An introduction to the category-theoretic solution of recursive domain equations

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by

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Contents

ABSTRACT

In advanced denotational semantics one frequently encounters equations of the form $D \cong F(D)$, where D ranges over e.g. cpo's or complete lattices and F involves constructors like $+$, $\times$ and $\rightarrow$. Researchers like Wand, Plotkin, Lehmann and Smyth have advocated a category-theoretical solution method for these equations. This paper presents a systematic introduction to the method without assuming any prior knowledge from category theory.

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0. Introduction

In advanced denotational semantics one frequently encounters equations like

(1) $D \cong D \to D$

or

(2) $D \cong U + A \times D$

where the variable D ranges over a class of domains such as cpo's or complete lattices. The first solution method for equation (1) was provided by Scott and led to the so-called D_∞ model of the untyped lambda calculus [Scott]. Later on the method has been extended and applied to various other classes of domains; see e.g. [Plotkin] or [Smyth].

In [Wand] Wand gave a more abstract treatment of these solution methods in terms of category theory. His main result was a theorem which isolated the common core of several methods and which could be applied to various categories of domains. The categorical framework has been further developed in [Smyth & Plotkin] and [Lehmann & Smyth].

All these papers require some background in category theory and this makes them rather inaccessible to those unfamiliar with that field. It is the purpose of the present paper to explain the category-theoretic solution of recursive domain equations without presupposing any knowledge of category theory from the reader. The material covered is roughly the same as that in section 1-4 of [Smyth & Plotkin] but the presentation is more elementary and systematic. All the necessary categorical notions are defined and explained and proofs are worked out in detail.

The structure of the paper is as follows.
In chapter 1 we summarize the part of fixed point theory up to and including the least fixed point theorem for continuous functions on cpo's.
Chapter 2 is a systematic generalization of this material to initial fixed points of continuous functors in ω-categories, resulting in theorem 2.23 which corresponds to theorem 3.6 in [Wand] and lemma 2 in [Smyth & Plotkin].
In chapter 3 this general solution method is specialized towards the kinds of domains needed in denotational semantics. The main purpose of this chapter is to derive simple 'local' conditions which imply the categorical conditions necessary for application of the theory of chapter 2. To this end chapter 3 is divided in five sections.
In section 3.1 the concept of an O-category is introduced. With an O-category one can associate a 'derived' category of projection pairs. A simple criterion is given for determining whether such a derived category is an ω-category.
Similarly in section 3.2 with a functor on O-categories a derived functor on categories of projection pairs is associated, and a simple condition is given for determining whether the derived functor is ω-continuous.
Section 3.3 contains some technical results.
Section 3.4 contains several examples of O-categories. Most of these examples have important applications in denotational semantics.

Section 3.5 contains examples of continuous functors. Most of these functors are closely related to domain constructors in denotational semantics.
The appendix contains some slight extensions to the basic category theory presented in the previous chapters. For a more extensive treatment of category theory the reader is referred to [Arbib & Manes], [Herrlich & Strecker] or [MacLane].

Some special remarks are in order concerning the format of proofs. Most proofs in this paper are presented in a rather rigid form, viz. as a sequence of numbered assertions together with references to other assertions from which they are derived. Such references or hints are enclosed in square brackets. Groups of related proof steps are often headed by an announcement of what is going to be proved, enclosed in square brackets and quotation marks. The main reasons for using this format are that it makes the proof structure very explicit and that it facilitates step by step comparison of proofs, which is important in relating the theory of chapter 2 to that of chapter 1.

1. A summary of fixed point theory for partially ordered sets.

In this chapter we present a short overview of the main notions and results concerning fixed points of functions on complete partial orders etc.. This overview mainly serves as preparation for the theory in chapter 2, which is presented as a systematic generalization of the material in this chapter. The formulation of some definitions, theorems and proofs, which might seem somewhat unusual in some cases, is tailored to that purpose. In particular the rigid format of proofs -a sequence of numbered assertions together with references to other assertions from which they are derived- serves to facilitate comparison with proofs of corresponding theorems in chapter 2. As we assume the reader's familiarity with the subject of the current chapter, we refrain from further comment.

Definition 1.1 [partially ordered set]

A partially ordered set (poset) is a pair $(C, \sqsubseteq)$, where C is a set and $\sqsubseteq$ is a binary relation on C satisfying:

1. $(\forall x \in C \mid x \sqsubseteq x)$ [reflexivity]
2. $(\forall x, y \in C \mid (x \sqsubseteq y \wedge y \sqsubseteq x) \Rightarrow x = y)$ [antisymmetry]
3. $(\forall x, y, z \in C \mid (x \sqsubseteq y \wedge y \sqsubseteq z) \Rightarrow x \sqsubseteq z)$ [transitivity]

□

Note 1.2

In addition to $\sqsubseteq$ we will also use relations $\sqsubset$, $\sqsupseteq$ and $\sqsupset$, defined by:

$$
\begin{aligned}
(\forall x, y \in C \mid \; & x \sqsubset y \Leftrightarrow (x \sqsubseteq y \wedge x \neq y), \\
& x \sqsupseteq y \Leftrightarrow y \sqsubseteq x , \\
& x \sqsupset y \Leftrightarrow y \sqsubset x \\
)
\end{aligned}
$$

□

Definition 1.3 [Least, minimal, greatest, maximal]

Let $(C, \sqsubseteq)$ be a partially ordered set; $x \in C$.

1. x is a least element of $C \Leftrightarrow (\forall y \in C \mid x \sqsubseteq y)$.
2. x is a minimal element of $C \Leftrightarrow \neg (\exists y \in C \mid y \sqsubset x)$.
3. x is a greatest element of $C \Leftrightarrow (\forall y \in C \mid y \sqsubseteq x)$.
4. x is a maximal element of $C \Leftrightarrow \neg (\exists y \in C \mid x \sqsubset y)$.

□

Note 1.4

From definition 1.3 and antisymmetry of $\sqsubseteq$ it follows that

1. $(C, \sqsubseteq)$ has at most one least element.
2. $(C, \sqsubseteq)$ has at most one greatest element.

3. a least element is also a minimal element.

4. a greatest element is also a maximal element.

□

Definition 1.5 [(least) upper bound, (greatest) lower bound]

Let $(C, \sqsubseteq)$ be a partially ordered set; $X \subseteq C$.

1.1. $UB(X) = \{y \in C \mid (\forall x \in X \mid x \sqsubseteq y)\}$.
The elements of $UB(X)$ are called upper bounds of X.

1.2. $\bigsqcup X$ = the least element of $UB(X)$, if it exists.
$\bigsqcup X$ is called the least upper bound of X.

2.1. $LB(X) = \{y \in C \mid (\forall x \in X \mid y \sqsubseteq x)\}$.
The elements of $LB(X)$ are called lower bounds of X.

2.2. $\bigsqcap X$ = the greatest element of $LB(X)$, if it exists.
$\bigsqcap X$ is called the greatest lower bound of X.

□

Definition 1.6 [directed set]

Let $(C, \sqsubseteq)$ be a partially ordered set; $X \subseteq C$.

X is directed $\Leftrightarrow$ every finite subset of X has an upper bound in X.

□

Definition 1.7 [ω-chain]

1. An ascending ω-chain in $(C, \sqsubseteq)$ is a sequence $<x_i>_{i=0}^{\infty}$ s.t.
   – $(\forall i \mid i \geq 0 \mid x_i \in C)$
   – $(\forall i \mid i \geq 0 \mid x_i \sqsubseteq x_{i+1})$

2. A descending ω-chain in $(C, \sqsubseteq)$ is a sequence $<x_i>_{i=0}^{\infty}$ s.t.
   – $(\forall i \mid i \geq 0 \mid x_i \in C)$
   – $(\forall i \mid i \geq o \mid x_i \sqsupseteq x_{i+1})$

□

Note 1.8

1. For a sequence $<x_i>_{i=0}^{\infty}$ the entities $\bigsqcup \{x_i \mid i \geq 0\}$ and $\bigsqcap \{x_i \mid i \geq 0\}$ (if they exist) will be denoted by $\bigsqcup_{i=0}^{\infty} x_i$ and $\bigsqcap_{i=0}^{\infty} x_i$ respectively.

2. Deletion of an initial segment of an ascending ω-chain does not affect its least upper bound, i.e.
   for all $k, l : 0 \leq k \leq l$: $\bigsqcup_{i=k}^{\infty} x_i = \bigsqcup_{i=l}^{\infty} x_i$.

□

Definition 1.9 [ω-cpo, d-cpo, cl]

1. An <u>ω-cpo</u> [ω-complete partial order] is a poset $(C, \sqsubseteq)$ s.t.
   – $(C, \sqsubseteq)$ has a least element.
   – every ascending ω-chain in C has a least upper bound in C.

2. A <u>d-cpo</u> [directed-complete partial order] is a poset $(C, \sqsubseteq)$ s.t.
   – $(C, \sqsubseteq)$ has a least element.
   – every directed subset of C has a least upper bound in C.

3. A <u>complete lattice</u> is a poset $(C, \sqsubseteq)$ s.t. every subset of C has a least upper bound in C.

□

<u>Note 1.10</u>

1. It can easily be shown that in a complete lattice every subset X also has a greatest lower bound, viz.
   $\bigsqcap X = \bigsqcup LB(X)$.

2. The least element of an ω-cpo, d-cpo, or cl $(C, \sqsubseteq)$ will be denoted by $\bot_C$, or just $\bot$.
   The greatest element of a cl $(C, \sqsubseteq)$ will be denoted by $\top_C$, or just $\top$.

3. From definition 1.9 it follows immediately, that
   – every cl is a d-cpo.
   – every d-cpo is an ω-cpo.

□

<u>Definition 1.11</u> [monotonic function]
Let $(C_1, \sqsubseteq_1)$ and $(C_2, \sqsubseteq_2)$ be posets.
A function $f \in C_1 \to C_2$ is <u>monotonic</u> $\Leftrightarrow$
$(\forall x, y \in C_1 \mid x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y))$
□

<u>Definition 1.12</u> [ω-continuous function]
Let $(C_1, \sqsubseteq_1)$ and $(C_2, \sqsubseteq_2)$ be ω-cpo's.
A function $f \in C_1 \to C_2$ is (upward) <u>ω-continuous</u> $\Leftrightarrow$

– f is monotonic.
– for each ascending ω-chain $<x_i>_{i=0}^{\infty}$ in $(C_1, \sqsubseteq_1)$:

$$f(\bigsqcup_{i=0}^{\infty} x_i) = \bigsqcup_{i=0}^{\infty} f(x_i)$$

□

<u>Definition 1.13</u> [(least)(pre-) fixed point]
Let $(C, \sqsubseteq)$ be a poset; $f \in C \to C$; $y \in C$.

1.1 $FP(f) = \{x \in C \mid f(x) = x\}$.

1.2 y is a fixed point (f.p.) of f *iff* $y \in FP(f)$.

1.3 y is the least fixed point (l.f.p.) of f *iff* y is the least element of $FP(f)$.

2.1 $PFP(f) = \{x \in C \mid f(x) \sqsubseteq x\}$.

2.2 y is a prefixed point (p.f.p.) of f *iff* $y \in PFP(f)$.

2.3 y is the least prefixed point (l.p.f.p.) of f *iff* y is the least element of $PFP(f)$.

□

Lemma 1.14 [least prefixed point is also least fixed point]
Let $(C, \sqsubseteq)$ be a poset; $f \in C \to C$ and f monotonic; $x \in C$.
If x is the l.p.f.p of f, then x is the l.f.p. of f.

□

Proof

1. f is monotonic
2. x is l.p.f.p. of f

["x is f.p. of f"]

3.	$f(x) \sqsubseteq x$	[2, def1.13]
4.	$f(f(x)) \sqsubseteq f(x)$	[3, 1]
5.	$f(x)$ is p.f.p. of f	[4, def1.13]
6.	$x \sqsubseteq f(x)$	[5, 2]
7.	$x = f(x)$	[6, 3]

["x is l.f.p. of f"]

8.	$(\forall y \in PFP(f) \mid x \sqsubseteq y)$	[2, def1.13]
9.	$(\forall y \in FP(f) \mid y \in PFP(f))$	[def1.13]
10.	$(\forall y \in FP(f) \mid x \sqsubseteq y)$	[9, 8]
11.	x is l.f.p. of f	[10, 7]

□

Theorem 1.15 [Knaster - Tarski]
Let $(C, \sqsubseteq)$ be a cl; $f \in C \to C$ and f monotonic.
Then $FP(f)$ is a cl, and l.f.p. of f is $\bigsqcap PFP(f)$.

□

Proof
Omitted. See e.g. [Tarski]

□

Theorem 1.16 [least fixed point theorem]
Let $(C, \sqsubseteq)$ be a cpo;

$f \in C \rightarrow C$; f ω-continuous.

Then

a. $<f^n(\perp)>_{n=0}^{\infty}$ is an ascending ω-chain.

b. Let $z = \bigsqcup_{n=0}^{\infty} f^n(\perp)$.

z is the least fixed point of f.

□

<u>Proof</u>

a. Induction on n:

<u>base step</u>

1. $\perp \sqsubseteq f(\perp)$
2. $f^0(\perp) \sqsubseteq f^1(\perp)$ [1]

<u>induction step</u>

3. Let $n \geq 0$
4. $f^n(\perp) \sqsubseteq f^{n+1}(\perp)$ [ind. hyp]
5. $f(f^n(\perp)) \sqsubseteq f(f^{n+1}(\perp))$ [4 , f monotonic]
6. $f^{n+1}(\perp) \sqsubseteq f^{n+2}(\perp)$ [5]

7. $(\forall n \mid n \geq 0 \mid f^n(\perp) \sqsubseteq f^{n+1}(\perp))$ [2 , 3−6 , induction]

b.

b1. ["z is fixed point"]

8. $z = \bigsqcup_{n=0}^{\infty} f^n(\perp)$
9. $z = \bigsqcup_{n=1}^{\infty} f^n(\perp)$ [note 1 . 8 . 2]
10. $z = \bigsqcup_{n=0}^{\infty} f^{n+1}(\perp)$ [9]
11. $f(z) = f(\bigsqcup_{n=0}^{\infty} f^n(\perp))$ [8]
12. $f(z) = \bigsqcup_{n=0}^{\infty} f^{n+1}(\perp)$ [11 , f continuous]
13. $f(z) = z$ [10, 12]
14. z is a fixed point of f . [13]

b2. ["z is least fixed point"]

15. Let $y \in FP(f)$.

[by induction we show : $(\forall n \mid n \geq 0 \mid f^n(\perp) \sqsubseteq y)$]

<u>base step</u>

16. $\perp \sqsubseteq y$.

<u>induction step</u>

17. Let $n \geq 0$
18. $f^n(\perp) \sqsubseteq y$ [ind. hyp]

19. $f^{n+1}(\bot) \sqsubseteq f(y)$ [18, f monotonic]
20. $f^{n+1}(\bot) \sqsubseteq y$ [19, 15]

21. $(\forall n \mid n \geq 0 \mid f^n(\bot) \sqsubseteq y)$ [16, 17 – 20, induction]
22. $\bigsqcup_{n=0}^{\infty} f^n(\bot) \sqsubseteq y$ [21, def1.5.1]
23. $(\forall y \in FP(f) \mid z \sqsubseteq y)$ [15, 8, 22]
24. z is l.f.p. of f [14, 23]

□

2. Fixed point theory for categories

In chapter 1 we have summarized the construction of solutions for equations of the form $x = f(x)$, where x should be an element of an ω-cpo, d-cpo, or cl C, and f is a continuous function from C to C. Our end goal is the solution of equations like

$$D \cong A + B \times D \ ,$$
$$D \cong D \rightarrow D$$

or generally

(1) $\qquad D \cong F(D)$

where D should be an entire ω-cpo, d-cpo, cl or something similar, and F prescribes a way of combining objects of these kinds. An appropriate and very general mathematical framework for studying equations like (1) is provided by category theory. In this chapter we shall introduce some elements of category theory and present a solution method for equations like (1) which is a systematic generalization of the theory in chapter 1. To make this generalization explicit, the structure of this chapter is very similar to that of chapter 1. Where appropriate, definitions, theorems and proof steps are provided with references to their counterparts in chapter 1.

A "category" is an abstraction of "a collection of sets and functions between them". A category consists of a collection of "objects", which are abstractions of sets, and for each ordered pair of objects a set of "arrows" or "morphisms", which are abstractions of functions. For morphisms there exist associative composition operators and identity morphisms. The complete definition follows:

<u>Definition 2.1</u> [category]

A <u>category</u> K consists of:

1. a class $Obj(K)$, called the <u>objects</u> of K.
2. for all A, B in $Obj(K)$, a set $Hom_K(A, B)$. Such a set is called a hom-set, and its elements are called the <u>arrows</u> or <u>morphisms</u> from A to B.
3. for all A, B, C in $Obj(K)$, a map
   $\circ_{ABC} : Hom_K(B, C) \times Hom_K(A, B) \rightarrow Hom_K(A, C)$
   called <u>composition</u>

such that the following conditions are satisfied :

- composition is associative
- for all A in $Obj(K)$

  there exists an I_A in $Hom_K(A, A)$,

  called the <u>identity</u> of A, such that
  for all B in $Obj(K)$; $f \in Hom_K(A, B)$, $g \in Hom_K(B, A)$:

$f \circ I_A = f$ and $I_A \circ g = g$

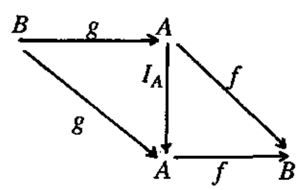
□

Note 2.2 Where context supplies sufficient information, the following abbreviations will be used:

- $\circ_{ABC}$ becomes $\circ$
- $Hom_K(A, B)$ becomes $Hom(A, B)$
- $f \in Hom(A, B)$ becomes $f : A \to B$

□

Statements about categories are often in terms of diagrams, with nodes representing objects and edges representing morphisms. Equality between morphism compositions then amounts to commutativity of such diagrams.
For example, the last line of Def. 2.1 is equivalent to saying that the following diagram commutes:



Example 2.3
Some examples of categories are :

SET, where the objects are sets and the morphisms are total functions between them.

PFN, where the objects are sets and the morphisms are partial functions between them.

VECT, where the objects are vector spaces and the morphisms are linear maps.

CPO, where the objects are ω-cpo's and the morphisms are continuous functions.

□

In particular, with any poset $(C, \sqsubseteq)$ there corresponds a category $K_{(C, \sqsubseteq)}$ as follows:

$$Obj\ (K_{(C,\sqsubseteq)}) = C$$

$$\text{for all } x, y \in C : Hom\ (x, y) = \begin{cases} \{x \to y\} & \text{if } x \sqsubseteq y \\ \varnothing & \text{otherwise} \end{cases}$$

where $\{x \to y\}$ stands for a set consisting of a single arrow from x to y. Composition and identity are defined by:

$$\text{for all } x, y, z \in C \text{ s.t. the arrows exist} : (y \to z) \circ (x \to y) = x \to z$$

$$\text{for all } x \in C \quad : \quad I_x = x \to x$$

Note that these two definitions correspond to transitivity and reflexivity of $\sqsubseteq$.

This correspondence forms the basis for the generalization from the theory of chapter 1 to that of chapter 2. E.g. the familiar notions of subset of a poset and dual of a poset generalize to definition 2.4.:

Definition 2.4 [(full) subcategory]
Let K and L be categories.

1. L is a subcategory of K *fif*
   – $Obj\ (L)$ is a subclass of $Obj\ (K)$
   – for all A, B in $Obj\ (L)$: $Hom_L\ (A, B) \subseteq Hom_K\ (A, B)$
2. L is a full subcategory of K *fif*
   – $Obj\ (L)$ is a subclass of $Obj\ (K)$
   – for all A, B in $Obj\ (L)$: $Hom_L\ (A, B) = Hom_K\ (A, B)$
3. K^{op} is the category with
   – $Obj\ (K^{op}) = Obj\ (K)$
   – for all A, $B \in Obj\ (K^{op})$: $Hom_{K^{op}}\ (A, B) = Hom_K\ (B, A)$

□

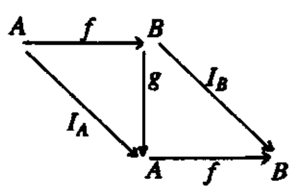
The example category $K_{(C, \sqsubseteq)}$ also shows that a category is a weaker structure than a poset. Transitivity and reflexivity can be modelled by composition and identity, but there is no counterpart of antisymmetry. This makes it almost impossible to prove that objects are equal. Usually the best one can prove is that objects are isomorphic, as defined below:

Definition 2.5 [isomorphism]
Let K be a category; A, $B \in Obj\ (K)$; $f \in Hom\ (A, B)$.
f is an isomorphism *fif* there exists a $g \in Hom\ (B, A)$
such that $f \circ g = I_B$ and $g \circ f = I_A$.
Such a g is called an inverse of f .

□

Note 2.6

1. It can easily be verified that an inverse of f , if it exists, is unique. It will be denoted by f^{-1} .
2. Diagrammatically, isomorphism amounts to commutativity of the following diagram:



□

In Def. 1.3 we defined a least element of a poset $(C, \sqsubseteq)$ to be an $x \in C$ such that $(\forall y \in C \mid x \sqsubseteq y)$. In terms of the category $K_{(C, \sqsubseteq)}$ this amounts to an object x such that for all objects y there exists a unique arrow from x to y. This correspondence suggests the following definition:

Definition 2.7 [initial, terminal ; compare with Def. 1.3]

Let K be a category; $A \in Obj(K)$.

1. A is an initial object of K *fif* for every $X \in Obj(K)$ $Hom(A, X)$ has exactly one element.
2. A is a terminal object of K *fif* for every $X \in Obj(K)$ $Hom(X, A)$ has exactly one element.

□

Note 2.8

It can easily be verified that if A and B are both initial objects of K, they are isomorphic. Similarly for terminal objects. Also, initial object and terminal object are dual notions, in the sense that any initial object in K is a terminal object in K^{op}.

□

According to Def. 1.7 an ascending ω-chain in a poset $(C, \sqsubseteq)$ is a sequence $<x_i>_{i=0}^{\infty}$ such that $(\forall i \mid i \geq 0 \mid x_i \sqsubseteq x_{i+1})$. Its counterpart in the corresponding category $K_{(C,\sqsubseteq)}$ would be a sequence of objects $<x_i>_{i=0}^{\infty}$ such that for all i there exists an arrow from x_i to x_{i+1}. In $K_{(C,\sqsubseteq)}$ the arrow from x_i to x_{i+1} is unique, but in general it is not. Therefore it is necessary to mention the arrows as well. This leads to:

Definition 2.9 [ω-chain, ω^{op}-chain; compare with Def. 1.7]

Let K be a category.

1. An ω-chain in K is a sequence $< (D_i, f_i) >_{i=0}^{\infty}$ *s.t.*
   - $(\forall i \mid i \geq 0 \mid D_i \in Obj(K))$
   - $(\forall i \mid i \geq 0 \mid f_i \in Hom(D_i, D_{i+1}))$
2. An ω^{op}-chain in K is a sequence $< (D_i, f_i) >_{i=0}^{\infty}$ *s.t.*
   - $(\forall i \mid i \geq 0 \mid D_i \in Obj(K))$
   - $(\forall i \mid i \geq 0 \mid f_i \in Hom(D_{i+1}, D_i))$

□

Note 2.10

Diagrammatically an ω-chain $< (D_i, f_i) >_{i=0}^{\infty}$ is represented as an infinite diagram

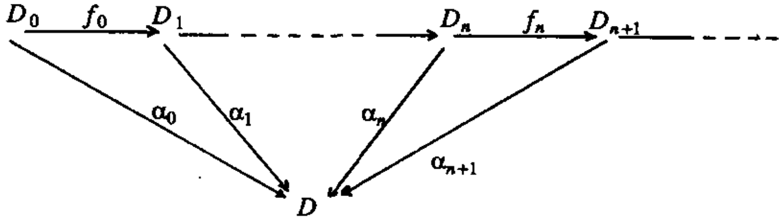
$$D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} --- D_n \xrightarrow{f_n} D_{n+1} \xrightarrow{f_{n+1}} ---$$

and an ω^{op}-chain $< (D_i, f_i) >_{i=0}^{\infty}$ as

$$D_0 \xleftarrow{f_0} D_1 \xleftarrow{f_1} --- D_n \xleftarrow{f_n} D_{n+1} \xleftarrow{f_{n+1}} ---$$

□

In a poset $(C, \sqsubseteq)$ an upper bound of an ascending ω-chain $<x_n>_{n=0}^{\infty}$ is an $y \in C$ such that $(\forall n \mid n \geq 0 \mid x_n \sqsubseteq y)$. In the category $K_{(C,\sqsubseteq)}$ this would correspond to an object D for an ω-chain $< (D_n, f_n) >_{n=0}^{\infty}$ such that for all n there exists an arrow from D_n to D. Diagrammatically:



In $K_{(C,\sqsubseteq)}$ the α_n are unique and because of the way composition is defined in $K_{(C,\sqsubseteq)}$ we have that for all n $\alpha_n = \alpha_{n+1} \circ f_n$, i.e. the diagram commutes. In an arbitrary category there may be more arrows from D_n to D however and commutativity is not implied. Hence the following definition:

Definition 2.11 [(co-)cone]

Let K be a category;

1. Let $\Delta = < (D_i, f_i) >_{i=0}^{\infty}$ be an ω-chain in K.
   A co - cone for Δ is a pair (D, α) such that
   $- D \in obj(K)$
   $- \alpha$ is a sequence $<\alpha_i>_{i=0}^{\infty}$ such that

   $(\forall i \mid i \geq 0 \mid \alpha_i \in Hom(D_i, D))$
   $(\forall i \mid i \geq 0 \mid \alpha_i = \alpha_{i+1} \circ f_i)$

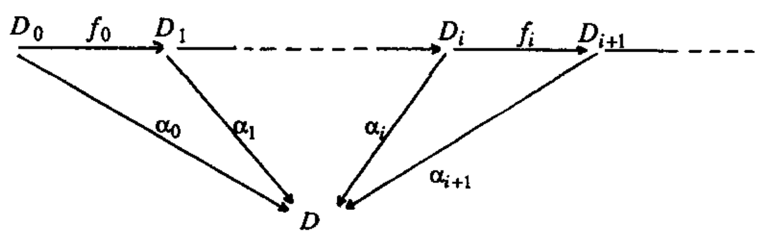
2. Let $\Delta = < (D_i, f_i) >_{i=0}^{\infty}$ be an ω^{op}-chain in K.
   A cone for Δ is a pair (D, α) such that
   $- D \in Obj(K)$
   $- \alpha$ is a sequence $<\alpha_i>_{i=0}^{\infty}$ such that

   $(\forall i \mid i \geq 0 \mid \alpha_i \in Hom(D, D_i))$
   $(\forall i \mid i \geq 0 \mid \alpha_i = f_i \circ \alpha_{i+1})$

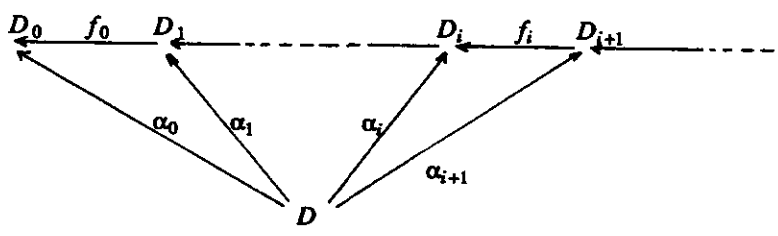
□

Note 2.12

Diagrammatically, a co-cone (D, α) for an ω-chain $< (D_i, f_i) >_{i=0}^{\infty}$ is represented by the following infinite commutative diagram:



A cone (D, α) for an ω^{op}-chain $< (D_i, f_i) >_{i=0}^{\infty}$ is represented by the following diagram:

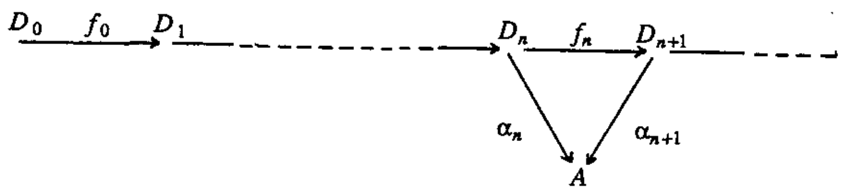


□

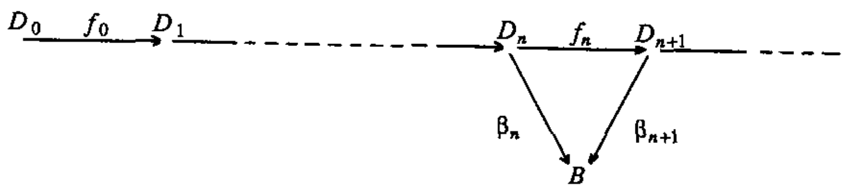
According to Def. 1.5 and note 1.8 the least upper bound of an ascending ω-chain $x = \langle x_i \rangle_{i=0}^{\infty}$ is the least element of $UB\ (x)$, the set of upper bounds of x. I.e. if a is the least upper bound of x, then for all $b \in UB(x) : a \sqsubseteq b$.

Let us now consider the counterpart of this notion for the category $K_{(C,\sqsubseteq)}$. Let $\Delta = \langle (D_i, f_i) \rangle_{i=0}^{\infty}$ be an ω-chain, and let (A, α) and (B, β) be two cocones for Δ. Then the following diagrams (of which only the parts near f_n have been drawn completely) commute:

(1)

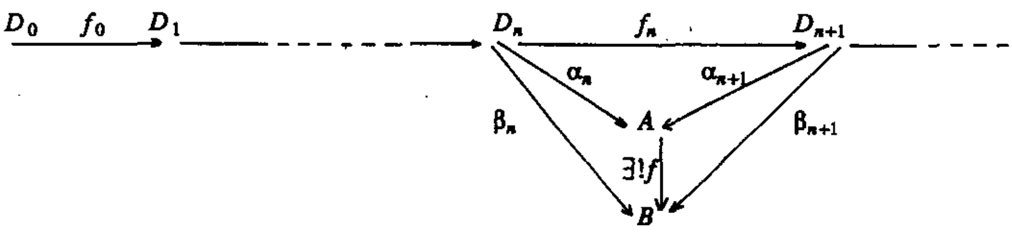


(2)



If A is "at most" B, there is a unique arrow f from A to B. Because of the way composition has been defined in $K_{(C,\sqsubseteq)}$, we have that for all $n \geq 0 : \beta_n = f \circ \alpha_n$. I.e. the following diagram commutes (again we only draw the part near f_n):

(3)



In particular, if A is "least", then for any "upper bound" B there should be a unique arrow from A to B such that diagram (3) commutes; i.e. A is initial among the "upper bounds" for Δ.

Note that in the the above, because of the special structure of $K_{(C,\sqsubseteq)}$, for given Δ, A and B the α_n, β_n and f are uniquely determined. In an arbitrary category L however, for given Δ and A there may be many α such that (A, α) is a cocone for Δ, and there may be many arrows from A to B, so commutativity as in (3) is not guaranteed. Therefore, when considering two cocones (A, α) and (B, β) for A, rather than taking all arrows from A to B, we should restrict ourselves to those arrows f for which (3) commutes. It can easily be verified that in this way we obtain a category $UB(\Delta)$ of cocones for Δ, and we can take the initial objects of $UB(\Delta)$ as the proper generalization of the notion "least upper bound of an ascending ω-chain". This completes the motivation of the following definition:

Definition 2.13 [UB, LB, limit, colimit; compare with Def. 1.5]
Let K be a category.

1. Let Δ be an ω-chain in K.
    1. $UB(\Delta)$ is the category X such that
       – $Obj(X)$ is the class of co-cones for Δ.
       – for all $(A, \alpha), (B, \beta)$, in $Obj(X)$:
       $Hom_X((A, \alpha), (B, \beta)) = \{f \in Hom_K(A, B) \mid (\forall n \mid n \geq 0 \mid \beta_n = f \circ \alpha_n)\}$
    2. A colimit of Δ in an initial object of $UB(\Delta)$.
2. Let Δ be an ω^{op}-chain in K.
    1. $LB(\Delta)$ is the category X such that
       – $Obj(X)$ is the class of cones for Δ.
       – for all $(A, \alpha), (B, \beta)$ in $Obj(X)$:
       $Hom_X((A, \alpha), (B, \beta)) = \{f \in Hom_K(A, B) \mid (\forall n \mid n \geq 0 \mid \beta_n = \alpha_n \circ f)\}$
    2. A limit of Δ is a terminal object of $LB(\Delta)$.

□

Note 2.14

1. It can easily be verified that $UB(\Delta)$ and $LB(\Delta)$ are categories indeed. Composition and identities are as in K.
2. The unique arrow f such that diagram (3) commutes is called the mediating morphism from the colimit (A, α) to the cocone (B, β).
3. Colimits and limits are unique up to isomorphism.
4. Deletion of an initial part of an ω-chain does not affect the colimit property, i.e. for all $k, l : 0 \leq k \leq l$:

   if $(A, <\alpha_n>_{n=k}^{\infty})$ is a colimit for $<(D_n, f_n)>_{n=k}^{\infty}$,
   then $(A, <\alpha_n>_{n=l}^{\infty})$ is a colimit for $<(D_n, f_n)>_{n=l}^{\infty}$.
   [Compare with note 1.8.2]

□

Given the above definitions the generalization of the notion ω-cpo is straightforward.

Definition 2.15 [ω-category, ω^{op}-category]

1. An <u>ω-complete category</u> (<u>ω-category</u> for short) is a category which has an initial object and in which each ω-chain has a colimit.
2. An <u>ω^{op}-complete category</u> (<u>ω^{op}-category</u> for short) is a category which has a terminal object and in which each ω^{op}-chain has a limit.

□

Given two posets $(C_1, \sqsubseteq_1)$ and $(C_2, \sqsubseteq_2)$ one could consider arbitrary functions from C_1 to C_2 but more interesting are the monotonic functions, i.e. those functions f satisfying the additional requirement $(\forall x, y \in C_1 \mid x \sqsubseteq_1 y \Rightarrow f(x) \sqsubseteq_2 f(y))$.
One could say that a monotonic function is in a sense "structure preserving". For categories there exists a similar notion. Mappings between categories should at least map objects to objects and arrows to arrows. The "structure" of a category is determined by its commuting diagrams, and if we are interested in "structure preserving" mappings we should consider mappings with the additional property that commuting diagrams are mapped to commuting diagrams. Such mappings are called functors and are defined as follows:

<u>Definition 2.16</u> [functor]
Let K and L be categories.
A <u>functor</u> F from K to L consists of

1. A map $Obj(K) \rightarrow Obj(L)$
2. For all $A, B \in Obj(K)$, a map $Hom_K(A, B) \rightarrow Hom_L(F(A), F(B))$
   such that
   – if $g \circ f$ is defined in K, then $F(g \circ f) = F(g) \circ F(f)$
   – for all $A \in Obj(K) : F(I_A) = I_{F(A)}$

□

Note that in the special case of the category $K_{(C,\sqsubseteq)}$ a functor from $K_{(C_1,\sqsubseteq_1)}$ to $K_{(C_2,\sqsubseteq_2)}$ corresponds to a monotonic function from $(C_1, \sqsubseteq_1)$ to $(C_2, \sqsubseteq_2)$.

From Def. 2.16 it follows immediately that cocones are mapped to cocones. If we require the additional property that colimits are mapped to colimits, we obtain the counterpart of an ω-continuous function:

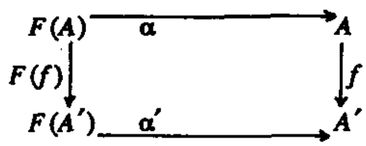
<u>Definition 2.17</u> [ω-cocontinuous functor; compare with Def. 1.12]
Let K and L be ω-categories.
A functor F from K to L is <u>ω-cocontinuous</u> *iff* F preserves colimits, i.e.
for each ω-chain $<(D_i, f_i)>_{i=0}^{\infty}$ in K with colimit $(D, <\alpha_i>_{i=0}^{\infty})$, the pair $(F(D), <F(\alpha_i)>_{i=0}^{\infty})$ is colimit of the ω-chain (in L) $<(F(D_i), F(f_i))>_{i=0}^{\infty}$.

□

<u>Note 2.18</u>
In the sequel we shall often abbreviate "ω-cocontinuous" to "continuous".

Given a poset $(C, \sqsubseteq)$ and a function $f \in C \to C$, a prefixed point of f is an $x \in C$ such that $f(x) \sqsubseteq x$. In terms of the category $K_{(C,\sqsubseteq)}$ and a functor F from $K_{(C,\sqsubseteq)}$ to $K_{(C,\sqsubseteq)}$ a prefixed point of F would be an object A for which there exists an arrow from $F(A)$ to A. In $K_{(C,\sqsubseteq)}$ such an arrow is unique but in an arbitrary category there may be many arrows from $F(A)$ to A. Therefore it is more appropriate to define a prefixed point of F as a pair (A, α), where A is an object and $\alpha \in Hom(F(A), A)$.

The least prefixed point of a function f has been defined as the least element of the set $PFP(f)$ of prefixed points of f. The obvious generalization of this notion is that of an initial object of a category $PFP(F)$ of prefixed points of a functor F. The objects of this category are the pairs (A, α) as mentioned above. For the morphisms from (A, α) to (A', α') we could take all morphisms from A to A' in the original category, but this seems too general as it does not involve α and α'. The obvious restriction is to those morphisms f for which the following diagram commutes.



For fixed points the reasoning is similar. As equality of objects cannot be proved in general, we should content ourselves with isomorphism and define a fixed point of a functor F as a pair (A, α) such that α is an isomorphism from $F(A)$ to A. The complete definitions follow.

Definition 2.19 [(pre-) fixed point]

Let K be a category; F a functor from K to K.

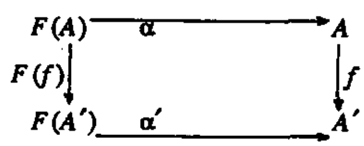
1. A prefixed point of F is a pair (A, α), where $A \in Obj(K)$ and $\alpha \in Hom(F(A), A)$.
2. A fixed point of F is a pair (A, α), where (A, α) is a prefixed point of F and α is an isomorphism.

□

Definition 2.20 [$PFP(F), FP(F)$]

Let K be a category; F a functor from K to K.

1. The category of prefixed points of F, denoted $PFP(F)$, is the category X with
   - $Obj(X)$ is the class of prefixed points of F.
   - for each $(A, \alpha), (A', \alpha') \in Obj(X)$, $Hom_X((A, \alpha), (A', \alpha'))$ is the set of arrows f in $Hom_K(A, A')$, for which the following diagram commutes:



2. The category of fixed points of F, denoted $FP\ (f)$, is the category X with
   – $Obj\ (X)$ is the class of fixed points of F
   – arrows defined similarly to 1.

□

Note 2.21

1. It can easily be verified that both $PFP\ (F)$ and $FP\ (f)$ are categories indeed. Identity and composition are inherited from K.

2. As each fixed point of F is also a prefixed point of F, it follows that $FP\ (F)$ is a full subcategory of $PFP\ (F)$.

3. The objects of $PFP\ (F)$ are also called F-algebras.

□

We are now ready to formulate and prove Lemma 2.22, which is a generalization of Lemma 1.14. The structure of the proof parallels that of Lemma 1.14 as much as possible:

Steps	1 - 6	correspond exactly
Steps	7 - 14	correspond to step 7 in the proof of Lemma 1.14: proving an arrow to be an isomorphism is the counterpart of using the antisymmetry of $\sqsubseteq$.
Steps	15 - 18	correspond to steps 8 - 11 of Lemma 1.14.

Lemma 2.22 [initial prefixed point is also initial fixed point; compare with Lemma 1.14]
Let K be a category; F a functor from K to K.
If (A, α) is an initial object of $PFP\ (F)$, then it is an initial object of $FP\ (F)$.

□

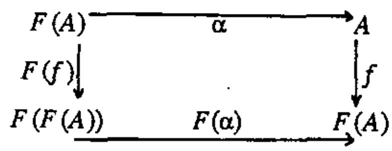
Proof

1. F is a functor from K to K.
2. (A, α) is an initial object of $PFP\ (F)$.

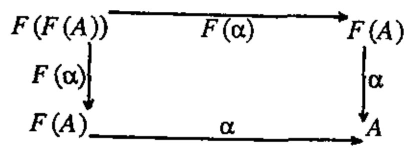
   ["$(A, \alpha) \in Obj\ (FP\ (F))$"]

3. $A \in Obj\ (K) \wedge \alpha : F(A) \rightarrow A$ [2, Def. 2.20.1]
4. $F(A) \in Obj\ (K) \wedge F(\alpha) : F(F(A)) \rightarrow F(A)$ [3,1]
5. $(F(A), F(\alpha)) \in Obj\ (PFP\ (F))$ [4, Def. 2.20.1]
6. $(\exists ! f \in (A, \alpha) \rightarrow (F(A), F(\alpha)))$ [5,2]
7. Let $f \in (A, \alpha) \rightarrow (F(A), F(\alpha))$ [6]
8. diagram (1) commutes [7, Def. 2.20.1]
9. diagram (2) commutes [trivial]
10. $\alpha : (F(A), F(\alpha)) \rightarrow (A, \alpha)$ [9, Def. 2.20.1]

11.	$\alpha \circ f = I_A$	[2,6,7,10]
12.	$f \circ \alpha$	
	$= F(\alpha) \circ F(f)$	[8]
	$= F(\alpha \circ f)$	[1]
	$= F(I_A)$	[11]
	$= I_{F(A)}$	[1]
13.	α is an isomorphism with inverse f	[11,12]
14.	$(A, \alpha) \in Obj(FP(F))$	[3,13, Def. 2.20.2]



(1)



(2)

["(A, α) is an initial object of $FP(F)$"]

15.	$(\forall (B, \beta) \in Obj(PFP(F)) \mid (\exists! f : (A, \alpha) \to (B, \beta)))$	[2]
16.	$Obj(FP(F))$ is a subclass of $Obj(PFP(F))$	[Def. 2.19, 2.20]
17.	$(\forall (B, \beta) \in Obj(FP(F)) \mid (\exists! f : (A, \alpha) \to (B, \beta)))$	[15,16]
18.	(A, α) is an initial object of $FP(F)$	[14,17]

□

Finally we present the main result of this chapter, viz. a generalization to categories of the least fixed point theorem 1.16. The proof of theorem 2.23 resembles that of theorem 1.16 as much as possible. Steps 1-14 are in exact correspondence; the remaining part deals with the initiality of the fixed point and is of course more involved than its counterpart in theorem 1.16.

Theorem 2.23 [initial fixed point theorem; compare with Theorem 1.16]

Let K be an ω-category;

F an ω-cocontinuous functor from K to K;

U an initial object of K;

u the unique arrow from U to $F(U)$.

Then

a. $<(F^n(U), F^n(u))>_{n=0}^{\infty}$ is an ω-chain.

b. Let (A, μ) be a colimit of $<(F^n(U), F^n(u))>_{n=0}^{\infty}$.
$(\exists \alpha \in Hom(F(A), A) \mid (A, \alpha)$ is an initial object in $FP(F))$.

□

Proof

a. Induction on n:

base step

1. $u \in Hom(U, F(U))$
2. $F^0(u) \in Hom(F^o(U), F^1(U))$

induction step

3. Let $n \geq 0$
4. $F^n(u) \in Hom(F^n(U), F^{n+1}(U))$ [ind. hyp.]
5. $F(F^n(u)) \in Hom(F(F^n(U)), F(F^{n+1}(U)))$ [4, F functor]
6. $F^{n+1}(u) \in Hom(F^{n+1}(U), F^{n+2}(U))$ [5]

7. $(\forall n \mid n \geq 0 \mid F^n(u) \in Hom(F^n(U), F^{n+1}(U)))$ [2, 3-6, induction]

b.

b1. ["$(\exists \alpha \in Hom(F(A), A) \mid (A, \alpha) \in Obj(FP(F)))$"]

8. $(A, <\mu_n>_{n=0}^{\infty})$ is a colimit of $<(F^n(U), F^n(u))>_{n=0}^{\infty}$
9. $(A, <\mu_n>_{n=1}^{\infty})$ is a colimit of $<(F^n(U), F^n(u))>_{n=1}^{\infty}$ [note 2.14.4]
10. $F(A), <\mu_{n+1}>_{n=0}^{\infty})$ is a colimit of $<(F^{n+1}(U), f^{n+1}(u))>_{n=0}^{\infty}$ [9]

12. $(F(A), <F(\mu)_n>_{n=0}^{\infty})$ is a colimit of $<(F^{n+1}(U), F^{n+1}(u))>_{n=0}^{\infty}$ [8, F ω-cocont.]

13. $(\exists ! \alpha \in Hom(F(A), A) \mid \alpha$ is mediating and isomorphism) [10, 12]
14. $(\exists \alpha \in Hom(F(A), A) \mid (A, \alpha) \in Obj(FP(F)))$ [13]

b2. ["(A, α) is initial in $FP(F)$"]

15. Let α be the mediating isomorphism between $F(A)$ and A. [13]
16. Let $(B, \beta) \in Obj(FP(F))$.

[We have to show : $(\exists ! \xi \in Hom_{FP(F)}((A, \alpha), (B, \beta)))$.
The proof proceeds in three steps.
- In step $c1$ we construct a cocone (B, ν) for Δ and take the mediating morphism from A to B as candidate for ξ.
- In step $c2$ we show $\xi \in Hom_{FP(F)}((A, \alpha), (B, \beta))$
- In step $c3$ we show that ξ is the only element of $Hom_{FP(F)}((A, \alpha), (B, \beta))$
]

c1. ["construction of a cone (B, ν) for Δ"]

17. Let ν_0 be the unique arrow from U to B.
18. For all $n \geq 0$, let $\nu_{n+1} = \beta \circ F(\nu_n)$

[We show by induction : $(\forall n \mid n \geq 0 \mid \nu_{n+1} \circ F^n(u) = \nu_n)$]

base step

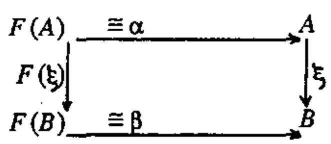
19. $\nu_1 \circ F^0(u)$
$= \beta \circ F(\nu_0) \circ u$ [18]
$= \nu_0$ [U initial]

induction step

20. Let $n \geq 0$
21. $\nu_{n+1} \circ F^n(u) = \nu_n$ [ind. hyp.]
22. $\nu_{n+2} \circ F^{n+1}(u)$
$= \beta \circ F(\nu_{n+1}) \circ F^{n+1}(u)$ [18]
$= \beta \circ F(\nu_{n+1} \circ F^n(u))$ [F functor]
$= \beta \circ F(\nu_n)$ [21]
$= \nu_{n+1}$ [18]

23. $(\forall n \mid n \geq 0 \mid \nu_{n+1} \circ F^n(u) = \nu_n)$ [19, 20-22, ind.]
24. (B, ν) is a cocone for Δ [23, Def. 2.11]
25. $(\exists! \xi \in Hom(A, B) \mid (\forall n \mid n \geq 0 \mid \nu_n = \xi \circ \mu_n))$ [24, 8, colimit prop.]
26. Let ξ be the mediating morphism from A to B [25]

c2. [To show; $\xi \in Hom_{FP(F)}((A, \alpha), (B, \beta))$, i.e. commutativity of the following diagram:



]

27. $(\forall n \mid n \geq 0 \mid \alpha \circ F(\mu_n) = \mu_{n+1})$ [15]
28. $(\forall n \mid n \geq 0 \mid \beta \circ F(\nu_n) = \nu_{n+1})$ [18]

29. Let $n \geq 0$.
30. $\beta \circ F(\xi) \circ \alpha^{-1} \circ \mu_{n+1}$
$= \beta \circ F(\xi) \circ F(\mu_n)$ [27]
$= \beta \circ F(\xi \circ \mu_n)$ [F functor]
$= \beta \circ F(\nu_n)$ [25/26]
$= \nu_{n+1}$ [18]
31. $(\forall n \mid n \geq 1 \mid \beta \circ F(\xi) \circ \alpha^{-1} \circ \mu_n = \nu_n)$ [29, 30]
32. $\beta \circ F(\xi) \circ \alpha^{-1} \circ \mu_0 = \nu_0$ [17, U initial]
33. $(\forall n \mid n \geq 0 \mid \nu_n = \beta \circ F(\xi) \circ \alpha^{-1} \circ \mu_n)$ [31, 32]
34. $\beta \circ F(\xi) \circ \alpha^{-1} = \xi$ [25/26, 33]
35. $\xi \in Hom_{FP(F)}(A, \alpha), (B, \beta))$ [34, Def. 2.20.2]

c3. [To show : ξ is the only element of $Hom_{FP(F)}((A, \alpha), (B, \beta))$]

36. Let $\lambda \in Hom_{FP(F)}((A, \alpha), (B, \beta))$

37. $\lambda \circ \alpha = \beta \circ F(\lambda)$ [36, Def. 2.20.2]

[We show by induction : $(\forall n \mid n \geq 0 \mid \lambda \circ \mu_n = \nu_n)$]

base step

38. $\lambda \circ \mu_0 = \nu_0$ [17, U initial]

induction step

39. Let $n \geq 0$

40. $\lambda \circ \mu_n = \nu_n$ [ind. hyp.]

41. $\lambda \circ \mu_{n+1}$

$= \lambda \circ \alpha \circ F(\mu_n)$ [27]

$= \beta \circ F(\lambda) \circ F(\mu_n)$ [37]

$= \beta \circ F(\lambda \circ \mu_n)$ [F functor]

$= \beta \circ F(\nu_n)$ [40]

$= \nu_{n+1}$ [18]

42. $(\forall n \mid n \geq 0 \mid \lambda \circ \mu_n = \nu_n)$ [38, 39-41, ind.]

43. $\lambda = \xi$ [25/26, 42]

□

§ 3. O-categories and local criteria for initiality and continuity

Since in general it is difficult to verify whether a given category is an ω-category and similarly whether a functor is ω-continuous we shall introduce the concept of an O-category.

With an O-category we can associate a "derived" category of so-called projection pairs. Our main interest is in these categories, for there are relatively easy (local) criteria to see whether these categories are ω-categories and also whether functors between these categories are ω-continuous.

In sections 3.1, 3.2 and 3.3 many theoretical results are given. In section 3.4 these results are applied to present some examples of ω-categories while in section 3.5 several functors are shown to be ω-continuous.

3.1. O-categories and the initiality theorem

Definition 3.1 [O-category]
An O-category is a category s.t.

- every hom-set is a poset in which every ascending ω-chain has a l.u.b.
- composition is ω-continuous in the sense of definition 1.12.

□

Definition 3.2 [projection pair]
Let K be an O-category; $A, B \in Obj(K)$.
A projection pair from A to B is a pair (f, g) s.t.

- $f \in Hom_K(A, B)$
- $g \in Hom_K(B, A)$
- $g \circ f = I_A$
- $f \circ g \sqsubseteq I_B$

□

Definition 3.3 [embedding, projection]
Let (f, g) be a projection pair. Then f is called an embedding and g a projection.

□

Notation 3.4
The first coordinate of a projection pair α is denoted by α^L and the second coordinate by α^R.

□

Lemma 3.5
Let K be an O-category; $A, B \in Obj(K)$; (f, g) and (f', g') projection pairs from A to B. Then

1. $f \sqsubseteq f' \Leftrightarrow g \sqsupseteq g'$

2. $f = f' \Leftrightarrow g = g'$

□

Proof

1. Assume $f \sqsubseteq f'$. Since composition is continuous, hence monotonic we find:
   $g = g \circ I_B \sqsupseteq g \circ (f' \circ g') = (g \circ fsupprime) \circ g' \sqsupseteq (g \circ f) \circ g' = I_A \circ g' = g'$.
   Assume $g \sqsupseteq g'$.
   $f = f \circ I_A = f \circ (g' \circ f') \sqsubseteq f \circ (g \circ f') = (f \circ g) \circ f' \sqsubseteq I_B \circ f' = f'$.
2. $f = f' \Leftrightarrow f \sqsubseteq f' \wedge f' \sqsubseteq f \Leftrightarrow g \sqsupseteq g' \wedge g' \sqsupseteq g \Leftrightarrow g = g'$.

□

Definition 3.6 [K_{PR}, category of projection pairs]
Let K be an O-category.
The category K_{PR} is the category M s.t.

- $Obj(M) = Obj(K)$
- For all $A, B \in Obj(M)$:
  $Hom_M(A, B)$ is the set consisting of all projection pairs from A to B.
- The composition of $\beta \in Hom_M(B, C)$ and $\alpha \in Hom_M(A, B)$ is the pair $(\beta^L \circ \alpha^L, \alpha^R \circ \beta^R)$. This is indeed a projection pair from A to C.
- The identity of $Hom_M(A, A)$ is (I_A, I_A).

□

Note 3.7
It is easily verified that K_{PR} is indeed a category.

□

Remark 3.8
Since by lemma 3.5 a projection pair is uniquely determined by its embedding part, the category K_{PR} is isomorphic to the subcategory of K consisting of all objects of K but with as morphisms only the embeddings. This category is sometimes denoted by K^E. A similar remark holds when we replace embeddings by projections (not: projection pairs!).

The reason for choosing K_{PR} instead of K^E is two-fold: First of all the treatment becomes more symmetric. Secondly, if we would have chosen K^E there would be more chance to confuse morphisms from K^E with those of K.

□

Lemma 3.9
Let K be an O-category; Δ an ω-chain in K_{PR}; (D, α) and (E, β) cocones for Δ.
Then $<\alpha_i^L \circ \beta_i^R>_{i \geq 0}$ is an ascending chain in $Hom_K(E, D)$.
Moreover, if $f = \bigsqcup_{i \geq 0} (\alpha_i^L \circ \beta_i^R)$ then for all $n \geq 0$:

$\alpha_n^R \circ f = \beta_n^R$ and

$f \circ \beta_n^L = \alpha_n^L$.

□

Proof

Let $\Delta = <(D_i, f_i)>_{i \geq 0}$ and $n \geq 0$.

1. $\alpha_n^L \circ \beta_n^R$
$= (\alpha_{n+1} \circ f_n)^L \circ (\beta_{n+1} \circ f_n)^R$
$= \alpha_{n+1}^L \circ f_n^L \circ f_n^R \circ \beta_{n+1}^R$
$\sqsubseteq \alpha_{n+1}^L \circ I_{D_{n+1}} \circ \beta_{n+1}^R$
$= \alpha_{n+1}^L \circ \beta_{n+1}^R$.

2. $\bigsqcup_{i \geq 0} (\alpha_i^L \circ \beta_i^R)$ exists. [1 , K is an O-category]

3. Let $f = \bigsqcup_{i \geq 0} (\alpha_i^L \circ \beta_i^R)$ and let $n \geq 0$.
$\alpha_n^R \circ f$
= [3]
$\alpha_n^R \circ \bigsqcup_{i \geq 0} (\alpha_i^L \circ \beta_i^R)$
= [1 , ω-continuity of ∘]
$\bigsqcup_{i \geq n} (\alpha_n^R \circ \alpha_i^L \circ \beta_i^R)$
= [$\alpha_n = \alpha_i \circ f_{i-1} \circ \cdots \circ f_n$ for all $i \geq n$]
$\bigsqcup_{i \geq n} \beta_n^R$
=
β_n^R .
Similarly $f \circ \beta_n^L = \alpha_n^L$.

□

If we take $(E, \beta) = (D, \alpha)$ then $f = \bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R)$.

Of course $f \sqsubseteq I_D$. In the "limit-case" where equality holds we have the following result :

Theorem 3.10

Let K be an O-category; $\Delta = <(D_i, f_i)>_{i \geq 0}$ an ω-chain in K_{PR} ; (D, α) a cocone for Δ . Suppose $\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R) = I_D$.

Then (D, α) is a colimit for Δ .

Moreover, if (E, β) is a cocone for Δ then

$$\left(\bigsqcup_{i \geq 0} (\beta_i^L \circ \alpha_i^R), \bigsqcup_{i \geq 0} (\alpha_i^L \circ \beta_i^R)\right)$$

is the mediating morphism from D to E.

□

Proof

1. $\bigsqcup_{i\geq 0} (\alpha_i^L \circ \alpha_i^R) = I_D$ [assumption]
2. Let $f = \bigsqcup_{i\geq 0} (\alpha_i^L \circ \beta_i^R)$; $g = \bigsqcup_{i\geq 0} (\beta_i^L \circ \alpha_i^R)$
3. $f \in Hom_K(E, D)$ and $g \in Hom_K(D, E)$ [2, lemma 3.9]
4. $g \circ f$
   = [ω-continuity of $\circ$, 2]
   $\bigsqcup_{i\geq 0} (\beta_i^L \circ \alpha_i^R \circ \alpha_i^L \circ \beta_i^R)$
   = [$\alpha_i^R \circ \alpha_i^L = I_{D_i}$]
   $\bigsqcup_{i\geq 0} (\beta_i^L \circ \beta_i^R)$
5. $g \circ f \sqsubseteq I_E$ [4]
6. $f \circ g = \bigsqcup_{i\geq 0} (\alpha_i^L \circ \alpha_i^R)$ [4, symmetry]
7. $f \circ g = I_D$ [1, 6]
8. (g, f) is a projection pair from D to E [3, 5, 7]

[To show: (g, f) is the unique morphism η satisfying
(∗) $(\forall n \mid n \geq 0 \mid \eta \circ \alpha_n = \beta_n)$]

a. [" (g, f) satisfies (∗) "]
   9. $(g, f) \circ \alpha_n$
   = [def. of composition in K_{PR}]
   $(g \circ \alpha_n^L, \alpha_n^R \circ f)$
   = [Lemma 3.9]
   (β_n^L, β_n^R)
   $= \beta_n$

b. ["η satisfies (∗) => $\eta = (g, f)$ "]
   10. η
   =
   $(\eta^L \circ I_D, I_D \circ \eta^R)$
   = [1 , ω-continuity of $\circ$]
   $(\bigsqcup_{i\geq 0} (\eta^L \circ \alpha_i^L \circ \alpha_i^R), \bigsqcup_{i\geq 0} (\alpha_i^L \circ \alpha_i^R \circ \eta^R))$
   = [η satisfies (∗)]
   $(\bigsqcup_{i\geq 0} (\beta_i^L \circ \alpha_i^R), \bigsqcup_{i\geq 0} (\alpha_i^L \circ \beta_i^R))$
   = [2]
   (g, f)

□

We would like to have the converse of theorem 3.10, possibly under certain conditions. To present such a condition we need:

Definition 3.11 [localized category]

An O-category K is called localized fif for any ω-chain Δ in K_{PR} and for any Δ-colimit (D, α) there exists $E \in Obj(K)$ and a projection pair (i, g) from E to D s.t.

$$\bigsqcup_{n\geq 0} (\alpha_n^L \circ \alpha_n^R) = i \circ g$$

□

Theorem 3.12 [Initiality theorem]

Let K be a localized O-category;

$\Delta = \langle (D_i, f_i) \rangle_{i\geq 0}$ an ω-chain in K_{PR};

(D, α) a co-cone for Δ.

Then: (D, α) is a colimit for $\Delta \Leftrightarrow \bigsqcup_{i\geq 0} (\alpha_i^L \circ \alpha_i^R) = I_D$.

□

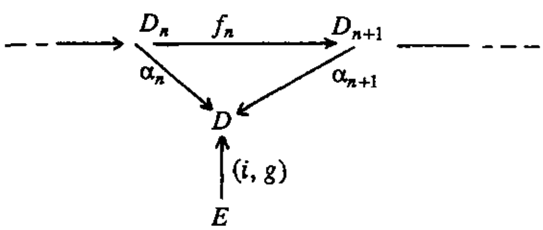
Proof

The " $\Leftarrow$ " part has already been proved (see Theorem 3.10).

[" $\Rightarrow$ "]

1. Let $f = \bigsqcup_{i\geq 0} (\alpha_i^L \circ \alpha_i^R)$
2. Let $(i, g) : E \to D$ be a projection pair s.t. $f = i \circ g$ [K is localized]

[We want to show that $f = I_D$. In order to do this, consider the following diagram:



Since (i, g) is a projection pair we have $g \circ i = I_E$. Thus to say that $f = i \circ g = I_D$ means that g is the inverse of i, i.e. that (i, g) has an inverse. Conversely, if we could prove that (i, g) has an inverse, then it is easy to show that $f = I_D$. In fact it suffices to show that there exists a projection

pair η s.t. $(i, g) \circ \eta = (I_D, I_D)$. The proof has the following structure:
In part a. we construct a sequence of projection pairs $\beta_n : D_n \rightarrow E$.
In part b. we show that $(E, <\beta_n>_{n\geq 0})$ is a co-cone for Δ.
In part c. we use the hypothesis that $(D, <\alpha_n>_{n\geq 0})$ is a colimit for Δ to show the existence of an η s.t. $(i, g) \circ \eta = (I_D, I_D)$.
In part d. we show $f = I_D$.
]

a. ["construction of projection pairs $\beta_n : D_n \rightarrow E$"]

3. Let for all $n \geq 0$, $\beta_n = (g \circ \alpha_n^L, \alpha_n^R \circ i)$

4. $\beta_n^R \circ \beta_n^L$
 $=$ $(\alpha_n^R \circ i) \circ (g \circ \alpha_n^L)$
 $=$ [2]
 $\alpha_n^R \circ f \circ \alpha_n^L$
 $=$ [Lemma 3.9]
 $\alpha_n^R \circ \alpha_n^L$
 $=$ [α_n is a projection pair from D_n to D]
 I_{D_n}

5. $\beta_n^L \circ \beta_n^R$
 $=$ $(g \circ \alpha_n^L) \circ (\alpha_n^R \circ i)$
 $\sqsubseteq$ [α_n is a projection pair]
 $g \circ i$
 $=$ [2]
 I_E

6. For all $n \geq 0$, β_n is a projection pair from D_n to E [4, 5]

b. ["$(E, <\beta_n>_{n\geq 0})$ is a co-cone for Δ"]

7. $\beta_{n+1} \circ f_n$
 $=$ [3]
 $(g \circ \alpha_{n+1}^L \circ f_n^L, f_n^R \circ \alpha_{n+1}^R \circ i)$
 $=$
 $(g \circ (\alpha_{n+1} \circ f_n)^L, (\alpha_{n+1} \circ f_n)^R \circ i)$
 $=$ [$\alpha_{n+1} \circ f_n = \alpha_n$]
 $(g \circ \alpha_n^L, \alpha_n^R \circ i)$
 $=$ [3]
 β_n

8. $(E, <\beta_n>_{n\geq 0})$ is a co-cone for Δ [7]

c. ["existence of an η such that $(i, g) \circ \eta = (I_D, I_D)$"]

9. Let $\eta : D \rightarrow E$ be the unique projection pair
 s.t. $(\forall n \mid n \geq 0 \mid \eta \circ \alpha_n = \beta_n)$ [8, (D, α) is a colimit]

10. $(i, g) \circ \eta$ is a projection pair from D to D [2, 9]

11. $(i, g) \circ \eta \circ \alpha_n$

$=$ [9]

$(i, g) \circ \beta_n$

$=$ [3]

$(i \circ g \circ \alpha_n^L, \alpha_n^R \circ i \circ g)$

$=$ [2]

$(f \circ \alpha_n^L, \alpha_n^R \circ f)$

$=$ [Lemma 3.9]

(α_n^L, α_n^R)

$=$

α_n

12. $(i, g) \circ \eta = (I_D, I_D)$ [10, 11, (D, α) is colimit]

d. ["$f = I_D$"]

13. I_D

$=$

$I_D \circ I_D$

$=$ [12]

$i \circ \eta^L \circ \eta^R \circ g$

$\sqsubseteq$

$i \circ g$

$=$ [2]

f

14. $f = I_D$ [13, $f \sqsubseteq I_D$]

15. $\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R) = I_D$ [1, 14]

□

3.2. Functors and the continuity theorem

Let K, L be two O-categories and $F : K \to L$ a functor. Since morphisms of K_{PR} are pairs of morphisms of K, it seems likely that F induces a functor, say F_{PR} from K_{PR} to L_{PR}. However, this is not quite true. If we define

$$F_{PR}(A) = F(A) \quad (A \in Obj(K_{PR}))$$

and

$$F_{PR}((f, g)) = (F(f), F(g)) \quad ((f, g) \text{ a projection pair in } K)$$

then we would like to show that $(F(f), F(g))$ is a projection pair in L.
Supposing that (f, g) goes from D to E we know $g \circ f = I_D$, hence $F(g) \circ F(f) = F(g \circ f) = F(I_D) = I_{F(D)}$; but from $f \circ g \sqsubseteq I_E$ we can in general not deduce that

$F(f) \circ F(g) \sqsubseteq I_{F(E)}$. However, if F is locally monotonic (see Def. below), then the latter holds.

Definition 3.13 [local monotonicity]

Let K, L be O-categories; F a functor from K to L.

F is locally monotonic *fif*

for all $A, B \in Obj(K)$:

F, viewed as a map: $Hom_K(A, B) \to Hom_L(F(A), F(B))$, is monotonic.

□

Suppose that F is a locally monotonic functor from K to L where K and L are O-categories. Then F_{PR} mapping an object A of K_{PR} to $F(A)$ and a projection pair η to

$$F_{PR}(\eta) = (F(\eta^L), F(\eta^R)) \qquad (1)$$

is indeed a functor: $K_{PR} \to L_{PR}$.

We are interested in establishing a relatively easy criterion for ω-continuity of F_{PR}. For this the initiality theorem looks promising. However in order to apply this theorem we have to assume that the O-category K is localized. So let's make this assumption. To investigate ω-continuity of F_{PR} let's also fix an ω-chain $\Delta = <(D_i, f_i)>_{i \geq 0}$ in K_{PR} with colimit $(D, <\alpha_i>_{i \geq 0})$.

Since F_{PR} is a functor $(F_{PR}(D), <F_{PR}(\alpha_i)>_{i \geq 0})$ is a co-cone for the ω-chain $\Delta' = <(F_{PR}(D_i), F_{PR}(f_i)>_{i \geq 0}$ in L_{PR}.

As $(D, <\alpha_i>_{i \geq 0})$ is a colimit for Δ the initiality theorem shows

$$\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R) = I_D \qquad (2)$$

Moreover, we have

$(F_{PR}(D), <F_{PR}(\alpha_i)>_{i \geq 0})$ is a colimit for Δ'

$\Leftarrow$ [theorem 3.10]

$$\bigsqcup_{i \geq 0} (F_{PR}(\alpha_i)^L \circ F_{PR}(\alpha_i)^R) = I_{F(D)}$$

$\Leftrightarrow$ [by (1) above $F_{PR}(\alpha_i)^L = F(\alpha_i^L)$ and $F_{PR}(\alpha_i)^R = F(\alpha_i^R)$]

$$\bigsqcup_{i \geq 0} (F(\alpha_i^L) \circ F(\alpha_i^R)) = I_{F(D)}$$

$\Leftrightarrow$ [F is a functor]

$$\bigsqcup_{i \geq 0} (F(\alpha_i^L \circ \alpha_i^R)) = F(I_D)$$

$\Leftrightarrow$ [2]

$$\bigsqcup_{i \geq 0} (F(\alpha_i^L \circ \alpha_i^R)) = F(\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R))$$

This derivation motivates the following definition.

<u>Definition 3.14</u> [local continuity]

Let K and L be O-categories;

F a functor from K to L.

F is <u>locally continuous</u> *iff*

for all $A, B \in Obj(K)$:

F, viewed as a map: $Hom_K(A, B) \to Hom_L(F(A), F(B))$,

is ω-continuous.

□

An immediate consequence of the derivation preceding this definition is the following result.

<u>Theorem 3.15</u> [continuity theorem]

Let K and L be O-categories; F a functor from K to L.

Suppose that F is locally continuous and K is localized.

Then $F_{PR} : K_{PR} \to L_{PR}$ is ω-continuous.

□

As an application, suppose K is a localized O-category and $F : K \to K$ is a locally continuous functor. Then $F_{PR} : K_{PR} \to K_{PR}$ is ω-continuous. Suppose in addition that K_{PR} is an ω-category with initial object $\perp$.

Then F_{PR} has an initial fixed point (A, α) where α is an isomorphism from $F(A)$ to A in K_{PR}. Our goal is to describe this isomorphism more explicitly. Recalling the proof of theorem 2.23 we know there exists a colimit $(A, <\mu_n : F^n(\perp) \to A>_{n\geq 1})$ for a certain ω-chain Δ. Also, $(F(A), <F_{PR}(\mu_{n-1}) : F^n(\perp) \to A>_{n\geq 1})$ is a colimit for Δ. Since (A, μ) is a colimit for Δ the initiality theorem implies $\bigsqcup_{n\geq 1} (\mu_n^L \circ \mu_n^R) = I_A$ and theorem 3.10 then shows that the mediating morphism from A to $F(A)$ is given by

$(\bigsqcup_{n\geq 1} (F_{PR}(\mu_{n-1})^L \circ \mu_n^R), \bigsqcup_{n\geq 1} (\mu_n^L \circ F_{PR}(\mu_{n-1})^R))$ hence by

$(\bigsqcup_{n\geq 1} (F(\mu_{n-1}^L) \circ \mu_n^R), \bigsqcup_{n\geq 1} (\mu_n^L \circ F(\mu_{n-1}^R)))$. Thus we have shown

<u>Corollary 3.16</u>

Let K be a localized O-category;

F a locally continuous functor from K to K.

Suppose that K_{PR} is an ω-category with initial object $\perp$ and that $(A, <\mu_n : F^n(\perp) \to A>_{n\geq 1})$ is a colimit as in theorem 2.23.b.

Then $(\bigsqcup_{n\geq 1} (F(\mu_{n-1}^L) \circ \mu_n^R), \bigsqcup_{n\geq 1} (\mu_n^L \circ F(\mu_{n-1}^R)))$ is the mediating isomorphism from A to $F(A)$. Its inverse is

$(\bigsqcup_{n\geq 1} (\mu_n^L \circ F(\mu_{n-1}^R)), \bigsqcup_{n\geq 1} (F(\mu_{n-1}^L) \circ \mu_n^R))$.

□

One of the conditions of the continuity theorem is that the O-category K is localized. In proposition 3.18 below we shall present an easy criterion to guarantee this.

Definition 3.17 [idempotent; split]

Let C be a category; $D \in Obj(C)$; $f \in Hom(D, D)$.

f is called an idempotent when $f \circ f = f$.

f is called split when there exist $E \in Obj(C)$, $g \in Hom(D, E)$, $h \in Hom(E, D)$ such that $f = h \circ g$ and $g \circ h = I_E$.

□

Note that any split morphism is idempotent.

Proposition 3.18

Let K be an O-category.

Suppose that every idempotent in K is split.

Then K is localized.

□

Proof

Let Δ be an ω-chain in K_{PR};

(D, α) a co-cone for Δ;

$f = \bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R)$

1. $f \circ f$

$=$

$(\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R)) \circ (\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R))$

$=$ [ω-continuity of $\circ$]

$\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R \circ \alpha_i^L \circ \alpha_i^R)$

$=$ [$\alpha_i^R \circ \alpha_i^L = I_{D_i}$]

$\bigsqcup_{i \geq 0} (\alpha_i^L \circ \alpha_i^R)$

$= f$

2. There exist $E \in Obj(K)$, $g \in Hom(D, E)$, $h \in Hom(E, D)$ with $g \circ h = I_E$ and $h \circ g = f$ [1, idempotents split]
3. $f \sqsubseteq I_D$
4. (h, g) is a projection pair from E to D s.t. $f = h \circ g$ [2, 3]
5. K is localized [Def3.11]

□

3.3. Some technical results

We are interested in constructing new localized O-categories from old ones. Below we shall present two such constructions. Some notions used in these constructions are defined in the Appendix.

Let us start with an arbitrary category K. By definition 2.4.3 K^{op} is the category which has the same objects as K, but with the arrows reversed: i.e. given

$A, B \in Obj(K^{op}) = Obj(K)$, $Hom_{K^{op}}(A, B) = Hom_K(B, A)$.

Composition of two arrows f and g in K^{op} is defined fif g and f can be composed in K. Moreover, if we denote composition in K by $\circ$ and in K^{op} by $*$, then $f * g = g \circ f$ (provided the latter is defined). Clearly K^{op} is a category. Moreover, if K is an O-category then so is K^{op}.

We are interested in the relation between $(K^{op})_{PR}$ and K_{PR}.

So let us consider a projection pair (f, g) from A to B in K^{op}, i.e. $f \in Hom_{K^{op}}(A, B)$, $g \in Hom_{K^{op}}(B, A)$, $g * f = I_A$ and $f * g \sqsubseteq I_B$. Then

$g \in Hom_K(A, B)$, $f \in Hom_K(B, A)$, $f \circ g = I_A$ and $g \circ f \sqsubseteq I_B$, i.e. $(g, f) : A \to B$ is a projection pair in K. It can now be verified that we get a functor $S : (K^{op})_{PR} \to K_{PR}$ by defining

$S(A) = A$ for an object A, and

$S(f, g) = (g, f)$ for a projection pair (f, g).

Indeed, if $\eta_i = (f_i, g_i)$ is a projection pair of K^{op} $(i = 1, 2)$ then $\eta_1 * \eta_2 = (f_1 * f_2, g_2 * g_1)$ hence $S(\eta_1 * \eta_2) = (g_1 \circ g_2, f_2 \circ f_1) = (g_1, f_1) \circ (g_2, f_2) = S(\eta_1) \circ S(\eta_2)$.

Since $(K^{op})^{op} = K$ we also have a functor going from $K_{PR} = ((K^{op})^{op})_{PR}$ to $(K^{op})_{PR}$ which is the inverse (see appendix, definition A.5) of S.

Summarizing the above we have

<u>Proposition 3.19</u>

Let K be an O-category. Then so is K^{op}. Moreover, there is an isomorphism $S : (K^{op})_{PR} \to K_{PR}$ acting as the identity on objects and interchanging the left and right part of projection pairs.

□

The functor S can be used to show that K^{op} is a localized O-category if K is. First of all by exercise A.6 S preserves colimits since S is an isomorphism. Secondly, if we keep the notation as in definition 3.11 we have to show the existence of a projection pair (i, g) in K^{op} s.t. $\bigsqcup_{n \geq 0} (\alpha_n^L * \alpha_n^R) = i * g$. Now using that K is localized we know that there exists a projection pair (g, i) in K s.t. $\bigsqcup_{n \geq 0} (S(\alpha_n)^L \circ S(\alpha_n)^R) = g \circ i$.

Since $S(\alpha_n) = (\alpha_n^R, \alpha_n^L)$ this means that

$\bigsqcup_{n \geq 0} (\alpha_n^L * \alpha_n^R) = i * g$. Also $(i, g) = S^{-1}(g, i)$ is a projection pair in K^{op} so we are done. Thus we have proved

<u>Corollary 3.20</u>

Suppose K is a localized O-category. Then so is K^{op}.

□

Next suppose we are given two categories K and L. Then we can form the category $K \times L$ (see Def. A.1) and if we assume that K and L are O-categories, then the same holds for $K \times L$ if we give the hom-sets
$Hom_{K \times L}((A, B), (A', B')) = Hom_K(A, A') \times Hom_L(B, B')$ the coordinate-wise ordering
$((f, g) \sqsubseteq (f', g')$ fiff $f \sqsubseteq f'$ and $g \sqsubseteq g')$.
Then such a hom-set is a poset with l.u.b.'s for ascending chains. Also composition is continuous since it is defined coordinate-wise.
To investigate the relation between $(K \times L)_{PR}$ and $K_{PR} \times L_{PR}$, use definition A.3 where the projection functors $\pi_1 : K \times L \to K$ and $\pi_2 : K \times L \to L$ are introduced. These functors are easily seen to be locally continuous. Hence we get functors $\pi_{1,PR} : (K \times L)_{PR} \to K_{PR}$ and $\pi_{2,PR} : (K \times L)_{PR} \to L_{PR}$. Of course if $K \times L$ were localized then $\pi_{j,PR}$ would be ω-continuous. On the other hand, in order to prove that $K \times L$ is localized, only assuming that K and L are localized, we will need that $\pi_{j,PR}$ is ω-continuous. To prove the latter, consider the functor $(\pi_{1,PR} \times \pi_{2,PR}) \circ \Delta_{(K \times L)_{PR}}$ (see Def. A.2, A.7) from $(K \times L)_{PR}$ to $K_{PR} \times L_{PR}$ acting as the identity on objects and mapping a morphism $\alpha = (\alpha^L, \alpha^R)$ to $((\pi_1(\alpha^L), \pi_1(\alpha^R)), (\pi_2(\alpha^L), \pi_2(\alpha^R)))$. This functor is an isomorphism : its inverse maps a morphism (f, g) to the projection pair $((f^L, g^L), (f^R, g^R))$ in $K \times L$.
We also have projection functors from $K_{PR} \times L_{PR}$ to K_{PR}, resp. L_{PR}. Composing these by $(\pi_{1,PR} \times \pi_{2,PR}) \circ \Delta_{(K \times L)_{PR}}$ we get $\pi_{1,PR}$, resp. $\pi_{2,PR}$ (by definition of $\pi_{1,PR} \times \pi_{2,PR}$).
Now projection functors preserve colimits (see exercise A.4) and the same holds for functors which are isomorphisms (A.6).
Hence $\pi_{j,PR}$, being the composition of two ω-continuous functors, is also ω-continuous ($j = 1, 2$).
We can now prove that $K \times L$ is localized if both K and L are : given an ω-chain Δ in $(K \times L)_{PR}$ with colimit $(D, <\alpha_n>_{n \geq 0})$ we have to show that $\bigsqcup_{n \geq 0} (\alpha_n^L \circ \alpha_n^R) = i \circ g$ for a suitable projection pair (i, g) in $K \times L$. We proceed as follows: since $\pi_{1,PR}$ is ω-continuous $(\pi_1(D), <\pi_{1,PR}(\alpha_n)>_{n \geq 0})$ is a colimit for some ω-chain in K_{PR}. Since K is localized it follows that $\bigsqcup_{n \geq 0} (\pi_{1,PR}(\alpha_n)^L \circ \pi_{1,PR}(\alpha_n)^R) = i_1 \circ g_1$ for some projection pair (g_1, i_1) in K. Since $\pi_{1,PR}(\alpha_n) = (\pi_1(\alpha_n^L), \pi_1(\alpha_n^R))$ and since π_1 is a locally continuous functor we obtain

$$\begin{aligned} i_1 \circ g_1 &= \bigsqcup_{n \geq 0} (\pi_{1,PR}(\alpha_n)^L \circ \pi_{1,PR}(\alpha_n)^R) \\ &= \bigsqcup_{n \geq 0} \pi_1(\alpha_n^L \circ \alpha_n^R) \\ &= \pi_1(\bigsqcup_{n \geq 0} (\alpha_n^L \circ \alpha_n^R)) \end{aligned}$$

Similarly
$i_2 \circ g_2 = \pi_2(\bigsqcup_{n \geq 0} (\alpha_n^L \circ \alpha_n^R))$, hence $\bigsqcup_{n \geq 0} (\alpha_n^L \circ \alpha_n^R) = (i_1 \circ g_1, i_2 \circ g_2) = (i_1, i_2) \circ (g_1, g_2)$ and $((i_1, i_2), (g_1, g_2))$ is a projection pair in $K \times L$.
Summarizing we have

<u>Proposition 3.21</u>

Suppose K and L are localized O-categories. Then so is $K \times L$. There is an isomorphism from $(K \times L)_{PR}$ to $K_{PR} \times L_{PR}$ mapping an object A to A and a morphism f to $((\pi_1(f^L), \pi_1(f^R)), (\pi_2(f^L), \pi_2(f^R)))$ where π_1 and π_2 are the projection functors from $K \times L$ to K, resp. L.

□

§ 4. Examples of localized O-categories

Let K be the category CPO where the objects are ω-cpo's and the morphisms are ω-continuous maps. In this case the hom-sets are again ω-cpo's and composition is ω-continuous.
Thus is K an O-category. We even have

Theorem 3.22

CPO is a localized O-category.

□

Proof

[By proposition 3.18 it suffices to show that every idempotent splits.]

1. Let $(D, \sqsubseteq)$ be an ω-cpo;
   $f : D \to D$ ω-continuous and $f \circ f = f$;
   $E = f(D)$;
   $\leq$ the restriction of $\sqsubseteq$ to E.

a. ["$(E, \leq)$ is an ω-cpo"]

   2. $(E, \leq)$ is a poset with smallest element $f(\bot_D)$. [$E \subseteq D$, f is monotonic]
   3. Let $\langle x_i \rangle_{i \geq 0}$ be an ascending chain in $(E, \leq)$.
   4. $\langle x_i \rangle_{i \geq 0}$ is an ascending chain in $(D, \sqsubseteq)$ [3, 1]
   5. $\bigsqcup_{i \geq 0} {}_D\, x_i$ exists [4, D is an ω-cpo]
   6. $\bigsqcup_{i \geq 0} {}_D\, x_i$
      = [$f \circ f = f$ hence f acts as the identity on $x_i \in E = f(D)$]
      $\bigsqcup_{i \geq 0} {}_D\, f(x_i)$
      = [f is ω-continuous]
      $f(\bigsqcup_{i \geq 0} {}_D\, x_i)$
      $\in$ [$E = f(D)$]
      E
   7. $\bigsqcup_{i \geq 0} {}_E\, x_i = \bigsqcup_{i \geq 0} {}_D\, x_i$ [6, $(E, \leq)$ is a subposet of $(D, \sqsubseteq)$]

8. Define $g : D \to E, x \mapsto f(x)$;
   $h : E \to D, x \mapsto x$.
9. g and h are ω-continuous [1, 7, 8]
10. $h \circ g = f$ and $g \circ h = I_E$ [8, $f \mid E = I_E$]
11. f is split [9, 10]
12. CPO is localized [1, 11, proposition 3.18]

□

Given an O-category K we can form K_{PR} and we have a nice criterion (theorem 3.10) to see if certain cocones in K_{PR} are actually colimits. We shall use this criterion now to prove

Theorem 3.23

$\underline{\text{CPO}}_{PR}$ is an ω-category.

□

Proof

a. ["existence of initial object"]

1. Let A be a cpo consisting of a single element a.
2. Let D be any cpo.
3. Define $f : A \to D$ by $f(a) = \bot_D$ and $g : D \to A$ by $g(x) = a$ $(\forall x \in D)$.
4. $g \circ f = I_A$ and $f \circ g \sqsubseteq I_D$. [3]
5. (f, g) is a projection pair from A to D. [f and g are ω-continuous, 4]
6. Let (f', g') be a projection pair from A to D.
7. $g' = g$. [$A = \{a\}$]
8. $(f', g') = (f, g)$. [Lemma 3.5]
9. A is an initial object of $\underline{\text{CPO}}_{PR}$. [5, 6, 8]

b. ["existence of colimits"]

10. Let $\Delta = <(D_n, f_n)>_{n \geq 0}$ be an ω-chain in $\underline{\text{CPO}}_{PR}$.
11. Let for all n, r with $n \leq r$ the projection pair $f_{r-1} \circ \cdots \circ f_n : D_n \to D_r$ be a denoted by f_{nr}.
12. Let $D = \{ <x_n>_{n \geq 0} \mid (\forall n \mid n \geq 0 \mid x_n \in D_n \wedge x_n = f_n^R(x_{n+1})) \}$ and let $\sqsubseteq$ be the componentwise ordering on D.

b1. ["$(D, \sqsubseteq)$ is an ω-cpo."]

13. $(D, \sqsubseteq)$ is a poset. [by definition of $\sqsubseteq$]
14. $<\bot_{D_n}>_{n \geq 0}$ is the least element of D. [$f_n^R(\bot_{D_{n+1}}) = \bot_{D_n}$, as will be shown in lemma 3.31]
15. Let $<x^{(i)}>_{i \geq 0}$ be an ascending chain in D and let $x^{(i)} = <x_n^{(i)}>_{n \geq 0}$.
16. $(\forall n \mid n \geq 0 \mid <x_n^{(i)}>_{i \geq 0}$ is an ascending chain in $D_n)$.
17. Let $x_n = \bigsqcup_{i \geq 0} x_n^{(i)}$ for all $n \geq 0$. [16, D_n is an ω-cpo]
18. $(\forall n \mid n \geq 0 \mid x_n \in D_n \wedge x_n = f_n^R(x_{n+1}))$. [17, f_n^R is ω-continuous]
19. Let $x = <x_n>_{n \geq 0}$.
20. $x \in D$ and $x = \bigsqcup_{i \geq 0} x^{(i)}$. [19, 18, 17, 15]
21. $(D, \sqsubseteq)$ is an ω-cpo. [13, 14, 15, 20]

b2. ["construction of projection pairs $\alpha_n : D_n \to D$ for all n."]

22. Define $\pi_n : D \to D_n$ by $\pi_n(<x_i>_{i \geq 0}) = x_n$ and

$\rho_n : D_n \to D$ by $\rho_n(t) = <y_i>_{i\geq 0}$ where
$y_i = f^L_{ni}(t)$ if $i \geq n$ and $y_i = f^R_{in}(t)$ if $i \leq n$.

23. π_n is ω-continuous for all n. [15, 20, 19, 17]

24. ρ_n is ω-continuous for all n. [f^L_{ni} and f^R_{in} are all ω-continuous]

25. $\pi_n \circ \rho_n = I_{D_n}$ [$f_n = (I_{D_n}, I_{D_n})$]

[To show $\rho_n \circ \pi_n \sqsubseteq I_D$]

26. Let $x = <x_i>_{i\geq 0} \in D$.

27. For all $i \leq n$

$f^R_{in}(x_n)$

$=$ [$f_{in} = f_{n-1} \circ \cdots \circ f_i$]

$f^R_i \circ \cdots \circ f^R_{n-1}(x_n)$

$=$ [$x \in D$, 12]

x_i

28. For all $i \geq n$

$f^L_{ni}(x_n)$

$=$ [$n \leq i$ so by 27 $f^R_{ni}(x_i) = x_n$]

$f^L_{ni}(f^R_{ni}(x_i))$.

$\sqsubseteq$ [f_{ni} is a projection pair]

x_i

29. $\rho_n \circ \pi_n(x)$

$=$ [22,26]

$\rho_n(x_n)$

$\sqsubseteq$ [22, 27, 28]

$<x_i>_{i\geq 0}$

30. $\rho_n \circ \pi_n \sqsubseteq I_D$.

31. $\alpha_n = (\rho_n, \pi_n) \in Hom_{\underline{CPO}_{pr}}(D_n, D)$ [23, 24, 25, 30]

32. α^R_n

$=$ [31]

π_n

$=$ [22, 12]

$f^R_n \circ \pi_{n+1}$

$=$ [31]

$(\alpha_{n+1} \circ f_n)^R$

33. $\alpha_n = \alpha_{n+1} \circ f_n$ [Lemma 3.5]

34. $(D, <\alpha_n>_{n\geq 0})$ is a cocone for Δ. [10, 31, 33]

35. For all $i \geq 0$

$\pi_i \circ (\bigsqcup_{n\geq 0} (\rho_n \circ \pi_n))$

$=$ [ω-continuity of $\circ$]

$\bigsqcup_{n\geq 0} (\pi_i \circ \rho_n \circ \pi_n)$

$=$

$\bigsqcup_{n\geq i} ((\pi_i \circ \rho_n) \circ \pi_n)$

$=$ [22]

$\bigsqcup_{n\geq i} (f_{in}^R \circ \pi_n)$

$=$ [11, 12]

$\bigsqcup_{n\geq i} \pi_i$

$=$

π_i

36. Let $h = \bigsqcup_{n\geq 0} (\rho_n \circ \pi_n)$;

$x = \langle x_j \rangle_{j\geq 0} \in D$, $h(x) = \langle y_j \rangle_{j\geq 0}$

Then for all $i \geq 0$

x_i

$=$

$\pi_i(x)$

$=$ [35]

$(\pi_i \circ h)(x)$

$=$

y_i

37. $\bigsqcup_{n\geq 0} (\rho_n \circ \pi_n) = I_D$ [36]

38. $(D, \langle \alpha_n \rangle_{n\geq 0})$ is a colimit for Δ [34, 37, theorem 3.10]

□

There are certain subcategories of CPO for which theorem 3.22 and 3.23 also hold. The objects of these subcategories are posets which have l.u.b.'s for more subsets then just the ascending chains. Also, morphisms preserve more l.u.b.'s than in the case of ω-cpo's.

Before we give the precise definition of these categories we first give

Definition 3.24 [d-continuity]

Let A, B be either d-cpo's or cl's

A function $f : A \to B$ is d-continuous fif

– f is monotonic

– for each directed subset (see definition 1.6.)

D of A : $f(\bigsqcup D) = \bigsqcup f(D)$.

□

The reason why we want f to be monotonic is that in case B is a d-cpo we only know that $f(D)$ has a l.u.b. when $f(D)$ is directed. It is easily seen that the latter holds when f is monotonic.

On the other hand, if B is a cl, then we can deduce that f is monotonic assuming only the 2nd part of the definition: for suppose $x, y \in A$ and $x \sqsubseteq y$. Let $D = \{x, y\}$. Then D is directed, hence

$f(y) = f(\bigsqcup D)$

$= \bigsqcup f(D)$

$= \bigsqcup \{f(x), f(y)\}$

Thus $f(y) \sqsupseteq f(x)$ and f is monotonic.

Definition 3.25 [DCPO, CL]

- DCPO is the category with as objects d-cpo's and as morphisms the d-continuous maps between d-cpo's.
- CL is the category with as objects cl's and as morphisms the d-continuous maps between cl's.

□

It is trivially verified that DCPO and CL are categories. Also we have $Obj(\underline{CL}) \subseteq Obj(\underline{DCPO}) \subseteq Obj(\underline{CPO})$ and if A, B are cl's then $Hom_{\underline{CL}}(A, B) = Hom_{\underline{DCPO}}(A, B)$ while if C, D are d-cpo's we only know that $Hom_{\underline{DCPO}}(C, D) \subseteq Hom_{\underline{CPO}}(C, D)$.
(In other words, DCPO and CL are subcategories of CPO, while CL is a full subcategory of DCPO.)
Just as for CPO the hom-sets are ω-cpo's we have for $K = \underline{DCPO}$ or $K = \underline{CL}$ that hom-sets of K are objects of K. Hence in particular these hom-sets are ω-cpo's.
Moreover, composition of morphisms of K is d-continuous. The verification of this, together with the verification that $Hom_K(A, B) \in Obj(K)$ $(A, B \in Obj(K))$, is left to the reader. In any case these facts show that both DCPO and CL are O-categories.

Theorem 3.26

DCPO and CL are localized O-categories.

□

Proof

In the case of DCPO we get a proof by using the proof of theorem 3.22 only replacing 'ascending chain' by 'directed subset' and 'ω' by 'd'.
In the case of CL we have to make the additional observation that E, as defined in the proof of theorem 3.22, is a cl. The latter follows by Knaster-Tarski (theorem 1.15), since $E = \{x \in D \mid f(x) = x\}$ and f is monotonic.

□

Theorem 3.27

$\underline{DCPO}_{PR}$ and $\underline{CL}_{PR}$ are ω-categories.

□

Proof

Again, a proof for DCPO is obtained by a trivial modification of the proof of theorem 3.23. And again the proof for CL requires more, for we have to show that D as defined in the proof of theorem 3.23 (def.no. 12 of the proof) is a cl.
This can be seen as follows:
It is easy to show that the cartesian product $\prod_{n \geq 0} D_n$ with the coordinate-wise ordering is a cl. Moreover, D is a subposet of this product and if the maps f_n^R would preserve l.u.b.'s of arbitrary subsets

of D_{n+1} then l.u.b.'s of arbitrary subsets of D would exist (and equal the 'component-wise' l.u.b.). Since f_n^R is only known to preserve l.u.b.'s of directed subsets of D_{n+1} we need another application of Knaster-Tarski: observe that D is the set of fixed points of the monotonic map $f: \prod_{n\geq 0} D_n \to \prod_{n\geq 0} D_n$ given by $f(<x_n>_{n\geq 0}) = <f_n^R(x_{n+1})>_{n\geq 0}$.

□

Both categories DCPO and CL have fewer objects than CPO in the sense that there are ω-cpo's which are no d-cpo's, hence also no cl's.

A different way to obtain a new category is to form smaller hom-sets. To present an example we need

Definition 3.28 [strict]

Let C, D be cpo's; $f: C \to D$ a function.

f is called strict iff $f(\bot_C) = \bot_D$.

□

Definition 3.29 [$\text{CPO}_\bot$]

$\text{CPO}_\bot$ is the category of all cpo's with as morphisms the ω-continuous functions which are also strict.

□

Since the composition of strict functions is again strict, $\text{CPO}_\bot$ is a subcategory of CPO and since the l.u.b. of an ascending chain of strict functions is also strict, $\text{CPO}_\bot$ is an O-category.

If we are only interested in projection pairs then we can restrict ourselves to $\text{CPO}_\bot$, for we have

Proposition 3.30

$(\text{CPO}_\bot)_{PR} = \text{CPO}_{PR}$.

□

Proof

It suffices to show that a projection pair α in CPO is also a projection pair in $\text{CPO}_\bot$. Thus we only have to show that α^L and α^R are strict. This follows from the lemma below.

□

Lemma 3.31

Let X, Y be posets with least elements $\bot_X$, resp. $\bot_Y$. Let $f: X \to Y$, $g: Y \to X$. Suppose g is monotonic and $g \circ f \sqsubseteq I_X$. Then g is strict.

□

Proof

$g(\bot_Y) \sqsubseteq g(f(\bot_X)) \sqsubseteq \bot_X$.

□

Recalling the definition of localized O-category and knowing that CPO is such a category, it is now clear from the proposition above that $\text{CPO}_\bot$ is also such a category. As our interest is in CPO_{PR} (rather than in CPO) it may be convenient to start with $\text{CPO}_\bot$ instead of CPO (see for example proposition 3.34).

3.5. Examples of locally continuous functors

Now that we have a few examples of localized O-categories we shall present several functors which turn out to be locally continuous.

We shall restrict ourselves to CPO and $\text{CPO}_\perp$ (both localized) though everything carries over to DCPO and CL.

Let $K = \text{CPO}$ and let $H : K^{op} \times K \to K$ be the following functor:

Given two ω-cpo's X, Y define $H(X, Y) = Hom_K(X, Y)$ (i.e. the set of ω-continuous functions from X to Y, which, with point-wise ordering is an ω-cpo).

Given a morphism $(f, g) : (X, Y) \to (X', Y')$ of $K^{op} \times K$,

define $H(f, g) : Hom_K(X, Y) \to Hom_K(X', Y')$ by

$$H(f, g)(h) = g \circ h \circ f$$

(Note that this is well-defined since $f \in Hom_K(X', X)$, $h \in Hom_K(X, Y)$ and $g \in Hom_K(Y, Y')$.)

It is easy to check that H is indeed a functor (the so-called Hom functor). Moreover, $K^{op} \times K$ is a localized O-category (by corollary 3.20 and proposition 3.21) and $H(f, g)$ is ω-continuous (since composition is ω-continuous). By definition 3.14 this means that H is a locally continuous functor. Hence, by the continuity theorem 3.15

$H_{PR} : (K^{op} \times K)_{PR} \to K_{PR}$ is an ω-continuous functor.

We would like to compose H_{PR} with the isomorphism: $K_{PR} \times K_{PR} \to (K^{op} \times K)_{PR}$ which can be obtained by applying proposition 3.19 and proposition 3.21. As this isomorphism acts as the identity on objects, we shall describe it explicitly only on morphisms. To do so, let $F_1 : K_{PR} \to (K^{op})_{PR}$ be the isomorphism of proposition 3.19. Thus $F_1(f, g) = (g, f)$ for a projection pair (f, g) of K. Now form $F_1 \times I : K_{PR} \times K_{PR} \to (K^{op})_{PR} \times K_{PR}$ where I is the identity functor on K_{PR}. For a pair of projection pairs $((f, g), (h, k))$ of K we then have

$$F_1 \times I\,((f, g), (h, k)) = ((g, f), (h, k)).$$

Also, let $F_2 : (K^{op})_{PR} \times K_{PR} \to (K^{op} \times K)_{PR}$ be the isomorphism of proposition 3.21. Then

$$F_2\,((g, f), (h, k)) = ((g, h), (f, k)).$$

Thus $F_2 \circ (F_1 \times I) : K_{PR} \times K_{PR} \to (K^{op} \times K)_{PR}$ maps

$((f, g), (h, k))$ to $((g, h), (f, k))$.

Composing with H_{PR} gives the following result:

Proposition 3.32 ['Arrow' functor]

Let $K = \text{CPO}$. Define $A : K_{PR} \times K_{PR} \to K_{PR}$ on an object (X, Y) as $A(X, Y) = Hom_K(X, Y)$ and on a morphism $((f, g), (h, k))$ as

$A((f, g), (h, k)) = H_{PR}((g, h), (f, k)) = (H(g, h), H(f, k))$.

In more detail: if $(f, g) : X \to X'$ and $(h, k) : Y \to Y'$ are both projection pairs of K, then $A((f, g), (h, k)) : Hom_K(X, Y) \to Hom_K(X', Y')$ is the projection pair with as first coordinate the map $H(g, h) : Hom_K(X, Y) \to Hom_K(X', Y')$, $s \mapsto h \circ s \circ g$ and as second coordinate the map

$H(f,k) : Hom_K(X', Y') \rightarrow Hom_K(X, Y), t \mapsto k \circ t \circ f$.
Then A is ω-continuous.

□

We could also have started with $K = \underline{\text{CPO}_\perp}$ and the Hom-functor SH on K. In this way we would obtain SA which differs from A only on objects: $SA(X, Y)$ is the ω-cpo of all strict and ω-continuous functions from X to Y. (SA stands for 'strict-arrow' functor.)

As another example, consider the functor

$Prod : K \times K \rightarrow K$, mapping object (X, Y) to $X \times Y$ (the cartesian product of X and Y)
and morphism $(f, g) : (X, Y) \rightarrow (X', Y')$ to
$Prod(f, g) : X \times Y \rightarrow X' \times Y', (x, y) \mapsto (f(x), g(y))$.

Since $Prod(f, g)$ is ω-continuous, $Prod$ is locally continuous, hence induces $Prod_{PR} : (K \times K)_{PR} \rightarrow K_{PR}$. Composing with the isomorphism from $K_{PR} \times K_{PR}$ to $(K \times K)_{PR}$ we obtain

Proposition 3.33 ['Product' functor]
The functor $P : K_{PR} \times K_{PR} \rightarrow K_{PR}$ defined by

$P(X, Y) = X \times Y$ ((X, Y) an object) ; and

$P(\alpha, \beta) = (Prod(\alpha^L, \beta^L), Prod(\alpha^R, \beta^R))$ ((α, β) a morphism)

is ω-continuous.

□

Now let $K = \underline{\text{CPO}_\perp}$ and define

$S\,Prod : K \times K \rightarrow K$, mapping object
(X, Y) to $\{(x, y) \in X \times Y \mid x \neq \perp_X \wedge y \neq \perp_Y\} \cup \{(\perp_X, \perp_Y)\}$
and morphism $(f, g) : (X, Y) \rightarrow (X', Y')$ to
$S\,Prod(f, g) : S\,Prod(X, Y) \rightarrow S\,Prod(X', Y')$,

$$(x, y) \mapsto \begin{cases} (f(x), g(y)) & \text{if } f(x) \neq \perp_{X'} \wedge g(y) \neq \perp_{Y'} \\ (\perp_{X'} \perp_{Y'},) & \text{if } f(x) = \perp_{X'} \vee g(y) = \perp_{Y'} \end{cases}$$

It is easily verified that $S\,Prod$ is a functor (which is false if we try to extend $S\,Prod$ to $\underline{\text{CPO}}$!) which is locally continuous.
Hence it induces $S\,Prod_{PR} : (K \times K)_{PR} \rightarrow K_{PR}$.
Composing with the isomorphism from $K_{PR} \times K_{PR}$ to $(K \times K)_{PR}$ gives

Proposition 3.34 ['Smashed Product' functor]
The functor $SP : K_{PR} \times K_{PR} \rightarrow K_{PR}$ (where $K = \underline{\text{CPO}_\perp}$) defined by

$SP(X, Y) = S\,Prod(X, Y)$ ((X, Y) an object) ; and
$SP(\alpha, \beta) = (S\,Prod(\alpha^L, \beta^L), S\,Prod(\alpha^R, \beta^R))$ ((α, β) a morphism)

is ω-continuous.

□

Turning to $K = \underline{\mathrm{CPO}}$, define

Plus : $K \times K \to K$, mapping object
(X, Y) to $\{ (0, x) \mid x \in X \} \cup \{ (1, y) \mid y \in Y \} \cup \{\bot\}$ where $\bot$ is a new symbol. The ordering on $Plus\,(X, Y)$ is such that $\bot$ is the least element, elements of the form $(0, x)$ and $(1, y)$ are incomparable while the ordering on $\{0\} \times X$, resp. $\{1\} \times Y$ is inherited from X, resp. Y.
Morphism $(f, g) : (X, Y) \to Plus\,(X', Y')$ is mapped to
$Plus\,(f, g) : Plus\,(X, Y) \to Plus\,(X', Y')$ mapping $\bot$ to $\bot$, $(0, x)$ to $(0, f(x))$ and $(1, y)$ to $(1, g(y))$.

One can easily see that *Plus* is a locally continuous functor. As with the preceding examples we get

Proposition 3.35 ['Separated Sum' functor]
The functor $SS : K_{PR} \times K_{PR} \to K_{PR}$ $(K = \underline{\mathrm{CPO}})$ mapping object (X, Y) to $Plus\,(X, Y)$ and morphism (α, β) to $(Plus\,(\alpha^L, \beta^L), Plus\,(\alpha^R, \beta^R))$ is ω-continuous.

□

As a slight variation of the above let $K = \underline{\mathrm{CPO}_\bot}$ and define

C Plus : $K \times K \to K$, mapping object
(X, Y) to $(\{0\} \times X) \cup (\{1\} \times Y)$ with identification $(0, \bot_X) = (1, \bot_Y)$. In the ordering this element is to be the least while other elements of the form $(0, x)$ and $(1, y)$ are incomparable and the ordering on $\{0\} \times X$, resp. $\{1\} \times Y$ is inherited from X, resp. Y.
Morphism $(f, g) : (X, Y) \to (X', Y')$ is mapped to
$C\,Plus\,(f, g) : C\,Plus\,(X, Y) \to C\,Plus\,(X', Y')$ mapping $(0, x)$ to $(0, f(x))$ and $(1, y)$ to $(1, g(y))$. (Note that this is well-defined : $(0, \bot_X) = (1, \bot_Y)$, but also $(0, f(\bot_X)) = (1, g(\bot_Y))$ since f and g are strict!)

Again this is a locally continuous functor.

Proposition 3.36 ['Coalesced Sum' functor]
The functor $CS : K_{PR} \times K_{PR} \to K_{PR}$ $(K = \underline{\mathrm{CPO}_\bot})$ mapping object (X, Y) to $C\,Plus\,(X, Y)$ and morphism (α, β) to $(C\,Plus\,(\alpha^L, \beta^L), C\,Plus\,(\alpha^R, \beta^R))$ is ω-continuous.

□

Finally we have the '*lift*'-functor

lift : $\underline{\mathrm{CPO}} \to \underline{\mathrm{CPO}_\bot}$ mapping object
X to $X \cup \{\bot\}$ where $\bot$ is a new symbol, which is to be the least element of $X \cup \{\bot\}$.
Morphism $f : X \to Y$ is mapped to $lift\,(f) : lift\,(X) \to lift\,(Y)$ mapping $\bot$ to $\bot$ and $x \in X$ to $f(x) \in Y$.

Note that the map $i : X \to lift\,(X), x \mapsto x$ is not strict, hence, even though i is continuous and injective, it is no embedding (see definition 3.3 and lemma 3.31).
Still, *lift* is a useful functor since it is locally continuous:

Proposition 3.37

$UP : \underline{\text{CPO}}_{PR} \rightarrow \underline{\text{CPO}}_{PR}$ mapping object X to $lift(X)$ and morphism α to $(lift(\alpha^L), lift(\alpha^R))$ is ω-continuous.

□

4. Appendix

In this appendix we slightly extend the basic category theory presented in the previous chapters. All of this will be very elementary; in fact just a couple of definitions which are given for completeness' sake.
The reason why we give these definitions here, rather than in chapter 2 or 3, is that we did not wish to disturb the main line of these chapters.

There are a few ways to define new categories from old ones. One construction is to take the product defined below.

Definition A.1. [product of categories]
Let K, L be two categories. The product of K and L is the category $K \times L$ with as objects all pairs (A, B) where $A \in Obj(K)$ and $B \in Obj(L)$, while the morphisms from (A, B) to (C, D) are all elements of $Hom_K(A, C) \times Hom_L(B, D)$. Composition and identity morphisms are defined coordinate-wise.

□

That $K \times L$ indeed is a category is trivially verified.
Also in the other definitions of this appendix, it's always very easy to check the (implicit) claim that something newly defined is a category or a functor.

Related to the product of categories we have

Definition A.2. [product of functors]
Let $F_i : K_i \to L_i$ be functors $(i=1,2)$. The product of F_1 and F_2 is the functor $F_1 \times F_2 : K_1 \times K_2 \to L_1 \times L_2$ mapping a pair (u_1, u_2) (which may be an object as well as a morphism) to $(F_1(u_1), F_2(u_2))$.

□

Definition A.3. [projection functors]
Let K_1, K_2 be categories. The projection functor $\pi_i : K_1 \times K_2 \to K_i$ is defined by taking the i-th coordinate of a pair (which may be an object or a morphism) $(i=1,2)$.

□

Exercise A.4.
Show that projection functors are ω-continuous. In fact, an ω-chain Δ in $K_1 \times K_2$ has a colimit iff $\pi_1 \Delta$ and $\pi_2 \Delta$ (defined in an obvious way) have colimits in K_1 resp. K_2.

□

Definition A.5. [composition, identity, isomorphism, inverse]
Let $F : K \to L$, $G : L \to M$ be two functors. The composition of F and G is the functor $G \circ F : K \to M$ obtained by first applying F and then G, both on objects and morphisms.
The identity functor on K is the functor $I_K : K \to K$ mapping an object and a morphism to itself.
F is an isomorphism iff there exists a functor $H : L \to K$ such that $H \circ F = I_K$ and $F \circ H = I_L$. Such a H is unique and is called the inverse of F.

□

Exercise A.6

Show that a functor which is an isomorphism is ω-continuous.

□

In case we have functors $F_i : K \rightarrow L_i$ $(i=1,2)$ we can also define $F : K \rightarrow L_1 \times L_2$ mapping u to $(F_1(u), F_2(u))$ where u may be an object or a morphism. We can relate F to $F_1 \times F_2$ via

Definition A.7. [diagonal functor]

Let K be a category. The diagonal functor is the functor $\Delta_K : K \rightarrow K \times K$ mapping u to (u, u) (u an object or a morphism).

□

Note that F above is just $(F_1 \times F_2) \circ \Delta_K$. Also note that (with the notations as above) $(\pi_1 \times \pi_2) \circ \Delta_{K_1 \times K_2} = I_{K_1 \times K_2}$.

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