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Document status and date:
Published: 01/01/2002

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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• The final published version features the final layout of the paper including the volume, issue and page numbers.

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Scale dependence of the entropy production in stochastic dynamics

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2nd July 2002

Abstract: We study the value of the mean entropy production as a function of the level of description. We prove that by coarsening the system, we obtain a lower entropy production. As main example we consider the Lorentz lattice gas and prove that the microscopic entropy production is strictly larger than the one predicted by the Boltzmann equation.

1 Introduction

The value of the entropy production of a physical system depends on the level of description and the choice of macro-variables. E.g. in [5] a reversible dynamics for fluctuations on a macroscopic scale is derived from an asymmetric exclusion process, i.e., from an irreversible stochastic dynamics (i.e., zero macroscopic entropy production and non-zero microscopic one).

Within the space-time Gibbsian formalism developed in [8],[9],[12], the mean entropy production is the relative entropy density of the distributions of the forward and the backward process. Consequently, the entropy production vanishes if the process is reversible, and the converse can be proved for various stochastic dynamics, see [11], [13].

From the convexity of the relative entropy density, we have as a general inequality

$$s(Q \circ T | P \circ T) \leq s(Q|P), \tag{1.1}$$

where T is any stochastic kernel. The transition from a microscopic description towards a more macroscopic description can be viewed in many cases as such a stochastic kernel (e.g., from actual positions of particles towards densities). We thus expect that going to a more macroscopic description of a system implies a decrease in entropy production.

In this paper, we give three examples illustrating this phenomenon, and we show that the inequality can be strict. Our first example is a 'fuzzy' stochastic process that can be obtained when we observe a stochastic process with "goggles" that cannot distinguish between certain states. Secondly, we consider an effective uncoupled Markovian dynamics obtained as a 'kinetic limit' of a coupled system. The stochastic Lorentz lattice gas is studied as a third example. We compute the entropy production for the environment process (the configuration of scatterers

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as seen from a particle starting at the origin), and prove that it is zero if and only if the scattering rule is such that forward and backward scatterings have the same probability. Next we consider the kinetic limit which is a linear Boltzmann equation. This equation corresponds to a Markov process of velocities, of which the entropy production can be computed. If the scattering rule depends in a non-trivial way on the configuration of scatterers, then the entropy production associated with the Boltzmann equation is strictly smaller than the microscopic one.

2 Entropy production for Markov processes

2.1 Setting

In this section we briefly review some definitions and results from [12]. Let Ω be a compact metric space, and $\{\sigma_t : t \geq 0\}$ a pure jump Markov process on Ω, with generator, acting on bounded Borel measurable functions, of the type

$$Lf(\sigma) = \sum_{\eta \in \Omega} c(\sigma, \eta)(f(\eta) - f(\sigma)) \qquad (2.1)$$

where $c(\sigma, \eta)$ is non-negative and for each σ, the set

$$\{\eta : c(\sigma, \eta) > 0\}$$

is finite. A probability measure μ on the Borel-σ-field of Ω is stationary for the process if for all f:

$$\int Lf d\mu = 0.$$

This implies that started with σ_0 distributed according to μ, $\{\sigma_t, t \geq 0\}$ is a stationary process. Trajectories of the process will be denoted by $\omega = \omega_{0 \leq t \leq T}$, and $\mathbb{P}_\mu$ denotes the path space measure of the process started with σ_0 distributed according to μ

2.2 Time reversal

For

$$\pi : \Omega \to \Omega$$

a homeomorphism such that $\pi \circ \pi = \mathrm{id}$, we define the π-time reversal Θ_π of a trajectory ω:

$$(\Theta_\pi(\omega))(t) = \pi(\omega(T - t)) \qquad (2.2)$$

For π the identity, this is the ordinary time reversal. However, if Ω contains variables such as velocities, then it is natural to combine the time-reversal with an involution which reverses the velocities.

Definition 2.3 *A probability measure μ on Ω invariant under π is called π-reversible if*

$$\mathbb{P}_\mu \circ \Theta_\pi = \mathbb{P}_\mu \qquad (2.4)$$

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For a given π-invariant probability measure μ, let $\tilde{L}$ denote the $L^2(\mu)$ generator of the reversed process $\{\pi(X_{T-t}) : 0 \le t \le T\}$.

Lemma 2.5 *For any π-invariant measure μ we have:*

1. *Let L^* be the adjoint of L in $L^2(\mu)$, then*

$$\tilde{L} = \pi L^* \pi \tag{2.6}$$

2. *μ is π-reversible if and only if*

$$\pi L^* \pi = L \tag{2.7}$$

3. *If μ is π-reversible, then μ is stationary*

Proof. The first two statements follow immediately from definition 2.3. For the third statement, let $f : \Omega \to \mathbb{R}$, then

$$\int Lf \, d\mu = \int f L^* 1 \, d\mu = \int f \left(\pi L (\pi 1) \right) d\mu = 0 \tag{2.8}$$

where in the last step we used $\pi 1 = 1$ and $L1 = 0$. ∎

2.3 Entropy production

For a given involution π, and a π-invariant stationary measure μ, the random variable entropy production is defined as a function on path space by

$$S_\pi^T(\omega) = \left(\log \frac{d\mathbb{P}_\mu \circ \Theta_\pi}{d\mathbb{P}_\mu} \right)(\omega) \tag{2.9}$$

Clearly, this is a priori not well-defined. However, we will restrict ourselves to the case where π is such that $c(x, y) > 0$ if and only if $c(\pi y, \pi x) > 0$ (this is usually called "dynamic reversibility"). In that case, with generator of the type (2.1), the Radon Nikodym derivative [7] in (2.9) can be spelled out as

$$
\begin{aligned}
\log S_\pi^T(\omega) &= \int_0^T \log \frac{c(\omega_{s-}, \omega_{s+})}{c(\pi \omega_{s+}, \pi \omega_{s-})} dN_s(\omega) \\
&\quad - \int_0^T \left(c(\omega_{s-}, \omega_{s+}) - c(\pi \omega_{s+}, \pi \omega_{s-}) \right) ds
\end{aligned} \tag{2.10}
$$

where $N_t(\omega)$ denotes the number of jumps of the trajectory ω in the time interval $[0, t]$. The mean entropy production $(MEP_\pi(\mu))$ is defined as

$$MEP_\pi(\mu, L) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}_\mu(S_\pi^T) \tag{2.11}$$

which in our set-up reads

$$MEP_\pi(\mu, L) = \sum_{\sigma, \eta \in \Omega} \mu(\sigma) c(\sigma, \eta) \log \frac{c(\sigma, \eta)}{c(\pi \eta, \pi \sigma)}. \tag{2.12}$$

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From (2.11) one sees that $MEP_\pi(\mu, L)$ is equal to the relative entropy density of the path space measure $\mathbb{P}_\mu \circ \Theta_\pi$ with respect to the path space measure $\mathbb{P}_\mu$. Therefore $MEP_\pi(\mu, L)$ is non-negative and equal to zero if and only if $\mathbb{P}_\mu \circ \Theta_\pi = \mathbb{P}_\mu$. In what follows we will often omit the dependence on L and simply write $MEP_\pi(\mu)$.

For a discrete time stochastic process $\{X_n : n \in \mathbb{N}\}$, with X_0 distributed according to μ, a π-invariant probability measure on Ω, the random variable entropy production is defined as follows

$$S_\pi^n(x_1, \dots, x_n) = \frac{\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)}{\mathbb{P}(X_1 = \pi x_n, X_2 = \pi x_{n-1}, \dots, X_n = \pi x_1)} \tag{2.13}$$

The mean entropy production is

$$MEP(X) = \lim_{n \to \infty} \frac{1}{n} \sum_{x_1, \dots, x_n} \mathbb{P}[X_1 = x_1, \dots, X_n = x_n] \log \frac{\mathbb{P}[X_1 = x_1, \dots, X_n = x_n]}{\mathbb{P}[X_1 = \pi x_n, \dots, X_n = \pi x_1]} \tag{2.14}$$

provided this limit exists.

3   Example 1: fuzzy processes

Let $\omega = \{1, \dots, N\}$ be the state space of a discrete time process $\{X_n : n \in \mathbb{Z}\}$. Consider a function $F : \{1, \dots, N\} \to \{1, \dots, K\}$ for some $K \leq N$. The F-fuzzy version of the X process is denoted $\{Y_n : n \in \mathbb{Z}\}$ defined via $Y_n = F(X_n)$. In case $K < N$ and X_n a Markov chain, Y_n is in general not Markovian. For the entropy production to both the processes X_n and Y_n, we now prove the following theorem:

Theorem 3.1

$$MEP(Y) \leq MEP(X) \tag{3.2}$$

with equality if and only if F is a bijection.

Proof. By definition

$$
\begin{aligned}
MEP(Y) &= \sum_{y_1, \dots, y_n} \mathbb{P}[Y_1 = y_1, \dots, Y_n = y_n] \log \frac{\mathbb{P}[Y_1 = y_1, \dots, Y_n = y_n]}{\mathbb{P}[Y_1 = y_n, \dots, Y_n = y_1]} \\
&= \sum_{y_1, \dots, y_n} \sum_{x_1 \in F^{-1}(y_1)} \dots \sum_{x_n \in F^{-1}(y_n)} \mathbb{P}[Y_1 = y_1, \dots, Y_n = y_n] \times \\
&\quad \times \log \frac{\sum_{x_1 \in F^{-1}(y_1)} \dots \sum_{x_n \in F^{-1}(y_n)} \mathbb{P}[Y_1 = y_1, \dots, Y_n = y_n]}{\sum_{x_n \in F^{-1}(y_n)} \dots \sum_{x_1 \in F^{-1}(y_1)} \mathbb{P}[Y_1 = y_n, \dots, Y_n = y_1]} \\
&\leq \sum_{x_1, \dots, x_n} \mathbb{P}[X_1 = x_1, \dots, X_n = x_n] \log \frac{\mathbb{P}[X_1 = x_1, \dots, X_n = x_n]}{\mathbb{P}[X_1 = x_n, \dots, X_n = x_1]} \\
&= MEP(X)
\end{aligned}
$$

The third step results from the concavity of $x \to \log x$, from which we obtain the inequality

$$
\begin{aligned}
\sum_i a_i \log \frac{\sum_i a_i}{\sum_i b_i} &= -\sum_i a_i \log \frac{\sum_i (b_i/a_i) a_i}{\sum_i a_i} \\
&\leq -\sum_i a_i \sum_j a_j \frac{\log(b_j/a_j)}{\sum_j a_j} = \sum_i a_i \log \frac{a_i}{b_i}
\end{aligned}
$$

∎

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4 Example 2: effective Markovian dynamics

We consider a Markov process on a state space of the form $\Omega = S \times A$ where S and A are finite sets. The generator of the process is of the form

$$L_\epsilon f(\sigma, \alpha) = \sum_{\sigma' \in S} k_\alpha(\sigma, \sigma')(f(\sigma', \alpha) - f(\sigma, \alpha)) + \epsilon^{-1} \sum_{\alpha' \in A} (f(\sigma, \alpha') - f(\sigma, \alpha)) \qquad (4.1)$$

In words, this means that the σ evolves with rates depending on α and the state α itself changes at rate ϵ^{-1}. To avoid trivialities, we assume that $k_\alpha(\sigma, \sigma') \neq k_{\alpha'}(\sigma, \sigma')$ for at least one pair $\alpha \neq \alpha', \sigma, \sigma'$. For simplicity we suppose that the rates satisfy

$$\sum_\sigma k_\alpha(\sigma, \sigma') = \sum_{\sigma'} k_\alpha(\sigma, \sigma') = 1, \qquad (4.2)$$

which implies that the stationary measure is uniform on $S \times A$. For the involution π we take the identity. The prefactor ϵ^{-1} indicates that the time-scale on which α varies (i.e., time between successive jumps of the α-process) is of order ϵ. Therefore we expect that in the limit $\epsilon \to 0$ the α-process decouples from the σ-process and the latter becomes Markovian with rates

$$\bar{k}(\sigma, \sigma') = \frac{1}{|A|} \sum_\alpha k_\alpha(\sigma, \sigma'). \qquad (4.3)$$

More precisely, on the time scale of the jumps of σ component, the α component is in equilibrium in the limit $\epsilon \to 0$. This is proved in the following lemma

Lemma 4.4 *In the limit $\epsilon \to 0$, the process $\{\sigma_t : t \geq 0\}$ becomes a Markov process with generator*

$$L_{eff} f(\sigma) = \sum_{\sigma'} \bar{k}(\sigma, \sigma')(f(\sigma') - f(\sigma)) \qquad (4.5)$$

Proof. Let us denote $\mathbb{E}^\epsilon_{\sigma, \alpha}$ expectation in the Markov process with generator L_ϵ, and $\bar{\mathbb{E}}_\sigma$ expectation in the Markov process with generator L_{eff}. Pick $f : S \to \mathbb{R}$ and denote $\{\alpha_t : t \geq 0\}$ the Markov process with generator

$$\hat{L} g(\alpha) = \sum_{\alpha'} (g(\alpha') - g(\alpha))$$

on $g : A \to \mathbb{R}$. Then we have, using the definition of the generator (4.1):

$$\mathbb{E}^\epsilon_{\sigma, \alpha} f(\sigma_t) - \mathbb{E}^\epsilon_{\sigma, \alpha} f(\sigma_0) = \mathbb{E}^\epsilon_{\sigma, \alpha} \int_0^t \sum_{\sigma'} k_{\alpha_{s\epsilon^{-1}}}(\sigma_s, \sigma')(f(\sigma') - f(\sigma_s)) ds. \qquad (4.6)$$

The distribution of α_t converges in total variation norm to the uniform measure $\mu_A = \frac{1}{|A|} \sum_{a \in A} \delta_a$, for every initial $\alpha_0 = \alpha$. Therefore,

$$\lim_{\epsilon \to 0} \left(\mathbb{E}^\epsilon_{\sigma, \alpha} f(\sigma_t) - \mathbb{E}^\epsilon_{\sigma, \alpha} f(\sigma_0^\epsilon) \right) = \int_0^t \bar{k}(\sigma_s, \sigma)(f(\sigma') - f(\sigma_s)) ds. \qquad (4.7)$$

At this stage, we have to prove that every limiting process of $\{\sigma_t^\epsilon : t \in [0,T]\}$ as $\epsilon \to 0\}$ is Markovian. Indeed, in that case we derive from (4.7) that the generator of this limiting Markov process is L_{eff} and finally the existence of a limit of the distributions of $\{\sigma_t^\epsilon : 0 \leq t \leq T\}$ follows from the tightness criterion, [4] Theorem 1.3, p. 51. To prove the Markov property, we need some more notation:

$$
\begin{aligned}
\mathcal{F}_t^\sigma &= \sigma\{\sigma_s : 0 \leq s \leq t\} \\
\mathcal{F}_t^\alpha &= \sigma\{\alpha_s : 0 \leq s \leq t\} \\
\mathcal{F}_t &= \sigma\{(\sigma_s, \alpha_s) : 0 \leq s \leq t\}
\end{aligned}
\tag{4.8}
$$

We have to prove that for $0 < s < t$:

$$
\lim_{\epsilon \to 0} \mathbb{E}_{\sigma,\alpha}^\epsilon \left[f(\sigma_t) | \mathcal{F}_s^\sigma \right] = \bar{\mathbb{E}}_\sigma \left[f(\sigma_t) | \sigma_s \right]
\tag{4.9}
$$

where $\bar{\mathbb{E}}_\sigma$ denotes expectation in the Markov process with generator L_{eff} started from σ. By the Markov property of $\{(\sigma_t, \alpha_t) : t \geq 0\}$, we have

$$
\begin{aligned}
\mathbb{E}_{\sigma,\alpha}^\epsilon \left(f(\sigma_t) | \mathcal{F}_s^\sigma \right) &= \mathbb{E}_{\sigma,\alpha}^\epsilon \left(\mathbb{E}_{\sigma,\alpha}^\epsilon \left(f(\sigma_t) | \mathcal{F}_s \right) | \mathcal{F}_s^\sigma \right) \\
&= \mathbb{E}_{\sigma,\alpha}^\epsilon \left(\mathbb{E}_{\sigma,\alpha}^\epsilon \left(f(\sigma_t) | \sigma_s, \alpha_s \right) | \mathcal{F}_s^\sigma \right)
\end{aligned}
\tag{4.10}
$$

Hence, it suffices to prove that

$$
\lim_{\epsilon \to 0} \mathbb{E}_{\sigma,\alpha}^\epsilon \left(f(\sigma_t) | \sigma_s = \xi, \alpha_s = \beta \right) = \bar{\mathbb{E}}_\sigma \left(f(\sigma_t) | \sigma_s = \xi \right)
\tag{4.11}
$$

The left hand side of (4.11) equals

$$
\lim_{\epsilon \to 0} \mathbb{E}_{\xi,\beta}^\epsilon \left(f(\sigma_{t-s}) \right) = \bar{\mathbb{E}}_\xi \left(f(\sigma_{t-s}) \right) = \bar{\mathbb{E}}_\sigma \left(f(\sigma_t) | \sigma_s = \xi \right)
\tag{4.12}
$$

where we used (4.7). ∎

Let μ_S denote uniform measure on S and μ_A uniform measure on A. $MEP^\epsilon(\mu_S \times \mu_A)$ denotes the entropy production of the process $\{(\sigma_t, \alpha_t) : t \geq 0\}$ and $MEP(\mu_S)$ denotes the entropy production for the limiting process with generator L_{eff}. We obtain the following proposition as a direct consequence of (2.11) the strict convexity of $x \to x \log(x)$ and the previous lemma.

Proposition 4.13 *1. For every $\epsilon > 0$:*

$$
MEP(\mu_S \times \mu_A, L_\epsilon) = \frac{1}{|A||S|} \sum_{\alpha \in A} \sum_{\sigma, \sigma' \in S} k_\alpha(\sigma, \sigma') \log \frac{k_\alpha(\sigma, \sigma')}{k_\alpha(\sigma', \sigma)}
\tag{4.14}
$$

2. The entropy production of the limiting process is given by

$$
MEP(\mu_S, L_{eff}) = \frac{1}{|S|} \sum_{\sigma, \sigma' \in S} \bar{k}(\sigma, \sigma') \log \frac{\bar{k}(\sigma, \sigma')}{\bar{k}(\sigma', \sigma)}
\tag{4.15}
$$

3. For every $\epsilon > 0$ we have the inequality:

$$
MEP(\mu_S) < MEP^\epsilon(\mu_S \times \mu_A)
\tag{4.16}
$$

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5 Example 3: the stochastic Lorentz lattice gas

In this section we compare the microscopic entropy production of a stochastic Lorentz lattice gas with the entropy production of its kinetic limit given by a linear Boltzmann equation, and show strict inequality in the non-reversible case.

5.1 The process

Let $\lambda : \mathbb{Z}^d \to \{0,1\}$ denote a configuration of scatterers on the regular lattice $\mathbb{Z}^d$ and write Ω for the set of all such configurations. For $x \in \mathbb{Z}^d$, $\lambda(x) = 0, 1$ is respectively interpreted as the absence and presence of a scatterer at site x. For $\lambda \in \Omega$, and $x \in \mathbb{Z}^d$ we denote $\tau_x \lambda(y) = \lambda(y + x)$. We suppose that λ are randomly distributed according to a translation invariant and ergodic probability measure μ on Ω Let $S_d \equiv \{e \in \mathbb{Z}^d : |e| = 1\}$ be the set of unit vectors in $\mathbb{Z}^d$. By means of any $\lambda \in \Omega$ and $x \in \mathbb{Z}^d$, a 'scattering law' $p_{\lambda,x} : S_d \times S_d \to [0,1]$ is introduced. The scattering law $p_{\lambda,x}$ depends locally on the scatterer configuration around $x \in \mathbb{Z}^d$ and this dependency is assumed to be translation invariant, i.e. $p_{\lambda,x} = p_{\tau_x \lambda, 0}$.

The discrete time scattering process $\{(X_n, V_n) : n \in \mathbb{N}\}$ on $K \equiv \mathbb{Z}^d \times S_d$, is defined by means of the transition operator

$$
\begin{aligned}
(P_\lambda f)(x, v) &= \mathbb{E}[f(x_k, v_k)|(x_{k-1}, v_{k-1}) = (x, v)] \qquad (5.1)\\
&= (1 - \lambda(x))f(x + v, v) + \lambda(x) \sum_{e \in S^d} p_{\lambda,x}(v, e) f(x + e, e)
\end{aligned}
$$

In words this means the following: a particle, moving in a scatterer configuration λ arrives at site $x \in \mathbb{Z}^d$ with a unit velocity v. If $\lambda(x) = 0$, then the particle moves one step in the direction of its incoming velocity v. If $\lambda(x) = 1$, then the particle is scattered according to the probability law $p_{\lambda,x}$, i.e., the particle gets a new velocity e with probability $p_{\lambda,x}(v, e)$ and moves one step in the direction of e. We assume that the probability law $p(v, e)$ is doubly stochastic, i.e., for all $\lambda \in \Omega, x \in \mathbb{Z}^d$

$$
\sum_{e \in S_d} p_{\lambda,x}(v, e) = \sum_{v \in S_d} p_{\lambda,x}(v, e) = 1
$$

The case $p_{\lambda,x}(v, e) = 1/2d$ is denoted as 'isotropic scattering'.

5.2 The environment process

The environment process $\{(\tau_{X_n} \lambda, V_n) : n \in \mathbb{N}\}$ is a Markov process defined by the transition operator

$$
\begin{aligned}
(Pf)(\lambda, v) &= \mathbb{E}[f(\tau_{X_k}\lambda, V_k)|(\tau_{X_{k-1}}\lambda, V_{k-1}) = (\eta, v)] \qquad (5.2)\\
&= (1 - \eta(0))f(\tau_v \eta, v) + \eta(0) \sum_{e \in S_d} p_{\lambda,0}(e, v) f(\tau_e \eta, e)
\end{aligned}
$$

In words this means that we follow the configuration of scatterers as seen from the scattered particle, and keep track of the velocity.

The continuous time version of the environment process is defined via the generator

$$
Lf(\lambda, v) = (1 - \lambda(0))(f(\tau_v \lambda, v) - f(\lambda, v)) - \lambda(0) \sum_{e \in S_d} p_{\lambda,0}(v, e)(f(\tau_e \lambda, e) - f(\lambda, v)) \qquad (5.3)
$$

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In this process the particle jumps on the event times of an independent mean one Poisson process.

The following lemma characterizes the scattering laws for which the environment process (EP) $\{(\tau_{X_t}, V_t) : t \in [0,T]\}$ is π-reversible. The proof, a straightforward computation, can be found in appendix A. Define $\hat{\mu} = \mu \times \frac{1}{2d} \sum_{e \in S_d} \delta_e$. The natural time reversal in our context is given by

$$\pi : \Omega \times S_d \to \Omega \times S_d : (\eta, v) \to (\tau_{-v}\eta, -v) \tag{5.4}$$

Lemma 5.5 *1. For every translation invariant doubly stochastic scattering rule $p_{\lambda,x}$, $\hat{\mu}$ is a stationary measure for the EP.*

2. $\hat{\mu}$ is a π-reversible measure for the EP iff

$$p_{\lambda,0}(v,w) = p_{\lambda,0}(-w,-v) \tag{5.6}$$

Remark 5.7 *1. The fact that $\hat{\mu}$ is stationary for the EP is a consequence of the assumption of a doubly stochastic scattering law $p_{\lambda,0}(v,e)$ i.e.*

$$\sum_e p_{\lambda,0}(v,e) = \sum_v p_{\lambda,0}(v,e) = 1$$

for any $\lambda \in \Omega$.

2. π-reversibility is established for the particular involution π defined by (5.4) as

$$\pi(\eta, v) = (\tau_{-v}\eta, -v).$$

This is easily understood as v denotes the incoming velocity becomes an outgoing velocity and changes sign under time-reversal.

3. The condition $p_{\lambda,0}(v,w) = p_{\lambda,0}(-w,-v)$ means that probabilities of forward and backward scattering are equal.

5.3   The random variable entropy production

The random variable entropy production is defined on all trajectories $\omega \equiv \{(\lambda_s, v_s) : s \in [0,T]\}$ by means of the time reversal involution Θ_π:

$$(\Theta_\pi(\omega))(t) = \pi\omega_{T-t} = (\tau_{-v_{T-t}}\lambda_{T-t}, -v_{T-t})$$

To compute the random variable entropy production $S_\pi^T(\omega)$ (cf. (2.9)), we introduce the reference process $\mathbb{P}_\mu^o$, corresponding to the π-reversible stochastic Lorentz gas where $p(v,e) = 1/2d$. The random walk is assumed to be started from $x = 0$, at unit speed v_0. We have

$$\log \frac{d\mathbb{P}_\mu}{d\mathbb{P}_\mu^o}(\omega) = \int_0^T \log[2dp_{\lambda_{s-},0}(v_{s-}, v_s)]dN_s(\omega)$$

$$- \int_0^T \lambda_{s-}(0)[p_{\lambda_{s-},0}(v_{s-}, v_s) - 1/2d]ds.$$

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from a straigtforward application of Girsanovs theorem [7]. The process $N_s(\omega) = \sum_{0 \le t \le s} I(v_{t-} \ne v_t)$ counts all scattering events that ocurred in ω before time s. The random variable entropy production is now found as a difference of two such expressions,

$$S_\pi^T(\omega) = \log \frac{d\mathbb{P}_\mu}{d\mathbb{P}_\mu^o}(\Theta_\pi \omega) - \log \frac{d\mathbb{P}_\mu}{d\mathbb{P}_\mu^o}(\omega) \tag{5.8}$$

$$= \int_0^T \log \frac{p_{\lambda_{s-},0}(v_{s-}, v_s)}{p_{\lambda_{s-},0}(-v_s, -v_{s-})} dN_s(\omega) \tag{5.9}$$

From this expression, we see that the random variable entropy production vanishes in the reversible case, where $p_{\lambda,0}(v, e) = p_{\lambda,0}(-e, -v)$. However, S_π^T does not have a fixed sign for all trajectories ω.

5.4 Mean entropy production

By taking the steady state expectation of (5.8), we obtain the mean entropy production $MEP_\pi(\mu)$ of the EP.

Proposition 5.10

$$MEP_\pi(\mu) = \frac{1}{2d} \int \sum_{e,v \in S_d} \lambda(0) p_{\lambda,0}(v, e) \log \frac{p_{\lambda,0}(v, e)}{p_{\lambda,0}(-e, -v)} \mu(d\lambda) \tag{5.11}$$

5.5 Kinetic limit: the Boltzmann equation

In this subsection we pass to the kinetic limit of the stochastic Lorentz gas which gives a linear Boltzmann equation. The limiting equation can then be interpreted as the master equation of a Markovian velocity process of which we can again compute the entropy production.

The kinetic limit is obtained as the $\epsilon \to 0$-limit of the processes $\{(X_t^\epsilon, V_t^\epsilon) : 0 \le t \le T\}$ defined by the generator

$$L_\lambda^\epsilon = L_o + \epsilon[L - L_o] \tag{5.12}$$

where L is given in 5.3 and L_o is obtained from the same expression, putting $\lambda = 0$, i.e., L_o corresponds to the free motion (no scattering). The introduction of the scaling parameter ϵ in the generator (5.12) makes the lapse of time between two successive scattering events of the order ϵ^{-1}. Afterwards, we have to consider this process on the time scale $\epsilon^{-1}t$ in order to obtain a non-trivial "kinetic" limit. More details on the derivation of the Boltzmann equation as a kinetic limit can be found in [2], [15].

Theorem 5.13   *1. There exists a Markov process (x_t, v_t) on $\mathbb{R} \times S_d$, such that*

$$(\epsilon X_{\epsilon^{-1}t}^\epsilon, V_{\epsilon^{-1}t}^\epsilon) \xrightarrow{\mathcal{D}} (x_t, v_t).$$

2. The limiting process satisfies

$$dx_t = v_t dt \tag{5.14}$$

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where v_t is the Markov process on S_d with generator

$$L_s f(v) = \sum_{e \in S_d} \bar{p}(v, e)[f(e) - f(v)] \tag{5.15}$$

for functions $f : S_d \to \mathbb{R}$.

3. The master equation of the limiting process (x_t, v_t) is the linear Boltzmann equation

$$\frac{\partial \rho}{\partial t}(x, v, t) - v\frac{\partial \rho}{\partial x}(x, v, t) = \sum_e \frac{\bar{p}(v, e)}{2d}[\rho(x, e, t) - \rho(x, v, t)], \tag{5.16}$$

with

$$\bar{p}(v, e) = \int \lambda(0) p_{\lambda,0}(e, v)\mu(d\lambda)$$

Proof. For the proof the reader is referred to appendix B. ■

The mean entropy production $\overline{MEP}_\pi(\mu)$ in the kinetic limit thus is the entropy production corresponding to the limiting velocity process with generator L_s. This yields, using (2.12), and strict convexity of $x \to x \log x$:

Theorem 5.17 *The entropy production of the limiting velocity process is given by*

$$\overline{MEP}_\pi(\mu) = \frac{1}{2d} \sum_{e,v \in S_d} \bar{p}(v, e) \log \frac{\bar{p}(v, e)}{\bar{p}(-e, -v)} \tag{5.18}$$

Moreover, $\overline{MEP}_\pi(\mu)$ is less than or equal to the entropy production of the EP in (5.11), with strict equality if and only if $p_{\lambda,0}$ depends non-trivially on λ and (5.6) is not fulfilled.

A   Proof of lemma (5.5)

Proof.
It is sufficent to show that $\int gPf d\hat{\mu} = \int fPg\hat{\mu}$. Observe that

$$\int (1 - \lambda(0))f(\tau_v \lambda, v)g(\lambda, v)\hat{\mu}(d\lambda, dv)$$

$$= \int (1 - \lambda(-v))f(\lambda, v)g(\tau_{-v}\lambda, v)\hat{\mu}(d\lambda, dv)$$

and

$$\int \lambda(0) \sum_e p_{\lambda,0}(v, e)f(\tau_e\lambda, e)g(\lambda, v)\hat{\mu}(d\lambda, dv)$$

$$= \frac{1}{2d}\int \lambda(0) \sum_{e,e'} p_{\lambda,0}(e', e)f(\tau_e\lambda, e)g(\lambda, e')\mu(d\lambda)$$

$$= \frac{1}{2d}\int \sum_{e,e'} p_{\tau_{-e}\lambda,0}(e', e)\lambda(-e)f(\lambda, e)g(\tau_{-e}\lambda, e')\mu(d\lambda)$$

$$= \int \lambda(-v) \sum_{e'} p_{\tau_{-v}\lambda,0}(e', v)f(\lambda, v)g(\tau_{-v}\lambda, e')\hat{\mu}(d\lambda, dv)$$

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Hence,

$$(P^*g)(\lambda, v) = (1 - \lambda(-v))g(\tau_{-v}\lambda, v) + \lambda(-v)\sum_e p_{\tau_{-v}\lambda, 0}(e, v)g(\tau_{-v}\lambda, e) \qquad (A.1)$$

On the other hand,

$$(\pi P\pi)(g)(\lambda, v) = (P(\pi g))(\tau_{-v}\lambda, -v) = (I) + (II) \qquad (A.2)$$

(I):

$$\begin{aligned}
(I) &= (1 - \tau_{-v}\lambda(0))(\pi g)(\tau_{-v}\tau_{-v}\lambda, -v) \\
&= (1 - \lambda(-v))g(\tau_v\tau_{-2v}\lambda, v) \\
&= (1 - \lambda(-v))g(\tau_{-v}\lambda, v)
\end{aligned}$$

(II):

$$\begin{aligned}
(II) &= \tau_{-v}\lambda(0)\sum_e p_{\tau_{-v}\lambda, 0}(-v, e)(\pi g)(\tau_e\tau_{-v}\lambda, e) \\
&= \lambda(-v)\sum_e p_{\tau_{-v}\lambda, 0}(-v, e)g(\tau_{-e}\tau_e\tau_{-v}\lambda, -e) \\
&= \lambda(-v)\sum_e p_{\tau_{-v}\lambda, 0}(-v, e)g(\tau_{-v}\lambda, -e) \\
&= \lambda(-v)\sum_e p_{\tau_{-v}\lambda, 0}(-e, v)g(\tau_{-v}\lambda, -e) \quad (*) \\
&= \lambda(-v)\sum_e p_{\tau_{-v}\lambda, 0}(e, v)g(\tau_{-v}\lambda, e)
\end{aligned}$$

which is the same as (A.1). In $(*)$ we used the π-reversibility condition: $p_{\lambda, 0}(-v, e) = p_{\lambda, 0}(-e, v)$. ∎

B Appendix B: Sketch of proof of the Boltzmann equation

Since we did not find a proof of the linear Boltzmann equation in our context, we give here a sketch of the proof. Proofs of the linear Boltzmann equation in other more complicated deterministic Lorentz gases can be found in [1], [3]. For the sake of notational simplicity we restrict to the one-dimensional case.

Let us denote by $\mathbb{E}^\epsilon_{x,v}$ expectation in the process with generator

$$L^\epsilon_\lambda = L_o + \epsilon(L_\lambda - L_o) \qquad (B.1)$$

Consider $f, \rho : \mathbb{R}^d \times S_d \to \mathbb{R}$ such that for any $v \in S_d$ $f(\cdot, v)$ and $\rho(\cdot, v)$ are Schwarz functions. We define

$$Y^\epsilon_t(v, f, \rho) = \sum_{x \in \mathbb{Z}} \epsilon\rho(\epsilon x, v)\mathbb{E}^\epsilon_{x,v}f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) \qquad (B.2)$$

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The aim is to prove that in the limit $\epsilon \to 0$, $Y_t^\epsilon(v, f, \rho)$ converges to $\int \rho(x, v) f(x, v, t) dx$, where $f(x, v, t)$ solves the linear Boltzmann equation with initial condition $f(x, v, 0) = f(x, v)$. Using (B.1) we have

$$
\begin{aligned}
\frac{d}{dt} & \left(Y_t^\epsilon(v, f, \rho) \right) \\
= & \sum_{x \in \mathbb{Z}} (1 - \epsilon \lambda(x)) \rho(\epsilon x, v) \left(\mathbb{E}_{x,v}^\epsilon f(\epsilon X_{\epsilon^{-1}t} + \epsilon v, V_{\epsilon^{-1}t}) - \mathbb{E}_{x,v}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) \right) \\
+ & \sum_x \epsilon \lambda(x) p_{\lambda,x}(v, w) \rho(\epsilon x, v) \left(\mathbb{E}_{x,w}^\epsilon f(\epsilon X_{\epsilon^{-1}t} + \epsilon w, V_{\epsilon^{-1}t}) - \mathbb{E}_{x,v}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) \right) \text{ (B.3)}
\end{aligned}
$$

Now we perform a Taylor expansion, and collect only the terms which will not vanish in the limit $\epsilon \to 0$, which gives

$$
\begin{aligned}
\frac{d}{dt} & \left(Y_t^\epsilon(v, f, \rho) \right) \\
= & \; o(\epsilon) + Y_t^\epsilon(v, v \cdot \frac{\partial f}{\partial x}, \rho) \\
+ & \sum_{x \in \mathbb{Z}} \sum_w \epsilon \lambda(x) p_{\lambda,x}(v, w) \rho(\epsilon x, v) \left(\mathbb{E}_{x,w}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) - \mathbb{E}_{x,v}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) \right) \text{ (B.4)}
\end{aligned}
$$

The first term corresponds to the free motion, the second term to the scattering. The second term has still to be cast into the form of a Y_t^ϵ field. This can be done by performing an extra average over x. Let $B_\epsilon(x)$ be a lattice interval containing x with a length of $[\epsilon^{-1/2}]$, then we write

$$
\begin{aligned}
& \sum_{x \in \mathbb{Z}} \sum_w \epsilon \lambda(x) p_{\lambda,x}(v, w) \rho(\epsilon x, v) \left(\mathbb{E}_{x,w}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) - \mathbb{E}_{x,v}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) \right) \\
= & \sum_x \epsilon \rho(\epsilon x, v) \left(\sum_{y \in B_\epsilon(x)} \frac{1}{|B_\epsilon x|} \lambda(y) p_{\tau_y \lambda, 0}(v, w) \right) \left(\mathbb{E}_{x,w}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) - \mathbb{E}_{x,v}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) \right) + o(\epsilon) \\
= & \sum_x \epsilon \rho(\epsilon x, v) \left(\int \lambda(0) p_{\lambda,0}(v, w) \, \mu(d\lambda) \right) \left(\mathbb{E}_{x,w}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) - \mathbb{E}_{x,v}^\epsilon f(\epsilon X_{\epsilon^{-1}t}, V_{\epsilon^{-1}t}) \right) + o(\epsilon) \text{ (B.5)}
\end{aligned}
$$

where in the first step we used translation invariance of the scattering rule and in the last step we used ergodicity of the distribution of scatterers. Combination of (B.3), (B.4) and (B.5) gives

$$
\begin{aligned}
\frac{d}{dt} & \left(Y_t^\epsilon(v, f, \rho) \right) \\
= & \; Y_t^\epsilon(v, v \frac{\partial f}{\partial x}, \rho) + \sum_w \bar{p}(v, w) \left(Y_t^\epsilon(w, f, \rho) - Y_t^\epsilon(v, f, \rho) \right) + o(\epsilon) \text{ (B.6)}
\end{aligned}
$$

which is exactly the weak form of the linear Boltzmann equation.

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