.2, it may even happen that domain evolutions do not continue up to complete extinction (i.e. the situation in which all fluid is sucked out).

Our stability analysis has a certain analogy with the stability analysis of the trivial travelling wave solution for the unbounded configuration with injection/suction at infinity that we discussed earlier. The role of the travelling wave solution is played by the trivial spherical solution in our work. Including surface tension via boundary condition (1.6) the suction problem is regularised in our case as well. As for the unbounded domain this can be concluded from the eigenvalues that appear in a linearised evolution problem. For the travelling wave a bound on the suction rate is necessary to make sure that all eigenvalues have the desired sign. Also in the stability analysis of the spherical solution for the bounded case in Hele-Shaw flow a similar condition is sometimes necessary. This strongly depends on the space dimension.

Moreover, another condition must be satisfied. In order to exclude some eigenvalues with positive sign we need to ensure that

$$\int_{\Omega(t)} x dx = 0. \tag{1.13}$$

Note that this means that the geometric centre of the domain is located at the position of the sink. Since the geometric centre is invariant, both for Hele-Shaw flow and Stokes flow, (1.13) holds for all t if it holds for $t = 0$. It has been proved by Tian [70], [71] for Hele-Shaw flow with boundary condition (1.6) that if (1.13) is not satisfied, then the solution breaks down before all liquid is sucked out or the domain becomes unbounded with zero area. It is interesting to investigate the reverse question. Can all liquid be sucked out in certain situations? This question is answered with "yes" in Chapter 3 for the 3D Hele-Shaw problem. We assume a nearly spherical initial domain that satisfies (1.13) at $t = 0$. Furthermore the suction rate must be lower than a certain value. This gives a partial solution to an open problem posed in 1993 [39].

In Chapter 5 we conclude that also for the 2D case this is true. Moreover, there is no bound on the suction rate. Also the case in which kinetic undercooling is included is discussed. An important consequence for this type of regularisation is that in general invariance of the geometric centre is lost. Therefore we want to know whether it is still possible to obtain similar results, forcing (1.13) to hold for all t, for instance by restricting ourselves to domains that are initially symmetric with respect to all coordinate planes.

The linear analysis of the trivial travelling wave solution, that we discussed in Section 1.2, shows that for the zero surface tension case with $\mu > 0$ perturbations with shorter wavelengths decay faster than those with longer wavelengths. We will prove that a similar result holds in the case of the trivial spherical solution. The eigenfunctions for the linearised problem are related to Richardson moments, which have invariance properties for the zero surface tension case. It is interesting to ask whether in the nonlinear evolution perturbations decay faster if certain moments vanish.

Another important aspect is to investigate suction outside the geometric centre and in particular stability with respect to the suction point. We want to know whether in a fixed initial domain there is a continuous dependence near the geometric centre between

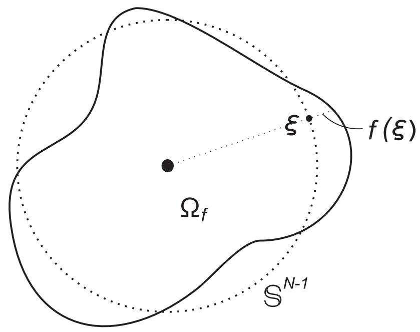


Figure 1.5: *Parametrisation of a domain* Ω_f *by means of a function* $f : \mathbb{S}^{N-1} \to (-1, \infty)$

the position of the suction point and the length of the maximal time interval on which a solution exists.

The trivial domain evolution shows self-similar behaviour,

$$\Omega(t) = \alpha(t)\Omega(0).$$

In other words, the size of the domain changes in time but the shape does not. It is interesting to investigate whether existence of more solutions with this self-similarity can be proved using linearisation and bifurcation theory. We will do this for 3D Hele-Shaw flow with surface tension. We will show that for certain negative values of μ, families of non-trivial self-similarly vanishing solutions exist that bifurcate from the trivial spherical solution. These solutions are domains that can be parameterised by approximations of zonal spherical harmonics in the way that is described in Section 1.5. After proving existence we ask ourselves whether some of the constructed non-trivial solutions are stable.

1.5 What methods will we use?

Domain evolutions are studied by means of scalar functions. Any continuous function $f : \mathbb{S}^{N-1} \to (-1, \infty)$ describes a star-shaped domain in $\mathbb{R}^N$ as follows:

$$\Omega_f := \left\{ x \in \mathbb{R}^N \setminus \{0\} : |x| < 1 + f\left(\frac{x}{|x|}\right) \right\} \cup \{0\}, \tag{1.14}$$

see also Figure 1.5. The domain evolutions $t \mapsto \Omega(t)$ that we consider in this thesis will be described by means of a function $R(\cdot, t)$ that satisfies $\Omega(t) = \Omega_{R(\cdot,t)}$.

For instance, the trivial solution is parameterised by

$$R(\xi, t) = \alpha(t) - 1.$$

In order to investigate stability it is convenient to regard the trivial domain evolution $t \mapsto \alpha(t)\mathbb{B}^N$ as a stationary solution. This is done by rescaling the moving domain $t \mapsto \Omega(t)$ by a factor $\alpha(t)$. The evolution of the rescaled domain is parameterised by

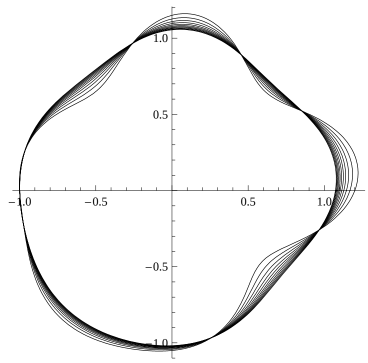


Figure 1.6: *Evolution $t \mapsto \Omega_{r(\cdot,t)}$ for Hele-Shaw flow with injection and $\gamma = 0$. This picture is generated from the linearised model (see Chapter 2). The moving domain converges to $\mathbb{B}^N$. In Figure 1.3, the evolution $t \mapsto \Omega_{R(\cdot,t)}$ starting from the same initial shape $\Omega(0) = \Omega_{R(\cdot,0)} = \Omega_{r(\cdot,0)}$ is plotted.*

$r(\cdot,t)$ given by

$$r(\cdot,t) = \frac{1 + R(\cdot,t)}{\alpha(t)} - 1. \tag{1.15}$$

As a consequence, $\Omega_{r(\cdot,t)} = \alpha(t)^{-1}\Omega_{R(\cdot,t)} = \alpha(t)^{-1}\Omega(t)$. In the case of the trivial solution we have $r \equiv 0$ for all t. In Figures 1.3 and 1.6 possible domain evolutions $t \mapsto \Omega_{R(\cdot,t)}$ and $t \mapsto \Omega_{r(\cdot,t)}$ are sketched for some initial domain $\Omega(0)$. Figure 1.6 suggests that $\Omega_{r(\cdot,t)}$ converges to $\mathbb{B}^N$ as t approaches infinity. In other words, r goes to zero as t tends to infinity.

For Hele-Shaw flow with surface tension an equation of type

$$\frac{\partial r}{\partial t} = \frac{\gamma}{\alpha(t)^3}\mathcal{F}_1(r) + \frac{\mu}{\alpha(t)^N}\mathcal{F}_2(r), \tag{1.16}$$

for some operators $\mathcal{F}_1$ and $\mathcal{F}_2$, will be derived in Chapter 3. For Stokes flow we find

$$\frac{\partial r}{\partial t} = \frac{\gamma}{\alpha(t)}\mathcal{G}_1(r) + \frac{\mu}{\alpha(t)^N}\mathcal{G}_2(r), \tag{1.17}$$

for some operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in Chapter 6.

An important property that will be used both for Hele-Shaw and Stokes flow is smoothness of these operators. They turn out to be analytic near zero. We briefly discuss the concept of analyticity for operators between function spaces and mention some properties. For details and proofs, see [15] or [60]. An operator $\mathcal{F} : X \to Y$, for X and Y Banach spaces, is called analytic near zero when for small $\|r\|$ it can be written as

$$\mathcal{F}(r) = \sum_{k=0}^{\infty} \mathcal{F}^k(r, r, \ldots, r),$$

where $\mathcal{F}^k$ are bounded symmetric k-linear mappings such that for some $\epsilon > 0$

$$\sum_{k=0}^{\infty} \|\mathcal{F}^k\| \epsilon^k < \infty,$$

where

$$\|\mathcal{F}^k\| := \sup_{\|x_1\|=\|x_2\|=\cdots=\|x_k\|=1} \|\mathcal{F}^k(x_1, x_2, \ldots, x_k)\|. \tag{1.18}$$

The term $\mathcal{F}^1(r)$ is called the Fréchet derivative of $\mathcal{F}$ at zero in the direction r or the linearisation around zero of $\mathcal{F}$. The following notation will be used:

$$\mathcal{F}'(0)[r] := \mathcal{F}^1(r).$$

Analyticity at $r \in X$ and Fréchet derivative at $r \in X$ (linearisation around r) for $r \neq 0$ are defined analogously. Analyticity is a useful property mainly because it enables us to study nonlinear problems by looking at the linear ones and using perturbation arguments. Often it is enough to demand lower regularity. However, we choose to show analyticity since it is often not more complicated than showing Fréchet differentiability.

Important properties of analytic operators that we will use are the following ones:

- Compositions of analytic operators are analytic.
- Point-wise multiplications of analytic operators (if well defined) are analytic.
- Fréchet derivatives of analytic operators are analytic.

The following lemma is an extension of the Implicit Function Theorem for functions in $\mathbb{R}^N$ to operators on function spaces.

Lemma 1.1. *Let X, Y, and Z be Banach spaces, let $f : X \times Y \to Z$ be analytic near (x_0, y_0), and suppose that $f(x_0, y_0) = 0$. Suppose that the Fréchet derivative w.r.t. the second argument at (x_0, y_0) given by*

$$h \mapsto f'(x_0, y_0)[0, h]$$

is bijective from Y to Z. Then there exists a unique analytic mapping $y : \mathcal{U} \to Y$, for $\mathcal{U}$ a small neighbourhood of x_0 in X, that satisfies

$$f(x, y(x)) = 0$$

and $y(x_0) = y_0$.

Proof. See [80, Ch. 8]. □

The function spaces in which we consider our evolutions must be closed under point-wise multiplication and they must be Banach algebras. This means that they are Banach spaces for which there exists a $C > 0$ such that for all elements r_1 and r_2 the point-wise product $r_1 r_2$ satisfies

$$\|r_1 r_2\| \leq C \|r_1\| \|r_2\|.$$

Two types of Banach algebras that will be considered in this thesis are Hölder spaces and Sobolev spaces.

- The Hölder spaces $\mathcal{C}^{k,\beta}(\mathbb{S}^{N-1})$ and $\mathcal{C}^{k,\beta}(\mathbb{B}^N)$ for $k \in \mathbb{N}_0$ and $\beta \in (0,1)$ are Banach algebras.
- The Sobolev spaces $\mathbb{H}^s(\mathbb{S}^{N-1})$ and $\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)$ for real $s > \frac{N-1}{2}$ are Banach algebras.

We call an element of a Banach algebra invertible if there exists another element such that the product is equal to the multiplicative identity. Banach algebras have the property that for each invertible element there exists a neighbourhood on which inversion is a well-defined analytic operation.

Let us now discuss some methods and concepts that we will use to prove our existence results. For classical Hele-Shaw flow ($\gamma = 0$), we can get rid of the factor $\alpha(t)^{-N}$ in (1.16) to make the evolution operator on the right-hand side of the equation autonomous. This is done by introducing a new time variable (see Chapter 2). For such autonomous operators the principle of linearised stability [53] will be used to prove global existence results in Hölder spaces. We need to prove that the Fréchet derivative $\mathcal{F}_2'(0)$ has certain properties. It must be a sectorial operator and its spectrum must be located in the left half-space of the complex plane away from the imaginary axis.

For $\gamma > 0$ only the three-dimensional version of Hele-Shaw flow with source/sink can be treated in this way. For $N \neq 3$ it is impossible to make the right-hand side of (1.16) autonomous. The eigenvalues are different from the zero surface tension case. Nevertheless, the linearisation of the evolution operators for these problems have a lot in common. In both cases it is possible to express the Fréchet derivatives in terms of the Dirichlet-to-Neumann operator $\mathcal{N}$ on the unit sphere. This is a mapping of order one that maps a function $f : \mathbb{S}^{N-1} \to \mathbb{R}$ to $g : \mathbb{S}^{N-1} \to \mathbb{R}$ where g is the normal derivative of the unique harmonic extension of f inside $\mathbb{B}^N$. Furthermore, we find the same set of eigenfunctions, namely the spherical harmonics.

Let $\mathfrak{H}_k^N$ be the vector space of harmonic homogeneous polynomials of degree k in N variables. Spherical harmonics are defined as the restrictions of these polynomials to the unit sphere,

$$\mathfrak{S}_k^N := \left\{ q\,|_{\mathbb{S}^{N-1}} : q \in \mathfrak{H}_k^N \right\}.$$

From [56, Lemma 4] we get

$$\nu(N,k) := \dim \mathfrak{S}_k^N = \begin{cases} \dfrac{(2k+N-2)(k+N-3)!}{k!(N-2)!} & k \in \mathbb{N}, \\ 1 & k = 0. \end{cases}$$

For example for $N = 2$ we have $\nu(2,k) = 2$ for $k \neq 0$ and $\nu(2,0) = 1$, while for $N = 3$ we have $\nu(3,k) = 2k+1$. For each $\mathfrak{S}_k^N$ we choose an orthonormal basis with respect to the $\mathbb{L}^2(\mathbb{S}^{N-1})$-inner product

$$\mathfrak{S}_k^N = \left\langle s_{k1}^N, s_{k2}^N, \ldots, s_{k\nu(N,k)}^N \right\rangle.$$

We will often omit the index N in s_{kj}^N and $\mathfrak{S}_k^N$.

In the literature the complex-valued spherical harmonics for $N = 3$ are often denoted by Y_{kj}, with $k \in \mathbb{N}_0$ and $j \in \{-k, -k+1, \ldots, k\}$. Some corresponding domains are plotted in Figure 1.7. For each $k \in \mathbb{N}_0$ the linear subspace of $\mathfrak{S}_k^3$ consisting of func-

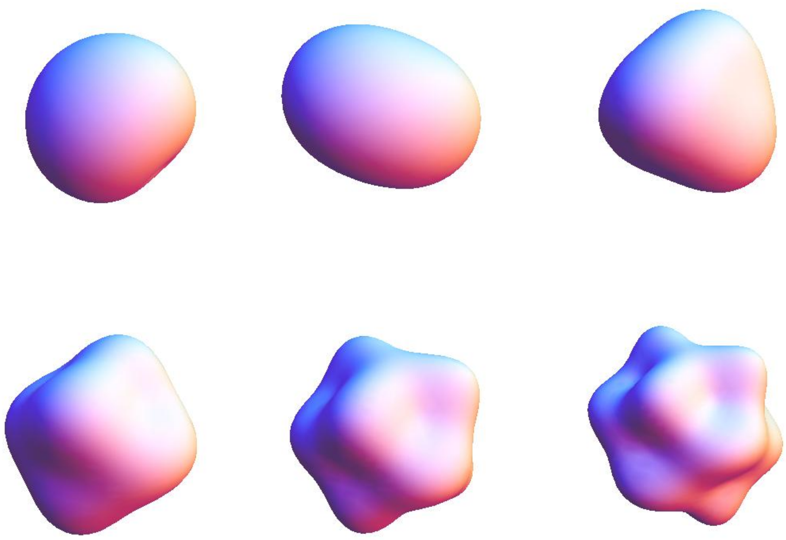


Figure 1.7: *Some three-dimensional domains that are parameterised by spherical harmonics of degree 1,2,3,4,5,6. The Spherical harmonics of degree zero are constants. Therefore they parameterise balls around the origin.*

tions that are axially symmetric around the vertical axis is one-dimensional and spanned by the so-called zonal harmonics Y_{k0}. In this thesis we will use the notation $Y_k := Y_{k0}$ for the sake of brevity.

The 2D spherical harmonics s^2_{kj} are the functions $\sin k\theta$ and $\cos k\theta$ where θ is the polar variable, see Figure 1.8.

The following two facts on the spherical harmonics will frequently be used in this thesis:

- The harmonic extension of s_{kj} inside $\mathbb{B}^N$ is given by $x \mapsto |x|^k s_{kj}\left(\frac{x}{|x|}\right)$.
- If $h \in \mathfrak{S}^N_k$, then $\mathcal{N}h = kh$.

Note that both properties follow immediately from the definition of spherical harmonics.

In order to treat the non-autonomous cases $N = 2$ and $N \geq 4$ we find estimates for $(r, \mathcal{F}(r,t))_s$, where $\mathcal{F}(r,t)$ denotes the time-dependent right-hand side of (1.16) and $(\cdot,\cdot)_s$ is the $\mathbb{H}^s$-inner product. The same method will be used for Stokes flow. In the case of Hele-Shaw flow with kinetic undercooling, time dependence appears in a much more complicated way than in (1.16). Therefore, finding useful estimates is much harder.

In Appendix A, we discuss a modification of a theorem by Kato and Lai [50, Thm. A] that will be used to derive existence results from these energy estimates.

An important tool that is necessary to obtain useful energy estimates is a generalised

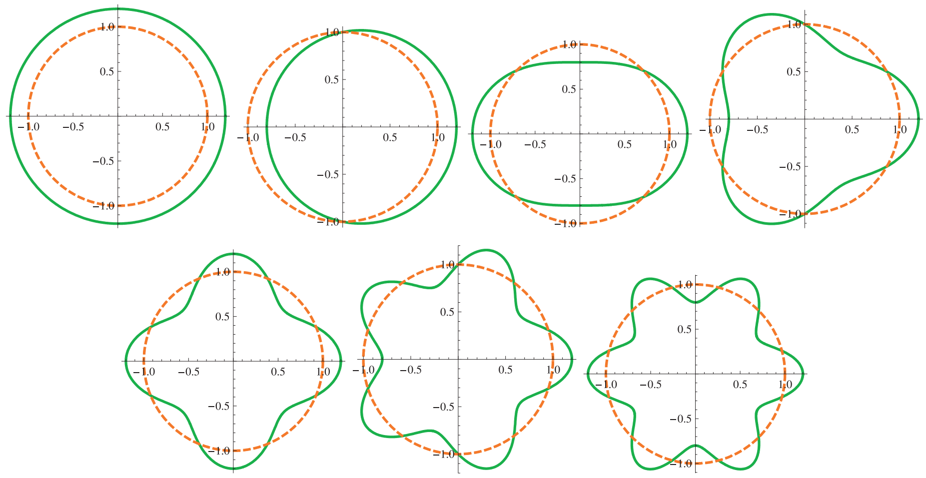


Figure 1.8: *The two-dimensional spherical harmonics are sines and cosines in the polar variable. Some domains that are parameterised by the cosines are plotted.*

version of the chain rule of differentiation. It says that for the differential operators

$$D_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \leq i < j \leq N,$$

one has for $r : \mathbb{S}^{N-1} \to \mathbb{R}$ smooth enough

$$D_{ij}\mathcal{F}(r) = \mathcal{F}'(r)[D_{ij}r], \tag{1.19}$$

for our evolution operators $\mathcal{F} = \mathcal{F}_k$, for $k = 1, 2$. This rule is based on equivariance properties of $\mathcal{F}_k$ with respect to rotations and it will be proved in Chapter 5. In the next example the power of this chain rule is demonstrated by means of an autonomous equation. The symbol $C > 0$ is a constant that may vary throughout the calculations.

Example 1.2. Consider an equation of type

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r),$$

where $\mathcal{F}$ is an analytic operator on functions on $\mathbb{S}^{N-1}$ that satisfies (1.19) and $\mathcal{F}(0) = 0$. As said before, Hele-Shaw flow with $\gamma = 0$ can be described by such an equation with an evolution operator that does not depend on τ. We assume $\mathcal{F}$ to be time-independent to keep this example as simple as possible. Assume further that $\mathcal{F}$ is of order one and suppose that one obtains from linear analysis that

$$(r, \mathcal{F}'(0)[r])_s \leq -\lambda \|r\|^2_{s+\frac{1}{2}},$$

for some $\lambda > 0$. We want to find a similar estimate for the nonlinear operator $\mathcal{F}$ for $r \in$

$\mathbb{H}^{s+1}(\mathbb{S}^{N-1})$ with $\|r\|_s < \delta$ for some $\delta > 0$ small enough, making use of the inequality

$$(y, \tilde{y})_s \leq \|y\|_{s+\frac{1}{2}} \|\tilde{y}\|_{s-\frac{1}{2}}$$

for $y \in \mathbb{H}^{s+\frac{1}{2}}(\mathbb{S}^{N-1})$ and $\tilde{y} \in \mathbb{H}^s(\mathbb{S}^{N-1})$. Note that analyticity implies that

$$\|\mathcal{F}(r) - \mathcal{F}'(0)r\|_{s-\frac{1}{2}} \leq C\|r\|^2_{s+\frac{1}{2}}, \tag{1.20}$$

for some $C > 0$ and $\|r\|_{s+\frac{1}{2}}$ small. However, we cannot conclude that $\|r\|_{s+\frac{1}{2}}$ is small from the fact that $\|r\|_s$ is small. If $\|r\|_{s+\frac{1}{2}}$ would be small, then we were able to derive

$$\begin{aligned} \frac{1}{2}\frac{d}{d\tau}\left(\|r(\tau)\|_s^2\right) &\leq (r, \mathcal{F}(r))_s = (r, \mathcal{F}'(0)[r])_s + (r, \mathcal{F}(r) - \mathcal{F}'(0)[r])_s \\ &\leq (-\lambda + C\|r\|_{s+\frac{1}{2}})\|r\|^2_{s+\frac{1}{2}}. \end{aligned} \tag{1.21}$$

This would automatically imply stability for $\|r\|_{s+\frac{1}{2}}$ small. However, we do not control $\|r\|_{s+\frac{1}{2}}$ and fail to get a useful estimate. Now we introduce the following inner product on $\mathbb{H}^s(\mathbb{S}^{N-1})$ that is equivalent to $(\cdot, \cdot)_s$:

$$(r, \tilde{r})_{s-1,1} := (r, \tilde{r})_{s-1} + \sum_{1 \leq i < j \leq N} (D_{ij}r, D_{ij}\tilde{r})_{s-1}. \tag{1.22}$$

Because $\|r\|_{s-\frac{1}{2}}$ is small it is allowed to apply (1.21), replacing s by $s-1$ and $s+\frac{1}{2}$ by $s-\frac{1}{2}$, to obtain

$$(r, \mathcal{F}(r))_{s-1} \leq (-\lambda + C\|r\|_{s-\frac{1}{2}})\|r\|^2_{s-\frac{1}{2}}. \tag{1.23}$$

The chain rule yields

$$\begin{aligned} (D_{ij}r, D_{ij}\mathcal{F}(r))_{s-1} &= (D_{ij}r, \mathcal{F}'(r)[D_{ij}r])_{s-1} \\ &= (D_{ij}r, \mathcal{F}'(0)[D_{ij}r])_{s-1} \\ &\quad +(D_{ij}r, \left\{\mathcal{F}'(r) - \mathcal{F}'(0)\right\}[D_{ij}r])_{s-1} \\ &\leq -\lambda\|D_{ij}r\|^2_{s-\frac{1}{2}} \\ &\quad +C\|D_{ij}r\|_{s-\frac{1}{2}} \left\|\left\{\mathcal{F}'(r) - \mathcal{F}'(0)\right\}[D_{ij}r]\right\|^2_{s-\frac{3}{2}} \\ &\leq -\lambda\|D_{ij}r\|^2_{s-\frac{1}{2}} + C\|r\|_{s-\frac{1}{2}}\|D_{ij}r\|^2_{s-\frac{1}{2}}, \end{aligned} \tag{1.24}$$

for some $C > 0$. In the last step we used local Lipschitz continuity of $\mathcal{F}'$ near zero. Now it follows from (1.22)-(1.24) that

$$\begin{aligned} (r, \mathcal{F}(r))_{s-1,1} &\leq -\lambda\|r\|^2_{s-\frac{1}{2},1} + C\|r\|_{s-\frac{1}{2}}\|r\|^2_{s-\frac{1}{2},1} \\ &= (-\lambda + C\|r\|_{s-\frac{1}{2}})\|r\|^2_{s-\frac{1}{2},1}. \end{aligned}$$

For any $\tilde{\lambda} \in (0, \lambda)$ there exists a $\delta > 0$ such that for $\|r\|_s < \delta$ we get $-\lambda + C\|r\|_{s-\frac{1}{2}} < -\tilde{\lambda}$. As a consequence

$$(r, \mathcal{F}(r))_{s-1,1} < -\tilde{\lambda}\|r\|_{s-\frac{1}{2},1},$$

for $\|r\|_s < \delta$. □

1.6 Outline of the thesis

This thesis consists of three parts. In the first part we focus on stability results for the problems that can be described by autonomous evolution operators.

- In Chapter 2 we treat Hele-Shaw flow for the case $\gamma = 0$. Only the injection problem is well-posed. We prove infinite lifetime for solutions assuming that the initial domain is nearly a ball. Perturbations of the spherical solution turn out to decay algebraically fast. We also show that convergence is faster if low Richardson moments vanish.
- Chapter 3 is dedicated to Hele-Shaw flow in $\mathbb{R}^3$ with $\gamma > 0$ for nearly spherical initial domains. Again the lifetime for the injection problem is infinite, but we also have global existence of solutions to the suction problem if the conditions on the geometry of the initial domain, that we mentioned before, are satisfied and $|\mu|/\gamma < 32\pi/5$.

In Chapter 4 we use bifurcation theory to find non-spherical solutions that vanish self-similarly from bifurcation theory. We also show that the ones that are approximated by $\Omega_{-\epsilon Y_2}$ for $\epsilon > 0$ are stable w.r.t. a certain class of perturbations.

In the third part we tackle the problems where time dependence in the evolution operator occurs.

- Hele-Shaw flow with $\gamma > 0$ for $N = 2$ and $N \geq 4$ is treated in Chapter 5. For the injection problems the lifetime is again infinite. For the suction problem we prove a similar result as for the 3D suction problem. There is no bound on the suction rate for $N = 2$ because for large time the effect of surface tension dominates the effect of the sink, as (1.16) shows. For $N \geq 4$ the sink is dominant and we have linear instability.
- In Chapter 6 Stokes flow with surface tension is treated. In comparison to Hele-Shaw flow determining the Fréchet derivative of the evolution operator is more complicated. To find this Fréchet derivative we use vector valued spherical harmonics to solve a boundary value problem on the ball. As (1.17) suggests, results for Stokes flow in dimensions 2 and 3 are similar to those for Hele-Shaw flow for $N \geq 4$.
- In Chapter 7 we return to Hele-Shaw flow and consider the case in which both surface tension and kinetic undercooling are present.

An article with the contents of Chapters 2 and 3 has been published (see [76]) and the contents of Chapters 4-6 have been submitted (see [62], [77], [75], and [74]).

Chapter 2

Classical Hele-Shaw flow

2.1 Introduction

The classical Hele-Shaw model, characterised by (1.1), (1.4), and zero pressure on the boundary, is the simplest model that we will discuss. The main goal of this chapter is to prove stability of the spherical solution when a source is located at the centre. The first step is to derive an equation for the evolution (see (2.17) and (2.29)). After that we show that the nonlinear non-local operator $\mathcal{F}$ that describes this evolution has the following properties:

- It is smooth.
- The spectrum of the linearisation $\mathcal{F}'(0)$ consists of negative numbers away from the imaginary axis.
- The linearisation is a sectorial operator.

In Section 2.2 we discuss the smoothness of the evolution operator in full detail. Its crucial ingredient is a solution operator for an elliptic boundary value problem. To show smoothness of the solution operator the Implicit Function Theorem is used.

In Section 2.3 the linearisation around the spherical solution is discussed. This linearisation is of first order and essentially given by the Dirichlet-to-Neumann operator on the unit ball. Based on the spectral properties a global existence result is derived and it is shown that perturbations of the spherical solution decay algebraically.

In Section 2.4 we show that convergence for domains for which low Richardson moments vanish, is faster. This is done by discussing the linearisation of the evolution operator restricted to the corresponding invariant manifolds.

Let us now define the moving boundary problem. The parameter μ in Chapter 1 will be fixed to 1 here, because situations with different injection rates are equivalent after rescaling time. We seek both a family of domains $t \mapsto \Omega(t) \subseteq \mathbb{R}^N$, $0 \in \Omega(t)$, parameterised by time and two functions $v(\cdot, t) : \Omega(t) \to \mathbb{R}^N$ and $p(\cdot, t) : \Omega(t) \to \mathbb{R}$

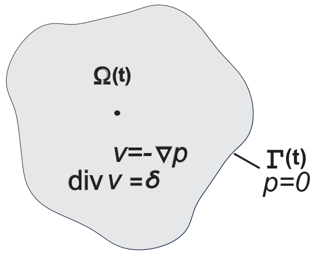


Figure 2.1: *The fixed time problem for classical Hele-Shaw flow with a source at the centre*

such that

$$\operatorname{div} v = \delta \quad \text{in } \Omega(t), \tag{2.1}$$

$$v = -\nabla p \quad \text{in } \Omega(t), \tag{2.2}$$

$$p = 0 \quad \text{on } \Gamma(t) := \partial\Omega(t), \tag{2.3}$$

see Figure 2.1.

The normal velocity v_n of the boundary $t \mapsto \Gamma(t)$ is given by

$$v_n = v \cdot n. \tag{2.4}$$

The fixed time problem given by (2.1), (2.2), and (2.3) can be reduced to

$$\begin{aligned} \Delta p &= -\delta \quad \text{in } \Omega(t), \\ p &= 0 \quad \text{on } \Gamma(t). \end{aligned}$$

On $\Gamma(t)$ we have

$$v_n = -\frac{\partial p}{\partial n}.$$

In Chapter 1 we already mentioned that if we start with $\Omega(0) = \mathbb{B}^N$ where $\mathbb{B}^N = \{x \in \mathbb{R}^N : |x| < 1\}$, then $\Omega(t) = \alpha(t)\mathbb{B}^N$ with $\alpha(t)$ given by

$$\alpha(t) = \sqrt[N]{\frac{Nt}{\sigma_N} + 1}, \tag{2.5}$$

where σ_N is the area of the unit sphere $\mathbb{S}^{N-1}$.

For the classical Hele-Shaw problem a variational inequality has been derived by Elliott and Janovsky [17] and Gustafsson [34]. Weak solutions have been investigated by Gustafsson [34] and Begehr and Gilbert [5].

Gustafsson [34] proved the following monotonicity result. If $t \mapsto \Omega(t)$ and $t \mapsto \Omega'(t)$ are two solutions such that $\Omega(0) \subseteq \Omega'(0)$, then $\Omega(t) \subseteq \Omega'(t)$ for all t. This result can be used to show a stability result as follows. Let $t \mapsto \Omega(t)$ be a solution to the problem (2.1)-(2.4) such that $\Omega(0)$ is a small perturbation of the unit ball $\mathbb{B}^N$, let us say in the $\mathcal{C}^0$-topology, such that there exists an $\epsilon > 0$ for which

$$(1-\epsilon)\mathbb{B}^N \subseteq \Omega(0) \subseteq (1+\epsilon)\mathbb{B}^N.$$

In other words, the initial domain is a subset of some ball of radius larger than 1 and there exists a ball with radius smaller than 1 that lies inside the initial domain. Because of monotonicity we have at time t:

$$f_{1-\epsilon}(t)\mathbb{B}^N \subseteq \Omega(t) \subseteq f_{1+\epsilon}(t)\mathbb{B}^N$$

with

$$f_{1\pm\epsilon}(t) := \sqrt[N]{\frac{Nt}{\sigma_N} + (1\pm\epsilon)^N}.$$

It is clear that for large time $f_{1+\epsilon}(t) - f_{1-\epsilon}(t)$ goes to zero. The gap between the boundaries of the two growing balls $f_{1\pm\epsilon}(t)\mathbb{B}^N$ becomes smaller and smaller and $\Gamma(t) := \partial\Omega(t)$ is forced to stay in between. This suggests convergence of $\Omega(t)$ to an expanding ball.

However, there is no guarantee yet that for an initial perturbation of a ball a classical solution with infinite lifetime indeed exists. In this chapter we prove a global existence result in time in terms of functions r that parameterise domain evolutions as explained in Section 1.5. We will consider classical solutions in the so-called little Hölder spaces. Existence of classical short-time solutions and uniqueness have been proved by Escher and Simonett [22].

2.2 The evolution equation and its linearisation

Define for any star-shaped domain evolution $t \mapsto \Omega(t)$ that solves (2.1)-(2.4) the continuous function $R : \mathbb{S}^{N-1} \times [0,\infty) \to (-1,\infty)$ such that $\Omega(t) = \Omega_{R(\cdot,t)}$ conform (1.14). Often we will write $R(t)$ for $R(\cdot,t)$. Let $r : \mathbb{S}^{N-1} \times [0,\infty) \to (-1,\infty)$ be given by

$$r(t) = \frac{1+R(t)}{\alpha(t)} - 1, \tag{2.6}$$

so that

$$\Omega_{r(t)} = \alpha(t)^{-1}\Omega_{R(t)}. \tag{2.7}$$

The trivial spherical solution is described by $r(t) \equiv 0$. We will often omit the argument t in $r(t)$ if we consider a fixed domain.

Define for any $r : \mathbb{S}^{N-1} \to (-1,\infty)$ the set Γ_r as the boundary of Ω_r and introduce for suitable r

- $\tilde{z}(r,\cdot) : \mathbb{S}^{N-1} \to \Gamma_r$ by
$$\tilde{z}(r,\xi) = (1+r(\xi))\,\xi, \tag{2.8}$$
- $n(r,\cdot)$ by the function that maps an element $\xi \in \mathbb{S}^{N-1}$ to the exterior unit normal vector on Γ_r at the point $\tilde{z}(r,\xi)$.

We will often use the notations $\tilde{z}(r)$ and $n(r)$ instead of $\tilde{z}(r,\cdot)$ and $n(r,\cdot)$.

Lemma 2.1. *Suppose that* $t \mapsto \Omega_{R(t)}$ *solves the moving boundary problem given by* (2.1)-(2.4) *and assume that R is differentiable with respect to both arguments. Then*

$$\frac{\partial R}{\partial t}(\xi) = \frac{v(\tilde{z}(R,\xi)) \cdot n(R,\xi)}{n(R,\xi)\cdot\xi}, \quad \xi \in \mathbb{S}^{N-1}.$$

Proof. Let t^* be a fixed value for t. Let $p(t^*)$ be the position of a particle on $\Gamma_{R(t^*)}$ at time t^* and let $p(t)$ be its position at time t near t^*. We have

$$p(t) = p(t^*) + \int_{t^*}^{t} v(p(\tilde{t}),\tilde{t})d\tilde{t}.$$

Define the function $f(\cdot,t) : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ that maps $\xi \in \mathbb{S}^{N-1}$ to $\xi' \in \mathbb{S}^{N-1}$ such that $\tilde{z}(R(t),\xi')$ is the position at time t of the particle that was located at $\tilde{z}(R(t^*),\xi)$ at time t^*. Then

$$\Big[1 + R\big(f(\xi,t),t\big)\Big]f(\xi,t) = \Big[1 + R\big(\xi,t^*\big)\Big]\xi + \int_{t^*}^{t} v\Big(\Big[1 + R\big(f(\xi,\tilde{t}),\tilde{t}\big)\Big]f(\xi,\tilde{t}),\tilde{t}\Big)d\tilde{t}, \tag{2.9}$$

because $f(\xi,t^*) = \xi$. Define for small $\epsilon > 0$ the mapping $F : \mathbb{S}^{N-1} \times (t^* - \epsilon, t^* + \epsilon) \to \Gamma_{R(t^*)}$ by

$$F(\xi,t) = \Big(1 + R(f(\xi,t),t^*)\Big)f(\xi,t).$$

Differentiating (2.9) with respect to t at $t = t^*$ one gets

$$\frac{\partial R}{\partial t}(\xi,t^*)\xi + \frac{\partial F}{\partial t}(\xi,t^*) = v\Big((1 + R(\xi,t^*))\xi,t^*\Big).$$

Taking the inner product with $n(R(t^*),\xi)$ and knowing that the term with $\frac{\partial F}{\partial t}$ is tangential to $\Gamma_{R(t^*)}$ we obtain

$$\frac{\partial R}{\partial t}(\xi,t^*)(n(R(t^*),\xi)\cdot\xi) = v\Big((1 + R(\xi,t^*))\xi,t^*\Big)\cdot n(R(t^*),\xi).$$

This proves the lemma. □

Define $\Phi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi}\ln|x| & N = 2, \\ \dfrac{1}{(N-2)\sigma_N|x|^{N-2}} - \dfrac{1}{(N-2)\sigma_N} & N \geq 3. \end{cases} \tag{2.10}$$

Up to a constant this function is the fundamental solution of the Laplacian. Define $U : \Omega_R \to \mathbb{R}$ by

$$U := p - \Phi. \tag{2.11}$$

Because $\Delta\Phi = -\delta$ we have

$$\begin{aligned} \Delta U &= 0, && \text{in } \Omega(t), \\ U &= -\Phi, && \text{on } \Gamma(t). \end{aligned}$$

Define for each continuous $f : \mathbb{S}^{N-1} \to (-1, \infty)$ the function $L_f : \Omega_f \to \mathbb{R}$ as the harmonic function that coincides with $-\Phi$ at the boundary Γ_f:

$$L_f(\xi) = -\Phi(\xi), \quad \xi \in \Gamma_f.$$

Lemma 2.2. *If $R(t)$ and $r(t)$ are related via* (2.6)*, then for $\xi \in \Omega_{r(t)}$*

$$L_{r(t)}(\xi) = \alpha(t)^{N-2} L_{R(t)}(\alpha(t)\xi) + c(t) \tag{2.12}$$

and

$$\nabla L_{r(t)}(\xi) = \alpha(t)^{N-1} \nabla L_{R(t)}(\alpha(t)\xi) \tag{2.13}$$

where $c(t)$ only depends on t.

Proof. It is clear that the right-hand side of (2.12) is an harmonic expression in ξ on $\Omega_{r(t)}$. Let $\xi \in \Gamma_{r(t)}$ such that $\alpha(t)\xi \in \Gamma_{R(t)}$. Due to the scaling behaviour of Φ we have on the boundary

$$\alpha(t)^{N-2} L_{R(t)}(\alpha(t)\xi) = -\alpha(t)^{N-2}\Phi(\alpha(t)\xi) = -\Phi(\xi) - c(t).$$

For $N = 2$ we have

$$c(t) = -\frac{1}{2\pi} \ln \alpha(t)$$

and for $N \geq 3$

$$c(t) = \frac{1 - \alpha(t)^{N-2}}{(N-2)\sigma_N}.$$

This proves (2.12) and (2.13) follows from (2.12). □

Since

$$\nabla\Phi(x) = -\frac{1}{\sigma_N |x|^N} x,$$

it follows from (2.2), (2.11), and Lemma 2.1 that

$$\begin{aligned} \frac{\partial R}{\partial t}(\xi) &= -\frac{\nabla U(\tilde{z}(R,\xi)) \cdot n(R,\xi)}{n(R,\xi)\cdot\xi} + \frac{1}{\sigma_N |\tilde{z}(R,\xi)|^{N-1}} \\ &= -\frac{\nabla U(\tilde{z}(R,\xi)) \cdot n(R,\xi)}{n(R,\xi)\cdot\xi} + \frac{1}{\sigma_N (1 + R(\xi))^{N-1}} \\ &= -\frac{\nabla U(\tilde{z}(R,\xi)) \cdot n(r,\xi)}{n(r,\xi)\cdot\xi} + \frac{1}{\sigma_N \alpha(t)^{N-1} (1 + r(\xi))^{N-1}}, \end{aligned} \tag{2.14}$$

because $r(\xi) > -1$ and $n(R,\xi) = n(r,\xi)$. It is clear that $U = L_R$ and from (2.13) it follows that

$$\nabla L_{r(t)}(\tilde{z}(r,\xi)) = \alpha(t)^{N-1}\nabla L_{R(t)}(\tilde{z}(R,\xi)).$$

As a consequence,

$$\nabla U(\tilde{z}(R,\xi)) = \alpha(t)^{1-N}\nabla L_r(\tilde{z}(r,\xi)). \tag{2.15}$$

Note that because of (2.6) and $\alpha' = \sigma_N^{-1}\alpha^{1-N}$ we have

$$\begin{aligned}\frac{\partial r}{\partial t} &= \frac{1}{\alpha}\frac{\partial R}{\partial t} - \frac{\alpha'(1+R)}{\alpha^2} = \frac{1}{\alpha}\frac{\partial R}{\partial t} - \frac{1+R}{\alpha^{N+1}\sigma_N} \\ &= \frac{1}{\alpha}\frac{\partial R}{\partial t} - \frac{1+r}{\alpha^N\sigma_N},\end{aligned} \tag{2.16}$$

where we omitted t arguments and $'$ denotes differentiation with respect to t. It follows from (2.14), (2.15), and (2.16) that

$$\frac{\partial r}{\partial t}(\xi) = \frac{1}{\alpha(t)^N}\left(-\frac{\nabla L_r(\tilde{z}(r,\xi))\cdot n(r,\xi)}{n(r,\xi)\cdot\xi} + \frac{1}{\sigma_N(1+r(\xi))^{N-1}} - \frac{1+r(\xi)}{\sigma_N}\right).$$

We rewrite this evolution equation as

$$\frac{\partial r}{\partial t} = \frac{1}{\alpha(t)^N}\mathcal{F}(r) \tag{2.17}$$

with

$$(\mathcal{F}(r))(\xi) = -\frac{\nabla L_r(\tilde{z}(r,\xi))\cdot n(r,\xi)}{n(r,\xi)\cdot\xi} + \frac{1}{\sigma_N(1+r(\xi))^{N-1}} - \frac{1+r(\xi)}{\sigma_N}. \tag{2.18}$$

Introducing the transformation $\tau = \tau(t)$, such that $\tau(0) = 0$ and

$$\frac{d\tau}{dt} = \frac{1}{\alpha(t)^N} = \frac{1}{\frac{Nt}{\sigma_N}+1},$$

which implies

$$\tau(t) = \frac{\sigma_N}{N}\ln\left(\frac{Nt}{\sigma_N}+1\right), \tag{2.19}$$

we get an autonomous evolution equation

$$\frac{\partial\bar{r}}{\partial\tau} = \mathcal{F}(\bar{r}) \tag{2.20}$$

where $\bar{r}(\tau) = r(t)$. In the sequel we will write r instead of $\bar{r}$.

Now we transform the problem to the fixed reference domain $\mathbb{B}^N$. Let for $k \in \mathbb{N}_0$ and $\beta \in (0,1)$ the little Hölder spaces $h^{k,\beta}(K)$ on a compact domain K be defined as the closure of $C^\infty(K)$ in the Hölder spaces $C^{k,\beta}(K)$. These spaces have the property that $h^{k,\beta}(K)$ is dense in $h^{k',\beta'}(K)$ if $k'+\beta' < k+\beta$. Furthermore, the embedding of $h^{k,\beta}(K)$ in

$h^{k',\beta'}(K)$ is compact (see [1, Thm. 8.6]). In this chapter we study domain evolutions by means of functions r in the little Hölder spaces $h^{2,\beta}(\mathbb{S}^{N-1})$. Endowed with the norm of $\mathcal{C}^{k,\beta}(\mathbb{S}^{N-1})$ the little Hölder spaces are Banach spaces.

By $\|\cdot\|_{k,\beta}$ we denote the standard norm of $\mathcal{C}^{k,\beta}(\mathbb{S}^{N-1})$ and $\|\cdot\|_k$ denotes the norm of $\mathcal{C}^k(\mathbb{S}^{N-1})$ for $k \in \mathbb{N}_0$. The norm of $\mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N})$ will be denoted by $\|\cdot\|_{\mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N})}$. All other norms will be denoted in a similar way, for example $\|\cdot\|_{\mathbb{L}^2(\mathbb{S}^{N-1})}$, $\|\cdot\|_{\mathcal{C}^{k,\beta}(\overline{\Omega_r})}$.

By [53, Thm. 0.3.2] there exists an extension operator $E \in \mathcal{L}(\mathcal{C}^{k,\beta}(\mathbb{S}^{N-1}), \mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N}))$ for $k \in \{0,1,2\}$ and $\beta \in [0;1)$, such that

$$E(r)|_{\mathbb{S}^{N-1}} = r.$$

Define $z : \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})\right)^N$ by

$$z(r,x) = (1 + E(r,x))\, x,$$

where $z(r,\cdot) = z(r)$ and $E(r,\cdot) = E(r)$. Note that $z(r)$ is an extension of $\tilde{z}(r)$ to $\mathbb{B}^N$.

Lemma 2.3. *There exists a $\delta > 0$ such that if $\|r\|_{2,\beta} < \delta$ then $z(r) : \overline{\mathbb{B}^N} \to \overline{\Omega_r}$ is bijective.*

Proof. In this proof we write $r_E(x)$ instead of $E(r,x)$. Let x and x' be in $\overline{\mathbb{B}^N} \setminus \{0\}$ such that $z(r,x) = z(r,x')$ but $x \neq x'$. If x and x' are linearly independent, then $r_E(x) = -1$. This is impossible if $\|r\|_{2,\beta}$ is small. So there exists a $\lambda \in \mathbb{R}$ such that $x' = \lambda x$. Without loss of generality we assume that $\lambda \in [-1,1)$, because the roles of x and x' can be interchanged. We get $1 - \lambda = \lambda r_E(\lambda x) - r_E(x)$. Define $f : [-1,1] \to \mathbb{R}$ by

$$f(\sigma) := -\sigma r_E(\sigma x).$$

We have $f(1) - f(\lambda) = 1 - \lambda$. By the Mean Value Theorem there exists a $\sigma^* \leq 1$ such that

$$1 = f'(\sigma^*) = -r_E(\sigma^* x) - \sigma^* \sum_i x_i \frac{\partial r_E(\sigma^* x)}{\partial x_i},$$

where f' is the derivative of f. This leads to a contradiction if $\|r\|_{2,\beta}$ is small. To complete the proof of injectivity, suppose that $z(r,x) = 0$. If $x \neq 0$, then we have again $r_E(x) = -1$ which is impossible. To prove surjectivity let $y \in \overline{\Omega_r}$. Since the case $y = 0$ is trivial we assume $y \neq 0$. Because $\overline{\Omega_r}$ is star-shaped there exists a $\lambda \in (0;1]$ such that

$$y = \lambda \tilde{z}\left(r, \frac{y}{|y|}\right) = \lambda \left(1 + r_E\left(\frac{y}{|y|}\right)\right) \frac{y}{|y|}. \tag{2.21}$$

Define $g : [0,1] \to \mathbb{R}$ by

$$g(\sigma) = \sigma \left(1 + r_E\left(\sigma \frac{y}{|y|}\right)\right).$$

We have $g(0) = 0$ and $g(1) = 1 + r_E\left(\frac{y}{|y|}\right)$. By continuity there exists a $\sigma^* \in [0,1]$ such that

$$g(\sigma^*) = \lambda\left(1 + r_E\left(\frac{y}{|y|}\right)\right). \tag{2.22}$$

We conclude from (2.21) and (2.22) that

$$\begin{aligned} z\left(r, \sigma^* \frac{y}{|y|}\right) &= \sigma^*\left(1 + r_E\left(\sigma^* \frac{y}{|y|}\right)\right)\frac{y}{|y|} = g(\sigma^*)\frac{y}{|y|} \\ &= \lambda\left(1 + r_E\left(\frac{y}{|y|}\right)\right)\frac{y}{|y|} = y. \end{aligned}$$

□

Define $\mathcal{J} : \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$ by

$$\mathcal{J}(r) = \frac{\partial z(r)}{\partial x}.$$

Again we make no difference between $\mathcal{J}(r)$ and $\mathcal{J}(r, \cdot)$.

Lemma 2.4. *There exists an $\delta > 0$ such that if $r \in \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ satisfies $\|r\|_{2,\beta} < \delta$, then $\mathcal{J}(r,x)$ is an invertible matrix for every $x \in \overline{\mathbb{B}^N}$ and $x \mapsto \mathcal{J}(r,x)^{-1} \in \left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$. Furthermore, $z(r)^{-1} \in \left(\mathcal{C}^{2,\beta}(\overline{\Omega_r})\right)^N$.*

Here, $z(r)^{-1}$ denotes the inverse of $z(r)$ as a mapping, whereas $\mathcal{J}(r,x)^{-1}$ is the inverse of $\mathcal{J}(r,x)$ as a matrix.

Proof. First of all, $\mathcal{J}(0,x) \equiv I$ (the identity matrix). We will make use of the fact that the spaces $\left(\mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$ are Banach algebras. The mapping $\mathcal{J}$ is continuous near zero from $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ to $\left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$. Invertible elements in $\left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$ form an open set. We conclude that $\mathcal{J}(r)$ is invertible for $\|r\|_{2,\beta}$ small and $\mathcal{J}(r)^{-1} \in \left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$. It is clear that $z(r)^{-1}$ is continuously differentiable and the components $\frac{\partial (z(r)^{-1})_i}{\partial x_j}$ satisfy

$$\frac{\partial (z(r)^{-1})_i}{\partial x_j} = [\mathcal{J}(r)^{-1}]_{ij} \circ z(r)^{-1}, \tag{2.23}$$

Differentiation leads to

$$\frac{\partial^2 (z(r)^{-1})_i}{\partial x_j \partial x_k} = \left(\nabla [\mathcal{J}(r)^{-1}]_{ij} \circ z(r)^{-1}\right) \cdot \frac{\partial z(r)^{-1}}{\partial x_k}. \tag{2.24}$$

Since the composition of an element of $\mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N})$ and an element of $\mathcal{C}^1(\overline{\Omega_r}, \overline{\mathbb{B}^N})$ is in $\mathcal{C}^{0,\beta}(\overline{\Omega_r})$ we get from (2.23) $\frac{\partial (z(r)^{-1})_i}{\partial x_j} \in \mathcal{C}^{0,\beta}(\overline{\Omega_r})$. Combining this and (2.24) it follows that $\frac{\partial^2 (z(r)^{-1})_i}{\partial x_j \partial x_k} \in \mathcal{C}^{0,\beta}(\overline{\Omega_r})$. This completes the proof. □

We denote the components of $\mathcal{J}(\cdot)^{-1}$ by $j^{i,k} : \mathcal{U} \to \mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})$. By Lemma 2.3 and Lemma 2.4 we see that there exists a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ and two mappings $\mathcal{A} : \mathcal{U} \to \mathcal{L}(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}), \mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N}))$ and $\mathcal{Q} : \mathcal{U} \to \mathcal{L}\left(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}), \left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^N\right)$ such that

$$\mathcal{A}(r)u = \left(\Delta\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k,l} j^{i,l}(r) \frac{\partial}{\partial x_i}\left(j^{k,l}(r)\frac{\partial u}{\partial x_k}\right) \tag{2.25}$$

and

$$\mathcal{Q}(r)u = \left(\nabla\left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k} j^{k,i}(r)\frac{\partial u}{\partial x_k} e_i, \tag{2.26}$$

where e_i is the i-th unit vector in $\mathbb{R}^N$. Let $\mathcal{S} : \mathcal{U} \to \mathcal{L}(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}), \mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}))$ be defined by

$$\mathcal{S}(r)u = \begin{pmatrix} \mathcal{A}(r)u \\ \mathrm{Tr}u \end{pmatrix}. \tag{2.27}$$

and introduce $\varphi : \mathcal{U} \to \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ by

$$\varphi(r, x) = \Phi((1 + r(x))x). \tag{2.28}$$

Using this notation, (2.18) formally can be written as

$$\mathcal{F}(r) := \frac{\mathrm{Tr}\left(\mathcal{Q}(r)\left[\mathcal{S}(r)^{-1}\begin{bmatrix} 0 \\ \varphi(r) \end{bmatrix}\right]\right) \cdot n(r)}{n(r) \cdot \mathrm{id}} + \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N}, \tag{2.29}$$

where id is the identity on $\mathbb{S}^{N-1}$. To show that $\mathcal{F}$ is well-defined in this way we need to show that $\mathcal{S}(r)$ is invertible for small r. We also show that $\mathcal{F}$ is analytic near zero by proving that the operators $\mathcal{A}$, $\mathcal{Q}$, n, and φ are analytic, using the Implicit Function Theorem and Banach algebra properties. Many of the following lemmas have already been proved in [60].

Lemma 2.5. *The operator φ is analytic around zero from $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ to $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$.*

Proof. Because of analyticity and radial symmetry of Φ, there exists an analytic function $f : G \to \mathbb{R}$, for G a neighbourhood of 0 in $\mathbb{R}$, such that

$$\varphi(r, x) = f(r(x)).$$

Hence for small r

$$\varphi(r) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} r^k.$$

Define for each $k \in \mathbb{N}_0$ the k-linear form $\varphi_k : \left(\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})\right)^k \to \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ by

$$\varphi_k(r_1, \ldots, r_k) = \frac{f^{(k)}(0)}{k!} \prod_{i=1}^{k} r_i.$$

Because $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ is a Banach algebra we have

$$\|\varphi_k\|_X = \sup_{\forall i: \|r_i\|_{2,\beta}=1} \|\varphi_k(r_1, \ldots, r_k)\|_{2,\beta} \le C^{k-1} \frac{|f^{(k)}(0)|}{k!},$$

for some constant $C > 0$. The norm $\|\cdot\|_X$ on $X = \mathcal{L}^k(\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}), \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}))$ is defined in (1.18). For small $\epsilon > 0$ the analyticity of f yields

$$\sum_{k=0}^{\infty} \|\varphi_k\|_X \epsilon^k < \infty.$$

This completes the proof. □

Lemma 2.6. *The operator n is analytic around zero from $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ to $\left(\mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})\right)^N$.*

Proof. This proof can also be found in [60]. First we take two open non-empty sets W_1 and W_2 in $\mathbb{R}^{N-1}$ and smooth regular parameterizations $\Xi_1 : W_1 \to \mathfrak{U}_1$ and $\Xi_2 : W_1 \to \mathfrak{U}_2$ of two subsets of the unit-sphere $\mathfrak{U}_1$ and $\mathfrak{U}_2$ such that $\mathfrak{U}_1 \cup \mathfrak{U}_2 = \mathbb{S}^{N-1}$. We also choose a smooth partition of unity $\{\chi_1, \chi_2\}$ subordinate to the covering $\{\mathfrak{U}_1, \mathfrak{U}_2\}$. Defining $n^{[k]}(r, \cdot) : W_k \to \mathbb{R}^N$ by $n^{[k]}(r) = n^{[k]}(r, \cdot) = n(r) \circ \Xi_k$ we have for all $\xi \in \mathbb{S}^{N-1}$

$$n(r, \xi) = \chi_1(\xi) n^{[1]}(r, \Xi_1^{-1}(\xi)) + \chi_2(\xi) n^{[2]}(r, \Xi_2^{-1}(\xi)).$$

Here we define $\chi_j(\xi) n^{[j]}(r, \Xi_j^{-1}(\xi)) = 0$ if ξ is not in $\mathfrak{U}_j$. We can reduce the problem to proving analyticity of $n^{[1]}$ and $n^{[2]}$ around zero. Let k be either 1 or 2. Introduce $\eta_k(r) : W_k \to \mathbb{R}^N$ by

$$\eta_k(r) : w \mapsto (1 + r(\Xi_k(w)))\, \Xi_k(w).$$

We define $F^{[k]} : \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \times \left(\mathcal{C}^{1,\beta}(W_k)\right)^N \to \left(\mathcal{C}^{1,\beta}(W_k)\right)^N$ by

$$F^{[k]}(r, \tilde{n}) := \begin{pmatrix} \left(\dfrac{\partial \eta_k(r)}{\partial w}\right)^T \\ \tilde{n}^T \end{pmatrix} \tilde{n} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$F^{[k]}$ is analytic because η_k is analytic. The derivative of $F^{[k]}$ with respect to the second argument at $(0, n^{[k]}(0))$ is given by

$$F^{[k]\prime}(0, n^{[k]}(0))[0, h] = \begin{pmatrix} \left(\dfrac{\partial \Xi_k}{\partial w}\right)^T \\ 2n^{[k]}(0)^T \end{pmatrix} h.$$

The matrix on the right-hand side in nonsingular since the first $N-1$ rows are independent vectors that are tangential to $\mathbb{S}^{N-1}$ and the last row is orthogonal to $\mathbb{S}^{N-1}$ since $n^{[k]}(0) = n(0) \circ \Xi_k = \Xi_k$. We now apply the Implicit Function theorem to

$$F^{[k]}(r, n(r)) = 0,$$

to complete the proof. □

Lemma 2.7. *The operator $\mathcal{S}$ is analytic near zero from $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ to $\mathcal{L}(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}), \mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}))$.*

Proof. It is clear that $z : \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})\right)^N$ is analytic. Hence, we also have analyticity of $\mathcal{J} : \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$ and the components $j^{i,k}$ of the inverse $\mathcal{J}(\cdot)^{-1} : \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$ for small $\|r\|_{2,\beta}$. This is due to analyticity of inversion near the multiplicative identity in the Banach algebra $\left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^{N\times N}$. From (2.25) it follows that $\mathcal{A}$ and $\mathcal{S}$ are analytic around zero. □

Lemma 2.8. *There is a neighbourhood $\mathcal{U}$ of zero in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ such that if $r \in \mathcal{U}$, then $\mathcal{S}(r) : \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}) \to \mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ is invertible. Furthermore, the mapping $\Pi : \mathcal{U} \to \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})$ defined by*

$$\Pi : r \mapsto \mathcal{S}(r)^{-1} \begin{pmatrix} 0 \\ -\varphi(r) \end{pmatrix} \tag{2.30}$$

is analytic around zero.

Proof. The first step is showing that $\mathcal{S}(0) = [\Delta, \mathrm{Tr}]^T$ is invertible. Injectivity is a direct consequence of the maximum principle. Let $(f, g) \in \mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$. Define $\tilde{g} \in \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})$ by $\tilde{g} = Eg$. Then by [32, Cor. 4.14] there is a unique $h \in \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})$ satisfying $\Delta h = f$ and $h(x) = \tilde{g}(x)$ for $x \in \mathbb{S}^{N-1}$. This proves surjectivity. Invertible operators form an open set in $\mathcal{L}(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}), \mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}))$. Combining this and continuity of $\mathcal{S}$ near zero, we see that $\mathcal{S}(r)$ is invertible for r small in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$.

Because of Lemma 2.7 we have analyticity of $(r, \psi) \mapsto \mathcal{S}(r)\psi$. This is easily derived from the definition of analyticity and the fact that $\mathcal{S}(r)$ is linear and bounded. Define $F : \mathcal{U} \times \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}) \to \mathcal{C}^{0,\beta}(\overline{\mathbb{B}^N}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ by

$$F(r, \psi) = \mathcal{S}(r)\psi - \begin{pmatrix} 0 \\ -\varphi(r) \end{pmatrix}.$$

Analyticity of this mapping follows from Lemma 2.5. The Fréchet derivative with respect to the second argument at $(0, 0)$ is

$$F'(0,0)[0, h] = \mathcal{S}(0)h = \begin{pmatrix} \Delta h \\ \mathrm{Tr} h \end{pmatrix}.$$

Since $\mathcal{S}(0)$ is an isomorphism, there exists by the Implicit Function Theorem a unique analytic mapping $\Pi : \mathcal{U} \to \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})$ that satisfies

$$F(r, \Pi(r)) = 0.$$

□

Lemma 2.9. *The operator $\mathcal{Q}$ is analytic from a neighbourhood $\mathcal{U}$ of zero in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ to $\mathcal{L}\left(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}), \left(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N})\right)^N\right)$ and the operator*

$$\Theta : r \mapsto \mathcal{Q}(r)\mathcal{S}(r)^{-1}\begin{pmatrix} 0 \\ -\varphi(r) \end{pmatrix}$$

is analytic from a neighbourhood $\mathcal{U}$ of zero in $\mathcal{C}^{2,\beta}(\overline{\mathbb{S}^{N-1}})$ to $(\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N}))^N$.

Proof. The first part follows from (2.26) and analyticity of $j^{i,k}$ that we obtained in the proof of Lemma 2.7. As a consequence, the mapping $F : \mathcal{U} \times \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N}) \to (\mathcal{C}^{1,\beta}(\overline{\mathbb{B}^N}))^N$ defined by

$$F(r, \psi) = \mathcal{Q}(r)\psi$$

is analytic. Define $G : \mathcal{U} \to \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \times \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})$ by

$$G(r) := \begin{pmatrix} r \\ \mathcal{S}(r)^{-1}\begin{pmatrix} 0 \\ -\varphi(r) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} r \\ \Pi(r) \end{pmatrix}.$$

This mapping is analytic by Lemma 2.8. Therefore $\Theta = F \circ G$ is analytic. □

Lemma 2.10. *The operator $\mathcal{F}$ is analytic from a neighbourhood $\mathcal{U}$ of zero in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ to $\mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})$.*

Proof. The composition of the trace operator and the operator Θ in Lemma 2.9 is analytic near zero. Taking the inner product with $n(r)$ results into a new analytic operator because of Lemma 2.6. Near $r = 0$ the operator $r \mapsto \frac{1}{n(r)\cdot \mathrm{id}}$ is analytic since it is the composition of two analytic operators namely inversion near 1 in the Banach algebra $\mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})$ and a pointwise product of the two analytic operators, n and $r \mapsto \mathrm{id}$. The analyticity of $r \mapsto \frac{1}{1+r}$ can be proved using the methods in the proof of Lemma 2.5. □

Lemma 2.11. *The operator $\mathcal{F}$ is analytic from a neighbourhood $\mathcal{U}$ of zero in $h^{2,\beta}(\mathbb{S}^{N-1})$ to $h^{1,\beta}(\mathbb{S}^{N-1})$.*

Proof. By Lemma 2.10 it is sufficient to show that the image of $h^{2,\beta}(\mathbb{S}^{N-1}) \cap \mathcal{U}$ under $\mathcal{F}$ is contained in $h^{1,\beta}(\mathbb{S}^{N-1})$. Let r be a small element of $h^{2,\beta}(\mathbb{S}^{N-1}) \cap \mathcal{U}$. Choose any positive ε such that $\beta + \varepsilon < 1$. By [53, Prop. 0.2.1] we have

$$h^{2,\beta}(\mathbb{S}^{N-1}) = \overline{\mathcal{C}^{2,\beta+\varepsilon}(\mathbb{S}^{N-1})}^{\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})}, \quad h^{1,\beta}(\mathbb{S}^{N-1}) = \overline{\mathcal{C}^{1,\beta+\varepsilon}(\mathbb{S}^{N-1})}^{\mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})}.$$

There are $r_n \in \mathcal{C}^{2,\beta+\varepsilon}(\mathbb{S}^{N-1})$ such that $r_n \to r$ in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$. By continuity of the mapping $\mathcal{F} : \mathcal{U} \to \mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})$ on a neighbourhood $\mathcal{U}$ of zero in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$ (this follows from Lemma 2.10), we have $\mathcal{F}(r_n) \to \mathcal{F}(r)$ in $\mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})$ and $\mathcal{F}(r_n) \in \mathcal{C}^{1,\beta+\varepsilon}(\mathbb{S}^{N-1})$. This implies that $\mathcal{F}(r) \in h^{1,\beta}(\mathbb{S}^{N-1})$. □

The next step is finding the linearisation of the evolution operator $\mathcal{F}$ around zero.

Lemma 2.12. *For $\Pi : \mathcal{U} \to \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})$ as defined in (2.30) we have*

$$\Delta\Pi'(0)[h] = 0 \quad \text{in } \mathbb{B}^N$$

and

$$\Pi'(0)[h] = \frac{1}{\sigma_N} h \quad \text{in } \mathbb{S}^{N-1}.$$

Proof. We have seen in Lemma 2.8 that Π is analytic near zero. From the definition of φ it follows that

$$\varphi'(0)[h] = -\frac{1}{\sigma_N} h. \tag{2.31}$$

Differentiating $\mathcal{S}(r)\Pi(r) = \begin{pmatrix} 0 \\ -\varphi(r) \end{pmatrix}$ one obtains

$$(\mathcal{S}'(0)[h])(\Pi(0)) + (\mathcal{S}(0))(\Pi'(0)[h]) = \begin{pmatrix} \Delta(\Pi'(0)[h]) \\ \mathrm{Tr}(\Pi'(0)[h]) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sigma_N} h \end{pmatrix},$$

where we used $\Pi(0) = 0$. □

Lemma 2.13. *The Fréchet derivative of $\mathcal{F}$ at 0 is*

$$\mathcal{F}'(0)[h] = -\frac{1}{\sigma_N}\mathcal{N}h - \frac{N}{\sigma_N}h, \tag{2.32}$$

where $\mathcal{N} : \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1}) \to \mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})$ is the Dirichlet-to-Neumann operator on the unit ball given by

$$\mathcal{N}h := \mathrm{Tr}\nabla\mathcal{S}(0)^{-1} \begin{pmatrix} 0 \\ h \end{pmatrix} \cdot n(0). \tag{2.33}$$

Proof. Write $\mathcal{F}$ as

$$\mathcal{F}(r) = -\frac{\mathrm{Tr}\mathcal{Q}(r)\Pi(r) \cdot n(r)}{n(r) \cdot \mathrm{id}} + \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N},$$

where id is the identity $\xi \mapsto \xi$ on $\mathbb{S}^{N-1}$. Using $\Pi(0) = 0$, $n(0) = \mathrm{id}$, and a Taylor expansion, we get

$$\mathcal{F}'(0)[h] = -\frac{\mathrm{Tr}\mathcal{Q}(0)\Pi'(0)[h] \cdot n(0)}{n(0) \cdot \mathrm{id}} - \frac{N}{\sigma_N}h = -\mathrm{Tr}\nabla\Pi'(0)[h] \cdot n(0) - \frac{N}{\sigma_N}h. \tag{2.34}$$

From Lemma 2.12 it follows that

$$\Pi'(0)[h] = \frac{1}{\sigma_N}\mathcal{S}(0)^{-1}\begin{pmatrix} 0 \\ h \end{pmatrix}. \tag{2.35}$$

The lemma follows from (2.33), (2.34), and (2.35). □

2.3 The spectrum of the linearisation and stability

In this section we apply the principle of linearised stability to the evolution equation, given by (2.20) and (2.29), in order to derive a stability result for the stationary solution $r \equiv 0$. For this purpose we study the spectral properties of the operator $\mathcal{F}'(0) : h^{2,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ given by (2.32). First we find the eigenvalues of the Dirichlet-to-Neumann operator $\mathcal{N} : h^{2,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$. We do this by studying the spherical harmonics s_{kj}, that form an orthonormal basis of eigenvectors of $\mathcal{N}$ for $\mathbb{L}^2(\mathbb{S}^{N-1})$ (see Chapter 1).

Lemma 2.14. *If $q \in \mathfrak{H}_k^N$, then for all $x \in \mathbb{S}^{N-1}$*

$$\frac{\partial q}{\partial n}(x) = kq(x), \tag{2.36}$$

where n is the normal on $\mathbb{S}^{N-1}$. For $s \in \mathfrak{S}_k^N$ we have

$$\mathcal{N}s = ks. \tag{2.37}$$

Proof. Define $\tilde{q} \in \mathfrak{S}_k^N$ by $\tilde{q} = q \mid_{\mathbb{S}^{N-1}}$ such that

$$q(x) = |x|^k \tilde{q}\left(\frac{x}{|x|}\right).$$

We obtain (2.36) differentiating this identity in radial direction and taking $x \in \mathbb{S}^{N-1}$. The second statement is a consequence of the first statement and the fact that any $s \in \mathfrak{S}_k^N$ has a unique harmonic extension in $\mathfrak{H}_k^N$ given by $x \mapsto |x|^k s(\frac{x}{|x|})$. □

Since the functions s_{kj} form a complete orthonormal set, the spectrum of $\mathcal{N}$ in $\mathbb{L}^2(\mathbb{S}^{N-1})$ consists entirely of eigenvalues and coincides with $\mathbb{N}_0$. Eigenvectors in $h^{2,\beta}(\mathbb{S}^{N-1})$ are also eigenvectors in $\mathbb{L}^2(\mathbb{S}^{N-1})$ and vice versa because spherical harmonics are smooth.

Corollary 2.15. *The set of eigenvalues of $\mathcal{N} : h^{2,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ is exactly $\mathbb{N}_0$. For $k \in \mathbb{N}_0$ the corresponding eigenspace is $\mathfrak{S}_k^N$. The point spectrum of $\mathcal{F}'(0)$ is therefore*

$$\pi(\mathcal{F}'(0)) = \left\{-\frac{N}{\sigma_N}, -\frac{N+1}{\sigma_N}, -\frac{N+2}{\sigma_N}, \dots\right\}$$

and the eigenspace for eigenvalue $-\frac{N+k}{\sigma_N}$ is $\mathfrak{S}_k^N$.

Lemma 2.16. *For each $\lambda \in \mathbb{C}$, the mapping $\lambda\mathcal{I} - \mathcal{F}'(0)$ maps $h^{2,\beta}(\mathbb{S}^{N-1})$ continuously into $h^{1,\beta}(\mathbb{S}^{N-1})$.*

Proof. This is a consequence of Lemma 2.11 and $h^{2,\beta}(\mathbb{S}^{N-1}) \hookrightarrow h^{1,\beta}(\mathbb{S}^{N-1})$. □

Lemma 2.17. *The spectrum of $\mathcal{N} : h^{2,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ consists entirely of eigenvalues,*

$$\mathrm{sp}(\mathcal{N}) = \pi(\mathcal{N}) = \mathbb{N}_0.$$

The resolvent $(\lambda\mathcal{I} - \mathcal{N})^{-1} : h^{1,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ is compact for all $\lambda \notin \mathrm{sp}(\mathcal{N})$. The spectrum of $\mathcal{F}'(0) : h^{2,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ also consists entirely of eigenvalues

$$\mathrm{sp}(\mathcal{F}'(0)) = \pi(\mathcal{F}'(0)) = \left\{ -\frac{N}{\sigma_N}, -\frac{N+1}{\sigma_N}, -\frac{N+2}{\sigma_N}, \dots \right\}$$

and the resolvent $(\lambda\mathcal{I} - \mathcal{F}'(0))^{-1} : h^{1,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ is compact for all $\lambda \notin \mathrm{sp}(\mathcal{F}'(0))$.

Proof. By [19, Thm. B.3, B.4], $\mathcal{F}'(0)$ generates an analytic semigroup on $h^{1,\beta}(\mathbb{S}^{N-1})$ with dense domain of definition $h^{2,\beta}(\mathbb{S}^{N-1})$. This implies that the resolvent set of $\mathcal{F}'(0)$ is not empty. There exists a $\lambda^* \in \mathbb{C}$ such that

$$\lambda^*\mathcal{I} - \mathcal{F}'(0) : h^{2,\alpha}(\mathbb{S}^{N-1}) \to h^{1,\alpha}(\mathbb{S}^{N-1})$$

is invertible and by the Open Mapping Theorem the inverse is bounded. Since $h^{2,\alpha}(\mathbb{S}^{N-1}) \hookrightarrow\hookrightarrow h^{1,\alpha}(\mathbb{S}^{N-1})$ (see [1, Thm. 8.6]),

$$\lambda^*\mathcal{I} - \mathcal{F}'(0) : h^{1,\alpha}(\mathbb{S}^{N-1}) \to h^{1,\alpha}(\mathbb{S}^{N-1})$$

is compact. From [49, Ch. 3 Thm. 6.29] we have $\mathrm{sp}(\mathcal{F}'(0)) = \pi(\mathcal{F}'(0))$ and the resolvent is compact for $\lambda \notin \pi(\mathcal{F}'(0))$. It is clear that similar results hold for $\mathcal{N}$. □

Now we apply these results for the linearisation to the nonlinear problem (2.20), using the principle of linearised stability.

Theorem 2.18. *Let $0 < \lambda_0 < \frac{N}{\sigma_N}$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in h^{2,\beta}(\mathbb{S}^{N-1})$ with $\|r_0\|_{2,\beta} < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = r_0,$$

has a solution $r \in \mathcal{C}\left([0,\infty), h^{2,\beta}(\mathbb{S}^{N-1})\right) \cap \mathcal{C}^1\left([0,\infty), h^{1,\beta}(\mathbb{S}^{N-1})\right)$ satisfying

$$\|r(\tau)\|_{2,\beta} \leq Me^{-\lambda_0\tau}\|r_0\|_{2,\beta}. \tag{2.38}$$

Proof. As mentioned before, $\mathcal{F}'(0)$ generates an analytic semigroup on $h^{1,\beta}(\mathbb{S}^{N-1})$. Because of Lemma 2.17 the spectrum is left of the imaginary axis and it has distance $-\frac{N}{\sigma_N}$ to it. Furthermore, since $\mathcal{I} + \mathcal{F}'(0)$ is an isomorphism between $h^{2,\beta}(\mathbb{S}^{N-1})$ and $h^{1,\beta}(\mathbb{S}^{N-1})$, the graph norm of $\mathcal{F}'(0)$ is equivalent to the norm of $h^{2,\beta}(\mathbb{S}^{N-1})$. We can apply [53, Thm. 9.1.2] to show the global existence of r in time and the estimate. □

Combining Theorem 2.18 and equation (2.19) we get the following estimate for the non-autonomous problem (2.17):

$$\|r(t)\|_{2,\beta} \leq M \left(\frac{Nt}{\sigma_N} + 1 \right)^{-\zeta} \|r_0\|_{2,\beta}. \tag{2.39}$$

for any $\zeta \in (0, 1)$.

2.4 Faster convergence in absence of low-order moments

In this section we show that if the integrals of harmonic polynomials with low degrees over the moving domain vanish, then convergence to the equilibrium will be faster than in Theorem 2.18.

Let h_{kj} be the unique harmonic extension of s_{kj} given by $h_{kj}(x) = |x|^k s_{kj}(\frac{x}{|x|})$. For a domain Ω, the Richardson moments of order k are defined by the quantities $\int_\Omega h_{kj} dx$ (see also [64]). Define for $K \in \mathbb{N}_0$

$$\mathfrak{M}_K^N := \left\{ r \in C^0(\mathbb{S}^{N-1}) \middle| \int_{\Omega_r} dx = \frac{\sigma_N}{N} \wedge \forall q \in \cup_{j=1}^K \mathfrak{H}_j^N : \int_{\Omega_r} q(x) dx = 0 \right\}. \tag{2.40}$$

We have $r \in \mathfrak{M}_K^N$ if the corresponding domain Ω_r has the same volume as the unit ball and the Richardson moments of order $1, 2 \ldots, K$ vanish.

Note that by a suitable length scaling we can at least achieve $r(0) \in \mathfrak{M}_0^N$.

Lemma 2.19. *Let r be a solution to (2.17). If $r(0)$ is in $\mathfrak{M}_K^N$, then $r(t) \in \mathfrak{M}_K^N$ for all $t \geq 0$.*

Proof. Let $n_{R(t)}$ be the normal vector field on $\Gamma_{R(t)}$. Because of (2.1)-(2.4) and Green's identities we have for $k \neq 0$

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_{R(t)}} h_{kj} dx &= \int_{\Gamma_{R(t)}} h_{kj} v \cdot n_{R(t)} dx \\ &= -\int_{\Gamma_{R(t)}} h_{kj} \frac{\partial p}{\partial n} dx = \int_{\Gamma_{R(t)}} \left(p \frac{\partial h_{kj}}{\partial n} - h_{kj} \frac{\partial p}{\partial n} \right) dx = \\ &= \int_{\Omega_{R(t)}} \left(p \Delta h_{kj} - h_{kj} \Delta p \right) dx = h_{kj}(0) = 0. \end{aligned}$$

Thus for $k \neq 0$

$$\begin{aligned} \int_{\Omega_{r(t)}} h_{kj}(x) dx &= \int_{\Omega_{R(t)}} h_{kj}(\alpha(t)^{-1} y) \alpha(t)^{-N} dy = \alpha(t)^{-N-k} \int_{\Omega_{R(t)}} h_{kj}(y) dy = \\ &= \alpha(t)^{-N-k} \int_{\Omega_{R(0)}} h_{kj}(y) dy = \alpha(t)^{-N-k} \int_{\Omega_{r(0)}} h_{kj}(y) dy. \end{aligned}$$

For $k \neq 0$ we conclude that if $\int_{\Omega_{r(0)}} h_{kj} dx = 0$, then $\int_{\Omega_{r(t)}} h_{kj} dx = 0$ for all $t \geq 0$. If

$\int_{\Omega_{r(0)}} dx = \frac{\sigma_N}{N}$, then it follows from (1.11) and (2.5) that

$$\int_{\Omega_{r(t)}} dx = \frac{1}{\alpha(t)^N} \int_{\Omega_{R(t)}} dx = \frac{1}{\alpha(t)^N} \left(\frac{\sigma_N}{N} + t \right) = \frac{\sigma_N}{N}.$$

□

Define for $K, L \in \mathbb{N}_0$ the vector spaces

$$h_K^{L,\beta}(\mathbb{S}^{N-1}) := \{ r \in h^{L,\beta}(\mathbb{S}^{N-1}) : (r, s_{kj})_{\mathbb{L}^2(\mathbb{S}^{N-1})} = 0, k \le K \}. \tag{2.41}$$

Now $h_K^{L,\beta}(\mathbb{S}^{N-1})$, equipped with the norm $\| \cdot \|_{L,\beta}$, is a closed subspace of $h^{L,\beta}(\mathbb{S}^{N-1})$ and therefore a Banach space. Define the index sets

$$\mathbb{I}_K := \{ (k,j) \in \mathbb{N}_0 \times \mathbb{N} : k \in \{0, 1, \ldots . K\}, j \in \{1, 2, \ldots, \nu(N,k)\} \},$$

and define the operators $f_K : h^{2,\beta}(\mathbb{S}^{N-1}) \to \mathbb{R}^{\mathbb{I}_K}$ by

$$f_K(r)_{kj} := \int_{\Omega_r} h_{kj} dx - \frac{\sqrt{\sigma_N}}{N} \delta_{k0}, \tag{2.42}$$

where $\delta_{k0} = 0$ if $k \neq 0$ and $\delta_{k0} = 1$ if $k = 0$. Since the constant function s_{00} satisfies $\|s_{00}\|_{\mathbb{L}^2(\mathbb{S}^{N-1})} = 1$ we have $s_{00} \equiv \frac{1}{\sqrt{\sigma_N}}$. Consequently $h_{00} \equiv \frac{1}{\sqrt{\sigma_N}}$. We conclude from (2.40) that $f_K(r) = 0$ if and only if $r \in \mathfrak{M}_K^N$. Let $\mathcal{P}_K : h^{2,\beta}(\mathbb{S}^{N-1}) \to h_K^{2,\beta}(\mathbb{S}^{N-1})$ be the orthogonal projections on $h_K^{2,\beta}(\mathbb{S}^{N-1})$ with respect to the $\mathbb{L}^2(\mathbb{S}^{N-1})$-inner product and define $\phi_K : h^{2,\beta}(\mathbb{S}^{N-1}) \to \mathbb{R}^{\mathbb{I}_K} \times h_K^{2,\beta}(\mathbb{S}^{N-1})$ by

$$\phi_K(r) := \begin{pmatrix} f_K(r) \\ \mathcal{P}_K r \end{pmatrix}. \tag{2.43}$$

Because $h_{kj}(x) = |x|^k s_{kj}(\frac{x}{|x|})$ we have

$$\begin{aligned} f_K(r)_{kj} &= \int_{\mathbb{S}^{N-1}} \int_0^{1+r(\varphi)} \rho^{k+N-1} s_{kj}(\varphi) d\rho d\varphi - \frac{\sqrt{\sigma_N}}{N} \delta_{k,0} \\ &= \int_{\mathbb{S}^{N-1}} \frac{(1 + r(\varphi))^{k+N}}{k+N} s_{kj}(\varphi) d\varphi - \frac{\sqrt{\sigma_N}}{N} \delta_{k,0}, \end{aligned}$$

where $\varphi = \frac{x}{|x|} \in \mathbb{S}^{N-1}$. The operators $f_K : h^{2,\beta}(\mathbb{S}^{N-1}) \to \mathbb{R}^{\mathbb{I}_K}$ are analytic because the components $(f_K(\cdot))_{kj}$ are finite sums of bounded multilinear operators. Frechét-differentiation of f_K leads to

$$f_K'(0)[r]_{kj} = (r, s_{kj})_{\mathbb{L}^2(\mathbb{S}^{N-1})}. \tag{2.44}$$

By linearity of $\mathcal{P}_K$ we have

$$\phi_K'(0)[r] = \begin{pmatrix} f_K'(0)[r] \\ \mathcal{P}_K r \end{pmatrix}.$$

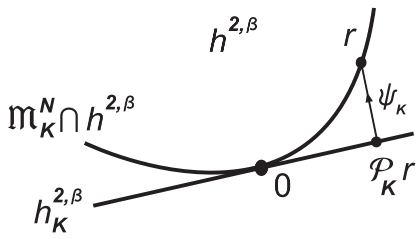


Figure 2.2: *A local bijection between $\mathfrak{M}_K^N$ and $h^{2,\beta}(\mathbb{S}^{N-1})$*

If $\phi_K'(0)[r] = 0$, then $(r, s_{kj})_{\mathbb{L}^2(\mathbb{S}^{N-1})} = 0$ for all k and j which implies $r = 0$. If $(v, \tilde{r}) \in \mathbb{R}^{\mathbb{I}_K} \times h_K^{2,\beta}(\mathbb{S}^{N-1})$, then $r := \tilde{r} + \sum_{k \le K} \sum_{j=1}^{\nu(k,N)} v_{kj} s_{kj}$ satisfies $\phi_K'(0)[r] = (v, \tilde{r})$. Therefore $\phi_K'(0)$ is an isomorphism between $h^{2,\beta}(\mathbb{S}^{N-1})$ and $\mathbb{R}^{\mathbb{I}_K} \times h_K^{2,\beta}(\mathbb{S}^{N-1})$. By the Implicit Function Theorem, ϕ_K is invertible in a neighbourhood $\mathcal{U}$ of 0 in $h^{2,\beta}(\mathbb{S}^{N-1})$. Define the analytic operator $\psi_K : h_K^{2,\beta}(\mathbb{S}^{N-1}) \cap \mathcal{U} \to \mathfrak{M}_K^N$ by

$$\psi_K(r) := \phi_K^{-1}(0, r),$$

see Figure 2.2. Differentiating $\phi_K(\psi_k(r)) = (0, r)$ results into

$$\phi_K'(\psi_K(0))[\psi_K'(0)[r]] = (0, r).$$

Since $\psi_K(0) = 0$ it follows that

$$f_K'(0)[\psi_K'(0)[r]] = 0$$

and

$$\mathcal{P}_K(\psi_K'(0)[r]) = r.$$

From (2.44) we conclude that $\psi_K'(0)$ is the identity on $h_K^{2,\beta}(\mathbb{S}^{N-1})$.

Corollary 2.20. *The tangent space of $\mathfrak{M}_K^N$ at 0 in $h^{2,\beta}(\mathbb{S}^{N-1})$ is equal to $h_K^{2,\beta}(\mathbb{S}^{N-1})$ and for $r \in h_K^{2,\beta}(\mathbb{S}^{N-1})$ we have*

$$\psi_K'(0)[r] = r.$$

Define the operator $\mathcal{G}_K : h_K^{2,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ as the restriction of $\mathcal{F}$ to $h_K^{2,\beta}(\mathbb{S}^{N-1})$. From Lemma 2.13 it follows that

$$\mathcal{G}_K'(0) = \left(-\frac{1}{\sigma_N}\mathcal{N} - \frac{N}{\sigma_N}\mathcal{I} \right) \Bigg|_{h_K^{2,\beta}(\mathbb{S}^{N-1})}.$$

In order to determine the spectrum of $\mathcal{G}_K'(0)$, we write

$$h^{2,\beta}(\mathbb{S}^{N-1}) = h_K^{2,\beta}(\mathbb{S}^{N-1}) \oplus \left(h_K^{2,\beta}(\mathbb{S}^{N-1}) \right)^{\perp},$$

where the orthoplement is taken with respect to the $\mathbb{L}^2(\mathbb{S}^{N-1})$-inner product. We have the following lemma, which holds in general for direct sums of closed subspaces of Banach spaces.

Lemma 2.21. *Let X be a Banach space and D a dense subspace of X. Let $A : D \to X$ be a linear closed sectorial operator on X and suppose that* $\mathrm{sp}(A) = \pi(A)$. *Let X_1 and X_2 be closed disjoint subspaces of X such that $X_1 \oplus X_2 = X$ and $(D \cap X_1) \oplus (D \cap X_2) = D$. Define for $i = 1, 2$, the operators $A_i : D \cap X_i \to X$ by $A_i = A\mid_{D\cap X_i}$. Suppose that $A_i(D \cap X_i) \subseteq X_i$ and suppose that all eigenspaces of A are subspaces of either X_1 or X_2. Regard A_i as operators on X_i with dense domain of definition $D \cap X_i$. The following statements are true:*

i) A_i *is closed ,*

ii) $\mathrm{sp}(A) = \mathrm{sp}(A_1) \cup \mathrm{sp}(A_2)$.

If furthermore D endowed with the graph norm is compactly embedded in X, then the following statements are true as well:

iii) $\mathrm{sp}(A_i) = \pi(A_i)$,

iv) $\mathrm{sp}(A_1) \cap \mathrm{sp}(A_2) = \emptyset$,

v) if $\lambda \in \rho(A)$, *then* $(\lambda\mathcal{I} - A)^{-1}\mid_{X_i} = (\lambda\mathcal{I} - A_i)^{-1}$,

vi) A_i *is sectorial.*

Proof. Statement i) follows immediately from the closedness of the operator A and the spaces X_i. To prove ii) we take $\lambda \in \rho(A)$. It is clear that $\lambda\mathcal{I} - A_i : D \cap X_i \to X_i$ is injective, for both $i = 1$ and $i = 2$. To prove surjectivity, take $g \in X_1$. There exists an $f \in D$ such that $(\lambda\mathcal{I} - A)f = g$. Then $f = f_1 + f_2$ with $f_i \in D \cap X_i$. Because $(\lambda\mathcal{I} - A_i)f_i \in X_i$ we have $(\lambda\mathcal{I} - A)f_1 = g$. Therefore $\lambda\mathcal{I} - A_i : D \cap X_i \to X_i$ is bijective. If $(\lambda\mathcal{I} - A_i)^{-1} : X_i \to D \cap X_i$ were unbounded, then there would be a sequence $(f_n)_{n=1}^{\infty}$ in $D \cap X_i$ with $\|f_n\| = 1$ (taking the graph norm) such that $\|(\lambda\mathcal{I} - A_i)f_n\| < \frac{1}{n}$. Since this contradicts $\lambda \in \rho(A)$, we conclude that $\lambda \in \rho(A_1) \cap \rho(A_2)$. To complete the proof of ii) we take a $\lambda \in \mathrm{sp}(A) = \pi(A)$. There exists an $f \in D \cap X$ not equal to zero such that $Af = \lambda f$. This f is in either X_1 or in X_2. We conclude that $\lambda \in \pi(A_1) \cup \pi(A_2) \subseteq \mathrm{sp}(A_1) \cup \mathrm{sp}(A_2)$. To prove iii), take a fixed $\lambda^* \in \rho(A)$ such that $\lambda^* \in \rho(A_i)$ for $i = 1, 2$. Now $\lambda^*\mathcal{I} - A_i$ is a continuous bijection from $D \cap X_i$ to X_i. By the Open Mapping Theorem the inverse is continuous as well. By compactness of the embedding of $D \cap X_i$ in X_i we conclude that $(\lambda^*\mathcal{I} - A_i)^{-1} : X_i \to X_i$ is compact. From [49, Ch. 3 Thm. 6.29] it follows that $\mathrm{sp}(A_i) = \pi(A_i)$. Now, iv) is a direct result from the fact that eigenspaces of A are subspaces of either X_1 or X_2. Statement v) follows from the fact that if $(\lambda\mathcal{I} - A_i)^{-1}g = f$ for $g \in X_i$, then $g = (\lambda\mathcal{I} - A_i)f = (\lambda\mathcal{I} - A)f$. Statement vi) follows immediately from statement v). □

Corollary 2.22. *$\mathcal{G}_K'(0)$ is a closed sectorial operator on $h_K^{1,\beta}(\mathbb{S}^{N-1})$ with $\mathcal{D}(\mathcal{G}_K'[0]) = h_K^{2,\beta}(\mathbb{S}^{N-1})$.*

Lemma 2.23. *The spectrum of $\mathcal{G}_K'(0)$ consists entirely of eigenvalues and*

$$\mathrm{sp}(\mathcal{G}_K'(0)) = \mathrm{sp}(\mathcal{F}'(0)) \setminus \left\{ -\frac{N}{\sigma_N}, -\frac{N+1}{\sigma_N}, \ldots, -\frac{N+K}{\sigma_N} \right\}.$$

Proof. An eigenvector of $\mathcal{F}'(0)$ is in $h^{2,\beta}_K(\mathbb{S}^{N-1})$ if and only if the corresponding eigenvalue is in $\mathrm{sp}(\mathcal{F}'(0)) \setminus \left\{-\frac{N}{\sigma_N}, -\frac{N+1}{\sigma_N}, \dots, -\frac{N+K}{\sigma_N}\right\}$. Therefore $\pi(\mathcal{G}'_K(0)) = \mathrm{sp}(\mathcal{F}'(0)) \setminus \left\{-\frac{N}{\sigma_N}, -\frac{N+1}{\sigma_N}, \dots, -\frac{N+K}{\sigma_N}\right\}$. Now the lemma follows from Lemma 2.21. □

Theorem 2.24. *Let $0 < \lambda_0 < \frac{N+K+1}{\sigma_N}$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in h^{2,\beta}(\mathbb{S}^{N-1}) \cap \mathfrak{M}^N_K$ and $\|r_0\|_{2,\beta} < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = r_0,$$

has a solution $r \in \mathcal{C}\left([0,\infty), h^{2,\beta}(\mathbb{S}^{N-1})\right) \cap \mathcal{C}^1\left([0,\infty), h^{1,\beta}(\mathbb{S}^{N-1})\right) \cap \mathfrak{M}^N_K$ satisfying

$$\|r(\tau)\|_{2,\beta} \le M e^{-\lambda_0 \tau} \|r_0\|_{2,\beta}.$$

Proof. Existence follows from Theorem 2.18. Note that if $r \in \mathfrak{M}^N_K$ is small enough, then

$$\psi_K(\mathcal{P}_K r) = r.$$

The following evolution equation holds for $\mathcal{P}_K r$:

$$\frac{\partial\left(\mathcal{P}_K r\right)}{\partial \tau} = \mathcal{P}_K \frac{\partial r}{\partial \tau} = \mathcal{P}_K \mathcal{F}(r) = \left(\mathcal{P}_K \circ \mathcal{F} \circ \psi_K\right)(\mathcal{P}_K r).$$

Linearising the evolution operator $\mathcal{P}_K \circ \mathcal{F} \circ \psi_K : h^{2,\beta}_K(\mathbb{S}^{N-1}) \cap \mathcal{U} \to h^{1,\beta}_K(\mathbb{S}^{N-1})$ around zero leads to

$$\left(\mathcal{P}_K \circ \mathcal{F} \circ \psi_K\right)'(0) = \mathcal{P}_K \circ \mathcal{F}'(\psi_K(0)) \circ \psi'_K(0) = \mathcal{P}_K \circ \mathcal{F}'(0)\,|_{h^{2,\beta}_K(\mathbb{S}^{N-1})} = \mathcal{G}'_K(0). \quad (2.45)$$

We used Corollary 2.20 and $\psi_K(0) = 0$. Since $0 < \lambda_0 < \frac{N+K+1}{\sigma_N}$ we have by Corollary 2.22, Lemma 2.23, [53, Thm. 9.1.2], analyticity of ψ_K, and $\psi_K(0) = 0$

$$\begin{aligned} \|r(\tau)\|_{2,\beta} &= \|(\psi_K \circ \mathcal{P}_K) r(\tau)\|_{2,\beta} \le C \|\mathcal{P}_K r(\tau)\|_{2,\beta} \le \\ &\le C e^{-\lambda_0 \tau} \|\mathcal{P}_K r_0\|_{2,\beta} \le C e^{-\lambda_0 \tau} \|r_0\|_{2,\beta}. \end{aligned}$$

□

Combining Theorem 2.24 and (2.19) we get the following estimate for the non-autonomous problem (2.17):

$$\|r(t)\|_{2,\beta} \le M \left(\frac{Nt}{\sigma_N} + 1\right)^{-\zeta} \|r_0\|_{2,\beta}, \quad (2.46)$$

for any $\zeta \in \left(0, 1 + \frac{K+1}{N}\right)$.

Chapter 3

Hele-Shaw flow with surface tension in $\mathbb{R}^3$

3.1 Introduction

In Chapter 1 we briefly discussed how the suction problem is regularised by boundary condition (1.6) in the case of an unbounded domain and parallel suction at infinity. This condition includes the influence of surface tension forces on the free surface of the liquid in the model.

An example from physics where surface tension is present is a drop of mercury that contracts to a spherical drop due to surface tension. Also the process of viscous sintering in glass industry is modeled as a moving boundary problem with surface tension on the boundary (see e.g. [51]).

In Chapter 2 a stability result for the zero surface tension Hele-Shaw model with a source was derived based on linearisation. If we replace the source by a sink, then the linear problem becomes unstable. The suction problem is the reverse injection problem. Hence the negative eigenvalues for the operator (2.13), describing the linearised evolution, change sign.

If surface tension on the boundary is included, then the suction problem is no longer the reverse injection problem. The goal of this chapter is to show stability results for the spherical solution both for injection and suction in combination with surface tension. As in the previous chapter this is done by linearising a nonlinear parabolic equation around this trivial solution. Again the spherical harmonics are the eigenfunctions. Escher and Simonett [21] already proved existence of short-time solutions.

For the suction problem, Tian [70] proved that if the geometric centre and the suction point do not coincide, then the solution breaks down before all fluid is sucked out or the domain becomes unbounded with zero area. We answer the reverse question, by showing that all liquid can be removed under the conditions that suction takes place in the geometric centre and the ratio $\frac{|\mu|}{\gamma}$ is small enough. This gives a partial answer to an open problem posed in 1993 [39].

In Section 3.4 we prove that any smaller amount can be removed if the geometric centre is close enough to the suction point. Here we use the fact that the evolution induces a semiflow. This follows from the abstract theory of quasilinear parabolic equations [3].

In the previous chapter invariance of Richardson moments is used to show faster decay rates than (2.38) indicates. In this chapter this invariance property is crucial to obtain stability for the case of suction. Although all other moments are not invariant in the presence of surface tension, the zeroth and first Richardson moments still are. The eigenvalues corresponding to spherical harmonics of degree zero and one are the only ones that are positive for any combination of the surface tension coefficient γ and suction rate $\mu < 0$. Assuming that the zeroth and first moments are absent we study a projected version of the nonlinear evolution equation on the orthoplement of the spherical harmonics of degree zero and one. Positive eigenvalues can be excluded, making use of the local bijection, that we called ψ_1 in Chapter 2, between this orthoplement and the invariant manifold describing domains for which some Richardson moments are zero.

In general, the evolution equation cannot be transformed to an autonomous equation after introducing another time variable. Time dependence occurs in the parabolic equation (3.22) because of the rescaling of the domain. By rescaling a domain by a factor α, the mean curvature of its boundary is scaled by a factor α^{-1}. On the other hand the fundamental solution of the Laplacian scales as $\Phi(\alpha x) = \alpha^{2-N}\Phi(x)$ modulo a constant. Only in the three-dimensional case both effects (surface tension and strength of source/sink) scale in the same way.

Because of this difference in scaling behaviour, the right-hand side of (3.22) is the sum of two terms in which time dependence appears in different ways. The second term is equal to the evolution operator for classical Hele-Shaw flow (2.29) multiplied by the parameter μ. We have seen that this term is related to a solution operator of a boundary value problem with Φ as boundary data. In the other term this boundary data is related to the mean curvature of the moving domain.

Only the three-dimensional case can be treated as autonomous by introducing a new time variable to eliminate the factor α^{-3} in (3.22). The calculations up to (3.22) hold in any space dimension. After that we restrict ourselves to $N = 3$. The question of stability in dimensions 2 and $N \geq 4$ is answered in Chapter 5. There other methods are used that are based on energy estimates in Hilbert spaces.

The fixed time problem is modified in the following way:

$$\operatorname{div} v = \mu\delta \quad \text{in } \Omega(t), \tag{3.1}$$

$$v = -\nabla p \quad \text{in } \Omega(t), \tag{3.2}$$

$$p = -\gamma\kappa \quad \text{on } \Gamma(t) := \partial\Omega(t). \tag{3.3}$$

Here, $\kappa(\cdot, t) : \Gamma(t) \to \mathbb{R}$ is the mean curvature of the moving boundary $t \mapsto \Gamma(t)$ (taken negative if $\Omega(t)$ is convex), μ is the injection rate if $\mu > 0$ or the suction rate if $\mu < 0$, and $\gamma > 0$ is the surface tension coefficient. The normal velocity v_n of the moving boundary $\Gamma(t)$ is given by

$$v_n = v \cdot n. \tag{3.4}$$

From (3.1), (3.2), and (3.3) it follows that

$$\begin{aligned} \Delta p &= -\mu\delta \quad \text{in } \Omega(t), &(3.5)\\ p &= -\gamma\kappa \quad \text{on } \Gamma(t) = \partial\Omega(t). &(3.6) \end{aligned}$$

On $\Gamma(t)$ we have

$$v_n = -\frac{\partial p}{\partial n}.$$

In contrast to Chapter 2, μ may take other values than 1 here. Therefore, there are some modifications. Assume that the initial domain $\Omega(0)$ has a volume that is equal to the volume of the unit ball, which is $\frac{\sigma_N}{N}$. Then

$$\mathfrak{V}(t) = \frac{\sigma_N}{N} + \mu t. \tag{3.7}$$

Note that for negative μ, the problem only makes sense if

$$t \le T := -\frac{\sigma_N}{\mu N}.$$

If $\Omega(0) = \mathbb{B}^N$, then $\Omega(t) = \alpha(t)\mathbb{B}^N$, where

$$\alpha(t) = \sqrt[N]{\frac{\mu N t}{\sigma_N} + 1}.$$

This is in accordance with the notation in Chapter 2 if $\mu = 1$.

3.2 The evolution equation and its linearisation

In this section we derive a nonlinear non-local evolution equation describing the motion of the domain, in a similar way as we did in Chapter 2 for $\gamma = 0$. We introduce $R(\cdot, t) : \mathbb{S}^{N-1} \to (-1, \infty)$ such that $\Omega_{R(\cdot,t)} = \Omega(t)$ and rescale by introducing $r(\cdot, t) : \mathbb{S}^{N-1} \to (-1, \infty)$ such that (2.6) and (2.7) hold. We introduce the bijection $\tilde{z}(r) : \mathbb{S}^{N-1} \to \Gamma_r$ by (2.8). Note that Lemma 2.1 still holds for the problem with surface tension. Therefore,

$$\frac{\partial R}{\partial t}(\xi) = -\frac{\nabla p(\tilde{z}(R,\xi)) \cdot n(R,\xi)}{n(R,\xi)\cdot\xi}, \quad \xi \in \mathbb{S}^{N-1}. \tag{3.8}$$

Defining $U : \Omega_R \to \mathbb{R}$ by

$$U := p - \mu\Phi, \tag{3.9}$$

we get from (3.5) and (3.6)

$$\begin{aligned} \Delta U &= 0 && \text{in } \Omega(t),\\ U &= -\gamma\kappa_R - \mu\Phi && \text{on } \Gamma(t). \end{aligned}$$

Here $\kappa_R : \Gamma_R \to \mathbb{R}$ stands for the mean curvature of Γ_R. Define for each continuous $f : \mathbb{S}^{N-1} \to (-1, \infty)$ the function $K_f : \Omega_f \to \mathbb{R}$ as the harmonic function that meets

$-\kappa_f : \Gamma_f \to \mathbb{R}$ at the boundary Γ_f:

$$K_f(\xi) = -\kappa_f(\xi), \quad \xi \in \Gamma_f.$$

Lemma 3.1. *If $R(t)$ and $r(t)$ are related via* (2.6)*, then for $\xi \in \Omega_{r(t)}$*

$$K_{r(t)}(\xi) = \alpha(t) K_{R(t)}(\alpha(t)\xi), \tag{3.10}$$

and

$$\nabla K_{r(t)}(\xi) = \alpha(t)^2 \nabla K_{R(t)}(\alpha(t)\xi). \tag{3.11}$$

Proof. This follows directly from the scaling behaviour of the curvature

$$\kappa_{r(t)}(\xi) = \alpha(t)\kappa_{R(t)}(\alpha(t)\xi)$$

and the fact that $\xi \mapsto K_{R(t)}(\alpha(t)\xi)$ is harmonic in $\Omega_{r(t)}$. □

It follows that

$$U = \gamma K_{R(t)} + \mu L_{R(t)},$$

with $L_{R(t)}$ as defined in Section 2.2. From (2.13) and (3.11) it follows that

$$\nabla U(\tilde{z}(R,\xi)) = \frac{\gamma}{\alpha(t)^2} \nabla K_r(\tilde{z}(r,\xi)) + \frac{\mu}{\alpha(t)^{N-1}} \nabla L_r(\tilde{z}(r,\xi)). \tag{3.12}$$

In the same way as (2.14) we derive from (3.8) and (3.9)

$$\frac{\partial R}{\partial t}(\xi) = -\frac{\nabla U(\tilde{z}(R,\xi)) \cdot n(r,\xi)}{n(r,\xi)\cdot\xi} + \frac{\mu}{\sigma_N \alpha(t)^{N-1}(1+r(\xi))^{N-1}}. \tag{3.13}$$

Substituting (3.12) in (3.13) and using

$$\frac{\partial r}{\partial t} = \frac{1}{\alpha}\frac{\partial R}{\partial t} - \mu \frac{1+r}{\alpha^N \sigma_N} \tag{3.14}$$

we obtain the following evolution equation for r:

$$\begin{aligned} \frac{\partial r}{\partial t}(\xi) &= -\frac{\gamma}{\alpha(t)^3} \frac{\nabla K_r(\tilde{z}(r,\xi)) \cdot n(r,\xi)}{n(r,\xi)\cdot\xi} \\ &\quad + \frac{\mu}{\alpha(t)^N}\left(-\frac{\nabla L_r(\tilde{z}(r,\xi))\cdot n(r,\xi)}{n(r,\xi)\cdot\xi} + \frac{1}{\sigma_N(1+r(\xi))^{N-1}} - \frac{1+r(\xi)}{\sigma_N} \right). \end{aligned} \tag{3.15}$$

Definition 3.2. *For any r define $\kappa(r,\cdot)$ as the function that maps an element ξ of the unit sphere to the mean curvature of Γ_r at $\tilde{z}(r,\xi)$. We will often use the notation $\kappa(r)$ instead of $\kappa(r,\cdot)$.*

Let $\Xi : W \to \mathbb{S}^{N-1}$ be a smooth regular parametrisation of (a part of) $\mathbb{S}^{N-1}$ and define

$\mathcal{G}(r) : W \to \mathbb{R}^{(N-1)\times(N-1)}$ by

$$\mathcal{G}(r) = \left(\frac{\partial\left(\tilde{z}(r)\circ\Xi\right)}{\partial\omega}\right)^T \frac{\partial\left(\tilde{z}(r)\circ\Xi\right)}{\partial\omega}, \tag{3.16}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_{N-1})^T$ denotes an element in W. The Laplace-Beltrami operator on Γ_r is defined by

$$\Delta_r u = \left(\sum_{i,j}\frac{1}{\sqrt{g(r)}}\frac{\partial}{\partial\omega_i}\left(\sqrt{g(r)}g^{ij}(r)\frac{\partial(u\circ\tilde{z}(r)\circ\Xi)}{\partial\omega_j}\right)\right)\circ\Xi^{-1}\circ\tilde{z}(r)^{-1}. \tag{3.17}$$

Here $g^{ij}(r)$ are the coefficients of $\mathcal{G}(r)^{-1}$ and

$$g(r) = \det\mathcal{G}(r).$$

In fact, we need to introduce at least two parametrisations and a partition of unity as we did in the proof of Lemma 2.6. Since the operators that we define here are all local we allow ourselves to work with only one to simplify notation.

Lemma 3.3. *The Laplace-Beltrami operator Δ_r is symmetric on $\mathbb{L}^2(\Gamma_r)$.*

Proof. This is a straightforward calculation. □

Introduce for $u : \mathbb{S}^{N-1} \to \mathbb{R}$ smooth enough,

$$\begin{aligned}\mathcal{B}(r)u &:= \Delta_r\left(u\circ\tilde{z}(r)^{-1}\right)\circ\tilde{z}(r)\\ &= \left(\sum_{i,j}\frac{1}{\sqrt{g(r)}}\frac{\partial}{\partial\omega_i}\left(\sqrt{g(r)}g^{ij}(r)\frac{\partial(u\circ\Xi)}{\partial\omega_j}\right)\right)\circ\Xi^{-1}.\end{aligned} \tag{3.18}$$

Let n_r and κ_r be the normal and curvature on Γ_r such that $n(r) = n_r\circ\tilde{z}(r)$ and $\kappa(r) = \kappa_r\circ\tilde{z}(r)$. By [14, Sec. 2.5 Thm. 1] we have

$$\Delta_r\mathrm{id} = \kappa_r n_r, \tag{3.19}$$

where Δ_r acts on every component of $\tilde{z}(r)$ separately. This formula is equivalent to

$$\mathcal{B}(r)\tilde{z}(r) = \kappa(r)n(r). \tag{3.20}$$

Lemma 3.4. *There exists a neighbourhood $\mathcal{U}$ of zero in $\mathcal{C}^{4,\beta}(\mathbb{S}^{N-1})$ such that κ is analytic from $\mathcal{U}$ to $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$.*

Proof. By (3.20) we have

$$\kappa(r) = \left(\mathcal{B}(r)\tilde{z}(r)\right)\cdot n(r) \tag{3.21}$$

The operator n is analytic from a neighbourhood of zero in $\mathcal{C}^{4,\beta}(\mathbb{S}^{N-1})$ to $\left(\mathcal{C}^{1,\beta}(\mathbb{S}^{N-1})\right)^N$ (This follows from Lemma 2.6 and $\mathcal{C}^{4,\beta}(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$). Take two smooth regular parametrisations $\Xi_1 : W_2 \to \mathbb{S}^{N-1}$ and $\Xi_2 : W_2 \to \mathbb{S}^{N-1}$ and two operators $\mathcal{G}_1$ and $\mathcal{G}_2$

defined for these Ξ_1 and Ξ_2 by (3.16). From now on let k be either 1 or 2. Both $\mathcal{G}_k : \mathcal{C}^{4,\beta}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{3,\beta}(W_k)\right)^{(N-1)\times(N-1)}$ and $\det(\mathcal{G}_k(\cdot)) : \mathcal{C}^{4,\beta}(\mathbb{S}^{N-1}) \to \mathcal{C}^{3,\beta}(W_k)$ are analytic around zero, because $\mathcal{C}^{3,\beta}(W_k)$ is a Banach algebra. Since $\mathcal{G}_k(0) = \left(\frac{\partial \Xi_k}{\partial \omega}\right)^T \frac{\partial \Xi_k}{\partial \omega}$ is invertible on W_k (as a matrix) the mapping $\mathcal{G}_k(\cdot)^{-1} : h^{4,\beta}(\mathbb{S}^{N-1}) \to \left(\mathcal{C}^{3,\beta}(W_k)\right)^{(N-1)\times(N-1)}$ is analytic around zero. Since $\det(\mathcal{G}_k(0))$ is away from zero, $r \mapsto \sqrt{\det \mathcal{G}_k(r)}$ is a analytic near zero. For this see [60, Ch. 3 Lemma 7] that is based on the Implicit Function Theorem. We conclude that both $\mathcal{B}$ and n are analytic near zero. The lemma follows from this. □

Reintroducing the operators z, $\mathcal{A}$, $\mathcal{Q}$, $\mathcal{S}$, and φ from Section 2.2 on a neighbourhood $\mathcal{U}$ of zero in $\mathcal{C}^{2,\beta}(\mathbb{S}^{N-1})$, we obtain

$$\frac{\partial r}{\partial t} = \frac{\gamma}{\alpha(t)^3}\mathcal{F}_1(r) + \frac{\mu}{\alpha(t)^N}\mathcal{F}_2(r), \tag{3.22}$$

with

$$\mathcal{F}_1(r) = \frac{\operatorname{Tr}\left(\mathcal{Q}(r)\left[\mathcal{S}(r)^{-1}\begin{bmatrix} 0 \\ \kappa(r) \end{bmatrix}\right]\right)\cdot n(r)}{n(r)\cdot \mathrm{id}}$$

and

$$\mathcal{F}_2(r) = \frac{\operatorname{Tr}\left(\mathcal{Q}(r)\left[\mathcal{S}(r)^{-1}\begin{bmatrix} 0 \\ \varphi(r) \end{bmatrix}\right]\right)\cdot n(r)}{n(r)\cdot \mathrm{id}} + \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N},$$

conform (2.29).

Lemma 3.5. *The operators $\mathcal{F}_1 : h^{4,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ and $\mathcal{F}_2 : h^{4,\beta}(\mathbb{S}^{N-1}) \to h^{1,\beta}(\mathbb{S}^{N-1})$ are analytic in a neighbourhood $\mathcal{U}$ of zero in $h^{4,\beta}(\mathbb{S}^{N-1})$.*

Proof. Analyticity of $\mathcal{F}_2$ follows from Lemma 2.11 in combination with

$$h^{4,\beta}(\mathbb{S}^{N-1}) \hookrightarrow h^{2,\beta}(\mathbb{S}^{N-1}).$$

Analyticity of $\mathcal{F}_1$ can be proved in the same way as analyticity of $\mathcal{F}_2$, using Lemma 3.4. □

To find an expression for the linearisation of $\mathcal{F}_1$ around zero in terms of $\mathcal{N}$ we linearise κ.

Lemma 3.6. *The linearisation around zero of the curvature operator κ is given by*

$$\kappa'(0)[h] = \Delta_0 h + (N-1)h$$

where Δ_0 denotes the Laplace-Beltrami operator on the unit sphere, that satisfies

$$\Delta_0 h = -\mathcal{N}^2 h - (N-2)\mathcal{N}h. \tag{3.23}$$

Proof. See [60, Ch. 6] for the first part. For the second part see [56]. □

Lemma 3.7. *We have*

$$\begin{aligned}\mathcal{F}_1'(0)[h] &= \mathcal{N}(\kappa'(0)[h]) = \mathcal{N}\left(\Delta_0 h + (N-1)h\right) \\ &= \mathcal{N}\left(-\mathcal{N}^2 h - (N-2)\mathcal{N}h + (N-1)h\right)\end{aligned}$$

and

$$\mathcal{F}_2'(0)[h] = -\frac{1}{\sigma_N}\mathcal{N}h - \frac{N}{\sigma_N}h.$$

Proof. Note that we calculated $\mathcal{F}_2'(0)$ in Lemma 2.13. Introduce

$$K(r) = \mathcal{S}(r)^{-1}\begin{pmatrix} 0 \\ \kappa(r) \end{pmatrix}.$$

Following the proof of Lemma 2.12 and using the fact that $K(0)$ is constant we derive

$$\begin{pmatrix} \Delta K'(0)[h] \\ \mathrm{Tr}K'(0)[h] \end{pmatrix} = \begin{pmatrix} 0 \\ \kappa'(0)[h] \end{pmatrix}.$$

In other words,

$$K'(0)[h] = \mathcal{S}(0)^{-1}\begin{pmatrix} 0 \\ \kappa'(0)[h] \end{pmatrix}.$$

Now the linearisation of $\mathcal{F}_1$ can be found in the same way as the linearisation of $\mathcal{F}_2$ in Lemma 2.13, replacing φ by κ. We get

$$\mathcal{F}_1'(0)[h] = \frac{\mathrm{Tr}\,\mathcal{Q}(0)K'(0)[h]\cdot n(0)}{n(0)\cdot \mathrm{id}} = \frac{\partial K'(0)[h]}{\partial n} = \mathcal{N}\kappa'(0)[h].$$

The lemma follows from Lemma 3.6. □

From now on we consider the case $N = 3$. We get from (3.22)

$$\frac{\partial r}{\partial t} = \frac{1}{\alpha(t)^3}\left(\gamma\mathcal{F}_1(r) + \mu\mathcal{F}_2(r)\right). \tag{3.24}$$

Introducing the time variable $\tau = \tau(t)$ such that $\tau(0) = 0$ and

$$\frac{d\tau}{dt} = \frac{1}{\alpha(t)^3}, \tag{3.25}$$

implying that

$$\tau(t) = \frac{4\pi}{3\mu}\ln\left(\frac{3\mu t}{4\pi} + 1\right), \tag{3.26}$$

we obtain the autonomous equation

$$\frac{\partial \bar{r}}{\partial \tau} = \mathcal{F}(\bar{r}), \tag{3.27}$$

where $\bar{r}(\tau) = r(t)$ and

$$\mathcal{F}(\bar{r}) := \gamma \mathcal{F}_1(\bar{r}) + \mu \mathcal{F}_2(\bar{r}). \tag{3.28}$$

(This is in accordance with Chapter 2 if $\gamma = 0$ and $\mu = 1$.) For the suction problem, the vanishing time $t = T_\mu$ corresponds to $\tau = \infty$. From now on we write r instead of $\bar{r}$. Combining (3.28) and Lemma 3.7 one finds the following linearisation for $\mathcal{F}$ around zero:

$$\mathcal{F}'(0)[h] = \gamma \mathcal{N}\left(-\mathcal{N}^2 h - \mathcal{N}h + 2h\right) - \frac{\mu}{4\pi}(\mathcal{N}h + 3h). \tag{3.29}$$

3.3 The spectrum of the linearisation and stability for $N = 3$

In order to apply the principle of linearised stability to the evolution equation (3.27) for the three-dimensional problem we need to study the spectral properties of the operator $\mathcal{F}'(0) : h^{4,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ given by (3.29). Since the point spectrum of $\mathcal{N}$ is equal to $\mathbb{N}_0$, the eigenvalues of $\mathcal{F}'(0)$ are given by

$$g_k := \gamma k(-k^2 - k + 2) - \frac{\mu}{4\pi}(k+3), \quad k \in \mathbb{N}_0$$

and the corresponding eigenspaces are $\left(\mathfrak{S}_k^3\right)_{k=0}^{\infty}$. In the case $\mu > 0$, all g_k are negative. Let us consider the case $\mu < 0$ for which g_0 and g_1 are positive. If

$$\frac{|\mu|}{\gamma} = -\frac{\mu}{\gamma} < \frac{32\pi}{5}, \tag{3.30}$$

then all other eigenvalues are negative. This follows from the fact that for $k \geq 2$ the sequence $(g_k)_{k=0}^{\infty}$ decreases and $g_2 < 0$.

Corollary 3.8. *The point spectrum of $\mathcal{F}'(0) : h^{4,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ is*

$$\pi(\mathcal{F}'(0)) = \{g_0, g_1, g_2, \dots\}.$$

The eigenspace for eigenvalue g_k is $\mathfrak{S}_k^3$. If $\mu > 0$, then all eigenvalues of $\mathcal{F}'(0) : h^{4,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ are negative. If $\mu < 0$, then the eigenvalues g_0 and g_1 are positive. All other eigenvalues $g_2, g_3, \dots$ are negative if and only if (3.30) holds.

For two Banach spaces X and Y such that $X \hookrightarrow Y$ dense we define $\mathcal{H}(X,Y)$ as the collection of operators $A \in \mathcal{L}(X,Y)$ for which $-A$ is the infinitesimal generator of a strongly continuous analytic semigroup on Y with dense domain of definition X.

We will prove that $-\mathcal{F}'(0) \in \mathcal{H}\left(h^{4,\beta}(\mathbb{S}^2), h^{1,\beta}(\mathbb{S}^2)\right)$. For this we need the following two lemmas.

Lemma 3.9. *Let X and Y be Banach spaces such that $X \hookrightarrow Y$ and X dense in Y. Suppose that $F : X \to Y$ and $K : X \to Y$ are bounded linear operators such that $F \in \mathcal{H}(X,Y)$ and suppose that K is compact. Then $F + K \in \mathcal{H}(X,Y)$.*

Proof. See [8, Thm. 5.6]. □

Lemma 3.10. *The mapping* $\mathcal{N} : h^{k+1,\beta}(\mathbb{S}^{N-1}) \to h^{k,\beta}(\mathbb{S}^{N-1})$ *is continuous for all* $k \in \mathbb{N}$.

Proof. Define the Banach space X by

$$X = \{\psi \in \mathcal{C}^{k+1,\beta}(\overline{\mathbb{B}^N}) : \Delta\psi = 0\}.$$

Let it inherit the norm of $\mathcal{C}^{k+1,\beta}(\overline{\mathbb{B}^N})$. Because of the maximum principle, the mapping $\mathrm{Tr} : X \to \mathcal{C}^{k+1,\beta}(\mathbb{S}^{N-1})$ is injective. To prove that it is surjective as well, let g be an element of $\mathcal{C}^{k+1,\beta}(\mathbb{S}^{N-1})$. Let $f : [0,1] \to [0,1]$ be a smooth function such that $f(x) = 0$ for all $x \in [0, \frac{1}{3}]$ and $f(x) = 1$ for all $x \in [\frac{2}{3}, 1]$. Define $\tilde{g}(x) = f(|x|)g(\frac{x}{|x|})$. Then $\tilde{g} \in \mathcal{C}^{k+1,\beta}(\overline{\mathbb{B}^N})$. By [32, Cor. 4.14] there exists a $u \in \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^N})$, such that u is harmonic and u meets $\tilde{g}$ on the boundary. By [32, Thm. 6.19] we have $u \in \mathcal{C}^{k+1,\beta}(\overline{\mathbb{B}^N})$. Therefore, the bounded operator $\mathrm{Tr} : X \to \mathcal{C}^{k+1,\beta}(\mathbb{S}^{N-1})$ is surjective and by the Open Mapping Theorem it has a bounded inverse $\mathrm{Tr}^{-1} : \mathcal{C}^{k+1,\beta}(\mathbb{S}^{N-1}) \to X$. From

$$\mathcal{N} = \frac{\partial}{\partial n} \circ \mathrm{Tr}^{-1}$$

and the boundedness of $\frac{\partial}{\partial n} : X \to \mathcal{C}^{k,\beta}(\mathbb{S}^{N-1})$ we get the desired result. □

Lemma 3.11. *We have* $-\mathcal{F}'(0) \in \mathcal{H}\left(h^{4,\beta}(\mathbb{S}^2), h^{1,\beta}(\mathbb{S}^2)\right)$.

Proof. See Section 3.5. □

Lemma 3.12. *The spectrum of* $\mathcal{F}'(0) : h^{4,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ *consists entirely of eigenvalues,*

$$\mathrm{sp}(\mathcal{F}'(0)) = \{g_0, g_1, g_2, \dots\}.$$

The resolvent $\left(\lambda\mathcal{I} - \mathcal{F}'(0)\right)^{-1} : h^{1,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ *is compact for all* $\lambda \notin \mathrm{sp}(\mathcal{F}'(0))$.

Proof. Define for each $\lambda \in \mathbb{R}$ the polynomial

$$p_\lambda(X) = \gamma(X^3 + X^2 - 2X) + \frac{\mu}{4\pi}(X + 3) + \lambda.$$

Note that $p_\lambda(\mathcal{N}) = \lambda\mathcal{I} - \mathcal{F}'(0)$. Take λ^* large, such that p_{λ^*} has one negative zero ζ_1 and two zeros ζ_2 and $\zeta_3 = \overline{\zeta_2}$ in $\mathbb{C} \setminus \mathbb{R}$. Then

$$\left(\lambda^*\mathcal{I} - \mathcal{F}'(0)\right)^{-1} = -\frac{1}{\gamma}\left(\zeta_1\mathcal{I} - \mathcal{N}\right)^{-1}\left(\zeta_2\mathcal{I} - \mathcal{N}\right)^{-1}\left(\zeta_3\mathcal{I} - \mathcal{N}\right)^{-1}.$$

Since $\zeta_i \notin \mathbb{N}_0$ for $i \in \{1,2,3\}$ we conclude from Lemma 2.17 that $(\zeta_i\mathcal{I} - \mathcal{N})^{-1} : h^{1,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ is compact. As a consequence, $\left(\lambda^*\mathcal{I} - \mathcal{F}'(0)\right)^{-1} : h^{1,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ is compact. From the Hille-Yosida Theorem and Lemma 3.11 it follows that $\mathcal{F}'(0)$ is a closed operator on $h^{1,\beta}(\mathbb{S}^2)$ with domain $h^{4,\beta}(\mathbb{S}^2)$. Applying [49, Thm. III.6.29] we see that the spectrum of $\mathcal{F}'(0) : h^{2,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ consists entirely of eigenvalues and the resolvent $\left(\lambda\mathcal{I} - \mathcal{F}'(0)\right)^{-1} : h^{1,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ is compact for all $\lambda \notin \mathrm{sp}(\mathcal{F}'(0))$. □

Theorem 3.13. *Let $\mu > 0$ and $0 < \lambda_0 < \frac{3\mu}{4\pi}$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in h^{4,\beta}(\mathbb{S}^2)$ with $\|r_0\|_{4,\beta} < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = r_0,$$

has a solution $r \in \mathcal{C}\left([0,\infty), h^{4,\beta}(\mathbb{S}^2)\right) \cap \mathcal{C}^1\left([0,\infty), h^{1,\beta}(\mathbb{S}^2)\right)$ satisfying

$$\|r(\tau)\|_{4,\beta} \leq Me^{-\lambda_0 \tau}\|r_0\|_{4,\beta}. \tag{3.31}$$

Proof. Combining Lemma 3.11 with [63, Thm. 11.31] we see that $\mathcal{F}'(0)$ is sectorial. Note that $-\frac{3\mu}{4\pi}$ is the largest eigenvalue of $\mathcal{F}'(0)$. The theorem follows if we combine this, Lemmas 3.5, 3.12, and [53, Thm. 9.1.2]. □

From (3.26) and (3.31) we get for the non-autonomous problem (3.24),

$$\|r(t)\|_{4,\beta} \leq M\left(\frac{3\mu t}{4\pi} + 1\right)^{-\zeta} \|r_0\|_{4,\beta},$$

for $\zeta = \frac{4\pi}{3\mu}\lambda_0 \in (0,1)$.

The case $\mu < 0$ is more complicated. We restrict ourselves to evolutions in $\mathfrak{M}_1^3$ given by

$$\mathfrak{M}_1^3 = \left\{ r \in \mathcal{C}^0(\mathbb{S}^2) : \int_{\Omega_r} dx = \frac{4\pi}{3}, \int_{\Omega_r} x_j dx = 0,\ j \in \{1,2,3\} \right\}, \tag{3.32}$$

where x_j denotes the j-th component of x. Note that (3.32) follows from $\mathfrak{H}_1^3 = \langle x_1, x_2, x_3 \rangle$ and (2.40). We have $r \in \mathfrak{M}_1^3$ if and only if the corresponding domain Ω_r has the volume of the unit ball and its geometric centre is located at the origin.

In general, Lemma 2.19 does not hold if surface tension is included. However, for the Richardson moments of order zero and one we still have the following invariance property.

Lemma 3.14. *Let r solve* (3.27). *If $r_0 \in \mathfrak{M}_1^3$, then $r(t) \in \mathfrak{M}_1^3$ for all $t > 0$.*

Proof. It can be checked in the same way as in Lemma 2.19 that if $\Omega_{r(0)}$ has the same volume as the unit ball, then $\Omega_{r(t)}$ has the same volume as the unit ball for all t. By Green's second identity, (3.1)-(3.4), (3.19), and Lemma 3.3 we have for $j \in \{1,2,3\}$

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega_{R(t)}} x_j dx &= \int_{\Gamma_{R(t)}} x_j (v,n) dx = \int_{\Gamma_{R(t)}} -x_j \frac{\partial p}{\partial n} dx \\
&= \int_{\Omega_{R(t)}} p\Delta x_j dx - \int_{\Omega_{R(t)}} x_j \Delta p dx - \int_{\Gamma_{R(t)}} p \frac{\partial x_j}{\partial n} dx \\
&= -\int_{\Gamma_{R(t)}} p \frac{\partial x_j}{\partial n} dx = \gamma \int_{\Gamma_{R(t)}} \kappa n_j dx = \gamma \int_{\Gamma_{R(t)}} \Delta_{R(t)} x_j dx = 0.
\end{aligned}$$

The lemma follows from this. □

Theorem 3.15. *Let* $\mu < 0$ *satisfy* (3.30) and let $0 < \lambda_0 < \frac{5}{4\pi}\mu + 8\gamma$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in h^{4,\beta}(\mathbb{S}^2) \cap \mathfrak{M}_1^3$ and $\|r_0\|_{4,\beta} < \delta$, then the problem

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = r_0, \tag{3.33}$$

has a solution $r \in \mathcal{C}\left([0,\infty), h^{4,\beta}(\mathbb{S}^2)\right) \cap \mathcal{C}^1\left([0,\infty), h^{1,\beta}(\mathbb{S}^2)\right)$ satisfying

$$\|r(\tau)\|_{4,\beta} \leq Me^{-\lambda_0\tau}\|r_0\|_{4,\beta}. \tag{3.34}$$

Proof. Let for all $L \in \mathbb{N}_0$ the subspaces $h_1^{L,\beta}(\mathbb{S}^2)$ of $h^{L,\beta}(\mathbb{S}^2)$ be defined as in (2.41) and let $\hat{\mathcal{F}}$ be the restriction of $\mathcal{F}$ to $h_1^{4,\beta}(\mathbb{S}^2)$, such that $\hat{\mathcal{F}}'(0) = \mathcal{F}'(0)\mid_{h_1^{4,\beta}(\mathbb{S}^2)}$. Remember that $\mathcal{F}'(0)$ respects to the decomposition $h^{k,\beta}(\mathbb{S}^2) = h_1^{k,\beta}(\mathbb{S}^2) \oplus \mathfrak{S}_0^3 \oplus \mathfrak{S}_1^3$ such that

$$\hat{\mathcal{F}}'(0)\left[h_1^{4,\beta}(\mathbb{S}^2)\right] \subseteq h_1^{1,\beta}(\mathbb{S}^2).$$

Based on Lemma 2.21, $\hat{\mathcal{F}}'(0)$ is a closed sectorial operator on $h_1^{1,\beta}(\mathbb{S}^2)$ with dense domain of definition $h_1^{4,\beta}(\mathbb{S}^2)$ and

$$\mathrm{sp}(\hat{\mathcal{F}}'(0)) = \{g_2, g_3, g_4, \dots\}.$$

This spectrum consists of negative real numbers. The largest element is $-\frac{5}{4\pi}\mu - 8\gamma$.

Reintroduce $\mathcal{P}_1$, ϕ_1, and ψ_1 from Chapter 2 and consider the following equation for $\mathcal{P}_1 r$:

$$\frac{\partial(\mathcal{P}_1 r)}{\partial \tau} = (\mathcal{P}_1 \circ \mathcal{F} \circ \psi_1)(\mathcal{P}_1 r). \tag{3.35}$$

Since

$$(\mathcal{P}_1 \circ \mathcal{F} \circ \psi_1)'(0) = \hat{\mathcal{F}}'(0)$$

(as in (2.45)), we can apply [53, Thm. 9.1.2] to get a $\delta > 0$ such that if $\tilde{r}_0 = \mathcal{P}_1 r_0 \in h_1^{4,\beta}(\mathbb{S}^2)$ with $\|\tilde{r}_0\|_{4,\beta} < \delta$, then the problem

$$\frac{\partial \tilde{r}}{\partial \tau} = (\mathcal{P}_1 \circ \mathcal{F} \circ \psi_1)(\tilde{r}), \quad \tilde{r}(0) = \tilde{r}_0,$$

has a unique solution $\tilde{r} \in \mathcal{C}\left([0,\infty), h_1^{4,\beta}(\mathbb{S}^2)\right) \cap \mathcal{C}^1\left([0,\infty), h_1^{1,\beta}(\mathbb{S}^2)\right)$. Furthermore, there exists an $M' > 0$ independent of $\tilde{r}_0$ such that

$$\|\tilde{r}(\tau)\|_{4,\beta} \leq M'e^{-\lambda_0\tau}\|\tilde{r}_0\|_{4,\beta}.$$

Now construct

$$r := \psi_1(\tilde{r}).$$

We will show that this is a solution to the original problem. Because $\mathcal{P}_1$ is the local inverse of ψ_1 it is clear that $\mathcal{P}_1 r = \tilde{r}$. Therefore

$$\frac{\partial r}{\partial \tau} = \psi_1'(\tilde{r}) \left[\frac{\partial \tilde{r}}{\partial \tau}\right] = \psi_1'(\tilde{r}) \left[\mathcal{P}_1 \mathcal{F}(r)\right] = \psi_1'(\mathcal{P}_1 r)[\mathcal{P}_1 \mathcal{F}(r)]. \tag{3.36}$$

Because $\psi_1(\mathcal{P}_1 r) = r$ for all $r \in \mathfrak{M}_1^3$ with $\|r\|_{4,\beta}$ small, we have

$$\psi_1'(\mathcal{P}_1 r)[\mathcal{P}_1 h] = h, \tag{3.37}$$

for all $h \in T_r\mathfrak{M}_1^3$. Lemma 3.14 yields $\mathcal{F}(r) \in T_r\mathfrak{M}_1^3$. Therefore from (3.36) and (3.37) it follows that

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r).$$

Analyticity of ψ_1 (see Chapter 2) and $\psi_1(0) = 0$ imply existence of a $\delta > 0$ and an $M > 0$ such that for $r_0 \in \mathfrak{M}_1^3$ with $\|r_0\|_{4,\beta} < \delta$ we have

$$\begin{aligned} \|r(\tau)\|_{4,\beta} &= \|(\psi_1 \circ \mathcal{P}_1)(r(\tau))\|_{4,\beta} \leq M\|\mathcal{P}_1 r(\tau)\|_{4,\beta} \\ &\leq Me^{-\lambda_0 \tau}\|\mathcal{P}_1 r_0\|_{4,\beta} \leq Me^{-\lambda_0 \tau}\|r_0\|_{4,\beta}. \end{aligned}$$

This proves the theorem. □

If we combine (3.26) and (3.34) we get for the non-autonomous problem (3.24),

$$\|r(t)\|_{4,\beta} \leq M\left(\frac{3\mu t}{4\pi} + 1\right)^{\zeta} \|r_0\|_{4,\beta},$$

for $\zeta = -\frac{4\pi}{3\mu}\lambda_0$ and $t \in [0, T_\mu)$.

3.4 Stability with respect to perturbations of the suction point

If the suction point is not at the geometric centre of the initial domain then the result of Theorem 3.15 does not hold. As shown in [70] and [71], the solution either becomes unbounded or it breaks down before all liquid is sucked out. In this section we show that for an initial domain that satisfies the conditions of Theorem 3.15 (a nearly spherical shape with geometric centre located at origin) a slight modification of the place of suction leads to a small change of the maximal time of existence.

Let X be a metric space and let $T^+ : X \to (0, \infty) \cup \{\infty\}$ be some mapping. Define

$$V := \left\{(x, \tau) \in X \times [0, \infty) : \tau < T^+(x)\right\}.$$

A mapping $f : V \to X$ is called a semiflow on X if

1. V is open in $X \times (0, \infty)$;
2. $f \in \mathcal{C}(V, X)$;
3. $f(\cdot, 0) = \mathrm{id}$;

4. If $x \in X$, $\tau \in [0, T^+(x))$, and $\tau^* \in [0, T^+(f(x,\tau)))$, then $\tau + \tau^* < T^+(x)$ and $f(x, \tau + \tau^*) = f(f(x,\tau), \tau^*)$.

Define $\mathcal{E} : \mathcal{U} \to \mathcal{L}(\mathcal{C}^{2,\beta}(\mathbb{S}^2), \mathcal{C}^{1,\beta}(\mathbb{S}^2))$ and $l : \mathcal{U} \to \mathcal{C}^{2,\beta}(\mathbb{S}^2)$ by

$$\mathcal{E}(r)\psi = \frac{\operatorname{Tr}\left(\mathcal{Q}(r)\left[\mathcal{S}(r)^{-1}\left[\begin{array}{c}0\\ \psi\end{array}\right]\right]\right)\cdot n(r)}{n(r)\cdot \mathrm{id}} \tag{3.38}$$

and

$$l(r) = \frac{1}{4\pi}\left(\frac{1}{(1+r)^2} - 1 - r\right), \tag{3.39}$$

where $\mathcal{U}$ is a suitable neighbourhood of zero in $\mathcal{C}^{2,\beta}(\mathbb{S}^2)$. Now we rewrite (3.27) in this way:

$$\frac{\partial r}{\partial \tau} = \gamma \mathcal{E}(r)\kappa(r) + \mu \mathcal{E}(r)\varphi(r) + \mu l(r).$$

Lemma 3.16. *The mapping $\mathcal{E}$ is analytic around zero from $\mathcal{U}$ to $\mathcal{L}(\mathcal{C}^{2,\beta}(\mathbb{S}^2), \mathcal{C}^{1,\beta}(\mathbb{S}^2))$.*

Proof. First we prove analyticity of $r \mapsto \mathcal{S}(r)^{-1}$ via the Implicit Function Theorem. Define $f : \mathcal{C}^{2,\beta}(\mathbb{S}^2) \times \mathcal{L}\left(\mathcal{C}^{0,\beta}(\overline{\mathbb{B}^3}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^2), \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^3})\right) \to \mathcal{L}\left(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^3})\right)$ by

$$f(r, X) := X \circ \mathcal{S}(r) - \mathcal{I}.$$

We have $f(0, \mathcal{S}(0)^{-1}) = 0$ and f is analytic near $(0, \mathcal{S}(0)^{-1})$ because of Lemma 2.7. From differentiation we obtain

$$f'(0, \mathcal{S}(0)^{-1})[0, h] = h \circ \mathcal{S}(0).$$

Since $f'(0, \mathcal{S}(0)^{-1})[0, \cdot]$ is bijective from $\mathcal{L}\left(\mathcal{C}^{0,\beta}(\overline{\mathbb{B}^3}) \times \mathcal{C}^{2,\beta}(\mathbb{S}^2), \mathcal{C}^{2,\beta}(\overline{\mathbb{B}^3})\right)$ to $\mathcal{L}\left(\mathcal{C}^{2,\beta}(\overline{\mathbb{B}^3})\right)$, with inverse $h \mapsto h \circ \mathcal{S}(0)^{-1}$, there exists an analytic mapping $r \mapsto X(r)$ satisfying

$$f(r, X(r)) = 0.$$

Clearly $X = \mathcal{S}(\cdot)^{-1}$. Analyticity of n and $\mathcal{Q}$ are proved in Lemmas 2.6 and 2.9. $\square$

Lemma 3.17. *Let $\mu < 0$ and $\alpha_1 < \alpha < \beta < 1$. There exists a neighbourhood $\mathcal{U}$ of 0 in $h^{3,\beta}(\mathbb{S}^2)$ and a function $T^+ : \mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2) \to (0,\infty) \cup \{\infty\}$ such that if $r_0 \in \mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = r_0,$$

has a unique maximal solution

$$r \in \mathcal{C}([0, T^+(r_0)), h^{4,\alpha_1}(\mathbb{S}^2)) \cap \mathcal{C}^{0,\eta}([0, T^+(r_0)), h^{1,\alpha}(\mathbb{S}^2)),$$

where $\eta = 1 - \frac{\alpha - \alpha_1}{3}$. The mapping $(r_0, \tau) \mapsto r(\tau)$ is a semiflow on $\mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)$ (in the

$h^{4,\alpha_1}(\mathbb{S}^2)$*-topology) defined on the set*

$$V := \left\{(r_0, \tau) \in (\mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)) \times [0, \infty) : \tau < T^+(r_0)\right\}.$$

Proof. According to [21, Lemma 3.1], there exists a neighbourhood $\hat{\mathcal{U}}$ of 0 in $h^{2,\beta}(\mathbb{S}^2)$,

$$\kappa_1 \in \mathcal{C}^\omega(\hat{\mathcal{U}}, \mathcal{L}(h^{3,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2))),$$

and

$$\kappa_2 \in \mathcal{C}^\omega(\hat{\mathcal{U}}, h^{1,\beta}(\mathbb{S}^2))$$

such that

$$\kappa(r) = \kappa_1(r)r + \kappa_2(r). \tag{3.40}$$

From (3.19) it follows that κ_1 is a quasilinear differential operator of second order and κ_2 is of first order. There exists a small neighbourhood of zero $\mathcal{U} \subset \hat{\mathcal{U}}$ in $h^{3,\beta}(\mathbb{S}^2)$ such that

$$\kappa_1 \in \mathcal{C}^\omega(\mathcal{U}, \mathcal{L}(h^{4,\alpha}(\mathbb{S}^2), h^{2,\alpha}(\mathbb{S}^2)))$$

and

$$\kappa_2 \in \mathcal{C}^\omega(\mathcal{U}, h^{2,\beta}(\mathbb{S}^2)).$$

Combining this with Lemma 3.16 we can choose $\mathcal{U}$ such that

$$r \mapsto \mathcal{E}(r)\kappa_1(r) \in \mathcal{C}^\omega(\mathcal{U}, \mathcal{L}(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2))). \tag{3.41}$$

Calculating the Fréchet derivative of κ at zero using (3.40), we get from Lemma 3.6

$$\kappa_1(0) + \kappa_2'(0) = -\mathcal{N}^2 - \mathcal{N} + 2\mathcal{I},$$

from which it follows that

$$\mathcal{E}(0)\kappa_1(0) = \mathcal{N}\kappa_1(0) = -\mathcal{N}^3 + \mathcal{K},$$

where $\mathcal{K} : h^{4,\alpha}(\mathbb{S}^2) \to h^{1,\alpha}(\mathbb{S}^2)$ is compact. Lemma 3.9 and Lemma 3.11 yield

$$-\mathcal{E}(0)\kappa_1(0) \in \mathcal{H}(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2)),$$

because $-\frac{1}{\gamma}\mathcal{F}'(0)$ is in highest order equal to $\mathcal{N}^3$. By [3, Thm. I.1.3.1] $\mathcal{H}(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2))$ is open in $\mathcal{L}(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2))$. Combining this with (3.41) it follows that we can choose $\mathcal{U}$ such that

$$r \mapsto -\gamma\mathcal{E}(r)\kappa_1(r) \in \mathcal{C}^\omega(\mathcal{U}, \mathcal{H}(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2))). \tag{3.42}$$

We have proven analyticity near zero of φ and $\mathcal{E}$ in Lemmas 2.5 and 3.16. Analyticity of l can be obtained from the same arguments as in the proof of Lemma 2.5. It follows that we can choose $\mathcal{U}$ such that

$$r \mapsto \gamma\mathcal{E}(r)\kappa_2(r) + \mu\mathcal{E}(r)\varphi(r) + \mu l(r) \in \mathcal{C}^\omega(\mathcal{U}, h^{1,\beta}(\mathbb{S}^2)). \tag{3.43}$$

The little Hölder spaces satisfy

$$\left(h^{4,\alpha}(\mathbb{S}^2), h^{1,\alpha}(\mathbb{S}^2)\right)^0_{1-\frac{\alpha-\alpha_1}{3},\infty} = h^{4,\alpha_1}(\mathbb{S}^2).$$

For more information about continuous interpolation of Hölder spaces, see [53, Ch. 1]. The result follows from (3.42), (3.43), and [2, Thm. 12.1]. □

Theorem 3.18. *Let $\mu < 0$, $T \in (0,\infty)$ and $\eta \in (0,1)$. Define*

$$\alpha_1 = \alpha + 3(\eta - 1). \tag{3.44}$$

There exists a $\delta > 0$ such that if $r_0 \in h^{4,\alpha_1}(\mathbb{S}^2)$ and $\|r_0\|_{4,\alpha_1} < \delta$, then the problem

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = r_0,$$

has a solution $r \in C\left([0,T), h^{4,\alpha_1}(\mathbb{S}^2)\right) \cap C^{0,\eta}\left([0,T), h^{1,\alpha}(\mathbb{S}^2)\right)$.

Proof. Because of the semiflow property, that is proved in Lemma 3.17, the set

$$\left\{(r_0,\tau) \in (\mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)) \times [0,\infty) : \tau < T^+(r_0)\right\}$$

is open in $h^{4,\alpha_1}(\mathbb{S}^2) \times (0,\infty)$. Since

$$T^+(0) = \infty,$$

the point $(0,T)$ is an interior point of V. Therefore there exists a neighbourhood $\tilde{\mathcal{U}}$ of zero in $h^{4,\alpha_1}(\mathbb{S}^2)$ such that for all $r_0 \in \tilde{\mathcal{U}}$ we have

$$T^+(r_0) \geq T.$$

□

In the following theorems $\mathcal{U}$ denotes a sufficiently small neighbourhood of zero in $h^{3,\beta}(\mathbb{S}^2)$.

Theorem 3.19. *Let $T > 0$ and suppose that $r^* \in \mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)$ satisfies $T^+(r^*) = \infty$. There exists a $\delta > 0$ such that if $r_0 \in h^{4,\alpha_1}(\mathbb{S}^2)$ with $\|r_0 - r^*\|_{4,\alpha_1} < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = r_0,$$

has a solution $r \in C\left([0,T), h^{4,\alpha_1}(\mathbb{S}^2)\right) \cap C^{0,\eta}\left([0,T), h^{1,\alpha}(\mathbb{S}^2)\right)$, where η satisfies (3.44).

Proof. Since (r^*,T) is an interior point of V we can argue as in the proof of Theorem 3.18 to prove this theorem, see Figure 3.1 □

Now we will prove the main result of this section. Instead of changing the suction point in a fixed domain we translate the domain in the opposite direction and leave the

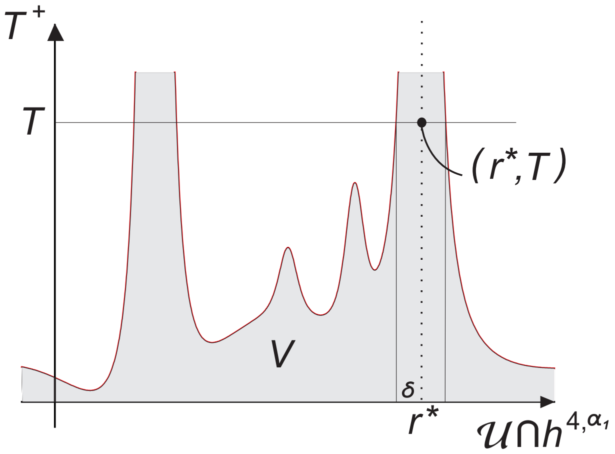


Figure 3.1: *Since* (r^*, T) *is an interior point of* V *there exists an open ball around* r^* *of initial functions* r_0 *with* $T^+(r_0) > T$.

suction point at the same position.

Theorem 3.20. *Let* $T > 0$ *and suppose that* $r^* \in \mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)$ *satisfies* $T^+(r^*) = \infty$. *There exists a* $\delta > 0$ *such that for all* $z \in \mathbb{R}^3$ *with* $|z| < \delta$ *there is a* $\tilde{r} \in \mathcal{U} \cap h^{4,\alpha_1}(\mathbb{S}^2)$ *such that* $\Omega_{\tilde{r}} = \Omega_{r^*} + \{z\}$ *(i.e. a translation of* Ω_{r^*}*). Furthermore, the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r), \quad r(0) = \tilde{r},$$

has a solution $r \in \mathcal{C}\left([0, T), h^{4,\alpha_1}(\mathbb{S}^2)\right) \cap \mathcal{C}^{0,\eta}\left([0, T), h^{1,\alpha}(\mathbb{S}^2)\right)$, *where* η *satisfies* (3.44).

Proof. Define for small $\delta > 0$ the ball $B = \{z \in \mathbb{R}^N : |z| < \delta\}$. It is known that Ω_r is convex for r small in $h^{3,\beta}(\mathbb{S}^2)$. Consequently, for small $\|r\|_{3,\beta}$ and small z the domain $\Omega_r + \{z\}$ is convex. As a result, there is a $\tilde{r} \in \mathcal{C}^0(\mathbb{S}^{N-1})$ for which $\Omega_{\tilde{r}} = \Omega_r + \{z\}$. Define on B the mapping S by $S : z \mapsto S_z$ where

$$S_z : r \mapsto \tilde{r},$$

such that $\tilde{r}$ has the property $\Omega_{\tilde{r}} = \Omega_r + \{z\}$. By Theorem 3.19, it is sufficient to show that $z \mapsto S_z r^*$ is continuous near zero from B to $h^{4,\alpha_1}(\mathbb{S}^2)$. In [10, Lemma 4.3] continuity from B to Besov spaces is proved. The same arguments can be used for little Hölder spaces. □

3.5 Proof of Lemma 3.11

The structure of this proof is as follows. We relate $\mathcal{N}^3$ to a Fourier multiplier operator $\hat{\mathcal{N}}_0^3$ on $\mathbb{R}^2$. The operator $-\hat{\mathcal{N}}_0^3$ generates a strongly continuous analytic semigroup. Using techniques from [19], [21], and [22] together with additional perturbation arguments

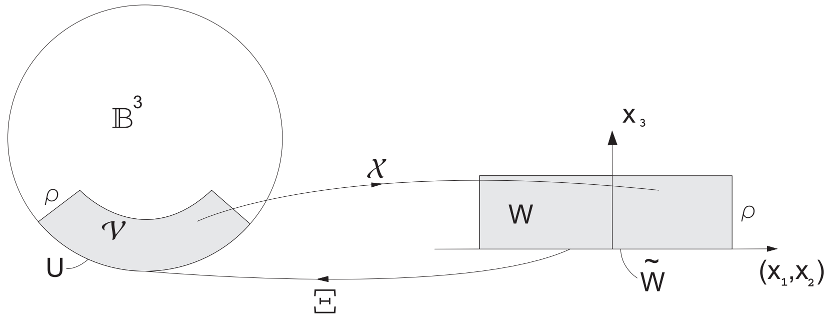


Figure 3.2: *Chart of* $\mathcal{V}$

we see that $\mathcal{N}^3 \in \mathcal{H}\left(h^{4,\beta}(\mathbb{S}^2), h^{1,\beta}(\mathbb{S}^2)\right)$. Since $-\mathcal{F}'(0)$ is in highest order equal to $\mathcal{N}^3$ the lemma follows from Lemma 3.9.

1. Let $(\mathfrak{U}_i, \Xi_i^{-1})_{i=1}^M$ be an atlas of $\mathbb{S}^2$ consisting of smooth regular charts, with $\tilde{W}_i := \Xi_i^{-1}(\mathfrak{U}_i)$ such that $0 \in \tilde{W}_i$. Define

$$W_i := \tilde{W}_i \times (0, \rho),$$

for some $\rho < 1$,

$$\mathcal{V}_i := \{x \in \overline{\mathbb{B}^3} : 1 - \rho < |x| < 1, \frac{x}{|x|} \in \mathfrak{U}_i\}$$

and introduce $\mathcal{X}_i : \mathcal{V}_i \to W_i$ by

$$\mathcal{X}_i(x) := \left(\Xi_i^{-1}\left(\frac{x}{|x|}\right), 1 - |x|\right)^T,$$

see Figure 3.2. Let $\hat{\mathcal{A}}_i : h^{2,\beta}(\overline{W}_i) \to h^{0,\beta}(\overline{W}_i)$ and $\hat{\mathcal{Q}}_i : h^{2,\beta}(\overline{W}_i) \to h^{1,\beta}(\overline{\tilde{W}}_i)$ be defined by

$$\hat{\mathcal{A}}_i p := \Delta\left(p \circ \mathcal{X}_i\right) \circ \mathcal{X}_i^{-1},$$
$$\hat{\mathcal{Q}}_i p := \frac{\partial}{\partial n}\left(p \circ \mathcal{X}_i\right) \circ \mathcal{X}_i^{-1} = -\frac{\partial p}{\partial x_3},$$

where n is the normal on $\mathbb{S}^2$. From now on we restrict our attention to one chart and omit the index i in $\tilde{W}_i$, W_i, $\mathcal{X}_i$, $\mathcal{V}_i$, $\mathfrak{U}_i$, $\hat{\mathcal{A}}_i$ and $\hat{\mathcal{Q}}_i$. There exist functions $\hat{a}_{jk}$, $\hat{a}_j \in \mathcal{C}^\infty(\overline{W})$ such that

$$\hat{\mathcal{A}} = \sum_{j,k=1}^{3} \hat{a}_{jk} \frac{\partial^2}{\partial x_j \partial x_k} - \sum_{j=1}^{3} \hat{a}_j \frac{\partial}{\partial x_j}.$$

The little Hölder spaces on open sets are defined as the closure of the Schwarz space in the topology of the bounded uniformly Hölder continuous functions. Here we will only work with functions with compact support. Define the half-space $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : x_3 \geq 0)\}$ and introduce $\hat{\mathcal{A}}_0 : h^{2,\beta}(\mathbb{R}^3_+) \to h^{0,\beta}(\mathbb{R}^3_+)$ by

$$\hat{\mathcal{A}}_0 := -1 + \sum_{j,k=1}^{3} \hat{a}_{jk}(0)\frac{\partial^2}{\partial x_j \partial x_k}.$$

Note that $\hat{a}_{33}(0) = 1$ and $\hat{a}_{13}(0) = \hat{a}_{23}(0) = \hat{a}_{31}(0) = \hat{a}_{32}(0) = 0$. In this proof Tr denotes the trace operator for functions on the halfspace $\mathbb{R}^3_+$. Define $\hat{\mathcal{R}}_0 : h^{1,\beta}(\mathbb{R}^2 \times \{0\}) \to h^{1,\beta}(\mathbb{R}^3_+)$ as the solution operator, $\hat{\mathcal{R}}_0 g = u$, of the problem

$$\begin{cases} -\hat{\mathcal{A}}_0 u &= 0 \qquad \text{in } \mathbb{R}^3_+ \\ \quad \mathrm{Tr} u &= g \qquad \text{in } \mathbb{R}^2 \times \{0\}. \end{cases}$$

Define the operator $\hat{\mathcal{N}}_0$ by

$$\hat{\mathcal{N}}_0 := \hat{\mathcal{Q}} \circ \hat{\mathcal{R}}_0.$$

From [22, eqn. (4.10)] we get

$$\mathbb{F}\hat{\mathcal{N}}_0\mathbb{F}^{-1} = \mathcal{M}_{f(\cdot,1)}, \tag{3.45}$$

where $\mathbb{F}$ denotes Fourier transform in $\mathbb{R}^2$, $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x,y) = \sqrt{y^2 + \sum_{j,k=1}^{2} \hat{a}_{jk}(0)x_j x_k},$$

and $\mathcal{M}_{f(\cdot,1)}$ stands for multiplication with the function $f(\cdot,1)$. Because $(f(x,y))^3$ is positively homogeneous of degree 3 and its derivatives are bounded on $|x|^2 + y^2 = 1$ it follows that $\hat{\mathcal{N}}_0^3 \in \mathcal{H}\left(h^{4,\beta}(\mathbb{R}^2 \times \{0\}), h^{1,\beta}(\mathbb{R}^2 \times \{0\})\right)$, see [4]. For positively homogeneous functions of degree 1, see [20, Thm. A.2]. In [21, Cor. 5.2], the same argument is used for a different operator of order three.

2. The next step is relating $\hat{\mathcal{N}}_0^3$ to $\mathcal{N}^3$ if the chart domains are small. For convenience we use the notation

$$\mathcal{X}_* g := g \circ \mathcal{X}^{-1},$$

for functions g on $\mathcal{V}$. The following statement holds true. For any $\varepsilon > 0$ and $\zeta \in (0,\beta)$ there is a $\rho > 0$, an atlas $(\mathfrak{U}_i, \Xi_i^{-1})_{i=1}^M$, a partition of unity $(\psi_i)_{i=1}^M$ subordinate to $(\mathcal{V}_i)_{i=1}^M$, and a $C_\varepsilon > 0$ such that for $l \in \{1,2,3\}$ and $\hat{\mathcal{N}}_0$ constructed from the atlas as described above we have for all $p \in h^{l+1,\beta}(\mathbb{S}^2)$ and for all charts

$$\|\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{l,\beta}(\mathbb{R}^2)} \leq \varepsilon\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{l+1,\beta}(\mathbb{R}^2)} + C_\varepsilon\|p\|_{l+1,\zeta}. \tag{3.46}$$

To see this, we argue as in the proof of [19, Thm. B.4] and choose ρ sufficiently small, depending on ε. Here and in the sequel we identify $\mathcal{C}^{l,\beta}(\mathbb{R}^2)$ and $\mathcal{C}^{l,\beta}(\mathbb{R}^2 \times \{0\})$. Functions $\mathcal{X}_*(\psi p)$ can be extended to the entire $\mathbb{R}^2$ without losing regularity because of the smoothness of the partition of unity. We want to show that for any $\delta > 0$ and fixed $\zeta \in (0,\beta)$ the atlas can be chosen such that there is a $C_\delta > 0$ such that for all $p \in h^{4,\beta}(\mathbb{S}^2)$ and all charts

$$\|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} \leq \delta\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + C_\delta\|p\|_{4,\zeta}. \tag{3.47}$$

In the sequel, we will often use the fact that for each $k \in \mathbb{N}$

$$\mathcal{N} \in \mathcal{L}(h^{k+1,\beta}(\mathbb{S}^2), h^{k,\beta}(\mathbb{S}^2))$$

and

$$\hat{\mathcal{N}}_0 \in \mathcal{L}(h^{k+1,\beta}(\mathbb{R}^2), h^{k,\beta}(\mathbb{R}^2)).$$

For this, see Lemma 3.10 and [19, App. B]. First we show that there exists a suitable atlas and a constant C' independent of p such that

$$\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} \leq C'\left(\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + \|p\|_{4,\zeta}\right) \tag{3.48}$$

and

$$\|\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{2,\beta}(\mathbb{R}^2)} \leq C'\left(\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + \|p\|_{4,\zeta}\right). \tag{3.49}$$

Let us start with the first estimate. Apply (3.46) with $\varepsilon = 1$ and $l = 3$. Let $C > 0$ denote a varying constant. We get

$$\begin{aligned}
&\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} \\
\leq\ &\|\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} + \|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} \\
\leq\ &\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + C\|p\|_{4,\zeta} + C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)}.
\end{aligned}$$

Estimate (3.48) follows. Replace p by $\mathcal{N}p$ in (3.46) and take $\varepsilon = 1$ and $l = 2$. From (3.48) it follows that

$$\begin{aligned}
&\|\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{2,\beta}(\mathbb{R}^2)} \\
\leq\ &\|\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{2,\beta}(\mathbb{R}^2)} + \|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{2,\beta}(\mathbb{R}^2)} \\
\leq\ &\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} + C\|\mathcal{N}p\|_{3,\zeta} + C\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} \\
\leq\ &C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + C\|p\|_{4,\zeta}.
\end{aligned}$$

Now we prove estimate (3.47). Let $\delta > 0$. Let $\eta > 0$ be a small number to be fixed later. By the triangle inequality,

$$\begin{aligned}
\|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} \leq\ &\|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} + \\
&+\|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} + \\
&+\|\hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)}.
\end{aligned}$$

We will estimate the three terms on the right separately, denoting by C_η constants that depend on η while C denotes constants that are independent of η. Applying

(3.46) to $\mathcal{N}^2 p$ with $l = 1$ and using (3.49) we get

$$\begin{aligned} & \|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} \\ \leq \;& \eta\|\mathcal{X}_*(\psi\mathcal{N}^2 p)\|_{\mathcal{C}^{2,\beta}(\mathbb{R}^2)} + C_\eta\|\mathcal{N}^2 p\|_{2,\zeta} \\ \leq \;& \eta C\left(\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + \|p\|_{4,\zeta}\right) + C_\eta\|p\|_{4,\zeta} \\ \leq \;& \eta C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + C_\eta\|p\|_{4,\zeta}. \end{aligned} \tag{3.50}$$

Applying (3.46) with $l = 2$, replacing p by $\mathcal{N}p$, gives us

$$\begin{aligned} & \|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} \\ \leq \;& C\|\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{2,\beta}(\mathbb{R}^2)} \\ \leq \;& \eta C\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} + C_\eta\|\mathcal{N}p\|_{3,\zeta} \\ \leq \;& \eta C\|\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} + C_\eta\|p\|_{4,\zeta}. \end{aligned} \tag{3.51}$$

From (3.48) it follows that

$$\|\hat{\mathcal{N}}_0\mathcal{X}_*(\psi\mathcal{N}^2 p) - \hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} \leq \eta C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + C_\eta\|p\|_{4,\zeta}.$$

Analogously,

$$\begin{aligned} & \|\hat{\mathcal{N}}_0^2\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} \\ \leq \;& C\|\mathcal{X}_*(\psi\mathcal{N}p) - \hat{\mathcal{N}}_0\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{3,\beta}(\mathbb{R}^2)} \\ \leq \;& \eta C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + C_\eta\|p\|_{4,\zeta}. \end{aligned} \tag{3.52}$$

From (3.50)-(3.52) one obtains

$$\|\mathcal{X}_*(\psi\mathcal{N}^3 p) - \hat{\mathcal{N}}_0^3\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{1,\beta}(\mathbb{R}^2)} \leq \eta C\|\mathcal{X}_*(\psi p)\|_{\mathcal{C}^{4,\beta}(\mathbb{R}^2)} + C_\eta\|p\|_{4,\zeta}.$$

We take $\eta = \frac{\delta}{C}$ and get the desired result (3.47).

3. The next step is proving that for all $\lambda > 0$,

$$\lambda\mathcal{I} + \mathcal{N}^3 : h^{4,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$$

is an isomorphism. Note that

$$\lambda\mathcal{I} + \mathcal{N}^3 = (\sqrt[3]{\lambda}\mathcal{I} + \mathcal{N})(\sqrt[3]{\lambda}e^{\frac{2\pi}{3}i}\mathcal{I} + \mathcal{N})(\sqrt[3]{\lambda}e^{-\frac{2\pi}{3}i}\mathcal{I} + \mathcal{N}). \tag{3.53}$$

Parallel to the proof of Lemma 2.17 we can derive surjectivity of $\mu\mathcal{I} + \mathcal{N} : h^{k+1,\beta}(\mathbb{S}^2) \to h^{k,\beta}(\mathbb{S}^2)$, for $\mu \in \mathbb{C} \setminus -\mathbb{N}_0$ and for all $k \in \mathbb{N}$. Surjectivity of $\lambda\mathcal{I} + \mathcal{N}^3 : h^{4,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$ follows if we apply this result for $k = 1, 2, 3$ and $\mu = \sqrt[3]{\lambda}, \sqrt[3]{\lambda}e^{\frac{2\pi}{3}i}, \sqrt[3]{\lambda}e^{-\frac{2\pi}{3}i}$.

4. There exist $C > 0$ and $\lambda_* > 0$ such that for all $r \in h^{4,\beta}(\mathbb{S}^2)$ and $\lambda \in \mathbb{C}$ with

Re $\lambda \geq \lambda_*$ we have

$$|\lambda| \|r\|_{1,\beta} + \|r\|_{4,\beta} \leq C \|(\lambda \mathcal{I} + \mathcal{N}^3) r\|_{1,\beta}. \tag{3.54}$$

This can be obtained from (3.47) via exactly the same procedure that is used in the proof of [19, Thm. B.4]. Estimate (3.54) and the fact that

$$\lambda_* \mathcal{I} + \mathcal{N}^3 : h^{4,\beta}(\mathbb{S}^2) \to h^{1,\beta}(\mathbb{S}^2)$$

is an isomorphism imply that $\mathcal{N}^3 \in \mathcal{H}\left(h^{4,\beta}(\mathbb{S}^2), h^{1,\beta}(\mathbb{S}^2)\right)$, see [3, Remark I.1.2.1.(a)].

Chapter 4

Bifurcation solutions

4.1 Introduction

In the previous chapter we showed that (3.30) is a necessary condition for $r = 0$ to be an asymptotically stable stationary solution to (3.24) for the 3D suction problem. If $\mu/\gamma = -32\pi/5$, then zero is an eigenvalue for the linearised evolution problem and if $\mu/\gamma < -32\pi/5$, then $r = 0$ is linearly unstable. This motivates us to find stationary solutions to the nonlinear problem that appear as transcritical bifurcation solutions. More generally, for any $k \in \mathbb{N} \setminus \{1\}$ there is a negative value for μ for which the spherical harmonics of degree k are the kernel of the linearised evolution problem. We investigate existence of branches of stationary solutions $r = \rho_k(\sigma)$ to the nonlinear problem that are approximated by some elements in $\mathfrak{S}_k$ for $\mu = m_k(\sigma)$.

For such families of solutions, investigation of their stability is a natural and important issue. Therefore we also present a stability result.

Our existence result is given in Section 4.2, where we apply a well-known result on "bifurcation from a simple eigenvalue". To ensure that the eigenvalue under consideration is simple, we have to restrict our basic space, thereby introducing a symmetry breaking. The new basic space consists of functions corresponding to domains that are symmetric with respect to rotations around the vertical axis. Similar approaches are used for the study of viscous drops in electric fields and for tumour models in [24], [26], and [28].

In Section 4.3 we obtain a stability result for one class of stationary solutions that we found. This concept of stability is weaker than the stability in Chapter 2 and 3 in the following sense. Only perturbations that respect the axial symmetry are allowed. Furthermore, the perturbed domain must satisfy the conditions on the geometry that were crucial in the theory for the suction problem in Section 3.3.

In Section 4.4 a technical result is proved that characterises the local bifurcation picture. For μ slightly smaller than μ_2 we find disc-like shapes and for μ slightly larger than μ_2 we have cigar-like shapes, see Figure 4.2. Our main stability result is given in Theorem 4.5. It states that the branch of bifurcation solutions consisting of disc-like shapes is nonlinearly stable (near the bifurcation point). This relies on analysing the linearisation of the evolution operator around the bifurcation solution and applying the results of Crandall and Rabinowitz [9]. This method is also used in [24] and [26].

We would like to point out that the results of this chapter crucially depend on the space dimension 3. As we have seen, the evolution operator in other dimensions is essentially time dependent. Hence, its kernel is time dependent for $N \neq 3$.

Any solution r to (3.24) parameterises a domain $\Omega_{R(\cdot,t)}$ that solves the Hele-Shaw problem (3.1)-(3.4), where r and R are related via (2.6). This implies that stationary solutions r to (3.24) parameterise domains that have the property

$$\Omega_{R(\cdot,t)} = \alpha(t)\Omega_{R(\cdot,0)},$$

with $\alpha : [0, \frac{4\pi}{3|\mu|}) \to (0,1]$ given by

$$\alpha(t) := \sqrt[3]{1 - \frac{3|\mu|t}{4\pi}}.$$

We call domain evolutions with this property self-similarly vanishing. An example is the trivial domain evolution $\Omega(t) = \alpha(t)\mathbb{B}^3$ that is represented by $r = 0$.

Let $\mathbb{H}^s(\mathbb{S}^2)$ be the Sobolev space of order s of functions on the unit sphere $\mathbb{S}^2$. Let $(\cdot,\cdot)_s$ denote its inner product defined by

$$(r, \tilde{r})_s := \sum_{k,j} (k^2+1)^s r_{kj}\tilde{r}_{kj},$$

with $r_{kj} = (r, s_{kj})_0$ and $\tilde{r}_{kj} = (\tilde{r}, s_{kj})_0$, where $(\cdot,\cdot)_0$ is the $\mathbb{L}^2(\mathbb{S}^2)$-inner product and s_{kj} are the spherical harmonics that are chosen to be $\mathbb{L}^2(\mathbb{S}^2)$-orthonormal (as in Chapter 1). In the following, $\mathcal{U}$ is a suitable small neighbourhood of zero in $\mathbb{H}^s(\mathbb{S}^2)$ for $s > 5$. It is possible to define $\mathcal{F}_1 : \mathcal{U} \to \mathbb{H}^{s-3}(\mathbb{S}^2)$ and $\mathcal{F}_2 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^2)$ in the same way as in Chapter 3, replacing Hölder spaces by Sobolev spaces. For more detail, see Chapter 5. It is clear that stationary solutions to (3.24) satisfy

$$\gamma\mathcal{F}_1(r) + \mu\mathcal{F}_2(r) = 0. \tag{4.1}$$

The linearisations of $\mathcal{F}_1$ and $\mathcal{F}_2$ around $r \equiv 0$ are given in Lemma 3.7 where $N = 3$ and $\mathcal{N} : \mathbb{H}^\sigma(\mathbb{S}^2) \to \mathbb{H}^{\sigma-1}(\mathbb{S}^2)$ for $\sigma > 1$ is defined in the same way as in (2.33). Because $\mathcal{N}h = kh$ for $h \in \mathfrak{S}_k$ it is clear that if $\mu = \mu_k := -\gamma\zeta_k$ with

$$\zeta_k := 4\pi\frac{k^3+k^2-2k}{k+3}, \quad k = 2,3,4\ldots,$$

then $\ker \mathcal{F}'(0) = \mathfrak{S}_k$.

4.2 Non-trivial stationary solutions via bifurcation

Define for $\sigma \geq 0$ the subspace $\mathbb{H}^\sigma_\times(\mathbb{S}^2)$ of $\mathbb{H}^\sigma(\mathbb{S}^2)$ consisting of those functions r that parameterise domains Ω_r that are invariant with respect to rotations around the z-axis. It is well known that

$$\mathfrak{S}_l \cap \mathbb{H}^\sigma_\times(\mathbb{S}^2) = \langle Y_l \rangle,$$

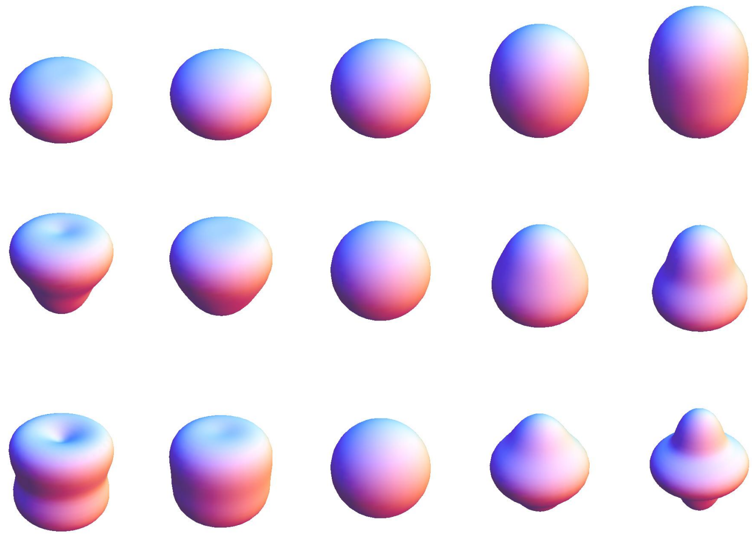


Figure 4.1: *The domains Ω_{cY_k}, for $k = 2, 3, 4$ and several values of $c \in \mathbb{R}$. For $|c|$ small but nonzero, these domains approximate non-trivial stationary solutions to the rescaled problem, i.e. shapes of self-similarly vanishing domains.*

where Y_l are the zonal harmonics given by

$$Y_l(\theta) = P_l(\cos\theta),$$

where θ denotes the polar coordinate on $\mathbb{S}^2$ and P_l are the Legendre polynomials.

The mappings $\mathcal{F}_1$ and $\mathcal{F}_2$ respect rotational symmetries. Therefore, on a suitable neighbourhood $\mathcal{U}_\times$ of zero in $\mathbb{H}^s_\times(\mathbb{S}^2)$, we have a smooth mapping $\mathcal{F}_{\times,\mu} : \mathcal{U}_\times \to \mathbb{H}^{s-3}_\times(\mathbb{S}^2)$ given by

$$\mathcal{F}_{\times,\mu} := (\gamma\mathcal{F}_1 + \mu\mathcal{F}_2)|_{\mathcal{U}_\times}.$$

We shall now state the existence result. We keep s and γ fixed and denote by X_k the orthoplement of $\langle Y_k \rangle$ in $\mathbb{H}^s_\times(\mathbb{S}^2)$.

Theorem 4.1. *(Existence of bifurcation solutions) Let $k \geq 2$ be an integer. There exists a $\delta_k > 0$ and a curve $(\rho_k, m_k) : (-\delta_k, \delta_k) \to \mathbb{H}^s_\times(\mathbb{S}^2) \times \mathbb{R}$ with $\rho_k(0) = 0$ and $m_k(0) = \mu_k$ such that for all $\sigma \in (-\delta_k, \delta_k)$*

$$\mathcal{F}_{\times,m_k(\sigma)}(\rho_k(\sigma)) = 0. \tag{4.2}$$

Furthermore, there exist C^1-functions $\nu_k : (-\delta_k, \delta_k) \to X_k$ with $\nu_k(0) = 0$ such that for $\sigma \in (-\delta_k, \delta_k)$ we have

$$\rho_k(\sigma) = \sigma Y_k + \sigma\nu_k(\sigma). \tag{4.3}$$

Moreover, there is a neighbourhood of $(0, \mu_k)$ in $\mathbb{H}^s_\times(\mathbb{S}^2) \times \mathbb{R}$ on which any zero of $(r, \mu) \mapsto$

$\mathcal{F}_{\times,\mu}(r)$ is either of the form $\big(\sigma Y_k + \sigma v_k(\sigma), m_k(\sigma)\big)$ or of the form $(0, \mu)$.

This theorem ensures the existence of non-trivial stationary solutions to (4.1). In particular, Y_k gives the direction in which these solutions bifurcate from the trivial solution $r = 0$, see Figure 4.1. The proof of Theorem 4.1 uses the following lemma.

Lemma 4.2. *Let $k \geq 2$ be an integer. We have $\ker \mathcal{F}'_{\times,\mu_k}(0) = \langle Y_k \rangle$ and $R(\mathcal{F}'_{\times,\mu_k}(0))$ is the orthoplement of $\langle Y_k \rangle$ in $\mathbb{H}^{s-3}_{\times}(\mathbb{S}^2)$.*

Proof. The zonal harmonics form a complete orthogonal system in $\mathbb{H}^s_\times(\mathbb{S}^2)$. Consequently, we get from Lemma 3.7 and the fact that $\mathcal{N} Y_l = l Y_l$

$$\mathcal{F}'_{\times,\mu_k}(0)[h] = \sum_{l=0}^{\infty} g_{l,\mu_k}(h, Y_l)_0 Y_l, \qquad h \in \mathbb{H}^s_\times(\mathbb{S}^2),$$

where $g_{l,\mu}$ are the eigenvalues of $\mathcal{F}'_{\times,\mu}(0)$:

$$g_{l,\mu} := -\gamma(l^3 + l^2 - 2l) - \frac{\mu}{4\pi}(l+3). \tag{4.4}$$

As $g_{l,\mu_k} = 0$ if and only if $l = k$ and $g_{l,\mu_k} \sim -\gamma l^3$ for large l, both statements follow immediately. □

The proof of Theorem 4.1 follows if we combine Lemma 4.2, [67, Thm. 13.5], and the fact that

$$\partial_\mu(\mathcal{F}'_{\times,\mu}(0))|_{\mu=\mu_k}[Y_k] = \mathcal{F}'_2(0)[Y_k] = -\frac{k+3}{4\pi} Y_k \notin R(\mathcal{F}'_{\times,\mu_k}(0)).$$

4.3 Stability of bifurcation solutions

The mappings ψ_1 and $\mathcal{P}_1$ defined in Chapter 2 form a local bijection between the manifold $\mathfrak{M}^3_1$ and its tangent space at $r = 0$. In order to study stability with respect to perturbations that preserve axial symmetry we need to introduce a suitable local bijection between the submanifold of $\mathfrak{M}^3_1$ consisting of axially symmetric functions and its tangent space in $\mathbb{H}^s_\times(\mathbb{S}^2)$. After that, we obtain an evolution equation for the projection of r on this tangent space.

Introduce

$$\mathfrak{M}^3_{\times,1} := \mathfrak{M}^3_1 \cap \mathbb{H}^s_\times(\mathbb{S}^2) = \left\{ r \in \mathbb{H}^s_\times(\mathbb{S}^2) : \int_{\Omega_r} dx = \frac{4\pi}{3}, \int_{\Omega_r} x_3 dx = 0 \right\}, \tag{4.5}$$

where x_3 denotes the third component of the spacial variable x. This manifold consists of the functions r in $\mathbb{H}^s_\times(\mathbb{S}^2)$ for which the corresponding domains Ω_r have the volume of the unit ball and a geometric centre that is located at the origin. It is invariant under the nonlinear evolution, i.e. if r solves (3.24) and $r(0) \in \mathfrak{M}^3_{\times,1}$ then $r(t) \in \mathfrak{M}^3_{\times,1}$ for all t. This is a consequence of Lemma 3.14 and preservation of axial symmetry. Since the solutions $\rho_k(\sigma)$ exists up to complete extinction it is clear from the theory of Tian [71] that $\rho_k(\sigma) \in \mathfrak{M}^3_{\times,1}$. Introduce

$$\mathbb{H}^s_{\times,1}(\mathbb{S}^2) := \{ r \in \mathbb{H}^s_\times(\mathbb{S}^2) : (r, Y_0)_0 = (r, Y_1)_0 = 0 \}$$

and let $X_{k,1}$ be the orthoplement of $\langle Y_k \rangle$ in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$. On a sufficiently small neighbourhood $\mathcal{U}_\times$ of zero in $\mathbb{H}^s_\times(\mathbb{S}^2)$ we introduce the operator $\phi_{\times,1} : \mathcal{U}_\times \to \mathbb{R}^2 \times \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ by

$$\phi_{\times,1}(r) := (f_{\times,1}(r), \mathcal{P}_1 r)^T,$$

where $\mathcal{P}_1$ is the $\mathbb{L}^2(\mathbb{S}^2)$-orthogonal projection on $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ and

$$f_{\times,1}(r) := \left(\int_{\Omega_r} dx - \frac{4\pi}{3}, \int_{\Omega_r} x_3 dx \right)^T. \tag{4.6}$$

The mapping $\phi_{\times,1}$ is a local bijection between a neighbourhood of zero in $\mathbb{H}^s_\times(\mathbb{S}^2)$ and a neighbourhood of zero in $\mathbb{R}^2 \times \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$. This follows in the same way as analytic bijectivity of ϕ_1 making use of the Implicit Function Theorem (see Chapter 2),

$$f'_{\times,1}(0)[h] = \Big((h, x \mapsto 1)_0, (h, x \mapsto x_3)_0 \Big)^T,$$

$\langle Y_0 \rangle = \langle 1 \rangle$, and $\langle Y_1 \rangle = \langle x_3 \rangle$. Note that in contrast to the definition of f_1 in (2.42), integrals over x_1 and x_2 are left out in the definition of $f_{\times,1}$.

On a neighbourhood $\mathcal{U}_{\times,1}$ of zero in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ we define the analytic bijection $\psi_{\times,1}$: $\mathcal{U}_{\times,1} \to \mathfrak{M}^3_{\times,1}$ by

$$\psi_{\times,1}(r) = \phi^{-1}_{\times,1}(0, r). \tag{4.7}$$

Define $\tilde{\mathcal{F}}_{\times,\mu} : \mathcal{U}_{\times,1} \to \mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ by

$$\tilde{\mathcal{F}}_{\times,\mu} := \mathcal{P}_1 \circ \mathcal{F}_{\times,\mu} \circ \psi_{\times,1}.$$

From the argument that we used to prove Corallary 2.20, we see that $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ is the tangent space of $\mathfrak{M}^3_{\times,1}$ at zero and $\psi'_{\times,1}(0) : \mathbb{H}^s_{\times,1}(\mathbb{S}^2) \to \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ is the identity. Consequently for $h \in \mathbb{H}^s_{\times,1}(\mathbb{S}^2)$

$$\tilde{\mathcal{F}}'_{\times,\mu}(0)[h] = \mathcal{P}_1 \mathcal{F}'_{\times,\mu}(0)[\psi'_{\times,1}(0)[h]] = \mathcal{F}'_{\times,\mu}(0)[h]$$

and for the spectra of the operators $\tilde{\mathcal{F}}'_{\times,\mu}(0)$ and $\mathcal{F}'_{\times,\mu}(0)$ we have

$$\mathrm{sp}(\tilde{\mathcal{F}}'_{\times,\mu}(0)) = \mathrm{sp}(\mathcal{F}'_{\times,\mu}(0)) \setminus \{g_{0,\mu}, g_{1,\mu}\} = \{g_{2,\mu}, g_{3,\mu}, \dots\} \tag{4.8}$$

and $\tilde{\mathcal{F}}'_{\times,\mu}(0)[Y_k] = g_{k,\mu} Y_k$, where $g_{k,\mu}$ is defined by (4.4). Fix now $k \geq 2$. Introducing $\tilde{\rho}_k(\sigma) := \mathcal{P}_1 \rho_k(\sigma)$ one gets from (4.2)

$$\tilde{\mathcal{F}}_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma)) = 0.$$

From Lemma 4.2 it is known that

$$\ker \tilde{\mathcal{F}}'_{\times,\mu_k}(0) = \big(R(\tilde{\mathcal{F}}'_{\times,\mu_k}(0)) \big)^\perp = \langle Y_k \rangle .$$

Here the orthoplement is taken in $\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$. From Theorem 4.1 we know that in a neighbourhood of $(0,\mu_k)$ in $\mathbb{H}^s_\times(\mathbb{S}^2)\times\mathbb{R}$ any zero of $(r,\mu)\mapsto\mathcal{F}_{\times,\mu}(r)$ is either of the form $(0,\mu)$ or $(\rho_k(\sigma),m_k(\sigma))$. As a consequence, in a neighbourhood of $(0,\mu_k)$ in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)\times\mathbb{R}$, any zero of $(r,\mu)\mapsto\tilde{\mathcal{F}}_{\times,\mu}(r)$ is either of the form $(0,\mu)$ or $(\tilde{\rho}_k(\sigma),m_k(\sigma))$. Combining these results one applies [9, Cor. 1.13] to find an interval I_k containing μ_k, a $\delta_k>0$, and continuously differentiable functions $\xi_k:I_k\to\mathbb{R}$, $\eta_k:(-\delta_k,\delta_k)\to\mathbb{R}$, $u_k:I_k\to\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$, and $w_k:(-\delta_k,\delta_k)\to\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ such that for $\tilde{\sigma}\in I_k$ and $\sigma\in(-\delta_k,\delta_k)$

$$\tilde{\mathcal{F}}'_{\times,\tilde{\sigma}}(0)[u_k(\tilde{\sigma})]=\xi_k(\tilde{\sigma})u_k(\tilde{\sigma})$$

and

$$\tilde{\mathcal{F}}'_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma))[w_k(\sigma)]=\eta_k(\sigma)w_k(\sigma).$$

Moreover, $\xi_k(\mu_k)=\eta_k(0)=0$, $u_k(\mu_k)=w_k(0)=Y_k$, $u_k(\tilde{\sigma})-Y_k\in X_{k,1}$ for $\tilde{\sigma}\in I_k$ and $w_k(\sigma)-Y_k\in X_{k,1}$ for $\sigma\in(-\delta_k,\delta_k)$. However, from (4.4), (4.8), and continuity of ξ_k it follows that u_k is constant:

$$\forall\tilde{\sigma}\in I_k:\xi_k(\tilde{\sigma})=g_{k,\tilde{\sigma}},\ u_k(\tilde{\sigma})=Y_k.$$

Consequently for all $k\geq 2$ and $\tilde{\sigma}\in I_k$

$$\xi_k'(\tilde{\sigma})=-\frac{k+3}{4\pi}<0. \tag{4.9}$$

Lemma 4.3. *We have $m_k'(0)>0$ for k even and $m_k'(0)=0$ for k odd.*

Proof. This is proved in Section 4.4. □

Now we restrict our attention to $k\in 2\mathbb{N}$. Because of Lemma 4.3, $m_k(\sigma)$ is nonzero for small nonzero $|\sigma|$. From [9, Thm. 1.16] and (4.9) we get for small $|\sigma|$

$$\operatorname{sgn}(\eta_k(\sigma))=-\operatorname{sgn}(\sigma m_k'(\sigma)\xi_k'(\mu_k))=\operatorname{sgn}(\sigma m_k'(\sigma)).$$

Lemma 4.3 yields

$$\operatorname{sgn}(\eta_k(\sigma))=\operatorname{sgn}\sigma. \tag{4.10}$$

We conclude that for small positive σ, $\eta_k(\sigma)$ is a positive eigenvalue of $\tilde{\mathcal{F}}'_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma))$ and for small negative σ, $\eta_k(\sigma)$ is a negative eigenvalue of $\tilde{\mathcal{F}}'_{\times,m_k(\sigma)}(\tilde{\rho}_k(\sigma))$. Now we consider the curve of bifurcation solutions for $k=2$. Observe that zero is the largest element of $\operatorname{sp}(\tilde{\mathcal{F}}'_{\times,m_2(0)}(0))$. We prove that for small negative σ, $\operatorname{sp}(\tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma)))$ is situated on the left of the imaginary axis to show stability of the disc-like shapes, see Figure 4.2.

Lemma 4.4. *There exists a positive δ such that for $\sigma\in(-\delta,0)$ the operator $\tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma))$ generates an analytic contraction semigroup on $\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ with domain of definition $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$.*

Proof. To shorten notation we introduce $F_\sigma:=\tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma))$, $Y:=\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$, and $X:=\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ with norms $\|\cdot\|_Y$ and $\|\cdot\|_X$.

It is not hard to prove that $-F_0\in\mathcal{H}(Y,X)$, because showing that (3.54) holds for Sobolev norms is straightforward. There are $M>0$, $\omega\in\mathbb{R}$, and $\vartheta\in(\frac{\pi}{2},\pi)$ such that

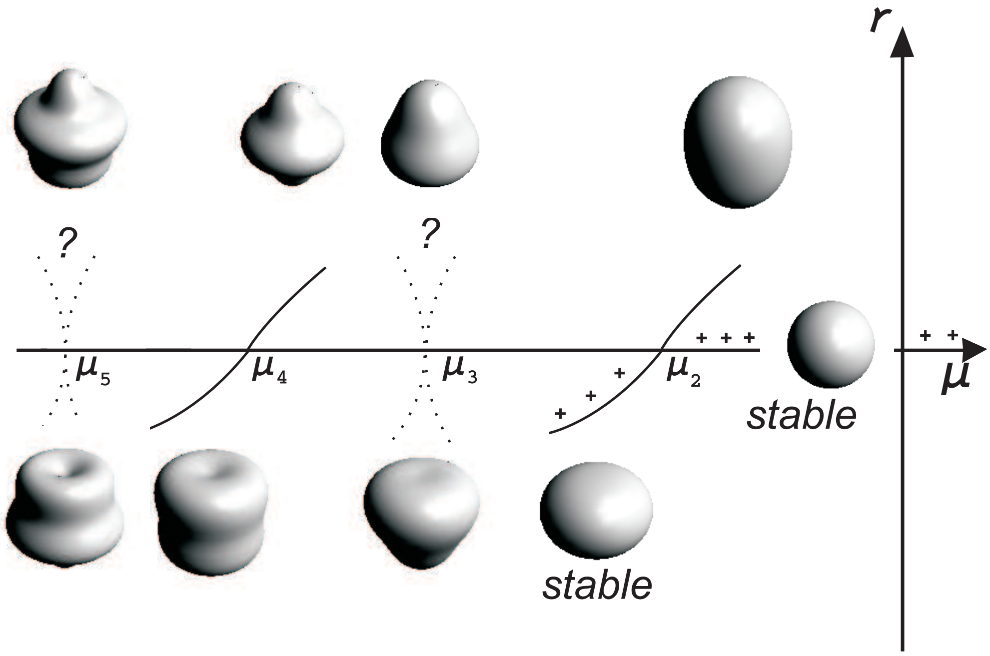


Figure 4.2: *A sketch of the curves of nontrivial stationary domains bifurcating from the unit ball. Domains of the type $\Omega_{\pm\epsilon Y_k}$ approximate these bifurcation solutions. The solutions approximated by $\Omega_{-\epsilon Y_2}$ for positive ϵ are stable. Due to the additional symmetry of the problem with respect to the reflection $(x, y, z) \mapsto (x, y, -z)$, if (ρ, μ) is a solution to the bifurcation problem then so is $(\hat{\rho}, \mu)$ where $\hat{\rho}$ is defined by $\Omega_{\hat{\rho}} = -\Omega_\rho$. For even k, uniqueness and Lemma 4.3 imply $\rho = \hat{\rho}$ on the bifurcation branches. Thus, Ω_ρ is symmetric with respect to the x_1x_2-plane. If k is odd, then the branch consists of pairs of different solutions ρ and $\hat{\rho}$ for the same value of μ, and a pitchfork bifurcation occurs. However, we do not know whether $\mu < \mu_k$ or $\mu > \mu_k$ on these branches.*

$S := \{z \in \mathbb{C} : |\arg(z - \omega)| < \vartheta\} \subseteq \rho(F_0)$ and for $\lambda \in S$ we have

$$\|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|}.$$

Here $\mathcal{R}(\lambda, F_0) : X \to X$ is the resolvent of F_0 and $(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}(X)})$ is the space of bounded operators on X. Endow Y with the graph norm

$$\|x\|_{\mathcal{D}(F_0)} := \|x\|_X + \|F_0 x\|_X,$$

which is equivalent to the norm $\|\cdot\|_Y$.

Fix $\kappa \in (0, 1)$. Note at first that due to continuity of ρ_2 and m_2 (see Theorem 4.1) the mapping $\sigma \mapsto F_\sigma$ is continuous with values in $\mathcal{L}(Y, X)$. Therefore, $F_\sigma \in \mathcal{H}(Y, X)$ and

$$\|F_\sigma - F_0\|_{\mathcal{L}(\mathcal{D}(F_0), X)} < \frac{\kappa}{1 + 2M}, \tag{4.11}$$

for $|\sigma|$ sufficiently small.

For $\lambda \in S$

$$\begin{aligned} \|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X, \mathcal{D}(F_0))} &\leq \|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} + \|F_0 \mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} \\ &\leq \|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} + \|(\lambda \mathcal{I} - F_0)\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} + |\lambda| \|\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} \\ &\leq 1 + \frac{(1 + |\lambda|)M}{|\lambda - \omega|}. \end{aligned}$$

Combining this and (4.11) one sees that there exists a $\Lambda > 0$ such that if $\lambda \in S_\Lambda := \{z \in S : |z| > \Lambda\}$, then

$$\|(F_\sigma - F_0)\mathcal{R}(\lambda, F_0)\|_{\mathcal{L}(X)} < \kappa. \tag{4.12}$$

For $\lambda \in S_\Lambda \subseteq \rho(F_0)$ and $f \in X$ we consider the problem

$$\lambda u - F_\sigma u = f, \tag{4.13}$$

that is to be solved for $u \in \mathcal{D}(F_0) = Y$. Introducing

$$v := \lambda u - F_0 u,$$

we get the following problem that is equivalent to (4.13):

$$v = (F_\sigma - F_0)\mathcal{R}(\lambda, F_0)v + f. \tag{4.14}$$

From (4.12) it follows that the mapping

$$\mathcal{Z} : v \mapsto (F_\sigma - F_0)\mathcal{R}(\lambda, F_0)v + f$$

defines a contraction on X. The Banach Contraction Theorem yields that there exists a unique $v' \in X$ such that $\mathcal{Z}(v') = v'$. As a consequence,

$$u' := \mathcal{R}(\lambda, F_0)v' \tag{4.15}$$

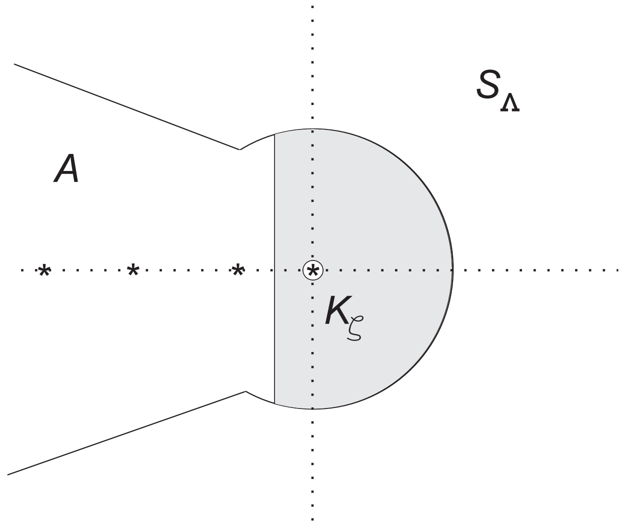


Figure 4.3: *The stars denote the elements in the spectrum of F_0. There is one zero eigenvalue. All other elements in the spectrum are in the region $A := \mathbb{C} \setminus (S_\Lambda \cup K)$. For negative values of σ close enough to zero, one element $\eta_2(\sigma)$ of the spectrum of F_σ is still located in the little ball $\{z \in \mathbb{C} : |z| < \zeta\}$. All other elements are in A. Since $\eta_2(\sigma)$ is negative all elements in $\mathrm{sp}(F_\sigma)$ have negative real part.*

solves (4.13) uniquely. Because of (4.12) and (4.14) we have $\|v'\|_X \leq \kappa \|v'\|_X + \|f\|_X$ and get

$$\|u'\|_X \leq \frac{M}{|\lambda - \omega|} \|v'\|_X \leq \frac{1}{1-\kappa} \frac{M}{|\lambda - \omega|} \|f\|_X.$$

Consequently, $S_\Lambda \subseteq \rho(F_\sigma)$ and $-F_\sigma \in \mathcal{H}(Y, X)$. It follows from the continuity of $\sigma \mapsto F_\sigma$ around 0 and the perturbation result given in [9, Lemma 1.3], that for sufficiently small $\delta, \zeta > 0$ we have

$$\mathrm{sp}(F_\sigma) \cap \{z \in \mathbb{C} : |z| < \zeta\} = \{\eta_2(\sigma)\}, \tag{4.16}$$

if $|\sigma| < \delta$. Define the compact sets

$$K := \left\{ z \in \mathbb{C} \setminus S_\Lambda : \operatorname{Re} z \geq \frac{1}{2} g_{3,\mu_2} \right\}, \quad K_\zeta := \{z \in K : |z| \geq \zeta\},$$

see Figure 4.3. We will show that for fixed ζ

$$\mathrm{sp}(F_\sigma) \cap K_\zeta = \emptyset \tag{4.17}$$

for all σ sufficiently close to zero. For $\sigma = 0$ this is obvious. From continuity of $(\lambda, \sigma) \mapsto \lambda\mathcal{I} - F_\sigma$ and the fact that isomorphisms form an open subset in $\mathcal{L}(Y, X)$ we get the following result. For all $\hat{\lambda} \in K_\zeta$ there exist $\chi(\hat{\lambda}) > 0$ and $\sigma(\hat{\lambda}) > 0$ such that for all $\lambda \in \mathbb{C}$ with $|\lambda - \hat{\lambda}| < \chi(\hat{\lambda})$ and for all σ with $|\sigma| < \sigma(\hat{\lambda})$ the operator $\lambda\mathcal{I} - F_\sigma$ is an isomorphism between Y and X. Since K_ζ is compact, the open covering of K_ζ consisting of the balls

$$B(\hat{\lambda}) := \{z \in \mathbb{C} : |z - \hat{\lambda}| < \chi(\hat{\lambda})\}$$

has a finite subcovering $\big(B(\hat{\lambda}_i)\big)_{i\in\mathbb{I}}$. Define $\sigma^* := \min_{i\in\mathbb{I}} \sigma(\hat{\lambda}_i)$. Let $|\sigma| < \sigma^*$ and $\lambda \in K_\zeta$. There exists an $i^* \in \mathbb{I}$ such that $\lambda \in B(\hat{\lambda}_{i^*})$. It is obvious that $|\sigma| < \sigma(\hat{\lambda}_{i^*})$. As a consequence $\lambda\mathcal{I} - F_\sigma$ is an isomorphism between Y and X. In other words $\lambda \notin \mathrm{sp}(F_\sigma)$. This proves (4.17).

From (4.16), (4.17), and (4.10) we conclude that for σ negative and near zero, $\mathrm{sp}(F_\sigma)$ is in the open left half-space of the complex plane at a positive distance from the imaginary axis. This completes the proof. □

Now we state the main result of this chapter. Again we regard r as a function of the time variable τ that we introduced in (3.26).

Theorem 4.5. *(Stability of bifurcation solutions) Suppose that $s > 5$. There exists a $\delta_1 > 0$, such that if $\sigma \in (-\delta_1, 0)$ and $\lambda_0 \in (0, -\operatorname{Re}\eta_2(\sigma))$, then there exists a $\delta_2 > 0$ and an $M > 0$ such that the following statement holds. If $r_0 \in \mathfrak{M}^3_{\times,1}$ and $\|r_0 - \rho_2(\sigma)\|_s < \delta_2$, then there exists a solution $r \in \mathcal{C}([0,\infty), \mathbb{H}^s(\mathbb{S}^2)) \cap \mathcal{C}^1([0,\infty), \mathbb{H}^{s-3}(\mathbb{S}^2))$ to*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}_{\times,m_2(\sigma)}(r), \quad r(0) = r_0.$$

Furthermore, for $\tau \in [0,\infty)$ we have $r(\tau) \in \mathfrak{M}^3_{\times,1}$ and

$$\|r(\tau) - \rho_2(\sigma)\|_s \le M e^{-\lambda_0 \tau} \|r_0 - \rho_2(\sigma)\|_s. \tag{4.18}$$

Proof. First we show solvability of the problem

$$\frac{\partial \tilde{r}}{\partial \tau} = \tilde{\mathcal{F}}_{\times,m_2(\sigma)}(\tilde{r}), \quad \tilde{r}(0) = \mathcal{P}_1 r_0, \tag{4.19}$$

for $\mathcal{P}_1 r_0$ near $\tilde{\rho}_2(\sigma)$. Since $\mathcal{I} + \tilde{\mathcal{F}}'_{\times,m_2(0)}(0)$ is an isomorphism between $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ and $\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$, $\mathcal{I} + \tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma))$ is an isomorphism between $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$ and $\mathbb{H}^{s-3}_{\times,1}(\mathbb{S}^2)$ as well for σ near zero. This implies that the graph norm of $\tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma))$ is equivalent to $\|\cdot\|_s$. Because $\tilde{\mathcal{F}}_{\times,m_2(\sigma)}$ is analytic near zero and $\tilde{\mathcal{F}}_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma)) = 0$, it follows from Lemma 4.4 and [53, Thm. 9.1.2] that there exists a solution to (4.19) that satisfies

$$\|\tilde{r}(\tau) - \tilde{\rho}_2(\sigma)\|_s \le \tilde{M} e^{-\lambda_0\tau} \|\mathcal{P}_1 r_0 - \tilde{\rho}_2(\sigma)\|_s,$$

for $\tilde{M}$ independent of $\mathcal{P}_1 r_0$. This estimate follows from the fact that $\mathrm{sp}(\tilde{\mathcal{F}}'_{\times,m_2(\sigma)}(\tilde{\rho}_2(\sigma)))$ is on the left of the line $\operatorname{Re} z = -\lambda_0$. Now $r := \psi_{\times,1}(\tilde{r})$ solves the original problem and we obtain (4.18) from the analyticity of $\psi_{\times,1}$. □

In view of (3.26), this exponential decay in τ translates into algebraic decay in t:

$$\|r(t) - \rho_2(\sigma)\|_s \le M\left(1 - \frac{3|\mu|t}{4\pi}\right)^{\lambda_0 \frac{4\pi}{3|\mu|}} \|r_0 - \rho_2(\sigma)\|_s. \tag{4.20}$$

4.4 Proof of Lemma 4.3

Parameterise $\mathbb{S}^2$ by spherical coordinates:

$$\Xi : (\theta, \phi) \mapsto \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{4.21}$$

Let ρ be the radial coordinate $\rho = |x|$.

In this appendix, $m_k'(0)$ will be calculated in terms of the Clebsch-Gordan coefficients I_k and J_k that are defined by

$$I_k := \int_{\mathbb{S}^2} (Y_k)^3 d\sigma, \quad J_k := \int_{\mathbb{S}^2} Y_k \left(\frac{\partial Y_k}{\partial\theta}\right)^2 d\sigma. \tag{4.22}$$

It is known that

$$I_k = \begin{cases} (4\pi)^{-1/2}(2k+1)^{3/2}\frac{k!^3(3k/2)!^2}{(3k+1)!(k/2)!^6} & k \text{ even}, \\ 0 & k \text{ odd}, \end{cases} \tag{4.23}$$

see e.g. [54, eqns. (C.16), (C.23)].

Define $\tilde{Y}_k : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by

$$\tilde{Y}_k(x) = Y_k\left(\frac{x}{|x|}\right).$$

Since $\tilde{Y}_k$ (regarded as a function of ρ, θ, and ϕ) only depends on θ we have

$$J_k = \int_{\mathbb{S}^2} \tilde{Y}_k|\nabla\tilde{Y}_k|^2 d\sigma$$

and

$$\Delta_0 Y_k = \Delta\tilde{Y}_k\mid_{\mathbb{S}^2},$$

where Δ_0 is the Laplace-Beltrami operator on the sphere. Integration by parts shows

$$\begin{aligned} J_k &= -\int_{\mathbb{S}^2} \operatorname{div}(\tilde{Y}_k\nabla\tilde{Y}_k)\tilde{Y}_k d\sigma \\ &= -\int_{\mathbb{S}^2} (\nabla\tilde{Y}_k\cdot\nabla\tilde{Y}_k)\tilde{Y}_k d\sigma - \int_{\mathbb{S}^2} \tilde{Y}_k^2\Delta\tilde{Y}_k d\sigma \\ &= -J_k - \int_{\mathbb{S}^2} Y_k^2\Delta_0 Y_k\, d\sigma = -J_k + k(k+1)I_k. \end{aligned}$$

In the last step we used (3.23). It follows that

$$J_k = \frac{1}{2}k(k+1)I_k. \tag{4.24}$$

Thus, I_k and J_k are positive for k even and zero for k odd. Lemma 4.3 will follow if we combine this fact with Lemmas 4.6 and 4.11.

For the operators $\mathcal{E}$, κ, φ, and l defined by (3.38), Definition 3.2, (2.28), and (3.39), we found the following identities in Chapter 2 and 3:

$$\begin{aligned} \mathcal{E}(0) &= \mathcal{N}, \quad (4.25)\\ \kappa'(0) &= -\mathcal{N}^2 - \mathcal{N} + 2\mathcal{I}, \quad (4.26)\\ \varphi'(0) &= -\frac{1}{4\pi}\mathcal{I}, \quad (4.27)\\ l'(0) &= -\frac{3}{4\pi}\mathcal{I}, \end{aligned}$$

where $\mathcal{I}$ is the identity operator.

Lemma 4.6. *We have*

$$m_k'(0) = \frac{2\pi}{k+3}(\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k], Y_k)_0.$$

Proof. Differentiating the expression

$$\gamma\mathcal{F}_1(\rho_k(\sigma)) + m_k(\sigma)\mathcal{F}_2(\rho_k(\sigma)) = 0$$

twice with respect to σ gives

$$\begin{aligned} 0 &= \gamma\mathcal{F}_1''(\rho_k(\sigma))[\rho_k'(\sigma), \rho_k'(\sigma)] + m_k(\sigma)\mathcal{F}_2''(\rho_k(\sigma))[\rho_k'(\sigma), \rho_k'(\sigma)]\\ &\quad +\gamma\mathcal{F}_1'(\rho_k(\sigma))[\rho_k''(\sigma)] + m_k(\sigma)\mathcal{F}_2'(\rho_k(\sigma))[\rho_k''(\sigma)]\\ &\quad +2m_k'(\sigma)\mathcal{F}_2'(\rho_k(\sigma))[\rho_k'(\sigma)] + m_k''(\sigma)\mathcal{F}_2(\rho_k(\sigma)). \end{aligned}$$

Setting $\sigma = 0$ and using $\rho_k(0) = 0$, $m_k(0) = \mu_k$, $\mathcal{F}_2(0) = 0$, and $\rho_k'(0) = Y_k$ one obtains

$$\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k] + \mathcal{F}'_{\times,\mu_k}(0)[\rho_k''(0)] = -2m_k'(0)\mathcal{F}_2'(0)[Y_k]. \quad (4.28)$$

From Lemma 3.7 it follows that

$$(\mathcal{F}_2'(0)[Y_k], Y_k)_0 = -\frac{1}{4\pi}(k+3).$$

Moreover, since $\rho_k''(0) = 2\nu_k'(0) \in \langle Y_k\rangle^\perp$ (see Theorem 4.1) and $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ respect the decomposition $\mathbb{H}^s_\times(\mathbb{S}^2) = \langle Y_k\rangle \oplus X_k$, we get

$$(\mathcal{F}'_{\times,\mu_k}(0)[\rho_k''(0)], Y_k)_0 = 0.$$

Taking the inner product with Y_k on both sides of (4.28) and applying the last two equations yields the result. □

Now we introduce the vector spherical harmonics $\vec{V}_k$, $\vec{W}_k : \mathbb{S}^2 \to \mathbb{R}^3$ conform [29] and

[38] (where they are denoted by $\vec{V}_{k0}$ and $\vec{W}_{k0}$) in the following way:

$$\begin{aligned} \vec{V}_k &:= -\sqrt{\frac{k+1}{2k+1}} Y_k e_\rho + \frac{1}{\sqrt{(k+1)(2k+1)}} \frac{\partial Y_k}{\partial \theta} e_\theta, \\ \vec{W}_k &:= \sqrt{\frac{k}{2k+1}} Y_k e_\rho + \frac{1}{\sqrt{k(2k+1)}} \frac{\partial Y_k}{\partial \theta} e_\theta, \end{aligned} \tag{4.29}$$

for $k \in \mathbb{N}$ and $\vec{V}_0 := -Y_0 e_\rho$. Here e_ρ and e_θ are the usual unit vectors corresponding to spherical coordinates.

Let ∇_0 be the surface gradient defined by

$$\nabla_0 y = \nabla E y - \frac{\partial E y}{\partial \rho} e_\rho, \tag{4.30}$$

where Ey is a smooth extension of $y : \mathbb{S}^2 \to \mathbb{R}$ inside $\mathbb{B}^3$. From [29] or [38] we have the following formulas:

$$Y_k e_\rho = -\sqrt{\frac{k+1}{2k+1}} \vec{V}_k + \sqrt{\frac{k}{2k+1}} \vec{W}_k, \tag{4.31}$$

$$\nabla_0 Y_k = k\sqrt{\frac{k+1}{2k+1}} \vec{V}_k + (k+1)\sqrt{\frac{k}{2k+1}} \vec{W}_k = \frac{\partial Y_k}{\partial \theta} e_\theta, \tag{4.32}$$

$$\nabla(\rho^k Y_k) = \rho^{k-1} \sqrt{k(2k+1)} \vec{W}_k, \tag{4.33}$$

Here and in the sequel the expression $\rho^k Y_k$ should be interpreted as the function that maps an element of $\mathbb{B}^3$ characterised by spherical coordinates to $\rho^k Y_k(\theta, \phi)$. Note that (4.33) can be obtained combining (4.31) and (4.32).

Lemma 4.7. *For the operators $\mathcal{A}$ and $\mathcal{Q}$ defined by* (2.25) *and* (2.26) *we have*

$$\mathcal{A}'(0)[y] = -\Delta((Ey)\mathrm{id} \cdot \nabla) + ((Ey)\mathrm{id} \cdot \nabla)\Delta,$$

$$\mathcal{Q}'(0)[y] = -(\nabla E y)(\mathrm{id} \cdot \nabla) - (Ey)\nabla.$$

Proof. One obtains the first identity taking the Fréchet derivative of the identity

$$\mathcal{A}(r)\,(u \circ z(r)) = (\Delta u) \circ z(r),$$

for any twice differentiable function u, at $r = 0$ and using $\mathcal{A}(0) = \Delta$, $z(0) = \mathrm{id}$, and $z'(0)[y] = (Ey)\mathrm{id}$. In a similar way one finds from $\mathcal{Q}(0) = \nabla$

$$\mathcal{Q}'(0)[y] = -\nabla((Ey)\mathrm{id} \cdot \nabla) + (Ey)\mathrm{id}^T H, \tag{4.34}$$

where H is the Hessian: $[Hu]_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. It follows from (4.34) that

$$\begin{aligned} \mathcal{Q}'(0)[y] &= -(\nabla E y)(\mathrm{id} \cdot \nabla) - (Ey)\nabla(\mathrm{id} \cdot \nabla) + (Ey)\mathrm{id}^T H \\ &= -(\nabla E y)(\mathrm{id} \cdot \nabla) - (Ey)\nabla - (Ey)\mathrm{id}^T H + (Ey)\mathrm{id}^T H \\ &= -(\nabla E y)(\mathrm{id} \cdot \nabla) - (Ey)\nabla. \end{aligned}$$

This completes the proof. □

Lemma 4.8. *We have*

$$(\mathcal{E}'(0)[Y_k]\kappa'(0)[Y_k], Y_k)_0 = (k^2 + k - 2)(kI_k + J_k).$$

Proof. From (4.26) we have

$$\kappa'(0)[Y_k] = -(k^2 + k - 2)Y_k.$$

The lemma will be proved by showing

$$(\mathcal{E}'(0)[Y_k]Y_k, Y_k)_0 = -(kI_k + J_k). \tag{4.35}$$

Differentiating the identity

$$n(r) \cdot n(r) = 1$$

one obtains for all h

$$n'(0)[h] \cdot n(0) = n'(0)[h] \cdot \mathrm{id} = 0. \tag{4.36}$$

By the product rule of differentiation, (4.36), and the definition of $\mathcal{E}$ (see (3.38)) we have

$$\mathcal{E}'(0)[Y_k]Y_k = A + B + C,$$

where A, B, and C are given by

$$\begin{aligned} A &= \nabla \mathcal{S}(0)^{-1}(0, Y_k)^T \cdot n'(0)[Y_k], \\ B &= \mathcal{Q}'(0)[Y_k]\mathcal{S}(0)^{-1}(0, Y_k)^T \cdot n(0), \\ C &= -\nabla \mathcal{S}(0)^{-1}\mathcal{S}'(0)[Y_k]\mathcal{S}(0)^{-1}(0, Y_k)^T \cdot n(0), \end{aligned}$$

with the trace operators suppressed for the sake of brevity. We used the fact that the Fréchet derivative of $r \mapsto \mathcal{S}(r)^{-1}$ at $r = 0$ in direction h is given by $-\mathcal{S}(0)^{-1}\mathcal{S}'(0)[h]\mathcal{S}(0)^{-1}$.

Introduce $U : \mathbb{B}^3 \to \mathbb{R}$ by

$$U := \mathcal{S}(0)^{-1}(0, Y_k)^T = \rho^k Y_k.$$

For later use we note that by (4.33) and (4.29)

$$\nabla U = \rho^{k-1}\sqrt{k(2k+1)}\vec{W}_k, \qquad \mathrm{id} \cdot \nabla U = k\rho^k Y_k. \tag{4.37}$$

From (6.30) we get

$$n'(0)[Y_k] = -\nabla_0 Y_k = -\frac{\partial Y_k}{\partial \theta} e_\theta. \tag{4.38}$$

Using (4.29) and (4.37) it follows that

$$A = \operatorname{Tr}\nabla U \cdot n'(0)[Y_k] = -\sqrt{k(2k+1)}\operatorname{Tr}(\rho^{k-1}\vec{W}_k) \cdot \left(\frac{\partial Y_k}{\partial \theta} e_\theta\right) = -\left(\frac{\partial Y_k}{\partial \theta}\right)^2.$$

Further, Lemma 4.7 and (4.37) yield

$$\mathcal{Q}'(0)[Y_k]U = -\nabla(EY_k)k\rho^k Y_k - EY_k\left[\rho^{k-1}\sqrt{k(2k+1)}\vec{W}_k\right]$$

and by (4.29)

$$B = \operatorname{Tr}(\mathcal{Q}'(0)[Y_k]U) \cdot e_\rho = -k\left(\frac{\partial}{\partial n}(EY_k) + Y_k\right)Y_k,$$

where $\frac{\partial}{\partial n}$ is the normal derivative on $\mathbb{S}^2$. Set

$$\Psi := (EY_k)\mathrm{id} \cdot \nabla U = k(EY_k)\rho^k Y_k.$$

From Lemma 4.7 and the fact that U is harmonic it follows that

$$\begin{aligned} C &= -\frac{\partial}{\partial n}\mathcal{S}(0)^{-1}(\mathcal{A}'(0)[Y_k]U,\ 0)^T \\ &= \frac{\partial}{\partial n}\mathcal{S}(0)^{-1}(\Delta\Psi,\ 0)^T = \frac{\partial}{\partial n}\left(\mathcal{S}(0)^{-1}(\Delta\Psi,\ \operatorname{Tr}\Psi)^T - \mathcal{S}(0)^{-1}(0,\ \operatorname{Tr}\Psi)^T\right) \\ &= \frac{\partial\Psi}{\partial n} - \mathcal{N}(\operatorname{Tr}\Psi) = k\frac{\partial}{\partial n}(EY_k)Y_k + k^2Y_k^2 - k\mathcal{N}(Y_k^2). \end{aligned}$$

Adding the results, taking the inner product with Y_k, and using

$$(Y_k, \mathcal{N}Y_k^2)_0 = (\mathcal{N}Y_k, Y_k^2)_0 = kI_k$$

(4.35) follows. □

Lemma 4.9. *We have*

$$(\mathcal{N}\kappa''(0)[Y_k, Y_k], Y_k)_0 = 4k(k^2 + k - 1)I_k.$$

Proof. Let $\mathcal{G}(r)$ be defined by (3.16), with Ξ given by (4.21) and $\omega_1 = \theta, \omega_2 = \phi$. Introduce

$$g(r) := \det \mathcal{G}(r).$$

Note that

$$\mathcal{G}(r) = (1+r)^2\mathcal{G}_0 + \frac{\partial r}{\partial \omega} \otimes \frac{\partial r}{\partial \omega},$$

where

$$\mathcal{G}_0 := \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

and

$$\frac{\partial r}{\partial \omega} \otimes \frac{\partial r}{\partial \omega} := \begin{pmatrix} \left(\frac{\partial r}{\partial \theta}\right)^2 & \frac{\partial r}{\partial \theta}\frac{\partial r}{\partial \phi} \\ \frac{\partial r}{\partial \theta}\frac{\partial r}{\partial \phi} & \left(\frac{\partial r}{\partial \phi}\right)^2 \end{pmatrix}.$$

From $\mathcal{G}(r)$ we construct $\mathcal{B}(r)$ as in (3.18). The Laplace-Beltrami operator on $\mathbb{S}^2$ satisfies

$$\Delta_0 = \mathcal{B}(0) = -\mathcal{N}^2 - \mathcal{N}. \tag{4.39}$$

The following expansions around $r = 0$ are easily derived:

$$\begin{aligned}
\mathcal{G}(r) &= \mathcal{G}_0\left(I + 2rI + r^2 I + \mathcal{G}_0^{-1}\frac{\partial r}{\partial \omega} \otimes \frac{\partial r}{\partial \omega}\right), \\
\mathcal{G}(r)^{-1} &= \mathcal{G}_0^{-1}\left(I - 2rI + 3r^2 I - \frac{\partial r}{\partial \omega} \otimes \frac{\partial r}{\partial \omega}\mathcal{G}_0^{-1} + \mathcal{O}(r^3)\right), \\
g(r) &= \sin^2\theta\left(1 + 4r + 6r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2 + \csc^2\theta\left(\frac{\partial r}{\partial \phi}\right)^2 + \mathcal{O}(r^3)\right), \\
\sqrt{g(r)} &= \sin\theta\left(1 + 2r + r^2 + \frac{1}{2}\left(\frac{\partial r}{\partial \theta}\right)^2 + \frac{1}{2}\csc^2\theta\left(\frac{\partial r}{\partial \phi}\right)^2 + \mathcal{O}(r^3)\right), \\
X(r) &:= \sqrt{g(r)}\mathcal{G}(r)^{-1} \\
&= \sin\theta\mathcal{G}_0^{-1}\left(I - \frac{\partial r}{\partial \omega} \otimes \frac{\partial r}{\partial \omega}\mathcal{G}_0^{-1} + \frac{1}{2}\left(\frac{\partial r}{\partial \theta}\right)^2 I + \frac{1}{2}\csc^2\theta\left(\frac{\partial r}{\partial \phi}\right)^2 I + \mathcal{O}(r^3)\right), \\
Z(r) &:= \frac{1}{\sqrt{g(r)}} = \csc\theta\left(1 - 2r + 3r^2 - \frac{1}{2}\left(\frac{\partial r}{\partial \theta}\right)^2 - \frac{1}{2}\csc^2\theta\left(\frac{\partial r}{\partial \phi}\right)^2 + \mathcal{O}(r^3)\right).
\end{aligned}$$

From these expansions it follows that for any $h \in \mathbb{H}^s(\mathbb{S}^2)$

$$\begin{aligned}
\mathcal{G}'(0)[h] &= 2h\mathcal{G}_0, \\
\mathcal{G}''(0)[h,h] &= 2h^2\mathcal{G}_0 + 2\frac{\partial h}{\partial \omega} \otimes \frac{\partial h}{\partial \omega}, \\
X'(0)[h] &= 0.
\end{aligned}$$

Since zonal harmonics do not depend on the azimuthal coordinate ϕ we have

$$X''(0)[Y_k, Y_k] = \left(\frac{\partial Y_k}{\partial \theta}\right)^2 \begin{pmatrix} -\sin\theta & 0 \\ 0 & \csc\theta \end{pmatrix}, \tag{4.40}$$

$$Z''(0)[Y_k, Y_k] = \csc\theta \left(6(Y_k)^2 - \left(\frac{\partial Y_k}{\partial \theta}\right)^2\right), \tag{4.41}$$

$$\mathcal{B}'(0)[Y_k] = -2Y_k\Delta_0, \tag{4.42}$$

$$\mathcal{B}''(0)[Y_k, Y_k] = \left(6(Y_k)^2 - \left(\frac{\partial Y_k}{\partial \theta}\right)^2\right)\Delta_0 + \csc\theta\frac{\partial}{\partial\theta}\left[-\sin\theta\left(\frac{\partial Y_k}{\partial \theta}\right)^2\frac{\partial}{\partial\theta}\right] \tag{4.43}$$

$$+\csc^2\theta\left(\frac{\partial Y_k}{\partial \theta}\right)^2\frac{\partial^2}{\partial\phi^2}.$$

From (3.21) it follows that

$$\kappa''(0)[Y_k, Y_k] = A + B + C + D + E + F,$$

with

$$\begin{aligned} A &= (\mathcal{B}''(0)[Y_k, Y_k]\tilde{z}(0)) \cdot n(0), \\ B &= (\Delta_0\tilde{z}''(0)[Y_k, Y_k]) \cdot n(0), \\ C &= (\Delta_0\tilde{z}(0)) \cdot n''(0)[Y_k, Y_k], \\ D &= 2(\mathcal{B}'(0)[Y_k]\tilde{z}'(0)[Y_k]) \cdot n(0), \\ E &= 2(\mathcal{B}'(0)[Y_k]\tilde{z}(0)) \cdot n'(0)[Y_k], \\ F &= 2(\Delta_0\tilde{z}'(0)[Y_k]) \cdot n'(0)[Y_k]. \end{aligned}$$

Since $\tilde{z}(0) = n(0) = \mathrm{id}$ and $\frac{\partial \mathrm{id}}{\partial\theta} \perp \mathrm{id}$, $\frac{\partial \mathrm{id}}{\partial\phi} \perp \mathrm{id}$ we obtain from (4.43)

$$\begin{aligned} A &= \left(6(Y_k)^2 - \left(\frac{\partial Y_k}{\partial \theta}\right)^2\right)(\Delta_0\mathrm{id})\cdot\mathrm{id} - \left(\frac{\partial Y_k}{\partial \theta}\right)^2\frac{\partial^2\mathrm{id}}{\partial\theta^2}\cdot\mathrm{id} + \csc^2\theta\left(\frac{\partial Y_k}{\partial \theta}\right)^2\frac{\partial^2\mathrm{id}}{\partial\phi^2}\cdot\mathrm{id} \\ &= \left(6(Y_k)^2 - \left(\frac{\partial Y_k}{\partial \theta}\right)^2\right)(\Delta_0\mathrm{id})\cdot\mathrm{id} = -12(Y_k)^2 + 2\left(\frac{\partial Y_k}{\partial \theta}\right)^2. \end{aligned}$$

The last step follows from (4.39) and the identity $\mathcal{N}\mathrm{id} = \mathrm{id}$. We have $B \equiv 0$, because $z''(0) \equiv 0$. Taking the second Fréchet derivative at zero of the expression

$$n(r) \cdot n(r) = 1$$

we obtain from (4.38)

$$n''(0)[Y_k, Y_k] \cdot \mathrm{id} = -n'(0)[Y_k] \cdot n'(0)[Y_k] = -\left(\frac{\partial Y_k}{\partial \theta}\right)^2.$$

Therefore it follows from (4.39) that

$$C = 2\left(\frac{\partial Y_k}{\partial \theta}\right)^2.$$

In the following identities, that follow from [38, eqns. (B-6 a), (B-6 c), (B-7)], Δ_0 has to be applied component-wise:

$$\Delta_0 \vec{V}_k = -(k+1)(k+2)\vec{V}_k, \quad \Delta_0 \vec{W}_k = -k(k-1)\vec{W}_k. \tag{4.44}$$

From (4.31), (4.42), and (4.44) we get

$$\begin{aligned} D &= 2\mathcal{B}'(0)[Y_k](Y_k \mathrm{id}) \cdot \mathrm{id} = -4Y_k \Delta_0 (Y_k \mathrm{id}) \cdot \mathrm{id} \\ &= -4Y_k \Delta_0 \left[-\sqrt{\frac{k+1}{2k+1}} \vec{V}_k + \sqrt{\frac{k}{2k+1}} \vec{W}_k \right] \cdot \mathrm{id} \\ &= 4 \left[\frac{(k+1)^2(k+2)}{2k+1} + \frac{k^2(k-1)}{2k+1} \right] (Y_k)^2 \\ &= 4(k^2+k+2)(Y_k)^2. \end{aligned}$$

Further, (4.39) and (4.36) yield

$$E = -4Y_k(\Delta_0 \mathrm{id}) \cdot n'(0)[Y_k] = 8Y_k \mathrm{id} \cdot n'(0)[Y_k] = 0.$$

Finally (4.38) implies

$$\begin{aligned} F &= 2(\Delta_0(Y_k \mathrm{id})) \cdot -\frac{\partial Y_k}{\partial \theta} e_\theta \\ &= 2\Delta_0 \left[-\sqrt{\frac{k+1}{2k+1}} \vec{V}_k + \sqrt{\frac{k}{2k+1}} \vec{W}_k \right] \cdot -\frac{\partial Y_k}{\partial \theta} e_\theta \\ &= \left[2(k+1)(k+2)\sqrt{\frac{k+1}{2k+1}} \vec{V}_k - 2k(k-1)\sqrt{\frac{k}{2k+1}} \vec{W}_k \right] \cdot -\frac{\partial Y_k}{\partial \theta} e_\theta \\ &= \left[-2\frac{(k+1)(k+2)}{2k+1} + 2\frac{k(k-1)}{2k+1} \right] \left(\frac{\partial Y_k}{\partial \theta}\right)^2 = -4\left(\frac{\partial Y_k}{\partial \theta}\right)^2. \end{aligned}$$

The lemma follows by adding the results. □

Lemma 4.10. *We have*

$$(\mathcal{E}'(0)[Y_k]\varphi'(0)[Y_k], Y_k)_0 = \frac{1}{4\pi}(kI_k + J_k),$$

$$(\mathcal{N}\varphi''(0)[Y_k, Y_k], Y_k)_0 = \frac{k}{2\pi} I_k.$$

Proof. The first statement follows from (4.27) and (4.35). The second statement follows from the definition of φ. □

Lemma 4.11. *We have*

$$\begin{aligned} &(\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k], Y_k)_0 \\ = \;& \frac{2k(k^3 + 7k^2 + 6k - 6)}{k+3} I_k + \frac{6(k^2 + k - 2)}{k+3} J_k. \end{aligned}$$

Proof. It is obvious that

$$\mathcal{F}''_{\times,\mu_k}(0) = \gamma\left(\mathcal{F}''_1(0) - \zeta_k \mathcal{F}''_2(0)\right).$$

Observe that $\varphi(0) = 0$ and as $\mathcal{E}(r)$ vanishes on constants, $\mathcal{E}''(0)[h,h]\kappa(0) = 0$. Therefore

$$\mathcal{F}''_1(0)[h,h] = 2\mathcal{E}'(0)[h]\kappa'(0)[h] + \mathcal{E}(0)\kappa''(0)[h,h],$$

$$\mathcal{F}''_2(0)[h,h] = 2\mathcal{E}'(0)[h]\varphi'(0)[h] + \mathcal{E}(0)\varphi''(0)[h,h] + l''(0)[h,h].$$

From the definition of l it follows that $l''(0)[h,h] = \frac{3}{2\pi}h^2$. Combining the previous lemmas and (4.25) we obtain

$$\begin{aligned} &(\mathcal{F}''_{\times,\mu_k}(0)[Y_k, Y_k], Y_k)_0 \\ = \;& \gamma\big(2(\mathcal{E}'(0)[Y_k]\kappa'(0)[Y_k], Y_k)_0 - 2\zeta_k(\mathcal{E}'(0)[Y_k]\varphi'(0)[Y_k], Y_k)_0\big) \\ & + \gamma\big((\mathcal{N}\kappa''(0)[Y_k, Y_k], Y_k)_0 - \zeta_k(\mathcal{N}\varphi''(0)[Y_k, Y_k], Y_k)_0 - \zeta_k(l''(0)[Y_k, Y_k], Y_k)_0\big) \\ = \;& \gamma\left(2\left(k^2 + k - 2 - \zeta_k\frac{1}{4\pi}\right)(kI_k + J_k) + 4k(k^2 + k - 1)I_k - \zeta_k\frac{k}{2\pi}I_k - \zeta_k\frac{3}{2\pi}I_k\right) \\ = \;& \gamma\left(\frac{6(k^2 + k - 2)}{k+3}(kI_k + J_k) + 4k(k^2 + k - 1)I_k - 2k(k^2 + k - 2)I_k\right) \\ = \;& \gamma\left(\frac{6(k^2 + k - 2)}{k+3}(kI_k + J_k) + 2k^2(k+1)I_k\right). \end{aligned}$$

□

Lemma 4.3 follows from (4.23), (4.24), Lemma 4.6, and Lemma 4.11.

Chapter 5

Hele-Shaw flow with surface tension in $\mathbb{R}^N$

5.1 Introduction

Until now we only discussed the three-dimensional version of the Hele-Shaw moving boundary problem with surface tension and one source/sink (3.1)-(3.4). Since the linearisations of both $\mathcal{F}_1$ and $\mathcal{F}_2$ are negative operators (see Lemma 3.7), the spherical solution is asymptotically stable in the case of injection. For the suction problem it is important that $\mathcal{F}_1$ is a third order operator whereas $\mathcal{F}_2$ is of order one. Surface tension dominates the strength of the sink and the spherical solution is stable with respect to certain perturbations.

This Chapter is aimed at extending the theory for $N = 3$ to other space dimensions. More precisely, we will prove the results that are shown in Table 5.1. For dimensions unequal to 3, the local-in-time version of the suction problem is again well-posed because the qualitative properties of $\mathcal{F}_1(r)$ and $\mathcal{F}_2(r)$ are the same in all dimensions. However, (3.22) is essentially time dependent because of the scaling properties of curvature and Φ that we discussed earlier. For large time one of the two terms in $\gamma\alpha(t)^{-3}\mathcal{F}_1(r) + \mu\alpha(t)^{-N}\mathcal{F}_2(r)$ grows faster than the other one. Hence, large-time behaviour depends on the dimension.

An application of the two-dimensional version of the Hele-Shaw model is liquid flow in a Hele-Shaw cell. Our results for $N \geq 4$ turn out to be useful in Chapter 6 where we study the Stokes moving boundary problem.

For the injection problems we find existence for all $t > 0$. In the two-dimensional suction problem the geometric centre of the initial domain and the suction point must coincide in order to remove all liquid. The domain vanishes "as an asymptotically circular point". In contrast to the three-dimensional case, there is no restriction on the suction rate in the two-dimensional case. For $N \geq 4$ with suction, the spherical solution is linearly unstable. However, it is possible to derive a result like Theorem 3.18 for any $\mu < 0$ both for $N = 2$ and for $N \geq 4$. An arbitrarily large portion of liquid smaller than the entire domain can be removed if the initial domain is close enough to a ball. We call this "almost global existence".

dimension	$\mu > 0$	$\mu < 0$
2	global existence and decay (see Theorem 5.6)	• global existence and decay for correct geometric centre (see Theorem 5.8) • almost global existence (see Theorem 5.13)
3	global existence and decay (see Theorem 3.13)	• global existence and decay for correct geometric centre when $\frac{\|\mu\|}{\gamma} < \frac{32\pi}{5}$ (see Theorem 3.15) • almost global existence (see Theorem 3.18)
≥ 4	global existence and decay (see Theorem 5.10)	almost global existence (see Theorem 5.13)

Table 5.1: *Large-time behaviour of Hele-Shaw flow with surface tension and injection or suction in one point*

Instead of using the principle of linearised stability we apply Theorem A.1 in Appendix A to deal with the non-autonomous character of the problems. From the linearisations, that we calculated before, and perturbation results we derive estimates for $(r, \mathcal{F}(r,t))_s$ where $\mathcal{F}$ is the time-dependent evolution operator. Since Hölder spaces have no Hilbert space structure we define our evolution problem in Sobolev spaces. As a consequence, we need to demand higher regularity to make sure that our Sobolev spaces are embedded in a Hölder space of sufficiently high order.

As mentioned before, finding non-trivial stationary solutions to (3.22) by the method presented in Chapter 4 is not possible for $N \neq 3$ since the kernels of the evolution operators are time dependent.

5.2 The evolution equation and its linearisation

In this section the evolution problem in $\mathbb{R}^N$ that we discussed in Chapter 3 is formulated in Sobolev spaces. Let $(\cdot,\cdot)_0$ be the usual $\mathbb{L}^2(\mathbb{S}^{N-1})$-inner product and define for each $r \in \mathbb{L}^2(\mathbb{S}^{N-1})$ the numbers r_{kj} by

$$r_{kj} := (r, s_{kj})_0,$$

where s_{kj} denote the spherical harmonics as defined in Section 1.5. For all $s > 0$, equip the Sobolev space $\mathbb{H}^s(\mathbb{S}^{N-1})$ with the inner product

$$(r, \tilde{r})_s = \sum_{k,j} (k^2+1)^s r_{kj} \tilde{r}_{kj}.$$

In the sequel we will use the Sobolev Embedding Theorem: If $k \in \mathbb{N}_0$, $\beta \in (0,1)$, and $s > \frac{N-1}{2} + k + \beta$, then

$$\mathbb{H}^s(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^{k,\beta}(\mathbb{S}^{N-1})$$

and

$$\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N) \hookrightarrow \mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N}).$$

We will also use the fact that for $s > \frac{N-1}{2}$, $\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^s(\mathbb{S}^{N-1})$ are Banach algebras.

Reintroduce the functions $R(\cdot,t) : \mathbb{S}^{N-1} \to (-1,\infty)$ and $r(\cdot,t) : \mathbb{S}^{N-1} \to (-1,\infty)$ that parameterise the moving domain and its rescaled version as explained in Section 1.5. We will often write $r(t)$ instead of $r(\cdot,t)$ and if we consider a fixed domain, then the argument t will be suppressed.

From now on we assume that $r \in \mathbb{H}^s(\mathbb{S}^{N-1})$ where

$$s > \frac{N+7}{2}. \tag{5.1}$$

On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ the operators $n : \mathcal{U} \to \left(\mathbb{H}^{s-1}(\mathbb{S}^{N-1})\right)^N$ and $\kappa : \mathcal{U} \to \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ as defined in Chapters 2 and 3 are analytic (see [60, Ch. 3 Lemma 16]). This is proved in the same way as we did for Hölder spaces in Chapter 2.

By [63, Thm. 6.108], there exists an extension operator $E \in \mathcal{L}(\mathbb{H}^s(\mathbb{S}^{N-1}), \mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N))$, such that

$$Er|_{\mathbb{S}^{N-1}} = r. \tag{5.2}$$

Define $z : \mathbb{H}^s(\mathbb{S}^{N-1}) \to \left(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)\right)^N$ by

$$z(r,x) := (1 + E(r,x))\, x.$$

Here and hereafter we identify $z(r,\cdot)$ with $z(r)$, $E(r,\cdot)$ with $E(r)$, etc. Let $j^{k,i}(r)$ be the coefficients of the inverse of the matrix

$$\mathcal{J}(r) = \frac{\partial z(r)}{\partial x} \in \left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^{N\times N}.$$

Lemma 5.1. *Let $s > \frac{N+7}{2}$. There exists a $\delta > 0$ such that if $\|r\|_s < \delta$, then $z(r) : \overline{\mathbb{B}^N} \to \overline{\Omega}_r$ is bijective and $z(r)^{-1} \in \left(\mathcal{C}^2(\overline{\Omega_r})\right)^N$.*

Proof. From $\mathbb{H}^s(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^4(\mathbb{S}^{N-1})$ it follows that r is small in $\mathcal{C}^4(\mathbb{S}^{N-1})$. The result follows from Lemmas 2.3 and 2.4. □

On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ we define the following operators:

- $\mathcal{A} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N))$ and $\mathcal{Q} : \mathcal{U} \to \mathcal{L}\left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N\right)$ by (2.25) and (2.26). Note that $\mathcal{J}$ is continuous near zero, $\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^{N\times N}$ is a Banach algebra, and $\mathcal{J}(0) = I$. As a result, $\mathcal{J}(r)$ is invertible (in the sense of matrices) for $\|r\|_s$ small and the elements $j^{k,i}(r)$ are in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$. Because of (5.1) the space $\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N)$ is a Banach algebra. Therefore the operators $\mathcal{A}$ and $\mathcal{Q}$ are well-defined.

- $\mathcal{S} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$ as in (2.27).
- $\varphi : \mathcal{U} \to \mathbb{H}^s(\mathbb{S}^{N-1})$ as in (2.28).

Lemma 5.2. *For $\|r\|_s$ small, $\mathcal{S}(r)$ is an isomorphism between $\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$.*

Proof. It is known that $\mathcal{S}(0) = (\Delta, \mathrm{Tr})^T$ is an isomorphism between $\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^{s-\frac{7}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$. Isomorphisms form an open set in the space of linear bounded operators. Because $\mathcal{S}$ is smooth (by the methods in Lemma 2.7), $\mathcal{S}(r)$ is an isomorphism for small r. □

Suppose that (5.1) holds. Introduce on a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ the operators $\mathcal{E} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-2}(\mathbb{S}^{N-1}), \mathbb{H}^{s-3}(\mathbb{S}^{N-1}))$ and $l : \mathcal{U} \to \mathbb{H}^s(\mathbb{S}^{N-1})$ by

$$\mathcal{E}(r)\psi := \frac{\mathrm{Tr}\left(\mathcal{Q}(r)\left[\mathcal{S}(r)^{-1}\left[\begin{array}{c}0\\ \psi\end{array}\right]\right]\right)\cdot n(r)}{n(r)\cdot \mathrm{id}} \tag{5.3}$$

and

$$l(r) := \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N}. \tag{5.4}$$

Since $\mathcal{S}$ also maps $\mathcal{U}$ to $\mathcal{L}(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \times \mathbb{H}^s(\mathbb{S}^{N-1}))$, $\mathcal{E}$ also defines a mapping from $\mathcal{U}$ to $\mathcal{L}(\mathbb{H}^s(\mathbb{S}^{N-1}), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$. As in Chapter 3 we have for r in a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$

$$\frac{\partial r}{\partial t} = \frac{1}{\alpha(t)^3}\mathcal{F}(r,t), \tag{5.5}$$

where

$$\mathcal{F}(r,t) = \gamma\mathcal{F}_1(r) + \mu\alpha(t)^{3-N}\mathcal{F}_2(r), \tag{5.6}$$

for a third order operator $\mathcal{F}_1 : \mathcal{U} \to \mathbb{H}^{s-3}(\mathbb{S}^{N-1})$ and a first order operator $\mathcal{F}_2 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ given by

$$\mathcal{F}_1(r) = \mathcal{E}(r)\kappa(r),$$
$$\mathcal{F}_2(r) = \mathcal{E}(r)\varphi(r) + l(r),$$

In fact, $\mathcal{F}_2$ also maps a neighbourhood of zero in $\mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ to $\mathbb{H}^{s-3}(\mathbb{S}^{N-1})$.

Lemma 5.3. *(Analyticity of the evolution operator)*
Suppose that $s > \frac{N+7}{2}$.

- *The mapping $\mathcal{F}_1$ is analytic from a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ to $\mathbb{H}^{s-3}(\mathbb{S}^{N-1})$.*
- *The mapping $\mathcal{F}_2$ is analytic from a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ to $\mathbb{H}^{s-3}(\mathbb{S}^{N-1})$.*

Proof. These statements can be proved in the same way as in Lemma 3.5 (see also [60, Ch. 3 Lemma 20]). □

Introduce a new time variable $\tau = \tau(t)$ such that (3.25) holds and $\tau(0) = 0$. For $N \neq 3$ this gives

$$\tau(t) = \frac{\sigma_N}{\mu\,(N-3)}\left(\left(\frac{\mu N t}{\sigma_N}+1\right)^{1-\frac{3}{N}}-1\right). \tag{5.7}$$

Define

$$T := \frac{\sigma_N}{|\mu| N}.$$

In the case of suction we have $\alpha(T) = 0$. If the initial volume is equal to the volume of $\mathbb{B}^N$, then at $t = T$ all liquid is sucked out provided that the domain evolution continues up to complete extinction.

The original time interval on which the moving boundary problem is defined is infinite for the injection problems and finite for the suction problems. Considering (5.7), this does not change for the new time variable τ in the case $N \geq 4$. For $N = 2$ however, the new injection problem is defined on a finite time interval $(0, \tau_{\max})$ while the suction problem is defined on $(0, \infty)$. This is illustrated in Figure 5.1. We have

$$\begin{aligned}
\lim_{t\to\infty}\tau(t) &= \frac{2\pi}{\mu}, && \text{for } N=2,\ \mu>0,\\
\lim_{t\to T}\tau(t) &= \infty, && \text{for } N=2,\ \mu<0,\\
\lim_{t\to\infty}\tau(t) &= \infty, && \text{for } N\geq 4,\ \mu>0,\\
\lim_{t\to T}\tau(t) &= \frac{\sigma_N}{|\mu|(N-3)}, && \text{for } N\geq 4,\ \mu<0.
\end{aligned}$$

We denote these limit values for τ by $\tau_{\max}$. From (3.25), (5.5), and (5.6) it follows that

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, t(\tau)) = \gamma \mathcal{F}_1(r) + \mu\alpha(t(\tau))^{3-N}\mathcal{F}_2(r).$$

Here $t(\tau)$ is the value of t that corresponds to τ. To simplify notation, we will write from now on $\alpha(\tau)$ instead of $\alpha(t(\tau))$ and $\mathcal{F}(r,\tau)$ instead of $\mathcal{F}(r, t(\tau))$, thus

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r,\tau) := \gamma \mathcal{F}_1(r) + \mu\alpha(\tau)^{3-N}\mathcal{F}_2(r). \tag{5.8}$$

By Lemma 3.7 we have for all dimensions N the following expressions for $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ in terms of the Dirichlet-to-Neumann mapping $\mathcal{N}$:

$$\mathcal{F}_1'(0)[r] = -p_1(\mathcal{N})r \tag{5.9}$$

and

$$\mathcal{F}_2'(0)[r] = -p_2(\mathcal{N})r \tag{5.10}$$

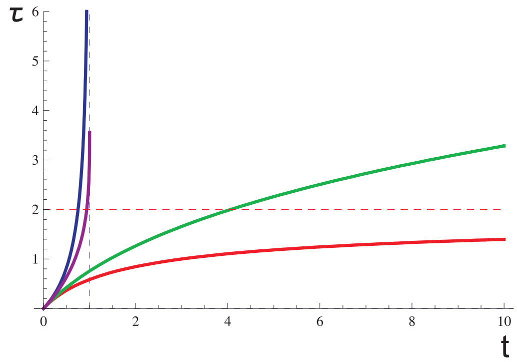


Figure 5.1: *The relation between t and τ; red: $N = 2$ with $\mu > 0$, blue: $N = 2$ with $\mu < 0$, green: $N = 4$ with $\mu > 0$, purple: $N = 4$ with $\mu < 0$. The red line has a horizontal asymptote and the blue graph a vertical asymptote. In the case of injection we plotted the case in which $\mu = \frac{\sigma_N}{N}$, for suction $\mu = -\frac{\sigma_N}{N}$*

where p_1 and p_2 are the polynomials

$$\begin{aligned} p_1(k) &:= k^3 + (N-2)k^2 - (N-1)k, \\ p_2(k) &:= \frac{1}{\sigma_N}k + \frac{N}{\sigma_N}. \end{aligned}$$

5.3 Energy estimates and global existence results

In this section we find estimates for $(r, \mathcal{F}(r,\tau))_s$ to prove a stability result for the stationary solution $r \equiv 0$. We make use of Theorem A.1 in Appendix A. Three cases will be considered, namely the injection problems for both $N = 2$ and $N \geq 4$ and the suction problem for $N = 2$. Estimates for $(r, \mathcal{F}_1'(0)[r])_s$ and $(r, \mathcal{F}_2'(0)[r])_s$ are easily obtained from (5.9) and (5.10). To show that the remaining "nonlinear parts" $(r, \mathcal{F}_1(r) - \mathcal{F}_1'(0)[r])_s$ and $(r, \mathcal{F}_2(r) - \mathcal{F}_2'(0)[r])_s$ are controlled by the "linear parts" we make use of Lemma 5.3.

For the injection problems we prove that for small $r(0)$ there exists a global solution $r(t)$ to (5.5) that converges to zero as t tends to infinity. For the two-dimensional suction problem we need to restrict ourselves to domains with certain geometric properties, in order to get a global existence result.

In Example 1.2 in Chapter 1 we made use of formula (1.19) to close a regularity gap for a first order evolution operator. Since $\mathcal{F}_1$ is of order three, the regularity gap is larger here. Therefore we derive a second order chain rule in Lemma 5.4.

Let ω be a bijection between $\{(l,m) \in \mathbb{N}^2 : 1 \leq l < m \leq N\}$ and $\{1, 2, \dots, \binom{N}{2}\}$. We define the following differential operators on functions on $\mathbb{S}^{N-1}$:

$$D_{\omega(l,m)} := x_l \frac{\partial}{\partial x_m} - x_m \frac{\partial}{\partial x_l}. \tag{5.11}$$

For each $i \in \{1, 2, \ldots, \binom{N}{2}\}$, D_i is the infinitesimal generator of a semigroup of operators $h \mapsto R_h$:

$$R_h f = e^{hD_i} f = f \circ g_h, \qquad f \in \mathbb{L}^2(\mathbb{S}^{N-1}),$$

where $g_h : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ are rotations of the unit sphere.

Lemma 5.4. *Let $s > \frac{N+7}{2}$. For $r \in \mathbb{H}^{s+1}(\mathbb{S}^{N-1})$ with $\|r\|_s$ small, we have the generalised chain rule of differentiation:*

$$D_i \mathcal{F}_k(r) = \mathcal{F}_k'(r)[D_i r], \qquad k = 1, 2. \tag{5.12}$$

If in addition $r \in \mathbb{H}^{s+2}(\mathbb{S}^{N-1})$, then the second order generalised chain rule of differentiation holds,

$$D_i D_j \mathcal{F}_k(r) = \mathcal{F}_k'(r)[D_i D_j r] + \mathcal{F}_k''(r)[D_i r, D_j r], \qquad k = 1, 2. \tag{5.13}$$

Proof. Because $\mathcal{F}_k$ ($k = 1, 2$) commutes with rotations, i.e.

$$\mathcal{F}_k(r) \circ g_h = \mathcal{F}_k(r \circ g_h),$$

we get

$$\begin{aligned} D_i \mathcal{F}_k(r) &= \lim_{h \to 0} \frac{1}{h}(R_h - \mathcal{I})\mathcal{F}_k(r) = \lim_{h \to 0} \frac{1}{h}\left(\mathcal{F}_k(r) \circ g_h - \mathcal{F}_k(r)\right) \\ &= \lim_{h \to 0} \frac{1}{h}\left(\mathcal{F}_k(r \circ g_h) - \mathcal{F}_k(r)\right) = \lim_{h \to 0} \frac{1}{h} \mathcal{F}_k'(r)[r \circ g_h - r] \\ &= \mathcal{F}_k'(r) \left[\lim_{h \to 0} \frac{r \circ g_h - r}{h} \right] = \mathcal{F}_k'(r)[D_i r], \end{aligned}$$

where $\mathcal{I}$ is the identity. Further, if

$$R_l f = e^{hD_j} f = f \circ g_l, \qquad f \in \mathbb{L}^2(\mathbb{S}^{N-1}),$$

for some rotation g_l, then

$$\begin{aligned} D_i D_j \mathcal{F}_k(r) &= \lim_{h \to 0} \lim_{l \to 0} \frac{1}{hl}(R_h - \mathcal{I})(R_l - \mathcal{I})\mathcal{F}_k(r) \\ &= \lim_{h \to 0} \lim_{l \to 0} \frac{1}{hl}\Big(\mathcal{F}_k(r \circ g_l \circ g_h) - \mathcal{F}_k(r \circ g_l) - \Big(\mathcal{F}_k(r \circ g_h) - \mathcal{F}_k(r) \Big) \Big) \\ &= \lim_{l \to 0} \frac{1}{l}\left(\mathcal{F}_k'(r \circ g_l)[D_i(r \circ g_l)] - \mathcal{F}_k'(r)[D_i r] \right) \\ &= \lim_{l \to 0} \frac{1}{l}\left\{ \mathcal{F}_k'(r \circ g_l) - \mathcal{F}_k'(r) \right\}[D_i(r \circ g_l)] \\ &\quad + \lim_{l \to 0} \frac{1}{l} \mathcal{F}_k'(r)[D_i(r \circ g_l - r)] \\ &= \mathcal{F}_k''(r)[D_i r, D_j r] + \mathcal{F}_k'(r)[D_i D_j r]. \end{aligned}$$

This proves the lemma. □

Let for $\sigma > 0$, $\|\cdot\|_{\sigma-2,2}$ be the norm on $\mathbb{H}^\sigma(\mathbb{S}^{N-1})$ induced by the inner product

$$(r, \tilde{r})_{\sigma-2,2} := (r, \tilde{r})_{\sigma-2} + \sum_{i,j}(D_iD_jr, D_iD_j\tilde{r})_{\sigma-2}. \tag{5.14}$$

This norm is equivalent to the norm $\|\cdot\|_\sigma$ that we introduced earlier (see [33, Sec. 4]).

Lemma 5.5. *If $r \in \mathfrak{S}_k^N$, then $D_ir \in \mathfrak{S}_k^N$.*

Proof. The spaces $\mathfrak{S}_k^N$ are invariant under rotations. The lemma follows from this and the fact that D_i generates a semigroup of rotations. □

In the following theorems $\mathcal{C}_w$ indicates weak continuity.

Theorem 5.6. *Let $N = 2$, $\mu > 0$, and $\lambda_0 \in \left(0, \frac{\mu}{2\pi}\right)$. Suppose that $s > 5$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^1)$ with $\|r_0\|_s < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0, \tag{5.15}$$

has a solution $r \in \mathcal{C}_w([0, \tau_{\max}), \mathbb{H}^s(\mathbb{S}^1)) \cap \mathcal{C}_w^1([0, \tau_{\max}), \mathbb{H}^{s-3}(\mathbb{S}^1))$. Furthermore, $((\xi, \tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^1 \times (0, \tau_{\max}))$. If we regard r as a function of the original time variable t (where the relation between t and τ is given by (5.7)), then

$$\|r(t)\|_s \leq M\left(\frac{\mu t}{\pi} + 1\right)^{-\frac{\pi\lambda_0}{\mu}} \|r_0\|_s, \quad t \in [0, \infty). \tag{5.16}$$

Proof. The theorem follows from the inequality

$$(r, \mathcal{F}(r, \tau))_{s-2,2} \leq -\lambda_0 \alpha(\tau)\|r\|_{s-2,2}^2, \tag{5.17}$$

for all $r \in \mathbb{H}^{s+3}(\mathbb{S}^1)$ with $\|r\|_s$ small. First we find a similar estimate for the Fréchet derivatives $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$. Perturbation arguments and the chain rule (5.13) lead to (5.17). Combining (3.25) and (5.17) we get algebraic decay of r as a function of t, given by (5.16).

Throughout the proof we will assume that $r \in \mathbb{H}^{s+3}(\mathbb{S}^1)$ with $\|r\|_s < \delta$, where δ is sufficiently small. The symbol C always denotes a constant that may vary throughout the proof. This constant is independent of r.

1. Take $\eta > 0$ such that $\lambda_0 < \frac{(1-\eta)\mu}{2\pi}$. Define

$$c_1 := \inf_{k \in \mathbb{N}_0, \tau \geq 0} \frac{\gamma p_1(k) + \eta\mu\alpha(\tau)p_2(k)}{(k^2+1)^{\frac{3}{2}}} = \inf_{k \in \mathbb{N}_0} \frac{\gamma p_1(k) + \eta\mu p_2(k)}{(k^2+1)^{\frac{3}{2}}} > 0,$$

and define $\varepsilon := \min\{c_1, \frac{(1-\eta)\mu}{2\pi} - \lambda_0\}$.

2. Let $\mathcal{F}'$ be the Fréchet derivative of $\mathcal{F}$ with respect to the first argument. From (5.9)

and (5.10) we have the following estimate for the linear part of $\mathcal{F}(r,\tau)$:

$$
\begin{aligned}
& (r, \mathcal{F}'(0,\tau)[r])_{s-2} \\
= \;& \gamma(r, \mathcal{F}_1'(0)[r])_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2'(0)[r])_{s-2} \\
= \;& \gamma(r, \mathcal{F}_1'(0)[r])_{s-2} + \eta\mu\alpha(\tau)(r, \mathcal{F}_2'(0)[r])_{s-2} + (1-\eta)\mu\alpha(\tau)(r, \mathcal{F}_2'(0)[r])_{s-2} \\
= \;& \sum_{k,j} (k^2+1)^{s-2+\frac{3}{2}} \frac{-\gamma p_1(k) - \eta\mu\alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} r_{kj}^2 \\
& +(1-\eta)\alpha(\tau) \sum_{k,j} (k^2+1)^{s-2+\frac{1}{2}} \frac{-\mu p_2(k)}{(k^2+1)^{\frac{1}{2}}} r_{kj}^2 \\
\leq \;& -c_1 \|r\|_{s-\frac{1}{2}}^2 - \frac{(1-\eta)\mu}{2\pi} \alpha(\tau) \|r\|_{s-\frac{3}{2}}^2 .
\end{aligned}
\tag{5.18}
$$

In the last step we used

$$
-\frac{p_2(k)}{\sqrt{k^2+1}} \leq -\frac{1}{2\pi}. \tag{5.19}
$$

3. Now we find an estimate for the remaining nonlinear part. From the analyticity of $\mathcal{F}_1$ and $\mathcal{F}_2$ near zero and the fact that $\mathcal{F}_1(0) = \mathcal{F}_2(0) = 0$, we have for r near zero in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1)$

$$
\begin{aligned}
\|\mathcal{F}_1(r) - \mathcal{F}_1'(0)[r]\|_{s-\frac{7}{2}} &\leq C\|r\|_{s-\frac{1}{2}}^2, \\
\|\mathcal{F}_2(r) - \mathcal{F}_2'(0)[r]\|_{s-\frac{5}{2}} &\leq C\|r\|_{s-\frac{3}{2}}^2 .
\end{aligned}
$$

Here the demand $s > 5$ is crucial. Now we get

$$
\begin{aligned}
& \gamma(r, \mathcal{F}_1(r) - \mathcal{F}_1'(0)[r])_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2(r) - \mathcal{F}_2'(0)[r])_{s-2} \\
\leq \;& C\left(\|r\|_{s-\frac{1}{2}}^3 + \alpha(\tau)\|r\|_{s-\frac{3}{2}}^3 \right).
\end{aligned}
\tag{5.20}
$$

4. From the chain rule (5.13) it follows that

$$
\begin{aligned}
& (r, \mathcal{F}(r,\tau))_{s-2,2} \\
= \;& \gamma(r, \mathcal{F}_1(r))_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2(r))_{s-2} \\
& +\gamma \sum_{i,j} (D_i D_j r, \mathcal{F}_1'(r)[D_i D_j r])_{s-2} + \mu\alpha(\tau) \sum_{i,j} (D_i D_j r, \mathcal{F}_2'(r)[D_i D_j r])_{s-2} \\
& +\gamma \sum_{i,j} (D_i D_j r, \mathcal{F}_1''(r)[D_i r, D_j r])_{s-2} + \mu\alpha(\tau) \sum_{i,j} (D_i D_j r, \mathcal{F}_2''(r)[D_i r, D_j r])_{s-2}.
\end{aligned}
\tag{5.21}
$$

We divide the right-hand side into three parts and we estimate these parts separately.

5. Adding (5.18) and (5.20) we get for the first part of (5.21)

$$\begin{aligned}
&\gamma(r, \mathcal{F}_1(r))_{s-2} + \mu\alpha(\tau)(r, \mathcal{F}_2(r))_{s-2} \\
\leq\ & -c_1\|r\|^2_{s-\frac{1}{2}} - \frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|r\|^2_{s-\frac{3}{2}} + C\left(\|r\|^3_{s-\frac{1}{2}} + \alpha(\tau)\|r\|^3_{s-\frac{3}{2}}\right) \\
\leq\ & -c_1\|r\|^2_{s-\frac{1}{2}} - \frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|r\|^2_{s-\frac{3}{2}} + C\delta\left(\|r\|^2_{s-\frac{1}{2}} + \alpha(\tau)\|r\|^2_{s-\frac{3}{2}}\right). \quad (5.22)
\end{aligned}$$

6. For the second part of (5.21) we get from similar arguments

$$\begin{aligned}
&\gamma(D_iD_jr, \mathcal{F}_1'(r)[D_iD_jr])_{s-2} + \mu\alpha(\tau)(D_iD_jr, \mathcal{F}_2'(r)[D_iD_jr])_{s-2} \\
=\ & \gamma(D_iD_jr, \mathcal{F}_1'(0)[D_iD_jr])_{s-2} + \mu\alpha(\tau)(D_iD_jr, \mathcal{F}_2'(0)[D_iD_jr])_{s-2} \\
& +\gamma(D_iD_jr, \left\{\mathcal{F}_1'(r) - \mathcal{F}_1'(0)\right\}[D_iD_jr])_{s-2} \\
& +\mu\alpha(\tau)(D_iD_jr, \left\{\mathcal{F}_2'(r) - \mathcal{F}_2'(0)\right\}[D_iD_jr])_{s-2} \\
\leq\ & -c_1\|D_iD_jr\|^2_{s-\frac{1}{2}} - \frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|D_iD_jr\|^2_{s-\frac{3}{2}} \\
& +C\delta\left(\|D_iD_jr\|^2_{s-\frac{1}{2}} + \alpha(\tau)\|D_iD_jr\|^2_{s-\frac{3}{2}}\right). \quad (5.23)
\end{aligned}$$

In the last step we used analyticity of $\mathcal{F}_1$ near zero in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ and analyticity of $\mathcal{F}_2$ near zero in $\mathbb{H}^{s-\frac{3}{2}}(\mathbb{S}^{N-1})$.

7. Because of Lemma 5.3, there exists a $C > 0$ such that for r near zero in $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1)$ we have $\|\mathcal{F}_1''(r)\|_{X_1} \leq C$, for $X_1 = \mathcal{L}^2(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1) \times \mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1), \mathbb{H}^{s-\frac{7}{2}}(\mathbb{S}^1))$ and $\|\mathcal{F}_2''(r)\|_{X_2} \leq C$, for $X_2 = \mathcal{L}^2(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{S}^1) \times \mathbb{H}^{s-\frac{3}{2}}(\mathbb{S}^1), \mathbb{H}^{s-\frac{5}{2}}(\mathbb{S}^1))$. Therefore, the third part of (5.21) can be estimated as follows:

$$\begin{aligned}
&\gamma(D_iD_jr, \mathcal{F}_1''(r)[D_ir, D_jr])_{s-2} + \mu\alpha(\tau)(D_iD_jr, \mathcal{F}_2''(r)[D_ir, D_jr])_{s-2} \\
\leq\ & C\left(\|r\|_{s+\frac{3}{2}}\|r\|^2_{s+\frac{1}{2}} + \alpha(\tau)\|r\|_{s+\frac{1}{2}}\|r\|^2_{s-\frac{1}{2}}\right) \\
\leq\ & C\left(\|r\|_{s-\frac{1}{2}}\|r\|^2_{s+\frac{3}{2}} + \alpha(\tau)\|r\|_{s-\frac{3}{2}}\|r\|^2_{s+\frac{1}{2}}\right) \\
\leq\ & C\delta\left(\|r\|^2_{s+\frac{3}{2}} + \alpha(\tau)\|r\|^2_{s+\frac{1}{2}}\right). \quad (5.24)
\end{aligned}$$

Here we used the following interpolation inequalities:

$$\|r\|^2_{s+\frac{1}{2}} \leq C\|r\|_{s-\frac{1}{2}}\|r\|_{s+\frac{3}{2}},$$

$$\|r\|^2_{s-\frac{1}{2}} \leq C\|r\|_{s-\frac{3}{2}}\|r\|_{s+\frac{1}{2}}.$$

8. Adding (5.22), (5.23), and (5.24) and using equivalence of the norms $\|\cdot\|_\sigma$ and

$\|\cdot\|_{\sigma-2,2}$ we get

$$\begin{aligned}(r,\mathcal{F}(r,\tau))_{s-2,2} &\leq -c_1\|r\|^2_{s-\frac{1}{2},2}-\frac{(1-\eta)\mu}{2\pi}\alpha(\tau)\|r\|^2_{s-\frac{3}{2},2}\\ &\quad +C\delta\left(\|r\|^2_{s-\frac{1}{2},2}+\alpha(\tau)\|r\|^2_{s-\frac{3}{2},2}\right).\end{aligned} \tag{5.25}$$

If we choose $\delta<\frac{\varepsilon}{C}$, then we get

$$\begin{aligned}(r,\mathcal{F}(r,\tau))_{s-2,2} &\leq -(c_1-\varepsilon)\|r\|^2_{s-\frac{1}{2},2}-\left(\frac{(1-\eta)\mu}{2\pi}-\varepsilon\right)\alpha(\tau)\|r\|^2_{s-\frac{3}{2},2}\\ &\leq -\lambda_0\alpha(\tau)\|r\|^2_{s-\frac{3}{2},2}\leq -\lambda_0\alpha(\tau)\|r\|^2_{s-2,2}.\end{aligned} \tag{5.26}$$

9. Applying Theorem A.1 to (5.26) we obtain a solution r to (5.15) on the entire interval $[0,\tau_{\max})$. The fact that $((\xi,\tau)\mapsto r(\tau)(\xi))\in\mathcal{C}^\infty(\mathbb{S}^1\times(0,\tau_{\max}))$ follows from [60, Ch. 6 Prop. 9, 10], where local existence and uniqueness results for Stokes flow with injection or suction are proved. Hele-Shaw flow can be treated in a similar way. Furthermore, we have $\|r(\tau)\|^2_{s-2,2}\leq y(\tau)$ where $y:[0,\tau_{\max})\to\mathbb{R}$ satisfies
$$\frac{dy}{d\tau}=-2\lambda_0\alpha(\tau)y,$$
with $y(0)=\|r_0\|^2_{s-2,2}$. Solving this ODE we get for $\tau>0$
$$y(\tau)=e^{-2\lambda_0\int_0^\tau\alpha(\tilde{\tau})d\tilde{\tau}}\|r_0\|^2_{s-2,2}.$$
After reintroducing the original time variable by (3.25) we get
$$\begin{aligned}y &= e^{-2\lambda_0\int_0^t\frac{1}{(\alpha(\tilde{t}))^2}d\tilde{t}}\|r_0\|^2_{s-2,2}\\ &= \left(\frac{\mu t}{\pi}+1\right)^{-\frac{2\pi\lambda_0}{\mu}}\|r_0\|^2_{s-2,2}.\end{aligned}$$
This proves the theorem.

□

Define $\mathfrak{M}_1^N$ as in (2.40) and introduce the Hilbert spaces $\mathbb{H}_1^\sigma(\mathbb{S}^{N-1})$ by

$$\mathbb{H}_1^\sigma(\mathbb{S}^{N-1}):=\{r\in\mathbb{H}^\sigma(\mathbb{S}^{N-1}):(r,s)_0=0,\ \forall s\in\mathfrak{S}_0^N\oplus\mathfrak{S}_1^N\}. \tag{5.27}$$

Define on a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ the operator $f_1:\mathcal{U}\to\mathbb{R}\times\mathbb{R}^N$ by

$$f_1(r):=\begin{pmatrix}\int_{\Omega_r}dx-\frac{\sigma_N}{N}\\ \int_{\Omega_r}xdx\end{pmatrix}. \tag{5.28}$$

Let $\mathcal{P}_1:\mathbb{H}^s(\mathbb{S}^{N-1})\to\mathbb{H}_1^s(\mathbb{S}^{N-1})$ be the orthogonal projection onto $\mathbb{H}_1^s(\mathbb{S}^{N-1})$ with respect to the $\mathbb{L}^2(\mathbb{S}^{N-1})$-inner product and define the local analytic bijection $\phi_1:\mathcal{U}\to\mathbb{R}\times\mathbb{R}^N\times$

$\mathbb{H}_1^s(\mathbb{S}^{N-1})$ by

$$\phi_1(r) := \begin{pmatrix} f_1(r) \\ \mathcal{P}_1 r \end{pmatrix}.$$

On a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}_1^s(\mathbb{S}^{N-1})$ we define $\psi_1 : \mathcal{U} \to \mathfrak{M}_1^N$ by

$$\psi_1(\tilde{r}) := \phi_1^{-1}(0, \tilde{r}). \tag{5.29}$$

Lemma 5.7. *Let $s > \frac{N+7}{2}$. For $\tilde{r} \in \mathbb{H}_1^{s+1}(\mathbb{S}^{N-1})$ with $\|\tilde{r}\|_s$ small, we have*

$$D_i \psi_1(\tilde{r}) = \psi_1'(\tilde{r})[D_i \tilde{r}].$$

For $\tilde{r} \in \mathbb{H}_1^{s+2}(\mathbb{S}^{N-1})$ with $\|\tilde{r}\|_s$ small, we have

$$D_i D_j \psi_1(\tilde{r}) = \psi_1'(\tilde{r})[D_i D_j \tilde{r}] + \psi_1''(\tilde{r})[D_i \tilde{r}, D_j \tilde{r}]. \tag{5.30}$$

Proof. In view of the proof of Lemma 5.4 it is sufficient to show that ψ_1 commutes with rotations. If $\tilde{r} \in \mathbb{H}_1^{s+2}(\mathbb{S}^{N-1})$, then we have $\tilde{r} \circ g \in \mathbb{H}_1^{s+2}(\mathbb{S}^{N-1})$ for any rotation $g : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$. Since $\psi_1(\tilde{r}) \in \mathfrak{M}_1^N$ we also have $\psi_1(\tilde{r}) \circ g \in \mathfrak{M}_1^N$. Because rotations and $\mathcal{P}_1$ commute we have $\mathcal{P}_1(\psi_1(\tilde{r}) \circ g) = (\mathcal{P}_1 \psi_1(\tilde{r})) \circ g = \tilde{r} \circ g$. Therefore

$$\psi_1(\tilde{r}) \circ g = \phi_1^{-1}\left(0, \mathcal{P}_1(\psi_1(\tilde{r}) \circ g)\right) = \phi_1^{-1}\left(0, \tilde{r} \circ g\right) = \psi_1(\tilde{r} \circ g).$$

This proves the lemma. □

Now we derive a global existence result for the suction case for $N = 2$. Like in the proof of Theorem 5.6, we get this result from energy estimates. The suction case is more complicated then the injection case, first of all because we need to restrict ourselves to evolutions in $\mathfrak{M}_1^2$ and consider an equivalent problem on $\mathbb{H}_1^s(\mathbb{S}^1)$ given by equation (5.34).

The second complication here is that we need to split up the time interval $[0, \infty)$ in two parts, $[0, \hat{T}]$ and $[\hat{T}, \infty)$. On the first interval, the norm of the solutions that we find might grow up to a value δ'. On the second interval we need an energy estimate, that is sharper than the one that we found on the first interval, in order to obtain exponential decay for solutions to (5.34). For any ratio of $|\mu|$ to γ, a suitable $\hat{T}$ exists, because for large time surface tension dominates suction. For the three-dimensional problem (see Chapter 3) this is not the case, because eigenvalues of the linearisations of the evolution operators do not change in time.

Theorem 5.8. *Let $N = 2$, $\mu < 0$, and take $\lambda_0 \in \left(0, \dfrac{6\gamma}{5\sqrt{5}}\right)$. Suppose that $s > 5$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^1) \cap \mathfrak{M}_1^2$ with $\|r_0\|_s < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0, \tag{5.31}$$

has a solution $r \in \mathcal{C}_w([0, \infty), \mathbb{H}^s(\mathbb{S}^1)) \cap \mathcal{C}_w^1([0, \infty), \mathbb{H}^{s-3}(\mathbb{S}^1))$ that is in $\mathfrak{M}_1^2$ for all t and

$((\xi,\tau)\mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^1\times(0,\infty))$. *Furthermore,*

$$\|r(\tau)\|_{s-2,2} \le Me^{-\lambda_0\tau}\|r_0\|_{s-2,2}. \tag{5.32}$$

Proof. Again, the symbol C is used for a constant that may vary throughout the proof.

1. Note that $-\frac{p_1(k)}{(k^2+1)^{\frac{3}{2}}}$ decreases in k. As a consequence, for $k\ge 2$

$$-\frac{\gamma p_1(k)}{(k^2+1)^{\frac{3}{2}}} \le -\frac{6\gamma}{5\sqrt{5}} < -\lambda_0. \tag{5.33}$$

   Furthermore $\frac{p_2(k)}{(k^2+1)^{\frac{3}{2}}}$ is bounded and $\lim_{\tau\to\infty}\alpha(\tau)=0$. Therefore there exists a $\hat{T}$ such that for $\tau\ge\hat{T}$ and $k\ge 2$

$$\frac{-\gamma p_1(k)+|\mu|\alpha(\tau)p_2(k)}{(k^2+1)^{\frac{3}{2}}} < -\lambda_0.$$

   Choose $K\in\mathbb{N}$ such that for $k>K$ we have $-\gamma p_1(k)+|\mu|p_2(k)<0$ and let $\mathcal{P}_K:\mathbb{L}^2(\mathbb{S}^1)\to\mathbb{L}^2(\mathbb{S}^1)$ be the orthogonal projection with respect to the $\mathbb{L}^2(\mathbb{S}^1)$-inner product onto the orthoplement of $\bigoplus_{k=0}^K \mathfrak{S}_k^2$. Define $c_1>0$ and $c_2>0$ by

$$c_1 := \inf_{k\ge2,\tau\ge\hat{T}} \frac{\gamma p_1(k)-|\mu|\alpha(\tau)p_2(k)}{\left(k^2+1\right)^{\frac{3}{2}}} = \inf_{k\ge2}\frac{\gamma p_1(k)-|\mu|\alpha(\hat{T})p_2(k)}{\left(k^2+1\right)^{\frac{3}{2}}} > \lambda_0$$

   and

$$c_2 := \inf_{k>K,\tau\ge0} \frac{\gamma p_1(k)-|\mu|\alpha(\tau)p_2(k)}{\left(k^2+1\right)^{\frac{3}{2}}} = \inf_{k>K}\frac{\gamma p_1(k)-|\mu|p_2(k)}{\left(k^2+1\right)^{\frac{3}{2}}}.$$

   The positivity of c_2 follows from the fact that $\frac{\gamma p_1(k)-|\mu|p_2(k)}{(k^2+1)^{\frac{3}{2}}}$ converges to γ if k goes to infinity. Define $\varepsilon:=\min\{c_1-\lambda_0,c_2\}$.

2. Assume for the moment that r satisfies (5.31). Then $\tilde{r}:=\mathcal{P}_1 r$ satisfies

$$\frac{\partial\tilde{r}}{\partial\tau} = \mathcal{P}_1\mathcal{F}(\psi_1(\tilde{r}),\tau). \tag{5.34}$$

   First we will prove solvability of this equation, finding estimates for $(\tilde{r},\mathcal{P}_1\mathcal{F}(\psi_1(\tilde{r}),\tau))_{s-2,2}$ for $\tilde{r}\in\mathbb{H}_1^{s+3}(\mathbb{S}^1)$ and $\|\tilde{r}\|_s<\delta'$ with δ' small enough.

3. Introduce on a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}_1^s(\mathbb{S}^1)$ the operators $\tilde{\mathcal{F}}_1:\mathcal{U}\to\mathbb{H}_1^{s-3}(\mathbb{S}^1)$ and $\tilde{\mathcal{F}}_2:\mathcal{U}\to\mathbb{H}_1^{s-1}(\mathbb{S}^1)$ by $\tilde{\mathcal{F}}_1:=\mathcal{P}_1\circ\mathcal{F}_1\circ\psi_1$ and $\tilde{\mathcal{F}}_2:=\mathcal{P}_1\circ\mathcal{F}_2\circ\psi_1$. Since these operators are compositions of analytic operators, they are analytic themselves. Because $\psi_1'(0)$ is the identity on $\mathbb{H}_1^s(\mathbb{S}^1)$ (see Corollary 2.20) we have for all $\tilde{r}\in\mathbb{H}_1^s(\mathbb{S}^1)$

$$\tilde{\mathcal{F}}_k'(0)[\tilde{r}] = \mathcal{F}_k'(0)[\tilde{r}],$$

for $k = 1, 2$. As a result,

$$\begin{aligned}
& \gamma(\tilde{r}, \tilde{\mathcal{F}}_1'(0)[\tilde{r}])_{s-2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2'(0)[\tilde{r}])_{s-2} \\
= \; & \sum_{k \le K} (k^2+1)^{s-2+\frac{3}{2}} \frac{-\gamma p_1(k) + |\mu|\alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} \tilde{r}_{kj}^2 \\
& + \sum_{k > K} (k^2+1)^{s-2+\frac{3}{2}} \frac{-\gamma p_1(k) + |\mu|\alpha(\tau) p_2(k)}{(k^2+1)^{\frac{3}{2}}} \tilde{r}_{kj}^2 \\
\le \; & C\|\tilde{r}\|_0^2 - c_2 \|\mathcal{P}_K \tilde{r}\|_{s-\frac{1}{2}}^2 \\
= \; & C\|\tilde{r}\|_0^2 + c_2 \|(\mathcal{I} - \mathcal{P}_K)\tilde{r}\|_{s-\frac{1}{2}}^2 - c_2 \|\tilde{r}\|_{s-\frac{1}{2}}^2 \\
\le \; & C\|\tilde{r}\|_0^2 - c_2 \|\tilde{r}\|_{s-\frac{1}{2}}^2 .
\end{aligned} \tag{5.35}$$

Here we used the fact that $\mathcal{I} - \mathcal{P}_K : \mathbb{L}^2(\mathbb{S}^1) \to \mathbb{H}^{s-\frac{1}{2}}(\mathbb{S}^1)$ is bounded.

4. For the suction problem we have $\alpha(\tau) \le 1$. Hence, the nonlinear parts can be estimated in the following way:

$$\gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}) - \tilde{\mathcal{F}}_1'(0)[\tilde{r}])_{s-2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}) - \tilde{\mathcal{F}}_2'(0)[\tilde{r}])_{s-2} \le C\|\tilde{r}\|_{s-\frac{1}{2}}^3 . \tag{5.36}$$

Here we used the analyticity of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$, as we did for $\mathcal{F}_1$ and $\mathcal{F}_2$ in the proof of Theorem 5.6.

5. Because of Lemma 5.7 and the fact that $\mathcal{P}_1$ commutes with rotations, the chain rule holds for $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ as well:

$$\begin{aligned}
& \gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2} \\
= \; & \gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}))_{s-2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}))_{s-2} \\
& + \gamma \sum_{i,j} (D_i D_j \tilde{r}, \tilde{\mathcal{F}}_1'(\tilde{r})[D_i D_j \tilde{r}])_{s-2} + \mu\alpha(\tau) \sum_{i,j} (D_i D_j \tilde{r}, \tilde{\mathcal{F}}_2'(\tilde{r})[D_i D_j \tilde{r}])_{s-2} \\
& + \gamma \sum_{i,j} (D_i D_j \tilde{r}, \tilde{\mathcal{F}}_1''(\tilde{r})[D_i \tilde{r}, D_j \tilde{r}])_{s-2} + \mu\alpha(\tau) \sum_{i,j} (D_i D_j \tilde{r}, \tilde{\mathcal{F}}_2''(\tilde{r})[D_i \tilde{r}, D_j \tilde{r}])_{s-2} .
\end{aligned} \tag{5.37}$$

The right-hand side consists of three parts that will be estimated separately on both intervals $[0, \hat{T}]$ and $[\hat{T}, \infty)$. We start with $[0, \hat{T}]$.

6. From (5.35) and (5.36), we have for the first part

$$\begin{aligned}
& \gamma(\tilde{r}, \tilde{\mathcal{F}}_1(\tilde{r}))_{s-2} + \mu\alpha(\tau)(\tilde{r}, \tilde{\mathcal{F}}_2(\tilde{r}))_{s-2} \\
\le \; & C\|\tilde{r}\|_0^2 - c_2 \|\tilde{r}\|_{s-\frac{1}{2}}^2 + C\delta' \|\tilde{r}\|_{s-\frac{1}{2}}^2 .
\end{aligned} \tag{5.38}$$

7. The second part can be treated in the same way as in the proof of Theorem 5.6. We

use (5.35), the boundedness of $\alpha(\tau)$, and the analyticity of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ to obtain

$$\begin{aligned}
&\gamma(D_iD_j\tilde{r},\tilde{\mathcal{F}}_1'(\tilde{r})[D_iD_j\tilde{r}])_{s-2}+\mu\alpha(\tau)(D_iD_j\tilde{r},\tilde{\mathcal{F}}_2'(\tilde{r})[D_iD_j\tilde{r}])_{s-2}\\
=\;&\gamma(D_iD_j\tilde{r},\tilde{\mathcal{F}}_1'(0)[D_iD_j\tilde{r}])_{s-2}+\mu\alpha(\tau)(D_iD_j\tilde{r},\tilde{\mathcal{F}}_2'(0)[D_iD_j\tilde{r}])_{s-2}\\
&+\gamma(D_iD_j\tilde{r},\{\tilde{\mathcal{F}}_1'(\tilde{r})-\tilde{\mathcal{F}}_1'(0)\}[D_iD_j\tilde{r}])_{s-2}\\
&+\mu\alpha(\tau)(D_iD_j\tilde{r},\{\tilde{\mathcal{F}}_2'(\tilde{r})-\tilde{\mathcal{F}}_2'(0)\}[D_iD_j\tilde{r}])_{s-2}\\
\leq\;&C\|D_iD_j\tilde{r}\|_0^2-c_2\|D_iD_j\tilde{r}\|_{s-\frac{1}{2}}^2+C\delta'\|D_iD_j\tilde{r}\|_{s-\frac{1}{2}}^2. \qquad (5.39)
\end{aligned}$$

Here, we also used the fact that $D_iD_j\tilde{r}\in\mathbb{H}_1^s(\mathbb{S}^1)$ if $\tilde{r}\in\mathbb{H}_1^{s+3}(\mathbb{S}^1)$. This follows from Lemma 5.5.

8. The third part is treated in the same way as in the proof of Theorem 5.6 as well. Using boundedness of $\alpha(\tau)$ and analyticity of $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ we find

$$\gamma(D_iD_j\tilde{r},\tilde{\mathcal{F}}_1''(\tilde{r})[D_i\tilde{r},D_j\tilde{r}])_{s-2}+\mu\alpha(\tau)(D_iD_j\tilde{r},\tilde{\mathcal{F}}_2''(\tilde{r})[D_i\tilde{r},D_j\tilde{r}])_{s-2}\leq C\delta'\|\tilde{r}\|_{s+\frac{3}{2}}^2. \qquad (5.40)$$

9. Combining (5.38), (5.39), and (5.40) and using equivalence of the norms $\|\cdot\|_{s+\frac{3}{2}}$ and $\|\cdot\|_{s-\frac{1}{2},2}$ we get on the interval $[0,\hat{T}]$

$$\begin{aligned}
&\gamma(\tilde{r},\tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2}+\mu\alpha(\tau)(\tilde{r},\tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2}\\
\leq\;&C\|\tilde{r}\|_{0,2}^2-c_2\|\tilde{r}\|_{s-\frac{1}{2},2}^2+C\delta'\|\tilde{r}\|_{s-\frac{1}{2},2}^2.
\end{aligned}$$

If we take $\delta'<\frac{\varepsilon}{C}$, then we get

$$\begin{aligned}
&\gamma(\tilde{r},\tilde{\mathcal{F}}_1(\tilde{r}))_{s-2,2}+\mu\alpha(\tau)(\tilde{r},\tilde{\mathcal{F}}_2(\tilde{r}))_{s-2,2}\\
\leq\;&C\|\tilde{r}\|_{0,2}^2\leq C\|\tilde{r}\|_{s-2,2}^2. \qquad (5.41)
\end{aligned}$$

Define $\tilde{r}_0:=\mathcal{P}_1r_0$, take $\delta<e^{-C\hat{T}}\delta'$ and assume that $\|\tilde{r}_0\|_{s-2,2}\leq\delta$. By Theorem A.1 there exists a solution to (5.34) on the interval $[0,\hat{T}]$ that satisfies

$$\|\tilde{r}(\tau)\|_{s-2,2}\leq e^{C\tau}\|\tilde{r}_0\|_{s-2,2},$$

such that

$$\|\tilde{r}(\hat{T})\|_{s-2,2}\leq e^{C\hat{T}}\delta<\delta'.$$

Smoothness on $(0,\hat{T}]$ follows again from [60, Prop. 9, 10].

10. Now we treat the interval $[\hat{T},\infty)$. Again we consider the chain rule and distinguish between three parts in (5.37). Because of the boundedness of α we have for

the first part

$$\begin{aligned}
&\gamma(\tilde r, \tilde{\mathcal{F}}_1(\tilde r))_{s-2} + \mu\alpha(\tau)(\tilde r, \tilde{\mathcal{F}}_2(\tilde r))_{s-2} \\
\leq\ & \gamma(\tilde r, \tilde{\mathcal{F}}_1'(0)[\tilde r])_{s-2} + \mu\alpha(\tau)(\tilde r, \tilde{\mathcal{F}}_2'(0)[\tilde r])_{s-2} + C\|\tilde r\|^3_{s-\frac12} \\
=\ & \sum_{k\geq 2}(k^2+1)^{s-2+\frac32}\frac{-\gamma p_1(k) + |\mu|\alpha(\tau)p_2(k)}{(k^2+1)^{\frac32}}\tilde r_{kj}^2 + C\|\tilde r\|^3_{s-\frac12} \\
\leq\ & -c_1\|\tilde r\|^2_{s-\frac12} + C\delta'\|\tilde r\|^2_{s-\frac12}. \qquad (5.42)
\end{aligned}$$

Note that in the summation we start counting from $k = 2$ because $\tilde r \in \mathbb{H}^s_1(\mathbb{S}^1)$.

11. For the second part we use the same strategy as for the first time interval to obtain

$$\begin{aligned}
&\gamma(D_iD_j\tilde r, \tilde{\mathcal{F}}_1'(\tilde r)[D_iD_j\tilde r])_{s-2} + \mu\alpha(\tau)(D_iD_j\tilde r, \tilde{\mathcal{F}}_2'(\tilde r)[D_iD_j\tilde r])_{s-2} \\
&\qquad \leq -c_1\|D_iD_j\tilde r\|^2_{s-\frac12} + C\delta'\|D_iD_j\tilde r\|^2_{s-\frac12}. \qquad (5.43)
\end{aligned}$$

12. For the third part, we get exactly the same result as for the first time interval, cf. (5.40).

13. Adding (5.42), (5.43), and (5.40) and using equivalence of the norms $\|\cdot\|_{s+\frac32}$ and $\|\cdot\|_{s-\frac12,2}$ we get

$$\gamma(\tilde r, \tilde{\mathcal{F}}_1(\tilde r))_{s-2,2} + \mu\alpha(\tau)(\tilde r, \tilde{\mathcal{F}}_2(\tilde r))_{s-2,2} \leq -c_1\|\tilde r\|^2_{s-\frac12,2} + C\delta'\|\tilde r\|^2_{s-\frac12,2}.$$

Taking $\delta' < \frac{\varepsilon}{C}$ we find

$$\gamma(\tilde r, \tilde{\mathcal{F}}_1(\tilde r))_{s-2,2} + \mu\alpha(\tau)(\tilde r, \tilde{\mathcal{F}}_2(\tilde r))_{s-2,2} \leq -\lambda_0\|\tilde r\|^2_{s-\frac12,2} \leq -\lambda_0\|\tilde r\|^2_{s-2,2}.$$

Applying Theorem A.1 again, we extend the solution $\tilde r$ that we found on $[0, \hat T]$ to $[\hat T, \infty)$, such that for $\tau \in [\hat T, \infty)$

$$\|\tilde r(\tau)\|_{s-2,2} \leq e^{-\lambda_0(\tau-\hat T)}\|\tilde r(\hat T)\|_{s-2,2}.$$

Combining the results on both intervals, we get existence of an $M' > 0$ independent of $\tilde r(0)$ such that for any $\tau \in [0, \infty)$

$$\|\tilde r(\tau)\|_{s-2,2} \leq M'e^{-\lambda_0\tau}\|\tilde r(0)\|_{s-2,2}.$$

Define

$$r = \psi_1(\tilde r).$$

From the smoothness of ψ_1 and the fact that $\psi_1(0) = 0$ we see that there exists an $M > 0$ such that if r_0 is small enough, then r is a solution to (5.31) that satisfies (5.32).

□

Theorem 5.9.

- *If $\lambda_0 \in \left(0, \frac{\mu}{\pi}\right)$ in Theorem 5.6, then the results still hold.*
- *If $\lambda_0 \in (0, 6\gamma)$ in Theorem 5.8, then the results still hold.*

Proof. Let us introduce the inner product

$$(r, \tilde{r})_s^* := \sum_{k,j} (\theta k^2 + 1)^s r_{kj} \tilde{r}_{kj} \tag{5.44}$$

in the proof of Theorem 5.6, where $\theta > 0$ is small. This inner product is equivalent to $(\cdot, \cdot)_s$ for all $\theta > 0$. Instead of (5.19) we have

$$-\frac{p_2(k)}{\sqrt{\theta k^2 + 1}} \leq -\frac{1}{\pi},$$

for θ small enough. Therefore, if we replace $(\cdot, \cdot)_\sigma$ by $(\cdot, \cdot)_\sigma^*$, $\| \cdot \|_\sigma$ by $\| \cdot \|_\sigma^*$, and $\frac{\mu}{2\pi}$ by $\frac{\mu}{\pi}$ in (5.18), (5.22), (5.23), (5.25), and (5.26), then the proof is still correct.

In the proof of Theorem 5.8 we use the inner product

$$(r, \tilde{r})_s^* := \sum_{k,j} \zeta_k^s r_{kj} \tilde{r}_{kj},$$

with $\zeta_0 = \zeta_1 = \zeta_2 = 1$ and $\zeta_k = \theta k^2 + 1$ for $k \geq 3$ where $\theta > 0$ is small. This allows us to replace (5.33) by the inequality

$$-\frac{\gamma p_1(k)}{\zeta_k^{\frac{3}{2}}} \leq -6\gamma, \quad k \geq 2,$$

if $\theta > 0$ is small enough. □

Now we derive a theorem for global existence for the higher-dimensional case. For injection, we have to deal with the problem that eigenvalues of the linearisation corresponding to spherical harmonics of degree zero and one go to zero for large time. In order to deal with this, we use the bijection ϕ_1 near the origin between $\mathbb{H}^s(\mathbb{S}^{N-1})$ and $\mathbb{R} \times \mathbb{R}^N \times \mathbb{H}_1^s(\mathbb{S}^{N-1})$ (see Figure 5.2) and consider the evolution of $\mathcal{P}_1 r$ on $\mathbb{H}_1^s(\mathbb{S}^{N-1})$ and the evolution of $f_1(r)$, the zeroth and first Richardson moments, separately. We write down an equation for $\mathcal{P}_1 r$ and find an energy estimate for its evolution operator. This equation differs from the one that we found in the proof of Theorem 5.8 because we also allow evolutions that are not in $\mathfrak{M}_1^N$. We use the fact that the zeroth and first Richardson moments as function of time are known beforehand. For the suction problem for $N \geq 4$ we do not get any global existence result because when t approaches T (or when τ approaches $\tau_{\max}$), more and more eigenvalues of the linearised evolution equation become positive. In other words: For large t, $\mathcal{F}_2$ dominates $\mathcal{F}_1$. Suction can no longer be controlled by surface tension.

Theorem 5.10. *Let $N \geq 4$ and $\mu > 0$. Suppose that $s > \frac{N+8}{2}$. There exists a $\delta > 0$ and an*

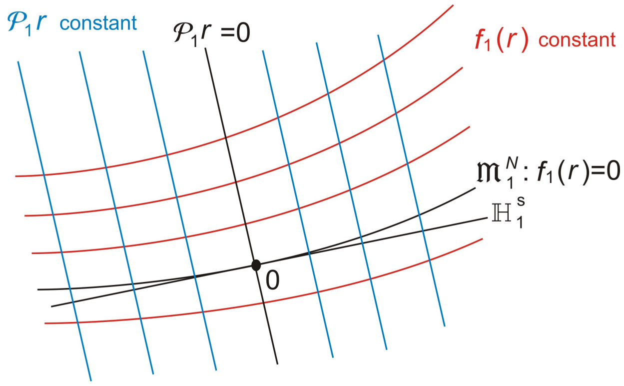


Figure 5.2: *A sketch of $\mathbb{H}^s(\mathbb{H}^{N-1})$ and the local bijection ϕ_1 near zero. Each element on $\mathfrak{M}_1^N$ is mapped to zero by f_1. The red lines are manifolds on which f_1 takes some other values. The blue lines, that are orthogonal to $\mathbb{H}_1^s(\mathbb{S}^{N-1})$, denote subsets on which $\mathcal{P}_1 r$ is constant.*

$M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with $\|r_0\|_s < \delta$, then the problem

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0, \tag{5.45}$$

has a solution $r \in \mathcal{C}_w([0,\infty), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}_w^1([0,\infty), \mathbb{H}^{s-3}(\mathbb{S}^{N-1}))$. Furthermore, $((\xi,\tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0,\infty))$. If we regard r as a function of t (where the relation between t and τ is given by (5.7)), then

$$\|r(t)\|_s \le M \left(\frac{\mu N t}{\sigma_N} + 1 \right)^{-1} \|r_0\|_s.$$

Proof. 1. Introduce the number $c_1 > 0$ by

$$c_1 := \inf_{k \ge 2} \frac{\gamma p_1(k)}{(k^2+1)^{\frac{3}{2}}}.$$

Choose $\lambda_0 \in (0, \frac{c_1}{2})$ and define $\varepsilon := \frac{c_1}{2} - \lambda_0$.

2. From the calculations in the proof of Lemma 3.14 we know that if $t \mapsto \Omega_{R(t)}$ solves (3.1)-(3.4) then the geometric centre of $\Omega_{R(t)}$ is constant and its volume increases linearly with rate μ. From this and (5.28) it follows that solutions r to (5.45) satisfy

$$f_1(r(\tau)) = \left(\frac{V_0}{\alpha(\tau)^N}, \frac{1}{\alpha(\tau)^{N+1}} m_0 \right)^T =: (V_\tau, m_\tau)^T, \tag{5.46}$$

where

$$(V_0, m_0)^T := f_1(r_0).$$

For notational convenience we introduce $q_\tau := (V_\tau, m_\tau)^T$. Assume for the moment that r satisfies (5.45). Then $\tilde{r} := \mathcal{P}_1 r$ satisfies

$$\frac{\partial \tilde{r}}{\partial \tau} = \mathcal{P}_1 \mathcal{F}\left(\phi_1^{-1}(q_\tau, \tilde{r}), \tau\right). \tag{5.47}$$

First we prove solvability of this equation, finding estimates for $\left(\tilde{r}, \mathcal{P}_1 \mathcal{F}\left(\phi_1^{-1}(q_\tau, \tilde{r}), \tau\right)\right)_{s-2,2}$, assuming that $|q_0|$ is small, $\tilde{r} \in \mathbb{H}_1^{s+3}(\mathbb{S}^{N-1})$ and $\|\tilde{r}\|_s < \delta$, with δ small enough.

3. Since $\tilde{r} \in \mathbb{H}_1^s(\mathbb{S}^{N-1})$ we have

$$\begin{aligned}\gamma(\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-2} + \mu\alpha(\tau)^{3-N}(\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-2} &\leq \gamma(\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-2} \\ &\leq -c_1\|\tilde{r}\|^2_{s-\frac{1}{2}}.\end{aligned} \tag{5.48}$$

4. Because of (5.29) and local Lipschitz continuity of $\mathcal{F}_k \circ \phi_1^{-1}$, for $k = 1, 2$, we have

$$\begin{aligned}&\|\mathcal{P}_1\mathcal{F}_k\left(\phi_1^{-1}(q_\tau, \tilde{r})\right) - \mathcal{P}_1\mathcal{F}_k(\psi_1(\tilde{r}))\|_{s-\frac{7}{2}} \\ =\ &\|\mathcal{P}_1\mathcal{F}_k\left(\phi_1^{-1}(q_\tau, \tilde{r})\right) - \mathcal{P}_1\mathcal{F}_k(\phi_1^{-1}(0, \tilde{r}))\|_{s-\frac{7}{2}} \\ \leq\ & C|q_\tau|.\end{aligned} \tag{5.49}$$

Since $\psi_1'(0)$ is the identity on $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$, the restriction of $\mathcal{F}_k'(0)$ to $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ is the Fréchet derivative around zero of the analytic mapping $\mathcal{P}_1 \circ \mathcal{F}_k \circ \psi_1$ on $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$. As a consequence,

$$\|\mathcal{P}_1\mathcal{F}_k(\psi_1(\tilde{r})) - \mathcal{F}_k'(0)[\tilde{r}]\|_{s-\frac{7}{2}} \leq C\|\tilde{r}\|^2_{s-\frac{1}{2}}. \tag{5.50}$$

Combining (5.49) and (5.50) we get the following estimate:

$$\begin{aligned}&\gamma\left\{\left(\tilde{r}, \mathcal{P}_1\mathcal{F}_1\left(\phi_1^{-1}(q_\tau, \tilde{r})\right)\right)_{s-2} - (\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-2}\right\} \\ &+\mu\alpha(\tau)^{3-N}\left\{\left(\tilde{r}, \mathcal{P}_1\mathcal{F}_2\left(\phi_1^{-1}(q_\tau, \tilde{r})\right)\right)_{s-2} - (\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-2}\right\} \\ \leq\ & C\Big(|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2}} + \alpha(\tau)^{3-N}|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2}} + \|\tilde{r}\|^3_{s-\frac{1}{2}} + \alpha(\tau)^{3-N}\|\tilde{r}\|^3_{s-\frac{1}{2}}\Big) \\ \leq\ & C\Big(|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2}} + \|\tilde{r}\|^3_{s-\frac{1}{2}}\Big).\end{aligned} \tag{5.51}$$

Here we used the fact that $\alpha(\tau)^{3-N} \leq 1$.

5. From the chain rule (5.13) we get

$$\begin{aligned}&\left(\tilde{r}, \mathcal{P}_1 \mathcal{F}\left(\phi_1^{-1}\left(q_\tau, \tilde{r}\right), \tau\right)\right)_{s-2,2} \\ =\;& \gamma\left(F_1+G_1+H_1\right)+\mu \alpha(\tau)^{3-N}\left(F_2+G_2+H_2\right), \qquad (5.52)\end{aligned}$$

where for $k=1,2$

$$\begin{aligned}F_k &= \left(\tilde{r}, \mathcal{P}_1 \mathcal{F}_k\left(\phi_1^{-1}\left(q_\tau, \tilde{r}\right)\right)\right)_{s-2}, \\ G_k &= \sum_{i,j}\left(D_i D_j \tilde{r}, \mathcal{P}_1 \mathcal{F}_k^{\prime}\left(\phi_1^{-1}\left(q_\tau, \tilde{r}\right)\right)\left[D_i D_j \phi_1^{-1}(q_\tau, \tilde{r})\right]\right)_{s-2}, \\ H_k &= \sum_{i,j}\left(D_i D_j \tilde{r}, \mathcal{P}_1 \mathcal{F}_k^{\prime\prime}\left(\phi_1^{-1}\left(q_\tau, \tilde{r}\right)\right)\left[D_i \phi_1^{-1}(q_\tau, \tilde{r}), D_j \phi_1^{-1}(q_\tau, \tilde{r})\right]\right)_{s-2}.\end{aligned}$$

We will estimate the terms containing F_k, G_k, and H_k separately.

6. It follows from (5.48) and (5.51) that

$$\begin{aligned}\gamma F_1+\mu \alpha(\tau)^{3-N} F_2 &\leq -c_1\|\tilde{r}\|_{s-\frac{1}{2}}^2+C|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2}}+C\|\tilde{r}\|_{s-\frac{1}{2}}^3 \\ &\leq -c_1\|\tilde{r}\|_{s-\frac{1}{2}}^2+C|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2}}+C \delta\|\tilde{r}\|_{s-\frac{1}{2}}^2. \qquad (5.53)\end{aligned}$$

7. Now we find an estimate for the terms of G_1. Because of (5.29) we have

$$G_1=\sum_{i,j}(D_i D_j \tilde{r}, I_{ij}+J_{ij}+K_{ij})_{s-2}, \qquad (5.54)$$

where

$$\begin{aligned}I_{ij} &= \mathcal{P}_1 \mathcal{F}_1^{\prime}\left(\phi_1^{-1}\left(q_\tau, \tilde{r}\right)\right)\left[D_i D_j \phi_1^{-1}(q_\tau, \tilde{r})\right]-\mathcal{P}_1 \mathcal{F}_1^{\prime}\left(\phi_1^{-1}(0, \tilde{r})\right)\left[D_i D_j \phi_1^{-1}(0, \tilde{r})\right], \\ J_{ij} &= \mathcal{P}_1 \mathcal{F}_1^{\prime}\left(\psi_1(\tilde{r})\right)\left[\psi_1^{\prime}(\tilde{r})[D_i D_j \tilde{r}]\right], \\ K_{ij} &= \mathcal{P}_1 \mathcal{F}_1^{\prime}\left(\psi_1(\tilde{r})\right)\left[\psi_1^{\prime\prime}(\tilde{r})[D_i \tilde{r}, D_j \tilde{r}]\right].\end{aligned}$$

Here we used Lemma 5.7. Because $\|\tilde{r}\|_s$ is small it follows from interpolation inequalities that

$$\begin{aligned}\|D_i D_j \psi_1(\tilde{r})\|_{s-\frac{1}{2}} &\leq \|\psi_1^{\prime}(\tilde{r})[D_i D_j \tilde{r}]\|_{s-\frac{1}{2}}+\|\psi_1^{\prime\prime}(\tilde{r})[D_i \tilde{r}, D_j \tilde{r}]\|_{s-\frac{1}{2}} \\ &\leq C\left(\|\tilde{r}\|_{s+\frac{3}{2}}+\|\tilde{r}\|_{s+\frac{1}{2}}^2\right) \leq C\left(\|\tilde{r}\|_{s+\frac{3}{2}}+\|\tilde{r}\|_{s-\frac{1}{2}}\|\tilde{r}\|_{s+\frac{3}{2}}\right) \\ &\leq C\|\tilde{r}\|_{s+\frac{3}{2}}, \qquad (5.55)\end{aligned}$$

Note that $\phi_1^{-1}(q_\tau, \tilde{r})-\phi_1^{-1}(0, \tilde{r}) \in \mathfrak{S}_0^N \oplus \mathfrak{S}_1^N$. Since ϕ_1 is a local analytic bijection between $\mathbb{H}^s(\mathbb{S}^{N-1})$ and $\mathbb{R}^{N+1}\times\mathbb{H}^s_1(\mathbb{S}^{N-1})$ we have

$$\|\phi_1^{-1}(q_\tau,\tilde r)-\phi_1^{-1}(0,\tilde r)\|_{s+\frac{3}{2}}\le C\|\phi_1^{-1}(q_\tau,\tilde r)-\phi_1^{-1}(0,\tilde r)\|_s\le C|q_\tau|. \tag{5.56}$$

Making use of Lipschitz continuity of $\mathcal{F}'_k\circ\phi_1^{-1}$, (5.55), (5.56), and the fact that $(q_\tau,\tilde r)^T$ is small in $\mathbb{R}^{N+1}\times\mathbb{H}^s_1(\mathbb{S}^{N-1})$ we get

$$\begin{aligned}\|I_{ij}\|_{s-\frac{7}{2}} &\le \left\|\left\{\mathcal{P}_1\mathcal{F}'_1\left(\phi_1^{-1}(q_\tau,\tilde r)\right)-\mathcal{P}_1\mathcal{F}'_1\left(\phi_1^{-1}(0,\tilde r)\right)\right\}\left[D_iD_j\psi_1(\tilde r)\right]\right\|_{s-\frac{7}{2}}\\ &\quad+\left\|\mathcal{P}_1\mathcal{F}'_1\left(\phi_1^{-1}(q_\tau,\tilde r)\right)\left[D_iD_j\left(\phi_1^{-1}(q_\tau,\tilde r)-\phi_1^{-1}(0,\tilde r)\right)\right]\right\|_{s-\frac{7}{2}}\\ &\le C|q_\tau|\|D_iD_j\psi_1(\tilde r)\|_{s-\frac{1}{2}}+C|q_\tau|\\ &\le C|q_\tau|\|\tilde r\|_{s+\frac{3}{2}}+C|q_\tau|.\end{aligned}$$

As a result

$$(D_iD_j\tilde r,I_{ij})_{s-2}\le C|q_\tau|\|\tilde r\|^2_{s+\frac{3}{2}}+C|q_\tau|\|\tilde r\|_{s+\frac{3}{2}}. \tag{5.57}$$

Because $\mathcal{F}'_1(0)$ is the Fréchet derivative at zero of the local analytic operator $\mathcal{P}_1\circ\mathcal{F}_1\circ\psi_1$,

$$\begin{aligned}&(D_iD_j\tilde r,J_{ij})_{s-2}\\ \le\ &\gamma(D_iD_j\tilde r,\mathcal{F}'_1(0)[D_iD_j\tilde r])_{s-2}\\ &+\gamma(D_iD_j\tilde r,\mathcal{P}_1\mathcal{F}'_1(\psi_1(\tilde r))[\psi'_1(\tilde r)[D_iD_j\tilde r]]-\mathcal{F}'_1(0)[D_iD_j\tilde r])_{s-2}\\ \le\ &-c_1\|D_iD_j\tilde r\|^2_{s-\frac{1}{2}}+C\|\tilde r\|_{s-\frac{1}{2}}\|\tilde r\|^2_{s+\frac{3}{2}}\\ \le\ &-c_1\|D_iD_j\tilde r\|^2_{s-\frac{1}{2}}+C\delta\|\tilde r\|^2_{s+\frac{3}{2}}.\end{aligned} \tag{5.58}$$

There exists a $C>0$, such that for $\tilde r$ near the origin in $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$ we have $\|\mathcal{P}_1\circ\mathcal{F}'_1(\psi_1(\tilde r))\circ\psi''_1(\tilde r)\|_X\le C$ for $X=\mathcal{L}^2(\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})\times\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1}),\mathbb{H}_1^{s-\frac{7}{2}}(\mathbb{S}^{N-1}))$. Therefore we have

$$\|K_{ij}\|_{s-\frac{7}{2}}\le C\|\tilde r\|^2_{s+\frac{1}{2}}.$$

By an interpolation inequality we have

$$(D_iD_j\tilde r,K_{ij})_{s-2}\le C\|\tilde r\|_{s+\frac{3}{2}}\|\tilde r\|^2_{s+\frac{1}{2}}\le C\|\tilde r\|_{s-\frac{1}{2}}\|\tilde r\|^2_{s+\frac{3}{2}}\le C\delta\|\tilde r\|^2_{s+\frac{3}{2}}. \tag{5.59}$$

Adding (5.57), (5.58), and (5.59) we get

$$\begin{aligned}&(D_iD_j\tilde r,I_{ij}+J_{ij}+K_{ij})_{s-2}\\ \le\ &-c_1\|D_iD_j\tilde r\|^2_{s-\frac{1}{2}}+C\Big(\delta\|\tilde r\|^2_{s+\frac{3}{2}}+|q_\tau|\|\tilde r\|_{s+\frac{3}{2}}+|q_\tau|\|\tilde r\|^2_{s+\frac{3}{2}}\Big).\end{aligned}$$

For the terms of G_2 we get from similar arguments

$$\begin{aligned}&\left(D_iD_j\tilde{r}, \mathcal{P}_1\mathcal{F}_2'\left(\phi_1^{-1}(q_\tau,\tilde{r})\right)\left[D_iD_j\phi_1^{-1}(q_\tau,\tilde{r})\right]\right)_{s-2}\\ \leq\ & C\left(\delta\|\tilde{r}\|^2_{s+\frac{1}{2}} + |q_\tau|\|\tilde{r}\|_{s+\frac{1}{2}} + |q_\tau|\|\tilde{r}\|^2_{s+\frac{1}{2}}\right).\end{aligned}$$

Here, we used the estimate $(D_iD_j\tilde{r}, \mathcal{P}_1\mathcal{F}_2'(0)[D_iD_j\tilde{r}])_{s-2} \leq 0$ that follows from (5.10). Because $\alpha(\tau)^{3-N} \leq 1$, we have

$$\begin{aligned}&\gamma G_1 + \mu\alpha(\tau)^{3-N}G_2\\ \leq\ & \sum_{i,j} -c_1\|D_iD_j\tilde{r}\|^2_{s-\frac{1}{2}} + C\left(\delta\|\tilde{r}\|^2_{s+\frac{3}{2}} + |q_\tau|\|\tilde{r}\|_{s+\frac{3}{2}} + |q_\tau|\|\tilde{r}\|^2_{s+\frac{3}{2}}\right).\end{aligned} \tag{5.60}$$

8. Note that

$$\|D_i\phi_1^{-1}(0,\tilde{r})\|_{s-\frac{1}{2}} = \|D_i\psi_1(\tilde{r})\|_{s-\frac{1}{2}} = \|\psi_1'(\tilde{r})[D_i\tilde{r}]\|_{s-\frac{1}{2}} \leq C\|\tilde{r}\|_{s+\frac{1}{2}}.$$

Combining this and (5.56) it follows that

$$\|D_i\phi_1^{-1}(q_\tau,\tilde{r})\|_{s-\frac{1}{2}} \leq C\left(\|\tilde{r}\|_{s+\frac{1}{2}} + |q_\tau|\right).$$

Consequently,

$$\begin{aligned}&\gamma H_1 + \mu\alpha(\tau)^{3-N}H_2\\ \leq\ & \sum_{i,j} C\|\tilde{r}\|_{s+\frac{3}{2}}\|D_i\phi_1^{-1}(q_\tau,\tilde{r})\|_{s-\frac{1}{2}}\|D_j\phi_1^{-1}(q_\tau,\tilde{r})\|_{s-\frac{1}{2}}\\ \leq\ & C\|\tilde{r}\|_{s+\frac{3}{2}}\left(\|\tilde{r}\|^2_{s+\frac{1}{2}} + |q_\tau|^2\right)\\ \leq\ & C\delta\|r\|^2_{s+\frac{3}{2}} + C|q_\tau|^2\|\tilde{r}\|_{s+\frac{3}{2}}.\end{aligned} \tag{5.61}$$

Again we used an interpolation inequality.

9. Adding (5.53), (5.60), and (5.61) we get for (5.52), taking $|q_\tau| \leq |q_0| < \delta$,

$$\begin{aligned}&\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi_1^{-1}(q_\tau,\tilde{r}),\tau\right)\right)_{s-2,2}\\ \leq\ & -c_1\|\tilde{r}\|^2_{s-\frac{1}{2},2} + C\left(|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2},2} + \delta\|\tilde{r}\|^2_{s-\frac{1}{2},2} + |q_\tau|\|\tilde{r}\|^2_{s-\frac{1}{2},2} + |q_\tau|^2\|\tilde{r}\|_{s-\frac{1}{2},2}\right)\\ \leq\ & -c_1\|\tilde{r}\|^2_{s-\frac{1}{2},2} + C\left(|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2},2} + \delta\|\tilde{r}\|^2_{s-\frac{1}{2},2}\right)\\ \leq\ & (-c_1 + C\delta)\|\tilde{r}\|^2_{s-\frac{1}{2},2} + C|q_\tau|^2 + \frac{c_1}{2}\|\tilde{r}\|^2_{s-\frac{1}{2},2}\\ \leq\ & \left(-\frac{c_1}{2} + C\delta\right)\|\tilde{r}\|^2_{s-\frac{1}{2},2} + C|q_\tau|^2.\end{aligned}$$

Here we used Cauchy's inequality and the fact that $|q_\tau|^2 \leq |q_\tau|$ for small $|q_0|$. If we choose $\delta < \frac{\varepsilon}{C}$, then by (5.46)

$$\begin{aligned}\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi_1^{-1}\left(q_\tau, \tilde{r}\right), \tau\right)\right)_{s-2,2} &\leq -\lambda_0\|\tilde{r}\|^2_{s-\frac{1}{2},2} + C|q_\tau|^2 \\ &\leq -\lambda_0\|\tilde{r}\|^2_{s-\frac{1}{2},2} + C\frac{|q_0|^2}{\alpha(\tau)^{2N}}.\end{aligned}$$

10. Arguing as in the proof of Theorems 5.6 and 5.8 we get global existence of a solution $\tilde{r}$ to (5.47) for fixed q_0 and for $\tilde{r}(0) = \mathcal{P}_1 r_0$ small enough. Furthermore, we have $\|\tilde{r}(\tau)\|_s^2 \leq y(\tau)$ where $y : [0, \infty) \to \mathbb{R}$ satisfies

$$\frac{dy}{d\tau} = -2\lambda_0 y + C\frac{|q_0|^2}{\alpha(\tau)^{2N}},$$

with $y(0) = \|\mathcal{P}_1 r_0\|_s^2$. This ODE can be solved using the variation of constants formula:

$$y(\tau) = e^{-2\lambda_0\tau}y(0) + C|q_0|^2\int_0^\tau \frac{e^{2\lambda_0(\tilde{\tau}-\tau)}}{\alpha(\tilde{\tau})^{2N}}d\tilde{\tau}.$$

We have

$$\begin{aligned}\int_0^\tau \frac{e^{2\lambda_0(\tilde{\tau}-\tau)}}{\alpha(\tilde{\tau})^{2N}}d\tilde{\tau} &\leq \int_0^{\frac{\tau}{2}} e^{2\lambda_0(\tilde{\tau}-\tau)}d\tilde{\tau} + \frac{1}{\alpha(\frac{\tau}{2})^{2N}}\int_{\frac{\tau}{2}}^\tau e^{2\lambda_0(\tilde{\tau}-\tau)}d\tilde{\tau} \\ &\leq \frac{1}{2\lambda_0}\left(e^{-\lambda_0\tau} - e^{-2\lambda_0\tau} + \frac{1}{\alpha(\frac{\tau}{2})^{2N}}\right) \\ &\leq \frac{C}{\alpha(\frac{\tau}{2})^{2N}} \leq \frac{C}{\alpha(\tau)^{2N}}.\end{aligned}$$

We omitted the exponential terms because they are smaller than a multiple of the algebraic terms. The result is

$$\|\tilde{r}(\tau)\|_s \leq Ce^{-\lambda_0\tau}\|\mathcal{P}_1 r_0\|_s + \frac{C}{\alpha(\tau)^N}|q_0|.$$

11. Now we construct a solution r to the original problem by setting

$$r(\tau) := \phi_1^{-1}(q_\tau, \tilde{r}(\tau)).$$

From Lipschitz continuity of ϕ_1^{-1} near the origin we get

$$\|r(\tau)\|_s \leq Ce^{-\lambda_0\tau}\|\mathcal{P}_1 r_0\|_s + \frac{C}{\alpha(\tau)^N}|q_0| \tag{5.62}$$

or

$$\begin{aligned}\|r(t)\|_s &\leq Ce^{-\lambda_0\tau(t)}\|\mathcal{P}_1 r_0\|_s + \frac{C}{\alpha(t)^N}|q_0| \\ &\leq \frac{C}{\alpha(t)^N}\left(|q_0| + \|\mathcal{P}_1 r_0\|_s\right) \\ &\leq \frac{C}{\alpha(t)^N}\|r_0\|_s.\end{aligned}$$

□

Remark 5.11. *Note that because of (5.62), if we restrict ourselves to the case $r_0 \in \mathfrak{M}_1^{N-1}$, which means $q_0 = 0$, then we have faster convergence.*

Remark 5.12. *Theorems 3.13, 5.9, and 5.10 show that for the injection problems $\|r(t)\|_s$ decays faster than C/t^ζ for $\zeta < 1$ in all dimensions.*

5.4 Almost global existence results for the suction problems

In this section we find almost global existence results for the suction problems. Both cases $N = 2$ and $N \geq 4$ will be treated. For almost global existence we do not need to restrict ourselves to evolutions in $\mathfrak{M}_1^N$. Remember that

$$\tau_{\max} := \begin{cases} \infty & \text{for } N = 2, \\ \frac{\sigma_N}{|\mu|(N-3)} & \text{for } N \geq 4. \end{cases}$$

Theorem 5.13. *Let $N = 2$ or $N \geq 4$ and $\mu < 0$. Let $T_+ \in (0, \tau_{\max})$ and $s > \frac{N+8}{2}$. There exists a $\delta > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with $\|r_0\|_s < \delta$, then there exists a solution $r \in \mathcal{C}_w([0, T_+), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}_w^1([0, T_+), \mathbb{H}^{s-3}(\mathbb{S}^{N-1}))$ to*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0. \tag{5.63}$$

Furthermore, $((\xi, \tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0, T_+))$.

Proof. For the case $N = 2$, we argue as in the proof of Theorem 5.8. There, we split up the time interval in two parts. A different approach for the second time interval was necessary there, because we wanted to show global existence and exponential decay in τ assuming that $r \in \mathfrak{M}_1^2$. Here we only consider the first time interval $[0, \hat{T}]$ and choose $\hat{T} \geq T_+$. In the estimates in steps 3-8 of the proof of Theorem 5.8 we replace $\tilde{r}$ by r and $\tilde{\mathcal{F}}_k$ by $\mathcal{F}_k$. All estimates that are found for the evolution operators $\tilde{\mathcal{F}}_k$ ($k = 1, 2$) on the first interval hold for the operators $\mathcal{F}_k$ as well, because up to equation (5.41) we did not use the fact that $r \in \mathfrak{M}_1^2$. In this way we derive that if $\|r\|_s < \delta'$, for δ' small, then we have

$$(r, \mathcal{F}(r, \tau))_{s-2,2} \leq C\|r\|_{s-2,2}^2.$$

We choose $\delta < \delta' e^{-CT_+}$ and use local existence results as before to prove the theorem for $N = 2$.

For $N \geq 4$, $\alpha(\tau)^{3-N}$ goes to infinity if τ approaches $\tau_{\max}$. However, on the time interval $[0, T_+]$, $\alpha(\tau)^{3-N}$ is bounded. Therefore we can use the same strategy as in the proof of Theorem 5.8 on the first of the two intervals to prove the theorem. $\square$

Chapter 6

Stokes flow

6.1 Introduction

In this Chapter the Stokes moving boundary problem with surface tension and with injection or suction is analysed by the methods that we used for the Hele-Shaw problem in Chapter 5. This Stokes problem is formulated as follows: find a family of domains $t \to \Omega(t)$ in $\mathbb{R}^N$ and two functions $p(\cdot,t) : \Omega(t) \to \mathbb{R}$ and $v(\cdot,t) : \Omega(t) \to \mathbb{R}^N$ that satisfy for each time

$$
\begin{aligned}
-\Delta v + \nabla p &= 0 \quad \text{in } \Omega(t), && (6.1)\\
\operatorname{div} v &= \mu\delta \quad \text{in } \Omega(t), && (6.2)\\
\left(\nabla v + \nabla v^T - pI\right) n &= \gamma\kappa n \quad \text{on } \Gamma(t) := \partial\Omega(t). && (6.3)
\end{aligned}
$$

The family $t \mapsto \Omega(t)$ models a liquid that moves under influence of injection or suction and surface tension. The functions v and p denote dimensionless velocity and pressure, respectively, μ is the injection rate ($\mu > 0$) or suction rate ($\mu < 0$), $\gamma > 0$ the surface tension coefficient, κ is the mean curvature (taken negative for convex domains), n is the outer normal on the boundary, I the identity matrix and δ denotes the delta distribution. The evolution of the boundary $t \mapsto \Gamma(t)$ is specified by the requirement that its normal velocity v_n satisfies

$$v_n = v \cdot n. \tag{6.4}$$

The velocity component in the fixed time problem (6.1)-(6.3) is determined only up to rigid body motions. The problem becomes uniquely solvable after adding two extra conditions, namely

$$\int_{\Omega(t)} v \, dx = 0, \tag{6.5}$$

which implies that the geometric centre of $\Omega(t)$ is constant in time (see Lemma 6.14) and

$$\int_{\Omega(t)} \operatorname{rot} v \, dx = 0. \tag{6.6}$$

Here, the operator rot in N dimensions should be interpreted in the following way. Let ω be any bijection between $\{(i,j) \in \mathbb{N}^2 : 1 \le i < j \le N\}$ and $\{1, 2, \ldots, \binom{N}{2}\}$. We define

$$\operatorname{rot} u := \sum_{1 \le i < j \le N} \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right) e_{\omega(i,j)}, \tag{6.7}$$

where e_k is the k-th unit vector in $\mathbb{R}^{\binom{N}{2}}$.

Equations (6.1) and (6.2) can be derived from the Navier-Stokes equations if one assumes a fluid with low Reynolds number. A closely related model is used to study the growth of certain tumours, for which the tissue can be modeled as a fluid (see [25], [26], and [27]). The process of viscous sintering in glass technology is modeled by Stokes flow as well (see [51]). More industrial applications are given in [68].

Short-time existence of solutions for the problem without injection or suction is proved in [33]. In the same work, global existence results have been found for the case that the initial domain is close to a ball. Joint spatial and temporal analyticity of the moving boundary for the problem without injection or suction has been proved in [19].

For the problem with injection or suction, short-time existence results and smoothness of the boundary have been proved in [60]. Exact solutions for the suction case are found in [11] from complex variable theory.

We start by identifying the trivial solution, where $\Omega(0) = \mathbb{B}^N := \{x \in \mathbb{R}^N : |x| < 1\}$. The trivial domain evolution is given by $\Omega(t) = \alpha(t)\mathbb{B}^N$, with α defined by (1.12). For this special solution, the functions v and p will be denoted by v_0 and p_0. From radial symmetry we obtain

$$v_0 = \frac{\mu}{\sigma_N |x|^N} x. \tag{6.8}$$

The mean curvature of $\mathbb{S}^{N-1}$ is $1 - N$. Therefore the mean curvature of $\Gamma(t)$ is equal to $\frac{1-N}{\alpha(t)}$. It follows from (6.1) and (6.3) that

$$p_0 = \mu\delta + \gamma \frac{N-1}{\alpha(t)} - 2\mu \frac{N-1}{\sigma_N \alpha(t)^N}.$$

Note that outside the origin p_0 only depends on t (not on x).

To investigate the stability of the trivial solution we rescale again by $\alpha(t)^{-1}$ and describe perturbations by means of a function $r(\cdot, t) : \mathbb{S}^{N-1} \to \mathbb{R}$.

In Section 6.3 a nonlinear non-local evolution equation for r is derived and linearised around $r = 0$.

In Section 6.4 the spectrum of the linearisation is determined. This linearisation is again characterised in terms of the Dirichlet-to-Neumann operator given by (2.33). This is done by solving a boundary value problem on $\mathbb{B}^N$ in terms of (scalar) spherical harmonics and vector-valued spherical harmonics (see [29], [30], and [38]). For $N \ge 4$ calculating the spectrum is more complicated and the problem is less interesting for applications. Therefore we restrict ourselves to the cases $N = 2, 3$.

In Section 6.5 global existence in time of solutions r is derived for the case of injection. We also show that the corresponding moving domain converges to a ball as time goes to infinity. This is done by finding energy estimates in Sobolev spaces. The first order chain rule, that we found in Lemma 5.4, is used to close a regularity gap.

In Section 6.6 we consider the case of suction. Because the eigenvalues of the linearisation go to infinity as t tends to T, we cannot derive global existence results. However, it is still possible to derive an almost global existence result as in Theorem 5.13.

6.2 Comparison between Stokes flow and Hele-Shaw flow

In this section we compare the Stokes moving boundary problem (6.1)-(6.6) to the corresponding problem for the Hele-Shaw problem that we discussed in the previous chapters.

In both cases we have an elliptic system at each time and the evolution of a moving boundary that follows from (6.4). The fixed time problem for Hele-Shaw flow is reduced to one scalar equation and a boundary condition for pressure only in (3.5) and (3.6). The system (6.1)-(6.3) cannot straightforwardly be decoupled. As a result, the components $\mathcal{F}_1$ and $\mathcal{F}_2$ of the evolution equation (6.29) are more complicated than those in equation (5.8) for Hele-Shaw flow.

The linearisations around the trivial solution to both problems are related to solution operators for boundary value problems on $\mathbb{B}^N$ (see (6.38)-(6.42)) and the (scalar) spherical harmonics (see Section 6.4) are eigenfunctions. This is not surprising since both evolution operators are equivariant with respect to rotations and therefore the eigenspaces have a corresponding invariance property. In contrast to Hele-Shaw flow, in order to solve the coupled Stokes system (6.38)-(6.42) we need to introduce vector-valued spherical harmonics as well.

For Stokes flow only the evolution problem for the uninteresting case $N = 1$ can be regarded as autonomous. Hence the methods of Chapters 2 and 3 cannot be used for the more relevant space dimensions 2 and 3. Neither can we apply the methods that we used in Chapter 4 to find non-trivial self-similar solutions. The existence results for $N = 2, 3$ (see Theorems 6.15 and 6.18) turn out to be similar to those for Hele-Shaw flow with $N \geq 4$.

The evolution operator in (6.29) is of first order whereas the operator for Hele-Shaw flow is of order three. Therefore one can apply a first order chain rule of differentiation (5.12) to obtain useful energy estimates in the existence proofs. For Hele-Shaw flow it was necessary to work with a second order chain rule.

6.3 The evolution equation and its linearisation

Let $\mathbb{H}^s(\mathbb{S}^{N-1})$ be the Sobolev space of real order s as defined in Chapter 5 and let the functions R and r be as defined in Section 1.5. Define for continuous $f : \mathbb{S}^{N-1} \to (-1, \infty)$ the domain Ω_f as in (1.14). In contrast to Chapter 5, for Stokes flow we demand that r is a small element of $\mathbb{H}^s(\mathbb{S}^{N-1})$ with

$$s > \frac{N+5}{2}. \tag{6.9}$$

Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be the function defined by (2.10) that satisfies $\Delta\Phi = -\delta$ and vanishes on $\mathbb{S}^{N-1}$. Define the functions V and P by

$$V := v + \mu\nabla\Phi = v - v_0 \tag{6.10}$$

and

$$P := p - \mu\delta.$$

Since v and p satisfy (6.1)-(6.2) we have

$$-\Delta V + \nabla P = 0 \quad \text{on } \Omega_R, \tag{6.11}$$
$$\operatorname{div} V = 0 \quad \text{on } \Omega_R. \tag{6.12}$$

The boundary condition (6.3) is equivalent to

$$(\nabla V + \nabla V^T - PI)n = \gamma\kappa n + 2\mu H n \qquad \text{on } \Gamma_R. \tag{6.13}$$

Here $H : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is the Hessian of Φ given by

$$H(x) = \frac{1}{\sigma_N |x|^N}\left(-I + \frac{N}{|x|^2} x \otimes x\right),$$

where $x \otimes x$ denotes the matrix with coefficients $x_i x_j$. The extra conditions (6.5) and (6.6) are equivalent to

$$\int_{\Omega_R} V\,dx = \int_{\Omega_R} \mu\nabla\Phi\,dx, \quad \int_{\Omega_R} \operatorname{rot} V\,dx = 0. \tag{6.14}$$

Define

- $(V_{1,f}, P_{1,f})^T : \Omega_f \to \mathbb{R}^N \times \mathbb{R}$ as the solution to (6.11) and (6.12) on the domain Ω_f with boundary condition

  $$(\nabla V_{1,f} + \nabla V_{1,f}^T - P_{1,f} I)n = \kappa n \qquad \text{on } \Gamma_f$$

  and extra conditions

  $$\int_{\Omega_f} V_{1,f}\,dx = 0, \quad \int_{\Omega_f} \operatorname{rot} V_{1,f}\,dx = 0,$$

- $(V_{2,f}, P_{2,f})^T : \Omega_f \to \mathbb{R}^N \times \mathbb{R}$ as the solution to (6.11) and (6.12) on the domain Ω_f with boundary condition

  $$(\nabla V_{2,f} + \nabla V_{2,f}^T - P_{2,f} I)n = 2Hn \qquad \text{on } \Gamma_f$$

  and extra conditions

  $$\int_{\Omega_f} V_{2,f}\,dx = \int_{\Omega_f} \nabla\Phi\,dx, \quad \int_{\Omega_f} \operatorname{rot} V_{2,f}\,dx = 0.$$

It is known (see e.g. [60, Ch. 3]), that the solutions $(V_{1,f}, P_{1,f})^T$ and $(V_{2,f}, P_{2,f})^T$ are uniquely defined for appropriate domains Ω_f. The solution $(V, P)^T$ to (6.11)-(6.14) can be written as follows:

$$(V, P)^T = \gamma(V_{1,R}, P_{1,R})^T + \mu(V_{2,R}, P_{2,R})^T. \tag{6.15}$$

Lemma 6.1. *If the relation between R and r is given by* (1.15), then

$$V_{1,r}(x) = V_{1,R}(\alpha(t)x), \tag{6.16}$$

$$P_{1,r}(x) = \alpha(t)P_{1,R}(\alpha(t)x), \tag{6.17}$$

$$V_{2,r}(x) = \alpha(t)^{N-1}V_{2,R}(\alpha(t)x), \tag{6.18}$$

$$P_{2,r}(x) = \alpha(t)^N P_{2,R}(\alpha(t)x). \tag{6.19}$$

Proof. Let $\hat{V}_{1,r}(x)$, $\hat{P}_{1,r}(x)$, $\hat{V}_{2,r}(x)$, and $\hat{P}_{2,r}(x)$ be the right-hand sides in (6.16)-(6.19). We will show that $\hat{V}_{1,r} = V_{1,r}$, $\hat{V}_{2,r} = V_{2,r}$, $\hat{P}_{1,r} = P_{1,r}$, and $\hat{P}_{2,r} = P_{2,r}$. Suppressing the time argument in $\alpha(t)$, we have for $x \in \Omega_r$

$$-\Delta\hat{V}_{1,r}(x) + \nabla\hat{P}_{1,r}(x) = \alpha^2\left(-\Delta V_{1,R}(\alpha x) + \nabla P_{1,R}(\alpha x)\right) = 0.$$

For $\hat{V}_{2,r}$ and $\hat{P}_{2,r}$ this can be done in a similar way. Let $x \in \Gamma_r$, such that $\alpha x \in \Gamma_R$ and define $\kappa_r : \Gamma_r \to \mathbb{R}$ and $\kappa_R : \Gamma_R \to \mathbb{R}$ as the mean curvature of these boundaries. We have

$$\begin{aligned}\left(\nabla\hat{V}_{1,r}(x) + \nabla\hat{V}_{1,r}^T(x) - \hat{P}_{1,r}(x)\right)n &= \alpha\left(\nabla V_{1,R}(\alpha x) + \nabla V_{1,R}^T(\alpha x) - P_{1,R}(\alpha x)\right)n \\ &= \alpha\gamma\kappa_R(\alpha x)n = \gamma\kappa_r(x)n.\end{aligned}$$

The corresponding boundary condition for $\hat{V}_{2,r}$ and $\hat{P}_{2,r}$ is checked in a similar way, using the fact that $H(x) = \alpha^N H(\alpha x)$. From scaling properties of $\nabla\Phi$ we get for the extra conditions

$$\begin{aligned}\int_{\Omega_r}\hat{V}_{2,r}(x)\,dx &= \int_{\Omega_r}\alpha^{N-1}V_{2,R}(\alpha x)\,dx = \int_{\Omega_R}\alpha^{-1}V_{2,R}(x)\,dx \\ &= \int_{\Omega_R}\alpha^{-1}\nabla\Phi(x)\,dx = \int_{\Omega_r}\alpha^{N-1}\nabla\Phi(\alpha x)\,dx = \int_{\Omega_r}\nabla\Phi(x)\,dx.\end{aligned}$$

Verifying the other conditions is straightforward. □

Let the operators $\tilde{z}, n, \kappa$, and z be defined as in Section 5.2. Again we identify $\tilde{z}(r)$ and $\tilde{z}(r,\cdot)$, $n(r)$ and $n(r,\cdot)$, etc.

Combining Lemma 2.1, (6.8), (6.10), and (6.15) we get

$$\frac{\partial R}{\partial t} = \gamma\frac{(V_{1,R}\circ\tilde{z}(R))\cdot n(R)}{n(R)\cdot \mathrm{id}} + \mu\left(\frac{(V_{2,R}\circ\tilde{z}(R))\cdot n(R)}{n(R)\cdot \mathrm{id}} + \frac{1}{\sigma_N(1+R)^{N-1}}\right).$$

By (3.14), Lemma 6.1, and $n(R) = n(r)$ we obtain

$$\frac{\partial r}{\partial t} = \frac{\gamma}{\alpha(t)} \frac{(V_{1,r} \circ \tilde{z}(r)) \cdot n(r)}{n(r) \cdot \mathrm{id}} + \frac{\mu}{\alpha(t)^N} \left(\frac{(V_{2,r} \circ \tilde{z}(r)) \cdot n(r)}{n(r) \cdot \mathrm{id}} + \frac{1}{\sigma_N (1+r)^{N-1}} - \frac{1+r}{\sigma_N} \right). \tag{6.20}$$

From the structure of the evolution equation we expect that the results for Stokes flow in space dimension two or higher are similar to those for Hele-Shaw flow in dimension four or higher.

Now we transform our moving boundary problem to the fixed reference domain $\mathbb{B}^N$.

Lemma 6.2. *Let $s > \frac{N+5}{2}$. There exists a $\delta > 0$ such that if $\|r\|_s < \delta$, then $z(r) : \overline{\mathbb{B}^N} \to \overline{\Omega}_r$ is bijective and $z(r)^{-1} \in \left(\mathcal{C}^2(\overline{\Omega_r})\right)^N$.*

Proof. From $\mathbb{H}^s(\mathbb{S}^{N-1}) \hookrightarrow \mathcal{C}^3(\mathbb{S}^{N-1})$ it follows that r is small in $\mathcal{C}^3(\mathbb{S}^{N-1})$. The result follows from Lemmas 2.3 and 2.4. □

Introduce the bilinear mapping $\star : \mathbb{R}^{\binom{N}{2}} \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$u \star v = \sum_{i=1}^{N} \left(\sum_{j=1}^{i-1} u_{\omega(j,i)} v_j - \sum_{j=i+1}^{N} u_{\omega(i,j)} v_j \right) e_i. \tag{6.21}$$

Here ω is the bijection that we introduced to define the operator rot in (6.7).

On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ with $s > \frac{N+5}{2}$ we define the following mappings:

- $\mathcal{A} : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N \right)$ componentwise by (2.25) and $\mathcal{Q} : \mathcal{U} \to \mathcal{L}\left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N), \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N \right)$ by (2.26). Because of (6.9) the space $\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)$ is a Banach algebra. Hence, $\mathcal{A}(r)$ and $\mathcal{Q}(r)$ are well-defined.
- $\mathcal{Q}^+ : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)\right)^{N \times N} \right)$,

  $b : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \right)$,

  and $\mathcal{R} : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N, \left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)\right)^{\binom{N}{2}} \right)$ by

$$\begin{aligned}
\mathcal{Q}^+(r)u &:= \left(\nabla \left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k,l} j^{k,i}(r) \frac{\partial u_l}{\partial x_k} e_l \otimes e_i, \\
b(r)u &:= \left(\operatorname{div} \left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{i,k} j^{k,i}(r) \frac{\partial u_i}{\partial x_k}, \\
\mathcal{R}(r)u &:= \left(\operatorname{rot} \left(u \circ z(r)^{-1}\right)\right) \circ z(r) = \sum_{1 \le i < k \le N} \sum_{l} \left(j^{l,i}(r) \frac{\partial u_k}{\partial x_l} - j^{l,k}(r) \frac{\partial u_i}{\partial x_l} \right) e_{\omega(i,k)},
\end{aligned}$$

with $j^{k,i}(r)$ as in Section 5.2.

- $\mathcal{S} : \mathcal{U} \to \mathcal{L}(\mathcal{X}_s, \mathcal{Y}_s)$, where

$$\begin{aligned}\mathcal{X}_s &:= \left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N \times \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}},\\ \mathcal{Y}_s &:= \left(\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)\right)^N \times \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \times \left(\mathbb{H}^{s-2}(\mathbb{S}^{N-1})\right)^N \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}},\end{aligned}$$

by

$$\mathcal{S}(r)(\tilde{v}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) = \left(\begin{array}{c} -\mathcal{A}(r)\tilde{v} + \mathcal{Q}(r)\tilde{p} + \tilde{\eta}_1 \\ b(r)\tilde{v} \\ \mathrm{Tr}(\mathcal{Q}^+(r)\tilde{v} + \mathcal{Q}^+(r)\tilde{v}^T - \tilde{p}I)n(r) + \tilde{\eta}_2 \star n(r) \\ \int_{\mathbb{B}^N} \tilde{v} \det \mathcal{J}(r)dx \\ \int_{\mathbb{B}^N} (\mathcal{R}(r)\tilde{v}) \det \mathcal{J}(r)dx \end{array} \right). \tag{6.22}$$

- $h : \mathcal{U} \to \left(\mathbb{H}^s(\mathbb{S}^{N-1})\right)^{N \times N}$ by

$$h(r, \xi) = H(\tilde{z}(r, \xi)) = \frac{1}{\sigma_N (1 + r(\xi))^N} \left(-I + N\xi \otimes \xi\right). \tag{6.23}$$

- $m : \mathcal{U} \to \mathbb{R}^N$ by

$$m(r) = \int_{\Omega_r} \nabla \Phi \, dx = -\frac{1}{\sigma_N} \int_{\mathbb{S}^{N-1}} r(\xi)\xi \, d\sigma. \tag{6.24}$$

Lemma 6.3. *The mapping $\mathcal{S}(0)$ is an isomorphism between $\mathcal{X}_s$ and $\mathcal{Y}_s$. The same holds for $\mathcal{S}(r)$ if r is small in $\mathbb{H}^s(\mathbb{S}^{N-1})$. Furthermore, $\mathcal{S}$ is analytic near zero.*

Proof. For the first statement we refer to [60, Ch. 3 Lemma 11]. Since $\mathcal{S}$ is continuous near zero, the second statement follows from the fact that isomorphisms form an open subset in $\mathcal{L}(\mathcal{X}_s, \mathcal{Y}_s)$. For the last statement we refer to [60, Ch. 3 Lemma 17]. □

Remark 6.4. *In the definition of $\mathcal{X}_s$, $\mathcal{Y}_s$, and $\mathcal{S}$, we included the components $\tilde{\eta}_1$ and $\tilde{\eta}_2$ because the equation $\mathcal{S}(r)f = g$ does not have a solution $f \in \mathcal{X}_s$ of the type $(\tilde{v}, \tilde{p}, 0, 0)$ for all $g \in \mathcal{Y}_s$.*

In the sequel we use the notation $\Pi_i f$ for the i-th component of any f. On a suitable neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ we define
$\mathcal{E} : \mathcal{U} \to \mathcal{L}\left(\left(\mathbb{H}^{s-2}(\mathbb{S}^{N-1})\right)^N \times \mathbb{R}^N, \mathbb{H}^{s-1}(\mathbb{S}^{N-1})\right)$ by

$$\left(\mathcal{E}(r)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right) := \frac{\left(\mathrm{Tr}\Pi_1 \mathcal{S}(r)^{-1}(0, 0, \psi_1, \psi_2, 0)^T\right) \cdot n(r)}{n(r) \cdot \mathrm{id}}. \tag{6.25}$$

The evolution equation (6.20) can be written in the following way:

$$\frac{\partial r}{\partial t} = \frac{\gamma}{\alpha(t)}\mathcal{F}_1(r) + \frac{\mu}{\alpha(t)^N}\mathcal{F}_2(r), \tag{6.26}$$

where $\mathcal{F}_1 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ and $\mathcal{F}_2 : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ are given by

$$\mathcal{F}_1(r) = \mathcal{E}(r)\begin{pmatrix} \kappa(r)n(r) \\ 0 \end{pmatrix}$$

and

$$\mathcal{F}_2(r) = \mathcal{E}(r)\begin{pmatrix} 2h(r)n(r) \\ m(r) \end{pmatrix} + \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N}.$$

Lemma 6.5. *Suppose that $s > \frac{N+5}{2}$. The mappings $\mathcal{F}_1$ and $\mathcal{F}_2$ are both analytic from a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ to $\mathbb{H}^{s-1}(\mathbb{S}^{N-1})$.*

Proof. In [60, Ch. 3 Lemma 19] this is proved for $\mathcal{F}_1$. Analyticity of $\mathcal{F}_2$ can be obtained in a similar way. The proof is based on local analyticity of $\mathcal{S}$, bijectivity of $\mathcal{S}(0)$ (see Lemma 6.3), and the Implicit Function theorem. □

Lemma 6.6. *If $\psi_1 = \kappa(r)n(r)$ or $\psi_1 = 2h(r)n(r)$ and ψ_2 is any element of $\mathbb{R}^N$, then $\mathcal{S}(r)(\tilde{v}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)^T = (0, 0, \psi_1, \psi_2, 0)^T$ implies*

$$\tilde{\eta}_1 := \Pi_3\mathcal{S}(r)^{-1}(0, 0, \psi_1, \psi_2, 0)^T = 0, \quad \tilde{\eta}_2 := \Pi_4\mathcal{S}(r)^{-1}(0, 0, \psi_1, \psi_2, 0)^T = 0.$$

Proof. We will consider the corresponding boundary value problem on Ω_r, defining $v = \tilde{v} \circ z(r)^{-1}$ and $p = \tilde{p} \circ z(r)^{-1}$. Let κ_r and n_r be the mean curvature and the normal on Γ_r. A variational formulation of

$$\begin{aligned} -\Delta v + \nabla p + \tilde{\eta}_1 &= 0, \quad \text{on } \Omega_\mathrm{r} \\ \operatorname{div} v &= 0, \quad \text{on } \Omega_\mathrm{r} \\ (\nabla v + \nabla v^T - pI)n_r + \tilde{\eta}_2 \star n_r &= \psi_1 \circ z(r)^{-1}, \quad \text{on } \Gamma_\mathrm{r} \\ \int_{\mathbb{B}^N} v\, dx &= \psi_2, \\ \int_{\mathbb{B}^N} \operatorname{rot} v\, dx &= 0, \end{aligned}$$

is given in [60, eqn. (3.24)]. From this variational formulation it follows that for all velocity fields w corresponding to rigid body motions in $\mathbb{R}^N$

$$\int_{\Omega_r} (\tilde{\eta}_1 \cdot w + \tilde{\eta}_2 \cdot \operatorname{rot} w)\, dx = \int_{\Gamma_r} (\psi_1 \circ \tilde{z}(r)^{-1}) \cdot w\, d\sigma.$$

Therefore, to prove this lemma it is sufficient to show that for all rigid body motions w we have

$$\int_{\Gamma_r} \kappa_r n_r \cdot w\, d\sigma = \int_{\Gamma_r} H n_r \cdot w\, d\sigma = 0.$$

Let Δ_r be the Laplace-Beltrami operator on Γ_r as defined in (3.17) and let ∇_r be defined by

$$\nabla_r f = \nabla f - (\nabla f \cdot n_r) n_r,$$

for any differentiable $f : \Omega_r \to \mathbb{R}$. From (3.19) and Green's formula for closed surfaces we derive

$$\begin{aligned} \int_{\Gamma_r} \kappa_r n_r \cdot w \, d\sigma &= \int_{\Gamma_r} \Delta_r \mathrm{id} \cdot w \, d\sigma = -\int_{\Gamma_r} \sum_i \nabla_r x_i \cdot \nabla_r w_i \, d\sigma = -\int_{\Gamma_r} \sum_i \nabla x_i \cdot \nabla_r w_i \, d\sigma \\ &= -\int_{\Gamma_r} \Big(\operatorname{div} w - \sum_{i,j} \frac{\partial w_i}{\partial x_j} (n_r \cdot e_i)(n_r \cdot e_j) \Big) \, d\sigma = 0. \end{aligned}$$

In the last step we used anti-symmetry of ∇w and $\operatorname{div} w = 0$. Because H is symmetric and $\Delta \Phi = -\delta$ we get

$$\begin{aligned} \int_{\Gamma_r} H n_r \cdot w \, d\sigma &= \int_{\Gamma_r} H w \cdot n_r \, d\sigma \\ &= \int_{\Omega_r} \operatorname{div}(Hw) \, dx = \int_{\Omega_r} (\Delta \nabla \Phi) \cdot w + \operatorname{tr}(H \nabla w) \, dx \\ &= \int_{\Omega_r} -\nabla \delta \cdot w + \operatorname{tr}(H \nabla w) \, dx = 0. \end{aligned}$$

In the last step we used $\operatorname{div} w = 0$ and the fact that the trace of the product of a symmetric matrix and an anti-symmetric matrix is zero. □

We introduce a new time variable $\tau = \tau(t)$ such that $\tau(0) = 0$ and

$$\frac{d\tau}{dt} = \frac{1}{\alpha(t)}. \tag{6.27}$$

From this we get for $N \geq 2$

$$\tau(t) = \frac{\sigma_N}{\mu(N-1)} \left(\left(\frac{\mu N t}{\sigma_N} + 1 \right)^{1-\frac{1}{N}} - 1 \right). \tag{6.28}$$

The injection problems are defined on an infinite time interval and the suction problems on a finite interval. In terms of the new time variable τ this is still the case, because

$$\begin{aligned} \lim_{t\to\infty} \tau(t) &= \infty, && \text{for } \mu > 0, \\ \lim_{t\to T} \tau(t) &= \frac{\sigma_N}{|\mu|(N-1)}, && \text{for } \mu < 0. \end{aligned}$$

Regarding r as a function of τ we get

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau) := \gamma \mathcal{F}_1(r) + \mu \alpha(\tau)^{1-N} \mathcal{F}_2(r). \tag{6.29}$$

For convenience we write here and in the sequel $\alpha(\tau)$ instead of $\alpha(t(\tau))$.

Now we determine the linearisation of the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ around $r = 0$.

Lemma 6.7. *Define* ∇_0 *by* (4.30). *We have*

$$\begin{aligned}\mathcal{F}_1'(0)[r] &= \mathcal{E}(0)\begin{pmatrix}\kappa'(0)[r]n(0)\\0\end{pmatrix},\\ \mathcal{F}_2'(0)[r] &= \mathcal{E}(0)\begin{pmatrix}\frac{2N(1-N)}{\sigma_N}rn(0)+\frac{2N}{\sigma_N}\nabla_0 r\\ m(r)\end{pmatrix}-\frac{N}{\sigma_N}r.\end{aligned}$$

Proof. First we show that

$$n'(0)[r] = -\nabla_0 r. \tag{6.30}$$

Fréchet differentiation of the expression $n(r)\cdot n(r)=1$ at $r=0$ leads to

$$n'(0)[r]\cdot n(0) = 0. \tag{6.31}$$

Let $\Xi=\Xi(\omega)$ be a regular parametrisation of a part of $\mathbb{S}^{N-1}$. Note that for $i=1,2,\ldots,N-1$ we have

$$0 = n(r)\cdot\frac{\partial\tilde{z}(r)}{\partial\omega_i} = n(r)\cdot\left((1+r)\frac{\partial\mathrm{id}}{\partial\omega_i}+\frac{\partial r}{\partial\omega_i}\mathrm{id}\right).$$

Fréchet differentiation of this expression at $r=0$ yields

$$0 \;=\; n'(0)[r]\cdot\frac{\partial\mathrm{id}}{\partial\omega_i}+n(0)\cdot\left(r\frac{\partial\mathrm{id}}{\partial\omega_i}+\frac{\partial r}{\partial\omega_i}\mathrm{id}\right) = n'(0)[r]\cdot\frac{\partial\mathrm{id}}{\partial\omega_i}+\frac{\partial r}{\partial\omega_i}. \tag{6.32}$$

Here we used the fact that $\frac{\partial\mathrm{id}}{\partial\omega_i}$ is orthogonal to $n(0)=\mathrm{id}$. Taking the vector fields $\frac{\partial\mathrm{id}}{\partial\omega_i}$, $i=1,2,\ldots N-1$, pointwise orthogonal we obtain

$$n'(0)[r] = \sum_{i=1}^{N-1}\left(n'(0)[r]\cdot\frac{\partial\mathrm{id}}{\partial\omega_i}\right)\left|\frac{\partial\mathrm{id}}{\partial\omega_i}\right|^{-2}\frac{\partial\mathrm{id}}{\partial\omega_i} = -\sum_{i=1}^{N-1}\frac{\partial r}{\partial\omega_i}\left|\frac{\partial\mathrm{id}}{\partial\omega_i}\right|^{-2}\frac{\partial\mathrm{id}}{\partial\omega_i} = -\nabla_0 r.$$

This follows from (6.31), (6.32), and the fact that $\frac{\partial\mathrm{id}}{\partial\omega_i}\perp n(0)$.

To shorten notation we introduce

$$\mathcal{G}_1(r)\psi_1 := \mathcal{S}(r)^{-1}(0,0,\psi_1,0,0)^T,\quad \mathcal{G}_2(r)\psi_2 := \mathcal{S}(r)^{-1}(0,0,0,\psi_2,0)^T. \tag{6.33}$$

Define $v_1:\mathcal{U}\to\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N$ and $p_1:\mathcal{U}\to\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)$ by

$$(v_1(r),p_1(r),0,0)^T := \mathcal{G}_1(r)\left(\kappa(r)n(r)\right)$$

Since $v_1(0)\equiv 0$ and $p_1(0)\equiv-\kappa(0)$ it follows that

$$\begin{aligned}v_1'(0)[r] &= \Pi_1\mathcal{G}_1'(0)[r]\big(\kappa(0)n(0)\big)+\Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big)\\ &= -\Pi_1\mathcal{S}(0)^{-1}\mathcal{S}'(0)[r]\mathcal{G}_1(0)\big(\kappa(0)n(0)\big)+\Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big)\\ &= -\Pi_1\mathcal{S}(0)^{-1}\mathcal{S}'(0)[r](0,-\kappa(0),0,0)^T+\Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big).\end{aligned} \tag{6.34}$$

Because $\mathcal{Q}'(0)[h]$ vanishes on constants we have by (6.22)

$$\mathcal{S}'(0)[r](0,-\kappa(0),0,0)^T = (0,0,\kappa(0)n'(0)[r],0,0)^T.$$

Thus by (6.34)

$$\begin{aligned} v_1'(0)[r] &= -\Pi_1\mathcal{G}_1(0)\big(\kappa(0)n'(0)[r]\big) + \Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)+\kappa(0)n'(0)[r]\big) \\ &= \Pi_1\mathcal{G}_1(0)\big(\kappa'(0)[r]n(0)\big). \end{aligned} \tag{6.35}$$

Since $v_1(0) = 0$, $n(0) = \mathrm{id}$, and $\mathcal{F}_1(r) = \frac{\mathrm{Tr} v_1(r)\cdot n(r)}{n(r)\cdot \mathrm{id}}$ we get

$$\mathcal{F}_1'(0)[r] = \mathrm{Tr} v_1'(0)[r]\cdot n(0). \tag{6.36}$$

The first part of the lemma follows from (6.25), (6.33), (6.35), and (6.36).

Now we calculate $\mathcal{F}_2'(0)[r]$ in a similar way. Define $v_2 : \mathcal{U} \to \left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N$ and $p_2 : \mathcal{U} \to \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)$ by

$$(v_2(r), p_2(r), 0, 0)^T := \mathcal{G}_1(r)\big(2h(r)n(r)\big) + \mathcal{G}_2(r)m(r).$$

From a simple calculation we obtain $v_2(0) \equiv 0$ and $p_2(0) \equiv 2\frac{1-N}{\sigma_N}$. Because m is linear we have $m(0) = 0$ and $m'(0)[r] = m(r)$. Combining this with (6.22), (6.23), and (6.31) we obtain

$$\begin{aligned} v_2'(0)[r] &= -\Pi_1\mathcal{S}(0)^{-1}\mathcal{S}'(0)[r]\mathcal{G}_1(0)\big(2h(0)n(0)\big) \\ &\quad +\Pi_1\mathcal{G}_1(0)\big(2h'(0)[r]n(0)+2h(0)n'(0)[r]\big) + \Pi_1\mathcal{G}_2(0)m(r) \\ &= -\Pi_1\mathcal{S}(0)^{-1}\mathcal{S}'(0)[r]\left(0,2\frac{1-N}{\sigma_N},0,0\right)^T \\ &\quad +\Pi_1\mathcal{G}_1(0)\left(\frac{2N(1-N)}{\sigma_N}rn(0) - \frac{2}{\sigma_N}n'(0)[r]\right) + \Pi_1\mathcal{G}_2(0)m(r) \\ &= -\Pi_1\mathcal{G}_1(0)\left(2\frac{N-1}{\sigma_N}n'(0)[r]\right) \\ &\quad +\Pi_1\mathcal{G}_1(0)\left(\frac{2N(1-N)}{\sigma_N}rn(0) - \frac{2}{\sigma_N}n'(0)[r]\right) + \Pi_1\mathcal{G}_2(0)m(r) \\ &= \Pi_1\mathcal{G}_1(0)\left(\frac{2N(1-N)}{\sigma_N}rn(0) - \frac{2N}{\sigma_N}n'(0)[r]\right) + \Pi_1\mathcal{G}_2(0)m(r). \end{aligned}$$

The lemma follows if we combine this, (6.30), and the Taylor expansion $\frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N} = -\frac{N}{\sigma_N}r + \mathcal{O}(r^2)$ around $r = 0$. □

Define the spherical harmonics of degree one by

$$s_{1j} := \sqrt{\frac{N}{\sigma_N}} x_j, \tag{6.37}$$

such that $(s_{1j})_{j=1}^N$ form an $\mathbb{L}^2$-orthonormal basis of $\mathfrak{S}_1^N$.

Lemma 6.8. *We have*

$$\mathcal{E}(0) \begin{pmatrix} 0 \\ m(r) \end{pmatrix} = -\frac{1}{\sigma_N} \sum_{j=1}^N r_{1j} s_{1j},$$

where $r_{kj} = (r, s_{kj})_0$.

Proof. Let $\tilde{v}$ and $\tilde{p}$ be defined by

$$(\tilde{v}, \tilde{p}, 0, 0)^T := \mathcal{G}_2(0) m(r),$$

with $\mathcal{G}_2$ as in (6.33). It is easy to check that $\tilde{p}$ is zero and $\tilde{v}$ is constant. Therefore $\tilde{v} \cdot n(0) = \tilde{v} \cdot \mathrm{id}$ can be written as a linear combination of $(s_{1j})_{j=1}^N$. Furthermore, by (6.37)

$$\begin{aligned} \int_{\mathbb{S}^{N-1}} (\tilde{v} \cdot n(0)) s_{1j} \, d\sigma &= \sqrt{\frac{N}{\sigma_N}} \int_{\mathbb{B}^N} \operatorname{div}(x_j \tilde{v}) \, dx = \sqrt{\frac{N}{\sigma_N}} \int_{\mathbb{B}^N} \tilde{v}_j \, dx \\ &= \sqrt{\frac{N}{\sigma_N}} m(r)_j = -\frac{1}{\sigma_N} r_{1j}. \end{aligned}$$

In the last step we used (6.24). It follows that

$$\mathcal{E}(0) \begin{pmatrix} 0 \\ m(r) \end{pmatrix} = \frac{\mathrm{Tr} \tilde{v} \cdot n(0)}{n(0) \cdot \mathrm{id}} = \tilde{v} \cdot n(0) = -\frac{1}{\sigma_N} \sum_{j=1}^N r_{1j} s_{1j}.$$

□

Combining Lemmas 6.7 and 6.8 we obtain the following result.

Corollary 6.9. *We have*

$$\begin{aligned} \mathcal{F}_1'(0)[r] &= \mathcal{E}(0) \begin{pmatrix} \kappa'(0)[r] n(0) \\ 0 \end{pmatrix}, \\ \mathcal{F}_2'(0)[r] &= \mathcal{E}(0) \begin{pmatrix} \frac{2N(1-N)}{\sigma_N} r n(0) + \frac{2N}{\sigma_N} \nabla_0 r \\ 0 \end{pmatrix} - \frac{N}{\sigma_N} r - \frac{1}{\sigma_N} \sum_{j=1}^N r_{1j} s_{1j}. \end{aligned}$$

6.4 Explicit solution for the linearised problem

In this section we characterise $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ in terms of the Dirichlet-to-Neumann operator $\mathcal{N}$ for the Laplacian on $\mathbb{B}^N$. The spectrum and the eigenfunctions of $\mathcal{F}_1'(0)$ and

$\mathcal{F}_2'(0)$ are easily derived from the spectral properties of $\mathcal{N}$. We restrict ourselves to the cases $N = 2$ and $N = 3$.

It follows from Corollary 6.9 that we need to solve the following boundary value problem on $\mathbb{B}^N$ to find this characterisation. Find for $f : \mathbb{S}^{N-1} \to \mathbb{R}^N$ functions v and p such that

$$-\Delta v + \nabla p = 0, \quad \text{on } \mathbb{B}^N \tag{6.38}$$

$$\operatorname{div} v = 0, \quad \text{on } \mathbb{B}^N \tag{6.39}$$

$$(\nabla v + \nabla v^T - pI)n = f, \quad \text{on } \mathbb{S}^{N-1} \tag{6.40}$$

$$\int_{\mathbb{B}^N} v \, dx = 0, \tag{6.41}$$

$$\int_{\mathbb{B}^N} \operatorname{rot} v \, dx = 0. \tag{6.42}$$

6.4.1 The two-dimensional boundary value problem

For the two-dimensional problem we introduce polar coordinates ρ and θ and unit vectors e_ρ and e_θ. Define for all $k \in \mathbb{Z}$ the functions $s_k : \mathbb{S}^1 \to \mathbb{C}$ by

$$s_k := \frac{1}{\sqrt{2\pi}} e^{ik\theta}.$$

Complexifying the spaces $\mathfrak{S}_k^2$ in Chapter 1, one can identify these functions with the spherical harmonics s_{kj} in the following way:

$$s_{k1} := s_k, \quad s_{k2} := s_{-k},$$

for $k > 0$ and $s_{01} = s_0$. We write

$$f = f^\rho e_\rho + f^\theta e_\theta, \quad f^\rho(\theta) = \sum_{k=-\infty}^{\infty} f_k^\rho s_k(\theta), \quad f^\theta(\theta) = \sum_{k=-\infty}^{\infty} f_k^\theta s_k(\theta),$$

$$v = v^\rho e_\rho + v^\theta e_\theta, \quad v^\rho(\rho,\theta) = \sum_{k=-\infty}^{\infty} v_k^\rho(\rho) s_k(\theta), \quad v^\theta(\rho,\theta) = \sum_{k=-\infty}^{\infty} v_k^\theta(\rho) s_k(\theta),$$

for $f^\rho, f^\theta : \mathbb{S}^1 \to \mathbb{R}$, $v^\rho, v^\theta : \mathbb{B}^2 \to \mathbb{R}$, $f_k^\rho, f_k^\theta \in \mathbb{C}$, and $v_k^\rho, v_k^\theta : [0,1] \to \mathbb{C}$. Because p is harmonic we have

$$p = \sum_{k=-\infty}^{\infty} p_k \rho^{|k|} s_k(\theta), \tag{6.43}$$

for certain $p_k \in \mathbb{C}$.

Lemma 6.10. *For $N = 2$, the system (6.38)-(6.42) is solvable if and only if $f_0^\theta = 0$, $f_1^\rho = i f_1^\theta$, and $f_{-1}^\rho = -i f_{-1}^\theta$. For the components of the normal velocity on $\mathbb{S}^1$ we have for $k \notin \{-1, 0, 1\}$*

$$v_k^\rho(1) = \frac{|k|}{2(k^2-1)} f_k^\rho - \frac{i \operatorname{sgn} k}{2(k^2-1)} f_k^\theta. \tag{6.44}$$

For $k \in \{-1, 0, 1\}$ we have $v_k^\rho = 0$.

Proof. Parallel to [30], we write (6.38) in polar coordinates,

$$v_k^{\rho\prime\prime} + \frac{1}{\rho} v_k^{\rho\prime} - \frac{k^2+1}{\rho^2} v_k^\rho - \frac{2ik}{\rho^2} v_k^\theta = |k| p_k \rho^{|k|-1},$$

$$v_k^{\theta\prime\prime} + \frac{1}{\rho} v_k^{\theta\prime} - \frac{k^2+1}{\rho^2} v_k^\theta + \frac{2ik}{\rho^2} v_k^\rho = ik p_k \rho^{|k|-1}.$$

For $k \neq 0$, the general regular solution to these equations is given by

$$v_k^\rho = \frac{1}{2} p_k \rho^{|k|+1} + A_k \rho^{|k|+1} + B_k \rho^{|k|-1}, \tag{6.45}$$

$$v_k^\theta = i \operatorname{sgn} k \big(- A_k \rho^{|k|+1} + B_k \rho^{|k|-1} \big), \tag{6.46}$$

for some constants A_k and B_k. For $k = 0$ we get

$$v_0^\rho = \frac{1}{2} p_0 \rho + A_0 \rho, \tag{6.47}$$

$$v_0^\theta = B_0 \rho. \tag{6.48}$$

For each $k \in \mathbb{Z}$ we have to determine three constants: p_k, A_k, and B_k. These follow from boundary condition (6.40), incompressibility condition (6.39) and extra conditions (6.41) and (6.42). In polar coordinates conditions (6.40) and (6.39) are given by

$$2 \frac{\partial v^\rho}{\partial \rho} - p \quad = \quad f^\rho, \tag{6.49}$$

$$\frac{\partial v^\theta}{\partial \rho} + \frac{\partial v^\rho}{\partial \theta} - v^\theta \quad = \quad f^\theta \tag{6.50}$$

and

$$\frac{\partial v^\rho}{\partial \rho} + \frac{1}{\rho} v^\rho + \frac{1}{\rho} \frac{\partial v^\theta}{\partial \theta} = 0. \tag{6.51}$$

We distinguish between three cases: $k = 0$, $k = \pm 1$, and $k \notin \{-1, 0, 1\}$.

1. For $k = 0$, (6.43), (6.47), (6.48), and (6.49)-(6.51) give the underdetermined system

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} f_0^\rho \\ f_0^\theta \\ 0 \end{pmatrix}.$$

From this system, B_0 cannot be determined. However, condition (6.42) implies

$$\int_{\mathbb{S}^1} v^\theta \, d\sigma = \pm \int_{\mathbb{B}^2} \operatorname{rot} v \, dx = 0.$$

From (6.48) we get $B_0 = 0$. We conclude

$$p_0 = -f_0^\rho, \;\; A_0 = \frac{1}{2} f_0^\rho, \;\; B_0 = 0.$$

Combining this and (6.47) we get $v_0^\rho = 0$. There is also a condition on f, namely

$$f_0^\theta = 0. \tag{6.52}$$

2. For $k = \pm 1$, we derive from (6.43), (6.45), (6.46), and (6.49)-(6.51)

$$\begin{pmatrix} 1 & 4 & 0 \\ \pm 1 & 0 & 0 \\ 3 & 8 & 0 \end{pmatrix} \begin{pmatrix} p_{\pm 1} \\ A_{\pm 1} \\ B_{\pm 1} \end{pmatrix} = \begin{pmatrix} f_{\pm 1}^\rho \\ -2if_{\pm 1}^\theta \\ 0 \end{pmatrix}. \tag{6.53}$$

From the first and second equation in the system (6.53) it follows that

$$p_{\pm 1} = \mp 2if_{\pm 1}^\theta, \;\; A_{\pm 1} = \frac{1}{4} f_{\pm 1}^\rho \pm \frac{1}{2} if_{\pm 1}^\theta.$$

We cannot determine $B_{\pm 1}$ from (6.53). However, (6.39) and (6.41) imply for $j = 1, 2$

$$\begin{aligned} \int_{\mathbb{S}^1} x_j v \cdot n \, d\sigma &= \int_{\mathbb{B}^2} \operatorname{div}(x_j v) \, dx = \int_{\mathbb{B}^2} x_j \operatorname{div} v \, dx + \int_{\mathbb{B}^2} \nabla x_j \cdot v \, dx \\ &= \int_{\mathbb{B}^2} v_j \, dx = 0. \end{aligned} \tag{6.54}$$

This yields

$$v_{\pm 1}^\rho(1) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} (x_1 \pm ix_2) v \cdot n \, d\sigma = 0. \tag{6.55}$$

Combining this and (6.45) it follows that

$$B_{\pm 1} = -\frac{1}{4} f_{\pm 1}^\rho \pm \frac{1}{2} if_{\pm 1}^\theta.$$

From the third equation in (6.53) we derive two more conditions on f:

$$f_{\pm 1}^\rho = \pm if_{\pm 1}^\theta. \tag{6.56}$$

3. For $k \notin \{-1, 0, 1\}$ we get from (6.43), (6.45), (6.46), and (6.49)-(6.51) the following system of equations:

$$\begin{pmatrix} |k| & 2(|k|+1) & 2(|k|-1) \\ k & 0 & 4(k - \operatorname{sgn} k) \\ |k|+2 & 4(|k|+1) & 0 \end{pmatrix} \begin{pmatrix} p_k \\ A_k \\ B_k \end{pmatrix} = \begin{pmatrix} f_k^\rho \\ -2if_k^\theta \\ 0 \end{pmatrix}. \tag{6.57}$$

The matrix on the left-hand side is invertible for $k \notin \{-1, 0, 1\}$ and the solution to

(6.57) is given by

$$\begin{aligned} p_k &= -f_k^\rho - i\,\mathrm{sgn}\,k f_k^\theta, \\ A_k &= \frac{(|k|+2)(f_k^\rho + i\,\mathrm{sgn}\,k f_k^\theta)}{4(|k|+1)}, \\ B_k &= \frac{k f_k^\rho + i(|k|-2) f_k^\theta}{4(k-\mathrm{sgn}\,k)}. \end{aligned}$$

We are interested in the normal component of the velocity v^ρ on $\mathbb{S}^1$. For $k \notin \{-1,0,1\}$ we get from (6.45)

$$v_k^\rho(1) = \frac{1}{2}p_k + A_k + B_k$$

and (6.44) follows.

□

Since $s_k, s_{-k} \in \mathfrak{S}_k^2$ we have

$$\mathcal{N} s_k = |k| s_k, \tag{6.58}$$

with $\mathcal{N}$ as defined in (2.33). Now we write for $N = 2$ the expressions for $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ that we found in Corollary 6.9 in terms of $\mathcal{N}$.

- Consider (6.38)-(6.42) with $f = \kappa'(0)[r]n(0)$. By Lemma 3.6

  $$\begin{aligned} f_k^\rho &= (-k^2+1) r_k, \\ f_k^\theta &= 0, \end{aligned}$$

  with $r_k = (r, s_k)_0$. Note that the conditions in Lemma 6.10 are satisfied. As a consequence, for $k \notin \{-1,0,1\}$

  $$v_k^\rho(1) = -\frac{1}{2}|k| r_k$$

  and $v_0^\rho(1) = v_{\pm 1}^\rho(1) = 0$. Corollary 6.9 and (6.58) imply

  $$\mathcal{F}_1'(0)[r] = -\frac{1}{2}\mathcal{N}\mathcal{P}_1 r, \tag{6.59}$$

  where $\mathcal{P}_1 : \mathbb{L}^2(\mathbb{S}^1) \to \mathbb{L}^2(\mathbb{S}^1)$ is the orthogonal projection along $\langle s_{-1}, s_0, s_1 \rangle = \mathfrak{S}_0^2 \oplus \mathfrak{S}_1^2$.

- Consider (6.38)-(6.42) with $f = \frac{2N(1-N)}{\sigma_N} r n(0) + \frac{2N}{\sigma_N}\nabla_0 r = -\frac{2}{\pi} r n(0) + \frac{2}{\pi}\nabla_0 r$. From $\nabla_0 s_k = \frac{\partial s_k}{\partial \theta} e_\theta = i k s_k e_\theta$ it follows that

  $$f_k^\rho = -\frac{2}{\pi} r_k$$

and

$$f_k^\theta = \frac{2ik}{\pi} r_k.$$

We see that the conditions in Lemma 6.10 are satisfied. We get for all $k \in \mathbb{Z}$

$$v_k^\rho(1) = 0.$$

Corollary 6.9 yields

$$\mathcal{F}_2'(0)[r] = -\frac{1}{\pi} r - \frac{1}{2\pi}(r_1 s_1 + r_{-1} s_{-1}). \tag{6.60}$$

6.4.2 The three-dimensional boundary value problem

For the three-dimensional problem we introduce the spherical harmonics $Y_{km} : \mathbb{S}^2 \to \mathbb{C}$ for each $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \dots, 0, \dots, k-1, k\}$ by means of spherical coordinates in the following way:

$$Y_{km} = (-1)^m \sqrt{\frac{(2k+1)(k-m)!}{4\pi(k+m)!}} P_k^m(\cos\theta) e^{im\phi},$$

where θ is the polar coordinate, ϕ the azimuthal coordinate and P_k^m the Legendre polynomial given by

$$P_k^m(x) = \frac{\sqrt{(1-x^2)^m}}{2^k k!} \frac{d^{k+m}}{dx^{k+m}} \left[(x^2-1)^k\right].$$

Complexifying the spaces $\mathfrak{S}_k^3$ in Chapter 1, one can identify Y_{km} with the spherical harmonics s_{kj} with $j = m + k + 1$.

We introduce the vector spherical harmonics $\vec{V}_{km}$, $\vec{X}_{km}$, $\vec{W}_{km} : \mathbb{S}^2 \to \mathbb{C}^3$ conform [29] and [38] in the following way:

$$\begin{aligned}
\vec{V}_{km} :&= -\sqrt{\frac{k+1}{2k+1}} Y_{km} e_\rho + \frac{1}{\sqrt{(k+1)(2k+1)}} \frac{\partial Y_{km}}{\partial\theta} e_\theta \\
&\quad + \frac{im}{\sqrt{(k+1)(2k+1)}\sin\theta} Y_{km} e_\phi, \\
\vec{X}_{km} :&= -\frac{m}{\sqrt{k(k+1)}\sin\theta} Y_{km} e_\theta - \frac{i}{\sqrt{k(k+1)}} \frac{\partial Y_{km}}{\partial\theta} e_\phi, \\
\vec{W}_{km} :&= \sqrt{\frac{k}{2k+1}} Y_{km} e_\rho + \frac{1}{\sqrt{k(2k+1)}} \frac{\partial Y_{km}}{\partial\theta} e_\theta + \frac{im}{\sqrt{k(2k+1)}\sin\theta} Y_{km} e_\phi,
\end{aligned}$$

for $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \dots, 0, \dots, k-1, k\}$. The functions Y_{km} form a complete orthonormal set in $\mathbb{L}^2(\mathbb{S}^2)$ and $\vec{V}_{km}$, $\vec{X}_{km}$, $\vec{W}_{km}$, excluding $\vec{X}_{00} \equiv \vec{W}_{00} \equiv 0$, form a complete orthonormal set in $\left(\mathbb{L}^2(\mathbb{S}^2)\right)^3$. From [38, eqn. (B-13)] it is easily checked that the functions $\vec{W}_{1-1}$, $\vec{W}_{10}$, and $\vec{W}_{11}$ are three independent constant vector fields. Therefore, if we also take $\mathbb{L}^2(\mathbb{S}^2)$-inner products of the constant vector fields $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$, and $e_3 = (0,0,1)^T$ with other vector spherical harmonics we find

$$\int_{\mathbb{S}^2} \vec{W}_{km}\, d\sigma = 0, \quad k \in \mathbb{N}_0 \setminus \{1\}, \quad \int_{\mathbb{S}^2} \vec{W}_{1m}\, d\sigma \neq 0 \tag{6.61}$$

$$\int_{\mathbb{S}^2} \vec{V}_{km}\, d\sigma = \int_{\mathbb{S}^2} \vec{X}_{km}\, d\sigma = 0, \quad k \in \mathbb{N}_0, \tag{6.62}$$

for all $m \in \{-k, -k+1, \ldots, 0, \ldots, k-1, k\}$.

We will use the following identities:

$$Y_{km} e_\rho = -\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + \sqrt{\frac{k}{2k+1}}\vec{W}_{km}, \tag{6.63}$$

$$\nabla_0 Y_{km} = k\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + (k+1)\sqrt{\frac{k}{2k+1}}\vec{W}_{km}, \tag{6.64}$$

$$\text{rot}\,(g(\rho)V_{km}(\theta,\phi)) = i\sqrt{\frac{k}{2k+1}}\left(\frac{dg}{d\rho} + \frac{k+2}{\rho}g\right)\vec{X}_{km}, \tag{6.65}$$

$$\text{rot}\,(g(\rho)X_{km}(\theta,\phi)) = i\sqrt{\frac{k}{2k+1}}\left(\frac{dg}{d\rho} - \frac{k}{\rho}g\right)\vec{V}_{km} + i\sqrt{\frac{k+1}{2k+1}}\left(\frac{dg}{d\rho} + \frac{k+1}{\rho}g\right)\vec{W}_{km}, \tag{6.66}$$

$$\text{rot}\,(g(\rho)W_{km}(\theta,\phi)) = i\sqrt{\frac{k+1}{2k+1}}\left(\frac{dg}{d\rho} - \frac{k-1}{\rho}g\right)\vec{X}_{km}, \tag{6.67}$$

for any g depending on ρ only (see [29] or [38]). Introduce functions $\alpha_{km}, \beta_{km}, \gamma_{km} : [0,1] \to \mathbb{C}$ such that

$$v(\rho,\theta,\phi) = \sum_{k,m} \alpha_{km}(\rho)\vec{V}_{km}(\theta,\phi) + \beta_{km}(\rho)\vec{X}_{km}(\theta,\phi) + \gamma_{km}(\rho)\vec{W}_{km}(\theta,\phi) \tag{6.68}$$

and introduce $f^V_{km}, f^X_{km}, f^W_{km} \in \mathbb{C}$ such that

$$f(\theta,\phi) = \sum_{k,m} f^V_{km}\vec{V}_{km}(\theta,\phi) + f^X_{km}\vec{X}_{km}(\theta,\phi) + f^W_{km}\vec{W}_{km}(\theta,\phi).$$

Here and in the sequel summations are over all $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \ldots, 0, \ldots, k-1, k\}$, excluding terms containing $\vec{X}_{00}$ and $\vec{W}_{00}$. Because p is harmonic, there exist $p_{km} \in \mathbb{C}$ such that

$$p(\rho,\theta,\phi) = \sum_{k,m} p_{km}\rho^k Y_{km}(\theta,\phi). \tag{6.69}$$

Lemma 6.11. *For $N = 3$, the system* (6.38)-(6.42) *is solvable if and only if $f^X_{1m} = f^W_{1m} = 0$ for*

$m \in \{-1, 0, 1\}$. *Furthermore*

$$v \cdot n(0) = \sum_{k \neq 1, m} \left[-\frac{k}{2k^2 + 4k + 3} \sqrt{\frac{k+1}{2k+1}} f^V_{km} + \frac{1}{2(k-1)} \sqrt{\frac{k}{2k+1}} f^W_{km} \right] Y_{km}. \quad (6.70)$$

Proof. From (6.63), (6.64), and (6.69) we get

$$\nabla p = \sum_{k,m} p_{km} \rho^{k-1} \sqrt{k(2k+1)} \vec{W}_{km}.$$

This can also be derived from [29, eqn. (3.5)]. Combining (6.68) and [29, eqn. (2.16)] we obtain

$$\Delta v = \sum_{k,m} (\Lambda_{k+1} \alpha_{km}) \vec{V}_{km} + (\Lambda_k \beta_{km}) \vec{X}_{km} + (\Lambda_{k-1} \gamma_{km}) \vec{W}_{km},$$

where

$$\Lambda_k : \psi \mapsto \frac{\partial^2 \psi}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial \psi}{\partial \rho} - \frac{k(k+1)}{\rho^2} \psi. \quad (6.71)$$

From (6.38) we get

$$\begin{aligned} \Lambda_{k+1} \alpha_{km} &= 0, \\ \Lambda_k \beta_{km} &= 0, \\ \Lambda_{k-1} \gamma_{km} &= p_{km} \rho^{k-1} \sqrt{k(2k+1)}. \end{aligned}$$

The general regular solution to these equations is given by

$$\begin{aligned} \alpha_{km}(\rho) &= A_{km} \rho^{k+1}, \\ \beta_{km}(\rho) &= B_{km} \rho^k, \\ \gamma_{km}(\rho) &= C_{km} \rho^{k-1} + \frac{1}{2} \sqrt{\frac{k}{2k+1}} p_{km} \rho^{k+1}. \end{aligned}$$

For each pair (k, m) we have to determine four constants: p_{km}, A_{km}, B_{km}, and C_{km}. As in the two-dimensional case, these constants follow from the boundary conditions (6.40), the incompressibility condition (6.39), and extra conditions (6.41) and (6.42). In [29, eqns. (4.3)-(4.6)], conditions (6.40) and (6.39) are written in terms of α_{km}, β_{km}, and γ_{km}. If we substitute the expressions above, then we get for $k \in \mathbb{N}_0$ and $m \in \{-k, -k+1, \ldots, 0, \ldots, k-1, k\}$

$$\begin{pmatrix} \left(\frac{k+1}{2k+1}\right)^{\frac{3}{2}} & \frac{2k^2+3k+2}{2k+1} & 0 & 0 \\ 0 & 0 & k-1 & 0 \\ \sqrt{\frac{k}{2k+1}} \frac{2k^2-1}{2k+1} & -\frac{\sqrt{k}\sqrt{k+1}(2k+3)}{2k+1} & 0 & 2(k-1) \\ \frac{k}{2k+1} & -\sqrt{\frac{k+1}{2k+1}}(2k+3) & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{km} \\ A_{km} \\ B_{km} \\ C_{km} \end{pmatrix} = \begin{pmatrix} f^V_{km} \\ f^X_{km} \\ f^W_{km} \\ 0 \end{pmatrix}. \quad (6.72)$$

Only for $k = 1$ the matrix on the left-hand side is not invertible. In this case we get

$$\begin{pmatrix} (\frac{2}{3})^{\frac{3}{2}} & \frac{7}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3\sqrt{3}} & -\frac{5}{3}\sqrt{2} & 0 & 0 \\ \frac{1}{3} & -5\sqrt{\frac{2}{3}} & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{1m} \\ A_{1m} \\ B_{1m} \\ C_{1m} \end{pmatrix} = \begin{pmatrix} f_{1m}^V \\ f_{1m}^X \\ f_{1m}^W \\ 0 \end{pmatrix}. \tag{6.73}$$

The rank of the matrix in (6.73) is two. Therefore there are six restrictions on f and six degrees of freedom. From the last three equations in the system (6.73), we obtain the following conditions:

$$f_{1m}^X = f_{1m}^W = 0, \tag{6.74}$$

for $m = -1, 0, 1$. From the first and the fourth equation in (6.73) we derive

$$A_{1m} = \frac{1}{9} f_{1m}^V \tag{6.75}$$

and

$$p_{1m} = \frac{5}{3}\sqrt{\frac{2}{3}} f_{1m}^V. \tag{6.76}$$

In order to calculate B_{1m} and C_{1m}, for $m = -1, 0, 1$, we consider (6.41) and (6.42). Writing condition (6.41) in spherical coordinates and substituting (6.68) we get

$$\begin{aligned} 0 &= \int_{\mathbb{B}^3} v \, dx \\ &= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 \alpha_{km}(\rho) \vec{V}_{km} + \rho^2 \beta_{km}(\rho) \vec{X}_{km} + \rho^2 \gamma_{km}(\rho) \vec{W}_{km} \, d\rho d\sigma \\ &= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 A_{km} \rho^{k+3} \vec{V}_{km} + B_{km} \rho^{k+2} \vec{X}_{km} \\ &\quad + \left(C_{km} \rho^{k+1} + \frac{1}{2}\sqrt{\frac{k}{2k+1}} p_{km} \rho^{k+3} \right) \vec{W}_{km} \, d\rho d\sigma \\ &= \sum_{k,m} \int_{\mathbb{S}^2} \frac{A_{km}}{k+4} \vec{V}_{km} + \frac{B_{km}}{k+3} \vec{X}_{km} + \left(\frac{C_{km}}{k+2} + \frac{1}{2}\sqrt{\frac{k}{2k+1}} \frac{p_{km}}{k+4} \right) \vec{W}_{km} \, d\sigma. \end{aligned}$$

It follows from (6.61) and (6.62) that

$$\frac{1}{3} C_{1m} + \frac{1}{10\sqrt{3}} p_{1m} = 0,$$

for $m = -1, 0, 1$. Combining this and (6.76) we get

$$C_{1m} = -\frac{\sqrt{2}}{6} f_{1m}^V, \tag{6.77}$$

for $m = -1, 0, 1$.

Note that rot $(\alpha_{km}(\rho)\vec{V}_{km})$ and rot $(\gamma_{km}(\rho)\vec{W}_{km})$ have no $\vec{W}_{km}$-components (see (6.65) and (6.67)). It follows from (6.62) that

$$\int_{\mathbb{S}^2} \text{rot}\,(\alpha_{km}(\rho)\vec{V}_{km})d\sigma = \int_{\mathbb{S}^2} \text{rot}\,(\gamma_{km}(\rho)\vec{W}_{km})d\sigma = 0. \tag{6.78}$$

Combining (6.42), (6.68), (6.78), and (6.66) we get

$$\begin{aligned} 0 &= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 \text{rot}\,(\beta_{km}(\rho)\vec{X}_{km})\; d\rho d\sigma \\ &= \sum_{k,m} \int_{\mathbb{S}^2} \int_0^1 \rho^2 i \sqrt{\frac{k}{2k+1}} \left(\frac{\partial \beta_{km}}{\partial \rho} - \frac{k}{\rho}\beta_{km}\right) \vec{V}_{km} \\ &\quad + \rho^2 i \sqrt{\frac{k+1}{2k+1}} \left(\frac{\partial \beta_{km}}{\partial \rho} + \frac{k+1}{\rho}\beta_{km}\right) \vec{W}_{km}\; d\rho d\sigma. \end{aligned} \tag{6.79}$$

As a result of (6.61), (6.62), and (6.79) we get for all $m \in \{-, 1, 0, 1\}$

$$B_{1m} = 0. \tag{6.80}$$

For $k \neq 1$ the solution to (6.72) is given by

$$\begin{aligned} p_{km} &= \frac{(2k+3)\sqrt{(2k+1)(k+1)}}{2k^2+4k+3} f^V_{km}, \\ A_{km} &= \frac{k}{2k^2+4k+3} f^V_{km}, \\ B_{km} &= \frac{1}{k-1} f^X_{km}, \\ C_{km} &= \frac{1}{2(k-1)} \left[-\frac{\sqrt{k}\sqrt{k+1}(2k+3)(k-1)}{2k^2+4k+3} f^V_{km} + f^W_{km} \right]. \end{aligned}$$

For the normal component of v on the unit sphere we get

$$\begin{aligned} v \cdot n(0) &= \sum_{k,m} \alpha_{km}(1)\vec{V}_{km} \cdot e_\rho + \beta_{km}(1)\vec{X}_{km} \cdot e_\rho + \gamma_{km}(1)\vec{W}_{km} \cdot e_\rho \\ &= \sum_{k,m} \left[-\alpha_{km}(1)\sqrt{\frac{k+1}{2k+1}} + \gamma_{km}(1)\sqrt{\frac{k}{2k+1}} \right] Y_{km} \\ &= \sum_{k,m} \left[-\sqrt{\frac{k+1}{2k+1}} A_{km} + \sqrt{\frac{k}{2k+1}} C_{km} + \frac{1}{2}\frac{k}{2k+1} p_{km} \right] Y_{km} \\ &= \sum_{k\neq 1,m} \left[-\frac{k}{2k^2+4k+3}\sqrt{\frac{k+1}{2k+1}} f^V_{km} + \frac{1}{2(k-1)}\sqrt{\frac{k}{2k+1}} f^W_{km} \right] Y_{km}. \end{aligned}$$

In the last step we omitted the terms for $k = 1$. This is possible because from (6.75), (6.76), and (6.77) it follows that

$$-\sqrt{\frac{2}{3}}A_{1m} + \sqrt{\frac{1}{3}}C_{1m} + \frac{1}{6}p_{1m} = 0,$$

for $m = -1, 0, 1$. Note that the fact that the terms for $k = 1$ in (6.70) vanish also follows if we argue as in (6.54). □

The solution of the boundary value problem (6.38)-(6.42) is given in Appendix B.

Now we write $\mathcal{F}_1'(0)$ and $\mathcal{F}_2'(0)$ in terms of $\mathcal{N}$ for space dimension 3, making use of the formula

$$\mathcal{N}Y_{km} = kY_{km}. \tag{6.81}$$

As for the two-dimensional case, we do this by considering two special cases for f in the system (6.38)-(6.42).

- Let us consider the case $f = \kappa'(0)[r]n(0)$. By Lemma 3.6

  $$\kappa'(0)[r]n(0) = \left(-\mathcal{N}^2 r - \mathcal{N}r + 2r\right) n(0) = \sum_{k,m}(-k^2 - k + 2)r_{km}Y_{km}e_\rho,$$

  with $r_{km} = (r, Y_{km})_0$. It follows from (6.63) that

  $$\kappa'(0)[r]n(0) = \sum_{k,m}(-k^2 - k + 2)\left(-\sqrt{\frac{k+1}{2k+1}}\vec{V}_{km} + \sqrt{\frac{k}{2k+1}}\vec{W}_{km}\right) r_{km}.$$

  Consequently,

  $$\begin{aligned} f^V_{km} &= -\sqrt{\frac{k+1}{2k+1}}(-k^2 - k + 2)r_{km}, \\ f^X_{km} &= 0, \\ f^W_{km} &= \sqrt{\frac{k}{2k+1}}(-k^2 - k + 2)r_{km}. \end{aligned}$$

  Note that the condition in Lemma 6.11 is satisfied. From (6.70) we get

  $$v \cdot n(0) = \sum_{k \neq 1, m} -\frac{k(k+2)(k+\frac{1}{2})}{2k^2 + 4k + 3} r_{km}Y_{km}.$$

  Corollary 6.9 and (6.81) yield

  $$\mathcal{F}_1'(0)[r] = -\mathcal{N}\left(\mathcal{N} + 2\mathcal{I}\right)\left(\mathcal{N} + \frac{1}{2}\mathcal{I}\right)\left(2\mathcal{N}^2 + 4\mathcal{N} + 3\mathcal{I}\right)^{-1}\mathcal{P}_1 r. \tag{6.82}$$

- Now we consider the case $f = \frac{2N(1-N)}{\sigma_N}rn(0) + \frac{2N}{\sigma_N}\nabla_0 r = -\frac{3}{\pi}rn(0) + \frac{3}{2\pi}\nabla_0 r$. From formulas (6.63) and (6.64) we get

$$\begin{aligned} f &= \sum_{k,m} -\frac{3}{\pi} r_{km} \left[-\sqrt{\frac{k+1}{2k+1}} \vec{V}_{km} + \sqrt{\frac{k}{2k+1}} \vec{W}_{km} \right] \\ &\quad + \frac{3}{2\pi} r_{km} \left[k\sqrt{\frac{k+1}{2k+1}} \vec{V}_{km} + (k+1)\sqrt{\frac{k}{2k+1}} \vec{W}_{km} \right] \\ &= \sum_{k,m} \frac{3}{2\pi}(k+2)\sqrt{\frac{k+1}{2k+1}} r_{km} \vec{V}_{km} + \frac{3}{2\pi}(k-1)\sqrt{\frac{k}{2k+1}} r_{km} \vec{W}_{km}. \end{aligned}$$

In this case we have

$$\begin{aligned} f^V_{km} &= \frac{3}{2\pi}(k+2)\sqrt{\frac{k+1}{2k+1}} r_{km}, \\ f^X_{km} &= 0, \\ f^W_{km} &= \frac{3}{2\pi}(k-1)\sqrt{\frac{k}{2k+1}} r_{km}. \end{aligned}$$

From (6.70) we get

$$v \cdot n = \sum_{k \neq 1, m} -\frac{3}{4\pi} \frac{k}{2k^2+4k+3} r_{km} Y_{km}.$$

Corollary 6.9 implies

$$\begin{aligned} \mathcal{F}_2'(0)[r] = -\frac{3}{4\pi} \mathcal{N} \left(2\mathcal{N}^2 + 4\mathcal{N} + 3\mathcal{I} \right)^{-1} \mathcal{P}_1 r - \frac{3}{4\pi} r \\ -\frac{1}{4\pi} \left(r_{1-1} Y_{1-1} + r_{10} Y_{10} + r_{11} Y_{11} \right). \end{aligned} \tag{6.83}$$

We summarise the results for the linearisations for $N = 2, 3$.

Corollary 6.12. *For $N = 2$ we have*

$$\begin{aligned} \mathcal{F}_1'(0)[r] &= -\frac{1}{2} \mathcal{N} \mathcal{P}_1 r, \\ \mathcal{F}_2'(0)[r] &= -\frac{1}{\pi} r - \frac{1}{2\pi} \left(r_1 s_1 + r_{-1} s_{-1} \right). \end{aligned}$$

For $N = 3$ we have

$$\begin{aligned} \mathcal{F}_1'(0)[r] &= -\mathcal{N} \left(\mathcal{N} + 2\mathcal{I} \right) \left(\mathcal{N} + \frac{1}{2}\mathcal{I} \right) \left(2\mathcal{N}^2 + 4\mathcal{N} + 3\mathcal{I} \right)^{-1} \mathcal{P}_1 r, \\ \mathcal{F}_2'(0)[r] &= -\frac{3}{4\pi} \mathcal{N} \left(2\mathcal{N}^2 + 4\mathcal{N} + 3\mathcal{I} \right)^{-1} \mathcal{P}_1 r \\ &\quad -\frac{3}{4\pi} r - \frac{1}{4\pi} \left(r_{1-1} Y_{1-1} + r_{10} Y_{10} + r_{11} Y_{11} \right). \end{aligned}$$

For $N = 2$ we have

$$(s_k, \mathcal{F}_j'(0)[s_k])_0 = -p_j(|k|) \tag{6.84}$$

for $j = 1, 2$, $k \in \mathbb{Z}$ and

$$p_1(k) = \begin{cases} \dfrac{k}{2} & k \neq 1 \\ 0 & k = 1 \end{cases}$$

$$p_2(k) = \begin{cases} \dfrac{1}{\pi} & k \neq 1 \\ \dfrac{3}{2\pi} & k = 1. \end{cases}$$

For $N = 3$, we have

$$(Y_{km}, \mathcal{F}_j'(0)[Y_{km}])_0 = -p_j(k), \tag{6.85}$$

for $j = 1, 2$, $k \in \mathbb{Z}$, $m \in \{-k, -k+1, \ldots, 0, \ldots, k-1, k\}$ and

$$p_1(k) = \begin{cases} \dfrac{k(k+2)(k+\frac{1}{2})}{2k^2 + 4k + 3} & k \neq 1 \\ 0 & k = 1 \end{cases}$$

$$p_2(k) = \begin{cases} \dfrac{3}{4\pi} \dfrac{k}{2k^2 + 4k + 3} + \dfrac{3}{4\pi} & k \neq 1 \\ \dfrac{1}{\pi} & k = 1. \end{cases}$$

Lemma 6.13. *Let $N = 2$ or $N = 3$. There exists a $c_1 > 0$ such that for all $r \in \mathbb{H}^s(\mathbb{S}^{N-1})$*

$$(\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-1} \leq -c_1 \|\tilde{r}\|^2_{s-\frac{1}{2}}$$

and

$$(\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-1} \leq 0,$$

where $\tilde{r} := \mathcal{P}_1 r$.

Proof. Define

$$c_1 := \inf_{k \in \mathbb{N}\setminus\{1\}} \frac{\gamma p_1(k)}{(k^2+1)^{\frac{1}{2}}}.$$

All values for $p_1(k)$ with integer $k \geq 2$ are positive and $\lim_{k\to\infty} \frac{p_1(k)}{(k^2+1)^{\frac{1}{2}}}$ is positive. Therefore $c_1 > 0$. Furthermore, $p_2(k) \geq 0$. This proves the lemma. □

6.5 Energy estimates and global existence results for the injection problems

In this section we find a global existence result and decay properties for solutions to (6.29) with $\mu > 0$. As in the previous chapter this is done by combining the estimates in Lemma 6.13 for the linearisation and perturbation arguments to obtain a useful estimate for the nonlinear evolution operator in (6.29). In contrast to the Hele-Shaw problem, it is sufficient to use the first order chain rule in Lemma 5.4 to close the regularity gap, because $\mathcal{F}_1$ and $\mathcal{F}_2$ are first order operators.

Let for $\sigma > 1$ the norm $\|\cdot\|_{\sigma-1,1}$ on $\mathbb{H}^\sigma(\mathbb{S}^{N-1})$ be induced by the inner product

$$(r, \tilde{r})_{\sigma-1,1} := (r, \tilde{r})_{\sigma-1} + \sum_i (D_i r, D_i \tilde{r})_{\sigma-1}, \tag{6.86}$$

where the operators D_i for $i \in \left\{1, 2, \ldots, \binom{N}{2}\right\}$ are defined as in (5.11). This norm is equivalent to the norm $\|\cdot\|_\sigma$ that we introduced earlier (see [33, Sec. 4]).

Lemma 6.14. *If $\Omega_{R(t)}$ satisfies (6.1)-(6.6), then*

$$M(t) = \int_{\Omega_{R(t)}} x \, dx$$

is constant in t.

Proof. Let M_i be the ith component of M. Combining the divergence theorem, (6.2), (6.4), and (6.5) we get

$$\frac{dM_i(t)}{dt} = \int_{\Gamma_{R(t)}} (v \cdot n_R) x_i \, d\sigma = \int_{\Omega_{R(t)}} v_i \, dx + \int_{\Omega_{R(t)}} x_i \operatorname{div} v \, dx = 0,$$

where n_R is the normal on $\Gamma_{R(t)}$. □

Let $\mathfrak{M}_1^N$ be as defined in (2.40) and let $\mathbb{H}_1^\sigma(\mathbb{S}^{N-1})$, f_1, $\mathcal{P}_1$, ϕ_1 and ψ_1 be as defined in (5.27)-(5.29). From Lemma 6.14 we see that if r is a solution to (6.29) with $r_0 \in \mathfrak{M}_1^N$, then $r(t) \in \mathfrak{M}_1^N$ for all t.

As we have seen before, ϕ_1 is an analytic bijection between a neighbourhood of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ and a neighbourhood of zero in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{H}_1^s(\mathbb{S}^{N-1})$ (see Figure 5.2). We use this bijection to obtain a stability result for Stokes flow with injection in space dimensions 2 and 3 as we did for Hele-Shaw flow for $N \geq 4$. It follows from Lemma 6.14 that a solution r to (6.29) satisfies

$$f_1(r(\tau)) = \left(\frac{V_0}{\alpha(\tau)^N}, \frac{1}{\alpha(\tau)^{N+1}} m_0 \right)^T =: (V_\tau, m_\tau)^T,$$

where

$$(V_0, m_0)^T := f_1(r(0)).$$

For notational convenience we introduce $q_\tau := (V_\tau, m_\tau)^T$.

Theorem 6.15. *Let $N = 2$ or $N = 3$ and $\mu > 0$. Suppose that $s > \frac{N+6}{2}$. There exist a $\delta > 0$ and an $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with $\|r_0\|_s < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0, \tag{6.87}$$

has a solution $r \in \mathcal{C}_w([0, \infty), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}_w^1([0, \infty), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$. Furthermore,

$r \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0,\infty))$. *If we regard r as a function of the original time-variable t, then*

$$\|r(t)\|_s \le M \left(\frac{\mu N t}{\sigma_N} + 1 \right)^{-1} \|r_0\|_s. \tag{6.88}$$

Proof. Choose $\lambda_0 \in (0, \frac{c_1}{2})$ and define $\varepsilon := \frac{c_1}{2} - \lambda_0$, with c_1 as defined in Lemma 6.13.

1. If r satisfies (6.29), then $\tilde{r} := \mathcal{P}_1 r$ satisfies

$$\frac{\partial \tilde{r}}{\partial \tau} = \mathcal{P}_1 \mathcal{F}\left(\phi_1^{-1}(q_\tau, \tilde{r}), \tau \right). \tag{6.89}$$

   First we prove solvability of this equation. Assume that $|q_0|$ is small, $\tilde{r} \in \mathbb{H}^{s+1}(\mathbb{S}^{N-1})$, and $\|\tilde{r}\|_s < \delta'$, with δ' small enough. The symbol C is used for a constant that may vary throughout the proof.

2. As in the proof of Theorem 5.10 we have by Lemma 6.13

$$\gamma(\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-1} + \mu\alpha(\tau)^{1-N}(\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-1} \le -c_1 \|\tilde{r}\|^2_{s-\frac{1}{2}}. \tag{6.90}$$

3. Using $\alpha(\tau)^{1-N} \le 1$ and the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ are first order operators we derive parallel to (5.49)-(5.51) for $\|\tilde{r}\|_s$ small

$$\begin{aligned} & \gamma\left\{ \left(\tilde{r}, \mathcal{P}_1\mathcal{F}_1\left(\phi_1^{-1}(q_\tau,\tilde{r})\right)\right)_{s-1} - (\tilde{r}, \mathcal{F}_1'(0)[\tilde{r}])_{s-1} \right\} \\ & +\mu\alpha(\tau)^{1-N}\left\{ \left(\tilde{r}, \mathcal{P}_1\mathcal{F}_2\left(\phi_1^{-1}(q_\tau,\tilde{r})\right)\right)_{s-1} - (\tilde{r}, \mathcal{F}_2'(0)[\tilde{r}])_{s-1} \right\} \\ \le\ & C\left(|q_\tau| \|\tilde{r}\|_{s-\frac{1}{2}} + \|\tilde{r}\|^3_{s-\frac{1}{2}} \right). \end{aligned} \tag{6.91}$$

   Note that $s > \frac{N+6}{2}$ is necessary for analyticity (see Lemma 6.5).

4. From (5.12) we get

$$\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi_1^{-1}(q_\tau,\tilde{r}), \tau\right)\right)_{s-1,1} = \gamma(F_1 + G_1) + \mu\alpha(\tau)^{1-N}(F_2 + G_2), \tag{6.92}$$

   where for $k = 1, 2$

$$\begin{aligned} F_k &:= \left(\tilde{r}, \mathcal{P}_1\mathcal{F}_k\left(\phi_1^{-1}(q_\tau,\tilde{r})\right)\right)_{s-1}, \\ G_k &:= \sum_i \left(D_i\tilde{r}, \mathcal{P}_1\mathcal{F}_k'\left(\phi_1^{-1}(q_\tau,\tilde{r})\right)[D_i\phi_1^{-1}(q_\tau,\tilde{r})]\right)_{s-1}. \end{aligned}$$

   We will estimate the terms F_k and G_k, for $k = 1, 2$, separately.

5. From (6.90) and (6.91) it follows that for $\|\tilde{r}\|_s$ small

$$\begin{aligned}\gamma F_1 + \mu\alpha(\tau)^{1-N}F_2 &\le -c_1\|\tilde{r}\|^2_{s-\frac{1}{2}} + C\left(|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2}} + \|\tilde{r}\|^3_{s-\frac{1}{2}}\right)\\ &\le (-c_1 + C\delta')\|\tilde{r}\|^2_{s-\frac{1}{2}} + C|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2}}. \qquad (6.93)\end{aligned}$$

6. Now we find an estimate for G_1. From $\psi_1(r) = \phi_1^{-1}(0,\tilde{r})$ and Lemma 5.7 we obtain

$$G_1 = \sum_i (D_i\tilde{r}, I_i + J_i)_{s-1}, \qquad (6.94)$$

where

$$\begin{aligned}I_i &:= \mathcal{P}_1\mathcal{F}_1'\left(\phi_1^{-1}(q_\tau,\tilde{r})\right)\left[D_i\phi_1^{-1}(q_\tau,\tilde{r})\right] - \mathcal{P}_1\mathcal{F}_1'\left(\phi_1^{-1}(0,\tilde{r})\right)\left[D_i\phi_1^{-1}(0,\tilde{r})\right]\\ J_i &:= \mathcal{P}_1\mathcal{F}_1'(\psi_1(\tilde{r}))[\psi_1'(\tilde{r})[D_i\tilde{r}]].\end{aligned}$$

Making use of the triangular inequality, $\psi_1(r) = \phi_1^{-1}(0,\tilde{r})$, Lemma 5.7, (5.56), and Lipschitz continuity of $\mathcal{F}_1' \circ \phi_1^{-1}$ we derive for $\|\tilde{r}\|_s$ small

$$\begin{aligned}\|I_i\|_{s-\frac{3}{2}} &\le \left\|\left\{\mathcal{P}_1\mathcal{F}_1'\left(\phi_1^{-1}(q_\tau,\tilde{r})\right) - \mathcal{P}_1\mathcal{F}_1'\left(\phi_1^{-1}(0,\tilde{r})\right)\right\}[\psi_1'(\tilde{r})[D_i\tilde{r}]]\right\|_{s-\frac{3}{2}}\\ &\quad + \left\|\mathcal{P}_1\mathcal{F}_1'\left(\phi_1^{-1}(q_\tau,\tilde{r})\right)\left[D_i\left(\phi_1^{-1}(q_\tau,\tilde{r}) - \phi_1^{-1}(0,\tilde{r})\right)\right]\right\|_{s-\frac{3}{2}}\\ &\le C|q_\tau|\|\tilde{r}\|_{s+\frac{1}{2}} + C|q_\tau|.\end{aligned}$$

As a consequence,

$$(D_i\tilde{r}, I_i)_{s-1} \le C|q_\tau|\|\tilde{r}\|^2_{s+\frac{1}{2}} + C|q_\tau|\|\tilde{r}\|_{s+\frac{1}{2}}. \qquad (6.95)$$

Because $\psi_1(0) = 0$ and $\psi_1'(0)$ is the identity (see Corollary 2.20), $\mathcal{F}_1'(0)$ is the Fréchet derivative at zero of the local analytic operator $\mathcal{P}_1 \circ \mathcal{F}_1 \circ \psi_1$ on a neighbourhood of zero in $\mathbb{H}_1^{s-\frac{1}{2}}(\mathbb{S}^{N-1})$. As a result

$$\begin{aligned}\gamma(D_i\tilde{r}, J_i)_{s-1} &\le \gamma(D_i\tilde{r}, \mathcal{F}_1'(0)[D_i\tilde{r}])_{s-1}\\ &\quad + \gamma\left(D_i\tilde{r}, \left\{\mathcal{P}_1\mathcal{F}_1'(\psi_1(\tilde{r}))[\psi_1'(\tilde{r})[D_1\tilde{r}]] - \mathcal{F}_1'(0)\right\}[D_i\tilde{r}]\right)_{s-1}\\ &\le -c_1\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} + C\|\tilde{r}\|_{s-\frac{1}{2}}\|D_i\tilde{r}\|^2_{s-\frac{1}{2}}\\ &\le -c_1\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} + C\delta'\|\tilde{r}\|^2_{s+\frac{1}{2}}. \qquad (6.96)\end{aligned}$$

Combining (6.95) and (6.96) we get from (6.94)

$$\gamma G_1 \le \sum_i -c_1\|D_i\tilde{r}\|^2_{s-\frac{1}{2}} + C\left(\delta'\|\tilde{r}\|^2_{s+\frac{1}{2}} + |q_\tau|\|\tilde{r}\|^2_{s+\frac{1}{2}} + |q_\tau|\|\tilde{r}\|_{s+\frac{1}{2}}\right).$$

We estimate G_2 in the same way, replacing c_1 by zero (see Lemma 6.13). Because $\alpha(\tau)^{1-N} \leq 1$ we get

$$\begin{aligned}&\gamma G_1 + \mu\alpha(\tau)^{1-N} G_2 \\ \leq\ & \sum_i -c_1 \|D_i\tilde{r}\|^2_{s-\frac{1}{2}} + C\left(\delta'\|\tilde{r}\|^2_{s+\frac{1}{2}} + |q_\tau|\|\tilde{r}\|^2_{s+\frac{1}{2}} + |q_\tau|\|\tilde{r}\|_{s+\frac{1}{2}}\right). \end{aligned} \tag{6.97}$$

7. Adding (6.93) and (6.97) and assuming that $|q_\tau| \leq |q_0| < \delta'$, it follows from (6.92) that

$$\begin{aligned}&\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi_1^{-1}(q_\tau, \tilde{r}), \tau\right)\right)_{s-1,1} \\ \leq\ & (-c_1 + C\delta' + C|q_\tau|)\|\tilde{r}\|^2_{s-\frac{1}{2},1} + C|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2},1} \\ \leq\ & (-c_1 + C\delta')\|\tilde{r}\|^2_{s-\frac{1}{2},1} + C|q_\tau|\|\tilde{r}\|_{s-\frac{1}{2},1} \\ \leq\ & (-c_1 + C\delta')\|\tilde{r}\|^2_{s-\frac{1}{2},1} + \frac{c_1}{2}\|\tilde{r}\|^2_{s-\frac{1}{2},1} + C|q_\tau|^2 \\ \leq\ & -\frac{c_1}{2}\|\tilde{r}\|^2_{s-\frac{1}{2},1} + C\left(\delta'\|\tilde{r}\|^2_{s-\frac{1}{2},1} + |q_\tau|^2\right). \end{aligned}$$

Here we used Cauchy's inequality. Choosing $\delta' < \frac{\varepsilon}{C}$ we get

$$\begin{aligned}\left(\tilde{r}, \mathcal{P}_1\mathcal{F}\left(\phi_1^{-1}(q_\tau, \tilde{r}), \tau\right)\right)_{s-1,1} &\leq -\lambda_0\|\tilde{r}\|^2_{s-1,1} + C|q_\tau|^2 \\ &\leq -\lambda_0\|\tilde{r}\|^2_{s-1,1} + C\frac{|q_0|^2}{\alpha(\tau)^{2N}}. \end{aligned} \tag{6.98}$$

8. Let $\tilde{r}_0 := \mathcal{P}_1 r_0$ be small in $\mathbb{H}^s_1(\mathbb{S}^{N-1})$. Arguing as in the proof of Theorem 5.10 we obtain from (6.98) a solution $\tilde{r}$ to (6.89) with $\tilde{r}(0) = \tilde{r}_0$. This solution is smooth for positive time by [60, Ch. 6 Prop. 9, 10] and

$$\|\tilde{r}(\tau)\|_{s-1,1} \leq Ce^{-\lambda_0\tau}\|\tilde{r}_0\|_{s-1,1} + \frac{C}{\alpha(\tau)^N}|q_0|. \tag{6.99}$$

As in the proof of Theorem 5.10 we construct a solution to the original problem from

$$r(\tau) := \phi_1^{-1}(q_\tau, \tilde{r}(\tau)),$$

with the desired decay properties.

□

Remark 6.16. *In view of* (6.99), *if we start with a domain Ω_{r_0} for which the zeroth and first Richardson moments vanish, i.e. $q_0 = (0,0)^T$, then convergence will be faster.*

Remark 6.17. *In contrast to the problem of Hele-Shaw flow with injection (see [76]), we cannot treat the case of zero surface tension for Stokes flow by the methods of the proof of Theorem 6.15.*

The order of $\mathcal{F}_2'(0)$ is lower than the order of $\mathcal{F}_2$. Therefore, energy estimates of the linearisation, $(r, \mathcal{F}_2'(0)[r])_s \leq -C\|r\|_s^2$, for some $C > 0$, cannot control energy estimates for the nonlinear part, $(r, \mathcal{F}_2(r) - \mathcal{F}_2'(0)[r])_s \leq \epsilon\|r\|_{s+\frac{1}{2}}^2$, for some $\epsilon > 0$.

6.6 Almost global existence results for the suction problems

In this section we use energy estimates to get an existence result for the suction problems in 2D and 3D. Starting close enough to the unit ball, an arbitrarily large portion of liquid can be removed.

Theorem 6.18. *Let $N = 2$ or $N = 3$, $\mu < 0$, $T_+ \in [0, \frac{\sigma_N}{|\mu|(N-1)})$, and $s > \frac{N+6}{2}$. There exists a $\delta > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with $\|r_0\|_s < \delta$, then there exists a solution $r \in \mathcal{C}_w([0,T_+), \mathbb{H}^s(\mathbb{S}^{N-1})) \cap \mathcal{C}_w^1([0,T_+), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$ to*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r, \tau), \quad r(0) = r_0. \tag{6.100}$$

Furthermore $((\xi, \tau) \mapsto r(\tau)(\xi)) \in \mathcal{C}^\infty(\mathbb{S}^{N-1} \times (0, T_+))$.

Proof. We assume that $r \in \mathbb{H}^{s+1}(\mathbb{S}^{N-1})$ with $\|r\|_s < \delta'$ for δ' small enough.

1. If $\mu < 0$, then $\alpha(\tau)^{1-N}$ goes to infinity as τ approaches $\frac{\sigma_N}{|\mu|(N-1)}$. Nevertheless, on the time interval $[0, T_+]$, $\alpha(\tau)^{1-N} < A$ for some $A > 0$. Choose $K \in \mathbb{N}$ such that for $k \geq K$
$$-\gamma p_1(k) + |\mu| A p_2(k) < 0.$$
Define $c_2 > 0$ by
$$c_2 := \inf_{k \geq K} \frac{\gamma p_1(k) - |\mu| A p_2(k)}{(k^2+1)^{\frac{1}{2}}}.$$
The positivity of c_2 follows from the fact that $\frac{\gamma p_1(k) - |\mu| A p_2(k)}{(k^2+1)^{\frac{1}{2}}}$ converges to $\frac{\gamma}{2}$ as k tends to infinity.

2. Let $\mathcal{P}_K : \mathbb{L}^2(\mathbb{S}^{N-1}) \to \mathbb{L}^2(\mathbb{S}^{N-1})$ be the orthogonal projection on the orthoplement of $\bigoplus_{k=0}^K \mathfrak{S}_k^N$ with respect to the $\mathbb{L}^2(\mathbb{S}^{N-1})$-inner product and define $r_{kj} := (r, s_{kj})_0$. Analogously to (5.35) we derive
$$\begin{aligned} &\gamma(r, \mathcal{F}_1'(0)[r])_{s-1} + \mu\alpha(\tau)^{1-N}(r, \mathcal{F}_2'(0)[r])_{s-1} \\ = \;& \sum_{k \leq K} (k^2+1)^{s-1+\frac{1}{2}} \frac{-\gamma p_1(k) + |\mu|\alpha(\tau)^{1-N} p_2(k)}{(k^2+1)^{\frac{1}{2}}} r_{kj}^2 \\ &+ \sum_{k > K} (k^2+1)^{s-1+\frac{1}{2}} \frac{-\gamma p_1(k) + |\mu|\alpha(\tau)^{1-N} p_2(k)}{(k^2+1)^{\frac{1}{2}}} r_{kj}^2 \\ \leq \;& C\|r\|_0^2 - c_2\|r\|_{s-\frac{1}{2}}^2. \end{aligned} \tag{6.101}$$

3. By analyticity of $\mathcal{F}_1$ and $\mathcal{F}_2$ and boundedness of $\alpha(\tau)^{1-N}$ on $[0, T_+]$ we have
$$\gamma(r, \mathcal{F}_1(r) - \mathcal{F}_1'(0)[r])_{s-1} + \mu\alpha(\tau)^{1-N}(r, \mathcal{F}_2(r) - \mathcal{F}_2'(0)[r])_{s-1} \leq C\|r\|^3_{s-\frac{1}{2}}. \quad (6.102)$$

4. By (6.101) and (6.102) we have
$$\gamma(r, \mathcal{F}_1(r))_{s-1} + \mu\alpha(\tau)^{1-N}(r, \mathcal{F}_2(r))_{s-1} \leq (-c_2 + C\delta')\|r\|^2_{s-\frac{1}{2}} + C\|r\|^2_0.$$

5. Further,
$$\begin{aligned} & \gamma(D_i r, \mathcal{F}_1'(r)[D_i r])_{s-1} + \mu\alpha(\tau)^{1-N}(D_i r, \mathcal{F}_2'(r)[D_i r])_{s-1} \\ = \; & \gamma(D_i r, \mathcal{F}_1'(0)[D_i r])_{s-1} + \mu\alpha(\tau)^{1-N}(D_i r, \mathcal{F}_2'(0)[D_i r])_{s-1} \\ & +\gamma(D_i r, \{\mathcal{F}_1'(r) - \mathcal{F}_1'(0)\}[D_i r])_{s-1} \\ & +\mu\alpha(\tau)^{1-N}(D_i r, \{\mathcal{F}_2'(r) - \mathcal{F}_2'(0)\}[D_i r])_{s-1} \\ \leq \; & (-c_2 + C\delta')\|D_i r\|^2_{s-\frac{1}{2}} + C\|D_i r\|^2_0. \end{aligned}$$

6. Combining these two results, using (5.12), and taking $\delta' < \frac{c_2}{C}$ we get
$$\begin{aligned} \gamma(r, \mathcal{F}_1(r))_{s-1,1} + \mu\alpha(\tau)^{1-N}(r, \mathcal{F}_2(r))_{s-1,1} & \leq (-c_2 + C\delta')\|r\|^2_{s-\frac{1}{2},1} + C\|r\|^2_{0,1} \\ & \leq C\|r\|^2_{0,1} \leq C\|r\|^2_{s-1,1}. \quad (6.103) \end{aligned}$$

   Choose $\delta < \delta' e^{-CT_+}$ and suppose that $\|r_0\| < \delta$. By Theorem A.1 there exists a solution r to (6.100) on $[0, T_+]$ that satisfies
$$\|r(\tau)\|_{s-1,1} \leq e^{C\tau}\|r_0\|_{s-1,1}.$$

   This solution is smooth on $(0, T_+)$ by [60, Ch. 6 Prop. 9, 10].

□

Chapter 7

Hele-Shaw flow with kinetic undercooling

7.1 Introduction

In Chapter 2 we proved that the spherical solution to the classical Hele-Shaw problem with injection is stable. On the other hand, the suction problem turns out to be ill-posed. However, we showed in Chapters 3 and 5 that if one assumes surface tension on the boundary, then the suction problem is regularised and the spherical solution is stable under certain conditions.

In this chapter we consider another type of regularisation besides surface tension, replacing boundary condition (3.3) by

$$p + \beta \frac{\partial p}{\partial n} = -\gamma \kappa. \tag{7.1}$$

The parameter $\beta > 0$ is the so-called kinetic undercooling constant.

The Hele-Shaw problem is the formal limit of the Stefan problem, that describes e.g. melting of ice. The name kinetic undercooling originates from this problem, since condition (7.1) with $\gamma = 0$ and $\beta > 0$ is used there to model certain thermodynamic effects on the interface between ice and water.

A justification for using (7.1) in a Hele-Shaw setting was given by Romero [65]. He relates the term $\beta \frac{\partial p}{\partial n}$ to the second principal curvature in the Hele-Shaw cell. This is the curvature of the very thin meniscus of the liquid in the narrow gap between the two plates.

In this chapter we study the injection problem for $N = 2, 3$ and the suction problem for $N = 2$. For 2D suction with $\beta = 0$ (surface tension only) it was possible to exclude positive eigenvalues assuming that the first Richardson moment vanishes for the initial domain. For $\beta > 0$ we have to deal with the problem that Richardson moments are no longer preserved. Nevertheless, it is still possible to restrict ourselves to domains that are symmetric with respect to both axes. In this way, the geometric centre stays at the origin for all time for the evolution leaves our symmetry intact. Suction in higher dimensions $N \geq 3$ is always linearly unstable for large time just as the problem for

$N \geq 4$ with $\beta = 0$.

Another problem that we encounter is the time dependence. Instead of a simple structure as in (5.8), we find an evolution equation in which we distinguish between three terms that are essentially time-dependent (see (7.16)).

In Section 7.2 we derive this evolution equation and linearise it in terms of the Dirichlet-to-Neumann operator as we did before. We investigate how the domain of definition of the evolution operator changes as a function of time and we derive Lemma 7.7, that tells us how to deal with the time dependence when we derive energy estimates for the nonlinear problem to obtain existence results in Sections 7.3 and 7.4.

7.2 The evolution equation and its linearisation

We reintroduce the Sobolev spaces $\mathbb{H}^s(\mathbb{S}^{N-1})$ as in Chapter 5 and we define the functions r and R, the domains Ω_r and Ω_R, and their boundaries Γ_r and Γ_R as in Chapter 1.

We assume that $r \in \mathbb{H}^s(\mathbb{S}^{N-1})$ with

$$s > \frac{N+5}{2}. \tag{7.2}$$

Define the harmonic function

$$U := p - \mu\Phi, \tag{7.3}$$

with Φ as in (2.10). It follows that

$$U + \beta\frac{\partial U}{\partial n} = -\gamma\kappa - \mu\Phi - \beta\mu\frac{\partial\Phi}{\partial n}, \qquad \text{on } \Gamma_\mathrm{R} := \partial\Omega_\mathrm{R}.$$

Define $K_R^\beta, L_R^\beta, Z_R^\beta : \Omega_R \to \mathbb{R}$ as the harmonic functions that satisfy

$$\begin{aligned} K_R^\beta + \beta\frac{\partial K_R^\beta}{\partial n} &= -\kappa && \text{on } \Gamma_\mathrm{R}, \\ L_R^\beta + \beta\frac{\partial L_R^\beta}{\partial n} &= -\Phi && \text{on } \Gamma_\mathrm{R}, \\ Z_R^\beta + \beta\frac{\partial Z_R^\beta}{\partial n} &= -\frac{\partial\Phi}{\partial n} && \text{on } \Gamma_\mathrm{R}. \end{aligned}$$

It is known that these functions are uniquely defined for appropriate domains. Note that

$$U = \gamma K_R^\beta + \mu L_R^\beta + \beta\mu Z_R^\beta. \tag{7.4}$$

In the next lemma we investigate the scaling properties of K_R^β, L_R^β, and Z_R^β.

Lemma 7.1. *Let $\eta > 0$ and let $r := \eta(1+R) - 1$ such that $\Omega_r = \eta\Omega_R$. We have for $x \in \Omega_R$*

$$\begin{aligned} K_R^\beta(x) &= \eta K_r^{\eta\beta}(\eta x), \\ L_R^\beta(x) &= \eta^{N-2} L_r^{\eta\beta}(\eta x) + c(\eta), \\ Z_R^\beta(x) &= \eta^{N-1} Z_r^{\eta\beta}(\eta x), \end{aligned}$$

where $c(\eta)$ only depends on η. We also have

$$\begin{aligned} \nabla K_R^\beta(x) &= \eta^2 \nabla K_r^{\eta\beta}(\eta x), \\ \nabla L_R^\beta(x) &= \eta^{N-1} \nabla L_r^{\eta\beta}(\eta x), \\ \nabla Z_R^\beta(x) &= \eta^{N} \nabla Z_r^{\eta\beta}(\eta x). \end{aligned}$$

Proof. It is sufficient to prove the first part of the lemma, since the second part follows directly from the first part. It is clear that $\eta K_r^{\eta\beta}(\eta x)$ is an harmonic expression in x. Let n_f be the normal vector field on Γ_f, where f is either R or r, and let κ_f be the mean curvature of Γ_f. Let $x \in \Gamma_R$. Note that by the scaling of the curvature,

$$\kappa_R(x) = \eta\kappa_r(\eta x). \tag{7.5}$$

Introduce $\xi = \eta x \in \Gamma_r$ and denote the derivatives with respect to x and ξ by ∇_x and ∇_ξ. It follows from (7.5) that

$$\begin{aligned} \eta K_r^{\eta\beta}(\eta x) + \beta n_R(x) \cdot \nabla_x(\eta K_r^{\eta\beta}(\eta x)) &= \eta \left(K_r^{\eta\beta} + \eta\beta n_r \cdot \nabla_\xi K_r^{\eta\beta} \right)(\xi) \\ &= \eta\kappa_r(\xi) = \kappa_R(x). \end{aligned}$$

Here we used $n_r(\xi) = n_R(x)$. The identities for L_R^β and Z_R^β are obtained in a similar way, making use of the scaling behaviour of Φ and

$$\frac{\partial\Phi}{\partial n_R}(x) = \eta^{N-1} \frac{\partial\Phi}{\partial n_r}(\eta x).$$

It is easily checked that for L_R^β one has to include an extra term $c(\eta)$ as in Lemma 2.2. □

Let the operators $\tilde{z}, n, \kappa$, and z be defined as in Section 5.2.

By (7.3), (7.4), (2.2), and Lemma 2.1

$$\begin{aligned} \frac{\partial R}{\partial t}(\xi) &= -\gamma \frac{\nabla K_R^\beta(\tilde{z}(R,\xi)) \cdot n(R,\xi)}{n(R,\xi) \cdot \xi} \\ &\quad + \mu \left(-\frac{\nabla L_R^\beta(\tilde{z}(R,\xi)) \cdot n(R,\xi)}{n(R,\xi) \cdot \xi} + \frac{1}{\sigma_N (1+R(\xi))^{N-1}} \right) \\ &\quad - \beta\mu \frac{\nabla Z_R^\beta(\tilde{z}(R,\xi)) \cdot n(R,\xi)}{n(R,\xi) \cdot \xi}. \end{aligned}$$

It follows from (3.14) and Lemma 7.1 with $\eta = \alpha(t)^{-1}$ that

$$\begin{aligned}\frac{\partial r}{\partial t} &= -\frac{\gamma}{\alpha(t)^3}\frac{(\nabla K_r^{\hat\beta(t)} \circ \tilde z(r)) \cdot n(r)}{n(r)\cdot \mathrm{id}} \\ &\quad + \frac{\mu}{\alpha(t)^N}\left(-\frac{(\nabla L_r^{\hat\beta(t)} \circ \tilde z(r)) \cdot n(r)}{n(r)\cdot \mathrm{id}} + \frac{1}{\sigma_N(1+r)^{N-1}} - \frac{1+r}{\sigma_N}\right) \\ &\quad - \frac{\beta\mu}{\alpha(t)^{N+1}}\frac{(\nabla Z_r^{\hat\beta(t)} \circ \tilde z(r)) \cdot n(r)}{n(r)\cdot \mathrm{id}}, \end{aligned} \tag{7.6}$$

where $\hat\beta(t) := \frac{\beta}{\alpha(t)}$.

On a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ we define the following mappings:

- $\mathcal{A} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N))$ and $\mathcal{Q} : \mathcal{U} \to \mathcal{L}\left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \left(\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N)\right)^N\right)$ by (2.25) and (2.26).
- for each $\hat\beta > 0$, $\mathcal{S}_{\hat\beta} : \mathcal{U} \to \mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$ by

$$\mathcal{S}_{\hat\beta}(r)u := \begin{pmatrix} \mathcal{A}(r)u \\ \mathrm{Tr}u + \hat\beta n(r)\cdot \mathcal{Q}(r)u \end{pmatrix}. \tag{7.7}$$

- $\varphi : \mathcal{U} \to \mathbb{H}^s(\mathbb{S}^{N-1})$ by (2.28) and $l : \mathcal{U} \to \mathbb{H}^s(\mathbb{S}^{N-1})$ by (5.4).
- $\omega : \mathcal{U} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ by

$$\omega(r) := n(r)\cdot \mathcal{Q}(r)\,(\Phi \circ z(r)) = n(r)\cdot\left\{-\frac{1}{\sigma_N(1+r)^{N-1}}\mathrm{id}\right\}. \tag{7.8}$$

Note that $\mathcal{A}$ and $\mathcal{Q}$ are well-defined because of (7.2). The operator ω is well-defined because $\mathcal{Q}$ maps $\mathcal{U}$ to $\mathcal{L}\left(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N), \left(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)\right)^N\right)$ as well.

Lemma 7.2. *The operator $\mathcal{S}_{\hat\beta}(0)$ defines an isomorphism between $\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$ and also between $\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N)$ and $\mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$. For each $\hat\beta > 0$ there exists a neighbourhood $\mathcal{U}_{\hat\beta} > 0$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ such that for $r \in \mathcal{U}_{\hat\beta}$ the mapping $\mathcal{S}_{\hat\beta}(r)$ is an isomorphism as well.*

Proof. To prove the first part, let $(f, g) \in \mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$. By the Lax-Milgram Theorem there is a unique a weak solution u to $\mathcal{S}_{\hat\beta}(0)u = (f, g)$. By [79, Thm. 20.4] we have $u \in \mathbb{H}^2(\mathbb{B}^N)$. Hence by [72, Sec. 5.4.1] and continuous interpolation we have $u \in \mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$.

The second statement follows from continuity of $\mathcal{S}_{\hat\beta}$ near zero and the fact that isomorphisms form an open subset in the set of bounded linear mappings. □

From this lemma it follows that for each $\hat{\beta} > 0$ there exists a neighbourhood $\mathcal{U}_{\hat{\beta}}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ on which the operator $\mathcal{E}_{\hat{\beta}} : \mathcal{U}_{\hat{\beta}} \to \mathcal{L}(\mathbb{H}^{s-2}(\mathbb{S}^{N-1}), \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$ given by

$$\mathcal{E}_{\hat{\beta}}(r)\psi := \frac{\mathrm{Tr}\left(\mathcal{Q}(r)\mathcal{S}_{\hat{\beta}}(r)^{-1}\begin{bmatrix} 0 \\ \psi \end{bmatrix}\right)\cdot n(r)}{n(r)\cdot \mathrm{id}}$$

is well defined. By Lemma 7.2 $\mathcal{E}_{\hat{\beta}}$ also defines a mapping between $\mathcal{U}_{\hat{\beta}}$ and $\mathcal{L}(\mathbb{H}^{s-1}(\mathbb{S}^{N-1}), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$. The dependence of $\mathcal{U}_{\hat{\beta}}$ on $\hat{\beta}$ is investigated in Corollary 7.4.

Let $\mathcal{N}$ be the Dirichlet-to-Neumann operator defined by (2.33). It is clear that for each $\hat{\beta}$ the operator $\mathcal{I} + \hat{\beta}\mathcal{N}$ is bounded and invertible from $\mathbb{H}^{\sigma+1}(\mathbb{S}^{N-1})$ to $\mathbb{H}^{\sigma}(\mathbb{S}^{N-1})$ for any $\sigma > 0$. Furthermore,

$$\mathrm{Tr}\circ\mathcal{S}_{\hat{\beta}}(0)^{-1}\begin{pmatrix} 0 \\ \cdot \end{pmatrix} = \left(\mathcal{I} + \hat{\beta}\mathcal{N}\right)^{-1}.$$

Since for all $k \in \mathbb{N}_0$

$$\frac{\sqrt{1+k^2}}{1+\hat{\beta}k} \le C\max\{1, \hat{\beta}^{-1}\},$$

we have by (2.37)

$$\left\|\left(\mathcal{I} + \hat{\beta}\mathcal{N}\right)^{-1}\right\|_{\mathcal{L}(\mathbb{H}^{\sigma}(\mathbb{S}^{N-1}),\mathbb{H}^{\sigma+1}(\mathbb{S}^{N-1}))} \le C\max\{1, \hat{\beta}^{-1}\}. \tag{7.9}$$

Lemma 7.3. *There exists a $C > 0$ independent of $\hat{\beta}$ such that for all $\hat{\beta} > 0$*

$$\|\mathcal{S}_{\hat{\beta}}(0)^{-1}\|_{\mathcal{L}(Y,X)} \le C\max\left\{\hat{\beta}, \hat{\beta}^{-1}\right\},$$

where $X := \mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$, $Y := \mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1})$.

Proof. In this proof C is a varying constant that is independent of f, g, and $\hat{\beta}$. Let $(f,g) \in Y$. By Lemma 7.2, the problem

$$\mathcal{S}_{\hat{\beta}}(0)\psi = \begin{pmatrix} f \\ g \end{pmatrix}$$

is uniquely solvable for $\psi \in \mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$. Introduce the functions ψ_H and ψ_0 such that $\psi = \psi_H + \psi_0$ with

$$\Delta\psi_H = 0, \quad \Delta\psi_0 = f$$

and

$$\mathrm{Tr}\psi_H = \mathrm{Tr}\psi, \quad \mathrm{Tr}\psi_0 = 0.$$

Note that

$$\mathrm{Tr}\psi_H + \hat{\beta}\frac{\partial\psi_H}{\partial n} = g - \hat{\beta}\frac{\partial\psi_0}{\partial n} \tag{7.10}$$

and since ψ_H is harmonic we have

$$\text{Tr}\psi_H = \left(\mathcal{I} + \hat{\beta}\mathcal{N}\right)^{-1} \left(\text{Tr}\psi_H + \hat{\beta}\frac{\partial \psi_H}{\partial n}\right). \tag{7.11}$$

From [23, Sec. 6.3 Thm. 5] it follows that for $u \in \mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$

$$\|u\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \leq C\left(\|\text{Tr}u\|_{s-1} + \|\Delta u\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)}\right). \tag{7.12}$$

Hence

$$\|\psi_0\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \leq C\|f\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} \tag{7.13}$$

and therefore

$$\left\|\frac{\partial \psi_0}{\partial n}\right\|_{s-2} \leq C\|\psi_0\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \leq C\|f\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)}. \tag{7.14}$$

It follows from (7.12), (7.11), (7.9), (7.10), and (7.14) that

$$\begin{aligned}
\|\psi_H\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} &\leq C\|\text{Tr}\psi_H\|_{s-1} \\
&= C\left\|\left(\mathcal{I} + \hat{\beta}\mathcal{N}\right)^{-1}\left(\text{Tr}\psi_H + \hat{\beta}\frac{\partial \psi_H}{\partial n}\right)\right\|_{s-1} \\
&\leq C\max\{1, \hat{\beta}^{-1}\}\left(\left\|\hat{\beta}\frac{\partial \psi_0}{\partial n}\right\|_{s-2} + \|g\|_{s-2}\right) \\
&\leq C\max\{1, \hat{\beta}^{-1}\}\left(\hat{\beta}\,\|f\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} + \|g\|_{s-2}\right) \\
&= C\max\{1, \hat{\beta}\}\left(\|f\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} + \hat{\beta}^{-1}\|g\|_{s-2}\right).
\end{aligned}$$

Combining this with (7.13) one gets

$$\|\psi\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \leq C\max\{1, \hat{\beta}\}\left(\|f\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} + \hat{\beta}^{-1}\|g\|_{s-2}\right).$$

This proves the lemma. □

It is known (see e.g. [13, Sec. 2.3 Cor. 1]) that $\mathcal{S}_{\hat{\beta}}(r)$ is invertible if

$$\|\mathcal{S}_{\hat{\beta}}(r) - \mathcal{S}_{\hat{\beta}}(0)\|_{\mathcal{L}(X,Y)} \leq \|\mathcal{S}_{\hat{\beta}}(0)^{-1}\|^{-1}_{\mathcal{L}(Y,X)}.$$

Furthermore,

$$\begin{aligned}
&\left\|\mathcal{S}_{\hat{\beta}}(r) - \mathcal{S}_{\hat{\beta}}(0)\right\|_{\mathcal{L}(X,Y)} \\
\leq\ & C\,\|\mathcal{A}(r) - \Delta\|_{\mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N),\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N))} + \hat{\beta}\left\|n(r)\cdot\mathcal{Q}(r) - \frac{\partial}{\partial n}\right\|_{\mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N),\mathbb{H}^{s-2}(\mathbb{S}^{N-1}))} \\
\leq\ & C(1+\hat{\beta})\|r\|_s \leq C\max\{1, \hat{\beta}\}\|r\|_s.
\end{aligned}$$

Combining this with Lemma 7.3 and (7.15) we see that $\mathcal{S}_{\hat{\beta}}(r)$ is invertible if

$$\|r\|_s \le C \min\{\hat{\beta}, \hat{\beta}^{-2}\},$$

for some $C > 0$ independent of $\hat{\beta}$.

Corollary 7.4. *There exists an open neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ such that for each $\hat{\beta} > 0$, $\mathcal{S}_{\hat{\beta}}(r)$ is invertible for all $r \in \mathcal{U}_{\hat{\beta}}$ with*

$$\mathcal{U}_{\hat{\beta}} := \min\{\hat{\beta}, \hat{\beta}^{-2}\}\mathcal{U}.$$

We rewrite (7.6) for $r \in \mathcal{U}_{\frac{\beta}{\alpha(t)}}$ in the following way:

$$\frac{\partial r}{\partial t} = \frac{\gamma}{\alpha(t)^3}\mathcal{F}_1(r,t) + \frac{\mu}{\alpha(t)^N}\mathcal{F}_2(r,t) + \frac{\beta\mu}{\alpha(t)^{N+1}}\mathcal{F}_3(r,t), \tag{7.15}$$

where

$$\begin{aligned}
\mathcal{F}_1(\cdot,\tau) &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathbb{H}^{s-2}(\mathbb{S}^{N-1}),\\
\mathcal{F}_2(\cdot,\tau) &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1}),\\
\mathcal{F}_3(\cdot,\tau) &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1}),
\end{aligned}$$

are defined by

$$\begin{aligned}
\mathcal{F}_1(r,t) &:= \mathcal{E}_{\frac{\beta}{\alpha(t)}}(r)\kappa(r),\\
\mathcal{F}_2(r,t) &:= \mathcal{E}_{\frac{\beta}{\alpha(t)}}(r)\varphi(r) + l(r),\\
\mathcal{F}_3(r,t) &:= \mathcal{E}_{\frac{\beta}{\alpha(t)}}(r)\omega(r).
\end{aligned}$$

In view of Corollary 7.4, to make sure that r stays in the domain of definition of $\mathcal{F}_k(\cdot,t)$ for $k = 1,2,3$, we need to show in the proof of the existence theorems for injection that $\|r(t)\|_s$ decays at least as fast as $\alpha(t)^{-1}$ as t tends to infinity and for suction $\|r(t)\|_s$ decays at least as fast as $\alpha(t)^2$ as t tends to $\frac{\sigma_N}{|\mu|N}$ (see also Figure 7.1).

Introduce the time variable $\tau = \tau(t)$ such that (3.25) holds and $\tau(0) = 0$. As in Chapter 5, for $N = 2$ the injection problem is now defined on a finite time interval whereas the suction problem is defined on an infinite time interval. For $N = 3$ both problems are defined on an infinite time interval as in Chapter 3. We get

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r,\tau) := \gamma\mathcal{F}_1(r,\tau) + \mu\alpha(\tau)^{3-N}\mathcal{F}_2(r,\tau) + \beta\mu\alpha(\tau)^{2-N}\mathcal{F}_3(r,\tau). \tag{7.16}$$

For convenience we write here and hereafter $\mathcal{F}_k(r,\tau)$ instead of $\mathcal{F}_k(r,t(\tau))$ $(k = 1,2,3)$ and $\alpha(\tau)$ instead of $\alpha(t(\tau))$. Note that for $N = 3$, the term with $\mathcal{F}_1$ scales in the same

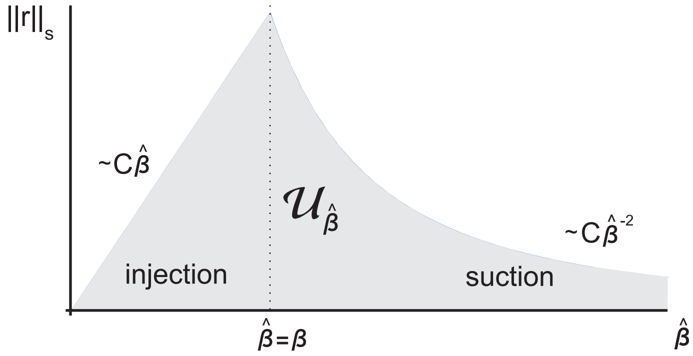


Figure 7.1: *Sketch of $\mathcal{U}_{\hat{\beta}}$. In our case we have $\hat{\beta} = \frac{\beta}{\alpha(t)}$ such that for injection $r(t)$ should decay as $\alpha(t)^{-1}$ and for suction $r(t)$ should decay as $\alpha(t)^2$ to make sure that r does not leave the domain of definition of the evolution operators.*

way as the term with $\mathcal{F}_2$, while the term with $\mathcal{F}_3$ is multiplied by $\alpha(\tau)^{-1}$. For the 2D case, the terms with $\mathcal{F}_1$ and $\mathcal{F}_3$ scale in the same way, whereas the term with $\mathcal{F}_2$ is multiplied by $\alpha(\tau)$. Because of this difference in scaling behaviour, the cases $N = 2$ and $N = 3$ must be treated in a different way.

It follows from (1.12), (3.26), and (5.7) that for $N = 2$

$$\alpha(\tau) = \left(1 - \frac{\mu\tau}{2\pi}\right)^{-1}$$

and for $N = 3$

$$\alpha(\tau) = e^{\frac{\mu}{4\pi}\tau}. \tag{7.17}$$

Lemma 7.5. *There exists a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ such that for each $\beta > 0$ and $\tau > 0$, the operators*

$$\begin{aligned}
\mathcal{E}_{\frac{\beta}{\alpha(\tau)}} &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathcal{L}(\mathbb{H}^{s-2}(\mathbb{S}^{N-1}), \mathbb{H}^{s-2}(\mathbb{S}^{N-1})),\\
\mathcal{E}_{\frac{\beta}{\alpha(\tau)}} &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathcal{L}(\mathbb{H}^{s-1}(\mathbb{S}^{N-1}), \mathbb{H}^{s-1}(\mathbb{S}^{N-1})),\\
\mathcal{F}_1(\cdot,\tau) &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathbb{H}^{s-2}(\mathbb{S}^{N-1}),\\
\mathcal{F}_2(\cdot,\tau) &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1}),\\
\mathcal{F}_3(\cdot,\tau) &: \mathcal{U}_{\frac{\beta}{\alpha(\tau)}} \to \mathbb{H}^{s-1}(\mathbb{S}^{N-1}),
\end{aligned}$$

are analytic where

$$\mathcal{U}_{\frac{\beta}{\alpha(\tau)}} := \min\left\{\frac{\beta}{\alpha(\tau)}, \frac{\alpha(\tau)^2}{\beta^2}\right\}\mathcal{U}.$$

Proof. It is clear that $\mathcal{S}_{\frac{\beta}{\alpha(\tau)}}$ is analytic from a fixed neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$

to both $\mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$ and $\mathcal{L}(\mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{3}{2}}(\mathbb{B}^N) \times \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$. Because of Corollary 7.4 the mapping $r \mapsto \mathcal{S}_{\frac{\beta}{\alpha(\tau)}}(r)^{-1}$ is analytic on $\mathcal{U}_{\frac{\beta}{\alpha(\tau)}}$. Since $\mathcal{Q}$ and n are analytic, $\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}$ is analytic from $\mathcal{U}_{\frac{\beta}{\alpha(\tau)}}$ (diminishing $\mathcal{U}$ if necessary) to both $\mathcal{L}(\mathbb{H}^{s-2}(\mathbb{S}^{N-1}), \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$ and $\mathcal{L}(\mathbb{H}^{s-1}(\mathbb{S}^{N-1}), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$. The lemma follows from this and analyticity of κ, φ, ω, and l. □

Lemma 7.6. *Let $'$ be Fréchet differentiation with respect to the first argument. We have*

$$\begin{aligned}
\mathcal{F}_1'(0,\tau)[h] &= \mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1} \kappa'(0)[h] \\
&= -\left(\mathcal{N}^3 + (N-2)\mathcal{N}^2 - (N-1)\mathcal{N}\right)\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1} h, \\
\mathcal{F}_2'(0,\tau)[h] &= \mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1} \varphi'(0)[h] + l'(0)[h] \\
&= -\frac{1}{\sigma_N}\mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1} h - \frac{N}{\sigma_N}h, \\
\mathcal{F}_3'(0,\tau)[h] &= \mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1} \omega'(0)[h] \\
&= \frac{N-1}{\sigma_N}\mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1} h.
\end{aligned}$$

Proof. Defining

$$u(r) := \mathcal{S}_{\frac{\beta}{\alpha(\tau)}}(r)^{-1}\begin{pmatrix} 0 \\ \kappa(r) \end{pmatrix}$$

we get

$$\mathcal{A}(r)u(r) = 0 \tag{7.18}$$

and

$$\left(\mathrm{Tr} + \frac{\beta}{\alpha(\tau)}n(r)\cdot\mathcal{Q}(r)\right)u(r) = \kappa(r). \tag{7.19}$$

Since $u(0)$ is constant we have $\mathcal{A}'(0)[h]u(0) = 0$ and $\mathcal{Q}'(0)[h]u(0) = 0$. Therefore, after differentiation of (7.18) and (7.19) we obtain

$$\Delta u'(0)[h] = 0,$$

$$\left(\mathrm{Tr} + \frac{\beta}{\alpha(\tau)}\frac{\partial}{\partial n}\right)u'(0)[h] = \kappa'(0)[h].$$

This implies

$$\mathrm{Tr}u'(0)[h] = \left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\kappa'(0)[h]$$

and since $\mathcal{Q}(0)u(0) = 0$ we have

$$\mathcal{F}_1'(0,\tau)[h] = \mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\kappa'(0)[h].$$

The first part of the lemma follows from this and Lemma 3.6. The expressions for $\mathcal{F}_2'(0,\tau)[h]$ and $\mathcal{F}_3'(0,\tau)[h]$ are obtained in the same way, using (2.31) and the identities

$$l'(0)[h] = -\frac{N}{\sigma_N}h, \quad \omega'(0)[h] = \frac{N-1}{\sigma_N}h,$$

that follow from (5.4), (7.8), and (6.31). □

Define the polynomial

$$p_1(X) = X^3 + (N-2)X^2 - (N-1)X.$$

For the case $N = 2$ we rewrite the linearisation of the evolution operator in the following way:

$$\begin{aligned}
\mathcal{F}'(0,\tau) &= \mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\left(\gamma\kappa'(0) + \beta\mu\omega'(0)\right) \\
&\quad + \mu\alpha(\tau)\left\{\mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\varphi'(0) + l'(0)\right\} \\
&= \left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\left(-\gamma p_1(\mathcal{N}) + \frac{\beta\mu}{2\pi}\mathcal{N}\right) \\
&\quad + \mu\alpha(\tau)\left\{\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\left(-\frac{1}{2\pi}\mathcal{N}\right) - \frac{1}{\pi}\mathcal{I}\right\} \\
&= \left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\left(-\gamma p_1(\mathcal{N}) + \frac{\beta\mu}{2\pi}\mathcal{N} - \frac{\mu\alpha(\tau)}{2\pi}\mathcal{N} - \frac{\mu\alpha(\tau)}{\pi}\mathcal{I} - \frac{\beta\mu}{\pi}\mathcal{N}\right) \\
&= \left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\left(-\gamma p_1(\mathcal{N}) - \frac{\beta\mu}{2\pi}\mathcal{N} - \frac{\mu\alpha(\tau)}{2\pi}(\mathcal{N} + 2\mathcal{I})\right). \qquad (7.20)
\end{aligned}$$

For the case $N = 3$ we rewrite the linearisation of the evolution operator in this way:

$$\begin{aligned}
\mathcal{F}'(0,\tau) &= \mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\left(\gamma\kappa'(0) + \mu\varphi'(0)\right) + \mu l'(0) \\
&\quad + \frac{\beta\mu}{\alpha(\tau)}\mathcal{N}\left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\omega'(0) \\
&= \left(\mathcal{I} + \frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\left(-\gamma p_1(\mathcal{N}) - \frac{\mu}{4\pi}(\mathcal{N} + 3\mathcal{I}) - \frac{\beta\mu}{4\pi\alpha(\tau)}\mathcal{N}\right). \qquad (7.21)
\end{aligned}$$

The following lemma is crucial in the energy estimates from which we derive existence results later on.

Lemma 7.7. *Let $s > \frac{N+5}{2}$. There exists a $C > 0$ independent of r and $\hat{\beta}$, a neighbourhood $\mathcal{U}$ of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$, and $\mathcal{U}_{\hat{\beta}}$ constructed as in Corollary 7.4, such that for all $\hat{\beta} > 0$ and $r \in \mathcal{U}_{\hat{\beta}}$ we have*

$$\|\mathcal{E}_{\hat{\beta}}(r) - \mathcal{E}_{\hat{\beta}}(0)\|_{\mathcal{L}(\mathbb{H}^{s-1}(\mathbb{S}^{N-1}),\mathbb{H}^{s-1}(\mathbb{S}^{N-1}))} \leq \frac{C}{\hat{\beta}}\|r\|_s$$

and

$$\|\mathcal{E}'_{\hat{\beta}}(r)[h] - \mathcal{E}'_{\hat{\beta}}(0)[h]\|_{\mathcal{L}(\mathbb{H}^{s-1}(\mathbb{S}^{N-1}),\mathbb{H}^{s-1}(\mathbb{S}^{N-1}))} \leq \frac{C}{\hat{\beta}}\|r\|_s\|h\|_s.$$

Proof. See Section 7.5. □

7.3 Energy estimates and global existence results for the injection problems

In this section we prove a global existence result for the 2D and the 3D injection case. In order to apply Lemma 7.7 assume that r is a small element of $\mathbb{H}^s(\mathbb{S}^{N-1})$ with

$$s > \frac{N+7}{2}.$$

As in Chapter 6, we use the first order chain rule of differentiation (5.12) and the inner product $(\cdot,\cdot)_{s-1,1}$ on $\mathbb{H}^s(\mathbb{S}^{N-1})$ as defined in (6.86).

Let us first discuss the 3D case. To obtain global existence of a solution r to (7.16) that decays as τ goes to infinity, assume that the parameters $\gamma > 0$, $\mu > 0$, and $\beta > 0$ satisfy the condition

$$\frac{\mu}{4\pi} \leq C_1, \tag{7.22}$$

where

$$C_1 := \inf_{k\in\mathbb{N}_0} \frac{\gamma p_1(k) + \frac{\mu}{4\pi}(k+3)}{(1+\beta k)\,(1+k^2)}.$$

This condition is satisfied if

$$\forall k \in \mathbb{N}_0 : \frac{4\pi\gamma}{\mu}(k^3 + k^2 - 2k) \geq \beta(k^3 + k) + k^2 - k - 2. \tag{7.23}$$

Let us first show that there are $\gamma > 0$, $\mu > 0$, and $\beta > 0$ that satisfy (7.23). Substituting $k = 1$ we see that β must be smaller than 1. If one takes $\beta \in (0,1)$ and γ and μ such that

$$\forall k \geq 2 : \frac{4\pi\gamma}{\mu}(k^3 + k^2 - 2k) > k^3 + k^2 - 2,$$

which is equivalent to

$$\frac{\mu}{\gamma} < \frac{16\pi}{5},$$

then (7.22) is satisfied.

Theorem 7.8. *Let $N = 3$ and let $\mu > 0$, $\gamma > 0$, and $\beta > 0$ satisfy (7.22). Suppose that $s > 5$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^2)$ with $\|r_0\|_s < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r,\tau), \quad r(0) = r_0, \tag{7.24}$$

has a solution $r \in \mathcal{C}_w([0,\infty),\mathbb{H}^s(\mathbb{S}^2)) \cap \mathcal{C}^1_w([0,\infty),\mathbb{H}^{s-2}(\mathbb{S}^2))$ that satisfies

$$\|r(\tau)\|_s \leq Me^{-C_1\tau}\|r_0\|_s. \tag{7.25}$$

Proof. Let $r \in \mathbb{H}^{s+2}(\mathbb{S}^2)$ with $\|r\|_s < \delta$ for δ small. In this proof C denotes a positive constant that does not depend on r and τ. We will prove the theorem making use of the same methods as in Chapters 5 and 6 and Lemma 7.7.

Note that we have to check whether $r(\tau)$ stays in $\mathcal{U}_{\frac{\beta}{\alpha(\tau)}}$ as τ goes to infinity. According to Lemma 7.4 this is the case if $\|r(\tau)\|_s$ decays faster than $\alpha(\tau)^{-1}$.

1. Because of (7.21) and (2.37) we have the following estimate for the linearisation:

$$\begin{aligned}(r,\mathcal{F}'(0,\tau)[r])_{s-1} &= \sum_{k,j} \frac{-\gamma p_1(k) - \frac{\mu}{4\pi}(k+3) - \frac{\beta\mu}{4\pi\alpha(\tau)}k}{\left(1+\frac{\beta}{\alpha(\tau)}k\right)(1+k^2)}(1+k^2)^s r_{kj}^2 \\ &\leq \sum_{k,j} \frac{-\gamma p_1(k) - \frac{\mu}{4\pi}(k+3)}{(1+\beta k)(1+k^2)}(1+k^2)^s r_{kj}^2 \\ &\leq -C_1\|r\|_s^2. \end{aligned} \tag{7.26}$$

2. Now we find an estimate for the nonlinear terms. We assume that $\|r\|_s \leq 2\delta/\alpha(\tau)$ for small $\delta > 0$, to make sure that r is in the domain of definition of $\mathcal{F}(\cdot,\tau)$. Note that

$$\|\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\|_X = \left\|\mathcal{N}\left(\mathcal{I}+\frac{\beta}{\alpha(\tau)}\mathcal{N}\right)^{-1}\right\|_X \leq \frac{\alpha(\tau)}{\beta}, \tag{7.27}$$

with $X = \mathcal{L}(\mathbb{H}^{s-2}(\mathbb{S}^{N-1}),\mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$. Since $\kappa(0)$ is constant we have $\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r)\kappa(0) = 0$. Combining this, (7.27), Lemma 7.7 (with s replaced by $s-1$), and analyticity of κ we get

$$\begin{aligned}\|\mathcal{F}_1(r,\tau) - \mathcal{F}_1'(0,\tau)[r]\|_{s-2} &= \left\|\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r)\kappa(r) - \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\kappa'(0)[r]\right\|_{s-2} \\ &\leq \left\|\left(\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r) - \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\right)(\kappa(r)-\kappa(0))\right\|_{s-2} \\ &\quad + \left\|\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\left(\kappa(r)-\kappa(0)-\kappa'(0)[r]\right)\right\|_{s-2} \\ &\leq C\alpha(\tau)\|r\|_s^2. \end{aligned} \tag{7.28}$$

Note that here the demand $s > 5$ is crucial. Let us now estimate the nonlinear terms of $\mathcal{F}_2(r,\tau) = \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r)\varphi(r) + l(r)$. We have

$$\|l(r) - l'(0)[r]\|_{s-1} \le C\|r\|_{s-1}^2. \tag{7.29}$$

The other nonlinear terms of $\mathcal{F}_2$ can be estimated in the same way as the terms of $\mathcal{F}_1$ (as in (7.28)) because they are of lower order. Making use of analyticity of ω we find

$$\begin{aligned}\|\mathcal{F}_3(r,\tau) - \mathcal{F}_3'(0,\tau)[r]\|_{s-\frac{3}{2}} &\le \left\|\left(\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r) - \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\right)(\omega(r) - \omega(0))\right\|_{s-\frac{3}{2}} \\ &\quad + \left\|\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\left(\omega(r) - \omega(0) - \omega'(0)[r]\right)\right\|_{s-\frac{3}{2}} \\ &\le C\alpha(\tau)\|r\|_{s-\frac{1}{2}}^2. \end{aligned} \tag{7.30}$$

Adding the results and using $\alpha(\tau) \ge 1$ we get from (7.16)

$$\begin{aligned} &(r, \mathcal{F}(r,\tau) - \mathcal{F}'(0,\tau)[r])_{s-1} \\ \le\ & C\alpha(\tau)\|r\|_s^3 + C\|r\|_{s-1}^3 + C\alpha(\tau)\alpha(\tau)^{-1}\|r\|_{s-\frac{1}{2}}^3 \le C\alpha(\tau)\|r\|_s^3. \end{aligned} \tag{7.31}$$

3. We conclude from (7.26) and (7.31) that

$$(r, \mathcal{F}(r,\tau))_{s-1} \le -C_1\|r\|_s^2 + C\alpha(\tau)\|r\|_s^3. \tag{7.32}$$

4. Define the operators D_i by (5.11). It follows from (7.26) that

$$(D_i r, \mathcal{F}'(0,\tau)[D_i r])_{s-1} \le -C_1\|D_i r\|_s^2. \tag{7.33}$$

5. Define $u_{\frac{\beta}{\alpha(\tau)}}$ and $w_{\frac{\beta}{\alpha(\tau)}}$ as in (7.67) and (7.68) such that

$$\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r) = \frac{1}{n(r)\cdot \mathrm{id}} w_{\frac{\beta}{\alpha(\tau)}}(r).$$

Because of (6.31) and (7.90)

$$\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(0)[h] = w'_{\frac{\beta}{\alpha(\tau)}}(0)[h] = -\frac{\alpha(\tau)}{\beta}\mathrm{Tr}u'_{\frac{\beta}{\alpha(\tau)}}(0)[h]$$

and therefore by (7.93)

$$\left\|\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(0)[h]\right\|_{\mathcal{L}(\mathbb{H}^\sigma(\mathbb{S}^{N-1}),\mathbb{H}^\sigma(\mathbb{S}^{N-1}))} \le C\alpha(\tau)\|h\|_{\sigma+1}, \tag{7.34}$$

for $\sigma > \frac{N+3}{2}$. From (7.27), (7.34), Lemma 7.7, analyticity of κ, and the fact that

$\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(r)[h]\kappa(0) = 0$ we get for $\|r\|_s < 2\delta/\alpha(\tau)$ for $\delta > 0$ small

$$\begin{aligned}
&\|\mathcal{F}_1'(r,\tau)[h] - \mathcal{F}_1'(0,\tau)[h]\|_{s-2} \\
= \;& \left\|\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(r)[h]\kappa(r) + \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r)\kappa'(r)[h] - \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\kappa'(0)[h]\right\|_{s-2} \\
\le \;& \left\|\left(\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(r)[h] - \mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(0)[h]\right)(\kappa(r)-\kappa(0))\right\|_{s-2} + \left\|\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(0)[h](\kappa(r)-\kappa(0))\right\|_{s-2} \\
&+ \left\|\left(\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r) - \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\right)\kappa'(r)[h]\right\|_{s-2} + \left\|\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\big(\kappa'(r)[h] - \kappa'(0)[h]\big)\right\|_{s-2} \\
\le \;& C\alpha(\tau)\|r\|_{s-1}\|h\|_{s-1}\|r\|_s + C\alpha(\tau)\|h\|_{s-1}\|r\|_s \\
&+ C\alpha(\tau)\|r\|_{s-1}\|h\|_s + C\alpha(\tau)\|r\|_s\|h\|_s \\
\le \;& C\alpha(\tau)\|r\|_s\|h\|_s
\end{aligned} \tag{7.35}$$

and

$$\|l'(r)[h] - l'(0)[h]\|_{s-1} \le C\|r\|_{s-1}\|h\|_{s-1}.$$

It also follows that

$$\begin{aligned}
&\|\mathcal{F}_3'(r,\tau)[h] - \mathcal{F}_3'(0,\tau)[h]\|_{s-\frac{3}{2}} \\
\le \;& \left\|\left(\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(r)[h] - \mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(0)[h]\right)(\omega(r)-\omega(0))\right\|_{s-\frac{3}{2}} \\
&+ \left\|\mathcal{E}'_{\frac{\beta}{\alpha(\tau)}}(0)[h](\omega(r)-\omega(0))\right\|_{s-\frac{3}{2}} + \left\|\left(\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(r) - \mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\right)\omega'(r)[h]\right\|_{s-\frac{3}{2}} \\
&+ \left\|\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0)\big(\omega'(r)[h] - \omega'(0)[h]\big)\right\|_{s-\frac{3}{2}} \\
\le \;& C\alpha(\tau)\|r\|_{s-\frac{1}{2}}\|h\|_{s-\frac{1}{2}}.
\end{aligned} \tag{7.36}$$

Here we used the estimate $\|\omega'(r)[h]\|_{s-\frac{3}{2}} \le C\|h\|_{s-\frac{1}{2}}$ that holds for small $\|r\|_s$. Since $\alpha(\tau) \ge 1$ we see from (7.16) that

$$\begin{aligned}
&\big(D_i r, \{\mathcal{F}'(r,\tau) - \mathcal{F}'(0,\tau)\}[D_i r]\big)_{s-1} \\
\le \;& C\alpha(\tau)\|r\|_s\|D_i r\|_s^2 + C\|r\|_{s-\frac{1}{2}}\|D_i r\|^2_{s-\frac{1}{2}} \\
\le \;& C\alpha(\tau)\|r\|_s\|D_i r\|_s^2.
\end{aligned}$$

Combining this with (7.33) we obtain

$$(D_i r, \mathcal{F}'(r,\tau)[D_i r])_{s-1} \le -C_1\|D_i r\|_s^2 + C\alpha(\tau)\|r\|_s\|D_i r\|_s^2. \tag{7.37}$$

6. It follows from (7.32) and (7.37) that for $\tau > 0$ and $\|r\|_{s-1,1} < 2\delta/\alpha(\tau)$ (taking δ

small enough)

$$\begin{aligned}(r, \mathcal{F}(r,\tau))_{s-1,1} &\leq -C_1\|r\|_{s,1}^2 + C\alpha(\tau)\|r\|_s\|r\|_{s,1}^2\\ &\leq -C_1\|r\|_{s,1}^2 + C\alpha(\tau)\|r\|_{s-1,1}\|r\|_{s,1}^2.\end{aligned}$$

On some finite time interval $[0,T]$ we have $-C_1 + 2C\alpha(\tau)\delta < 0$. Take δ small enough such that T can be chosen to satisfy

$$e^{\left(\frac{\mu}{4\pi}-C_1\right)T} < \frac{1}{3}. \tag{7.38}$$

For $\tau \in [0,T]$ and $\|r\|_{s-1,1} < 2\delta/\alpha(\tau)$ we have

$$\begin{aligned}(r, \mathcal{F}(r,\tau))_{s-1,1} &\leq (-C_1 + C\alpha(\tau)\|r\|_{s-1,1})\|r\|_{s,1}^2\\ &\leq (-C_1 + C\alpha(\tau)\|r\|_{s-1,1})\|r\|_{s-1,1}^2\\ &\leq -C_1\|r\|_{s-1,1}^2 + C\alpha(\tau)\|r\|_{s-1,1}^3. \end{aligned}\tag{7.39}$$

Theorem A.1 implies existence of a solution r to (7.24) on $[0,T]$ that satisfies $\|r(\tau)\|_{s-1,1}^2 \leq y(\tau)$, $y(0) = y_0 := \|r_0\|_{s-1,1}^2$, where $y : [0,\infty) \to \mathbb{R}$ satisfies the ODE

$$\frac{dy}{d\tau} = -2C_1 y + 2Ce^{\frac{\mu}{4\pi}\tau}y^{\frac{3}{2}}. \tag{7.40}$$

Here we used (7.17). We solve (7.40) to show that solutions y converge to zero as τ tends to infinity. Introducing $Y = e^{2C_1\tau}y$ we get

$$\frac{dY}{d\tau} = 2Ce^{\left(\frac{\mu}{4\pi}-C_1\right)\tau}Y^{\frac{3}{2}}$$

from which it follows that

$$Y(\tau) = \left(\frac{1}{\sqrt{Y(0)}} - \frac{C}{C_1 - \frac{\mu}{4\pi}}\left(1 - e^{\left(\frac{\mu}{4\pi}-C_1\right)\tau}\right)\right)^{-2}.$$

Since condition (7.22) is satisfied and y_0 is small it follows that for $\tau \in [0,\infty)$

$$\begin{aligned}y(\tau) &= \left(\frac{e^{C_1\tau}}{\sqrt{y_0}} - \frac{C}{C_1 - \frac{\mu}{4\pi}}\left(e^{C_1\tau} - e^{\frac{\mu}{4\pi}\tau}\right)\right)^{-2}\\ &= \left(e^{C_1\tau} - \frac{C\sqrt{y_0}}{C_1 - \frac{\mu}{4\pi}}\left(e^{C_1\tau} - e^{\frac{\mu}{4\pi}\tau}\right)\right)^{-2} y_0\\ &\leq \left(e^{C_1\tau} - \frac{C\sqrt{y_0}}{C_1 - \frac{\mu}{4\pi}}e^{C_1\tau}\right)^{-2} y_0\\ &\leq M^2 e^{-2C_1\tau}y_0,\end{aligned}$$

for some $M < 2$ that is independent of r. Consequently, for $\tau \in [0, T]$

$$\|r(\tau)\|_{s-1,1} \le Me^{-C_1\tau}\|r_0\|_{s-1,1}. \tag{7.41}$$

We see that $\|r(\tau)\|_{s-1,1} \le Me^{-\frac{\mu}{4\pi}\tau}\|r_0\|_{s-1,1} \le 2\delta/\alpha(\tau)$ for all $\tau \in [0, T]$. Furthermore,

$$\begin{aligned} -C_1 + C\alpha(\tau)\|r(\tau)\|_{s-1,1} &\le -C_1 + CMe^{\left(\frac{\mu}{4\pi}-C_1\right)\tau}\|r_0\|_{s-1,1} \\ &\le -C_1 + CM\|r_0\|_{s-1,1} \le 0 \end{aligned} \tag{7.42}$$

on the interval $[0, T]$. Note that $\|r(\tau)\|_{s-1,1}$ decays fast enough to make sure that (7.39) also holds for $\tau > T$ (with r replaced by $r(\tau)$). In fact, all calculations on $[0, T]$ can be extended to $[0, \infty)$. To prove this formally by contradiction, let $T^* > 0$ be the supremum of all $\tilde{T}$ such that there exists a solution r to (7.24) on $[0, \tilde{T}]$ and $-C_1 + C\alpha(\tau)\|r(\tau)\|_{s-1,1} < 0$ for all $\tau \in [0, \tilde{T}]$. Suppose that $T^* < \infty$. In view of (7.42) we have $T^* \ge T$ such that by (7.38)

$$e^{\left(\frac{\mu}{4\pi}-C_1\right)T^*} < \frac{1}{3}. \tag{7.43}$$

By the definition of T^*, (7.39) holds for $\tau \in [0, T^*)$ (with r replaced by $r(\tau)$) and so does (7.41). For small $\eta > 0$ it follows from (7.43) that $e^{\left(\frac{\mu}{4\pi}-C_1\right)(T^*-\eta)} < \frac{1}{2}$. Consequently,

$$\begin{aligned} -C_1 + C\alpha(T^* - \eta)\|r(T^* - \eta)\|_{s-1,1} &\le -C_1 + CMe^{\left(\frac{\mu}{4\pi}-C_1\right)\left(T^*-\eta\right)}\|r_0\|_{s-1,1} \\ &\le -C_1 + \frac{1}{2}CM\|r_0\|_{s-1,1} \\ &\le -C_1 + \frac{1}{2}CM\delta =: \Lambda < 0. \end{aligned}$$

Since $\Lambda < 0$ is independent of η and α is continuous, there exists an $\tilde{\eta} > 0$ such that for $\tau \in [T^* - \eta, T^* + \tilde{\eta}]$ we have $-C_1 + C\alpha(\tau)\|r(T^* - \eta)\|_{s-1,1} < 0$. Apparently we can extend our solution to an interval that is larger than $[0, T^*]$. This contradicts the definition of T^*.

We get global existence of a solution r on $[0, \infty)$ that satisfies (7.25). The condition in Corollary 7.4 is satisfied because of (7.25) and $\frac{\mu}{4\pi} \le C_1$.

□

Since $\alpha(\tau) = e^{\frac{\mu}{4\pi}\tau}$ (7.25) is equivalent to

$$\|r(\tau)\|_s \le M\alpha(\tau)^{-\frac{4\pi C_1}{\mu}}\|r_0\|_s.$$

Regarding r as a function of the original time variable t,

$$\|r(t)\|_s \le M\left(\frac{3\mu t}{4\pi} + 1\right)^{-\frac{4\pi C_1}{3\mu}}\|r_0\|_s. \tag{7.44}$$

Theorem 7.9. *Let $\gamma > 0$, $\mu > 0$, and $\beta \in (0,1)$. Define*

$$\tilde{C}_1 := \inf_{k\in\mathbb{N}_0} \frac{\gamma p_1(k) + \frac{\mu}{4\pi}(k+3)}{1+\beta k}. \tag{7.45}$$

The following two statements are true:

- $\tilde{C}_1$ *and* μ *satisfy* (7.22), *with* C_1 *replaced by* $\tilde{C}_1$.
- *The result of Theorem 7.8 and* (7.44) *still hold, with* C_1 *replaced by* $\tilde{C}_1$.

Proof. The first statement is obvious. To prove the second statement, we argue as in the proof of Theorem 5.9. Note that it is possible to show (7.26) for this C_1 in terms of the equivalent norm induced by the inner product

$$(r,\tilde{r})_s^* = \sum_{k,j} \zeta_k^s r_{kj}\tilde{r}_{kj},$$

with $\zeta_k = 1$ for k small and $\zeta_k = \theta k^2 + 1$ for k large. Here $\theta > 0$ needs to be a small number. The nonlinear terms can be estimated in exactly the same way as in the proof of Theorem 7.8. □

Now we discuss the two-dimensional problem. This case is more complicated than the 3D case, since lower order terms are multiplied by a factor $\alpha(\tau)$.

Theorem 7.10. *Let $N = 2$, $\mu > 0$, $\gamma > 0$, $\beta \in (0,\frac{1}{2})$, and $\lambda_0 \in (0,2)$. Suppose that $s > \frac{9}{2}$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in \mathbb{H}^s(\mathbb{S}^1)$ with $\|r_0\|_s < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r,\tau), \quad r(0) = r_0, \tag{7.46}$$

has a solution $r \in \mathcal{C}_w([0,\frac{2\pi}{\mu}), \mathbb{H}^s(\mathbb{S}^1)) \cap \mathcal{C}_w^1([0,\frac{2\pi}{\mu}), \mathbb{H}^{s-2}(\mathbb{S}^1))$ that satisfies

$$\|r(t)\|_s \le M\left(\frac{\mu t}{\pi}+1\right)^{-\frac{\lambda_0}{2}} \|r_0\|_s,$$

where we regard r as a function of the original time variable t.

Proof. As is the proof of Theorem 7.8 we take $r \in \mathbb{H}^{s+2}(\mathbb{S}^{N-1})$ with $\|r\|_s < 2\delta/\alpha(\tau)$ for δ small.

1. Let η be a small positive number that satisfies $\lambda_0 < 2(1-\eta)$ and define the positive constant
$$C_2 := \inf_{k\in\mathbb{N}_0} \frac{\gamma p_1(k) + \frac{\beta\mu}{2\pi}k + \frac{\eta\mu}{2\pi}(k+2)}{(1+\beta k)(1+k^2)}.$$

2. For the linearisation it follows that

$$\begin{aligned}
&(r,\mathcal{F}'(0,\tau)[r])_{s-1}\\
=\;& \sum_{k,j} \frac{-\gamma p_1(k) - \frac{\beta\mu}{2\pi}k - \frac{\mu\alpha(\tau)}{2\pi}(k+2)}{1+\frac{\beta}{\alpha(\tau)}k}(1+k^2)^{s-1} r_{kj}^2\\
=\;& \sum_{k,j} \frac{-\gamma p_1(k) - \frac{\beta\mu}{2\pi}k - \eta\frac{\mu\alpha(\tau)}{2\pi}(k+2)}{1+\frac{\beta}{\alpha(\tau)}k}(1+k^2)^{s-1} r_{kj}^2\\
&-(1-\eta)\frac{\mu\alpha(\tau)}{2\pi}\sum_{k,j}\frac{k+2}{1+\frac{\beta}{\alpha(\tau)}k}(1+k^2)^{s-1} r_{kj}^2\\
\le\;& \sum_{k,j} \frac{-\gamma p_1(k) - \frac{\beta\mu}{2\pi}k - \eta\frac{\mu}{2\pi}(k+2)}{1+\beta k}(1+k^2)^{s-1} r_{kj}^2\\
&-(1-\eta)\frac{\mu\alpha(\tau)}{2\pi}\sum_{k,j}\frac{k+2}{1+\beta k}(1+k^2)^{s-1} r_{kj}^2\\
\le\;& -C_2\|r\|_s^2 - (1-\eta)\frac{\mu\alpha(\tau)}{\pi}\|r\|_{s-1}^2. \qquad (7.47)
\end{aligned}$$

Note that in the last step we used $\beta < \frac{1}{2}$.

3. The qualitative properties of the nonlinear parts of $\mathcal{F}_k$, for $k = 1, 2, 3$, are the same in all space dimensions. Therefore, to estimate the nonlinear part we use (7.28), (7.30), and

$$\|\mathcal{F}_2(r,\tau) - \mathcal{F}_2'(0,\tau)[r]\|_{s-\frac{3}{2}} \le C\alpha(\tau)\|r\|_{s-\frac{1}{2}}^2, \qquad (7.48)$$

which can be derived in the same way as (7.30), replacing ω by φ. Note that by (7.29), the term $\|l(r) - l'(0)[r]\|_{s-1}$ plays no role because $\alpha(\tau) \ge 1$. Multiplying (7.48) by $\alpha(\tau)$ and adding the results it follows from (7.16) that

$$(r, \mathcal{F}(r,\tau) - \mathcal{F}'(0,\tau)[r])_{s-1} \le C\alpha(\tau)\|r\|_s^3 + C\alpha(\tau)^2\|r\|_{s-\frac{1}{2}}^3. \qquad (7.49)$$

4. From (7.47) and (7.49) we get

$$\begin{aligned}
(r,\mathcal{F}(r,\tau))_{s-1} \;\le\;& -C_2\|r\|_s^2 - (1-\eta)\frac{\mu\alpha(\tau)}{\pi}\|r\|_{s-1}^2\\
&+C\alpha(\tau)\|r\|_s^3 + C\alpha(\tau)^2\|r\|_{s-\frac{1}{2}}^3. \qquad (7.50)
\end{aligned}$$

5. In the same way it follows from (7.35), (7.36), and a similar estimate for the $\mathcal{F}_2$-terms that

$$\begin{aligned}
&(D_i r, \mathcal{F}'(r,\tau)[D_i r] - \mathcal{F}'(0,\tau)[D_i r])_{s-1}\\
\le\;& C\alpha(\tau)\|r\|_s\|D_i r\|_s^2 + C\alpha(\tau)^2\|r\|_{s-\frac{1}{2}}\|D_i r\|_{s-\frac{1}{2}}^2. \qquad (7.51)
\end{aligned}$$

6. Note that (7.47) yields

$$(D_i r, \mathcal{F}'(0,\tau)[D_i r])_{s-1} \le -C_2\|D_i r\|_s^2 - (1-\eta)\frac{\mu\alpha(\tau)}{\pi}\|D_i r\|_{s-1}^2.$$

Combining this with (7.51) we obtain

$$\begin{aligned} & (D_i r, \mathcal{F}'(r,\tau)[D_i r])_{s-1} \\ \le\ & -C_2\|D_i r\|_s^2 - (1-\eta)\frac{\mu\alpha(\tau)}{\pi}\|D_i r\|_{s-1}^2 \\ & + C\alpha(\tau)\|r\|_s\|D_i r\|_s^2 + C\alpha(\tau)^2\|r\|_{s-\frac{1}{2}}\|D_i r\|_{s-\frac{1}{2}}^2. \end{aligned} \tag{7.52}$$

7. It follows from (7.50) and (7.52) that

$$\begin{aligned} (r, \mathcal{F}(r,\tau))_{s-1,1} \ \le\ & -C_2\|r\|_{s,1}^2 - (1-\eta)\frac{\mu\alpha(\tau)}{\pi}\|r\|_{s-1,1}^2 \\ & + C\alpha(\tau)\|r\|_s\|r\|_{s,1}^2 + C\alpha(\tau)^2\|r\|_{s-\frac{1}{2}}\|r\|_{s-\frac{1}{2},1}^2. \end{aligned}$$

From an interpolation inequality and Cauchy's inequality it follows that

$$\begin{aligned} C\alpha(\tau)^2\|r\|_{s-\frac{1}{2}}\|r\|_{s-\frac{1}{2},1}^2 \ \le\ & C\alpha(\tau)^2\|r\|_{s-\frac{1}{2}}\|r\|_{s-1,1}\|r\|_{s,1} \\ \le\ & C\alpha(\tau)^4\|r\|_{s-\frac{1}{2}}^2\|r\|_{s-1,1}^2 + \frac{C_2}{2}\|r\|_{s,1}^2 \\ \le\ & C\alpha(\tau)^4\|r\|_{s-1,1}^4 + \frac{C_2}{2}\|r\|_{s,1}^2, \end{aligned}$$

such that

$$\begin{aligned} & (r, \mathcal{F}(r,\tau))_{s-1,1} \\ \le\ & \left(-\frac{C_2}{2} + C\alpha(\tau)\|r\|_s\right)\|r\|_{s,1}^2 - (1-\eta)\frac{\mu\alpha(\tau)}{\pi}\|r\|_{s-1,1}^2 + C\alpha(\tau)^4\|r\|_{s-1,1}^4. \end{aligned}$$

On some interval $[0,T] \subset [0,\frac{2\pi}{\mu})$ we have $-\frac{C_2}{2} + 2C\alpha(\tau)\delta < 0$. As in the proof of Theorem 7.8 we show that there exists a solution r that decays fast enough to make sure that $-\frac{C_2}{2} + C\alpha(\tau)\|r(\tau)\|_s < 0$ for all $\tau \in [0,\frac{2\pi}{\mu})$. For $\tau \in [0,T]$ we have

$$(r, \mathcal{F}(r,\tau))_{s-1,1} \le -(1-\eta)\frac{\mu\alpha(\tau)}{\pi}\|r\|_{s-1,1}^2 + C\alpha(\tau)^4\|r\|_{s-1,1}^4.$$

By Theorem A.1 there exists a solution r to (7.46) on $[0,T]$ with $\|r(\tau)\|_s^2 \le y(\tau)$, where $y: [0,\frac{2\pi}{\mu}) \to \mathbb{R}$ satisfies

$$\frac{dy}{d\tau} = -\frac{2(1-\eta)\mu}{\pi}\alpha(\tau)y + 2C\alpha(\tau)^4 y^2,$$

with $y(0) = y_0 := \|r_0\|_s^2$. To solve this ODE we introduce $\sigma := -4(1-\eta)\ln\left(1 - \frac{\mu\tau}{2\pi}\right) = -4(1-\eta)\ln\alpha(\tau)^{-1}$ such that

$$\frac{dy}{d\sigma} = -y + \frac{\pi C}{(1-\eta)\mu} e^{\frac{3}{4(1-\eta)}\sigma} y^2.$$

Substituting $y = e^{-\sigma}Y$ we get

$$\frac{dY}{d\sigma} = \frac{\pi C}{(1-\eta)\mu} e^{\left(\frac{3}{4(1-\eta)}-1\right)\sigma} Y^2$$

from which it follows that

$$Y(\sigma) = \left(\frac{1}{Y(0)} - \frac{\pi C}{(\frac{1}{4}-\eta)\mu}\left(1 - e^{\left(\frac{3}{4(1-\eta)}-1\right)\sigma}\right)\right)^{-1},$$

which is equivalent to

$$y(\sigma) = \left(e^{\sigma} - \frac{\pi C y_0}{(\frac{1}{4}-\eta)\mu}\left(e^{\sigma} - e^{\frac{3}{4(1-\eta)}\sigma}\right)\right)^{-1} y_0.$$

Since η and y_0 are small there exists an $M > 0$ such that

$$y(\tau) \leq M^2 e^{-\sigma} y_0 = M^2 \alpha(\tau)^{-4(1-\eta)} y_0.$$

As a consequence, for $\tau \in [0, T]$

$$\|r(\tau)\|_s \leq M\alpha(\tau)^{-2(1-\eta)}\|r_0\|_s \leq M\alpha(\tau)^{-\lambda_0}\|r_0\|_s.$$

Because r decays fast enough to make sure that $-\frac{C_2}{2} + C\alpha(\tau)\|r(\tau)\|_s < 0$ for all time, all calculations on $[0, T]$ can be extended to $[0, \frac{2\pi}{\mu})$, as in the proof of Theorem 7.8. This completes the proof.

□

Remark 7.11. *Since $\beta/\alpha(\tau)$ goes to zero as τ tends to $\tau_{\max}$, the restrictions on β in Theorems 7.8 − 7.10 can be omitted. This can be proved by arguing as in the proof of Theorem 5.8 where the time interval is split up in two parts. For any $\beta > 0$ there is a $T' > 0$ such that* (7.26) *and* (7.47) *hold on $[T', \infty)$.*

Moreover, this allows us to substitute a small positive number for β in (7.45) *to get $\tilde{C}_1 = \frac{3\mu}{4\pi}$, such that for $N = 3$ $\|r(t)\|_s$ decays as $C/t^{\zeta}\|r_0\|_s$ where $\zeta < 1$.*

7.4 Energy estimates and global existence results for a suction problem

The linearisation of the evolution operator given by (7.20) has positive eigenvalues for the suction problem. In Chapters 3 and 5 this problem was solved by restricting ourselves to perturbations on an invariant manifold.

For classical Hele-Shaw flow we found in Chapter 2 that Richardson moments are conserved. For Hele-Shaw flow with surface tension only (no kinetic undercooling) we proved in Chapter 3 that the geometric centre is conserved if the geometric centre is located at the suction point. If one includes kinetic undercooling regularisation, then the geometric centre may change.

In other words, the manifold $\mathfrak{M}_1^N$ defined by (2.40) is no longer invariant for $\beta > 0$. Nevertheless, there are subsets of $\mathfrak{M}_1^N$ such that solutions to the evolution problem, that are initially in this subset, stay in it until vanishing time. Define $\mathbb{H}^s_*(\mathbb{S}^{N-1})$ as the set of functions in $\mathbb{H}^s(\mathbb{S}^{N-1})$ that are even in all variables. These are the functions r for which the corresponding domains Ω_r are symmetric with respect to all coordinate planes. Define

$$\mathfrak{M}_*^N := \left\{ r \in \mathbb{H}^s_*(\mathbb{S}^{N-1}) : \int_{\Omega_r} dx = \frac{\sigma_N}{N} \right\}. \tag{7.53}$$

It is clear that this subset of $\mathfrak{M}_1^N$ has the desired invariance property. For the two-dimensional case $\mathbb{H}^s_*(\mathbb{S}^1)$ consists of those functions in $\mathbb{H}^s(\mathbb{S}^1)$ that can be written as

$$r = \sum_{k=0}^{\infty} a_k \cos 2k\theta,$$

where θ is the polar variable. We have $\dim(\mathbb{H}^s_*(\mathbb{S}^1) \cap \mathfrak{S}_k) = 0$ for k odd and $\dim(\mathbb{H}^s_*(\mathbb{S}^1) \cap \mathfrak{S}_k) = 1$ for k even. Introduce

$$\mathbb{H}^s_{*,1}(\mathbb{S}^{N-1}) := \left\{ r \in \mathbb{H}^s_*(\mathbb{S}^{N-1}) : \int_{\mathbb{S}^{N-1}} r dx = 0 \right\}.$$

The subspace $\mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})$ is the tangent space at zero of the manifold $\mathfrak{M}_*^N$ in $\mathbb{H}^s_*(\mathbb{S}^{N-1})$. Introduce on a neighbourhood $\mathcal{U}_*$ of zero in $\mathbb{H}^s_*(\mathbb{S}^{N-1})$ the operator $\phi_* : \mathcal{U}_* \to \mathbb{R} \times \mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})$ by

$$\phi_*(r) := (f_*(r), \mathcal{P}_1 r)^T,$$

where $\mathcal{P}_1$ is $\mathbb{L}^2(\mathbb{S}^{N-1})$-orthogonal projection on $\mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})$ and

$$f_*(r) := \int_{\Omega_r} dx - \frac{\sigma_N}{N}. \tag{7.54}$$

Arguing as in previous chapters we see that ϕ_* defines an analytic bijection between a neighbourhood of zero in $\mathbb{H}^s_*(\mathbb{S}^{N-1})$ and a neighbourhood of zero in $\mathbb{R} \times \mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})$. Now we introduce on a neighbourhood $\mathcal{U}_*$ of zero in $\mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})$ the analytic bijection $\psi_* : \mathcal{U}_* \to \mathfrak{M}_*^N$ by

$$\psi_*(r) = \phi_*^{-1}(0, r). \tag{7.55}$$

Define for $\tilde{r}$ near zero in $\mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})$

$$\tilde{\mathcal{F}}(\tilde{r}, \tau) := \mathcal{P}_1 \mathcal{F}(\psi_*(\tilde{r}), \tau)$$

and introduce $\tilde{\mathcal{F}}_j(\tilde{r},\tau) := \mathcal{P}_1\mathcal{F}_j(\psi_*(\tilde{r}),\tau)$ for $j = 1,2,3$. By methods that we have seen before, we have

$$\tilde{\mathcal{F}}_j'(0,\tau) = \mathcal{P}_1\mathcal{F}_j'(0,\tau) \mid_{\mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})} = \mathcal{F}_j'(0,\tau) \mid_{\mathbb{H}^s_{*,1}(\mathbb{S}^{N-1})} . \tag{7.56}$$

In the following theorem we assume that

$$\frac{|\mu|}{\pi} < C_3, \tag{7.57}$$

where

$$C_3 := \inf_{k\geq 2} \frac{\gamma p_1(k) - \frac{\beta|\mu|}{2\pi}k - \frac{|\mu|}{2\pi}(k+2)}{(1+\beta k)(k^2+1)}.$$

This condition is satisfied if

$$\forall k \geq 2 : \frac{\pi\gamma}{|\mu|}(k^3-k) \geq \beta\left(k^3 + \frac{3}{2}k\right) + k^2 + \frac{1}{2}k + 2.$$

Take for instance $\beta < 1$ and let $\gamma > 0$ and $\mu < 0$ satisfy

$$\forall k \geq 2 : \frac{\pi\gamma}{|\mu|} \geq \frac{k^3+k^2+2k+2}{k^3-k}. \tag{7.58}$$

Note that (7.58) holds when $\frac{|\mu|}{\gamma} < \frac{\pi}{3}$.

Theorem 7.12. *Suppose that $N = 2$, $\mu < 0$, $s > \frac{9}{2}$, and suppose that* (7.57) *holds. Let $\lambda_0 \in (\frac{|\mu|}{\pi}, C_3)$. There exists a $\delta > 0$ and an $M > 0$ such that if $r_0 \in \mathfrak{M}^2_*$ with $\|r_0\|_s < \delta$, then the problem*

$$\frac{\partial r}{\partial \tau} = \mathcal{F}(r,\tau), \quad r(0) = r_0,$$

has a solution $r \in \mathcal{C}_w([0,\infty),\mathbb{H}^s(\mathbb{S}^1)) \cap \mathcal{C}^1_w([0,\infty),\mathbb{H}^{s-2}(\mathbb{S}^1))$ that satisfies

$$\|r(t)\|_s \leq M\left(-\frac{|\mu|t}{\pi}+1\right)^{\frac{\pi\lambda_0}{|\mu|}} \|r_0\|,$$

where we regard r as a function of the original time-variable t.

Proof. Note that we have to show that $r(\tau)$ stays in $\mathcal{U}_{\frac{\beta}{\alpha(\tau)}}$ as τ goes to infinity. According to Corollary 7.4 this is the case if $\|r(\tau)\|_s$ decays faster than $\alpha(\tau)^2$.

1. Let $\tilde{r} := \mathcal{P}_1 r$ and $\tilde{r}_0 = \tilde{r}(0)$. The following evolution equation holds for $\tilde{r}$:

$$\frac{\partial \tilde{r}}{\partial \tau} = \tilde{\mathcal{F}}(\tilde{r},\tau). \tag{7.59}$$

First we prove solvability of this equation for $\tilde{r} \in \mathbb{H}^s_{*,1}(\mathbb{S}^1)$ with $\|\tilde{r}_0\|_s$ small enough. Let $\tilde{r} \in \mathbb{H}^{s+2}_{*,1}(\mathbb{S}^1)$ with $\|\tilde{r}\|_s < \delta'\alpha(\tau)^2$ for small $\delta' > 0$.

2. Since $\alpha(\tau) \leq 1$ it follows from (7.57) that

$$\begin{aligned}
(\tilde{r}, \tilde{\mathcal{F}}'(0,\tau)[\tilde{r}])_{s-1} &= \sum_{k\geq 2} \frac{-\gamma p_1(k) + \frac{\beta|\mu|}{2\pi}k + \frac{|\mu|\alpha(\tau)}{2\pi}(k+2)}{(1+\frac{\beta}{\alpha(\tau)}k)(k^2+1)}(k^2+1)^s \tilde{r}_{kj} \\
&\leq \sum_{k\geq 2} \frac{-\gamma p_1(k) + \frac{\beta|\mu|}{2\pi}k + \frac{|\mu|}{2\pi}(k+2)}{(1+\frac{\beta}{\alpha(\tau)}k)(k^2+1)}(k^2+1)^s \tilde{r}_{kj} \\
&\leq \alpha(\tau) \sum_{k\geq 2} \frac{-\gamma p_1(k) + \frac{\beta|\mu|}{2\pi}k + \frac{|\mu|}{2\pi}(k+2)}{(1+\beta k)(k^2+1)}(k^2+1)^s \tilde{r}_{kj} \\
&\leq -C_3 \alpha(\tau) \|\tilde{r}\|_s^2.
\end{aligned} \tag{7.60}$$

Note that C_3 is positive because of (7.57).

3. By Lemma 7.7, (7.27), and (7.56)

$$\begin{aligned}
&\|\tilde{\mathcal{F}}_1(\tilde{r},\tau) - \tilde{\mathcal{F}}_1'(0,\tau)[\tilde{r}]\|_{s-2} \\
= {} & \|\mathcal{P}_1\mathcal{F}_1(\psi_*(\tilde{r}),\tau) - \mathcal{F}_1'(0,\tau)[\tilde{r}]\|_{s-2} \\
\leq {} & \left\| \left\{ \mathcal{P}_1\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(\psi_*(\tilde{r})) - \mathcal{P}_1\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0) \right\} (\kappa(\psi_*(\tilde{r})) - \kappa(0)) \right\|_{s-2} \\
& + \left\| \mathcal{P}_1\mathcal{E}_{\frac{\beta}{\alpha(\tau)}}(0) \left\{ \kappa(\psi_*(\tilde{r})) - \kappa(0) - \kappa'(0)[\tilde{r}] \right\} \right\|_{s-2} \\
\leq {} & C\alpha(\tau)\|\tilde{r}\|_s^2.
\end{aligned}$$

Here and in the sequel we make use of local analyticity of ψ_* and $\psi_*(0) = 0$. For $\tilde{\mathcal{F}}_2$ and $\tilde{\mathcal{F}}_3$ we get similar estimates in lower norms. Adding the results and using $\alpha(\tau) \leq 1$ in (7.16) (while replacing $\mathcal{F}$ by $\tilde{\mathcal{F}}$ and $\mathcal{F}_j$ by $\tilde{\mathcal{F}}_j$, for $j = 1, 2, 3$) we get

$$\|\tilde{\mathcal{F}}(\tilde{r},\tau) - \tilde{\mathcal{F}}'(0,\tau)[\tilde{r}]\|_{s-2} \leq C\alpha(\tau)\|\tilde{r}\|_s^2.$$

Combining this and (7.60) we get

$$(\tilde{r}, \tilde{\mathcal{F}}(\tilde{r},\tau))_{s-1} \leq (-C_3 + C\|\tilde{r}\|_s)\alpha(\tau)\|\tilde{r}\|_s^2. \tag{7.61}$$

4. It follows from

$$\begin{aligned}
& \sup_{k\geq 2} \left| \frac{\gamma p_1(k) - \frac{\beta|\mu|}{2\pi}k - \frac{|\mu|\alpha(\tau)}{2\pi}(k+2)}{\left(1+\frac{\beta}{\alpha(\tau)}k\right)(1+k^2)} \right| \\
\leq {} & \alpha(\tau) \sup_{k\geq 2} \left| \frac{\gamma p_1(k) - \frac{\beta|\mu|}{2\pi}k - \frac{|\mu|\alpha(\tau)}{2\pi}(k+2)}{\beta k(1+k^2)} \right| \\
\leq {} & C\alpha(\tau)
\end{aligned}$$

that

$$\|\mathcal{P}_1\mathcal{F}'(0,\tau)\|_{\mathcal{L}(\mathbb{H}^s_{*,1}(\mathbb{S}^1),\mathbb{H}^{s-2}_{*,1}(\mathbb{S}^1))} < C\alpha(\tau). \tag{7.62}$$

For the 2D suction problem (7.35) holds as well. Estimating the lower order terms $\mathcal{F}_2$ and $\mathcal{F}_3$ in the same way and using $\alpha(\tau) \leq 1$, we obtain from (7.62)

$$\begin{aligned} & \|\tilde{\mathcal{F}}'(\tilde{r},\tau)[D_i\tilde{r}] - \tilde{\mathcal{F}}'(0,\tau)[D_i\tilde{r}]\|_{s-2} \\ \leq\ & \|\left\{\mathcal{P}_1\mathcal{F}'(\psi_*(\tilde{r}),\tau) - \mathcal{P}_1\mathcal{F}'(0,\tau)\right\}[\psi_*'(\tilde{r})[D_i\tilde{r}]]\|_{s-2} \\ & +\|\mathcal{P}_1\mathcal{F}'(0,\tau)[\psi_*'(\tilde{r})[D_i\tilde{r}] - \psi_*'(0)[D_i\tilde{r}]]\|_{s-2} \\ \leq\ & C\alpha(\tau)\|\tilde{r}\|_s\|D_i\tilde{r}\|_s. \end{aligned} \tag{7.63}$$

In the last step we used $\|\psi_*(\tilde{r})[D_i\tilde{r}]\|_s \leq C\|D_i\tilde{r}\|_s$ which holds for small $\|\tilde{r}\|_s$. By (7.60) we have

$$(D_i\tilde{r},\tilde{\mathcal{F}}'(0,\tau)[D_i\tilde{r}])_{s-1} \leq -C_3\alpha(\tau)\|D_i\tilde{r}\|_s^2.$$

Combining this and (7.63) one gets

$$(D_i\tilde{r},\tilde{\mathcal{F}}'(\tilde{r},\tau)[D_i\tilde{r}])_{s-1} \leq (-C_3 + C\|\tilde{r}\|_s)\alpha(\tau)\|D_i\tilde{r}\|_s^2. \tag{7.64}$$

5. Adding (7.61) and (7.64) we get for $\|\tilde{r}\|_s$ small

$$(\tilde{r},\tilde{\mathcal{F}}(\tilde{r},\tau))_{s-1,1} \leq (-C_3 + C\|\tilde{r}\|_s)\alpha(\tau)\|\tilde{r}\|_{s,1}^2 \leq -\lambda_0\alpha(\tau)\|\tilde{r}\|_{s,1}^2.$$

By Theorem A.1 there exists a solution $\tilde{r}$ to (7.59) with $\|\tilde{r}(\tau)\|_s^2 \leq y(\tau)$, where $y : [0,\infty) \to \mathbb{R}$ satisfies

$$\frac{dy}{d\tau} = -2\lambda_0\alpha(\tau)y,$$

with $y(0) = \|\tilde{r}(0)\|_s = \|\mathcal{P}_1 r\|_s$. We have seen in the proof of Theorem 5.6 that this implies

$$\|\tilde{r}(t)\|_s \leq \left(-\frac{|\mu|t}{\pi} + 1\right)^{\frac{\pi\lambda_0}{|\mu|}} \|\tilde{r}_0\|,$$

where we reintroduced the original time variable t. Now we construct

$$r := \psi_*(\tilde{r}).$$

There exists a $\delta > 0$ and an $M > 0$ such that if $\|r_0\|_s \leq \delta$ then

$$\|r(t)\|_s \leq M\left(-\frac{|\mu|t}{\pi} + 1\right)^{\frac{\pi\lambda_0}{|\mu|}} \|r_0\|$$

or

$$\|r(t)\|_s \leq M\alpha(t)^{\frac{2\pi\lambda_0}{|\mu|}} \|r_0\|.$$

The condition in Corollary 7.4 is satisfied since $\lambda_0 > \frac{\mu}{\pi}$.

□

In contrast to the two-dimensional suction problem in Theorem 5.8 there is a bound on the suction speed. If the suction speed is too large, then some eigenvalues of $\mathcal{F}'(0,\tau)$ are positive for all values of time.

We do not treat the three-dimensional suction problem because it is linearly unstable for long time. From the linearisation we see that as t tends to the vanishing time, more and more eigenvalues become positive like in the suction problem with only surface tension for $N \geq 4$.

7.5 Proof of Lemma 7.7

We assume that r is small in $\mathbb{H}^s(\mathbb{S}^{N-1})$ and C is a positive varying constant that does not depend on r and $\hat{\beta}$. Let for $\sigma > 0$ the mapping $\mathcal{P}_0 : \mathbb{H}^\sigma(\mathbb{S}^{N-1}) \to \mathbb{H}^\sigma(\mathbb{S}^{N-1})$ be defined as the projection along the spherical harmonic of degree zero,

$$\mathcal{P}_0 : r \mapsto r - \hat{r},$$

where $\hat{r} \in \mathbb{R}$ is defined by

$$\hat{r} = \frac{1}{\sigma_N} \int_{\mathbb{S}^{N-1}} r dx. \tag{7.65}$$

1. First we show that there exists a $C > 0$, independent of λ and ψ, such that

$$\|\mathcal{P}_0 \mathrm{Tr}\psi\|_{s-1} \leq C \left(\left\| \left(\lambda \mathrm{Tr} + \frac{\partial}{\partial n} \right) \psi \right\|_{s-2} + \|\Delta\psi\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} \right), \tag{7.66}$$

for any $\lambda > 0$ and $\psi \in \mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)$. To prove (7.66) we write $\psi = \psi_H + \psi_0$, where

$$\Delta\psi_H = 0, \quad \Delta\psi_0 = \Delta\psi$$

and

$$\mathrm{Tr}\psi_H = \mathrm{Tr}\psi, \quad \mathrm{Tr}\psi_0 = 0.$$

Introduce

$$\hat{\psi} := \frac{1}{\sigma_N} \int_{\mathbb{S}^{N-1}} \psi dx.$$

It follows from

$$\mathrm{Tr}\psi_H - \hat{\psi} = \sum_{k\geq 1} (\mathrm{Tr}\psi_H, s_{kj})_0 s_{kj}$$

that

$$\|\mathrm{Tr}\psi - \hat{\psi}\|_{s-1} = \|\mathrm{Tr}\psi_H - \hat{\psi}\|_{s-1} \leq \sqrt{2}\, \|(\lambda\mathcal{I} + \mathcal{N})\mathrm{Tr}\psi_H\|_{s-2}.$$

Here we used the fact that for $k \geq 1$

$$\frac{\sqrt{1+k^2}}{\lambda + k} \leq \sqrt{2}.$$

We get

$$\begin{aligned}
\|\mathrm{Tr}\psi - \hat{\psi}\|_{s-1} &\leq \sqrt{2}\left\|\left(\lambda\mathrm{Tr} + \frac{\partial}{\partial n}\right)\psi_H\right\|_{s-2} \\
&\leq \sqrt{2}\left\|\left(\lambda\mathrm{Tr} + \frac{\partial}{\partial n}\right)\psi\right\|_{s-2} + \sqrt{2}\left\|\left(\lambda\mathrm{Tr} + \frac{\partial}{\partial n}\right)\psi_0\right\|_{s-2} \\
&= \sqrt{2}\left\|\left(\lambda\mathrm{Tr} + \frac{\partial}{\partial n}\right)\psi\right\|_{s-2} + \sqrt{2}\left\|\frac{\partial\psi_0}{\partial n}\right\|_{s-2}.
\end{aligned}$$

Combining this with (7.14) we obtain (7.66).

2. Define $u_{\hat{\beta}} : \mathcal{U}_{\hat{\beta}} \to \mathcal{L}(\mathbb{H}^{s-1}(\mathbb{S}^{N-1}), \mathbb{H}^{s+\frac{1}{2}}(\mathbb{B}^N))$ by

$$u_{\hat{\beta}}(r) := \mathcal{S}_{\hat{\beta}}(r)^{-1}\begin{pmatrix} 0 \\ \cdot \end{pmatrix} \tag{7.67}$$

and introduce $w_{\hat{\beta}} : \mathcal{U}_{\hat{\beta}} \to \mathcal{L}(\mathbb{H}^{s-1}(\mathbb{S}^{N-1}), \mathbb{H}^{s-1}(\mathbb{S}^{N-1}))$ by

$$w_{\hat{\beta}}(r) := n(r)\cdot\mathcal{Q}(r)u_{\hat{\beta}}(r). \tag{7.68}$$

Since $u_{\hat{\beta}}(0)f$ is harmonic we get from (7.12)

$$\|u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \leq C\|\mathrm{Tr}u_{\hat{\beta}}(0)f\|_{s-1} = C\|(\mathcal{I}+\hat{\beta}\mathcal{N})^{-1}f\|_{s-1} \leq C\|f\|_{s-1} \tag{7.69}$$

and

$$\|w_{\hat{\beta}}(0)f\|_{s-1} = \|\mathcal{N}(\mathcal{I}+\hat{\beta}\mathcal{N})^{-1}f\|_{s-1} \leq \frac{1}{\hat{\beta}}\|f\|_{s-1}. \tag{7.70}$$

3. Introduce

$$\mathcal{R}(r) := n(r)\cdot\mathcal{Q}(r) - n(0)\cdot\mathcal{Q}(0). \tag{7.71}$$

It is clear from the definition of $\mathcal{S}_{\hat{\beta}}$ (see (7.7)) that

$$\mathrm{Tr}(u_{\hat{\beta}}(r) - u_{\hat{\beta}}(0)) = -\hat{\beta}(w_{\hat{\beta}}(r) - w_{\hat{\beta}}(0)). \tag{7.72}$$

As a result, for any $f \in \mathbb{H}^{s-1}(\mathbb{S}^{N-1})$ and $r \in \mathcal{U}_{\hat{\beta}}$

$$\begin{aligned}
\mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f) &= -\hat{\beta}\left(n(r)\cdot\mathcal{Q}(r)u_{\hat{\beta}}(r)f - n(0)\cdot\mathcal{Q}(0)u_{\hat{\beta}}(0)f\right) \\
&= -\hat{\beta}\left(\mathcal{R}(r)u_{\hat{\beta}}(r)f + n(0)\cdot\mathcal{Q}(0)\big(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\big)\right).
\end{aligned}$$

We rewrite this as follows:

$$\begin{aligned}
\big(\mathrm{Tr} + \hat{\beta}n(0)\cdot\mathcal{Q}(0)\big)\left(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\right) &= -\hat{\beta}\mathcal{R}(r)u_{\hat{\beta}}(r)f \\
= -\hat{\beta}\mathcal{R}(r)\left(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f + u_{\hat{\beta}}(0)f\right).
\end{aligned} \tag{7.73}$$

The mapping $\mathcal{R}$ is analytic from a neighbourhood of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ to $\mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$. It follows from (7.73) and (7.69) that

$$\begin{aligned} & \left\| \left(\mathrm{Tr} + \hat{\beta} n(0) \cdot \mathcal{Q}(0)\right) \left(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\right) \right\|_{s-2} \\ \leq \; & C\hat{\beta}\|r\|_s \left(\|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} + \|f\|_{s-1} \right). \end{aligned} \tag{7.74}$$

Setting $\psi = u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f$ and $\lambda = \hat{\beta}^{-1}$ in (7.66) we get from (7.74) and $n(0) \cdot \mathcal{Q}(0) = \frac{\partial}{\partial n}$ the estimate

$$\begin{aligned} & \|\mathcal{P}_0 \mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{s-1} \\ \leq \; & C \Big(\|r\|_s \|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \\ & + \|r\|_s \|f\|_{s-1} + \|\Delta(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} \Big). \end{aligned} \tag{7.75}$$

Since $\Delta u_{\hat{\beta}}(0)f = 0$ and $\mathcal{A}(r)u_{\hat{\beta}}(r)f = 0$ we have

$$\begin{aligned} \Delta(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f) &= (\Delta - \mathcal{A}(r))u_{\hat{\beta}}(r)f \\ &= (\Delta - \mathcal{A}(r))(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f) + (\Delta - \mathcal{A}(r))u_{\hat{\beta}}(0)f. \end{aligned}$$

It follows from (7.69) that

$$\begin{aligned} & \|\Delta(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} \\ \leq \; & \|\Delta - \mathcal{A}(r)\|_{\mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N))} (\|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} + C\|f\|_{s-1}) \\ \leq \; & C\|r\|_s (\|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} + \|f\|_{s-1}). \end{aligned} \tag{7.76}$$

Here we used Lipschitz continuity of $r \mapsto \mathcal{A}(r)$ from a neighbourhood of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ to $\mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N))$ and the fact that $\mathcal{A}(0) = \Delta$. From (7.75) and (7.76) we conclude

$$\|\mathcal{P}_0 \mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{s-1} \leq C\|r\|_s (\|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} + \|f\|_{s-1}). \tag{7.77}$$

It follows from (7.12) and (7.76) that

$$\begin{aligned} & \|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \\ \leq \; & C\|\mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{s-1} + C\|\Delta(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} \\ \leq \; & C\|\mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{s-1} + C\|r\|_s \left(\|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} + \|f\|_{s-1} \right). \end{aligned}$$

Consequently, for $\|r\|_s$ small

$$\|u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \leq C\|\mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{s-1} + C\|r\|_s\|f\|_{s-1} \quad (7.78)$$

Combining this with (7.77) we get

$$\|\mathcal{P}_0\mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{s-1} \leq C\|r\|_s \left(\|\mathrm{Tr}(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\|_{s-1} + \|f\|_{s-1}\right). \quad (7.79)$$

It follows from (7.72) and (7.79) that

$$\|\mathcal{P}_0(w_{\hat{\beta}}(r)f - w_{\hat{\beta}}(0)f)\|_{s-1} \leq C\|r\|_s\|w_{\hat{\beta}}(r)f - w_{\hat{\beta}}(0)f\|_{s-1} + \frac{C}{\hat{\beta}}\|r\|_s\|f\|_{s-1}. \quad (7.80)$$

4. Define for $\|r\|_s$ small $\tilde{j}(r) := \sqrt{\frac{g(r)}{g(0)}} \circ \Xi^{-1}$, with g and Ξ as in Section 3.2, such that for all $\psi : \mathbb{S}^{N-1} \to \mathbb{R}$

$$\int_{\mathbb{S}^{N-1}} \psi \tilde{j}(r) dx = \int_{\Gamma_r} \psi \circ z(r)^{-1} dx.$$

Since $u_{\hat{\beta}}(0)f$ is harmonic on $\mathbb{B}^N$ and $(u_{\hat{\beta}}(r)f) \circ z(r)^{-1}$ is harmonic on Ω_r we have

$$\int_{\mathbb{S}^{N-1}} w_{\hat{\beta}}(0) f dx = \int_{\mathbb{S}^{N-1}} \frac{\partial u_{\hat{\beta}}(0)f}{\partial n} dx = 0$$

and

$$\int_{\mathbb{S}^{N-1}} (w_{\hat{\beta}}(r)f)\tilde{j}(r) dx = \int_{\Gamma_r} n_r \cdot \nabla((u_{\hat{\beta}}(r)f) \circ z(r)^{-1}) dx = 0,$$

where n_r is the normal vector field on Γ_r. It follows that

$$\begin{aligned}
& \left| \int_{\mathbb{S}^{N-1}} w_{\hat{\beta}}(r)f - w_{\hat{\beta}}(0) f dx \right| \\
= & \left| \int_{\mathbb{S}^{N-1}} (w_{\hat{\beta}}(r)f)(1 - \tilde{j}(r)) dx \right| \\
\leq & \|w_{\hat{\beta}}(r)f\|_0 \|1 - \tilde{j}(r)\|_0 \\
\leq & \|w_{\hat{\beta}}(r)f - w_{\hat{\beta}}(0)f\|_0 \|1 - \tilde{j}(r)\|_0 + \|w_{\hat{\beta}}(0)f\|_0 \|1 - \tilde{j}(r)\|_0 \\
\leq & C\|r\|_s \|w_{\hat{\beta}}(r)f - w_{\hat{\beta}}(0)f\|_0 + C\|r\|_s \|w_{\hat{\beta}}(0)f\|_0 \\
\leq & C\|r\|_s \|w_{\hat{\beta}}(r)f - w_{\hat{\beta}}(0)f\|_0 + \frac{C}{\hat{\beta}}\|r\|_s \|f\|_{s-1}. \quad (7.81)
\end{aligned}$$

We used $\tilde{j}(0) = 1$, Lipschitz continuity of $\tilde{j}$ near zero from $\mathbb{H}^s(\mathbb{S}^{N-1})$ to $\mathbb{L}^2(\mathbb{S}^{N-1})$ and (7.70).

5. Since

$$\begin{aligned} & \|w_{\hat\beta}(r)f - w_{\hat\beta}(0)f\|_{s-1} \\ \leq \ & C\left(\|\mathcal{P}_0(w_{\hat\beta}(r)f - w_{\hat\beta}(0)f)\|_{s-1} + \left|\int_{\mathbb{S}^{N-1}} w_{\hat\beta}(r)f - w_{\hat\beta}(0)f dx\right|\right) \end{aligned}$$

we get from adding (7.80) and (7.81)

$$\|w_{\hat\beta}(r)f - w_{\hat\beta}(0)f\|_{s-1} \leq C\|r\|_s\|w_{\hat\beta}(r)f - w_{\hat\beta}(0)f\|_{s-1} + \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}.$$

For $\|r\|_s$ small it follows that

$$\|w_{\hat\beta}(r)f - w_{\hat\beta}(0)f\|_{s-1} \leq \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1} \tag{7.82}$$

and (7.72) yields

$$\|\mathrm{Tr}(u_{\hat\beta}(r)f - u_{\hat\beta}(0)f)\|_{s-1} \leq C\|r\|_s\|f\|_{s-1}. \tag{7.83}$$

Combining (7.78) and (7.83) we get

$$\|u_{\hat\beta}(r)f - u_{\hat\beta}(0)f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \leq C\|r\|_s\|f\|_{s-1}. \tag{7.84}$$

From (7.70) and (7.82) it follows that for $\|r\|_s$ small

$$\begin{aligned} & \|\mathcal{E}_{\hat\beta}(r)f - \mathcal{E}_{\hat\beta}(0)f\|_{s-1} \\ = \ & \left\|\frac{w_{\hat\beta}(r)f}{n(r)\cdot \mathrm{id}} - \frac{w_{\hat\beta}(0)f}{n(0)\cdot \mathrm{id}}\right\|_{s-1} \\ = \ & \left\|\frac{w_{\hat\beta}(r)f}{n(r)\cdot \mathrm{id}} - w_{\hat\beta}(0)f\right\|_{s-1} \\ \leq \ & \left\|w_{\hat\beta}(r)f - w_{\hat\beta}(0)f\right\|_{s-1} + \left\|\left(\frac{1}{n(r)\cdot \mathrm{id}} - 1\right) w_{\hat\beta}(r)f\right\|_{s-1} \\ \leq \ & \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1} + \|r\|_s\|w_{\hat\beta}(r)f\|_{s-1} \\ \leq \ & \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1} + \|r\|_s\|w_{\hat\beta}(r)f - w_{\hat\beta}(0)f\|_{s-1} + \|r\|_s\|w_{\hat\beta}(0)f\|_{s-1} \\ \leq \ & \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}. \end{aligned}$$

This proves the first part of the lemma.

6. For the second part, we first differentiate the identity

$$\int_{\mathbb{S}^{N-1}} (w_{\hat\beta}(r)f)\tilde{j}(r)dx = 0$$

to obtain

$$\int_{\mathbb{S}^{N-1}} (w'_{\hat\beta}(r)[h]f)\tilde{j}(r) + (w_{\hat\beta}(r)f)\tilde{j}'(r)[h]dx = 0. \tag{7.85}$$

In particular, since $\tilde{j}(0) = 1$, it follows that

$$\int_{\mathbb{S}^{N-1}} w'_{\hat\beta}(0)[h]f dx = -\int_{\mathbb{S}^{N-1}} (w_{\hat\beta}(0)f)\tilde{j}'(0)[h]dx$$

and therefore by (7.70)

$$\left|\int_{\mathbb{S}^{N-1}} w'_{\hat\beta}(0)[h]f dx\right| \le \|w_{\hat\beta}(0)f\|_0\|\tilde{j}'(0)[h]\|_0 \le \frac{C}{\hat\beta}\|h\|_s\|f\|_{s-1}. \tag{7.86}$$

Now we differentiate

$$\mathcal{A}(r)u_{\hat\beta}(r)f = 0$$

to obtain

$$\mathcal{A}'(r)[h]u_{\hat\beta}(r)f + \mathcal{A}(r)u'_{\hat\beta}(r)[h]f = 0 \tag{7.87}$$

and in particular

$$\mathcal{A}'(0)[h]u_{\hat\beta}(0)f + \Delta u'_{\hat\beta}(0)[h]f = 0. \tag{7.88}$$

In the same way it follows from

$$(\mathrm{Tr} + \hat\beta n(r)\mathcal{Q}(r))u_{\hat\beta}(r)f = f$$

that

$$\left(\mathrm{Tr} + \hat\beta\frac{\partial}{\partial n}\right)u'_{\hat\beta}(0)[h]f = -\hat\beta\left\{n'(0)[h]\cdot\nabla + n(0)\cdot\mathcal{Q}'(0)[h]\right\}u_{\hat\beta}(0)f. \tag{7.89}$$

From (7.66), (7.69), (7.88), and (7.89) we get

$$\begin{aligned}
&\|\mathcal{P}_0\mathrm{Tr}u'_{\hat\beta}(0)[h]f\|_{s-1}\\
\le\ & \frac{C}{\hat\beta}\left\|\left(\mathrm{Tr} + \hat\beta\frac{\partial}{\partial n}\right)u'_{\hat\beta}(0)[h]f\right\|_{s-2} + C\|\Delta u'_{\hat\beta}(0)[h]f\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)}\\
\le\ & C\|h\|_s\|f\|_{s-1}
\end{aligned}$$

and since (7.72) implies

$$\mathrm{Tr}u'_{\hat\beta}(r)[h]f = -\hat\beta w'_{\hat\beta}(r)[h]f \tag{7.90}$$

it follows that

$$\|\mathcal{P}_0 w'_{\hat\beta}(0)[h]f\|_{s-1} \le \frac{C}{\hat\beta}\|h\|_s\|f\|_{s-1}. \tag{7.91}$$

Adding (7.86) and (7.91) we get

$$\|w'_{\hat\beta}(0)[h]f\|_{s-1} \le \frac{C}{\hat\beta}\|h\|_s\|f\|_{s-1} \tag{7.92}$$

and therefore by (7.90)

$$\|\mathrm{Tr}u'_{\hat\beta}(0)[h]f\|_{s-1} \le C\|h\|_s\|f\|_{s-1}. \tag{7.93}$$

From (7.12), (7.69), (7.88), and (7.93) we derive

$$\begin{aligned}\|u'_{\hat\beta}(0)[h]f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} &\le C\left(\|\mathrm{Tr}u'_{\hat\beta}(0)[h]f\|_{s-1} + \|\Delta u'_{\hat\beta}(0)[h]f\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)}\right)\\ &\le C\|h\|_s\|f\|_{s-1}. \end{aligned}\tag{7.94}$$

7. From (7.70), (7.82), (7.85), (7.92), the identity $\tilde{j}(0) = 1$, and analyticity of $\tilde{j}$, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{S}^{N-1}} w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f dx\right|\\
=& \left|\int_{\mathbb{S}^{N-1}} (w'_{\hat\beta}(r)[h]f)(1-\tilde{j}(r)) - (w_{\hat\beta}(r)f)\tilde{j}'(r)[h] - w'_{\hat\beta}(0)[h]f dx\right|\\
=& \left|\int_{\mathbb{S}^{N-1}} (w'_{\hat\beta}(r)[h]f)(1-\tilde{j}(r)) - (w_{\hat\beta}(r)f - w_{\hat\beta}(0)f)\tilde{j}'(r)[h]\right.\\
&\left. -(w_{\hat\beta}(0)f)(\tilde{j}'(r)[h] - \tilde{j}'(0)[h])dx\right|\\
\le& C\|r\|_s\left(\|w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f\|_0 + \|w'_{\hat\beta}(0)[h]f\|_0\right)\\
&+C\|h\|_s\|w_{\hat\beta}(r)f - w_{\hat\beta}(0)f\|_0 + \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}\|h\|_s\\
\le& C\|r\|_s\|w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f\|_0 + \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}\|h\|_s. \end{aligned}\tag{7.95}$$

We also used $\|\tilde{j}'(r)[h]\|_0 \le C\|h\|_s$ which holds since $\|r\|_s$ is small and $\tilde{j}$ is analytic near zero.

8. Differentiating (7.73) we obtain

$$\left(\mathrm{Tr} + \hat\beta n(0)\cdot\mathcal{Q}(0)\right)u'_{\hat\beta}(r)[h]f = -\hat\beta\mathcal{R}'(r)[h]u_{\hat\beta}(r)f - \hat\beta\mathcal{R}(r)u'_{\hat\beta}(r)[h]f.$$

Since $\mathcal{R}(0) = 0$ we get

$$\left(\mathrm{Tr} + \hat\beta n(0)\cdot\mathcal{Q}(0)\right)u'_{\hat\beta}(0)[h]f = -\hat\beta\mathcal{R}'(0)[h]u_{\hat\beta}(0)f$$

and therefore

$$\begin{aligned}
& \left(\mathrm{Tr} + \hat{\beta} n(0) \cdot \mathcal{Q}(0)\right) (u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f) \\
= \quad & -\hat{\beta} \left(\mathcal{R}'(r)[h]u_{\hat{\beta}}(r)f - \mathcal{R}'(0)[h]u_{\hat{\beta}}(0)f\right) - \hat{\beta}\mathcal{R}(r)u'_{\hat{\beta}}(r)[h]f \\
= \quad & -\hat{\beta} \Big((\mathcal{R}'(r)[h] - \mathcal{R}'(0)[h])(u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f + u_{\hat{\beta}}(0)f) \\
& +\mathcal{R}'(0)[h](u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f)\Big) \\
& -\hat{\beta}\mathcal{R}(r)(u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f + u'_{\hat{\beta}}(0)[h]f).
\end{aligned}$$

From (7.69), (7.84), (7.94), analyticity of $\mathcal{R}$ from a neighbourhood of zero in $\mathbb{H}^s(\mathbb{S}^{N-1})$ to $\mathcal{L}(\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N), \mathbb{H}^{s-2}(\mathbb{S}^{N-1}))$, and the fact that $\mathcal{R}(0) = 0$, it follows that for $\|r\|_s$ small

$$\begin{aligned}
& \left\| \left(\mathrm{Tr} + \hat{\beta} n(0) \cdot \mathcal{Q}(0)\right) (u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f) \right\|_{s-2} \\
\leq \quad & C\hat{\beta}\|r\|_s\|f\|_{s-1}\|h\|_s + C\hat{\beta}\|r\|_s\|u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)}.
\end{aligned}$$

From (7.66) we get

$$\begin{aligned}
& \|\mathcal{P}_0\mathrm{Tr}(u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f)\|_{s-1} \\
\leq \quad & C\|r\|_s\|f\|_{s-1}\|h\|_s + C\|r\|_s\|u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} \\
& +C\|\Delta(u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f)\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)}.
\end{aligned} \tag{7.96}$$

Using (7.87) and (7.88) we deduce

$$\begin{aligned}
& \Delta(u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f) \\
= \quad & (\Delta - \mathcal{A}(r))u'_{\hat{\beta}}(r)[h]f + \mathcal{A}(r)u'_{\hat{\beta}}(r)[h]f + \mathcal{A}'(0)[h]u_{\hat{\beta}}(0)f \\
= \quad & (\Delta - \mathcal{A}(r))u'_{\hat{\beta}}(r)[h]f - \mathcal{A}'(r)[h]u_{\hat{\beta}}(r)f + \mathcal{A}'(0)[h]u_{\hat{\beta}}(0)f \\
= \quad & (\Delta - \mathcal{A}(r))(u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f + u'_{\hat{\beta}}(0)[h]f) \\
& -\mathcal{A}'(r)[h](u_{\hat{\beta}}(r)f - u_{\hat{\beta}}(0)f) + (\mathcal{A}'(0)[h] - \mathcal{A}'(r)[h])u_{\hat{\beta}}(0)f.
\end{aligned}$$

From (7.69), (7.84), (7.94) and analyticity of $\mathcal{A}$ we derive

$$\begin{aligned}
& \|\Delta(u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f)\|_{\mathbb{H}^{s-\frac{5}{2}}(\mathbb{B}^N)} \\
\leq \quad & C\|r\|_s\|u'_{\hat{\beta}}(r)[h]f - u'_{\hat{\beta}}(0)[h]f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} + C\|r\|_s\|f\|_{s-1}\|h\|_s.
\end{aligned} \tag{7.97}$$

It follows from (7.12) and (7.97) that

$$\begin{aligned}
&\|u'_{\hat\beta}(r)[h]f - u'_{\hat\beta}(0)[h]f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)}\\
\leq\ & C\|\mathrm{Tr}(u'_{\hat\beta}(r)[h]f - u'_{\hat\beta}(0)[h]f)\|_{s-1}\\
&+C\|r\|_s\|u'_{\hat\beta}(r)[h]f - u'_{\hat\beta}(0)[h]f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)} + C\|r\|_s\|f\|_{s-1}\|h\|_s
\end{aligned}$$

and therefore as $\|r\|_s$ is small

$$\begin{aligned}
&\|u'_{\hat\beta}(r)[h]f - u'_{\hat\beta}(0)[h]f\|_{\mathbb{H}^{s-\frac{1}{2}}(\mathbb{B}^N)}\\
\leq\ & C\|\mathrm{Tr}(u'_{\hat\beta}(r)[h]f - u'_{\hat\beta}(0)[h]f)\|_{s-1} + C\|r\|_s\|f\|_{s-1}\|h\|_s. \qquad (7.98)
\end{aligned}$$

Combining (7.96), (7.97), and (7.98) we get

$$\begin{aligned}
&\|\mathcal{P}_0\mathrm{Tr}(u'_{\hat\beta}(r)[h]f - u'_{\hat\beta}(0)[h]f)\|_{s-1}\\
\leq\ & C\|r\|_s\|\mathrm{Tr}(u'_{\hat\beta}(r)[h]f - u'_{\hat\beta}(0)[h]f)\|_{s-1} + C\|r\|_s\|f\|_{s-1}\|h\|_s.
\end{aligned}$$

Hence (7.72) implies

$$\begin{aligned}
&\|\mathcal{P}_0(w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f)\|_{s-1}\\
\leq\ & C\|r\|_s\|(w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f)\|_{s-1}\\
&+\frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}\|h\|_s.
\end{aligned}$$

Combining this estimate and (7.95) we find

$$\begin{aligned}
\|(w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f)\|_{s-1} \leq\ & C\|r\|_s\|(w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f)\|_{s-1}\\
&+\frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}\|h\|_s
\end{aligned}$$

and therefore as $\|r\|_s$ is small

$$\|(w'_{\hat\beta}(r)[h]f - w'_{\hat\beta}(0)[h]f)\|_{s-1} \leq \frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}\|h\|_s. \qquad (7.99)$$

9. Using analyticity of n, $n(0) = \mathrm{id}$, (6.31), (7.99), (7.92), (7.82), (7.70), and the identity

$$\mathcal{E}_{\hat\beta}(r)f = \frac{w_{\hat\beta}(r)f}{n(r)\cdot\mathrm{id}},$$

we derive

$$\begin{aligned}
&\left\|\mathcal{E}_{\hat\beta}'(r)[h]f-\mathcal{E}_{\hat\beta}'(0)[h]f\right\|_{s-1}\\
\leq\;&\left\|\frac{w_{\hat\beta}'(r)[h]f}{n(r)\cdot\mathrm{id}}-w_{\hat\beta}'(0)[h]f\right\|_{s-1}+\left\|\frac{n'(r)[h]\cdot\mathrm{id}}{(n(r)\cdot\mathrm{id})^2}w_{\hat\beta}(r)f\right\|_{s-1}\\
\leq\;&\left\|\frac{1}{n(r)\cdot\mathrm{id}}\left(w_{\hat\beta}'(r)[h]f-w_{\hat\beta}'(0)[h]f\right)\right\|_{s-1}+\left\|\left(\frac{1}{n(r)\cdot\mathrm{id}}-1\right)w_{\hat\beta}'(0)[h]f\right\|_{s-1}\\
&+\left\|\frac{n'(r)[h]\cdot\mathrm{id}}{(n(r)\cdot\mathrm{id})^2}\left(w_{\hat\beta}(r)f-w_{\hat\beta}(0)f\right)\right\|_{s-1}+\left\|\frac{n'(r)[h]\cdot\mathrm{id}}{(n(r)\cdot\mathrm{id})^2}w_{\hat\beta}(0)f\right\|_{s-1}\\
\leq\;&\frac{C}{\hat\beta}\|r\|_s\|f\|_{s-1}\|h\|_s.
\end{aligned}$$

This proves the second statement in the lemma.

Appendix A

Existence results from energy estimates

Theorem A.1. *(Kato and Lai) Let $V \hookrightarrow H \hookrightarrow X$ be densely injected Hilbert spaces for which there exists a continuous bilinear form $\langle \cdot, \cdot \rangle : V \times X \to \mathbb{R}$ that satisfies*

$$\langle r_1, r_2 \rangle = (r_1, r_2)_H$$

for $r_1 \in V$ and $r_2 \in H$. Let $\mathcal{F}$ be a weakly continuous map on $H \times [0, T]$ into X and on $V \times [0, T]$ into H such that for all $t \in [0, T]$ and $r \in V$

$$(r, \mathcal{F}(r, t))_H \leq f(\|r\|_H^2, t), \tag{A.1}$$

where the function f is a differentiable function on $[0, \infty) \times [0, T]$. Let $r_0 \in V$ and let the function $y : [0, T') \to \mathbb{R}$ be defined as the solution to

$$\frac{dy}{dt} = 2f(y, t), \quad y(0) = \|r_0\|_H^2,$$

where T' is the maximal time value for which a solution to this ODE can be defined. If the solution to the ODE is not unique, then one has to take the maximal solution. Suppose that y is bounded on $[0, T')$. Then, there is a solution $r \in \mathcal{C}_w([0, T'), H) \cap \mathcal{C}_w^1([0, T'), X)$ to

$$\frac{\partial r}{\partial t} = \mathcal{F}(r, t), \quad r(0) = r_0.$$

Here $\mathcal{C}_w$ indicates weak continuity. Moreover, one has for $t \in [0, T')$

$$\|r(t)\|_H^2 \leq y(t).$$

Proof. The difference between this theorem and the one in [50, Thm. A] is that f in (A.1) may depend on t here. However, the arguments in the proof, that is based on Galerkin approximations, of the original theorem still hold. □

In Chapters 5-7 we apply Theorem A.1 to obtain existence results on open time intervals. This can be achieved by applying the theorem infinite times.

Moreover, to apply this theorem we need to show that the evolution operators $\mathcal{F}$ in Chapters 5-7 that are continuous from let us say $V = \mathbb{H}^{\sigma}(\mathbb{S}^{N-1})$ to $H = \mathbb{H}^{\theta}(\mathbb{S}^{N-1})$ are also weakly continuous. There exists an $\epsilon > 0$ such that $\mathcal{F}$ is still well-defined and continuous from $V_\epsilon = \mathbb{H}^{\sigma-\epsilon}(\mathbb{S}^{N-1})$ to $H_\epsilon = \mathbb{H}^{\theta-\epsilon}(\mathbb{S}^{N-1})$. We have $V_\epsilon \hookrightarrow\hookrightarrow V$ and $H_\epsilon \hookrightarrow\hookrightarrow H$. Take a weakly convergent sequence $r_n \rightharpoonup r$ in V. Then $r_n \to r$ in V_ϵ. Hence $\mathcal{F}(r_n) \to \mathcal{F}(r)$ in H_ϵ. On the other hand, since $(r_n)_{n=1}^\infty$ is bounded in V, the sequence $(\mathcal{F}(r_n))_{n=1}^\infty$ is bounded in H. Let $\big(\mathcal{F}(r_{n_k})\big)_{k=1}^\infty$ be a subsequence of $(\mathcal{F}(r_n))_{n=1}^\infty$ for which there exists an $f \in H$ such that $\mathcal{F}(r_{n_k}) \rightharpoonup f$ in H. We have $\mathcal{F}(r_{n_k}) \to f$ in H_ϵ. As a consequence, $f = \mathcal{F}(r)$. We conclude that all weakly convergent subsequences of $(\mathcal{F}(r_n))_{n=1}^\infty$ have the same weak limit in H, namely $\mathcal{F}(r)$. From [80, Prop. 10.13] it follows that $\mathcal{F}(r_n) \rightharpoonup \mathcal{F}(r)$ in H.

Another problem is that the evolution operators $\mathcal{F}$ in Chapters 5, 6, and 7 are only defined on a neighbourhood of zero. Therefore we apply the above theorem to an operator that is equal to $\mathcal{F}$ on some neighbourhood of zero and extend it smoothly outside this neighbourhood. The energy estimates that we find force r to stay in the region where $\mathcal{F}$ is equal to the operator on which we apply the theorem.

Appendix B

Solution to the Stokes BVP

f	$v\mid_{\mathbb{S}^2}$	v
$\vec{V}_{km}, k \geq 2$	$\frac{k}{2k^2+4k+3}\vec{V}_{km}$	$\frac{k\rho^{k+1}}{2k^2+4k+3}\vec{V}_{km} + \frac{1}{2}\frac{\sqrt{k}\sqrt{k+1}(2k+3)(\rho^{k+1}-\rho^{k-1})}{2k^2+4k+3}\vec{W}_{km}$
$\vec{X}_{km}, k \geq 2$	$\frac{1}{k-1}\vec{X}_{km}$	$\frac{1}{k-1}\rho^k\vec{X}_{km}$
$\vec{W}_{km}, k \geq 2$	$\frac{1}{2(k-1)}\vec{W}_{km}$	$\frac{1}{2(k-1)}\rho^{k-1}\vec{W}_{km}$
$\vec{V}_{00}$	0	0
$\vec{V}_{1m}$	$\frac{1}{9}\vec{V}_{1m} + \frac{\sqrt{2}}{9}\vec{W}_{1m}$	$\frac{1}{9}\rho^2\vec{V}_{1m} + \left(-\frac{\sqrt{2}}{6} + \frac{5\sqrt{2}}{18}\rho^2\right)\vec{W}_{1m}$
$\vec{X}_{1m}, \vec{W}_{1m}$	$-$	$-$

f	p	$v \cdot n$
$\vec{V}_{km}, k \geq 2$	$\frac{(2k+3)\sqrt{(2k+1)(k+1)}}{2k^2+4k+3}\rho^k Y_{km}$	$-\frac{k}{2k^2+4k+3}\sqrt{\frac{k+1}{2k+1}}Y_{km}$
$\vec{X}_{km}, k \geq 2$	0	0
$\vec{W}_{km}, k \geq 2$	0	$\frac{1}{2(k-1)}\sqrt{\frac{k}{2k+1}}Y_{km}$
$\vec{V}_{00}$	Y_{00}	0
$\vec{V}_{1m}$	$\frac{5}{3}\sqrt{\frac{2}{3}}\rho Y_{1m}$	0
$\vec{X}_{1m}, \vec{W}_{1m}$	$-$	$-$

Table B.1: *This table shows the explicit solution to the boundary value problem* (6.38)-(6.42) *in three dimensions in terms of the eigenfunctions. An expression like $\rho^k Y_{km}$ should be interpreted as the function that maps an element of $\mathbb{B}^3$ characterised by spherical coordinates to $\rho^k Y_{km}(\theta, \phi)$.*

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Nomenclature

$\mathcal{A}(r)$	$u \mapsto \Delta\left(u \circ z(r)^{-1}\right) \circ z(r)$
$\mathcal{B}(r)$	$u \mapsto \Delta_r\left(u \circ \tilde{z}(r)^{-1}\right) \circ \tilde{z}(r)$
$\mathcal{C}^{k,\beta}(K)$	Hölder spaces on domain K
$\mathcal{C}^{k,\beta}(K)^N$	vectors with components in Hölder spaces
$\mathcal{C}^{k,\beta}(K)^{N\times N}$	matrices with components in Hölder spaces
$\mathcal{C}^{\omega}(X,Y)$	analytic mappings from X to Y
$\mathcal{C}_w(X,Y)$	weakly continuous mappings from X to Y
$\mathcal{C}_w^1(X,Y)$	weakly continuously differentiable mappings from X to Y
$\mathcal{D}(F)$	domain of definition of F
$\mathcal{E}$	for Hele-Shaw flow see (5.3), for Stokes flow see (6.25)
$\mathcal{F}$	evolution operator (different in each chapter)
$\tilde{\mathcal{F}}$	$\mathcal{P}_j \circ \mathcal{F} \circ \psi_j$ or $\mathcal{P}_j \circ \mathcal{F}(\cdot,\tau) \circ \psi_j$ often for $j=1$
$\hat{\mathcal{F}}$	restriction of $\mathcal{F}$ to a subspace of finite codimension orthogonal to spherical harmonics up to a certain degree (often 1)
$\mathcal{F}_{\times,\mu}$	restriction of $\mathcal{F}$ to functions with axial symmetry
$\tilde{\mathcal{F}}_{\times,\mu}$	$\mathcal{P}_1 \circ \mathcal{F}_{\times,\mu} \circ \psi_1$
$\mathcal{F}_1$	term in evolution operator (different in each chapter)
$\mathcal{F}_2$	term in evolution operator (different in each chapter)
$\mathcal{F}_3$	term in evolution operator in Chapter 7
$\mathcal{F}'(\rho)$	linearisation of $\mathcal{F}$ around ρ
$\mathcal{G}$	Gramm matrix, see (3.16)
$\mathcal{G}_K$	restriction of $\mathcal{F}$ to the orthoplement of $\cup_{k=0}^{K}\mathfrak{S}_k^N$
$\mathcal{H}(X,Y)$	operators A for which $-A$ generate analytic semigroups on Y with dense domain of definition X
$\mathcal{I}$	identity operator
$\mathcal{J}(r)$	derivative of $z(r)$
$\mathcal{L}(X)$	bounded linear mappings on X
$\mathcal{L}(X,Y)$	bounded linear mappings from X to Y
$\mathcal{M}_f$	multiplication with f from the left: $\mathcal{M}_f : g \mapsto fg$
$\mathcal{N}$	Dirichlet-to-Neumann operator on unit ball, see Section 1.5 or (2.33)
$\mathcal{P}_K$	projection on the $\mathbb{L}^2$-orthoplement of $\cup_{k=0}^{K}\mathfrak{S}_k^N$
$\mathcal{Q}(r)$	$u \mapsto \nabla\left(u \circ z(r)^{-1}\right) \circ z(r)$
$\mathcal{R}(r)$	in Chapter 6: $u \mapsto \operatorname{rot}\left(u \circ z(r)^{-1}\right) \circ z(r)$

$\mathcal{R}(r)$	in Chapter 7: $n(r) \cdot \mathcal{Q}(r) - n(0) \cdot \mathcal{Q}(0)$
$\mathcal{R}(\lambda, F)$	resolvent operator $(\lambda\mathcal{I} - F)^{-1}$
$\mathcal{S}(r)$	for Hele-Shaw flow see (2.27), for Stokes flow see (6.22)
$\mathcal{S}_{\hat{\beta}}(r)$	see (7.7)
$\mathcal{U}$	neighbourhood of zero in some function space
$\mathbb{B}^N$	unit ball in $\mathbb{R}^N$
$\mathbb{C}$	set of complex numbers
$\mathbb{F}$	Fourier transform
$\mathbb{H}^s$	Sobolev spaces (Spaces with vector/matrix-valued functions are indicated in the same way as the Hölder spaces.)
$\mathbb{H}^s_1$	orthoplement of $\mathfrak{S}^3_0 \oplus \mathfrak{S}^3_1$ in $\mathbb{H}^s$
$\mathbb{H}^s_\times$	subspace of $\mathbb{H}^s$ consisting of functions with z-axial symmetry
$\mathbb{H}^s_{\times,1}$	orthoplement of $\mathfrak{S}^3_0 \oplus \mathfrak{S}^3_1$ in $\mathbb{H}^s_\times$
$\mathbb{H}^s_*$	subspace of $\mathbb{H}^s$ consisting of functions that are even in all variables
$\mathbb{H}^s_{*,1}$	orthoplement of $\mathfrak{S}^3_0 \oplus \mathfrak{S}^3_1$ in $\mathbb{H}^s_*$
$\mathbb{I}$	an index set
$\mathbb{L}^p$	Lebesgue spaces
$\mathbb{N}$	set of natural numbers without 0
$\mathbb{N}_0$	set of natural numbers including 0
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{S}^{N-1}$	unit sphere in $\mathbb{R}^N$
$\mathbb{Z}$	set of integers
$\mathfrak{H}^N_k$	space of harmonic homogeneous polynomials of degree k in N variables
$\mathfrak{M}^N_K$	see (2.40)
$\mathfrak{M}^3_{\times,1}$	see (4.5)
$\mathfrak{M}^N_*$	see (7.53)
$\mathfrak{S}^N_k$	set of spherical harmonics of degree k on $\mathbb{S}^{N-1}$
$\mathfrak{S}_k$	shorter notation for $\mathfrak{S}^N_k$
$\mathfrak{V}$	volume of the domain
C	varying constant
D_i	differential operators on the unit sphere, see Section 5.3
E	extension operator of functions on the sphere to the ball
H	Hessian of Φ
I	identity matrix
K_r	harmonic function on Ω_r meeting $-\kappa$ on the boundary
K^β_r	harmonic function on Ω_r satisfying $K^\beta_r + \beta\frac{\partial K^\beta_r}{\partial n} = -\kappa_r$ on the boundary
L_r	harmonic function on Ω_r meeting $-\Phi$ on the boundary
L^β_r	harmonic function on Ω_r satisfying $L^\beta_r + \beta\frac{\partial L^\beta_r}{\partial n} = -\Phi$ on the boundary
N	dimension
R	function that parameterises the moving boundary
$R(F)$	range of F
Tr	trace operator
$T_x\mathcal{M}$	tangent space of manifold $\mathcal{M}$ at x
$\vec{V}_{km}$	vector spherical harmonic $\vec{V}_{km}$ (see Chapter 6)

$\vec{V}_k$	$\vec{V}_{k0}$
$\vec{W}_{km}$	vector spherical harmonic $\vec{W}_{km}$ (see Chapter 6)
$\vec{W}_k$	$\vec{W}_{k0}$
X_k	orthoplement of $\langle Y_k \rangle$ in $\mathbb{H}^s_\times(\mathbb{S}^2)$
$X_{k,1}$	orthoplement of $\langle Y_k \rangle$ in $\mathbb{H}^s_{\times,1}(\mathbb{S}^2)$
$\vec{X}_{km}$	vector spherical harmonic $\vec{X}_{km}$ (see Chapter 6)
Y_{km}	spherical harmonics in 3D
Y_k	zonal harmonics $Y_k := Y_{k0}$
Z_r^β	harmonic function on Ω_r satisfying $Z_r^\beta + \beta \frac{\partial Z_r^\beta}{\partial n} = -\frac{\partial \Phi}{\partial n}$ on the boundary
$b(r)$	$u \mapsto \operatorname{div} \left(u \circ z(r)^{-1} \right) \circ z(r)$
e_i	ith unit vector in cartesian coordinates
e_ρ, e_θ, e_ϕ	unit vectors in spherical coordinates
$f_K(r)$	Richardson moments of Ω_r up to order K, see (2.42)
$f_{\times,1}(r)$	see (4.6)
$f_*(r)$	see (7.54)
$h(r)$	$H \circ z(r)$
$h^{k,\beta}$	little Hölder spaces (Spaces with vector/matrix-valued functions are indicated in the same way as the Hölder spaces.)
$h_K^{k,\beta}$	orthoplement in $h^{k,\beta}$ of spherical harmonics s_{kj}^N with $k \leq K$
id	the identity ($\operatorname{id} x = x$)
$j^{i,j}(r)$	components of $\mathcal{J}(r)^{-1}$
$\ker F$	kernel of F
$l(r)$	$\frac{1}{\sigma_N (1+r)^{N-1}} - \frac{1+r}{\sigma_N}$
$m(r)$	see (6.24)
m_k	suction speed for bifurcation solution (taken negative), see Theorem 4.1
n	normal vector field on the boundary (This could be the unit sphere or a perturbation)
n_r	normal vector field on the boundary Γ_r
$n(r)$	$n_r \circ z(r)$
p	pressure
r	function that parameterises the rescaled moving boundary
r_{kj}	$(r, s_{kj})_0$
s	order of the Sobolev space
s_{kj}^N	spherical harmonics in N dimensions
s_{kj}	shorter notation for s_{kj}^N
sgn	sign-function: $\operatorname{sgn} x = 1$ for $x > 0$, $\operatorname{sgn} x = -1$ for $x < 0$, and $\operatorname{sgn} 0 = 0$
$\operatorname{sp}(F)$	spectrum of F
t	time
v	velocity
x	spacial variable
x_i	ith component of x
$z(r)$	$(1+r)\operatorname{id}$
Γ_f	the moving boundary parameterised by f
$\Gamma(t)$	the moving boundary
Δ	Laplacian

Δ_r	Laplace-Beltrami operator on Γ_r
Λ_k	see (6.71)
$\Pi_i f$	the ith component of f
Φ	see (2.10)
Ω_f	the moving domain parameterised by f
$\Omega(t)$	the moving domain
$\alpha(t)$	$\sqrt[N]{\frac{\mu N t}{\sigma_N} + 1}$
$\alpha(\tau)$	value of α at time t corresponding to τ
β	kinetic undercooling coefficient in Chapter 7
γ	surface tension coefficient
δ	small positive number
δ	Dirac delta distribution
δ_{kj}	Kronecker delta, equal to zero if $k \neq j$ and equal to one if $k = j$
ε	small positive number
θ	polar coordinate in polar and spherical coordinate system
κ	curvature of the boundary
κ_r	curvature of the boundary parameterised by r
$\kappa(r)$	$\kappa_r \circ z(r)$
μ	injection or suction speed
ν_k	see Theorem 4.1
$\nu(N,k)$	dimension of $\mathfrak{S}_k^N$
ξ	spacial variable
$\pi(F)$	point spectrum of operator F
ρ	radial coordinate in polar and spherical coordinate system
ρ_k	curve of bifurcation solutions, see Theorem 4.1
$\rho(F)$	resolvent set of operator F
σ_N	area of $\mathbb{S}^{N-1}$
τ	time variable defined in (3.26), (5.7), or (6.28)
ϕ	azimuthal coordinate in polar and spherical coordinate system
$\phi_k(r)$	the pair $(f_k(r), \mathcal{P}_k(r))^T$
$\phi_{\times,1}(r)$	the pair $(f_{\times,1}(r), \mathcal{P}_1 r)^T$
$\phi_*(r)$	the pair $(f_*(r), \mathcal{P}_1 r)^T$
$\varphi(r)$	$\Phi \circ z(r)$
$\psi_k(r)$	$\phi_k^{-1}(0,r)$
$\psi_{\times,1}(r)$	see (4.7)
$\psi_*(r)$	see (7.55)
$\omega(r)$	see (7.8)
$\|\cdot\|_{k,\beta}$	norm in $\mathcal{C}^{k,\beta}(\mathbb{S}^{N-1})$
$\|\cdot\|_{\mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N})}$	norm in $\mathcal{C}^{k,\beta}(\overline{\mathbb{B}^N})$
$\|\cdot\|_k$	norm in $\mathcal{C}^k(\mathbb{S}^{N-1})$ (in Chapter 2 and 3)
$\|\cdot\|_s$	norm in $\mathbb{H}^s(\mathbb{S}^{N-1})$ (in Chapter 4, 5, 6, and 7)
$\|\cdot\|_0$	norm in $\mathbb{L}^2(\mathbb{S}^{N-1})$
$\|\cdot\|_{s-1,1}$	see (6.86)
$\|\cdot\|_{s-2,2}$	see (5.14)
$\|\cdot\|_{\mathbb{H}^s(\mathbb{B}^N)}$	norm in $\mathbb{H}^s(\mathbb{B}^N)$

$(\cdot,\cdot)_s$	inner product in $\mathbb{H}^s(\mathbb{S}^{N-1})$
$(\cdot,\cdot)_0$	inner product in $\mathbb{L}^2(\mathbb{S}^{N-1})$
$(\cdot,\cdot)_{s-1,1}$	see (6.86)
$(\cdot,\cdot)_{s-2,2}$	see (5.14)
$(\cdot,\cdot)_{\mathbb{H}^s(\mathbb{B}^N)}$	inner product in $\mathbb{H}^s(\mathbb{B}^N)$
$X \hookrightarrow Y$	X is continuously embedded in Y.
$X \hookrightarrow\hookrightarrow Y$	X is compactly embedded in Y.
$f\mid_X$	restriction of f to X
$u \star v$	see (6.21)

Index

Summary

Qualitative aspects of mathematical models for the dynamics of liquids with a moving boundary are studied. These models describe for instance groundwater flow, extraction of oil, the growth of tumours and viscous sintering in the production of glass.

Stability of radially symmetric solutions and decay properties of perturbations are studied for the case that in a single point fluid is injected or extracted. For the motion of the moving boundary a nonlinear non-local evolution equation is derived. The domain is rescaled in such a way that the spherical solution is represented by a stationary solution. Because of this rescaling, the evolution operator is time dependent. The nonlinear stability results are based on linearisation, energy estimates and the principle of linearised stability.

The Hele-Shaw model is studied for several boundary conditions, describing various physical situations. In the case of zero pressure on the boundary, it is proved for the injection problem that balls around the injection point are asymptotically stable with respect to small star-shaped perturbations. If surface tension regularisation is included, then balls are stable even for the case of suction under additional assumptions on the initial geometry, suction speed and dimension. Moreover, perturbations turn out to decay algebraically fast.

For two dimensional suction, the influence of surface tension dominates the influence of the sink for large time. As a consequence, no condition on the suction speed is necessary. In contrast to the two dimensional problem there is a bound on the suction speed for the 3D problem. In dimensions higher or equal to four the influence of the sink dominates the influence of surface tension. This leads to linear instability for the spherical solution for any suction speed.

Making use of the autonomous character of the evolution equation, existence of non-trivial self-similarly vanishing solutions to the three dimensional suction problem with surface tension is proved. These solutions are found as bifurcation solutions from the trivial spherical solution. The suction speed plays the role of bifurcation parameter. Moreover, one branch of bifurcation solutions turns out to be stable with respect to a certain class of perturbations.

For the closely related Stokes flow stability of the spherical solution in the case of injection has been proved for dimensions two and three. For the suction problem for these dimensions the spherical solution is linearly unstable.

Samenvatting

Kwalitatieve aspecten van wiskundige modellen voor de dynamica van vloeistoffen met een bewegende rand worden bestudeerd. Deze modellen worden gebruikt om bijvoorbeeld de stroming van grondwater, de extractie van olie, de groei van tumoren en het sinteren van glas te beschrijven.

Voor het geval dat in één punt vloeistof wordt geïnjecteerd of geëxtraheerd, wordt stabiliteit van oplossingen met radiale symmetrie onderzocht en de snelheid waarmee verstoringen uitdoven wordt bepaald. Voor de evolutie van de bewegende rand wordt een niet-lineaire niet-lokale evolutievergelijking afgeleid. Het bewegende gebied wordt herschaald op een dusdanige wijze dat de bolvormige oplossing een stationaire oplossing is. De evolutie-operator is tijdsafhankelijk vanwege deze herschaling. De niet-lineaire stabiliteitsresultaten zijn gebaseerd op enerzijds linearisering en anderzijds energie-afschattingen of het principe van gelineariseerde stabiliteit.

Het Hele-Shaw model wordt bestudeerd in combinatie met verschillende randvoorwaarden, die veschillende natuurkundige en biologische situaties beschrijven. Voor het geval met injectie waarin de druk op de rand gelijk aan nul wordt verondersteld, wordt bewezen dat bollen, waarvan het middelpunt overeenkomt met de bron, asymptotisch stabiel zijn met betrekking tot kleine stervormige verstoringen. Wanneer oppervlaktespanning aanwezig is, zijn bollen ook stabiel in het geval van suctie onder bepaalde voorwaarden aangaande de aanvankelijke geometrie, suctie-snelheid en de dimensie waarin we het probleem beschouwen. Het blijkt dat verstoringen algebraïsch snel uitdoven.

In het tweedimensionale probleem met suctie domineert de invloed van de oppervlaktespanning de invloed van de suctie op den duur. Hierdoor hoeft geen restrictie opgelegd te worden op de suctie-snelheid. In tegenstelling tot het tweedimensionale geval is er wel een bovengrens voor deze suctie-snelheid in het driedimensionale geval. In dimensies vier of hoger is de invloed van de suctie dominant. Dit leidt tot lineaire instabiliteit voor de bolvormige oplossing voor elke suctie-snelheid.

Door gebruik te maken van het autonome karakter van de evolutievergelijking wordt existentie van niet-triviale gelijkvormig verdwijnende oplossingen bewezen voor het driedimensionale probleem met suctie en oppervlaktespanning. Deze oplossingen worden gevonden als bifurcatie-oplossingen van de triviale bolvormige oplossing. De suctiesnelheid speelt de rol van bifurcatieparameter. Verder blijkt een tak van bifurcatie-oplossingen stabiel te zijn met betrekking tot een bepaalde klasse van verstoringen.

Voor de sterk gerelateerde Stokes flow wordt stabiliteit van de bolvormige oplossing bewezen voor het geval van injectie in dimensies twee en drie. Voor het suctieprobleem is de bolvormige oplossing lineair onstabiel.

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Curriculum Vitae

Erwin Vondenhoff werd op 29 april 1981 geboren in Heerlen. In 1999 behaalde hij zijn VWO-diploma aan het St.-Janscollege te Hoensbroek. In hetzelfde jaar begon hij aan de studie Technische Wiskunde (Industrial and Applied Mathematics) aan de Technische Universiteit Eindhoven. In april 2005 behaalde hij zijn ingenieurs titel (ir.) na het schrijven van de afstudeerscriptie met de titel "Curves in Gaussian and Poisson spaces" onder begeleiding van prof. dr. ir. Jan de Graaf. Tijdens de studie liep hij een stage bij DSM te Geleen met als onderwerp endoscopische frictie. Sinds mei 2005 werkt hij onder begeleiding van dr. Georg Prokert als promovendus aan de Technische Universiteit Eindhoven aan een project dat gesponsord wordt door NWO (Nederlandse Organisatie voor Wetenschappelijk Onderzoek). Tevens heeft hij als promovendus onderwijs gegeven aan studenten van verschillende faculteiten van de Technische Universiteit Eindhoven. Verder heeft hij deelgenomen aan verschillende studiegroepen (wiskunde voor de industrie), die plaatsvonden aan universiteiten in Nederland (Eindhoven, Utrecht en Twente) en Denemarken (Odense en Lyngby).