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On Polytopic Approximations of Systems with Time-Varying Input Delays

Rob Gielen, Sorin Oлару, and Mircea Lazar

Abstract. Networked control systems (NCS) have recently received an increasing attention from the control systems community. One of the major problems in NCS is how to model the highly nonlinear terms caused by uncertain delays such as time-varying input delays. A straightforward solution is to employ polytopic approximations. In this paper we develop a novel method for creating discrete-time models for systems with time-varying input delays based on polytopic approximations. The proposed method is compared to several other existing approaches in terms of quality, complexity and scalability. Furthermore, its suitability for model predictive control is demonstrated.

Keywords: input delay, networked control systems, polytopic uncertainty.

1 Introduction

Recently networked control systems (NCS) have become one of the topics in control that receives a continuously increasing attention. This is due to the important role that transmission and propagation delay play in nowadays modern control applications. In [8], a survey on future directions in control, NCS were even indicated to be one of the emerging key topics in control. In NCS the connection between plant and controller is a network that is in general shared with other applications. The motivations for using NCS are mostly cost and efficiency related. In [6, 10] a comprehensive overview of the main difficulties within NCS and the recent developments in this field is given. Both papers present different setups and solutions

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for stabilizing controller design. In general two main issues can be distinguished: time-varying input delay and data packet dropout. The present paper focusses on the first issue. A general study on stability of NCS can be found in [1] and the references therein. More recently, in [3, 5] the problem of time-varying input delay was reformulated as a robust control problem and a static feedback controller was then synthesized by means of linear matrix inequalities (LMI). In [9], also by means of LMI, a model predictive control (MPC) scheme has been designed for systems with time-varying input delay.

One of the biggest challenges in stabilization and predictive control of NCS is to find a modeling framework that can handle time-varying input delays effectively. One of the most popular solutions to this problem, already employed in [3, 5, 9], is to model the delay-induced nonlinearity using a polytopic approximation. The advantage of this approach is that the resulting model is a linear parameter varying system for which efficient stabilization methods and control design techniques exist, see, for example, [7]. This paper proposes a new approach for deriving a polytopic approximation, based on the Cayley-Hamilton theorem. The method is compared with the above-mentioned techniques in terms of scalability, complexity and conservativeness. The suitability of all methods for predictive control is analyzed using the MPC strategy of [7].

2 Preliminaries

2.1 Basic Notation and Definitions

Let \mathbb{R} , \mathbb{R}_+ , \mathbb{Z} and \mathbb{Z}_+ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1, c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ | k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ | c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. A polyhedron, or a polyhedral set, in \mathbb{R}^n is a set obtained as the intersection of a finite number of open and/or closed half-spaces, a polytope is a compact polyhedron. Let $Co(\cdot)$ denote the convex hull. Let $\|A\|_2 := \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$ denote the induced matrix 2-norm. A well-known property is that $\|A\|_2^2 = \lambda_{\max}(A^T A)$, where $\lambda_{\max}(M)$ is the largest eigenvalue of $M \in \mathbb{R}^{n \times n}$.

2.2 Problem Definition

Consider the continuous time system with input delay

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t) \\ u(t) &= u_k, \quad \forall t \in [t_k + \tau_k, t_{k+1} + \tau_{k+1}] \quad \text{and} \quad u(t) = u_{\text{initial}}, \quad \forall t \in [0, \tau_0], \end{aligned} \quad (1)$$

where $t_k = kT_s$, $k \in \mathbb{Z}_+$ and $T_s \in \mathbb{R}_+$ is the sampling time. Furthermore $A_c \in \mathbb{R}^{n \times n}$, $B_c \in \mathbb{R}^{n \times m}$, $\tau_k \in \mathbb{R}_{[0, T_s]}$, $\forall k \in \mathbb{Z}_+$ is the network delay, $u_k \in \mathbb{R}^m$, $k \in \mathbb{Z}_+$ is the control action generated at $t = t_k$, $u(t) \in \mathbb{R}^m$ is the system input and $x(t) \in \mathbb{R}^n$ is the system state. The time-varying delay that affects the input signal is one of the

most important aspects of NCS. As in NCS the controller only has discrete time information, we employ next several algebraic manipulations to obtain a discrete time description of the system, i.e.

$$x_{k+1} = e^{A_c T_s} x_k + \int_0^{\tau_k} e^{A_c(T_s-\theta)} d\theta B_c u_{k-1} + \int_{\tau_k}^{T_s} e^{A_c(T_s-\theta)} d\theta B_c u_k. \quad (2)$$

The goal is to design a stabilizing controller that is robust in the presence of time-varying delays. However this is a highly nonlinear system and is in general not suitable for controller synthesis. To obtain a model more suitable for control design define

$$\Delta_k := \int_0^{\tau_k} e^{A_c(T_s-\theta)} d\theta B_c, \quad k \in \mathbb{Z}_+. \quad (3)$$

Furthermore, by manipulating (2) and introducing a new augmented state vector of the form $\xi_k^T = [x_k^T \ u_{k-1}^T]$ we obtain:

$$\xi_{k+1} = A(\Delta_k) \xi_k + B(\Delta_k) u_k, \quad (4)$$

with $A(\Delta_k) := \begin{bmatrix} A_d & \Delta_k \\ 0 & 0 \end{bmatrix}$, $B(\Delta_k) := \begin{bmatrix} B_d - \Delta_k \\ I_m \end{bmatrix}$, $B_d = \int_0^{T_s} e^{A_c(T_s-\theta)} d\theta B_c$ and $A_d = e^{A_c T_s}$. Here (4) is a *nonlinear parameter varying* system with unknown parameter τ_k . The challenge that remains is to find a polytopic approximation of this nonlinear uncertainty in order to reformulate (4) into a *linear parameter varying* system with unknown parameter Δ_k . To achieve this we define the following set of matrices:

$$\Delta := \text{Co}(\{\bar{\Delta}_l\}), \quad \bar{\Delta}_l \in \mathbb{R}^{n \times m}, \quad l \in \mathbb{Z}_{[0,L]}, \quad L \in \mathbb{Z}_+, \quad (5)$$

such that $\Delta_k \in \Delta, \forall \tau_k \in [0, \bar{\tau}]$, where $\bar{\tau}$ is the maximum input delay that can be introduced by the network. This is a model that can be handled by most robust control techniques, including MPC, as it will be shown later.

2.3 Existing Solutions

In [3, 5, 9] several methods for finding the generators of the set in (5) were derived. Here these methods are only explained briefly to obtain a self-contained assessment; further details and proofs can be found in the corresponding articles.

In [3] and the references therein an elementwise maximization is proposed where $\bar{\Delta}_l$ contain all possible combinations of maxima and minima for all entries of Δ_k . This approach will be referred to as the ME method.

Other methods, as the ones in [3] and [9], are based on the Jordan normal form (JNF), i.e. $A_c = VJV^{-1}$ with J block diagonal. Starting from (3), with a mild assumption on A_c and using the JNF yields:

$$\Delta_k = \sum_{i=1}^n A_c^{-1} V (e^{J_i T_s} - e^{J_i(T_s-\tau_k)}) V^{-1} B_c. \quad (6)$$

Filling in $\tau_k = 0$ and $\tau_k = \bar{\tau}$ gives the generators of the set Δ . The two papers differ in so far that in [9] a method is proposed to reduce the number of generators at the cost of a larger polytope. The method as presented in [3] will be referred to as JNF1 and the method from [9] as JNF2.

Another option was proposed in [5], which makes use of a Taylor series expansion of (3), i.e.:

$$\Delta_k = \left(- \sum_{i=1}^{\infty} \frac{(-\tau_k)^i}{i!} A_c^{i-1} e^{A_c T_s} \right) B_c. \quad (7)$$

The generators of Δ are also obtained for $\tau_k = 0$ and $\tau_k = \bar{\tau}$. The infinite sum is approximated by a finite number of terms p , which is also the number of generators for Δ , i.e. $L = p$. This method will be referred to as TA. Next we present a novel method for finding the generators of the set Δ .

3 Main Result

The method presented in this paper is based upon the Cayley-Hamilton theorem.

Theorem 1 (Cayley-Hamilton theorem). *If $p(\lambda) := \det(\lambda I_n - A)$ is the characteristic polynomial of a matrix $A \in \mathbb{R}^{n \times n}$ then $p(A) = 0$.*

The original proof of this theorem can be found in [2] and further details on the theorem are given in [4]. Using this theorem it is possible to express all powers of A of order n and higher as a combination of the first n powers, i.e.

$$A^i = c_{i,0}I + \dots + c_{i,n-1}A^{n-1}, \quad \forall i \in \mathbb{Z}_{\geq n}, \quad (8)$$

for some $c_{i,j} \in \mathbb{R}$, $j = 0, \dots, n-1$. Define now the functions

$$f_j(T_s - \theta) := \sum_{i=0}^{\infty} a_{i,j}(T_s - \theta)^i, \quad (9)$$

where $a_{i,j} := \frac{c_{i,j}}{i!}$. By Theorem 1 we can derive the following expression for Δ_k .

Lemma 1. *Let*

$$g_j(\tau_k) := \int_0^{\tau_k} f_j(T_s - \theta) d\theta, \quad (10)$$

for some $c_{i,j} \in \mathbb{R}$ and $f_j(T_s - \theta)$ as in (8) and (9). Then

$$\Delta_k = \sum_{j=0}^{n-1} g_j(\tau_k) A_c^j B_c. \quad (11)$$

Proof: Starting from (3) and using (8) we obtain:

$$\begin{aligned} \Delta_k &= \int_0^{\tau_k} \sum_{k=0}^{\infty} \frac{(T_s - \theta)^k}{k!} A_c^k d\theta B_c \\ &= \int_0^{\tau_k} \left(I_n + A_c(T_s - \theta) + \dots + A_c^n \frac{(T_s - \theta)^n}{n!} + \dots \right) d\theta B_c \\ &= \int_0^{\tau_k} \left(I_n + A_c(T_s - \theta) + \dots \right. \\ &\quad \left. \dots + (c_{n,0}I_n + \dots + c_{n,n-1}A_c^{n-1}) \frac{(T_s - \theta)^n}{n!} + \dots \right) d\theta B_c. \end{aligned} \tag{12}$$

Gathering all terms before the same matrices, writing them as a function of $T_s - \theta$ and using (9) yields:

$$\Delta_k = \int_0^{\tau_k} (f_0(T_s - \theta)I_n + \dots + f_{n-1}(T_s - \theta)A_c^{n-1}) d\theta B_c, \tag{13}$$

which concludes the proof. \square

Filling in the corresponding values for τ_k in $g_j(\tau_k)$ gives $g_{j,l} \in \mathbb{R}$ and $g_{j,u} \in \mathbb{R}$ such that $g_{j,l} \leq g_j(\tau_k) \leq g_{j,u}, \forall \tau_k \in [0, \bar{\tau}]$. By Lemma 1 it is possible to write all realizations of Δ_k as a convex combination of a finite number of matrices $\bar{\Delta}_l$, as stated in the next theorem.

Theorem 2. For any $\tau_k \in [0, \bar{\tau}]$, Δ_k satisfies:

$$\Delta_k \in Co(n\Delta_0, \dots, n\Delta_{2n-1}), \tag{14}$$

where

$$\Delta_j := g_{j,l}A_c^j B_c, \quad \Delta_{j+n} := g_{j,u}A_c^j B_c, \quad \forall j = 0, \dots, n-1. \tag{15}$$

Proof: Starting from Lemma 1, for any $\tau_k \in [0, \bar{\tau}]$ and $g_j(\tau_k)$ there exists a $v_j \in \mathbb{R}_{[0,1]}$ and $\mu_j = 1 - v_j$ such that:

$$\begin{aligned} \Delta_k &= (g_0(\tau_k)I_n + g_1(\tau_k)A_c + \dots + g_{n-1}(\tau_k)A_c^{n-1}) B_c \\ &= ((v_0g_{0,l} + \mu_0g_{0,u})I_n + \dots + (v_{n-1}g_{n-1,l} + \mu_{n-1}g_{n-1,u})A_c^{n-1}) B_c, \\ &= \left(\sum_{j=0}^{n-1} \left(\frac{v_j}{n}ng_{j,l} + \frac{\mu_j}{n}ng_{j,u} \right) A_c^j \right) B_c, \\ &= (\delta_0ng_{1,l}I + \delta_nng_{1,u}I + \dots + \delta_{n-1}ng_{n-1,l}A_c^{n-1} + \delta_{2n-1}ng_{n-1,u}A_c^{n-1}) B_c. \end{aligned} \tag{16}$$

As $\delta_i = \frac{v_j}{n}$, $\delta_{i+n} = \frac{\mu_j}{n}$ and hence, $\sum_{i=0}^{2n-1} \delta_i = 1$, concludes the proof. \square

Thus we have now found again the generators for the convex set as defined in (5). Throughout the remainder of the paper we will refer this approach as CH2, with the observation that the resulting polytope is spanned by $2n$ generators.

Remark 1. In Section 2.3 it was pointed out that both [3, 9] propose methods based upon the Jordan Normal Form with the difference that the method of [9] reduces the number of generators at the cost of a larger polytope, e.g. a square can always be contained in a triangle thus reducing the number of points spanning the polytope. A similar reasoning can also be applied to the method CH2 presented above, to obtain a polytope smaller than the one obtained via Theorem 2, but now with 2^n generators instead of $2n$. The method corresponding to this modification of CH2 will be referred to as CH1, to be consistent with the method JNF2 versus JNF1. \square

Observe that (9) is of infinite length and will in practice be approximated by a function of finite length p . The resulting polytopic embedding therefore has an error. Next, we provide an explicit upper bound on the 2-norm of the approximation error.

Theorem 3. *Let*

$$\rho := \frac{3\sqrt{\lambda_{\max}(A_c^T A_c)} T_s}{p}, \quad (17)$$

and suppose¹ $\rho < 1$. Then:

$$\left\| \int_0^{\tau_k} \sum_{k=p}^{\infty} \frac{A_c^k (T_s - \theta)^k}{k!} B_c d\theta \right\|_2 \leq \frac{\rho^p}{1-\rho} \bar{\tau} \sqrt{\lambda_{\max}(B_c^T B_c)}. \quad (18)$$

Proof:

$$\begin{aligned} & \left\| \int_0^{\tau_k} \sum_{k=p}^{\infty} \frac{A_c^k (T_s - \theta)^k}{k!} B_c d\theta \right\|_2 \leq \sum_{k=p}^{\infty} \left\| \int_0^{\tau_k} \frac{A_c^k (T_s - \theta)^k}{k!} B_c d\theta \right\|_2 \\ & \leq \sum_{k=p}^{\infty} \left\| \frac{A_c^k T_s^k}{\left(\frac{k}{3}\right)^k} B_c \bar{\tau} \right\|_2 \leq \sum_{k=p}^{\infty} \left(\frac{3T_s}{k}\right)^k \bar{\tau} \|A_c^k\|_2 \|B_c\|_2 \\ & \leq \sum_{k=p}^{\infty} \left(\frac{3\sqrt{\lambda_{\max}(A_c^T A_c)} T_s}{p}\right)^k \bar{\tau} \sqrt{\lambda_{\max}(B_c^T B_c)} = \frac{\rho^p}{1-\rho} \bar{\tau} \sqrt{\lambda_{\max}(B_c^T B_c)}, \quad (19) \end{aligned}$$

where the triangle and the Cauchy-Schwarz inequality were used. The inequality $\|A^k\|_2^2 \leq \|A\|_2^2 \times \dots \times \|A\|_2^2 = \lambda_{\max}^k(A^T A)$, which follows from the Cauchy-Schwarz inequality, was also employed. \square

Using Theorem 3 one can choose p such that the approximation error is small enough and then correct the resulting polytope accordingly. This can be done by performing a Minkowsky addition of the resulting polytope with the unit ball proportional to the size of the error bound.

¹ Note that p and T_s can always be chosen such that this requirement is satisfied.

4 Suitability for MPC

In this section we present an assessment of all modeling methods considered in this paper with focus on suitability for MPC. To do so, consider system (1)-(2) with $A_c = \begin{bmatrix} 1 & -1.2 \\ 4 & 6 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $T_s = 0.05$ and $\bar{\tau} = 0.045$. For CH1 and CH2 we chose $p = 15$ and thus, (18) yields $\frac{p^p}{1-p} \bar{\tau} \sqrt{\lambda_{\max}(B_c^T B_c)} \approx 4 \times 10^{-19}$. The approximation order needed by TA was $p = 8$. Each method has its polytope, as defined in (5), and generators spanning the polytope. In Figure 1 these polytopes are plotted. Notice that the accuracy of the methods ME, CH1, CH2 and TA is of the same order of magnitude, whereas for JNF1 and JNF2 the polytope is much larger (different axes).

We will now discuss the methods in terms of scalability, computational aspects and control performance. Firstly, note that the LMI used in [7] for stabilizing controller synthesis scales linearly with the number of generators of Δ . In Table 1 the number of generators for each method is shown.

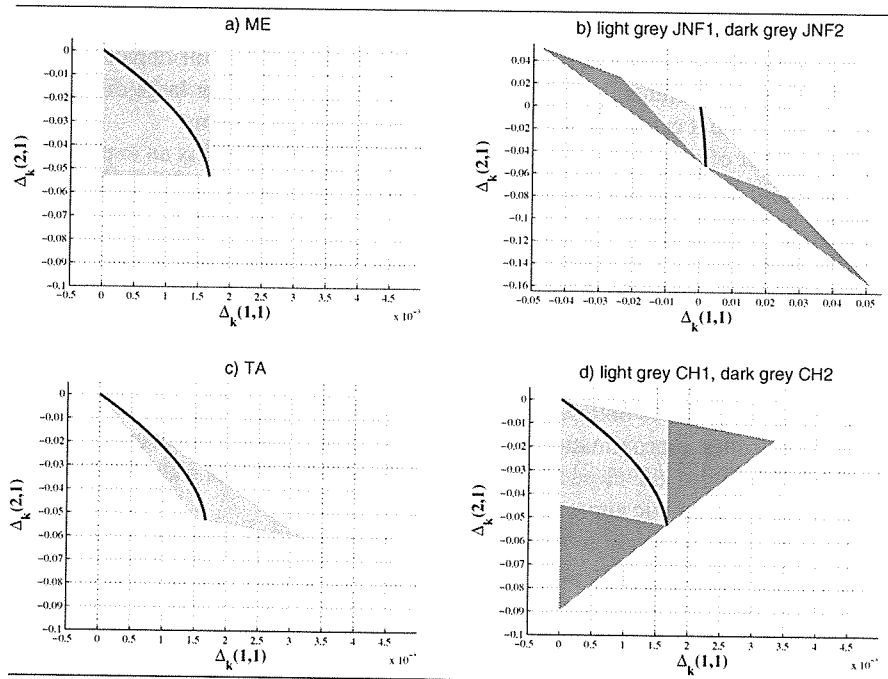


Fig. 1 Different polytopic approximations: along the axes are the values of $\bar{\Delta}_l(1,1)$ and $\bar{\Delta}_l(2,1)$ for $l = 1, 2$, in black all the possible realizations of Δ_k and the grey areas are the polytopes

Table 1 The number of generators per method (L)

method:	ME	JNF1	JNF2	TA	CH1	CH2
number of generators:	2^{nm}	2^n	$n + 1$	p	2^n	$2n$

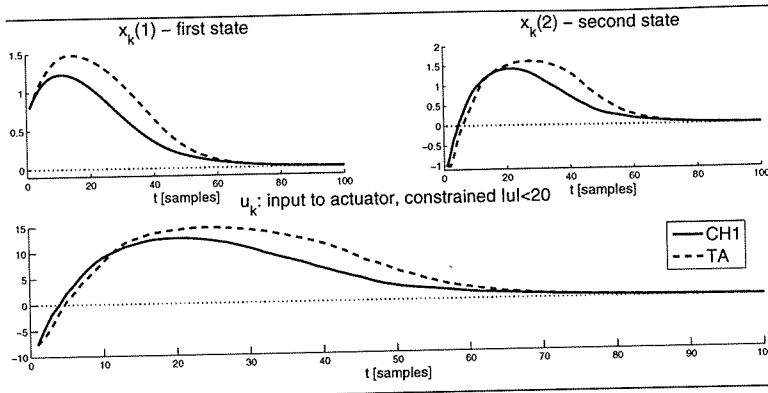


Fig. 2 Simulation of the same MPC scheme for two different models

A few further observations about the various methods are worth noticing:

- The number of generators yielded by the TA method does not depend on the state dimension. Hence, the TA approach seems well suited for large dimension systems, while it can be less efficient for low dimension systems.
- The number of generators for the methods ME, JNF1 and CH1 is an exponential function of the state dimension, which makes these approaches not suitable for large dimension systems.
- The ME method is not implementable because the extreme realizations of Δ_k are not necessarily obtained for $\tau_k = 0$ and $\tau_k = \bar{\tau}$.
- Both JNF methods have the disadvantage that they become complex when the JNF becomes complex, e.g. when A_c has complex values on the diagonal of J . Also, the JNF methods become more complex when A_c is not invertible.
- The TA method does not provide an upper bound on the estimation error due to the finite order approximation of the Taylor series. This means that one has to check stability of the closed-loop system a posteriori. If this stability test fails there is no systematic approach for finding a solution.
- CH1 and CH2 use an algorithm which calculates the determinant of a possibly large matrix and the roots of a high order polynomial.
- For CH1 and CH2, if p is chosen small this increases the number of generators, while if p is chosen very large a correction of the polytope becomes superfluous, but the influence on the computational complexity is insignificant.

Finally we can, by means of the MPC law from [7], compare the performance of the different approaches to see how they perform in the MPC context. At each time instant t_k a feedback gain $K(x_k)$ is calculated by solving a semi-definite programming problem, which yields the control action $u_k := K(x_k)x_k$. JNF2 and CH2 were not considered because they will never outperform their corresponding variants JNF1 and CH1, respectively. ME was not considered because this method is not really applicable due to numerical reasons mentioned above. In Figure 2 we plot the results of a simulation for the system under observation. The resulting closed

loop state and input trajectories of the simulations corresponding to CH1 and TA are plotted. Note that in the simulation CH1 uses less control effort and has less overshoot, even though the two methods achieve the same settling time. In the simulation corresponding to JNF1 the resulting MPC problem was not feasible. This indicates that overestimating the nonlinearity can lead to infeasibility.

5 Conclusions

A novel method for modeling uncertain time-varying input delays was presented. It has been shown that this method indeed creates a polytope that contains all possible realizations of the nonlinear terms induced by delays. Then it was shown how to upper bound the error made in the approximation of an infinite length polynomial and how to compensate for this error. It has been demonstrated that the approach presented in this paper can be more efficient compared with earlier presented methods, also in terms of suitability for MPC. Furthermore, the presented modeling method can be modified to allow for delays larger than the sampling time using techniques similar to the ones employed in e.g., [3, 9], which makes the developed method appealing for control of networked control systems.

References

1. Branicky, M.S., Phillips, S.M., Zhang, W.: Stability of Networked Control Systems: Explicit Analysis of Delay. In: Proceedings of the American Control Conference, vol. 4, pp. 2352–2357 (2000)
2. Cayley, A.: A memoir on the theory of matrices. *Philosophical Transactions London* 158, 17–37 (1857)
3. Cloosterman, M.B.G.: Control of Systems over Communication Networks: Modelling, Analysis and Design. Ph.D. dissertation, Eindhoven University of Technology, The Netherlands (2008)
4. Gantmacher, F.R.: *Theory of Matrices*, vols. 1, 2. Chelsea Publishing Company, New York (1959)
5. Hetel, L., Daafouz, J., Jung, C.: Stabilization of arbitrary switched linear systems with unknown time-varying delays. *IEEE Transactions on Automatic Control* 51(10), 1668–1674 (2006)
6. Hespanha, J., Naghshtabrizi, P., Xu, Y.: A survey of Recent Results in Networked Control Systems. *Proceedings of the IEEE* 95, 138–161 (2007)
7. Kothare, M.V., Balakrishnan, V., Morari, M.: Robust Constrained Model Predictive Control using Linear Matrix Inequalities. *Automatica* 32(10), 1361–1379 (1996)
8. Murray, R.M., Åström, K.J., Boyd, S.P., Brockett, R.W., Stein, G.: Future Directions in Control in an Information-Rich World. *IEEE Control Systems Magazine* 23(2), 20–33 (2003)
9. Olaru, S., Niculescu, S.I.: Predictive control for linear systems with delayed input subject to constraints. *IFAC World Congress*, Seoul, Korea (2008)
10. Tipsuwan, Y., Chow, M.-Y.: Control methodologies in networked control systems. *Control Engineering Practice* 11, 1099–1111 (2003)