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RANA 90-11
October 1990

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ABSTRACT

In this note it is shown how the theory of [1] can be modified to make it completely thermodynamically consistent. The theory presented here obeys both the principle of equipresence as well as a Clausius-Duhem inequality in extended form. The thus derived results are identical to those of [1]. It is shown that the final results depend explicitly on the choice made for the free energy functional.

1. Introduction

In their paper [1], Zhou and Miya derived a continuum theory for thermoelastic superconductors based on the principles of nonequilibrium thermodynamics and the Landau theory of phase transition. This paper is a pioneering one in the sense that it makes an attempt to incorporate the phenomenological theory of thermoelastic superconductors in the scope of modern continuum mechanics and thermodynamics. The theory results in a complete set of equations and boundary conditions for a thermoelastic superconducting medium. In my view the authors have made a fundamental first step in the set up of such a theory but, nevertheless, I have some comments on the way they have derived these results. These comments are:

i) The authors do not consequently use a local formulation for the balance equations.

ii) The principle of equipresence (cf. [2]) is not taken into account and neither is a list of independent variables given explicitly.

iii) The choice of $\mathbf{q}/T$ for the entropy flux is too simple; an extended formulation for this entropy flux is needed.

In this NOTE it is shown how by some small modifications the theory of [1] can be made completely thermodynamically consistent.
We emphasize that our results are the same as those of [1]. Hence, our only aim is to support the theory of [1] by giving it a more solid theoretical basis.

2. Derivation of the fundamental equations.

Our theory will be based upon an energy balance and an (extended) entropy inequality, both in local formulation, reading (see [1] for nomenclature)

$$\rho \dot{u} = \rho\, \dot{\Psi} + \rho\, \eta\, \dot{T} + \rho\, \dot{\eta}\, T = t_{ij}\, v_{i,j} - q_{k,k} + \rho\, r + J_i\, E_i \ , \tag{1}$$

(cf. [1], (2.8)), and

$$\rho\, \dot{\eta} + \phi_{k,k} - \rho\, \frac{r}{T} \geq 0 \ . \tag{2}$$

Here, ϕ_k is an *à priori unknown* entropy flux, which must be determined in due course of the exploration of the Clausius-Duhem inequality (2).
Hence, in contrast to [1] we do not put $\phi_k = q_k/T$; however, to include this classical term we write

$$\phi_{k,k} = -\ \frac{1}{T^2}\, q_k\, T_{,k} + \frac{1}{T}\, q_{k,k} + \frac{1}{T}\, K_{k,k} \ , \tag{3}$$

with now K_k unknown.
Substituting (1) and (3) into (2), multiplyed by T $(T > 0)$, we obtain

$$-\rho\, \dot{\Psi} - \rho\, \eta\, \dot{T} + t_{ij}\, v_{i,j} + J_i\, E_i - \frac{1}{T}\, q_k\, T_{,k} + K_{k,k} \geq 0 \ . \tag{4}$$

Note that the difference with [1], (2.9a), is, apart from the local form, the occurrence of the term $K_{k,k}$.

As in [1] we put

$$J_i = J_i^{(n)} + J_i^{(s)} \ , \ \text{ and } \ E_i = -\phi_{,i} - \dot{A}_i \ , \ \ (B_i = e_{ijk}\, A_{k,j}) \ , \tag{5}$$

yielding

$$\begin{aligned} J_i\, E_i \ &= J_i^{(n)}\, E_i - J_k^{(s)}\, (\phi_{,k} + \dot{A}_k) = \\ &= J_i^{(n)}\, E_i - (J_k^{(s)}\, \phi)_{,k} + J_{k,k}^{(s)}\, \phi - J_k^{(s)}\, \dot{A}_k \ . \end{aligned} \tag{6}$$

We define

$$\hat{K}_k = K_k - J_k^{(s)}\, \phi \ , \tag{7}$$

and we use the short-hand notation

$$J_i^{(n)}\, E_i - \frac{1}{T}\, q_k\, T_{,k} =: Q_d \geq 0 \ , \tag{8}$$

so that Q_d is the (classical) dissipation term due to normal current and thermal conduction. With (6) - (8) the inequality (4) becomes

$$-\rho\,\dot{\Psi} - \rho\,\eta\,\dot{T} + t_{ij}\,v_{i,j} + J^{(s)}_{k,k}\,\phi - J^{(s)}_k\,\dot{A}_k + \hat{K}_{k,k} + Q_d \geq 0 \ . \tag{9}$$

This is the basic form of the entropy inequality on which our theory will be built. We shall use the classical continuum-mechanics approach (Coleman and Noll, [3]), *including the principle of equipresence*, in exploring this inequality.
As in [1], we restrict ourselves to linear theory (small deformations, implying a.o. $E_{\alpha,\beta} \to \varepsilon_{ij}$, $\rho \approx \rho_0$, $d/dt \approx \partial/\partial t$).

We proceed with the following steps:

(i) choose a set of independent variables $\mathcal{Q}$;

(ii) assume, until the contrary is proven, that all dependent variables depend upon all elements of $\mathcal{Q}$, i.e. (equipresence!)

$$\begin{aligned} &\Psi = \Psi(\mathcal{Q}) \ , \quad \hat{K}_k = \hat{K}_k(\mathcal{Q}) \ , \quad \eta = \eta(\mathcal{Q}) \ , \\ &t_{ij} = t_{ij}(\mathcal{Q}) \ , \quad J^{(s)}_k = J^{(s)}_k(\mathcal{Q}) \ , \quad \text{etc.} \end{aligned} \tag{10}$$

(iii) derive, in the "classical way", from (9) constitutive equations for: η, t_{ij}, $J^{(s)}_k$ and $\hat{K}_k$, together with (as we shall show) an evolution equation for ψ of the form

$$\dot{\psi} = I(\mathcal{Q}) \ . \tag{11}$$

The following set of independent variables is chosen here, ($\bar{\psi}$ is the complex conjugate of ψ)

$$\mathcal{Q} = \{T,\ \varepsilon_{ij},\ T_{,k},\ \phi,\ A_i,\ \psi,\ \bar{\psi},\ \psi_{,k},\ \bar{\psi}_{,k}\} \ . \tag{12}$$

In this, T and ε_{ij} represent thermal and elastic interaction, $T_{,k}$ is included to allow for thermal conduction, ϕ and A_i stand for the electromagnetic effects and ψ, $\bar{\psi}$, $\psi_{,k}$, $\bar{\psi}_{,k}$ for the superconducting ones (as will turn out furtheron, we have to include the first derivatives of ψ and $\bar{\psi}$).
In the common way (see also [1]) it follows from (9) that

$$t_{ij} = \rho\,\frac{\partial\Psi}{\partial\varepsilon_{ij}} \quad , \quad \eta = -\,\frac{\partial\Psi}{\partial T} \ , \tag{13}$$

and

$$\frac{\partial\Psi}{\partial T_{,k}} = \frac{\partial\Psi}{\partial\phi} = 0 \ . \tag{14}$$

Then, (9) reduces to

$$
\begin{aligned}
&-\left(\rho\,\frac{\partial\Psi}{\partial A_k}+J_k^{(s)}\right)\dot{A}_k-\rho\,\frac{\partial\Psi}{\partial\psi}\,\dot{\psi}-\rho\,\frac{\partial\Psi}{\partial\bar{\psi}}\,\dot{\bar{\psi}}+\\
&-\rho\,\frac{\partial\Psi}{\partial\psi_{,k}}\,\dot{\psi}_{,k}-\rho\,\frac{\partial\Psi}{\partial\bar{\psi}_{,k}}\,\dot{\bar{\psi}}_{,k}+J_{k,k}^{(s)}\phi+\hat{K}_{k,k}+Q_d\geq 0\ .
\end{aligned}
\tag{15}
$$

In what follows the following statement is essential:

> we do not consider $\dot{\psi}$ (or $\dot{\bar{\psi}}$, $\dot{\psi}_{,k}$, $\dot{\bar{\psi}}_{,k}$) to be independent of ψ or $\psi_{,k}$; on the contrary, we assume the existence of an (evolution) relation for ψ of the form (11), i.e.

$$
\dot{\psi}=I(\ldots,\ \phi,\ A_i,\ \psi,\ \bar{\psi},\ \psi_{,k},\ \bar{\psi}_{,k})\ . \tag{16}
$$

We shall derive an explicit expression for the functional relation I and we shall show that this relation depends upon the form of the free energy Ψ.
However, first we conclude from (15), since the coefficient of $\dot{A}_k$ must be zero, that

$$
J_k^{(s)}=-\rho\,\frac{\partial\Psi}{\partial A_k}\ . \tag{17}
$$

Assuming

$$
\rho\,\Psi(Q)=\rho\,\Psi_n(T,\,\varepsilon_{ij},\,T_{,k})+F_s(Q)\ , \tag{18}
$$

with

$$
F_s\Big|_{\psi\equiv 0}=0\ , \tag{19}
$$

we can rewrite (17) as

$$
J_k^{(s)}=-\,\frac{\partial F_s}{\partial A_k}\ . \tag{20}
$$

Since Ψ, and hence also F_s, is independent of ϕ this constitutive equation implies that $J_k^{(s)}$ is independent of ϕ. Next, we write

$$
-\rho\,\frac{\partial\Psi}{\partial\psi_{,k}}\,\dot{\psi}_{,k}=\left(-\frac{\partial F_s}{\partial\psi_{,k}}\,\dot{\psi}\right)_{,k}+\left(\frac{\partial F_s}{\partial\psi_{,k}}\right)_{,k}\,\dot{\psi}\ , \tag{21}
$$

and we do the same with the term with $\dot{\bar{\psi}}_{,k}$, and, moreover, we define

$$
\tilde{K}_k=\hat{K}_k-\frac{\partial F_s}{\partial\psi_{,k}}\,\dot{\psi}-\frac{\partial F_s}{\partial\bar{\psi}_{,k}}\,\dot{\bar{\psi}}=\tilde{K}_k(Q,\ \dot{\psi},\ \dot{\bar{\psi}})\ . \tag{22}
$$

Thus we obtain from (15)

$$-\left[\frac{\partial F_s}{\partial \psi} - \left(\frac{\partial F_s}{\partial \psi_{,k}}\right)_{,k}\right] \dot{\psi} - \left[\frac{\partial F_s}{\partial \bar{\psi}} - \left(\frac{\partial F_s}{\partial \bar{\psi}_{,k}}\right)_{,k}\right] \dot{\bar{\psi}} + J^{(s)}_{k,k}\, \phi + \tilde{K}_{k,k} + Q_d \geq 0 \ . \quad (23)$$

NOTE: If $\dot{\psi}$ would be free to choose, i.e. if there would not exist a relation like (16), then (23) would result in

$$\frac{\partial F_s}{\partial \psi} - \left(\frac{\partial F_s}{\partial \psi_{,k}}\right)_{,k} = 0 \ , \quad \text{and} \quad J^{(s)}_{k,k} = 0 \ . \quad (24)$$

In this case there is no dissipation due to the supercurrent.

From (17) and (14)[2] we conclude that the third term in (23) is linear in ϕ. Since F_s, $\tilde{\mathrm{K}}$ and Q_d are independent of ϕ, the function I must then also be linear in ϕ, i.e.

$$\dot{\psi} = I = I_0 + I_1\, \phi \ , \quad (25)$$

where I_0 and I_1 are complex functions of, amongst others, A_k, ψ and $\bar{\psi}$, but independent of ϕ. We shall show how they can be determined from the entropy inequality once a specific choice for F_s is made.
Introducing

$$\bar{\Gamma} = \frac{\partial F_s}{\partial \psi} - \left(\frac{\partial F_s}{\partial \psi_{,k}}\right)_{,k} \quad (26)$$

and using (25) in (23) we can write the latter as

$$-\bar{\Gamma}\, I_0 - \Gamma\, \bar{I}_0 + \{J^{(s)}_{k,k} - \bar{\Gamma}\, I_1 - \Gamma\, \bar{I}_1\}\, \phi + \tilde{K}_{k,k} + Q_d \geq 0 \ . \quad (27)$$

Since the coefficient of ϕ is independent of ϕ, as are the remaining terms in (27), it must be zero. Hence, the following relation must hold

$$J^{(s)}_{k,k} = \bar{\Gamma}\, I_1 + \Gamma\, \bar{I}_1 \ . \quad (28)$$

We shall show how this relation restricts the possible form of F_s. With (28) the inequality (27) is satisfied if

$$I_0 = -\tfrac{1}{2}\, k\, \Gamma \ , \quad \text{with} \quad k > 0 \ , \quad \text{and} \quad \tilde{K}_k = 0 \ , \quad (29)$$

because it then takes the form

$$k\,|\Gamma|^2 + Q_d \geq 0\;. \tag{30}$$

Up till now, we did not make any assumption about the specific form of F_s; hence the above results hold generally. However, at this point we shall make a more special choice for F_s; a choice that will satisfy all our purposes here. We take

$$\begin{aligned} F_s &= \alpha\,\psi\,\bar\psi + \tfrac{1}{2}\,\beta\,\psi^2\,\bar\psi^2 + a_1\,\psi_{,k}\,\bar\psi_{,k} + a_2(\psi\,\bar\psi_{,k} + \bar\psi\,\psi_{,k})\;A_k \\ &+ i\,a_3(\psi\,\bar\psi_{,k} - \bar\psi\,\psi_{,k})\;A_k + a_4\,\psi\,\bar\psi\,A_k\,A_k\;, \end{aligned} \tag{31}$$

where α and a_i are real coefficients (since F_s must be real). With (31) the constitutive equation (20) yields

$$J_k^{(s)} = -a_2(\psi\,\bar\psi_{,k} + \bar\psi\,\psi_{,k}) - i\,a_3(\psi\,\bar\psi_{,k} - \bar\psi\,\psi_{,k}) - 2a_4\,\psi\,\bar\psi\,A_k\;, \tag{32}$$

while (28) gives

$$\bar\Gamma = -a_1\,\bar\psi_{,kk} - a_2\,\bar\psi\,A_{k,k} + i\,a_3(2\bar\psi_{,k}\,A_k + \bar\psi\,A_{k,k}) + (a_4\,A_k\,A_k + \hat\alpha)\;\bar\psi \tag{33}$$

where $\hat\alpha = \alpha + \beta\,\psi\,\bar\psi$.
Substitution of (32) and (33) into (28) results in the following relation which must be satisfied for arbitrary ψ and A_k,

$$\begin{aligned} &-2a_2\,\psi_{,k}\,\bar\psi_{,k} + [-(a_2 + i\,a_3)\;\bar\psi_{,kk} - a_4(2\bar\psi_{,k}\,A_k + \bar\psi\,A_{k,k})]\;\psi \\ &+[(-a_2 + i\,a_3)\;\psi_{,kk} - a_4(2\psi_{,k}\,A_k + \psi\,A_{k,k})]\;\bar\psi = \\ &= [-a_1\,\bar\psi_{,kk} - a_2\,\bar\psi\,A_{k,k} + i\,a_3(2\bar\psi_{,k}\,A_k + \bar\psi\,A_{k,k}) \\ &+(a_4\,A_k\,A_k + \alpha)\;\bar\psi]\;I_1 + [-a_1\,\psi_{,kk} - a_2\,\psi\,A_{k,k} \\ &-i\,a_3(2\bar\psi_{,k}\,A_k + \psi\,A_{k,k}) + (a_4\,A_k\,A_k + \alpha)\;\psi]\;\bar I_1\;. \end{aligned} \tag{34}$$

This relation is satisfied if and only if

(i) $\quad a_2 = 0\;,$ (35.1)

(ii) $\quad I_1 = -i\,c\,\psi \quad , \quad c \in \mathbb{R}\;,$ (35.2)

for convenience, we take $c = e/\hbar$, ($e \in \mathbb{R}$, $\hbar$: Planck's constant)

(iii) $\quad -i\,a_3 = -a_1\,I_1 = i\,\dfrac{e}{\hbar}\,a_1 \quad \Longrightarrow \quad a_3 = -\dfrac{e}{\hbar}\,a_1\;,$ (35.3)

(iv) $$a_4 = -i\,a_3\,I_1 = \left(\frac{e}{\hbar}\right)^2 a_1 \ . \tag{35.4}$$

With the conditions (35) the energy expression (31) becomes

$$\begin{aligned} F_s &= \alpha\,\psi\,\bar\psi + \tfrac{1}{2}\,\beta\,\psi^2\,\bar\psi^2 + a_1\,[\psi_{,k}\,\bar\psi_{,k} - \frac{i\,e}{\hbar}\,(\psi\,\bar\psi_{,k} - \bar\psi\,\psi_{,k})\,A_k \\ &+ \left(\frac{e}{\hbar}\right)^2 \psi\,\bar\psi\,A_k\,A_k] = \\ &= F'_{sn}\,(|\psi|) + a_1\,\left|\left(i\,\frac{\partial}{\partial x_k} + \frac{e}{\hbar}\,A_k\right)\,\psi\right|^2 . \end{aligned} \tag{36}$$

Comparing this result with [1], (2.13) we see that we have a complete agreement if we only take

$$a_1 = \frac{\hbar^2}{2m^*}\ , \quad \text{and} \quad e = e^*\ . \tag{37}$$

Furthermore, (32) then becomes

$$J_k^{(s)} = \frac{\hbar^2}{2m^*}\,\left[\frac{i\,e^*}{\hbar}\,(\psi\,\bar\psi_{,k} - \bar\psi\,\psi_{,k}) - 2\left(\frac{e^*}{\hbar}\right)^2 \psi\,\bar\psi\,A_k\right]\ , \tag{38}$$

again in agreement with [1], (33).

Finally, from (25) and (33) and with (35) and (37) we find

$$\begin{aligned} &\Lambda = \dot\psi + i\,\frac{e^*}{\hbar}\,\phi\,\psi = -\tfrac{1}{2}\,k\,\Gamma = \\ &= -\frac{k}{2}\left\{\frac{\hbar^2}{2m^*}\,\left[-\,\psi_{,kk} + \frac{i\,e^*}{\hbar}\,(2\psi_{,k}\,A_k + \psi\,A_{k,k}) + \left(\frac{e^*}{\hbar}\right)^2 \psi\,A_k\,A_k\right]\right. \\ &\left. +\alpha\,\psi + \beta\,\bar\psi\,\psi^2\right\}\ , \end{aligned} \tag{39}$$

and this is exactly relation [1], (3.8), if we take

$$k = \frac{2}{\hbar\,\gamma_R}\ \ (>0)\ . \tag{40}$$

Thus, we have derived the evolution relation for ψ we are looking for. The underlying derivation absolutely shows that this relation is related to a specific form of the free energy term F_s.
The residual entropy inequality becomes (compare with [1], (3.11))

$$\frac{2}{\hbar}\,\gamma_R\,|(\hbar\,\dot\psi + i\,e^*\,\phi\,\psi)|^2 + Q_d \geq 0\ . \tag{41}$$

3. Conclusions

In this NOTE we have presented a thermodynamically consistent (i.e. satisfying equipresence and an extended entropy inequality) theory for thermoelastic superconductors. In formula (20) a constitutive relation for the superconducting current, directly written as a derivative of the free energy F_s, is presented. This, and the other basical relations as (40) and (41), show that the ultimate results depend on the explicit choice of the expression for the free energy functional. After a specific choice of the free energy, the associated results completely agree with those of [1].

In contrast to [1], we needed a more general (extended) formula for the entropy flux $\underline{\phi}$ (see (3), i.e. $\phi_k \neq q_k/T$). With F_s according to (36), it can be shown that $(29)^2$ implies that the extra entropy flux $\underline{K}$ (defined by (3); see also (7) and (22)) must be of the form

$$K_k = J_k^{(s)}\,\phi + \frac{\hbar^2}{2m^*}\,[\bar{\psi}_{,k}\,\dot{\psi} + \psi_{,k}\,\dot{\bar{\psi}} + \frac{i\,e^*}{\hbar}\,(\bar{\psi}\,\dot{\psi} - \psi\,\dot{\bar{\psi}})\,A_k]\,, \tag{44}$$

(then $\tilde{K}_k = 0$). This entropy flux is exactly the same as the one expressed in the two surface integrals in [1], (2.14). Moreover, if we require $(\mathbf{J}^{(s)}, \mathbf{n})$ and $(\mathbf{K}, \mathbf{n})$ to be zero at the boundary of the superconductor we arrive at the boundary condition [1], (3.4).

As in [1], we have used here a linear, small deformation theory. However, a generalization to a finite strain theory can be set up along exactly the same lines.

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