Generalized stochastic processes

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A. Abstract

In this report we develop a theory of smooth stochastic processes, as well as their generalization to generalized stochastic processes. Our point of departure is De Bruijn's theory of generalized functions and the Wigner distribution. We apply that framework to definitions and theorems concerning smooth and generalized stochastic processes, and we present a theory of linear transforms in the space of these processes. Furthermore we introduce the notions of autocorrelation function and Wigner distribution of stochastic processes (smooth or generalized).

The theory presented in this report serves mainly as a preparation for a study of the phenomenon of noise. We devote a section (section 1.6) to the relation between generalized stochastic processes and noise, and we announce a few results of the theory of noise. Furthermore we shall briefly comment on alternative approaches in existing literature.

The author intends to devote a later publication to a more elaborate study on noise theory. This will not only discuss white, time stationary or frequency stationary noise, but also non-stationary noise. In particular this will contain a discussion on the simulation of noise by showers of noise quanta over the time-frequency plane.

The present report further contains two appendices. The first one gives a number of theorems concerning linear operators of the spaces of smooth and generalized functions, preceded by a survey of the fundamental notions and theorems of De Bruijn's theory that we use in this paper. The second appendix gives a theorem about convergence in the space of smooth functions.

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B. Notation.

We use Church's lambda calculus notation, but instead of his λ we have Ψ, as suggested by Freudenthal: if S is a set then putting $\Psi_{x\in S}$ in front of an expression (usually containing x) means to indicate the function with domain S and with the function values given by the expression. We write Ψ_x instead of $\Psi_{x\in S}$, if it is clear from the context which set S is meant.

In this paper the symbol $\mathbb{R}$ is used for the set of all real numbers, and we use the symbol $\mathbb{C}$ for the set of all complex numbers. If $z \in \mathbb{C}$ then Re z (Im z) denotes the real (imaginary) part of z. The overhead bar is used for complex conjugates. We shall write $\mathbb{N}$ ($\mathbb{N}_0$) for the set of all positive (non-negative) integers.

If $A_1,\ldots,A_n$ ($n \in \mathbb{N}$) are sets, then we denote by $A_1 \times\ldots\times A_n$ the set of all n-tuples $(a_1,\ldots,a_n)$ where $a_1 \in A_1,\ldots,a_n \in A_n$. In the case $V = \mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$ or $\mathbb{N}_0$ we write V^n instead of $V\times\ldots\times V$ (n times).

If V is a set and f and g are mappings of V into $\mathbb{C}$, then we write $f(v) = O(g(v))$ ($v \in V$) if there exists an $M > 0$ such that $\forall_{v\in V}[|f(v)| \le M|g(v)|]$.

If (Ω,Λ,P) is a σ-finite measure space (i.e. Ω is a set, Λ is a σ-algebra on Ω containing Ω itself, and P is a σ-finite positive measure on (Ω,Λ)), then $\mathcal{L}(\Omega)$ denotes the set of all mappings of Ω into $\mathbb{C}$ which are measurable (in $\mathbb{C}$ we have the σ-algebra of all Borel sets). In $\mathcal{L}(\Omega)$ we have an equivalence: $f = g$ if $f(\omega) = g(\omega)$ a.e. ($f \in \mathcal{L}(\Omega)$, $g \in \mathcal{L}(\Omega)$). If $1 \le p \le \infty$, then $\mathcal{L}_p(\Omega)$ denotes the set of all elements f of $\mathcal{L}(\Omega)$ for which

$$\int_\Omega |f|^p \, dP < \infty \qquad (1 \le p < \infty)$$

$$\operatorname{esssup}|f| := \sup\{a \in \mathbb{R} \mid |f(\omega)| \le a \text{(a.e.)}\} < \infty \qquad (p = \infty).$$

In $\mathcal{L}_p(\Omega)$ we have the p-norm $\|\ \|_{p,\Omega}$ (or $\|\ \|_p$ if it is clear which set Ω is meant) defined by

$$\|f\|_{p,\Omega} := \left(\int_\Omega |f|^p \, dP\right)^{\frac{1}{p}} \qquad (f \in \mathcal{L}_p(\Omega),\ 1 \le p < \infty)$$

$$\|f\|_{\infty,\Omega} := \operatorname{esssup}|f| \qquad (f \in \mathcal{L}_\infty(\Omega)) .$$

It is known that $(\mathcal{L}_p(\Omega), \|\ \|_p)$ is a Banach space.

If $1 \leq p \leq \infty$, then we define $q = 1$ if $p = \infty$, $q = \infty$ if $p = 1$, and $q = (1 - \frac{1}{p})^{-1}$ if $1 < p < \infty$ (note that always $1 \leq q \leq \infty$).
If $1 \leq p \leq \infty$ then we denote

$$(f,g)_\Omega = \int_\Omega f\bar{g}\, dP \qquad (f \in \mathcal{L}_p(\Omega),\ g \in \mathcal{L}_q(\Omega))$$

(or (f,g) if it is clear which set Ω is meant).
Note that for $1 \leq p \leq \infty$, $f \in \mathcal{L}_p(\Omega)$, $g \in \mathcal{L}_q(\Omega)$ (Hölder's inequality)

$$|(f,g)| \leq \|f\|_p \|g\|_q .$$

If $p = 2$, then $(\ ,\ )$ is an inner product, so $\mathcal{L}_2(\Omega)$ is a Hilbert space.

If (Ω,Λ,P) is a probability space (i.e. a measure space with $P(\Omega) = 1$) we sometimes write for $f \in \mathcal{L}_1(\Omega)$

$$E(f) := \int_\Omega f\, dP .$$

Note that in case of a probability space $\mathcal{L}_p(\Omega) \supset \mathcal{L}_r(\Omega)$ if $1 \leq p \leq r \leq \infty$.

In the case $\Omega = \mathbb{R}^n$ $(n \in \mathbb{N})$, Λ is the class of Borel sets of $\mathbb{R}^n$, and P is the Lebesgue measure we usually take $p = 2$ (unless otherwise stated), and often we write $[\ ,\ ]_n$ (or $[\ ,\ ]$) in situations where $(\ ,\ )$ occurs already with a different meaning.

0. Introduction.

0.1. In the theory of stochastic processes one usually (see [D], [P]) works with (a form of) the following

Definition. Let X and T be two non-empty sets, and let $\mathcal{B}$ be a σ-algebra on X. Let (Ω,Λ,P) be a probability space, and $\underline{x}$ a mapping of $T\times\Omega$ into X such that for every $t \in T$ and every $A \in \mathcal{B}$

$$\{\omega \in \Omega \mid \underline{x}(t,\omega) \in A\} \in \Lambda .$$

Then the seven-tuple $(X,\mathcal{B},T,\Omega,\Lambda,P,\underline{x})$ is called a stochastic process.

0.2. The stochastic processes of 0.1 are related to, but not equivalent to, distributions of time series as introduced by Wiener [W]. We can explain this as follows. Let (Ω,Λ,P) be a probability space, and assume that to every $\omega \in \Omega$ there is given a measurable mapping ξ_ω of $\mathbb{R}$ into $\mathbb{C}$. We then consider the tuple (Ω,Λ,P,ξ). The functions ξ_ω $(\omega \in \Omega)$ are Wiener's time series. The related case where Ω is the set of all generalized functions and $\xi_F = F$ for every generalized function F will be considered in 1.6, where the tuple is called a noise.

0.3. In this paper we shall mainly work with a definition of type 0.1. We take $X = T = \mathbb{C}$, and for $\mathcal{B}$ we take the set of all Borel sets of $\mathbb{C}$. If the probability space (Ω,Λ,P) is specified, then we just denote the stochastic process by $\underline{x}$ (note that $\Upsilon_{\omega\in\Omega}\, \underline{x}(t,\omega) \in \mathcal{L}(\Omega)$ for every $t \in \mathbb{C}$).

1. Smooth and generalized stochastic processes.

1.1. Smooth processes.

1.1.1. Let (Ω,Λ,P) be a probability space, and let $1 \le p \le \infty$.

1.1.2. Definition. Let $\underline{x}$ be a stochastic process in the sense of 0.3. If

$$1.\ \underline{x}(t) := \forall_{\omega\in\Omega}\ \underline{x}(t,\omega) \in \mathcal{L}_p(\Omega) \qquad (t \in \mathbb{C})$$

$$2.\ (\underline{x},f) := \forall_{t\in\mathbb{C}}\ (\underline{x}(t),f) \in S \qquad (f \in \mathcal{L}_q(\Omega))$$

(see appendix 1, 0.2), then we call $\underline{x}$ a smooth stochastic process of order p. The set of all smooth stochastic processes of order p is denoted by $S_{\Omega,p}$. In $S_{\Omega,p}$ we have an equivalence:

$$\underline{x} = \underline{y} :\Leftrightarrow \forall_{t\in\mathbb{C}}[\underline{x}(t) = \underline{y}(t)] \qquad (\underline{x} \in S_{\Omega,p},\ \underline{y} \in S_{\Omega,p}) .$$

1.1.3. Theorem. Let $\underline{x} \in S_{\Omega,p}$. There exist positive constants M, A and B such that

$$\|\underline{x}(t)\|_p \le M \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi B(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C}) .$$

Proof. We use theorem 1.3 of appendix 1. For every $t \in \mathbb{C}$ and $f \in \mathcal{L}_q(\Omega)$ we have by Hölder's inequality

$$|(f,\underline{x}(t))| \le \|f\|_q\ \|\underline{x}(t)\|_p .$$

So $\forall_{f\in\mathcal{L}_q(\Omega)}\ (f,\underline{x}(t))$ is a bounded linear functional of the Banach space $\mathcal{L}_q(\Omega)$ for every $t \in \mathbb{C}$. Furthermore there exist for every $f \in \mathcal{L}_q(\Omega)$ positive constants M and A such that

$$|(f,\underline{x}(t))| \le M \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi A^{-1}(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C})$$

(this follows from the fact that $(\underline{x},f) \in S$). Application of theorem 1.3 of appendix 1 yields: there exist positive constants M and A such that

$$|(f,\underline{x}(t))| \le M\|f\|_q \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi A^{-1}(\mathrm{Im}\ t)^2)$$

for every $f \in \mathcal{L}_q(\Omega)$ and every $t \in \mathbb{C}$. This means by [Z], Ch.12, §50, theorem 2 that

$$\|\underline{x}(t)\|_p \le M \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi A^{-1}(\mathrm{Im}\ t)^2)$$

for every $t \in \mathbb{C}$. $\square$

1.1.4. <u>Definition</u>. If $\underline{x} \in S_{\Omega,2}$, then its <u>autocorrelation functions</u> $R_{\underline{x}}$ is defined by

$$R_{\underline{x}} := \Psi_{(t,s)\in\mathbb{C}^2} \ (\underline{x}(t),\underline{x}(\bar{s})) \ .$$

We often write R instead of $R_{\underline{x}}$.

1.1.5. We list some properties of R.

(i) if $t \in \mathbb{C}$, $s \in \mathbb{C}$, then $\overline{R(t,s)} = R(\bar{s},\bar{t})$

(ii) if $\alpha_1,\ldots,\alpha_n$ are complex numbers, and $t_1 \in \mathbb{C},\ldots,t_n \in \mathbb{C}$ $(n \in \mathbb{N})$, then $\sum_{i,j} \alpha_i\bar{\alpha}_j \ R(t_i,\bar{t}_j) \geq 0$. This follows from the fact that

$$\sum_{i,j} \alpha_i\bar{\alpha}_j \ (\underline{x}(t_i),\underline{x}(t_j)) = \| \sum_i \alpha_i \ \underline{x}(t_i) \|_2^2 \ .$$

(iii) if $t \in \mathbb{C}, s \in \mathbb{C}$, then $|R(t,s)| \leq \|\underline{x}(t)\|_2 \ \|\underline{x}(\bar{s})\|_2$. This follows from Hölder's inequality.

1.1.6. <u>Theorem</u>. Let $1 \leq p \leq \infty$, and let $\underline{x} \in S_{\Omega,p}$, $\underline{y} \in S_{\Omega,q}$, then $\Psi_{(t,s)\in\mathbb{C}^2} \ (\underline{x}(t),\underline{y}(\bar{s})) \in S^2$

<u>Proof</u>. Let $s \in \mathbb{C}$ be fixed. It follows from the definition of $S_{\Omega,p}$ that $\Psi_{t\in\mathbb{C}} \ (\underline{x}(t),\underline{y}(\bar{s}))$ is an analytic function. This is also true for the function $\Psi_{s\in\mathbb{C}} \ (\underline{x}(t),\underline{y}(\bar{s}))$ if $t \in \mathbb{C}$ is fixed. By a theorem of Hartogs ([BT], III, §4, satz 15) the function $\Psi_{(t,s)\in\mathbb{C}^2} \ (\underline{x}(t),\underline{y}(\bar{s}))$ is analytic in both variables.

We may complete the proof by showing that there exist positive numbers M, A and B such that

$$|(\underline{x}(t),\underline{y}(\bar{s}))| \leq M \exp(-\pi A((\mathrm{Re}\ t)^2 + (\mathrm{Re}\ s)^2) + \pi B((\mathrm{Im}\ t)^2 + (\mathrm{Im}\ s)^2)$$

for every $t \in \mathbb{C}$, $s \in \mathbb{C}$. This easily follows from theorem 1.1.3 and Hölder's inequality. □

<u>Corollary</u>. If $\underline{x} \in S_{\Omega,2}$, then $R \in S^2$.

<u>Remark</u>. Suppose $\underline{x}$ is a mapping of $\mathbb{C} \times \Omega$ into $\mathbb{C}$ which satisfies $\underline{x}(t) \in \mathcal{L}_2(\Omega)$ for every $t \in \mathbb{C}$ and $\Psi_{(t,s)\in\mathbb{C}^2} \ (\underline{x}(t),\underline{x}(\bar{s})) \in S^2$. Then $\underline{x} \in S_{\Omega,2}$.

For, if U is the closure in $\mathcal{L}_2(\Omega)$ of the set $\{\sum_{i=1}^n \alpha_i \ \underline{x}(t_i) \mid n \in \mathbb{N},$ $\alpha_1 \in \mathbb{C},\ldots,\alpha_n \in \mathbb{C},\ t_1 \in \mathbb{C},\ldots,t_n \in \mathbb{C}\}$, and $f \in \mathcal{L}_2(\Omega)$, then f is the sum of an

f_1 in U and an f_2 in $\mathcal{L}_2(\Omega)$ which satisfies $(f_2,g) = 0$ $(g \in U)$. Now we may prove the smoothness of $(\underline{x},f) = (\underline{x},f_1)$ by using theorem 1 of appendix 2.

1.1.7. Let $n \in \mathbb{N}$, and let $1 \leq p \leq \infty$. We can introduce in an obvious way the notion of smooth stochastic process of order p of n variables, and the space $S^n_{\Omega,p}$. It is possible to prove theorem 1.1.3 and theorem 1.1.5 for the n-dimensional case, and we can define in the case $p = 2$ an autocorrelation function which turns out to be an element of S^{2n}.

1.1.8. We conclude this section with some examples.

(i) Let $n \in \mathbb{N}$. If $f \in S^n$, and $1 \leq p \leq \infty$, then we can define the embedding $\tilde{f}$ of f in $S^n_{\Omega,p}$ by

$$\tilde{f} := \Psi_{(t,\omega)\in\mathbb{C}^n\times\Omega} f(t) \ .$$

It is trivial that $\tilde{f} \in S^n_{\Omega,p}$.

(ii) If $\underline{x} \in S_{\Omega,2}, \underline{y} \in S_{\Omega,2}$, then $\underline{x} \otimes \underline{y} := \Psi_{((t,s),\omega)\in\mathbb{C}^2\times\Omega} \underline{x}(t,\omega)y(s,\omega)$ is an element of $S^2_{\Omega,1}$.

Smoothness of $(\underline{x} \otimes \underline{y},f)$ for $f \in \mathcal{L}_\infty(\Omega)$ may be proved by using the method of the proof of theorem 1.1.5.

(iii) If $1 \leq p \leq \infty$, and $\varepsilon > 0$, and if $(q_k)_{k\in\mathbb{N}_0}$ is a sequence on $\mathcal{L}_p(\Omega)$ such that $\|q_k\|_p = O(e^{-k\varepsilon})$ $(k \in \mathbb{N}_0)$ then

$$\underline{x} := \Psi_{(t,\omega)\in\mathbb{C}\times\Omega} \sum_{k=0}^{\infty} q_k(\omega)\psi_k(t)$$

defines an element of $S_{\Omega,p}$. In order to prove this we note that the series $\sum_{k=0}^{\infty} \|q_k\|_p \, |\psi_k(t)|$ is convergent for fixed $t \in \mathbb{C}$ (see appendix 1, 0.5 (iv)e)), so $\underline{x}(t) \in \mathcal{L}_p(\Omega)$ $(t \in \mathbb{C})$. Furthermore we have for $f \in \mathcal{L}_q(\Omega)$ by Lebesgue's theorem on dominated convergence

$$(\underline{x}(t),f) = \sum_{k=0}^{\infty} (q_k,f)\psi_k(t) \qquad (t \in \mathbb{C}) \ ,$$

and $(q_k,f) = O(e^{-k\varepsilon})$ $(k \in \mathbb{N}_0)$ by Hölder's inequality.

Now the smoothness of $(\underline{x},f)$ follows from appendix 1, 0.5 (iv)c). This means that $\underline{x} \in S_{\Omega,p}$. See 1.2.8 (iii) for a converse.

(iv) If $1 \le p \le r \le \infty$, then $S_{\Omega,p} \supset S_{\Omega,r}$. This may be proved by using twice the fact that $1 \le x \le y \le \infty \Rightarrow \mathcal{L}_x(\Omega) \supset \mathcal{L}_y(\Omega)$.

(v) If $1 \le p \le \infty$, and $\underline{x} \in S_{\Omega,p}$, then $\overline{\underline{x}} := \Psi_{(t,\omega)\in\mathbb{C}\times\Omega}\overline{x(\overline{t},\omega)} \in S_{\Omega,p}$.

1.2. Theory of linear functionals and linear operators of $S_{\Omega,p}$.

1.2.1. Introduction.

Let (Ω,Λ,P) be a probability space, and let $1 \le p \le \infty$. The aim of this section is to extend quasi-bounded linear functionals and quasi-bounded linear operators of S to linear mappings of $S_{\Omega,p}$ into $\mathcal{L}_p(\Omega)$ and $S_{\Omega,p}$ respectively (see appendix 1, section 2). We first extend quasi-bounded linear functionals of S, and then we reduce the extension of quasi-bounded linear operators of S to the extension of quasi-bounded linear functionals of S.

1.2.2. First suppose that $1 < p \le \infty$ (then $1 \le q < \infty$). We have the following <u>Theorem</u>. Let L be a quasi-bounded linear functional of S (see appendix 1, 2.1). If $\underline{x} \in S_{\Omega,p}$, then there is exactly one $g \in \mathcal{L}_p(\Omega)$ such that

$$L(\underline{x},f) = (g,f) \qquad (f \in \mathcal{L}_q(\Omega)) .$$

<u>Proof</u>. Let $\underline{x} \in S_{\Omega,p}$. It is easily seen that $\overline{L(\underline{x},f)}$ depends linearly on $f \in \mathcal{L}_q(\Omega)$. We show that $\Psi_{f\in\mathcal{L}_q(\Omega)}\ \overline{L(\underline{x},f)}$ is a bounded linear functional of $\mathcal{L}_q(\Omega)$. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence on $\mathcal{L}_q(\Omega)$ with $\| f_n \|_q \to 0$. It is not hard to prove from theorem 1.1.3 that $(\underline{x},f_n) \overset{S}{\to} 0$ (see appendix 1, 0.9).
By appendix 1, 0.9 we can write $(\underline{x},f_n) = N_\alpha \varphi_n$ $(n \in \mathbb{N})$ with some positive α and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ on S such that $\varphi_n \overset{S}{\to} 0$. Now $|\overline{L(\underline{x},f_n)}| = |L(N_\alpha\varphi_n)| \to 0$ by the definition of quasi-boundedness and by [B], theorem 23.2. This proves the boundedness of $\Psi_{f\in\mathcal{L}_q(\Omega)}\overline{L(\underline{x},f)}$.

From the fact that $\Psi_{f\in\mathcal{L}_q(\Omega)}\overline{L(\underline{x},f)}$ is a bounded linear functional of $\mathcal{L}_q(\Omega)$, we conclude from [Z], Ch. 12, §50, Theorem 2 that there exists exactly one

$g \in \mathcal{L}_p(\Omega)$ such that

$$L(\underline{x},f) = (f,g) \qquad (f \in \mathcal{L}_q(\Omega)) . \qquad \square$$

Remark (without proof). If (Ω,Λ,P) is a probability space such that $\mathcal{L}_2(\Omega)$ has infinite dimension, and if L is a linear functional of S such that $\Psi_{f\in\mathcal{L}_2(\Omega)}\overline{L(\underline{x},f)}$ is a bounded linear functional for every $\underline{x} \in S_{\Omega,2}$, then L is a quasi-bounded linear functional of S.

1.2.3. Now we consider the more delicate case $p = 1$.
Let $\underline{x} \in S_{\Omega,1}$. We prove the following

Lemma. Let L be a quasi-bounded linear functional of S.
The set function $\Psi_{A\in\Lambda}L(\underline{x},A)$ is absolutely continuous and completely additive (here we have written A for both the characteristic function of the set A and the set A itself).

Proof. We have to show that

1) if $A \in \Lambda$ and $P(A) = 0$, then $L(\underline{x},A) = 0$.

2) if $A, A_n \in \Lambda$ $(n \in \mathbb{N})$, and A is disjoint union of the A_n's then

$$L(\underline{x},A) = \sum_{n=1}^{\infty} L(\underline{x},A_n).$$

It is easy to prove 1): take an $A \in \Lambda$ with $P(A) = 0$. Now $(\underline{x},A) = 0$, so $L(\underline{x},A) = 0$.

Now we prove 2). Let $A, A_n \in \Lambda$ $(n \in \mathbb{N})$, and suppose that A is disjoint union of the A_n's. For $N \in \mathbb{N}$ we have

$$L(\underline{x},A) - \sum_{n=1}^{N} L(\underline{x},A_n) = L(\underline{x}, \bigcup_{n=N+1}^{\infty} A_n) .$$

We show that $(\underline{x}, \bigcup_{n=N+1}^{\infty} A_n) \overset{S}{\to} 0$. It follows from Lebesgue's theorem that for every $t \in \mathbb{C}$

$$\lim_{n\to\infty}(\underline{x}(t), \bigcup_{n=N+1}^{\infty} A_n) = 0 ,$$

and furthermore we have for every $t \in \mathbb{C}$

$$|(\underline{x}(t), \bigcup_{n=N+1}^{\infty} A_n)| \le \|\underline{x}(t)\|_1 .$$

We conclude from theorem 1.1.3 and theorem 1 of appendix 3 that $(\underline{x}, \bigcup_{n=N+1}^{\infty} A_n) \overset{S}{\to} 0$. From the fact that L is a quasi-bounded linear functional of S we infer that $L(\underline{x}, \bigcup_{n=N+1}^{\infty} A_n) \to 0$ if $N \to \infty$ (see also the proof of theorem 1.2.2). Therefore

$$\sum_{n=1}^{\infty} L(\underline{x},A_n) = \lim_{N\to\infty} \sum_{n=1}^{N} L(\underline{x},A_n) = L(\underline{x},A) . \qquad \square$$

1.2.4. Theorem. Let L be a quasi-bounded linear functional of S. If $\underline{x} \in S_{\Omega,1}$, then there is exactly one $g \in \mathcal{L}_1(\Omega)$ such that

$$L(\underline{x},f) = (g,f) \qquad (f \in \mathcal{L}_\infty(\Omega)) .$$

Proof. Let $\underline{x} \in S_{\Omega,1}$. We apply the Radon-Nikodym theorem (complex version, see [Z], Ch. 11, §45, Theorem 3) to the set function $\Psi_{A\in\Lambda} L(\underline{x},A)$ which is absolutely continuous and completely additive. There exists a $g \in \mathcal{L}_1(\Omega)$ such that

$$L(\underline{x},A) = (g,A) \qquad (A \in \Lambda) .$$

Now let $f \in \mathcal{L}_\infty(\Omega)$. There is a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions of the form $\sum_{i=1}^{m} \alpha_i A_i$ $(m \in \mathbb{N},\ \alpha_i \in \mathbb{C},\ A_i \in \Lambda)$ such that $\|f_n\|_\infty \le \|f\|_\infty$, and such that $f_n \to f$ almost everywhere in Ω. We can prove that $(\underline{x},f_n) \overset{S}{\to} (\underline{x},f)$ in the same way as we proved lemma 1.2.3. It follows that

$$L(\underline{x},f_n) \to L(\underline{x},f) \qquad (n \to \infty) .$$

It is not hard to prove that $L(\underline{x},f_n) = (g,f_n)$, and furthermore it is easy to see that $(g,f_n) \to (g,f)$ $(n \to \infty)$. This proves that $L(\underline{x},f) = (g,f)$. The uniqueness of g is trivial. $\square$

1.2.5. Let $1 \le p \le \infty$, let L be a quasi-bounded linear functional of S, let $\underline{x} \in S_{\Omega,p}$, and let g be the unique $\mathcal{L}_p(\Omega)$ function of theorem 1.2.2 (if $1 \le p \le \infty$) or of theorem 1.2.4 (if $p = \infty$).

Definition. We define $L_p\underline{x} := g$.

Remark 1. This definition is such that $(L_p\underline{x},f) = L(\underline{x},f)$ for every $f \in \mathcal{L}_q(\Omega)$.

Remark 2. Let $1 \le p \le r \le \infty$. If $\underline{x} \in S_{\Omega,p}$ and $\underline{x} \in S_{\Omega,r}$, then $L_p\underline{x} = L_r\underline{x}$. This follows from the fact that $\mathcal{L}_\infty(\Omega)$ is dense in $\mathcal{L}_p(\Omega)$, $\mathcal{L}_\infty(\Omega)$ is dense in $\mathcal{L}_r(\Omega)$, and from $(L_p\underline{x},f) = L(\underline{x},f) = L_r(\underline{x},f)$ for every $f \in \mathcal{L}_\infty(\Omega)$. Therefore, we rather write $L\underline{x}$ instead of $L_p\underline{x}$ or $L_r\underline{x}$. It is obvious that L is linear.

Remark 3. If $f \in S$, then we have $L\,\tilde{f} = \Psi_{\omega\in\Omega}\, L\, f$ (see 1.1.8(i)).

1.2.6. Let $1 \le p \le \infty$. We now describe the extension of a quasi-bounded linear operator T of S to a linear mapping of $S_{\Omega,p}$ into $S_{\Omega,p}$.

Let $t \in \mathbb{C}$, and consider the linear functional

$$L_t := \Psi_{h\in S}(Th)(t) \;.$$

According to appendix 1, 2.2 this functional is quasi-bounded. Let $\underline{x} \in S_{\Omega,p}$. We define

$$T\underline{x} := \Psi_{(t,\omega)\in\mathbb{C}\times\Omega}(L_t\underline{x})(\omega) \;.$$

Now we have $(T\underline{x})(t) \in \mathcal{L}_p(\Omega)$, and for every $f \in \mathcal{L}_q(\Omega)$ and $t \in \mathbb{C}$ the equation $T(\underline{x},f)(t) = ((T\underline{x})(t),f)$ holds. This means that $(T\underline{x},f) = \Psi_{t\in\mathbb{C}}T(\underline{x},f)(t) \in S$ for every $f \in \mathcal{L}_q(\Omega)$. So $T\underline{x} \in S_{\Omega,p}$.

Remark 1. The definition of $T\underline{x}$ is such that $(T\underline{x},f) = T(\underline{x},f)$ for every $f \in \mathcal{L}_q(\Omega)$.

Remark 2. If $f \in S$, then we have $T\,\tilde{f} = \Psi_{(t,\omega)\in\mathbb{C}\times\Omega}(Tf)(t)$ (see 1.1.8(i)).

Remark 3. If $Tg = 0$ for every $g \in S$, then $T\underline{x} = 0$ for every $\underline{x} \in S_{\Omega,p}$. For, if $f \in \mathcal{L}_q(\Omega)$, then $(T\underline{x},f) = T(\underline{x},f) = 0$.

1.2.7. We say a few things about the extension of linear functionals and linear operators of S^n. All preceding theorems can be stated and proved (with the proper modifications) for the n-dimensional case.

Theorem. Let T_1 and T_2 be quasi-bounded linear operators of S, and let T_1 and T_2 resp. $T_1 \otimes T_2$ (see appendix 1, 2.13) be extended according to 1.2.6 to linear operators of $S_{\Omega,2}$ resp. $S^2_{\Omega,1}$. If $\underline{x}_1 \in S_{\Omega,2}$, $\underline{x}_2 \in S_{\Omega,2}$, then $(T_1 \otimes T_2)(\underline{x}_1 \otimes \underline{x}_2) = T_1\underline{x}_1 \otimes T_2\underline{x}_2$ (see 1.1.8(ii)).

Proof. It is sufficient to show that for every $f \in \mathcal{L}_\infty(\Omega)$

$$((T_1 \otimes T_2)(\underline{x}_1 \otimes \underline{x}_2), f) = (T_1\underline{x}_1 \otimes T_2\underline{x}_2, f) \ .$$

Let $f \in \mathcal{L}_\infty(\Omega)$, and note that by definition $((T_1 \otimes T_2)(\underline{x}_1 \otimes \underline{x}_2), f) =$ $(T_1 \otimes T_2)(\underline{x}_1 \otimes \underline{x}_2, f)$. Furthermore we have for $t \in \mathbb{C}$, $s \in \mathbb{C}$

$$(T_1\underline{x}_1 \otimes T_2\underline{x}_2, f)(t,s) = ((T_1\underline{x}_1)(t)(T_2\underline{x}_2)(s), f) =$$

$$= T_1(\Upsilon_u(\underline{x}_1(u)(T_2\underline{x}_2)(s), f))(t) \ ,$$

and for $u \in \mathbb{C}$ we have

$$(\underline{x}_1(u)(T_2\underline{x}_2)(s), f) = T_2(\Upsilon_v(\underline{x}_1(u)\underline{x}_2(v), f))(s)$$

$$= T_2(\Upsilon_v((\underline{x}_1 \otimes \underline{x}_2)(u,v), f))(s) \ .$$

Now the theorem follows from appendix 1, 2.13. □

1.2.8. We conclude this section with a number of examples.

(i) The smoothing operators N_α $(\alpha > 0)$, the Fouriertransform F and its inverse F^*, the shift operators T_a and R_b $(a \in \mathbb{C},\ b \in \mathbb{C})$, and the operators P and Q can be extended to linear operators of $S_{\Omega,p}$ for $1 \le p \le \infty$, because these operators are quasi-bounded (see appendix 1, 2.10(iii)). We have by [B], 8.2 and 1.2.6 remark 3 $(FN_\alpha - N_\alpha F)\underline{x} = 0$, $FN_\alpha\underline{x} = N_\alpha F\underline{x}$ $(\alpha > 0,\ 1 \le p \le \infty,\ \underline{x} \in S_{\Omega,p})$. By a similar argument we have $FF^*\underline{x} = \underline{x}$, $T_aT_b\underline{x} = T_{a+b}\underline{x}$ $(a \in \mathbb{C}, b \in \mathbb{C})$, etc. for $\underline{x} \in S_{\Omega,p}$ $(1 \le p \le \infty)$.

(ii) If $\underline{x} \in S_{\Omega,2}$, then we have for $t \in \mathbb{C}$, $s \in \mathbb{C}$

$$R_{T\underline{x}}(t,s) = T(\Upsilon_u \overline{T(\Upsilon_v R(v,\overline{u}))(\overline{s})})(t)$$

(see also 1.1.4 and 1.2.6). As examples we have

$$R_{T_a\underline{x}} = (T_a \otimes T_a)R,\ R_{R_b\underline{x}} = (R_b \otimes R_{-b})R$$

for $a \in \mathbb{R}$, $b \in \mathbb{R}$ (see appendix 1, 2.13).

(iii) Let $1 \le p \le \infty$, and let $F \in S^*$. The linear functional $L_F := \Upsilon_{f \in S}[f,F]$ is quasi-bounded ([B], 22.1). So it is possible to extend L_F to $S_{\Omega,p}$. We denote $L_F\underline{x} =: [\underline{x},F]$ $(\underline{x} \in S_{\Omega,p})$.

If $f \in S$, $\underline{x} \in S_{\Omega,p}$, then we denote $[\underline{x},f] := [\underline{x},\mathrm{emb}(f)]$ (see appendix 1, 0.7). Now we may prove the converse of 1.1.8(iii): if $\underline{x} \in S_{\Omega,p}$, then there is an $\varepsilon > 0$ and a sequence $(q_k)_{k\in\mathbb{N}_0}$ in $\mathcal{L}_p(\Omega)$ such that $\| q_k \|_p = O(e^{-k\varepsilon})$ $(k \in \mathbb{N}_0)$ and $\underline{x} = \sum_{k=0}^{\infty} q_k\psi_k$. For, if $\underline{x} \in S_{\Omega,p}$ then there exist by appendix 1, 0.5 (iii) and theorem 1.1.3 constants $\varepsilon > 0$, $M > 0$, $A > 0$, $B > 0$ such that for every $f \in \mathcal{L}_q(\Omega)$ there is exactly one $g \in S$ such that

$$(\underline{x},f) = N_\varepsilon g, \quad |g(t)| \leq M\|f\|_q \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi B(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C}) .$$

Now we find that for some $M' > 0$

$$|([\underline{x},\psi_k],f)| = |[(\underline{x},f),\psi_k]| \leq M'\|f\|_q\ e^{-k\varepsilon} \qquad (f \in \mathcal{L}_q(\Omega),\ k \in \mathbb{N}_0) .$$

It follows from [Z], Ch. 12, §50, Theorem 2 that

$$\| [\underline{x},\psi_k] \|_p = O(e^{-k\varepsilon}) \qquad (k \in \mathbb{N}_0) .$$

Furthermore, we have for $f \in \mathcal{L}_q(\Omega)$ by appendix 1, 0.5(iv) d) and e)

$$(\underline{x},f) = \sum_{k=0}^{\infty} [(\underline{x},f),\psi_k]\psi_k = \sum_{k=0}^{\infty} ([\underline{x},\psi_k],f)\psi_k = (\sum_{k=0}^{\infty} [\underline{x},\psi_k]\psi_k,f)$$

(here we used Lebesgue's theorem on dominated convergence). This proves that $\underline{x} = \sum_{k=0}^{\infty} [\underline{x},\psi_k]\psi_k$. It is not hard to show that there is at most one sequence $(q_k)_{k\in\mathbb{N}_0}$ on $\mathcal{L}_p(\Omega)$ such that $\| q_k \|_p = O(e^{-k\varepsilon})$ $(k \in \mathbb{N}_0)$ for some $\varepsilon > 0$, and such that $\underline{x} = \sum_{k=0}^{\infty} q_k\psi_k$.

(iv) If $1 \leq p \leq \infty$, $\underline{x} \in S_{\Omega,p}$, and T is a quasi-bounded linear operator S, then $T\underline{x} = \sum_{k=0}^{\infty} [\underline{x},\psi_k]T\psi_k$. The proof of this fact is similar to that of 1.2.8(iii). See also appendix, 1.2.10(iv).

(v) Inverse smoothing theorem. Let $1 \leq p \leq \infty$. It is not hard to prove that for every $\underline{x} \in S_{\Omega,p}$ and every $\alpha > 0$ there is at most one $\underline{y} \in S_{\Omega,p}$ such that $\underline{x} = N_\alpha\underline{y}$ (see [B], 10.1(i)). If follows from 1.2.8(iii) and (iv) that for every $\underline{x} \in S_{\Omega,p}$ there is an $\alpha > 0$ and a $\underline{y} \in S_{\Omega,p}$ such that $\underline{x} = N_\alpha\underline{y}$.

1.3. Generalized stochastic processes.

1.3.1. Let (Ω,Λ,P) be a probability space, and let $1 \leq p \leq \infty$.

Definition. A generalized stochastic process of order p is a mapping $\underline{X}$ of the positive real numbers into $S_{\Omega,p}$ such that $N_\alpha \underline{X}_\beta = \underline{X}_{\alpha+\beta}$ for every $\alpha > 0$, $\beta > 0$. The class of all generalized stochastic processes of order p is denoted by $S^*_{\Omega,p}$. In $S^*_{\Omega,p}$ we have an equivalence: if $\underline{X} \in S^*_{\Omega,p}$, $\underline{Y} \in S^*_{\Omega,p}$, then we say that $\underline{X}$ and $\underline{Y}$ are equal ($\underline{X} = \underline{Y}$) if $\underline{X}_\alpha = \underline{Y}_\alpha$ for every $\alpha > 0$.

If $f \in \mathcal{L}_q(\Omega)$, then it is easy to verify that

$$(\underline{X},f) := \Psi_{\alpha>0}\ (\underline{X}_\alpha,f) \in S^*$$

for every $\underline{X} \in S^*_{\Omega,p}$, $f \in \mathcal{L}_q(\Omega)$ (see appendix 1, 0.7). As a consequence we have $(\underline{X}\ ,f) \in N_\alpha(S^*)$ for every $\underline{X} \in S^*_{\Omega,p}$, $f \in \mathcal{L}_q(\Omega)$, $\alpha > 0$.

1.3.2. Definition. If $\underline{X} \in S^*_{\Omega,2}$, then its autocorrelation function $R_{\underline{X}}$ is defined by

$$R_{\underline{X}} := \Psi_{\alpha>0}\ \Psi_{(t,s)\in\mathbb{C}^2}(\underline{X}_\alpha(t),\underline{X}_\alpha(\overline{s}))\ .$$

We often write R instead of $R_{\underline{X}}$.

1.3.3. Theorem. If $\underline{X} \in S^*_{\Omega,2}$, then $R \in S^2$ (see appendix 1, 0.10).

Proof. Let $\underline{X} \in S^*_{\Omega,2}$. It is easily seen that $R_\alpha \in S^2$ for every $\alpha > 0$ (see 1.1.6. corollary). If $\alpha > 0$, $\beta > 0$, $t \in \mathbb{C}$, $s \in \mathbb{C}$, then we have by 1.2.8(ii) (using $R_\beta(u,v) = \overline{R_\beta(v,u)}$ for $u \in \mathbb{R}$, $v \in \mathbb{R}$, see 1.1.5(i))

$$R_{\alpha+\beta}(t,s) = N_\alpha(\Psi_u\ \overline{N_\alpha(\Psi_v\ R_\beta(v,\overline{u}))(\overline{s})})(t) = (N_{\alpha,2}\ R_\beta)(t,s)$$

(see appendix 1, 0.10 and 2.13). So $R_{\alpha+\beta} = N_{\alpha,2}R_\beta \qquad (\alpha > 0,\ \beta > 0)$. □

1.3.4. In an obvious way we can define generalized stochastic processes of order p of n ($n \in \mathbb{N}$) variables (class $S^{n*}_{\Omega,p}$). It is also possible to define the autocorrelation function of an element of $S^{n*}_{\Omega,p}$. Theorem 1.3.3 remains valid for the n-dimensional case.

1.3.5. We give some examples.

(i) If $1 \leq p \leq \infty$, and $n \in \mathbb{N}_0$, then we can define for $F \in S^{n*}$ its embedding in

$S^{n*}_{\Omega,p}$ by

$$\tilde{F} := \Upsilon_{\alpha>0}\ \Upsilon_{(t,\omega)\in\mathbb{C}^n\times\Omega} F_\alpha(t) \ .$$

It is easily seen from 1.2.6 remark 2 and definition 1.3.1 that $\tilde{F} \in S^{n*}_{\Omega,p}$.

(ii) Let $\underline{X} \in S^*_{\Omega,2}$, $\underline{Y} \in S^*_{\Omega,2}$, and define $\underline{X} \otimes \underline{Y} := \Upsilon_{\alpha>0}\ \underline{X}_\alpha \otimes \underline{Y}_\alpha$ (see 1.1.8(ii)). Now $\underline{X} \otimes \underline{Y} \in S^{2*}_{\Omega,1}$. This follows from 1.1.8(ii) and 1.2.7 since

$$N_{\alpha,2}(\underline{X}_\beta \otimes \underline{Y}_\beta) = N_\alpha \underline{X}_\beta \otimes N_\alpha \underline{Y}_\beta = \underline{X}_{\alpha+\beta} \otimes \underline{Y}_{\alpha+\beta}$$

for $\alpha > 0$, $\beta > 0$ (see also appendix 1, 2.13).

(iii) Let $1 \le p \le \infty$, and let $(q_k)_{k\in\mathbb{N}_0}$ be a sequence on $\mathcal{L}_p(\Omega)$ which satisfies $\forall_{\varepsilon>0}[\| q_k \|_p = O(e^{k\varepsilon})\ (k \in \mathbb{N}_0)]$. It follows from 1.1.8(iii), 1.2.8(iv) and appendix 1, 0.5(iv)a) that $\Upsilon_{\alpha>0} \sum_{k=0}^{\infty} q_k N_\alpha \psi_k \in S^*_{\Omega,p}$. See also 1.4.6(iii) for a converse.

(iv) If $1 \le p \le r \le \infty$, then $S^*_{\Omega,p} \supset S^*_{\Omega,r}$. This follows from 1.1.8(iv).

(v) Let $1 \le p \le \infty$, and let $\underline{x} \in S_{\Omega,p}$. Define

$$\mathrm{emb}(\underline{x}) := \Upsilon_{\alpha>0}\ N_\alpha \underline{x} \ .$$

It is trivial that $\mathrm{emb}(\underline{x}) \in S^*_{\Omega,p}$.

1.4. Theory of linear functionals and linear operators of $S^*_{\Omega,p}$.

1.4.1. Introduction. Let (Ω,Λ,P) be a probability space, and let $1 \le p \le \infty$. In this section we shall extend a certain class of linear functionals and linear operators of S^* to linear mappings of $S^*_{\Omega,p}$ into $\mathcal{L}_p(\Omega)$ and $S^*_{\Omega,p}$ respectively. We restrict ourselves to continuous linear functionals of S^* (see appendix 1, 3.6), and to linear operators of S^* which are extensions of linear operators of S with an adjoint (see appendix 1, section 3). We shall not give many details of the proofs of the theorems in this section, because they have much in common with those in section 1.2.

1.4.2. <u>Theorem</u>. Let L be a continuous linear functional of S^*, let $1 \le p \le \infty$, and let $X \in S^*_{\Omega,p}$. There exists exactly one $g \in \mathcal{L}_p(\Omega)$ such that (see 1.3.1)

$$L(\underline{X},f) = (g,f) \qquad (f \in \mathcal{L}_q(\Omega)) \ .$$

<u>Proof</u>. In the case $1 < p \le \infty$ we can use the fact that for every sequence $(f_n)_{n\in\mathbb{N}}$ on $\mathcal{L}_q(\Omega)$ with $\|f_n\|_q \to 0$

$$(\underline{X},f_n) \overset{S^*}{\to} 0 \ .$$

From this it follows (continuity of L) that

$$\Upsilon_{f\in\mathcal{L}_q(\Omega)} \overline{L(\underline{X},f)}$$

is a bounded linear functional of $\mathcal{L}_q(\Omega)$. The remaining part of the proof is similar to the second part of the proof of theorem 1.2.2.

If $p = 1$ we use the fact that for $f,f_n \in \mathcal{L}_\infty(\Omega)$ $(n \in \mathbb{N})$, which are uniformly bounded in $\|\ \|_\infty$, and which satisfy $f_n \to f$ almost everywhere in Ω, $(\underline{X},f_n) \overset{S^*}{\to} (\underline{X},f)$. Now we can make use of the continuity of L. The remaining part of the proof is similar to the proofs of lemma 1.2.3 and theorem 1.2.4. $\square$

1.4.3. Let $1 \le p \le \infty$, let L be a continuous linear functional of S^*, let $\underline{X} \in S^*_{\Omega,p}$, and let g be the (unique) $\mathcal{L}_p(\Omega)$ function of theorem 1.4.2.

<u>Definition</u>. We define $L_p\underline{X} := g$.

<u>Remark 1</u>. This definition is such that $(L_p\underline{X},f) = L(\underline{X},f)$ for every $f \in \mathcal{L}_q(\Omega)$.

<u>Remark 2</u>. Let $1 \le p \le r \le \infty$. If $\underline{X} \in S_{\Omega,p}$ and $\underline{X} \in S_{\Omega,r}$, then $L_p\underline{X} = L_r\underline{X}$ (this follows as in 1.2.5 remark 2). Therefore we rather write $L\underline{X}$ instead of $L_p\underline{X}$ or $L_r\underline{X}$. It is obvious that L is linear.

<u>Remark 3</u>. If $F \in S^*$, then $L\widetilde{F} = \Upsilon_{\omega\in\Omega}\, LF$ (see 1.3.5(i)).

1.4.4. Let $1 \le p \le \infty$, and let T be a linear operator of S with an adjoint. This T is by appendix 1, 3.2 remark 1 and 3.3 extendable to S^* by means of a family $(Y_\alpha)_{\alpha>0}$ of linear mappings of $N_\alpha(S^*)$ into S such that

1. $Y_\alpha N_\alpha f = N_\alpha Tf \qquad (\alpha > 0,\ f \in S)$
2. $Y_{\alpha+\beta} N_{\alpha+\beta} F = N_\beta Y_\alpha N_\alpha F \qquad (\alpha > 0,\ \beta > 0,\ F \in S^*)$
3. $F_n \in S^*\ (n \in \mathbb{N}),\ F_n \overset{S^*}{\to} 0 \Rightarrow TF_n = \Upsilon_{\alpha>0} Y_\alpha N_\alpha F_n \overset{S^*}{\to} 0$
4. $[TF,g] = [F,T^* g] \qquad (F \in S^*,\ g \in S)$.

For every $\alpha > 0$ and every $t \in \mathbb{C}$

$$L_{t,\alpha} := \Upsilon_{F\in S^*} (TF)_\alpha(t) = \Upsilon_{F\in S^*} (Y_\alpha N_\alpha F)(t)$$

is a continuous linear functional of S^* (this follows from 3). So $L_{t,\alpha}$ is extendable to a linear mapping of $S^*_{\Omega,p}$ into $\mathcal{L}_p(\Omega)$ for every $\alpha > 0$ and every $t \in \mathbb{C}$ (1.4.3).

Let $\underline{X} \in S^*_{\Omega,p}$. We define

$$T\underline{X} := \Upsilon_{\alpha>0}\ \Upsilon_{(t,\omega)\in\mathbb{C}\times\Omega} (L_{t,\alpha}\underline{X})(\omega) \ .$$

<u>Theorem</u>. $T\underline{X} \in S^*_{\Omega,p}$.

<u>Proof</u>. We have to establish two things:

a) $(T\underline{X})_\alpha \in S_{\Omega,p}$ for every $\alpha > 0$.
b) $N_\beta (T\underline{X})_\alpha = (T\underline{X})_{\alpha+\beta}$ for every $\alpha > 0$, $\beta > 0$.

We prove a) by taking an $f \in \mathcal{L}_q(\Omega)$, and find for $t \in \mathbb{C}$ and $\alpha > 0$ (see 1.4.3 remark 1)

$$((T\underline{X})_\alpha(t),f) = (L_{t,\alpha}\underline{X},f) = L_{t,\alpha}(\underline{X},f) = (T(\underline{X},f))_\alpha(t) \ ,$$

and $(T(\underline{X},f))_\alpha \in S$. This proves that $(T\underline{X})_\alpha \in S_{\Omega,p}$. It also proves that $((T\underline{X})_\alpha,f) = (T(X,f))_\alpha$ for $f \in \mathcal{L}_q(\Omega)$, $\alpha > 0$.

We now prove b). Let $\alpha > 0$, $\beta > 0$, $f \in \mathcal{L}_q(\Omega)$. We find by 1.2.6 remark 1

$$\begin{aligned}(N_\beta (T\underline{X})_\alpha, f) &= N_\beta((T\underline{X})_\alpha, f) = N_\beta (T(\underline{X},f))_\alpha = \\ &= (T(\underline{X},f))_{\alpha+\beta} = ((T\underline{X})_{\alpha+\beta}, f) \ .\end{aligned}$$

And from this it follows that $N_\beta (T\underline{X})_\alpha = (T\underline{X})_{\alpha+\beta}$ $(\alpha > 0,\ \beta > 0)$. $\square$

Remark 1. The definition of $T\underline{X}$ is such that $(T\underline{X},f) = T(\underline{X},f)$ for every $f \in \mathcal{L}_q(\Omega)$. It is obvious that T is linear.

Remark 2. If $F \in S^*$, then we have $T\widetilde{F} = \forall_{\alpha>0} \forall_{(t,\omega)\in\mathbb{C}\times\Omega} (TF)_\alpha(t)$ (see 1.3.5(i)).

Remark 3. Let $\alpha > 0$. We can also extend the linear operator Y_α to a linear mapping of $N_\alpha(S^*_{\Omega,p})$, this is the set $\{\underline{X}_\alpha \mid \underline{X} \in S^*_{\Omega,p}\}$, into $S_{\Omega,p}$ by the method of 1.2.6 and 1.4.4. We have $Y_\alpha N_\alpha \underline{X} = (T\underline{X})_\alpha$ and $Y_\alpha(\underline{X}_\alpha, f) = (Y_\alpha \underline{X}_\alpha, f)$ for $f \in \mathcal{L}_q(\Omega)$.

Remark 4. If $Tf = 0$ for every $f \in S$, then (appendix 1, 3.2 remark 3) $TF = 0$ for every $F \in S^*$. From this it follows that $T\underline{X} = 0$ for every $\underline{X} \in S^*_{\Omega,p}$, because $(T\underline{X},g) = T(\underline{X},g) = 0$ for every $g \in \mathcal{L}_q(\Omega)$.

1.4.5. We make a few remarks about the extension of linear operators and linear functionals of S^{n*}. All preceding theorems can be proved for the n-dimensional case.

Theorem. Let T_1 and T_2 be linear operators of S with an adjoint, and let T_1 and T_2 resp. $T_1 \otimes T_2$ (see appendix 1, 3.12) be extended according to 1.4.4 to linear operators of $S^*_{\Omega,2}$ resp. $S^{2*}_{\Omega,1}$. If $\underline{X}_1 \in S^*_{\Omega,2}$, $\underline{X}_2 \in S^*_{\Omega,2}$, then $(T_1 \otimes T_2)(\underline{X}_1 \otimes \underline{X}_2) = T_1\underline{X}_1 \otimes T_2\underline{X}_2$ (see 1.3.5(ii)).

Proof. Let $\underline{X}_1 \in S^*_{\Omega,2}$, $\underline{X}_2 \in S^*_{\Omega,2}$, and let T_1 resp. T_2 be extendable by means of $(Y_{\alpha,1})_{\alpha>0}$ resp. $(Y_{\alpha,2})_{\alpha>0}$ (see appendix 1, 3.2 remark 1). According to appendix 1, 3.12 we have to show that for $\alpha > 0$

$$Y_{\alpha,1}N_\alpha\underline{X}_1 \otimes Y_{\alpha,2}N_\alpha\underline{X}_2 = (Y^{(1)}_{\alpha,1}Y^{(2)}_{\alpha,2}N_{\alpha,2})(\underline{X}_1 \otimes \underline{X}_2) .$$

Note that $N_{\alpha,2}(\underline{X}_1 \otimes \underline{X}_2) = N_\alpha\underline{X}_1 \otimes N_\alpha\underline{X}_2$ (see 1.3.5(ii)).
The remaining part of the proof is similar to the proof of theorem 1.2.7. □

1.4.6. We conclude this section with a number of examples.
(i) The smoothing operators N_α $(\alpha > 0)$. Fourier transform F and its inverse F^*, the shift operators T_a and R_b $(a \in \mathbb{C},\ b \in \mathbb{C})$, and the operators P and Q can be extended to linear operators of $S^*_{\Omega,p}$ for $1 \le p \le \infty$, because the operators have adjoints (see appendix 1, 3.9(i)). We have by 1.4.4 remark 4 and [B], 8.2 $(FN_\alpha - N_\alpha F)\underline{X} = 0$, $FN_\alpha\underline{X} = N_\alpha F\underline{X}$ $(\alpha > 0,\ 1 \le p \le \infty,\ \underline{X} \in S_{\Omega,p})$. By a similar argument we can prove $FF^*\underline{X} = \underline{X}$, $T_aT_b\underline{X} = T_{a+b}\underline{X}$ $(a \in \mathbb{C},\ b \in \mathbb{C})$, etc. for $\underline{X} \in S_{\Omega,p}$ $(1 \le p \le \infty)$.

(ii) Let T be a linear operator of S with an adjoint, and let T be extended according to 1.4.4 to a linear operator of $S^*_{\Omega,2}$. Suppose that T is extendable by means of the family $(Y_\alpha)_{\alpha>0}$ (see appendix 1, 3.2 remark). If $\underline{X} \in S^*_{\Omega,2}$, then

$$((T\underline{X})_\alpha(t),(T\underline{X})_\alpha(\bar{s})) = Y_\alpha(\curlyvee_u \overline{Y_\alpha(\curlyvee_v R_\alpha(v,\bar{u}))(s)})(t)$$

for $\alpha > 0$, $t \in \mathbb{C}$, $s \in \mathbb{C}$, where R is the autocorrelation function of $\underline{X}$ (see 1.3.2). It is not hard to prove this (see also 1.4.5). As special cases we have (after some calculation)

$$R_{T_a\underline{X}} = (T_a \otimes T_a)R, \quad R_{R_b\underline{X}} = (R_b \otimes R_{-b})R \qquad (a \in \mathbb{R},\ b \in \mathbb{R}) .$$

Here $R_{T_a\underline{X}}$ ($R_{R_b\underline{X}}$) denotes the autocorrelation function of $T_a\underline{X}$ ($R_b\underline{X}$). See also 1.2.8(ii).

(iii) Let $1 \le p \le \infty$, and let $g \in S$. Define $L_g := \curlyvee_{F\in S^*}[F,g]$. This L_g is a continuous linear functional of S^*, so it is possible (1.4.3) to extend L_g to a linear mapping of $S^*_{\Omega,p}$ into $\mathcal{L}_p(\Omega)$. We denote $L_g\underline{X} =: [\underline{X},g]$ ($\underline{X} \in S^*_{\Omega,p}$). We now prove the converse of 1.3.5(iii): if $\underline{X} \in S^*_{\Omega,p}$, then there is exactly one sequence $(q_k)_{k\in\mathbb{N}_0}$ on $\mathcal{L}_p(\Omega)$ such that

$$\forall_{\varepsilon>0}\ [\| q_k \|_p = O(e^{k\varepsilon}) \qquad (k \in \mathbb{N}_0)]$$

$$\forall_{\alpha>0}\ [\underline{X}_\alpha = \sum_{k=0}^{\infty} q_k N_\alpha \psi_k] .$$

For, if $\underline{X} \in S^*_{\Omega,p}$, and $\varepsilon > 0$, then (by appendix 1, 0.5(i) and 0.5(iv)a))

$$e^{-(k+\frac{1}{2})\varepsilon}([\underline{X},\psi_k],f) = [N_\varepsilon(\underline{X},f),\psi_k] \qquad (f \in \mathcal{L}_q(\Omega),\ k \in \mathbb{N}_0) ,$$

and for $t \in \mathbb{C}$, $f \in \mathcal{L}_q(\Omega)$ we have

$$|N_\varepsilon(\underline{X},f)(t)| \le \|\underline{X}_\varepsilon(t)\|_p \quad \|f\|_q .$$

We conclude that there is an $M > 0$ such that

$$|([\underline{X},\psi_k],f)| \le M\, e^{k\varepsilon}\, \|f\|_q \qquad (f \in \mathcal{L}_q(\Omega),\ k \in \mathbb{N}_0) .$$

By [Z], Ch. 12, §50, Theorem 2 it now follows that

$$\forall_{k\in\mathbb{N}_0}[\| [\underline{X},\psi_k] \|_p \le M\, e^{k\varepsilon}] .$$

This proves

$$\forall_{\varepsilon>0}[\| [\underline{X},\psi_k] \|_p = O(e^{k\varepsilon}) \qquad (k \in \mathbb{N}_0)] .$$

Now it is not hard to show that

$$\forall_{\alpha>0}[\underline{X}_\alpha = \sum_{k=0}^{\infty} [\underline{X},\psi_k] N_\alpha \psi_k] ,$$

and that there exists at most one sequence $(q_k)_{k\in\mathbb{N}_0}$ on $\mathcal{L}_p(\Omega)$ with the assigned properties (see also 1.2.8(iii)).

Remark. It is also possible to prove this theorem by making direct use of 1.2.8(iii).

(iv) Let T be a linear operator of S with an adjoint. We have $[T\underline{X},g] = [\underline{X},T^*g]$ for $\underline{X} \in S^*_{\Omega,p}$ $(1 \le p \le \infty)$ and $g \in S$. For, if $f \in \mathcal{L}_q(\Omega)$, then we have by definition 1.4.4 and appendix 1, 3.2

$$\begin{aligned}([T\underline{X},g],f) &= [(T\underline{X},f),g] = [T(\underline{X},f),g] \\ &= [(\underline{X},f),T^*g] = ([\underline{X},T^*g],f) .\end{aligned}$$

(v) It is an easy exercise to prove that for $\underline{X} \in S^*_{\Omega,p}$, $g \in S$

$$[\underline{X},g] = \sum_{k=0}^{\infty} [\underline{X},\psi_k][\psi_k,g] .$$

As an example we have $[\underline{X},\delta_\alpha(\overline{t})] = \underline{X}_\alpha(t)$ for $\alpha > 0$ and $t \in \mathbb{C}$ (see appendix 1, 0.7).

1.5. The Wigner distribution for smooth and generalized stochastic processes.

1.5.1. Let (Ω,Λ,P) be a probability space.

Definition. Let $\underline{x} \in S_{\Omega,2}$, $\underline{y} \in S_{\Omega,2}$. The Wigner distribution $V(\underline{x},\underline{y})$ of $\underline{x}$ and $\underline{y}$ is defined by

$$V(\underline{x},\underline{y}) := F^{(2)} Z_U(\underline{x} \otimes \overline{\underline{y}})$$

(see 1.1.8(ii) and (v), and appendix 1, 3.12; see also [B], 16).

Note that $V(\underline{x},\underline{y}) \in S^2_{\Omega,1}$.

1.5.2. We list some properties of $V(\underline{x},\underline{y})$ ($\underline{x} \in S_{\Omega,2}$, $\underline{y} \in S_{\Omega,2}$).

(i) For $a \in \mathbb{R}$, $b \in \mathbb{R}$ we have

$$V(T_a R_b \underline{x},\ T_a R_b \underline{y}) = (T_{a\sqrt{2}} \otimes T_{b\sqrt{2}}) V(\underline{x},\underline{y}) \ .$$

Proof. Note that for $a \in \mathbb{R}$ and $b \in \mathbb{R}$

$$\overline{T_a R_b \underline{y}} = T_a R_{-b} \overline{\underline{y}} \ .$$

For $f \in \mathcal{L}_\infty(\Omega)$ we have (see 1.2.7)

$$(F^{(2)} Z_U(T_a R_b \underline{x} \otimes T_a R_{-b} \overline{\underline{y}}), f) =$$

$$= F^{(2)} Z_U(T_a R_b \otimes T_a R_{-b})(\underline{x} \otimes \overline{\underline{y}}, f) =$$

$$= (T_{a\sqrt{2}} \otimes T_{b\sqrt{2}}) F^{(2)} Z_U(\underline{x} \otimes \overline{\underline{y}}, f) \ ,$$

because the latter relation holds on S^2. □

In the same way we may prove that for $a \in \mathbb{R}$, $b \in \mathbb{R}$

$$V(R_b T_a \underline{x},\ R_b T_a \underline{y}) = (T_{a\sqrt{2}} \otimes T_{b\sqrt{2}}) V(\underline{x},\underline{y}) \ .$$

(ii) We have for $t \in \mathbb{C}$, $\lambda \in \mathbb{C}$

$$V(F\underline{x},\ F\underline{y})(t,\lambda) = V(\underline{x},\underline{y})(-\lambda,t) \ .$$

This follows in the same way as the relations in (i).

(iii) Furthermore we have (the proofs are similar to the proof of (i))

$$(F^*)^{(2)} V(\underline{x},\underline{y}) = Z_U(\underline{x} \otimes \overline{\underline{y}}) \ ,$$

$$N_{\alpha,2} V(\underline{x},\underline{y}) = V(N_\alpha \underline{x},\ N_\alpha \underline{y}) \qquad (\alpha > 0) \ .$$

1.5.3. Theorem. If $\underline{x} \in S_{\Omega,2}$, then $E(V(\underline{x},\underline{x})) = F^{(2)} Z_U R$ (see 1.1.4).

Proof. This follows directly from the definition of $V(\underline{x},\underline{x})$ and from 1.2.6. $\square$

1.5.4. Definition. Let $\underline{X} \in S^*_{\Omega,2}$, $\underline{Y} \in S^*_{\Omega,2}$. We define the Wigner distribution $V(\underline{X},\underline{Y})$ of $\underline{X}$ and $\underline{Y}$ by

$$V(\underline{X},\underline{Y}) := \Psi_{\alpha>0}\, V(\underline{X}_\alpha,\underline{Y}_\alpha) \; .$$

Note that $V(\underline{X},\underline{Y}) \in S^{2*}_{\Omega,1}$ by 1.5.2(iii).

1.5.5. Theorem. If $\underline{X} \in S^*_{\Omega,2}$, then $\Psi_{\alpha>0}\,(V(\underline{X}_\alpha,\underline{X}_\alpha)) = F^{(2)}Z_u R$ (R is the autocorrelation function of $\underline{X}$, see 1.3.2).

Proof. By theorem 1.5.3 we have for $\alpha > 0$

$$E(V(\underline{X}_\alpha,\underline{X}_\alpha)) = F^{(2)}Z_U R_\alpha \; .$$

According to the proof of [B], theorem 16.1 and [B], 19 example (i) (the theorem stated there also holds for linear operators of S^n ($n \in \mathbb{N}$)), we have

$$F^{(2)}Z_U R_\alpha = N_{\alpha,2}F^{(2)}Z_U R \; .$$

$\square$

Remark. Note that R is completely determined by $F^{(2)}Z_U R$, for $F^{(2)}Z_U R = 0 \Leftrightarrow R = 0$.

1.6. Application to the theory of noise.

1.6.1. In this section we give a definition of noise, and we mention a number of results (the proofs of these results are to appear in a subsequent paper). Furthermore we shall briefly comment on existing literature on this subject.

1.6.2. Definition. (Compare 0.2.) Let Λ^* be the smallest σ-algebra on S^* such that $\Psi_{F\in S^*}[F,f]$ is measurable for every $f \in S$. If P^* is a probability measure on (S^*,Λ^*), then the triple (S^*,Λ^*,P^*) is called a noise.

1.6.3. The concepts of noise and generalized stochastic processes are related in the following sense. Let $1 \le p \le \infty$. If (S^*,Λ^*,P^*) is a p-noise (i.e. $f_n \overset{S}{\to} 0 \Rightarrow \| \Psi_{F\in S^*}[F,f_n] \|_p \to 0$ for every sequence $(f_n)_{n\in\mathbb{N}}$ on S), then

$\underline{X} := \forall_{\alpha>0} \forall_{(t,F)\in\mathbb{C}\times S^*} F_\alpha(t)$ is a generalized stochastic process of order p with probability space (S^*,Λ^*,P^*) (see 1.3.1). It can be proved that the stochastic vectors $([\underline{X},f_1],\ldots,[\underline{X},f_n])$ (see 1.4.6(iii)) and $\forall_{F\in S^*}([F,f_1],\ldots,[F,f_n])$ are equally distributed for every $n \in \mathbb{N}$ and every set $\{f_1,\ldots f_n\}$ of S. We call $\underline{X}$ the <u>associated generalized stochastic process</u>. On the other hand, if (Ω,Λ,P) is a probability space, then it can be proved that to every $\underline{X} \in S^*_{\Omega,p}$ there exists exactly one probability measure on (S^*,Λ^*) such that the stochastic vectors $([\underline{X},f_1],\ldots,[\underline{X},f_n])$ and $\forall_{F\in S^*}([F,f_1],\ldots,[F,f_n])$ are equally distributed for every $n \in \mathbb{N}$ and every set $\{f_1,\ldots,f_n\}$ of S.

If (S^*,Λ^*,P^*) is a p-noise $(1 \le p \le \infty)$, and if we have on S a linear operator T with an adjoint, then it can be shown that there exists exactly one probability measure P^*_T on (S^*,Λ^*) such that $\forall_{F\in S^*}([F,f_1],\ldots,[F,f_n])$ (in P^*_T-sense) and $\forall_{F\in S^*}([TF,f_1],\ldots,[TF,f_n])$ (in P^*-sense) are equally distributed for every $n \in \mathbb{N}$ and every set $\{f_1,\ldots,f_n\}$ of S. Furthermore (S^*,Λ^*,P^*_T) is a p-noise, and (if $\underline{X}$ denotes the generalized stochastic process associated with (S^*,Λ^*,P^*)) $T\underline{X}$ (see 1.4.4) is the generalized stochastic process associated with (S^*,Λ^*,P^*_T).

.6.4. We give the following definitions of white noise.

<u>Definiton 1</u>. If (S^*,Λ^*,P^*) is a noise such that $\forall_{F\in S^*}([F,f_1],\ldots,[F,f_n])$ is normally distributed with zero mean and if the variance-covariance matrix is $\mathrm{diag}(1,\ldots,1)$ for every orthonormal set $\{f_1,\ldots,f_n\}$ in S, then the noise is called <u>ideal white noise</u>.

<u>Definition 2</u>. If (S^*,Λ^*,P^*) is a noise such that $E(\forall_{F\in S^*}[F,g]) = 0$, $E(\forall_{F\in S^*}[F,g]\overline{[F,h]}) = [h,g]$ for every $g \in S$, $h \in S$, then the noise is called <u>second order white noise</u>.

Definition 1 appears to be more restrictive than definition 2. Suppose that (S^*,Λ^*,P^*) is a 2-noise (see 1.6.3), and denote its associated generalized stochastic process with $\underline{X}$. It is possible to prove that the noise is second order white noise if and only if the autocorrelation function R of $\underline{X}$ (see 1.3.2) is given by $\forall_{\alpha>0} \forall_{(t,s)\in\mathbb{C}^2}[\delta_\alpha(t),\delta_\alpha(\overline{s})]$, or equivalently, the averaged Wigner distribution $\mathsf{Y}_{\alpha>0}E(V(\underline{X}_\alpha,\underline{X}_\alpha))$ of $\underline{X}$ (see 1.5.6) is given by $\mathrm{emb}(\mathsf{Y}_{(t,\lambda)\in\mathbb{C}^2}\, 2^{-\frac{1}{2}})$ (see appendix I, 0.7 and 0.10).

1.6.5. Another interesting concept in noise theory is the concept of stationarity.

Definition 1. A noise (S^*,Λ^*,P^*) is called strict sense time stationary if $\mathsf{Y}_{F\in S^*}([T_aF,f_1],\dots,[T_aF,f_n])$ and $\mathsf{Y}_{F\in S^*}([F,f_1],\dots,[F,f_n])$ are equally distributed for every $a \in \mathbb{R}$, $n \in \mathbb{N}$ and $f_1 \in S,\dots,f_n \in S$.

Remark. Compare this definition with [D], Ch. II, §8(a).

Definition 2. If (S^*,Λ^*,P^*) is a 2-noise such that

$$E([T_a\underline{X},f]) = E([\underline{X},f]),\quad E([T_a\underline{X},f]\overline{[T_a\underline{X},g]}) = E([\underline{X},f]\overline{[\underline{X},g]})$$

for every $a \in \mathbb{R}$, $f \in S$, $g \in S$, then the noise is called wide sense time stationary ($\underline{X}$ denotes the associated generalized stochastic process).

Remark 1. Compare this definition with [D], Ch. II, §8(b).

Remark 2. In case of a 2-noise it can be proved that strict sense time stationary noise is also wide sense time stationary noise.

Remark 3. We can give analogous definitions for frequency stationary noise (then we have R_b $(b \in \mathbb{R})$ instead of T_a $(a \in \mathbb{R})$).

Wide sense stationarity properties of a noise have interesting consequences for the autocorrelation function R and the averaged Wigner distribution V of the generalized stochastic process associated with the noise. We mention in particular

(i) If the noise is wide sense time stationary, then we have for every $a \in \mathbb{R}$ (see appendix 1, 3.12)

$$T_a \otimes T_a R = R,\quad T_a^{(1)} V = V\ .$$

(ii) If the noise is wide sense frequency stationary, then we have for every $b \in \mathbb{R}$ (see appendix 1, 3.12)

$$R_b \otimes R_{-b} R = R,\quad T_b^{(2)} V = V\ .$$

On the other hand, if we have $T_a \otimes T_a R = R$ for every $a \in \mathbb{R}$ (or, equivalently, $T_a^{(1)} V = V$ for every $a \in \mathbb{R}$), then the noise is wide sense time stationary. A similar thing can be said for case (ii).

One of the most interesting results of the theory of noise (which may be proved from 1.6.4 and 1.6.5(i) and (ii)) is that noise is second order white noise if and only if it is wide sense time and frequency stationary noise. Up to now we were not able to prove an analogous theorem for the case of ideal white noise.

.6.6. We now devote attention to related theories in existing literature. We first consider an approach which starts from a definition of type 0.1 (see e.g. [D] or [P]). Let (Ω,Λ,P) be a probability space, and suppose that to every $t \in \mathbb{R}$ we have a complex valued measurable function $\underline{x}(t)$ defined on Ω.We may regard this as a stochastic process in the sense of definition 0.1. It is often supposed that the process satisfies certain stationarity conditions, e.g. strict sense time stationarity (i.e. the distribution of $(\underline{x}(t_1+a),\dots,\underline{x}(t_n+a))$ is independant of $a \in \mathbb{R}$ for every $n \in \mathbb{N}$, $t_1 \in \mathbb{R},\dots,t_n \in \mathbb{R}$) or wide sense time stationarity (in this case $\underline{x}(t)$ is supposed to be an element of $\mathcal{L}_2(\Omega)$ for every $t \in \mathbb{R}$, and $E(\underline{x}(t+a)\overline{\underline{x}(s+a)}) = E(\underline{x}(t)\overline{\underline{x}(s)})$ for every $t \in \mathbb{R}$, $s \in \mathbb{R}$, $a \in \mathbb{R}$).

If $\underline{x}(t) \in \mathcal{L}_2(\Omega)$ for every $t \in \mathbb{C}$, then its autocorrelation function R is defined by $R := \Upsilon_{(t,s)\in\mathbb{R}^2} E(\underline{x}(t)\overline{\underline{x}(s)})$. Compare this definition with 1.1.4 and 1.3.2.

Suppose that the process is wide sense time stationary (in the sense mentioned above), and specialize R to a function of the form $\Upsilon_{(t,s)\in\mathbb{R}^2} R(t-s)$. If $R \in \mathcal{L}_1(\mathbb{R})$, then the spectrum S of the process is defined to be the Fourier transform of R. We may compare the spectrum with the averaged Wigner distribution of the process by remarking that, in the time stationary case, the averaged Wigner distribution is, roughly spoken, the tensor product of the constant function and the spectrum (see 1.6.5(i)).

A few remarks about white noise. In literature on physical applications of the theory of stochastic processes, some authors (e.g. [P]) define white noise as a wide sense time stationary process with the δ-function as autocorrelation function. This coincides with what we called second order white noise. Others use the following definition: white noise is wide sense time stationary process with a "flat" spectrum (i.e. the spectrum is a constant function). This means that, in our terminology, the averaged Wigner distribution of the process is constant. Now 1.6.4 expresses the equivalence of both definitions.

We encounter ideal white noise when dealing with Brownian motion. In our theory we may define Brownian notion as a noise with an ideal white derivative (see 1.6.3). In [D] Brownian motion is defined as a process $\underline{x}$ for which $(\underline{x}(t_1) - \underline{x}(t_2),\ldots,\underline{x}(t_{n-1}) - \underline{x}(t_n))$ is normally distributed with zero mean and variance-covariance matrix $\mathrm{diag}(t_2 - t_1,\ldots,t_n - t_{n-1})$ for every $t_1 \in \mathbb{R},\ldots,t_n \in \mathbb{R}$, $t_1 \leq \ldots \leq t_n$. It is possible to prove the equivalence of both definitions.

We finally consider the theory of Wiener (see [W]). Wiener defines a stochastic process as a set of measurable real valued functions $\Psi_{t\in\mathbb{R}}\underline{x}(t,\alpha)$ $(\alpha \in [0,1])$ for which $\Psi_{\alpha\in[0,1]}\underline{x}(t,\alpha)$ is measurable for every $t \in \mathbb{R}$ (Wiener used the word "time series" instead of stochastic process). Note that this definition has features in common with both definition 0.1 and definition 0.2. It is furthermore supposed that $\int_0^1 (\underline{x}(0,\alpha))^2 d\alpha < \infty$. Wiener mainly considers processes which satisfy the so called ergodic hypothesis (i.e. a strong kind of strict sense time stationarity). Under this hypothesis it is possible to define the autocorrelation function (and the spectrum) of the process from the observation of a single time function $\Psi_{t\in\mathbb{R}}\underline{x}(t,\alpha)$:

$$\int_0^1 \underline{x}(\tau,\alpha)\underline{x}(0,\alpha)d\alpha = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \underline{x}(t + \tau,\beta)dt$$

for almost every $\beta \in [0,1]$. Wiener also studies Brownian motion (his definition of Brownian motion is essentially the same as the one used in [D]). We mention one of Wiener's results in particular: if $\underline{x}$ is a Brownian motion process, and if f and g are sufficiently smooth real valued functions defined on the reals, then

$$\int_0^1 \left(\int_{-\infty}^{\infty} f(t)d\underline{x}(t,\alpha)\right)\left(\int_{-\infty}^{\infty} g(s)d\underline{x}(s,\alpha)\right)d\alpha = \int_{-\infty}^{\infty} f(t)g(t)dt \,.$$

This is in accordance with the results of our theory: if (S^*,Λ^*,P^*) is

Brownian motion, and if $\underline{X}$ denotes the generalized stochastic process associated with (S^*,Λ^*,P^*), then (according to 1.6.3 and 1.6.4 and our definition of Brownian motion) $E([P\underline{X},f]\overline{[P\underline{X},g]}) = [g,f]$ for every $f \in S$, $g \in S$. Integrals of the type $\int_{-\infty}^{\infty} f(t)d\underline{X}(t)$ are often called Wiener integrals, and the word stochastic integrals is used for integrals like $\int_{-\infty}^{\infty} \underline{Y}(t)d\underline{X}(t)$ (here both $\underline{X}$ and $\underline{Y}$ are stochastic processes). We believe that it is possible to develop a theory of stochastic integrals with the formalism of section 1.2.

Appendix 1. Linear operators and linear functionals of S and S^*.

Summary.

In this appendix we prove a number of theorems concerning linear operators and linear functionals of the set of ordinary (i.e. non-stochastic) smooth functions (class S) and the set of generalized functions (class S^*). Several of these are used in the main text of this report.

First of all we give a survey of the main definitions and theorems of De Bruijn's theory which are used in this report. Next we introduce a class of linear operators of S, the quasi-bounded linear operators, which can be characterized in various ways. The second subject studied in this appendix is the extension of linear operators of S to linear operators of S^*. We shall show that an extended operator preserves convergence in S if and only if it has an adjoint (relative to the inner product of S). Incidentally we prove that every continuous linear functional of S^* can be represented as an element of S. We shall mainly deal with operators acting on functions of a single complex variable, but many results can be generalized straightforwardly to the higher dimensional case.

For notational conventions we refer to the notations section of this report (section B).

0. Introduction.

0.1. We give a short survey of the fundamental notions and theorems of De Bruijn's theory as far as relevant for this report. A detailed treatment can be found in [B].

0.2. If A and B are positive numbers we denote by $S_{A,B}$ the class of analytic functions f of one complex variable for which there exists a positive number M such that

$$|f(t)| \le M \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi B(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C}) .$$

The set of smooth functions of one complex variable is defined by $S := \bigcup_{A>0} \bigcup_{B>0} S_{A,B}$. See [B], (2.1).

0.3. In S we take the usual inner product and norm:

$$[f,g] := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx \qquad (f \in S,\ g \in S) ,$$

$$\| f \| := ([f,f])^{\frac{1}{2}} \qquad (f \in S) .$$

0.4. We consider a semigroup $(N_\alpha)_{\alpha>0}$ of linear operators of S (the smoothing operators). The N_α's satisfy $N_{\alpha+\beta} = N_\alpha N_\beta$ $(\alpha > 0,\ \beta > 0)$, where the product is the usual composition of mappings. These operators are defined as integral operators:

$$N_\alpha f := \forall_{z\in\mathbb{C}} \int_{-\infty}^{\infty} K_\alpha(z,t)f(t)dt \qquad (f \in S,\ \alpha > 0) ,$$

where the kernel K_α $(\alpha > 0)$ is given by

$$K_\alpha := \forall_{(z,t)\in\mathbb{C}^2} (\sinh\alpha)^{-\frac{1}{2}} \exp(\frac{-\pi}{\sinh\alpha}((z^2 + t^2)\cosh\alpha - 2zt)) .$$

See [B], section 3, 4, 5 and 6.

In fact these operators can be defined on the larger space S^+ consisting of all complex valued functions defined on the reals with the property that for every $\varepsilon > 0$

$$\int_{-x}^{x} |f(t)|dt = O(\exp(\varepsilon x^2)) \qquad (x \geq 0) .$$

See [B], section 20. It is easily seen that $\mathcal{L}_p(\mathbb{R}) \subset S^+$ for every p with $1 \leq p \leq \infty$. For $f \in S^+$, $\alpha > 0$ we have

$$N_\alpha f = \Psi_{z\in\mathbb{C}} \int_{-\infty}^{\infty} K_\alpha(z,t)f(t)dt ,$$

and this is a smooth function. So N_α maps S^+ into S $(\alpha > 0)$.

0.5. We summarize a number of properties of the $(N_\alpha)_{\alpha>0}$.

(i) $[N_\alpha f,g] = [f,N_\alpha g]$ for $\alpha > 0$, $f \in S$, $g \in S$ ([B], 6.5).

(ii) For every $\alpha > 0$ and every p $(1 \leq p \leq \infty)$ there are positive constants $C_{\alpha p}$, A and B such that for every $f \in \mathcal{L}_p(\mathbb{R})$

$$|(N_\alpha f)(t)| \leq C_{\alpha p} \|f\|_p \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi B(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C})$$

(this is a slight generalization of [B], 6.3).

(iii) If $f \in S$ and $\alpha > 0$, then there is at most one $g \in S$ satisfying $f = N_\alpha g$. In addition if $f \in S$, then there exists an $\alpha > 0$ and a $g \in S$ such that $f = N_\alpha g$. If furthermore $f \in S$, and the positive numbers M, A and B are such that

$$|f(t)| \leq M \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi B(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C}) ,$$

then we can find an $\alpha > 0$, $C > 0$, $A' > 0$, $B' > 0$, only depending on A and B, such that for the $g \in S$ with $f = N_\alpha g$

$$|g(t)| \leq MC \exp(-\pi A'(\mathrm{Re}\ t)^2 + \pi B'(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C}) .$$

See [B], 10.1.

(iv) We denote by ψ_k $(k \in \mathbb{N}_0)$ the Hermite functions (see [B], 27.6.3). For every $k \in \mathbb{N}_0$ we have $\psi_k \in S$. The set $\{\psi_k \mid k \in \mathbb{N}_0\}$ forms a complete orthonormal set in $\mathcal{L}_2(\mathbb{R})$ (see e.g. [K], 21.4). We list some properties of ψ_k $(k \in \mathbb{N}_0)$. For proofs and comments: see [B], 27.6.3.

a) The ψ_k are eigenfunctions of N_α for every $\alpha > 0$:

$$N_\alpha \psi_k = e^{-(k+\frac{1}{2})\alpha} \psi_k \qquad (k \in \mathbb{N}_0) .$$

b) If $\alpha > 0$, then

$$K_\alpha(z,t) = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\alpha} \psi_n(z)\psi_n(t) \qquad (z \in \mathbb{C},\ t \in \mathbb{C}) .$$

c) If $f \in S$, then there exists an $\varepsilon > 0$ such that

$$[f,\psi_k] = O(e^{-k\varepsilon}) \qquad (k \in \mathbb{N}_0) .$$

On the other hand, if $(c_k)_{k\in\mathbb{N}_0}$ is a sequence of complex numbers satisfying $c_k = O(e^{-k\varepsilon})$ for some $\varepsilon > 0$, then the function $\sum_{k=0}^{\infty} c_k\psi_k$ is an element of S.

d) If $f \in \mathcal{L}_2(\mathbb{R})$, $c_k := [f,\psi_k]$ $(k \in \mathbb{N}_0)$, then $\sum_{k=0}^{\infty} |c_k|^2 < \infty$, and $f = \sum_{k=0}^{\infty} c_k\psi_k$ in the sense of $\mathcal{L}_2(\mathbb{R})$. We have for such an f

$$N_\alpha f = \sum_{k=0}^{\infty} c_k\, e^{-(k+\frac{1}{2})\alpha} \psi_k$$

(this is an easy consequence of 0.5(ii) and 0.5(iv) a)). From this it is not hard to prove that $\| N_\alpha f \| \le e^{-\frac{1}{2}\alpha} \| f \|$ (see also [B], 6.2), and that $\lim_{\alpha \downarrow 0} \| N_\alpha f - f \| = 0$ for $f \in \mathcal{L}_2(\mathbb{R})$.

e) For every $t \in \mathbb{C}$ we have $\forall_{\varepsilon>0}\ [\psi_k(t) = O(e^{k\varepsilon})\ (k \in \mathbb{N}_0)]$.
This follows from 0.5(ii) and 0.5(iv) a).

0.6. We give a number of examples of linear operators of S.

(i) The smoothing operators N_α $(\alpha > 0)$.

(ii) The Fourier transform F

$$Ff := \Psi_{z\in\mathbb{C}} \int_{-\infty}^{\infty} e^{-2\pi i z t} f(t)\,dt \qquad (f \in S) .$$

For some properties of F we refer to [B], section 8 and 9.

(iii) The shift operators T_a and R_b ($a \in \mathbb{C}$, $b \in \mathbb{C}$)

$$T_a f := \Psi_{t\in\mathbb{C}}\, f(t + a) \qquad (f \in S) ,$$

$$R_b f := \Psi_{t\in\mathbb{C}}\, e^{-2\pi ibt} f(t) \qquad (f \in S) .$$

The operator P

$$Pf := \Psi_{t\in\mathbb{C}}\, \frac{f'(t)}{2\pi i} \qquad (f \in S) .$$

The operator Q

$$Qf := \Psi_{t\in\mathbb{C}}\, t\, f(t) \qquad (f \in S) .$$

For properties of these operators, see [B], section 11.

0.7. A <u>generalized function</u> F is a mapping of the set of positive real numbers into S such that

$$N_\alpha F_\beta = F_{\alpha+\beta} \qquad (\alpha > 0,\ \beta > 0) .$$

The set of all generalized functions is denoted by S^*. Instead of F_α we often write $N_\alpha F$ ($F \in S^*$, $\alpha > 0$).

If $f \in S^+$ (see 0.4) then its standard embedding in S^* is defined by

$$\mathrm{emb}(f) := \Psi_{\alpha>0}\, N_\alpha f$$

(this is a combination of [B], 17.2 and [B], 20.2).

If $F \in S^*$, $g \in S$ we can define the inner product $[F,g]$: write $g = N_\alpha h$ with some $\alpha > 0$, $h \in S$ (see 0.5(iii)). Now $[F,g] := [F_\alpha,h]$ (this depends only on F and g: see [B], section 18).

If $F \in S^*$, then we have for every $\varepsilon > 0$

$$[F,\psi_k] = O(e^{k\varepsilon}) \qquad (k \in \mathbb{N}_0) ,$$

$$F_\varepsilon = \sum_{k=0}^{\infty} [F,\psi_k] N_\varepsilon \psi_k ,$$

and, on the other hand, if $(c_k)_{k\in\mathbb{N}_0}$ is a sequence of complex numbers satisfying

$c_k = O(e^{k\varepsilon})$ $(k \in \mathbb{N}_0)$ for every $\varepsilon > 0$, then

$$\Upsilon_{\alpha>0} \sum_{k=0}^{\infty} c_k N_\alpha \psi_k$$

defines an element of S^* (see [B], 27.6.3). As an important example of a generalized function we have the delta function at the point t, defined by

$$\delta(t) := \Upsilon_{\alpha>0} \Upsilon_{z\in\mathbb{C}} K_\alpha(z,t) = \Upsilon_{\alpha>0} \Upsilon_{z\in\mathbb{C}} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\alpha} \psi_n(z)\psi_n(t) .$$

For $\alpha > 0$, $t \in \mathbb{C}$ we have $[F,\delta_\alpha(\overline{t})] = F_\alpha(t)$ $(F \in S^*)$. More generally: if $F \in S^*$ and $g \in S$, then

$$[F,g] = \sum_{k=0}^{\infty} [F,\psi_k][\psi_k,g] .$$

If $\alpha > 0$ then $N_\alpha(S^*) := \{F_\alpha \mid F \in S^*\}$. Obviously $N_\alpha(S^*) \subset S$.

0.8. Let T be a linear operator of S, and suppose that there exists a family $(Y_\alpha)_{\alpha>0}$ of linear mappings of $N_\alpha(S^*)$ into S such that

1) $Y_{\alpha+\beta} N_{\alpha+\beta} F = N_\beta Y_\alpha N_\alpha F$,

2) $Y_\alpha N_\alpha f = N_\alpha Tf$

for every $\alpha > 0$, $\beta \geq 0$, $F \in S^*$, $f \in S$. Then we call T extendable by means of $(Y_\alpha)_{\alpha>0}$ (see [B], (19.3) and (19.4)). It is possible to define a linear operator $\tilde{T}$ on S^* such that $\tilde{T}(\mathrm{emb}(f)) = \mathrm{emb}(Tf)$ $(f \in S)$. This $\tilde{T}$ is defined by

$$\tilde{T}F := \Upsilon_{\alpha>0} Y_\alpha N_\alpha F \qquad (F \in S^*)$$

(see [B], 19.2).

0.9. Convergence in S and S^*.

If $(f_n)_{n\in\mathbb{N}}$ is a sequence on S, then we write $f_n \overset{S}{\to} 0$ if there are positive numbers A and B such that

$$f_n(t)\exp(\pi A(\mathrm{Re}\ t)^2 - \pi B(\mathrm{Im}\ t)^2) \to 0$$

uniformly in $t \in \mathbb{C}$. If $f, f_n \in S$ $(n \in \mathbb{N})$, then we write $f_n \overset{S}{\to} f$ if $f_n - f \overset{S}{\to} 0$.

If $f_n \in S$ $(n \in \mathbb{N})$, $f_n \overset{S}{\to} 0$, then there exists an $\alpha > 0$ and a sequence $(g_n)_{n\in\mathbb{N}}$ on S such that $f_n = N_\alpha g_n$, $g_n \overset{S}{\to} 0$. For the proof we refer to [B], 23.1.

If $(F_n)_{n\in\mathbb{N}}$ is a sequence on S^*, then we write $F_n \overset{S^*}{\to} 0$ if $N_\alpha F_n \overset{S}{\to} 0$ for every $\alpha > 0$. If $F, F_n \in S^*$ $(n \in \mathbb{N})$, then we write $F_n \overset{S^*}{\to} F$ if $F_n - F \overset{S^*}{\to} 0$. For more details see [B], section 24.

0.10. We devote attention to smooth and generalized functions of n complex variables $(n \in \mathbb{N})$. The space S^n (see [B], section 7) is defined as the set of all complex valued functions f of n complex variables which are analytic in all variables, and for which there exist positive numbers M, A and B such that

$$|f(t_1,\dots,t_n)| \le M \exp\left(-\pi A \sum_{k=1}^{n} (\mathrm{Re}\, t_k)^2 + \pi B \sum_{k=1}^{n} (\mathrm{Im}\, t_k)^2\right)$$

for every $(t_1,\dots,t_n) \in \mathbb{C}^n$.

As an example of a smooth function of n variables we have

$$f_1 \otimes \dots \otimes f_n := \forall_{(t_1,\dots,t_n)\in\mathbb{C}^n}\; f_1(t_1)\cdot\ldots\cdot f_n(t_n) \;,$$

where $f_1 \in S, \dots, f_n \in S$.

The smoothing operators $N_{\alpha,n}$ $(\alpha > 0)$ are defined as integral operators with kernels (see [B], section 7)

$$K_{\alpha,n} := \forall_{(z_1,\dots,z_n;t_1,\dots,t_n)\in\mathbb{C}^n\times\mathbb{C}^n}\; \prod_{i=1}^{n} K_\alpha(z_i,t_i) \qquad (\alpha > 0) \;.$$

We have $N_{\alpha+\beta,n} = N_{\alpha,n} N_{\beta,n}$ for $\alpha > 0$, $\beta > 0$.

The inner product and norm in S^n is defined by

$$[f,g] := \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx \qquad (f \in S^n,\; g \in S^n) \;,$$

$$\|f\| := ([f,f])^{\frac{1}{2}} \qquad (f \in S^n) \;.$$

A generalized function F of n variables is a mapping of the positive real numbers into S^n such that $N_{\alpha,n}F_\beta = F_{\alpha+\beta}$ for every $\alpha > 0$, $\beta > 0$ (see [B], section 21). The set of all generalized functions of n variables is denoted by S^{n*}.

As an example of a generalized function of n variables we have

$$F_1 \otimes \ldots \otimes F_n := \curlyvee_{\alpha>0} (F_1)_\alpha \otimes \ldots \otimes (F_n)_\alpha ,$$

where $F_1 \in S^*,\ldots,F_n \in S^*$.

It is possible to give all preceding definitions and theorems (with the proper modifications) for S^n and S^{n*} (see e.g. [B], section 7, section 21, 27.4.1 and 27.26.1). As an example we mention that for every $f \in S^n$

$$f = \sum_{k\in\mathbb{N}_0^n} [f,\psi_{k,n}]\psi_{k,n}$$

(here $\psi_{k,n}$ denotes $\psi_{k_1} \otimes \ldots \otimes \psi_{k_n}$ for $k = (k_1,\ldots,k_n) \in \mathbb{N}_0^n$), and that for every multi-sequence $(c_k)_{k\in\mathbb{N}_0^n}$ on $\mathbb{C}$ which satisfies $c_k = O(e^{-\varepsilon(k_1 + \ldots + k_n)})$ $(k = (k_1,\ldots,k_n) \in \mathbb{N}_0^n)$ for some $\varepsilon > 0$ the function $\sum_{k\in\mathbb{N}_0^n} c_k\psi_{k,n}$ is an element of S^n. We have a similar result for generalized functions of n variables (see also 0.7).

1. Linear operators and linear functionals; A theorem about linear functionals of a Banach space.

1.1. Let V and W be linear spaces. A linear operator is a linear mapping of V into W. If $W = \mathbb{C}$, we speak of a linear functional instead of linear operator. If $W = V$, we say linear operator of V.

If V and W are normed linear spaces, and T is a linear operator, we say that T is bounded if $\sup_{f\neq0} \frac{\|Tf\|_W}{\|f\|_V}$ is finite. In this case the norm of T is defined by

$$\|T\| := \sup_{f\neq0} \frac{\|Tf\|_W}{\|f\|_V} .$$

If V is a linear space with an inner product [,], and T_1 and T_2 are linear operators of V, then we say that T_1 and T_2 are <u>adjoint operators</u> if $[T_1 f,g] = [f,T_2 g]$ for every $f \in V$, $g \in V$. It is easy to see that for every linear operator T of V there is at most one adjoint operator, which (if it exists) is denoted by T^*.

2. Many results of this report are based on the following

<u>Theorem</u>. Let $(B,\| \ \|)$ be a Banach space. Suppose that for every $\alpha > 0$ there is given a collection A_α of bounded linear functionals of B such that $\alpha_1 < \alpha_2 \Rightarrow A_{\alpha_1} \subset A_{\alpha_2}$ $(\alpha_1 > 0,\ \alpha_2 > 0)$. Then the following statements are equivalent:

$$\text{(i)}\quad \forall_{f\in B}\ \exists_{\alpha>0}\ \exists_{M>0}\ \forall_{L\in A_\alpha}\ [|Lf| \le M]$$

$$\text{(ii)}\quad \exists_{\alpha>0}\ \exists_{M>0}\ \forall_{f\in B}\ \forall_{L\in A_\alpha}\ [|Lf| \le M\|f\|]\ .$$

<u>Proof</u>. It is easily seen that (ii) $\Rightarrow$ (i): if $N > 0$ and $\beta > 0$ are such that $|Lf| \le N\|f\|$ for every $f \in B$, $L \in A_\beta$, then we can take $\alpha = \beta$ and $M = N\|f\|$ in (i).

We now suppose (i), and we shall show that

$$(*)\qquad \forall_{\alpha>0}\ \forall_{M>0}\ \exists_{f\in B}\ \exists_{L\in A_\alpha}\ [|Lf| > M\|f\|]$$

yields a contradiction. To every $n \in \mathbb{N}$ we can then find an $f_n \in B$ and an $L_n \in A_{1/n}$ such that $|L_n f_n| > n\|f_n\|$. Note that L_n is a bounded linear functional of B for every $n \in \mathbb{N}$, and that $(L_n f)_{n\in\mathbb{N}}$ is a bounded sequence for every $f \in B$. It follows therefore from the Banach-Steinhaus theorem that there is an $M > 0$ such that $|L_n f| \le M\|f\|$ for every $n \in \mathbb{N}$ and every $f \in B$. Contradiction. □

3. We use theorem 1.2 in the following form: If $\{L_t \mid t \in T\}$ is a set of bounded linear functionals of a Banach space B, and if for every $\alpha > 0$ there is a mapping g_α of T into $\mathbb{R}$ such that

$$\forall_{\alpha>0,\beta>0}\ \forall_{t\in T}\ [\alpha < \beta \Rightarrow 0 < g_\beta(t) \le g_\alpha(t)]\ ,$$

then the following statements are equivalent:

$$\forall_{f\in B}\ \exists_{\alpha>0}\ [L_t f = O(g_\alpha(t)) \qquad (t \in T)]$$

$$\exists_{\alpha>0}\ [\|L_t\| = O(g_\alpha(t)) \qquad (t \in T)]\ .$$

This follows directly from theorem 1.2 by taking

$$A_\alpha := \{\Psi_{f\in B} \frac{L_t f}{g_\beta(t)} \mid 0 < \beta \le \alpha,\ t \in T\} \qquad (\alpha > 0)\ .$$

As a special case of 1.3 we have the Banach-Steinhaus theorem: if B is a Banach space, and $\{L_t \mid t \in T\}$ is a set of bounded linear functionals of B such that

$$\forall_{f\in B}\ \exists_{M>0}\ \forall_{t\in T}\ [|L_t f| \le M]$$

then

$$\exists_{M>0}\ \forall_{f\in B}\ \forall_{t\in T}\ [|L_t f| \le M\|f\|]\ .$$

This follows from 1.3 by taking $g_\alpha(t) = 1$ $(\alpha > 0,\ t \in T)$.

Quasi-bounded linear operators of S.

Definition. A linear functional L of S is called quasi-bounded if LN_α is a bounded linear functional of S for every $\alpha > 0$.

Let T be a linear operator of S. We introduce the following types

type I. $\forall_{t\in\mathbb{C}}\ [\Psi_{f\in S}(Tf)(t)$ is a quasi-bounded linear functional of S].

type II. $\forall_{g\in S}[\Psi_{f\in S}\ [Tf,g]$ is a quasi-bounded linear functional of S].

type III. For every sequence $(\varphi_n)_{n\in\mathbb{N}}$ on S we have $\varphi_n \overset{S}{\to} 0 \Rightarrow T\varphi_n$ is pointwise bounded.

type IV. For every sequence $(\varphi_n)_{n\in\mathbb{N}}$ on S we have $\varphi_n \overset{S}{\to} 0 \Rightarrow T\varphi_n \overset{S}{\to} 0$.

type V. For every $\alpha > 0$ there exists a $\beta > 0$ and a bounded linear operator T_1 of S such that $TN_\alpha = N_\beta T_1$.

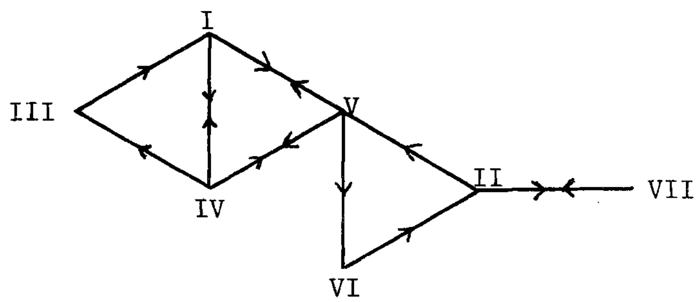
type VI. For every $\alpha > 0$ the linear operator TN_α is bounded.

type VII. For every $\alpha > 0$ the linear operator TN_α has an adjoint.

The most important result of this section is that all these types define the same class of linear operators of S which we shall call the set of quasi-bounded linear operators.

We shall prove successively that III $\Rightarrow$ I, IV $\Rightarrow$ III, I $\Leftrightarrow$ IV $\Leftrightarrow$ V, V $\Rightarrow$ VI, VI $\Rightarrow$ II, II $\Rightarrow$ V, II $\Leftrightarrow$ VII.

Figure.



3. Theorem. If T is a linear operator of S of type III, then T is of type I.

Proof. Suppose T is of type III, and let $t \in \mathbb{C}$. We have to show that for $\alpha > 0$ there is a $C_\alpha > 0$ such that

$$\forall_{g \in S} \; [\, |(TN_\alpha g)(t)| \le C_\alpha \|g\| \,] \; .$$

Suppose the contrary. Then there is a sequence $(g_n)_{n \in \mathbb{N}}$ on S such that $\|g_n\| \to 0$ and $|(TN_\alpha g_n)(t)| > n$ $(n \in \mathbb{N})$. It follows from [B], theorem 23.2 that $N_\alpha g_n \overset{S}{\to} 0$, and this means that $((TN_\alpha g_n)(t))_{n \in \mathbb{N}}$ is a bounded sequence. Contradiction. □

4. Theorem. If T is a linear operator of S of type IV, then T is of type III.

Proof. Trivial. □

5. Theorem. Let T be a linear operator of S. T is of type I if and only if T is of type IV. T is of type I if and only if T is of type V.

Proof. It is sufficient to prove that

a) V $\Rightarrow$ IV

b) IV $\Rightarrow$ I

c) I $\Rightarrow$ V .

a) Suppose that T is of type V, and let $\varphi_n \in S$ $(n \in \mathbb{N})$, $\varphi_n \overset{S}{\to} 0$. According to 0.9 we can find $\alpha > 0$ and $r_n \in S$ such that $\varphi_n = N_\alpha r_n$, $r_n \overset{S}{\to} 0$. Since T is of type V, there exists a $\beta > 0$ and a bounded linear operator T_1 of S such that $TN_\alpha = N_\beta T_1$. We find $T\varphi_n = TN_\alpha r_n = N_\beta T_1 r_n$, and by [B], 23.2 we conclude that $T\varphi_n \overset{S}{\to} 0$. This proves that T is of type IV.

b) Suppose that T is of type IV. It follows from 2.2 and 2.3 that T is of type I.

c) Finally suppose that T is of type I, and let $\alpha > 0$. For every $t \in \mathbb{C}$ we can find a $C > 0$ such that

$$|(TN_\alpha f)(t)| \leq C\|f\| \qquad (f \in \mathcal{L}_2(\mathbb{R}))$$

(this follows from 0.5(iv)d)). Therefore

$$\Upsilon_{f\in\mathcal{L}_2(\mathbb{R})}(TN_\alpha f)(t)$$

is a bounded linear functional of the Banach space $\mathcal{L}_2(\mathbb{R})$ for every $t \in \mathbb{C}$. Furthermore we can find for every $f \in \mathcal{L}_2(\mathbb{R})$ positive numbers M and A such that

$$|(TN_\alpha f)(t)| \leq M \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi A^{-1}(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C})$$

(this follows from the fact that $TN_\alpha f \in S$ for $f \in \mathcal{L}_2(\mathbb{R})$). We apply theorem 1.3 by taking

$$g_A := \Upsilon_{t\in\mathbb{C}} \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi A^{-1}(\mathrm{Im}\ t)^2) \qquad (A > 0)\ ,$$

and we find that there is an $M > 0$ and an $A > 0$ such that

$$|(TN_\alpha f)(t)| \leq M\|f\| \exp(-\pi A(\mathrm{Re}\ t)^2 + \pi A^{-1}(\mathrm{Im}\ t)^2)$$

for every $f \in \mathcal{L}_2(\mathbb{R})$, $t \in \mathbb{C}$.

We now prove V. Note that $S \subset \mathcal{L}_2(\mathbb{R})$. According to 0.5(iii) there exist numbers $\beta > 0$, $C > 0$, $A' > 0$, $B' > 0$ such that for every $f \in S$ there is exactly one $g \in S$ with $TN_\alpha f = N_\beta g$ and

$$|g(t)| \leq M\|f\|\, C \exp(-\pi A'(\mathrm{Re}\ t)^2 + \pi B'(\mathrm{Im}\ t)^2) \qquad (t \in \mathbb{C})\ .$$

If we define $T_1 f := g$, then it is easily seen that T_1 is a linear operator of S. It also follows that T_1 is bounded. This proves that T is of type V. $\square$

. Theorem. If T is a linear operator of S of type V, then T is of type VI.

Proof. This follows from the boundedness of the N_α for every $\alpha > 0$ (see 0.5(iv)d)). □

'. Theorem. If T is a linear operator of S of type VI, then T is of type II.

Proof. Let T be a linear operator of type VI. If $\alpha > 0$, $g \in S$, then we have by the boundedness of TN_α

$$|[TN_\alpha f,g]| \le \|TN_\alpha\| \, \|f\| \, \|g\| \qquad (f \in S)$$

This implies that T is of type II. □

3. Theorem. If T is a linear operator of S of type II, then T is of type V.

Proof. Suppose T is a linear operator of type II. For $\alpha > 0$, $k \in \mathbb{N}_0$

$$\Psi_{f\in\mathcal{L}_2(\mathbb{R})} \; [TN_\alpha f,\psi_k]$$

is a bounded linear functional of $\mathcal{L}_2(\mathbb{R})$ (this follows from 0.5(iv)d)). Here ψ_k is the k-th Hermite function (see 0.5(iv)). From the fact that $TN_\alpha f \in S$ $(f \in \mathcal{L}_2(\mathbb{R}))$ we conclude by 0.5(iv)c) that for every $f \in \mathcal{L}_2(\mathbb{R})$ there is an $M > 0$ and an $\varepsilon > 0$ such that

$$|[TN_\alpha f,\psi_k]| \le M\, e^{-k\varepsilon} \qquad (k \in \mathbb{N}_0) .$$

We apply theorem 1.3 by taking

$$g_\varepsilon := \Psi_{k\in\mathbb{N}_0} \; e^{-\varepsilon k} \qquad (\varepsilon > 0) ,$$

and find that there exist numbers $M > 0$ and $\varepsilon > 0$ such that

$$|[TN_\alpha f,\psi_k]| \le M\|f\|e^{-k\varepsilon} \qquad (k \in \mathbb{N}_0,\ f \in \mathcal{L}_2(\mathbb{R})) .$$

Now take $\beta = \frac{\varepsilon}{2}$, and define T_1 by

$$T_1 f := \sum_{k=0}^{\infty} [TN_\alpha f,\psi_k] e^{(k+\frac{1}{2})\beta} \psi_k \qquad (f \in S) .$$

According to 0.5(iv)c) T_1 maps S into S, and it is easy to see that T_1 is linear and bounded. Furthermore it follows from 0.5(iv)d) that

$$N_\beta T_1 f = \sum_{k=0}^{\infty} [TN_\alpha f, \psi_k]\psi_k = TN_\alpha f \qquad (f \in S) \ ,$$

so $N_\beta T_1 = TN_\alpha$. □

. Theorem. If T is a linear operator of S, then T is of type II if and only if T is of type VII.

Proof. First suppose T is a linear operator of type VII. We have for $g \in S$, $\alpha > 0$

$$|[TN_\alpha f, g]| = |[f, (TN_\alpha)^* g]| \le \| f \| \, \| (TN_\alpha)^* g \| \qquad (f \in S) \ ,$$

and this means that T is of type II.

Now we suppose that T is of type II. If $g \in S$, then

$$\Psi_{f \in S} \, [Tf, g]$$

is a quasi-bounded linear functional, and, according to [B], 22.2, there is exactly one $F_g \in S^*$ such that

$$[Tf, g] = [f, F_g] \qquad (f \in S) \ .$$

It is easy to see that F_g depends linearly on g. Now we have for $\alpha > 0$, $f \in S$, $g \in S$

$$[TN_\alpha f, g] = [f, N_\alpha F_g] \ ,$$

and this means that TN_α has an adjoint, viz. $\Psi_{g \in S} \, N_\alpha F_g$, so T is of type VII. □

. We give a number of examples

(i) If T is a linear operator of S with an adjoint T^*, then TN_α and $N_\alpha T^*$ are adjoint operators ($\alpha > 0$), so T is quasi-bounded. The converse is not true; e.g. the linear operator T defined by

$$Tf := \psi_0 \sum_{n=0}^{\infty} [f, \psi_n] \qquad (f \in S)$$

is quasi-bounded, but does not have an adjoint operator.

(ii) If T is a linear operator of S which is bounded, then T is quasi-bounded, for TN_α is bounded ($\alpha > 0$). The converse is not true; e.g. the linear operator T defined by

$$Tf := \Psi_{t\in\mathbb{C}}\, e^t f(t) \qquad (f \in S)$$

is quasi-bounded, but not bounded.

(iii) The operators N_α, F, T_a ($a \in \mathbb{C}$), R_b ($b \in \mathbb{C}$), P,Q (see 0.6) are quasi-bounded. In fact it is not easy to find examples of linear operators of S which are not quasi-bounded. We can give an example in a way similar to [B], 27.22.

(iv) If $f \in S$, and T is a quasi-bounded linear operator of S, then $\sum_{k=0}^{n} [f,\psi_k]\psi_k \overset{S}{\to} f$. Therefore we have $\sum_{k=0}^{n} [f,\psi_k]T\psi_k \overset{S}{\to} Tf$.

. Theorem. If $T_1,\dots,T_m$ ($m \in \mathbb{N}$) are quasi-bounded linear operators of S, and g is a polynomial in m variables, then $g(T_1,\dots,T_m)$ is quasi-bounded.

Proof. The theorem follows from the fact that finite sums and products of quasi-bounded operators are quasi-bounded operators: if $\varphi_n \overset{S}{\to} 0$, and T_1 and T_2 are quasi-bounded linear operators, and $\alpha_1,\alpha_2 \in \mathbb{C}$, then $(\alpha_1T_1 + \alpha_2T_2)\varphi_n \overset{S}{\to} 0$, $T_1T_2\varphi_n \overset{S}{\to} 0$. □

. Theorem. If T_n ($n \in \mathbb{N}$) are quasi-bounded linear operators, and if for every $g \in S$ the sequence $(T_ng)_{n\in\mathbb{N}}$ converges pointwise to an element $Tg \in S$, then T is a quasi-bounded linear operator of S.

Proof. For every $n \in \mathbb{N}$ and every $t \in \mathbb{C}$

$$\Psi_{g\in S}\, (T_ng)(t)$$

is a quasi-bounded linear functional of S. By [B], 22.2 there exists to every $n \in \mathbb{N}$ and every $t \in \mathbb{C}$ an $F_{n,t} \in S^*$ such that $(T_ng)(t) = [g,F_{n,t}]$ for $g \in S$.

Take a fixed $t \in \mathbb{C}$, and note that

$$\lim_{n\to\infty} [g, F_{n,t}] = (Tg)(t)$$

for every $g \in S$. This means by [B], 24.4 that the sequence $(F_{n,t})_{n\in\mathbb{N}}$ is S^*-convergent. Denote its S^*-limit by F_t. It follows that $(Tg)(t) = [g, F_t]$, so, again by [B], 22.2, $\Upsilon_{g\in S}\,(Tg)(t)$ is a quasi-bounded linear functional of S. This proves that T is a quasi-bounded linear operator of S. □

3. We make a few remarks about linear operators of S^n. It is not difficult to generalize the preceding definitions and theorems to the n-dimensional case. For the sake of elegance we restrict ourselves to the case $n = 2$. We give an important class of examples of quasi-bounded linear operators of S^2.

(i) Let T be a quasi-bounded linear operator of S. If $f \in S^2$, $z_2 \in \mathbb{C}$, then the series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} [f, \psi_n \otimes \psi_m]\psi_m(z_2)\psi_n$ is S-convergent (this follows from the fact that $\psi_m(z_2) = O(e^{m\varepsilon})$ for every $\varepsilon > 0$, see 0.5(iv)e) and c)). Therefore we have according to 2.10(iv) for $z_1 \in \mathbb{C}$

$$T(\Upsilon_{t\in\mathbb{C}}\, f(t,z_2))(z_1) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} [f, \psi_n \otimes \psi_m](T\psi_n)(z_1)\psi_m(z_2)\ .$$

It is not hard to prove that $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} [f, \psi_n \otimes \psi_m]T\psi_n \otimes \psi_m$ is an S^2-convergent series: write $f = N_{\varepsilon,2}g$ with some $\varepsilon > 0$, $g \in S^2$, note that

$$[f, \psi_n \otimes \psi_m]T\psi_n \otimes \psi_m = [g, \psi_n \otimes \psi_m]TN_\varepsilon\psi_n \otimes N_\varepsilon\psi_m$$

for $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, use the fact that there is a $\beta > 0$ and a bounded linear operator T_1 of S such that

$$TN_\varepsilon\psi_n = N_\beta T_1\psi_n \qquad (n \in \mathbb{N}_0)\ ,$$

and apply 0.5(ii).

So it follows that

$$T^{(1)} := \Upsilon_{f\in S^2}\ \Upsilon_{(z_1,z_2)\in\mathbb{C}^2}\ T(\Upsilon_{t\in\mathbb{C}}\, f(t,z_2))(z_1)$$

maps S^2 (linearly) into S^2. Furthermore it is not hard to prove that $T^{(1)}$ is a quasi-bounded linear operator of S^2. This follows from the fact that $T^{(1)}$ is limit in the sense of 2.12 of the quasi-bounded operators

$$\Psi_{f\in S^2} \sum_{n=0}^{N} \sum_{m=0}^{N} [f,\psi_n \otimes \psi_m] T\psi_n \otimes \psi_m \qquad (N \in \mathbb{N}).$$

In the same way we may prove that the linear operator $T^{(2)}$ defined by

$$T^{(2)} := \Psi_{f\in S^2} \Psi_{(z_1,z_2)\in\mathbb{C}^2} T(\Psi_{t\in\mathbb{C}} f(z_1,t))(z_2)$$

is a quasi-bounded linear operator of S^2.

If T_1 and T_2 are quasi-bounded linear operators of S, then, according to theorem 2.11, $T_1^{(1)}T_2^{(2)}$ is a quasi-bounded linear operator of S^2, and

$$(*) \qquad T_1^{(1)}T_2^{(2)}f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [f,\psi_n \otimes \psi_m] T_1\psi_n \otimes T_2\psi_m \qquad (f \in S^2) .$$

Note that it follows from $T_1^{(1)}T_2^{(2)}(f \otimes g) = T_1 f \otimes T_2 g = T_2^{(2)}T_1^{(1)}(f \otimes g)$ for $f \in S$, $g \in S$ (see 0.10), and from (*) that $T_1^{(1)}T_2^{(2)} = T_2^{(2)}T_1^{(1)}$ on S^2. We often write $T_1 \otimes T_2$ instead of $T_1^{(1)}T_2^{(2)}$. We call $T_1 \otimes T_2$ the <u>tensor product</u> of T_1 and T_2. As an example we have $N_{\varepsilon,2} = N_\varepsilon \otimes N_\varepsilon$ for $\varepsilon > 0$ (see 0.10 and [B], section 7).

(ii) As another example of a quasi-bounded linear operator of S^2 we mention

$$Z_U := \Psi_{f\in S^2} \Psi_{(t,s)\in\mathbb{C}^2} f(\tfrac{t+s}{\sqrt{2}},\tfrac{t-s}{\sqrt{2}})$$

(see [B], (7.1)).

. Some theorems about linear operators and linear functionals of S^*.

. In this section we prove some theorems concerning linear operators and functionals of S^*. Although the theorems are (with the proper modifications) valid for linear operators and functionals of S^{n*} ($n \in \mathbb{N}$), we only give the proofs for the one-dimensional case.

. Theorem. If T is a linear operator of S with an adjoint T^*, then it is possible to extend T to a linear operator $\tilde{T}$ of S^* such that

$$\tilde{T}(\mathrm{emb}(f)) = \mathrm{emb}(Tf) \qquad (f \in S) ,$$

$$[\tilde{T}F,g] = [F,T^*g] \qquad (F \in S^*,\ g \in S) .$$

Proof. Let $F \in S^*$, and consider the linear functional

$$(*) \qquad \Psi_{g\in S}\ [T^*g,F] .$$

From the fact that T^* is quasi-bounded (see 2.10(i)) we infer that for every $\alpha > 0$ there exists a $\beta > 0$ and a bounded linear operator T_1 of S such that $T^*N_\alpha = N_\beta T_1$. It easily follows that the linear functional in (*) is quasi-bounded.

According to [B], 22.2 there is exactly one $G \in S^*$ such that

$$[T^*g,F] = [g,G] \qquad (g \in S) .$$

It is easy to verify that G depends linearly on F. Now we define $\tilde{T}F = G$. From the definition of $\tilde{T}$ it readily follows that

$$[\tilde{T}F,g] = [F,T^*g] \qquad (F \in S^*,\ g \in S) .$$

We conclude the proof by showing that $\tilde{T}(\mathrm{emb}(f)) = \mathrm{emb}(Tf)$ for $f \in S$. This is very easy: $[\tilde{T}(\mathrm{emb}(f)),g] = [\mathrm{emb}(f),T^*g] = [f,T^*g] = [Tf,g] =$ $= [\mathrm{emb}(Tf),g]$ for every $g \in S$ (see [B], 18.1). This proves that $\tilde{T}(\mathrm{emb}(f)) = \mathrm{emb}(Tf)$. □

Remark 1. This theorem fits in with [B], section 19 in the following sense. It is possible to prove that Y_α defined by

$$Y_\alpha N_\alpha F := \sum_{k=0}^{\infty} [F,T^*\psi_k]N_\alpha\psi_k \qquad (F \in S^*)$$

maps $N_\alpha(S^*)$ linearly into S for $\alpha > 0$ (use the theorems of section 2 of this appendix). This $(Y_\alpha)_{\alpha>0}$ satisfies the relations (see also 0.8)

$$Y_{\alpha+\beta}N_{\alpha+\beta}F = N_\beta Y_\alpha N_\alpha F \qquad (F \in S^*,\ \alpha > 0,\ \beta > 0)\ ,$$

$$Y_\alpha N_\alpha f = N_\alpha Tf \qquad (f \in S,\ \alpha > 0)\ ,$$

$$\widetilde{T}F = \curlyvee_{\alpha>0}\ Y_\alpha N_\alpha F \qquad (F \in S^*)\ .$$

<u>Remark 2</u>. When dealing with a linear operator T with an adjoint, we work exclusively with the extension of T as described in theorem 3.2, which is again denoted by T.

<u>Remark 3</u>. There is just one linear operator V of S^* which satisfies $V(\mathrm{emb}(g)) = \mathrm{emb}(Tg)$ and $[VF,g] = [F,T^*g]$ for every $g \in S$, $F \in S^*$. From this it follows that

$$\forall_{f\in S}\ [Tf = 0] \Rightarrow \forall_{F\in S^*}\ [TF = 0]\ .$$

<u>Remark 4</u>. If $T_1,\dots,T_m$ are linear operators with an adjoint, and g is a polynomial in m variables, then $g(T_1,\dots,T_m)$ has an adjoint.

<u>Theorem</u>. Let T be a linear operator of S with an adjoint. If $F_n \overset{S^*}{\to} 0$, then $TF_n \overset{S^*}{\to} 0$.

<u>Proof</u>. Suppose $F_n \overset{S^*}{\to} 0$. If $g \in S$, then we have $[TF_n,g] = [F_n,T^*g] \to 0$ $(n \to \infty)$ according to the "only if" part of [B], 24.4 applied to the sequence $(F_n)_{n\in\mathbb{N}}$. But, according to the "if" part of [B], 24.4 applied to the sequence $(TF_n)_{n\in\mathbb{N}}$, this means that $TF_n \overset{S^*}{\to} 0$. □

<u>Remark</u>. If T is a linear operator of S, then there is at most one extension of T to a linear, convergence preserving operator of S^*, for which $T(\mathrm{emb}(f)) = \mathrm{emb}(Tf)$ $(f \in S)$. For, if T_1 and T_2 satisfy both the conditions mentioned, then both T_1F and T_2F is the S^*-limit of the sequence

$$\Big(\sum_{k=0}^{N} [F,\psi_k]\ \mathrm{emb}(T\psi_k)\Big)_{N\in\mathbb{N}_0} \qquad (F \in S^*)\ .$$

3.4. Suppose that T is a linear operator of S which is extended to a linear operator of S^* such that $T(\mathrm{emb}(f)) = \mathrm{emb}(Tf)$ for $f \in S$. In 3.8 we prove: if T preserves convergence in S^*, then T has an adjoint. In order to show this, we prove a general result about linear functionals of S^*, which states that every continuous linear functional of S^* is representable as an element of S.

3.5. Lemma. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence on $\mathbb{C}$ which satisfies

$$a_k b_k \to 0 \qquad (k \to \infty)$$

for every sequence $(b_k)_{k\in\mathbb{N}}$ with $\forall_{\varepsilon>0}\ [b_k = O(e^{k\varepsilon})\ (k \in \mathbb{N})]$. Then there is an $\varepsilon > 0$ such that $a_k = O(e^{-k\varepsilon})$ $(k \in \mathbb{N})$.

Proof. Suppose the contrary. There exists a sequence $(\delta_i)_{i\in\mathbb{N}}$, $\delta_i > 0$ $(i \in \mathbb{N})$, $\delta_i \downarrow 0$ $(i \to \infty)$ and a sequence of indices $(k_i)_{i\in\mathbb{N}}$, $k_i \in \mathbb{N}$ $(i \in \mathbb{N})$, $k_i \uparrow \infty$ $(i \to \infty)$ such that $|a_{k_i}| \geq e^{-k_i\delta_i}$ $(i \in \mathbb{N})$. If we define the sequence $(b_k)_{k\in\mathbb{N}}$ by

$$b_{k_i} = e^{k_i\delta_i} \qquad (i \in \mathbb{N}) ,$$

$$b_k = 0 \qquad (\text{if } \forall_{i\in\mathbb{N}}\ [k \neq k_i]) ,$$

then $b_k = O(e^{k\varepsilon})$ $(k \in \mathbb{N})$ for every $\varepsilon > 0$, but $\lim_{k\to\infty} a_k b_k \neq 0$. Contradiction. □

3.6. Definition. A linear functional L of S^* is called continuous if $F, F_n \in S^*$ $(n = 1,2,\ldots)$, $F_n \overset{S^*}{\to} F \Rightarrow LF_n \to LF$ $(n \to \infty)$.

3.7. Theorem. A linear functional L of S^* is continuous if and only if there exists a $g \in S$ such that $L(F) = [F,g]$ for $F \in S^*$. Such a g is uniquely determined (if it exists).

Proof. If there is a $g \in S$ such that $L(F) = [F,g]$ $(F \in S^*)$, then it easily follows from [B], 24.4 that L is continuous.

Now suppose that L is a continuous linear functional of S^*. It is not hard to prove that there is at most one $g \in S$ such that $L(F) = [F,g]$ $(F \in S^*)$.

In order to show the existence of a $g \in S$ such that $L(F) = [F,g]$ $(F \in S^*)$, we consider the sequence $(L(\mathrm{emb}(\psi_k)))_{k\in\mathbb{N}_0}$. If $(c_k)_{k\in\mathbb{N}_0}$ is a sequence satisfying $c_k = O(e^{k\varepsilon})$ $(k \in \mathbb{N}_0)$ for every $\varepsilon > 0$, then $c_k\ \mathrm{emb}(\psi_k) \overset{S^*}{\to} 0$, and therefore $c_k\ L(\mathrm{emb}(\psi_k)) \to 0$ $(k \to \infty)$. By lemma 3.5 we know that there is an $\varepsilon > 0$ such that

$$L(\mathrm{emb}(\psi_k)) = O(e^{-k\varepsilon}) \qquad (k \in \mathbb{N}_0)\ .$$

We define $g \in S$ (see 0.5(iv)c)) by

$$g := \sum_{k=0}^{\infty} \overline{L(\mathrm{emb}(\psi_k))}\psi_k\ .$$

If $F \in S^*$, then we have (according to 0.7)

$$[F,g] = \sum_{k=0}^{\infty} [F,\psi_k]L(\mathrm{emb}(\psi_k)) = L(F)\ ,$$

because $\sum_{k=0}^{N} [F,\psi_k]\mathrm{emb}(\psi_k) \overset{S^*}{\to} F.$ □

Remark. The g of theorem 3.7 is given by

$$g = \curlyvee_{t\in\mathbb{C}}\ \overline{L(\curlyvee_{\alpha>0}\ \curlyvee_{z\in\mathbb{C}}\ K_\alpha(z,\overline{t}))}\ .$$

See also [B], 22.2.

3.8. Theorem. If T is a linear operator of S which is extended to a linear operator of S^* such that $T(\mathrm{emb}(f)) = \mathrm{emb}(Tf)$ $(f \in S)$ and such that $F_n \in S^*$ $(n \in \mathbb{N})$, $F_n \overset{S^*}{\to} 0 \Rightarrow TF_n \overset{S^*}{\to} 0$, then T has an adjoint.

Proof. Let $g \in S$, and consider the linear functional

$$\curlyvee_{F\in S^*}\ [TF,g]$$

of S^*. This linear functional satisfies ([B], 24.4) $F, F_n \in S^*$ $(n = 1,2,\ldots)$, $F_n \overset{S^*}{\to} F \Rightarrow [TF_n,g] \to [TF,g]$. According to theorem 3.7 there is exactly one $g_1 \in S$ such that $[TF,g] = [F,g_1]$ for every $F \in S^*$. It is easily seen that this g_1 depends linearly on g. Furthermore we have for $f \in S$, $g \in S$ (see [B], 18.1)

$$[Tf,g] = [\mathrm{emb}(Tf),g] = [T(\mathrm{emb}(f)),g] =$$
$$= [\mathrm{emb}(f),g_1] = [f,g_1] .$$

So if we define $T^*g := g_1$, then T and T^* are adjoint operators. □

Remark. It is possible to prove the following theorem. If T is a linear operator of S^* which satisfies $F_n \in S^*$, $F_n \overset{S^*}{\to} 0 \Rightarrow \forall_{\alpha>0}\forall_{t\in\mathbb{C}}$ [$(TF_n)_\alpha(t)$ is bounded], then $F_n \in S^*$, $F_n \overset{S^*}{\to} 0 \Rightarrow TF_n \overset{S^*}{\to} 0$. The only available proof of this theorem is a bit tricky.

3.9. We give a few examples.

(i) The operators N_α, F, T_a ($a \in \mathbb{C}$), R_b ($b \in \mathbb{C}$), P,Q have adjoints, so they are extendable to linear operators of S^*. The families $(Y_\alpha)_{\alpha>0}$ corresponding to these operators (by [B], section 19) are the same as the one obtained via 3.2, remark 1.

(ii) If T is a linear operator of S with an adjoint, then

$$\sum_{k=0}^{N} [F,\psi_k]\mathrm{emb}(T\psi_k) \overset{S^*}{\to} TF ,$$

because

$$\sum_{k=0}^{N} [F,\psi_k]\mathrm{emb}(\psi_k) \overset{S^*}{\to} F ,$$

and, if $\alpha > 0$, we have (see 3.2, remark 1)

$$Y_\alpha N_\alpha F = (TF)_\alpha = \sum_{k=0}^{\infty} [F,\psi_k]Y_\alpha N_\alpha \psi_k .$$

3.10. We make a few remarks about linear operators of S^{n*} ($n \in \mathbb{N}$). The theorems proved in section 3 are (with the proper modifications) valid for the n-dimensional case. We restrict ourselves for the sake of elegance to the case n = 2.

3.11. We want to study tensor products of linear operators of S^* as in 2.13, but first we prove a

Lemma. If T is a linear operator of S with an adjoint, then for every $\alpha > 0$ there exists a $\beta > 0$, $M > 0$, $A > 0$, $B > 0$ such that

$$|(N_\alpha T\psi_k)(z)| \leq M\, e^{-k\beta}\exp(-\pi A(\mathrm{Re}\ z)^2 + \pi B(\mathrm{Im}\ z)^2)$$

for every $k \in \mathbb{N}_0$, $z \in \mathbb{C}$.

Proof. Let $\alpha > 0$, and write $\alpha = \delta + \gamma$ with some $\delta > 0$, $\gamma > 0$. Since T^* is quasi-bounded (2.10(i)) there exists a $\beta > 0$ and a bounded linear operator T_1 of S such that $T^* N_\gamma = N_\beta T_1$. Now

$$|[N_\gamma T\psi_k, g]| = |[\psi_k, T^* N_\gamma g]| = e^{-(k+\frac{1}{2})\beta}|[\psi_k, T_1 g]| \leq$$

$$\leq e^{-(k+\frac{1}{2})\beta}\|T_1\|\|g\|$$

for every $k \in \mathbb{N}_0$, $g \in S$. It follows that there is an $M > 0$ such that

$$\|N_\gamma T\psi_k\| \leq M\, e^{-k\beta} \qquad (k \in \mathbb{N}_0)\ .$$

The proof is easily completed by using $N_\alpha = N_\delta N_\gamma$ and 0.5(ii). □

3.12. If T is a linear operator of S with an adjoint T^*, then $T^{(1)}$ (considered as a linear operator of S^2, see 2.13) has an adjoint, viz. $(T^{(1)})^* = (T^*)^{(1)}$. This may be proved by applying Fubini's theorem to the integrals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (T^{(1)}f)(z_1,z_2)\overline{g(z_1,z_2)}dz_1dz_2$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z_1,z_2)\overline{((T^*)^{(1)}g)(z_1,z_2)}dz_1dz_2$$

for $f \in S^2$, $g \in S^2$. So $T^{(1)}$ is extendable to a linear operator of S^{2*} (theorem 3.2).

Let $(Y_\alpha)_{\alpha>0}$ be the family of linear operators which extends T to a linear operator of S^* (see 3.2, remark 1). We claim that

$$(*) \qquad T^{(1)}F = \Psi_{\alpha>0}\, Y_\alpha^{(1)} F_\alpha \qquad (F \in S^{2*}) ,$$

where

$$Y_\alpha^{(1)} F_\alpha := \Psi_{(z_1,z_2)\in\mathbb{C}^2}\, Y_\alpha(\Psi_{t\in\mathbb{C}}\, F_\alpha(t,z_2))(z_1)$$

for $F \in S^{2*}$ and $\alpha > 0$.

In order to prove this, we fix an $\alpha > 0$ and an $F \in S^{2*}$, and we note that (by 3.9(ii))

$$(T^{(1)}F)_\alpha = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [F,\psi_m \otimes \psi_n] N_\alpha T\psi_m \otimes N_\alpha \psi_n .$$

Take a $z_1 \in \mathbb{C}$ and a $z_2 \in \mathbb{C}$. We find

$$\begin{aligned}(T^{(1)}F)_\alpha(z_1,z_2) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} [F,\psi_m \otimes \psi_n](N_\alpha T\psi_m)(z_1)(N_\alpha\psi_n)(z_2) = \\ &= \sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} [F,\psi_m \otimes \psi_n](N_\alpha\psi_n)(z_2)](N_\alpha T\psi_m)(z_1)\end{aligned}$$

(by lemma 3.11 the double series is absolutely convergent). Note that

$$\sum_{n=0}^{\infty} [F,\psi_m \otimes \psi_n](N_\alpha\psi_n)(z_2) = O(e^{m\varepsilon}) \qquad (m \in \mathbb{N}_0)$$

for every $\varepsilon > 0$ (this follows from lemma 3.11 and 0.7). Again using 3.9(ii) yields

$$(T^{(1)}F)_\alpha(z_1,z_2) = Y_\alpha(\sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} [F,\psi_m \otimes \psi_n](N_\alpha\psi_n)(z_2)] N_\alpha\psi_m) ,$$

and (*) follows for

$$\Psi_{t\in\mathbb{C}}\, F_\alpha(t,z_2) = \sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} [F,\psi_m \otimes \psi_n](N_\alpha\psi_n)(z_2)] N_\alpha\psi_m .$$

After this (perhaps boring) analysis we can prove the following

<u>Theorem</u>. Let T_1 and T_2 be two linear operators of S with an adjoint, which are extendable by means of the families $(Y_{\alpha,1})_{\alpha>0}$ and $(Y_{\alpha,2})_{\alpha>0}$. We have

$$T_1^{(1)}T_2^{(2)}F = \Psi_{\alpha>0}\, Y_{\alpha,1}^{(1)}Y_{\alpha,2}^{(2)}F_\alpha \qquad (F \in S^{2*}) \;,$$

$$T_1^{(1)}T_2^{(2)}F = T_2^{(2)}T_1^{(1)}F \qquad (F \in S^{2*}) \;,$$

$$F_n \overset{S^{2*}}{\to} 0 \Rightarrow T_1^{(1)}T_2^{(2)}F_n \overset{S^{2*}}{\to} 0 \qquad (F_n \in S^{2*},\ n \in \mathbb{N}) \;.$$

<u>Proof</u>. Follows from the foregoing. □

<u>Examples</u>. (i) If $F \in S^{2*}$, then F is differentiable in both variables (see [B], 19, example (iv)), and $P^{(1)}P^{(2)}F = P^{(2)}P^{(1)}F$.

(ii) Z_U (see 2.13(ii)) is a self-adjoint linear operator of S^2. For $a \in \mathbb{R}$ and $F \in S^{2*}$ we have

$$Z_U T_{a/\sqrt{2}} \otimes T_{a/\sqrt{2}}\, F = T_a^{(1)} Z_U F \;,$$

since this relation holds on S^2 (see also 3.2, remark 3).

(iii) If $F_1 \in S^*$, $F_2 \in S^*$, and T_1 and T_2 are linear operators of S with an adjoint, then $(T_1 \otimes T_2)(F_1 \otimes F_2) = T_1F_1 \otimes T_2F_2$.

Appendix 2. A theorem on S-convergence.

1. Theorem. Let $\varphi_n \in S$ ($n \in \mathbb{N}$), and suppose that M, A and B are positive numbers such that

$$|\varphi_n(z)| \leq M \exp(-\pi A(\operatorname{Re} z)^2 + \pi B(\operatorname{Im} z)^2) \qquad (z \in \mathbb{C},\ n \in \mathbb{N})\ .$$

If $\varphi_n \to 0$ pointwise, then $\varphi_n \overset{S}{\to} 0$.

Proof. Suppose $\varphi_n \to 0$ pointwise. We first prove that for every $R > 0$

$$\lim_{n\to\infty} \varphi_n(z) = 0$$

uniformly in $|z| \leq R$: Note that for $R > 0$

$$|\varphi_n(z)| = \Big| \int_{|t|=2R} \frac{1}{2\pi i} \frac{\varphi_n(t)}{t-z}\, dt \Big| \leq \frac{1}{2\pi} \int_{|t|=2R} \frac{|\varphi_n(t)|}{|t-z|}\, |dt| \leq$$

$$\leq \frac{1}{2\pi R} \int_{|t|=2R} |\varphi_n(t)|\ |dt| \to 0$$

by Lebesgue's theorem on dominated convergence.

Now let $\varepsilon > 0$, and let $R > 0$ such that

$$|\varphi_n(z)| \leq \varepsilon \exp(-\tfrac{1}{2}\pi A(\operatorname{Re} z)^2 + 2\pi B(\operatorname{Im} z)^2)$$

for $|z| \geq R$. We can find a number $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |\varphi_n(z)| \leq \varepsilon \exp(-\tfrac{1}{2}\pi A(\operatorname{Re} z)^2 + 2\pi B(\operatorname{Im} z)^2)$$

for $|z| \leq R$ (we use the fact that $\varphi_n(z) \to 0$ uniformly in $|z| \leq R$). This proves that

$$|\varphi_n(z)| \leq \varepsilon \exp(-\tfrac{1}{2}\pi A(\operatorname{Re} z)^2 + 2\pi B(\operatorname{Im} z)^2)$$

for every $z \in \mathbb{C}$ if $n > N$. □

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