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The van der Waerden Conjecture: Two Proofs in One Year

J. H. van Lint

1. Introduction

One of the famous open problems in combinatorial theory was the van der Waerden conjecture on permanents of doubly stochastic matrices. After more than fifty years in which it managed to resist attacks it has finally been proved. As so often happens two mathematicians gave independent proofs nearly at the same time. At the end of 1980 the news that G. P. Egoritsjev (Г. П. Егорычев) [2] had found a proof spread quickly and within a few months translations and expositions of the proof were circulating (cf. D. E. Knuth [5], J. H. van Lint [6]). It came as quite a shock when Matematičeski Zametki *29* No. 6 appeared a few months ago with a paper by D. I. Falikman (Д. И. Фаликман) [3], submitted 14. 5. 1979 (!), with a completely different proof of the conjecture. Perhaps even more surprising is the fact that the two proofs have as a common feature that they use an elegant inequality, due to A. D. Alexandroff and W. Fenchel, which until quite recently seemed to be unknown to combinatorialists. From the literature it is obvious that geometers certainly know about the inequality.

In this note we shall describe and compare the two new proofs and the ideas that led to their discovery.

Let A be a square matrix of size n with entries a_{ij} $(1 \leqslant i, j \leqslant n)$. We define the *permanent* of A (notation: per A) by

$$\operatorname{per} A := \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}, \tag{1.1}$$

where σ runs through all permutations of $\{1, 2, \ldots, n\}$. One could say that the permanent is like the determinant but without all the minus signs.

In his book *Permanents* [9] H. Minc mentions that the name permanent is essentially due to Cauchy (1812) although the word as such was first used by Muir in 1882. Nevertheless a referee of one of Minc's earlier papers admonished him for inventing this ludicrous name!

Calculating permanents is very much more difficult than calculating determinants. However, there are some similarities. For instance, the permanent is a multilinear function of the columns of A. Let $\mathbf{a}_j = (a_{1j}, a_{2j}, \ldots, a_{nj})^T$ be the j-th column of A. We shall often write per A as per $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$. From (1.1) we have

$$\operatorname{per} A = \sum_{j=1}^{n} a_{ij} \operatorname{per} A(i \mid j) = \sum_{i=1}^{n} a_{ij} \operatorname{per} A(i \mid j), \tag{1.2}$$

where $A(i|j)$ is the matrix we obtain from A by deleting row i and column j. As usual, we call this procedure *developing* by a row or column.

The non-expert reader may wonder what purpose is served by defining permanents. Let us consider a well-known and not too easy example from combinatorics, the "*problème des ménages*". At a round table n couples are to be seated. The n wives have already occupied the odd-numbered seats $1, 3, \ldots, 2n-1$. No husband is allowed to sit next to his wife. In how many ways can the men be seated?

This is a typical counting problem for which the answer is given by a permanent. Let A be the matrix for which every entry is equal to 1 except if $i - j \equiv 0$ or $1 \pmod n$ in which case $a_{ij} = 0$. The permanent of this matrix, usually denoted by U_n, is the answer to our seating problem. A reader who at this point believes he is reading about a trivial area of mathematics may wish to prove that $U_n \sim e^{-2} n!$ $(n \to \infty)$. He might change his mind.

Let us now turn to van der Waerden's conjecture. Much of the work on permanents is in some way connected to this conjecture and about 75% of the work on permanents is less than 20 years old! First, a definition. A matrix A is called a *doubly-stochastic* matrix if every entry a_{ij} is nonnegative and all row sums and all column sums of A are equal to 1. One can consider the entries of A as conditional probabilities which accounts for the name. It seems that permanents do not play a rôle of any importance in probability theory, however. Trivial examples of doubly stochastic matrices are all permutation matrices and the matrix $J_n := n^{-1} J$ (in combinatorics the matrix for which all entries are 1 is usually denoted by J).

In 1926 B. L. van der Waerden [11] proposed as a problem (!) in Jber. D.M.V. *35* to determine the minimal permanent among all doubly stochastic matrices. It was natural to assume that this minimum is per $J_n = n!n^{-n}$. Let us denote by Ω_n the set of all doubly stochastic matrices. The assertion

$$(A \in \Omega_n \wedge A \neq J_n) \Rightarrow (\operatorname{per} A > \operatorname{per} J_n) \tag{1.3}$$

became known as the *van der Waerden conjecture*. Some-times just showing that $n!n^{-n}$ is the minimal value is referred to as the conjecture.

This note allows me to save for posterity a humorous experience of the late sixties. Van der Waerden, by then retired, had decided to attend a meeting on combinatorics, a field he had never seriously worked in. There was a talk by a young mathematician who was desperately trying to explain his complete thesis in 20 minutes. I was sitting in the front row, next to van der Waerden, when the famous conjecture was mentioned by the speaker and the alleged author inquired what this famous conjecture stated!! The exasperated speaker spent a few seconds of his precious time to explain and at the end of his talk wandered over to us to read the badge of the person who had asked this inexcusable question. I knew it was going to happen and still remember happily how he recoiled. Do not worry; he recovered and is now a famous combinatorialist. The lesson for the reader is the following. If you did not know of the "conjecture" then if is comforting to realize that it was 40 years old before van der Waerden heard that it had this name.

What is the origin of the problem? At my request van der Waerden went far back in his memory and came up with the following. One day in 1926 during the discussions, which took place daily in Hamburg. O. Schreier mentioned that G. A. Miller had proved that there is a mutual system of representatives for the right and left cosets of a subgroup H of a finite group G. At this point van der Waerden observed that this was a property of any two partitions of a set of size μn into μ subsets of size n. This theorem was published in the Hamburger Abhandlungen in 1927 [12]. In a note, added in proof, van der Waerden acknowledges that he has rediscovered the theorem which is now known as the König-Hall theorem. (Intermezzo: Among the many things which my thesis-supervisor F. van der Blij taught me was the good habit of looking at the titles of *all* the papers in a journal which one checks for a reference. The authors of the papers in reference [12] are E. Artin, M. Bauer, H. Behnke, W. Blaschke, E. Hecke, H. D. Kloosterman, H. Kneser, H. Petersson, H. Rademacher, J. Radon, K. Reidemeister, H. Schatz, O. Schreier, E. Sperner, B. L. van der Waerden. K. Zwirner; indeed: . . . mais où sont les neiges d'antan?).

In the terminology of permanents we can formulate the problem, Schreier and van der Waerden were considering as follows. Let $A_i (1 \leqslant i \leqslant \mu)$ and $B_k (1 \leqslant k \leqslant \mu)$ be the subsets in the two partitions and let $a_{ik} := |A_i \cap B_k|$. Then $A = (a_{ik})$ is a matrix with constant line sums (= n). The assertion that there is a mutual system of representatives of the sets A_i resp. B_k is the same as saying that per $A > 0$. At this point van der Waerden wondered what the minimal permanent, under the side condition that all line sums are 1, is. He posed this as a problem in Jber. d. D.M.V. *35* and thus the van der Waerden conjecture was born.

We pointed out above that this conjecture was responsible for much of the research on permanents. There were of course other interesting problems, e.g. an intriguing conjecture by Minc on the relation between permanents of (0, 1)-matrices and the line sums of these matrices. The reader interested in a survey of the knowledge up to 1978 is referred to Minc's book [9]. In fact much of the following introductory material is taken from this book.

Let us now have a look at the inequality which is essential in both of the new proofs of the van der Waerden conjecture. Consider convex sets $K_1, K_2, \ldots, K_m$ in $\mathbb{R}^n$. Let $x_1, x_2, \ldots, x_m$ be non-negative. We consider the set K consisting of all points $x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \ldots + x_m \mathbf{a}_m$, where $\mathbf{a}_i \in K_i$. As a function of the variables $x_1, \ldots, x_m$ the volume $V(K)$ of the set K is a homogeneous polynomial of degree n. We express this as follows:

$$V(K) = \sum_{i_1=1}^{m} \ldots \sum_{i_n=1}^{m} V_{i_1 \ldots i_n} x_{i_1} x_{i_2} \ldots x_{i_n},$$

where we require that the coefficients are symmetric with regard to subscripts. The coefficient $V_{i_1 \ldots i_n}$ is called the *mixed volume* of the sets $K_{i_1}, \ldots, K_{i_n}$. The following inequality was proved independently by A. D. Alexandroff [1] and W. Fenchel [4].

$$(V_{i_1 i_2 \ldots i_{n-1} i_n})^2 \geqslant V_{i_1 \ldots i_{n-1} i_{n-1}} \cdot V_{i_1 \ldots i_{n-2} i_n i_n}. \tag{1.4}$$

These historical comments on the geometry of convex sets were rather vague. They suffice to show the origin of the following inequality which was obtained in two steps from (1.4). Alexandroff reformulated (1.4) as a theorem on quadratic forms in n variables. By taking these forms to be given by diagonal matrices the inequality takes on a special form which allows us to state it as a theorem on permanents. We call this the *Alexandroff-Fenchel inequality* on permanents.

1.5. Theorem. *Let* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}$ *be vectors in* $\mathbb{R}^n$ *with positive coordinates and let* $\mathbf{b} \in \mathbb{R}^n$. *Then*

$$(\text{per}\,(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{b}))^2 \geqslant \text{per}\,(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{a}_{n-1}) \cdot \text{per}\,(\mathbf{a}_1, \ldots, a_{n-2}, \mathbf{b}, \mathbf{b}) \tag{1.6}$$

and equality holds iff $\mathbf{b} = \lambda \mathbf{a}_{n-1}$ *for some constant* λ.

Note that the inequality also holds if we only require that the $\mathbf{a}_i$ are non-negative but we cannot make the assertion concerning the consequence of equality unless all $\mathbf{a}_i$ are positive.

We shall give a proof of Theorem 1.5 in Section 5.

To my knowledge the only other appearance of the Alexandroff-Fenchel inequalities in the literature on combinatorial theory is in a paper by R. P. Stanley which appeared a few months ago in J. Combinatorial Theory (cf. [10]).

2. Notation and Definitions

We denote the set of all doubly stochastic matrices of size n by Ω_n. The subset consisting of matrices for which all entries are *positive* is denoted by Ω_n^*.

The matrix for which all entries are 1 is usually denoted by J. We define $J_n := n^{-1}J$. The vector $(1, 1, \ldots, 1)^T$ is denoted by $\mathbf{j}$.

A matrix $A \in \Omega_n$ such that $\text{per } A = \min \{\text{per } S \mid S \in \Omega_n\}$ is called a *minimizing matrix.*

The matrix obtained from A by deleting the i-th row and the j-th column is denoted by $A(i|j)$. We consider a matrix A as a sequence of n columns and write $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$. In Section 5 we consider permanents of matrices of size $n - 1$ but we wish to use the notation for matrices of size n. The trick is to write $\text{per}(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_{n-1}, \mathbf{e}_j)$ where $\mathbf{e}_j$ denotes the j-th standard basis vector. This permanent does not change value if the j-th row and n-th column are deleted:

$$\text{per}(\mathbf{a}_1, ..., \mathbf{a}_{n-1}, \mathbf{e}_j) = \text{per}(\mathbf{a}_1, ..., \mathbf{a}_{n-1}, \mathbf{e}_j) \quad (j|n).$$

The essential new idea introduced by Falikman is to study a function which differs only slightly from the permanent. In the following let $\epsilon > 0$. We first define

$$\Pi(A) := \prod_{i=1}^{n} \prod_{j=1}^{n} a_{ij} \quad (A \in \Omega_n). \tag{2.1}$$

The function F_ϵ which will be studied in Section 4 and Section 6 is given by

$$F_\epsilon(A) := \text{per } A + \epsilon/\Pi(A), \quad (A \in \Omega_n^*). \tag{2.2}$$

The idea is this; for fixed $\epsilon > 0$ it is clear that F_ϵ will be large near the boundary of Ω_n but on the other hand for fixed A we have $\lim_{\epsilon \to 0} F_\epsilon(A) = \text{per } A$.

3. Some Elementary Results on Permanents

There are a number of results on the set Ω_n and the structure of minimizing matrices which we shall need in our description of the two proofs. Most of these are fairly well-known and proofs can be found in many books, e.g. in [9]. Therefore we only mention the theorems and skip the proofs.

One of the most fundamental results in the theory of doubly stochastic matrices is Birkhoff's theorem which states that Ω_n is a convex polyhedron with the permutation matrices as vertices, i.e. a doubly stochastic matrix A can be expressed as $\sum_{i=1}^{m} \alpha_i P_i$ where the α_i are non-negative, $\sum_{i=1}^{m} \alpha_i = 1$, and the matrices P_i are permutation matrices. A consequence of this theorem is that if $A \in \Omega_n$ then per $A > 0$ (this was the fact which Schreier and van der Waerden needed in 1926).

It is clear that the boundary of Ω_n is going to give us difficulties when we try to find the minimal permanent. Therefore we would like to have some information on the zeros in an element of Ω_n. Suppose $A \in \Omega_n$ is a direct sum of an element $P \in \Omega_k$ and $Q \in \Omega_{n-k}$. From Birkhoff's theorem we know that per $P > 0$ and per $Q > 0$. So, by rearranging elements we may assume that the diagonal of A is positive. As an easy exercise the reader can show that we can now decrease two diagonal elements by ϵ and replace two of the 0's by ϵ in such a way that we find another element of Ω_n with a permanent *larger* than per A. This gives us the following lemma.

3.1. Lemma. *If $A \in \Omega_n$ is a minimizing matrix then A is not a direct sum of an element of Ω_k and an element of Ω_{n-k}.*

We shall use this lemma in two ways. First of all it implies that for any a_{ij} in a minimizing matrix A there is a permutation σ such that $\sigma(i) = j$ and $a_{s,\sigma(s)} > 0$ for $1 \leqslant s \leqslant n$, $s \neq i$. (This is not trivial but it can be derived from Lemma 3.1 without too much difficulty.) A second application of Lemma 3.1 is more direct: a row of a minimizing matrix has at least two positive elements.

4. How Far Do We Get with Calculus?

In 1959 a major attack on the van der Waerden conjecture was launched by M. Marcus and M. Newman [8]. Much of the work in subsequent years was stimulated by the following surprising theorem.

4.1. Theorem. *If $A \in \Omega_n$ is a minimizing matrix and $a_{hk} > 0$ then* per $A(h|k)$ = per A.

Notice that the matrix J_n indeed has the property that every subpermanent of size $n - 1$ is equal to per J_n.

The idea of the proof was to define a suitable set for which A is an interior point and then use differential calculus. The conditions on the row sums and column sums in Ω_n made it possible to introduce Lagrange multipliers.

One of Falikman's main lemmas is a result similar to Theorem 4.1 for the function F_ϵ instead of the permanent. His proof also uses differential calculus. We shall show below that in fact the same proof as used by Marcus and Newman can be applied. The most important consequence of Theorem 4.1 was the following theorem, also due to Marcus and Newman.

4.2 Theorem. *If there is a minimizing matrix A in Ω_n^* then* $\operatorname{per} A = \operatorname{per} J_n = n!/n^n$ *and in fact* $A = J_n$.

The proof, for which we refer to [9], depends on a trick which plays a central rôle in Egoritsjev's paper. Suppose A is an element of Ω_n for which $\operatorname{per} A(h|k) = \operatorname{per} A$ holds for all pairs h, k. If we replace any column of A by a vector x for which $\sum_{i=1}^{n} x_i = 1$ then, developing by this column we find that the new matrix has the same permanent as A. We shall refer to this idea as the *substitution principle.* If A is a minimizing matrix in Ω_n^* then Theorem 4.1 allows us to use the substitution principle to replace any two columns of A by their average and thus obtain a new minimizing matrix. In this way one constructs a sequence of minimizing matrices which tends to J_n. The uniqueness of the minimum takes a little extra work.

We now state Falikman's generalization of Theorem 4.1.

4.3. Theorem. *There is a matrix $A \in \Omega_n^*$ for which $F_\epsilon(A)$ is minimal. For this A and $1 \leqslant h, k \leqslant n$ we have*

$$\operatorname{per} A(h|k) = b + \frac{c}{a_{hk}},$$

where $c := \epsilon/\Pi(A)$ and $b := \operatorname{per} A - nc$.

Proof: Let α be the minimal entry in $X \in \Omega_n^*$. Then, applying the arithmetic-geometric mean inequality to the $n-1$ rows not containing α, we find

$$\Pi(X) \leqslant \alpha \left(\frac{1}{n}\right)^{n(n-1)}.$$

It follows that the matrices X with $F_\epsilon(X) \leqslant 2F_\epsilon(J_n)$ form a compact subset of Ω_n^*. So there is a matrix $A \in \Omega_n^*$ where F_ϵ attains its minimal value.

Now consider the function

$$\hat{F}_\epsilon(X) := F_\epsilon(X) - \sum_{i=1}^{n} \lambda_i \left(\sum_{k=1}^{n} x_{ik} - 1\right)$$

$$- \sum_{j=1}^{n} \mu_j \left(\sum_{k=1}^{n} x_{kj} - 1\right).$$

From (2.2) we find

$$\partial \hat{F}_\epsilon / \partial x_{ij} = \operatorname{per} X(i|j) - \frac{c}{x_{ij}} - \lambda_i - \mu_j$$

and hence we have

$$\operatorname{per} A(i|j) = \frac{c}{a_{ij}} + \lambda_i + \mu_j. \tag{4.4}$$

Multiplying both sides of (4.4) by a_{ij} and summing over j resp. i we find

$$\operatorname{per} A - nc = \lambda_i + \sum_{j=1}^{n} a_{ij}\mu_j = \mu_j + \sum_{i=1}^{n} a_{ij}\mu_i. \tag{4.5}$$

Introducing $\mathbf{j} := (1,1,\ldots,1)^T$, $\lambda := (\lambda_1, \ldots, \lambda_n)^T$, $\mu := (\mu_1, \ldots, \mu_n)^T$ we have from (4.5)

$$(\operatorname{per} A - nc)\mathbf{j} = \lambda + A\mu = \mu + A^T\lambda. \tag{4.6}$$

Multiplying by A^T gives us

$$(\operatorname{per} A - nc)\mathbf{j} = A^T\lambda + A^TA\mu,$$

i.e.

$$A^TA\mu = \mu.$$

Since $A^TA \in \Omega^*$ it has eigenvalue 1 with multiplicity 1 belonging to the eigenvector $\mathbf{j}$. Therefore both λ and μ are multiples of $\mathbf{j}$ and then (4.6) implies that $\lambda_i + \mu_j = \operatorname{per} A - nc$. □
This proves the assertion

To answer the title of this section we can say that we have nearly proved the conjecture. Nevertheless it took more than twenty years before the last step was found. One direction of research was to try to generalize Theorem 4.1 in such a way that the condition on a_{hk} could be removed. The first step in this direction was a theorem due to D. London [7] which we give below. The generalization was finally proved by Egoritsjev using algebraic methods.

Another way of settling the conjecture is to show that $\operatorname{per} A \geqslant \operatorname{per} J_n$ on Ω_n^* because it then follows from Theorem 4.2 that J_n is indeed a minimizing matrix. The second approach was used by Falikman. It is based on Theorem 4.3 but again algebraic methods are necessary. We discuss the algebraic tools in the next section.

We finish this section on methods from calculus by demonstrating London's generalization of Theorem 4.1.

4.7. Theorem. *If $A \in \Omega_n$ is a minimizing matrix then* per $A(i|j) \geqslant \operatorname{per} A$ *for all i and j.*

Proof: Given i and j there is a permutation σ such that $\sigma(i) = j$ and $a_{s,\sigma(s)} > 0$ for $1 \leqslant s \leqslant n$, $s \neq i$ (cf. Lemma 3.1). Let P be the corresponding permutation matrix. For $0 \leqslant \vartheta \leqslant 1$ we define $f(\vartheta) := \operatorname{per}((1-\vartheta)A + \vartheta P)$. Since A is a minimizing matrix $f'(0) \geqslant 0$, i.e.

$$0 \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} (-a_{ij} + p_{ij}) \operatorname{per} A(i|j) =$$

$$= -n \operatorname{per} A + \sum_{s=1}^{n} \operatorname{per} A(s \mid \sigma(s)).$$

By Theorem 4.1 we have $\operatorname{per} A(s \mid \sigma(s)) = \operatorname{per} A$ for $s \neq i$ and hence $\operatorname{per} A(i|j) \geqslant \operatorname{per} A$. □

5. A Contribution by Linear Algebra

In this section we shall give a direct proof of a theorem on symmetric bilinear forms (taken from [6]) which leads to the inequality that was derived by Egoritsjev from the Alexandroff-Fenchel inequalities. A corollary of this theorem is the result which Falikman needed to complete his proof.

We consider the space $\mathbb{R}^n$ with a symmetric inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T Q \mathbf{y}$. If Q has one positive eigenvalue and $n-1$ negative eigenvalues we shall speak of a *Lorentz space.* We use the standard terminology: a non-zero vector $\mathbf{x}$ is *isotropic* if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, *positive* resp. *negative* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive resp. negative.

If $\mathbf{a}$ is a positive and $\mathbf{b} \neq \mathbf{0}$ is not a multiple of $\mathbf{a}$ then by Sylvester's theorem the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ must contain a negative vector. Therefore the quadratic form in λ given by $\langle \mathbf{a} + \lambda \mathbf{b}, \mathbf{a} + \lambda \mathbf{b} \rangle$ must have a positive discriminant. Therefore we have the following inequality (Cauchy the wrong way around) which will lead to the inequality (1.6).

5.1. Lemma. *If* $\mathbf{a}$ *is a positive vector in a Lorentz space and* $\mathbf{b}$ *is arbitrary then*

$$\langle \mathbf{a}, \mathbf{b} \rangle^2 \geqslant \langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle$$

and equality holds iff $\mathbf{b} = \lambda \mathbf{a}$ *for some constant* λ.

The connection with permanents is provided by the following definition.

Consider vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-2}$ in $\mathbb{R}^n$ with *positive* coordinates. As usual let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$. We define an inner product on $\mathbb{R}^n$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \operatorname{per}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-2}, \mathbf{x}, \mathbf{y}), \tag{5.2}$$

i.e.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T Q \mathbf{y} \quad \text{where } Q \text{ is given by}$$

$$q_{ij} := \operatorname{per}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-2}, \mathbf{e}_i, \mathbf{e}_j). \tag{5.3}$$

Note that if A is a matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and if we delete the last two columns and the rows with index i and j, then the reduced matrix has permanent equal to q_{ij}.

5.4. Theorem. *The space* $\mathbb{R}^n$ *with the inner product defined by* (5.2) *is a Lorentz space.*

Proof: The proof is by induction. For $n = 2$ we have $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the assertion is true. Now assume the theorem is true for $\mathbb{R}^{n-1}$. In the first step of the proof we show that Q does not have the eigenvalue 0. Suppose $Q\mathbf{c} = \mathbf{0}$, i.e.

$$\operatorname{per}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-2}, \mathbf{c}, \mathbf{e}_j) = 0 \quad \text{for } 1 \leqslant j \leqslant n. \tag{5.5}$$

By deleting the last column and the j-th row we can consider (5.5) as a relation for vectors in $\mathbb{R}^{n-1}$. We consider the inner product given by

$$\operatorname{per}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-3}, \mathbf{x}, \mathbf{y}, \mathbf{e}_j) \quad (j \mid n) \tag{5.6}$$

and apply the induction hypothesis, (5.5) and Lemma 5.1. Substitution of $\mathbf{x} = \mathbf{a}_{n-2}$, $\mathbf{y} = \mathbf{a}_{n-2}$ in (5.6) gives a positive value and $\mathbf{x} = \mathbf{a}_{n-2}$, $\mathbf{y} = c$ gives the value 0. Therefore

$$\operatorname{per}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-3}, \mathbf{c}, \mathbf{c}, \mathbf{e}_j) \leqslant 0 \tag{5.7}$$

for $1 \leqslant j \leqslant n$ and for each j equality holds iff all coordinates of $\mathbf{c}$ except c_j are 0. If we multiply the left-hand side of (5.7) by the j-th coordinate of $\mathbf{a}_{n-2}$ and sum over j we find $\mathbf{c}^T Q \mathbf{c}$. Therefore the assumption $Q\mathbf{c} = \mathbf{0}$ implies that $\mathbf{c} = \mathbf{0}$.

For $0 \leqslant \vartheta \leqslant 1$ we define a matrix Q_ϑ by taking (5.2) and replacing every $\mathbf{a}_i$ by $\vartheta \mathbf{a}_i + (1 - \vartheta)\mathbf{j}$. From what we have shown above it follows that for every ϑ in $[0, 1]$ the matrix Q_ϑ does not have the eigenvalue 0. Therefore the number of positive eigenvalues is constant. Since this number is 1 for $\vartheta = 0$ it is also 1 for $\vartheta = 1$ which proves our assertion. □

In the proof of Theorem 5.4 we used Lemma 5.1 in the following form: If $\mathbf{a}$ is positive and $\mathbf{b} \neq 0$ then $\langle \mathbf{a}, \mathbf{b} \rangle = 0$ implies that $\mathbf{b}$ is negative.

It is this assertion which Falikman uses as one of his essential lemmas. His proof is also by induction.

The inequality (1.6) is of course a direct consequence of Theorem 5.4 and Lemma 5.1.

6. Falikman's Proof

The result of Section 5 makes it possible to show that the matrix A of Theorem 4.3 is in fact J_n. In order to do this we consider two columns **u** and **v** of A and show that they are equal. To be able to use the notation of Section 5 we take (w. l. o. g.) $\mathbf{u} = \mathbf{a}_{n-1}$ and $\mathbf{v} = \mathbf{a}_n$. We introduce two other vectors $\mathbf{t} := \mathbf{u} - \mathbf{v}$ and $\mathbf{s} := (s_1, ..., s_n)^T$ with $s_i := u_i v_i$. With the inner product (5.2) the vector **s** is clearly positive. From (5.2) and Theorem 4.3 we have

$$\langle \mathbf{t}, \mathbf{e}_i \rangle = \operatorname{per} A(i\,|\,n) - \operatorname{per} A(i\,|\,n-1) = \frac{ct_i}{u_i v_i}. \qquad (6.1)$$

From (6.1) we find $\langle \mathbf{t}, \mathbf{s} \rangle = c \sum_{i=1}^{n} t_i = 0$; Therefore **t** must be negative or **0**. From (6.1) we have

$$\langle \mathbf{t}, \mathbf{t} \rangle = c \sum_{i=1}^{n} \frac{t_i^2}{u_i v_i} \geqslant 0.$$

Therefore $\mathbf{t} = \mathbf{0}$, i.e. $\mathbf{u} = \mathbf{v}$.

So we now know that F_ϵ is minimal on Ω_n^* in J_n. Therefore every $X \in \Omega_n^*$ satisfies

$$\operatorname{per} X + \epsilon/\Pi(X) \geqslant \operatorname{per} J_n + \epsilon n^{n^2}$$

for any $\epsilon > 0$. So, in fact $\operatorname{per} X \geqslant \operatorname{per} J_n$ and in Section 4 we already saw that this completes the proof of the van der Waerden conjecture. This method does not show that J_n is the unique minimizing matrix in Ω_n.

7. Egoritsjev's Proof

Although Egoritsjev also uses the result of Section 5, his approach is different. He first proves the following generalization of Theorem 4.1. It was known that this result would imply the truth of the van der Waerden conjecture.

7.1. Theorem. *If $A \in \Omega_n$ is a minimizing matrix then* per $A(i\,|\,j) = \operatorname{per} A$ *for all i and j.*

Proof: Suppose the statement is false. Then by Theorem 4.7 there is a pair r, s such that $\operatorname{per} A(r\,|\,s) > \operatorname{per} A$. Choose t such that $a_{rt} > 0$. Consider the product of two factors per A. In the first of these we replace $\mathbf{a}_s$ by $\mathbf{a}_t$ and in the second we replace $\mathbf{a}_t$ by $\mathbf{a}_s$. Subsequently we develop the first factor by column s and the second permanent by column t. According to the inequality (4.7) we have

$$(\operatorname{per} A)^2 \geqslant \left(\sum_{k=1}^{n} a_{kt} \operatorname{per} A(k\,|\,s) \right) \left(\sum_{k=1}^{n} a_{ks} \operatorname{per} A(k\,|\,t) \right).$$

On the right-hand side every subpermanent is at least per A and $\operatorname{per} A(r\,|\,s) > \operatorname{per} A$. Since $\operatorname{per} A(r\,|\,s)$ is multiplied by a_{rt} which is positive we see that the right-hand side is larger than $(\operatorname{per} A)^2$, a contradiction. □

We have already observed in Section 4 that Theorem 7.1 allows us to use the substitution principle as follows. Take a minimizing matrix A and let **u** and **v** be two columns of A. Replace **u** and **v** by $\frac{1}{2}(\mathbf{u} + \mathbf{v})$. The new matrix is again a minimizing matrix.

Let A be any minimizing matrix and let **b** be column of A, say the last column. From Lemma 3.1 we know that in every row of A there are at least two positive elements. We now apply the substitution principle as sketched above a number of times but we never change the last column. In this way we can find a minimizing matrix $A' = (\mathbf{a}_1', ..., \mathbf{a}_{n-1}', \mathbf{b})$ for which $\mathbf{a}_1', ..., \mathbf{a}_{n-1}')$ all have *positive* coordinates. Now apply the inequality (1.6). By the substitution principle we have *equality*. Hence **b** is a multiple of $\mathbf{a}_i'$ for any i with $1 \leqslant i \leqslant n-1$. This implies that $\mathbf{b} = n^{-1}\mathbf{j}$ and hence $A = J_n$.

It is not unlikely that these ideas will lead to other even more simple proofs. One may certainly expect other applications of the Alexandroff-Fenchel inequality in combinatorics.

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