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Sheffer sequences, probability distributions and approximation operators

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Abstract

We present a new method to compute formulas for the action on monomials of a generalization of binomial approximation operators of Popoviciu type, or equivalently moments of associated discrete probability distributions with finite support. These quantities are necessary to check the assumptions of the Korovkin Theorem for approximation operators, or equivalently the Feller Theorem for convergence of the probability distributions. Our method unifies and simplifies computations of well-known special cases. It only requires a few basic facts from Umbral Calculus. We illustrate our method to well-known approximation operators and probability distributions, as well as to some recent q -generalizations of the Bernstein approximation operator introduced by Lewanowicz and Woźny, Lupas, and Phillips.

Key words: approximation operators of Popoviciu type, moments, Umbral Calculus, Sheffer sequences

1 Introduction

A generalization of binomial approximation operators of Popoviciu type was studied in [1]. Many well-known linear positive approximation operators belong to this class, like the Bernstein, Stancu and Cheney-Sharma operators.

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For every approximation operator of this type, there is a corresponding probability distribution with finite support which is a generalization of the distributions considered in [2, Section 3.3]. The actions on monomials of the operator are equivalent to the moments of the corresponding probability distribution. Moreover, if a sequence of approximation operators uniformly approximate continuous functions on $[0, 1]$, then the corresponding sequence of probability distributions obey a weak law of large numbers and vice-versa. The approximation property is usually checked using the Korovkin Theorem, which in this case has a probabilistic counterpart known as Feller's Theorem (see *e.g.*, [3, Section 5.2] for a detailed overview or [4,5] for gentle introductions). The conditions of the Korovkin and Feller Theorems are expressed in terms of the action of the approximation operators on monomials of degree not exceeding two and the first two moments of the corresponding probability distributions, respectively.

This paper contributes in two ways. We propose a new general way to compute the action on monomials of arbitrary order of all approximation operators in the class described above, or equivalently all moments of the corresponding probability distributions. Our approach yields general formulas from which we easily deduce the formulas for the action on monomials of degree not exceeding two obtained by Manole in [6] and Sablonnière in [7]. It is important to have more than one formula, because it depends on the case at hand which formula has a simpler form or more convenient to check the conditions of the Korovkin or Feller Theorems. Our method is based on the Umbral Calculus (see *e.g.*, [8,9]), but requires only knowledge of the basic definitions which we review in Section 2. Another advantage of our approach is that it is easily extended to operators based on polynomials of non-classical umbral calculi, *i.e.*, arising when using generalized differentiation operators of the type $D_c x^n = \frac{c_n}{c_{n-1}} x^{n-1}$. This includes the important case of q -Umbral Calculus, so that we are also able to treat q -generalizations of the Bernstein operators introduced by Lewanowicz and Woźny, Lupaş, and Phillips (see [10,11,12]).

The subject of our paper started with the seminal paper [13]. Umbral calculus methods for computing the action on monomials for this class of operators were introduced by Manole in [6]. Representations for the remainder terms and evaluation of the orders of approximation were obtained by Stancu and Occorsio in [14]. More references can be found in the survey papers [15,16] and the monograph [17].

This paper is organized as follows. In Section 2 we present the few basic facts from the Umbral Calculus that we need in this paper. The definitions of the approximation operators and the associated probability distributions are in Section 3, while Section 4 contains our main results for the standard Umbral Calculus. Section 5 contains explicit examples and calculations. The extension to non-classical umbral calculi and the q -Umbral Calculus in particular, is

discussed in Section 6. Conclusions can be found in Section 7.

2 Short introduction to Umbral calculus

In this section we give an introduction to the Umbral Calculus. Proofs of results without reference can be found in [9]. Note that the polynomials in the definitions below differ by a factor $n!$ from the definitions in [9].

A **polynomial sequence** is a sequence of polynomials in one variable with real coefficients such that the n th polynomial is of degree exactly n . Hence, every polynomial sequence is a basis for the vector space of polynomials. A polynomial sequence $(q_n)_{n \in \mathbb{N}}$ is a **sequence of convolution type** if the relations

$$q_n(x + y) = \sum_{k=0}^n q_k(x) q_{n-k}(y) \quad (1)$$

hold for all n , x , and y . If $(q_n)_{n \in \mathbb{N}}$ is of convolution type, then the sequence $(n!q_n)_{n \in \mathbb{N}}$ is said to be of **binomial type** because it satisfies an analogue of the Binomial Theorem. Sequences of convolution type are also characterized by the following formal generating function:

$$\sum_{n=0}^{\infty} q_n(x) z^n = e^{xg(z)}, \quad (2)$$

where g is a formal power series with $g(0) = 0$. A polynomial sequence $(s_n)_{n \in \mathbb{N}}$ is a **Sheffer sequence** if there exists a sequence $(q_n)_{n \in \mathbb{N}}$ of convolution type such that the relations

$$s_n(x + y) = \sum_{k=0}^n q_k(x) s_{n-k}(y) \quad (3)$$

hold for all n , x , and y . Obviously, a sequence of convolution type is a Sheffer sequence. Sheffer sequences are also characterized by the following formal generating function:

$$\sum_{n=0}^{\infty} s_n(x) z^n = \frac{1}{s(g(z))} e^{xg(z)}, \quad (4)$$

where g and s are formal power series with $g(0) = 0$ and $g'(0) \neq 0$, and $s(0) \neq 0$, respectively. Section 3 contains explicit examples of sequences of convolution type and Sheffer sequences.

In order to obtain powerful expansions theorems, the Umbral Calculus uses special classes of linear operators. A linear operator T on the vector space of polynomials is said to be **shift-invariant** if $TE^a = E^aT$ for all a , where

the shift operators E^a are defined by $(E^a p)(x) = p(x + a)$. If moreover Tx is a non-zero constant, then T is said to be a **delta operator** (also called theta operator in numerical analysis). Each delta operator Q possesses a **basic sequence**, *i.e.*, the unique sequence $(q_n)_{n \in \mathbb{N}}$ such that $q_0 = 1$, $q_n(0) = 0$ and $Qq_n = q_{n-1}$ for $n \geq 1$. It can be shown that each basic sequence is of convolution type. Conversely, each sequence of convolution type is the basic sequence of a delta operator. A sequence $(s_n)_{n \in \mathbb{N}}$ is a Sheffer sequence if and only if there exists a delta operator Q such that $Qs_n = s_{n-1}$ for $n \geq 1$. This delta operator Q is unique. The linear operator S defined by $Ss_n = q_n$, where $(q_n)_{n \in \mathbb{N}}$ is the basic sequence of Q , can be shown to be invertible and shift-invariant (and so is its inverse). Every delta operator has infinitely many Sheffer sequences, but a pair (Q, S) uniquely defines a Sheffer sequence. Sheffer sequences for $Q = D$ are called **Appell sequences**. The operator S is called the **invertible operator** of $(s_n)_{n \in \mathbb{N}}$. These operators are useful because they allow the following expansions for arbitrary polynomials p and shift-invariant operators T :

$$p = \sum_{k=0}^{\infty} [SQ^k p]_{x=0} s_k \quad (5)$$

$$T = \sum_{k=0}^{\infty} [T s_k]_{x=0} SQ^k \quad (6)$$

Taylor expansion of polynomials is a special case of (5) where $s_k(x) = x^k/k!$ and thus $Q = D$ and $S = I$. Moreover, (6) with the same choice yields that each shift-invariant operator can be expanded into a power series in D . Note that there are no convergence problems, since all infinite sums reduce to finite sums when applied to a polynomial. In fact, the **Isomorphism Theorem** of Umbral Calculus states that formal power series identities remain true when we substitute shift-invariant operators for the formal variable.

The **Pincherle derivative** T' of an arbitrary operator T acting on polynomials is defined by $T' := T\underline{x} - \underline{x}T$, where \underline{x} is the multiplication by x operator. If T is shift-invariant, then by (6) there exists a formal power series t such that $T = t(D)$. It can be shown that in this case $T' = t'(D)$. Hence, the Pincherle derivative of a shift-invariant operator is also a shift-invariant operator and the Pincherle derivative of a delta operator is an invertible shift-invariant operator. Moreover, the Pincherle derivative of a shift-invariant operator obeys the usual rules of differentiation. The Pincherle derivative enables us to recursively compute a basic sequence of a delta operator (called Rodrigues Formula in [9]):

$$n q_n = (x(Q')^{-1}) q_{n-1}. \quad (7)$$

The Pincherle derivative also appears in our main results in the next section.

The setup of the Umbral Calculus has been generalized in various ways. We discuss a particular generalization of the Umbral Calculus in Section 6.

3 Sheffer sequences, probability distributions and approximation operators

In this section we define a class of probability distributions of finite support and associate them to positive linear approximation operators. The class of approximation operators that we obtain generalizes the binomial approximation operators introduced by Popoviciu in [13].

In the following α and β are real numbers and n is a non-negative integer. Let $(s_n)_{n \in \mathbb{N}}$ be a Sheffer sequence for a delta operator Q with basic sequence $(q_n)_{n \in \mathbb{N}}$ as defined in the previous section. We throughout assume that $s_n(\alpha + \beta) \neq 0$ and $q_k(\alpha) s_{n-k}(\beta) / s_n(\alpha + \beta) \geq 0$. We now define a probability distribution $P_{n,\alpha,\beta}^{Q,S}$ on $\{0, 1, \dots, n\}$ by

$$P_{n,\alpha,\beta}^{Q,S}\{k\} = \frac{q_k(\alpha) s_{n-k}(\beta)}{s_n(\alpha + \beta)}. \quad (8)$$

The special case $S = I$ or equivalently, $s_k = q_k$ for $k \in \mathbb{N}$ was studied in [2]. Several well-known probability distributions with finite support can be written in this form as we will see in the next section. Note that in general we obtain a different probability distribution if we interchange k and $n - k$ in the numerator of (8). This form is briefly discussed at the end of Section 4.

The same convolution structure of polynomials lies at the heart of the following linear approximation operator studied in [1]:

$$(L_n^{Q,S} f)(x) = \frac{1}{s_n(1)} \sum_{k=0}^n q_k(x) s_{n-k}(1-x) f\left(\frac{k}{n}\right). \quad (9)$$

for all $f \in C[0, 1]$ and $x \in [0, 1]$. It is an important special case of the operators introduced in [18]. We discuss a variant of (9) in which x and $1 - x$ are interchanged at the right-hand in the last part of Section 4. Several well-known positive linear approximation operators can be written in the form (9) as we will see in the next section. If $S = I$, then $s_k = q_k$ for all $k \in \mathbb{N}$ and (9) reduces to the binomial operator introduced by Popoviciu in [13]:

$$(L_n^{Q,I} f)(x) = \frac{1}{q_n(1)} \sum_{k=0}^n q_k(x) q_{n-k}(1-x) f\left(\frac{k}{n}\right). \quad (10)$$

The operators defined by (10) were called binomial operators of Tiberiu Popoviciu type in [14], while the name Bernstein-Sheffer operators was used in [7]. We think the latter name is confusing; the name Sheffer approximation operator should be reserved for the operators defined by (9). A sufficient condition for positivity of the Sheffer-type operators $L_n^{Q,S}$ is $q_k'(0) \geq 0$ and $s_k(0) \geq 0$ for all $k \geq 0$ (see [1]), which reduces for binomial type operators to the condition $q_k'(0) \geq 0$ from [13].

The correspondence between the probability distributions and the approximation operators can now be described as follows. If $X^{\alpha,\beta}$ is a random variable with distribution $P_{n,\alpha,\beta}^{Q,S}$, then $(L_n^{Q,S}f)(x) = Ef\left(\frac{X^{x,1-x}}{n}\right)$, where E denotes expectation. In this paper we show how to compute the (factorial) moments of $P_{n,\alpha,\beta}^{Q,S}$ for the general choice of α and β . This allows us to include generalizations like the choices $\alpha = x/a$ and $\beta = (1-x)/a$, even for choices like $a = 1/n$ (see *e.g.*, [15,19,20]).

The operators (10) preserve the polynomials of degree up to 1, while the operators (9) only preserve the constants. Hence, the Korovkin convergence criteria for these operators reduce to $\lim_{n \rightarrow \infty} \|L_n^{Q,I}e_2 - e_2\|_\infty = 0$ for (10), and to $\lim_{n \rightarrow \infty} \|L_n^{Q,I}e_i - e_i\|_\infty = 0$ ($i = 1, 2$) for (9), where $\|\cdot\|_\infty$ is the supremum norm on $[0, 1]$. For the corresponding probability distributions, this corresponds to $\lim_{n \rightarrow \infty} E\left(\frac{(X^{x,1-x})^i}{n}\right) = x^i$ uniformly for $x \in [0, 1]$ (the condition for $i = 2$ may be rewritten as $\lim_{n \rightarrow \infty} \text{Var}\left(\frac{X^{x,1-x}}{n}\right) = 0$ if the condition for $i = 1$ is met). These conditions ensure that $\lim_{n \rightarrow \infty} \|L_n^{Q,S}f - f\|_\infty = 0$ or equivalently, $\lim_{n \rightarrow \infty} \|Ef\left(\frac{X^{x,1-x}}{n}\right) - f\|_\infty = 0$ for $f \in C[0, 1]$. The last condition can be shown to be equivalent to convergence in probability (see *e.g.*, [3, Section 5.2], [4] and [5]). Since random variables distributed according to our distributions have finite support, they cannot be written as sums of iid random variables except for the case of the binomial distribution. Hence, convergence in probability cannot be inferred from the weak law of large numbers.

4 Main results for standard Umbral Calculus

In this section we compute the moments of the probability distributions defined by (8), or equivalently the action on test functions of the approximation operators defined in (9). Some extensions will be discussed in Section 6. Since we are dealing with probability distributions on \mathbb{N} , it is easier to compute factorial moments than ordinary moments. In other words, we use the basis $\{x^{[\ell,1]}\}_{\ell \in \mathbb{N}}$, where $x^{[\ell,a]} = x(x-a)\dots(x-(\ell-1)a)$ instead of the standard basis $\{1, x^1, x^2, \dots\}$.

Definition 1 *The ordinary moments for the probability distribution $P_{n,\alpha,\beta}^{Q,S}$ will be denoted by $\mu_\ell^{Q,S}$ and the factorial moments will be denoted by $\mu_{(\ell)}^{Q,S}$.*

The action on monomials of the operator $L_n^{Q,S}$ follows from the relation

$$L_n^{Q,S}e_m = \frac{1}{n^m} \sum_{j=0}^m S(m,j)\mu_{(j)}^{Q,S}, \quad (11)$$

where $S(m, j)$ are the Stirling numbers of the second kind. In order to check the assumptions of the Korovkin or Feller Theorems, it is thus convenient to use $L_n^{Q,S} e_2 = (\mu_{(1)}^{Q,S} + \mu_{(2)}^{Q,S}) / n^2$ and $\text{Var} \left(\frac{X}{n} \right) = (\mu_{(2)}^{Q,S} + \mu_{(1)}^{Q,S} (1 - \mu_{(1)}^{Q,S})) / n^2$ when X is distributed according to $P_{n,\alpha,\beta}^{Q,S}$.

We now define an operator that helps us computing the factorial moments of $P_{n,\alpha,\beta}^{Q,I}$, *i.e.*, with $s_n = q_n$. All other formulas can be obtained as corollaries from this case.

Definition 2 Let $(q_n)_{n \in \mathbb{N}}$ be the basic sequence of a delta operator Q . For every $\ell \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we define a linear shift-invariant operator V_ℓ on the vector space of polynomials by

$$V_\ell q_n = \sum_{k=0}^n k^{[\ell]} q_k(\alpha) q_{n-k}. \quad (12)$$

Using (1) and (12) and interchanging the order of summation, it is easy to see that the operator V_ℓ is shift-invariant. It follows directly from (8) and (10) that $\mu_{(\ell)}^{Q,I} = (V_\ell q_n)(\beta) / q_n(\alpha + \beta)$.

The following lemma serves two purposes: it both links the generating functions (2) and (4) with the operator expansion (6) for the corresponding delta operator and it presents convenient forms for the operators used in our formulas.

Lemma 3 Let Q be a delta operator with basic sequence $(q_n)_{n \in \mathbb{N}}$ and let g be as in (2). Then $Q = g^{(-1)}(D)$ (where $g^{(-1)}$ denotes the compositional inverse of g), $g'(Q) = (Q')^{-1}$ and $g''(Q) = ((Q')^{-1})' = -Q''(Q')^{-3}$.

PROOF. By (2) and (6), we have $D = g(Q)$. Hence, $Q = g^{(-1)}(D)$, and thus $g'(Q) = g'(g^{(-1)}(D)) = \frac{1}{(g^{(-1)}(D))'} = (Q')^{-1}$. For the last statement we first write f for the compositional inverse of g . Differentiating the identity $g(f(t)) = t$ twice, we obtain $g''(f(t))(f'(t))^2 + g'(f(t))f''(t) = 0$. But $g'(f(t)) = (f'(t))^{-1}$ and thus $g''(f(t)) = -f''(t)(f'(t))^{-3}$. Combining everything, we finally arrive at $g''(Q) = (g'(Q))' = ((Q')^{-1})' = -Q''(Q')^{-3}$. \square

The following theorem is our main result. It is stated in terms of factorial moments of $P_{n,\alpha,\beta}^{Q,I}$, but by applying (11) and replacing α with x and β with $1 - x$ we obtain the actions on the monomials of the operators (10). Formula (16) was already obtained in [6] by repeated use of the Rodrigues Formula (7), while (14) implies the formula in [7] obtained from power series manipulations.

Theorem 4 The ℓ -th factorial moment of $P_{n,\alpha,\beta}^{Q,I}$ is given by

$$\mu_{(\ell)}^{Q,I} = \frac{1}{q_n(\alpha + \beta)} Q^\ell \frac{d^\ell}{dQ^\ell} e^{\alpha g(Q)} q_n(\beta). \quad (13)$$

For $\ell = 1$, this reduces to $\mu_1^{Q,I} = n\alpha/(\alpha + \beta)$, while for $\ell = 2$, we obtain the following three equivalent formulas:

$$\mu_{(2)}^{Q,I} = \frac{\alpha}{q_n(\alpha + \beta)} \left[E^\alpha \{g''(Q) + \alpha (g'(Q))^2\} q_{n-2} \right] (\beta) \quad (14)$$

$$= \frac{\alpha}{q_n(\alpha + \beta)} \left[E^\alpha (Q')^{-2} (\alpha I - Q''(Q')^{-1}) q_{n-2} \right] (\beta) \quad (15)$$

$$= \frac{\alpha}{q_n(\alpha + \beta)} \left[\frac{n(n-1) q_n(\alpha + \beta)}{\alpha + \beta} - \beta ((Q')^{-2}) q_{n-2}(\alpha + \beta) \right]. \quad (16)$$

PROOF. Since V_ℓ is shift-invariant, it follows from the expansion formula (6) that

$$V_\ell = \sum_{k=0}^{\infty} (V_\ell q_k)(0) Q^k = \sum_{k=0}^{\infty} k^{[\ell]} q_k(\alpha) Q^k = Q^\ell \frac{d^\ell}{dt^\ell} \left(\sum_{k=0}^{\infty} q_k(\alpha) t^k \right) \Big|_{t=Q}.$$

Using (2) we rewrite the last expression as $Q^\ell \frac{d^\ell}{dQ^\ell} e^{\alpha g(Q)}$. Because $e^{\alpha g(Q)} = e^{\alpha g(g^{(-1)}(D))} = e^{\alpha D} = E^\alpha$, for $\ell = 1$ we have $V_1 = \alpha g'(Q) Q E^\alpha = \alpha (Q')^{-1} Q E^\alpha$. Combining this with the Rodrigues Formula (7), we obtain $\mu_{(1)}^{Q,I} = (V_1 q_n)(\beta)/q_n(\alpha + \beta) = n\alpha/(\alpha + \beta)$. A similar computation yields the first formula for $\ell = 2$. An application of Lemma 3 yields the second formula for $\ell = 2$. By either differentiating $e^{xg(t)}$ twice (cf. [7]) or applying the Rodrigues Formula (7) twice (cf. [6]), we obtain $n(n-1) q_n(x) = xg''(Q) q_{n-2}(x) + x^2 (Q')^{-2} q_{n-2}(x)$. Using this to eliminate $g''(Q) q_{n-2}(x)$ from the first formula for $\mu_{(2)}^{Q,I}$, we obtain the third formula for $\mu_{(2)}^{Q,I}$. \square

Remark 5 The proofs of Lemma 3 and the equivalences between the three formulas for $\mu_{(2)}^{Q,I}$ in Theorem 4 can be rephrased as follows. Let $(q_n)_{n \in \mathbb{N}}$ be the basic sequence for a delta operator Q . Then $(Q')^{-2} q_n$ is a Sheffer sequence for Q with generating function $(g'(t))^2 e^{xg(t)}$. If moreover Q'' is an invertible operator, then $g''(Q) q_n$ is a Sheffer sequence for Q with generating function $g''(t) e^{xg(t)}$. The generating function part of these statements were used in [7].

Remark 6 Let Q be a delta operator with basic sequence $(q_n)_{n \in \mathbb{N}}$. In [21], the following generalization of the operators studied in the present paper was studied:

$$(S_{m,r,s}^Q f)(x) = \sum_{k=0}^{m-sr} p_{m-sr,k}(x) \sum_{j=0}^s p_{s,j}(x) f\left(\frac{k+jr}{m}\right)$$

with $p_{n,k}(x) = \frac{q_k(x)q_{n-k}(1-x)}{q_n(1)}$. If $s = 0$ or $r = 0$ then $S_{m,r,0}^Q = S_{m,0,s}^Q = T_m^Q$ reduces to the binomial operators of Popoviciu type defined by (10). The action of this operator on a monomial is a linear combination of the action on the same monomial of the operators of binomial type and hence can be treated using methods for the binomial operators of Popoviciu type (see [21]). A further generalization of this class of operators by generalizing both the nodes and the evaluations of f can be found in [22].

We now compute the factorial moments of $P_{n,\alpha,\beta}^{Q,S}$ for the general case with $S \neq I$. As before, let $(s_n)_{n \in \mathbb{N}}$ be a Sheffer sequence for a delta operator Q with basic sequence $(q_n)_{n \in \mathbb{N}}$. We will show that the formulas for the general case follow easily from Theorem 4 by suitably applying the invertible shift-invariant operator S defined by $Ss_n = q_n$.

Theorem 7 *The ℓ -th factorial moment of $P_{n,\alpha,\beta}^{Q,S}$ is given by*

$$\mu_{(\ell)}^{Q,S} = \frac{1}{s_n(\alpha + \beta)} Q^\ell \frac{d^\ell}{dQ^\ell} e^{\alpha g(Q)} s_n(\beta). \quad (17)$$

For $\ell = 1$, this reduces to

$$\mu_1^{Q,S} = \frac{\alpha [E^\alpha (Q')^{-1} s_{n-1}](\beta)}{s_n(\alpha + \beta)} = \frac{\alpha [E^\alpha S^{-1} \left(\frac{nq_n(x)}{x} \right)]_{x=\beta}}{s_n(\alpha + \beta)} \quad (18)$$

For $\ell = 2$, we obtain the following two equivalent formulas:

$$\mu_{(2)}^{Q,S} = \frac{\alpha}{s_n(\alpha + \beta)} [E^\alpha \{g''(Q) + \alpha (g'(Q))^2\} s_{n-2}](\beta) \quad (19)$$

$$= \frac{\alpha}{s_n(\alpha + \beta)} [E^\alpha (Q')^{-2} (\alpha I - Q'' (Q')^{-1}) s_{n-2}](\beta). \quad (20)$$

PROOF. Fixing α and using that the operator V_ℓ from Definition 2 is shift-invariant, we obtain

$$\sum_{k=0}^n k^{[\ell]} q_k(\alpha) s_{n-k} = S^{-1} \sum_{k=0}^n k^{[\ell]} q_k(\alpha) q_{n-k} = S^{-1} V_\ell q_n = V_\ell S^{-1} q_n = V_\ell s_n. \quad (21)$$

Hence, it follows from Theorem 4 that

$$\mu_{(\ell)}^{Q,S} = \frac{(V_\ell s_n)(\beta)}{s_n(\alpha + \beta)} = \frac{1}{s_n(\alpha + \beta)} Q^\ell \frac{d^\ell}{dQ^\ell} e^{\alpha g(Q)} s_n(\beta).$$

An application of the Rodrigues Formula (7) yields the second formula for $\ell = 1$. The proof of the equivalence of the formulas for $\ell = 2$ is similar to the proof of Theorem 4. \square

An analogue of Formula (16) for the general case exists but due to its complexity may not be very convenient to apply (see [1]).

The following corollary of Theorem 7 shows that for Appell sequences (*i.e.*, Sheffer sequences for the delta operator D , the moments admit a simple presentation.

Corollary 8 *The ℓ -th factorial moment of $P_{n,\alpha,\beta}^{D,S}$ is given by $(V_\ell A_n)(\beta) = \left(D^\ell \frac{d^\ell}{dD^\ell} e^{\alpha D} A_n\right)(\beta)$, where $(A_n)_{n \in \mathbb{N}}$ is an Appell sequence. In particular, we have*

$$\mu_1^{D,S} = \frac{\alpha A_{n-1}(\alpha + \beta)}{A_n(\alpha + \beta)} \quad (22)$$

$$\mu^{(2)D,S} = \frac{\alpha^2 A_{n-2}(\alpha + \beta)}{A_n(\alpha + \beta)}. \quad (23)$$

We conclude this section by remarking that our method also applies without difficulties to the following slightly different form of $P_{n,\alpha,\beta}^{Q,S}$:

$$\tilde{P}_{n,\alpha,\beta}^{Q,S}\{k\} = \frac{s_k(\alpha) q_{n-k}(\beta)}{s_n(\alpha + \beta)}. \quad (24)$$

Similar computations as in this section yield that the factorial moments $\tilde{\mu}_{(\ell)}^{Q,S}$ of $\tilde{P}_{n,\alpha,\beta}^{Q,S}$ may be obtained through the formula

$$\tilde{\mu}_{(\ell)}^{Q,S} = \frac{1}{s_n(\alpha + \beta)} Q^\ell \frac{d^\ell}{dQ^\ell} \left(\frac{e^\alpha g(Q)}{s(g(Q))} \right) q_n(\beta). \quad (25)$$

The operator corresponding to $\tilde{P}_{n,\alpha,\beta}^{Q,S}$ is given by

$$\left(\tilde{L}_n^{Q,S} f\right)(x) = \frac{1}{s_n(1)} \sum_{k=0}^n s_k(x) q_{n-k} (1-x) f\left(\frac{k}{n}\right). \quad (26)$$

This is a linear approximation operator on $[0, 1]$, which is positive under the same conditions as for $L_n^{Q,S}$. However, it has the interpolation property $\left(\tilde{L}_n^{Q,S} f\right)(1) = f(1)$, as opposed to $\left(L_n^{Q,S} f\right)(0) = f(0)$.

5 Explicit computations of factorial moments

In this section we compute moments for several explicit examples of probability distributions of the form (8) (note that in some cases our polynomials may differ by a factor $k!$ from classical definitions) and mention the related approximation operator.

5.1 Convolution type case

The convolution type case corresponds to $S = I$.

- If $q_k(x) = \frac{x^k}{k!}$, then $Q = D$ and

$$P_{n,\alpha,\beta}^{D,I}\{k\} = \binom{n}{k} \left(\frac{\alpha}{\alpha+\beta}\right)^k \left(\frac{\beta}{\alpha+\beta}\right)^{n-k}.$$

Hence, $P_{n,\alpha,\beta}^{D,I}$ is the binomial distribution with parameters n and $\alpha/(\alpha+\beta)$. The operator $L_n^{D,I}$ is known as the Bernstein operator B_n , which was used by Bernstein in his famous probabilistic proof of the Weierstrass approximation theorem [23].

- The delta operator $Q = \frac{1}{a}\nabla_a = \frac{1}{a}(I - E^{-a})$ has the basic sequence $q_k(x) = x^{[k,-a]}/k!$ and therefore for $\alpha, \beta \in \mathbb{N}$ we have

$$P_{n,\alpha,\beta}^{\frac{1}{a}\nabla_a,I}\{k\} = \binom{n}{k} \frac{\alpha^{[k,-a]}\beta^{[n-k,-a]}}{(\alpha+\beta)^{[n,-a]}}.$$

Hence, $P_{n,\alpha,\beta}^{\frac{1}{a}\nabla_a,I}$ is the Markov-Pólya urn scheme distribution. If $a = -1$, then $q_k(x) = \binom{x}{k}$ with $Q = \Delta = E^1 - I$, and thus

$$P_{n,\alpha,\beta}^{\Delta,I}\{k\} = \frac{\binom{\alpha}{k} \binom{\beta}{n-k}}{\binom{\alpha+\beta}{n}}.$$

Hence, if α, β are positive integers such that $\alpha + \beta \geq n$, then $P_{n,\alpha,\beta}^{Q,I}$ is the hypergeometric distribution with parameters n , α , and β . Since $q_n(1) = 0$ for $n \geq 1$, there is no approximation operator of the form (9) corresponding to the hypergeometric distribution.

If $a = 1$, then $q_n(x) = \binom{x+n-1}{n}$ with $Q = \nabla = I - E^{-1}$ and we have

$$P_{n,\alpha,\beta}^{\nabla,I}\{k\} = \frac{\binom{\alpha+k-1}{k} \binom{\beta+n-k-1}{n-k}}{\binom{\alpha+\beta+n-1}{n}}.$$

This special case is known as the Pólya-Eggenberger distribution (see [24, Chapter 9, Section 4]). The operator corresponding to the general Markov-Pólya urn scheme distribution is given by

$$(L_n^{\frac{1}{a}\nabla_a,I} f)(x) = \frac{1}{1^{[n,-a]}} \sum_{k=0}^n \binom{n}{k} x^{[k,-a]} (1-x)^{[n-k,-a]} f\left(\frac{k}{n}\right)$$

It has been introduced and investigated in detail by Stancu in [25,26]. There is an extensive literature on these approximation operators (see *e.g.*, the survey [27] or the recent paper [28]). We have $g^{-1}(t) = (1 - e^{-at})/a$, so $g(t) = -\ln(1 - at)/a$. Then $g'(Q) = (I - aQ)^{-1} = (E^{-a})^{-1} = E^a$ and $g''(Q) = a(I - aQ)^{-2} = aE^{2a}$. Because $\frac{(E^{2a+\alpha}q_{n-2})^{(\beta)}}{q_n^{(\alpha+\beta)}} = \frac{n(n-1)}{(\alpha+\beta)(\alpha+\beta+a)}$ we obtain that

$$\mu_{(2)}^{\frac{1}{a}\nabla_a, I} = \frac{n(n-1)\alpha(\alpha+a)}{(\alpha+\beta)(\alpha+\beta+a)} \quad (27)$$

It is easy to check that the assumptions of the Korovkin and Feller theorems are satisfied if we let a depend on n and tend to zero as $n \rightarrow \infty$.

- If $q_k(x) = x(x + bk)^{k-1}/k!$ ($b > 0$) (Abel polynomials with delta operator $A = E^{-b}D$) and $\alpha, \beta > 0$, then $P_{n,\alpha,\beta}^{A,I}$ is the quasi-binomial distribution II (see [29,30]). In this case the corresponding approximation operator is the second operator introduced by Cheney and Sharma in [31]

$$(L_n^{A,I}f)(x) = \frac{1}{(1+nb)^{n-1}} \sum_{k=0}^n \binom{n}{k} x(x+kb)^{k-1} (1-x)(1-x+(n-k)b)^{n-k-1} f\left(\frac{k}{n}\right).$$

Since the Abel polynomials are a special case of the Gould polynomials, we refer to the Gould example for the details of the moments.

- The exponential polynomials $t_n(x) = \frac{1}{n!} \sum_{k=1}^n S(n, k)x^k = \frac{1}{n!} e^{-x} \sum_{k=0}^{\infty} \frac{k^n x^k}{k!}$, where $S(n, k)$ denote the Stirling numbers of the second kind, are basic polynomials for the delta operator $T = \ln(I + D)$. The corresponding approximation operator

$$(L_n^T f)(x) = \frac{1}{t_n(1)} \sum_{k=0}^n t_k(x) t_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

was studied in [32] (see also [7] and [15]). In this case $(T')^{-2} = (I + D)^2$. Hence, using the Rodrigues Formula (7) and (16) we obtain

$$\mu_{(2)}^{T,I} = \frac{\alpha(n-1)}{(\alpha+\beta)^2} \left[\alpha n + \beta \frac{t_{n-1}(\alpha+\beta)}{t_n(\alpha+\beta)} \right] \quad (28)$$

Using the asymptotics of Stirling numbers, it can be shown that convergence is guaranteed for $\alpha + \beta = 1$ (see [7,32]), while for $\alpha = nx$ and $\beta = n - nx$ convergence is proved in [19] using a simple estimate based on the Rodrigues Formula (7).

- The delta operator $Q = G = \frac{1}{a}E^{-b}\nabla_a = \frac{1}{a}(E^{-b} - E^{-a-b})$ (Gould operator) has basic polynomials $q_k(x) = x(x + a + kb)^{[k-1, -a]}/k!$. The corresponding probability distribution for the special case $a = -1$ is known as the quasi-hypergeometric distribution II or quasi-Pólya II distribution (see [33]), de-

pending on the signs of α , β and b :

$$P_{n,\alpha,\beta}^{G,I}\{k\} = \binom{n}{k} \frac{\alpha(\alpha + kb - 1)^{[k-1,1]} \beta(\beta + (n-k)b - 1)^{[n-k-1,1]}}{(\alpha + \beta)(\alpha + \beta + nb - 1)^{[n-1,1]}}.$$

The corresponding approximation operator for general a given by

$$(L_n^G f)(x) = \frac{\sum_{k=0}^n \binom{n}{k} x(x+a+kb)^{[k-1,-a]} (1-x)(1-x+a+(n-k)b)^{[n-k-1,-a]} f\left(\frac{k}{n}\right)}{(1+a+nb)^{[n,-a]}}$$

has been studied by Stancu and Occorsio in [14] with the nodes $\frac{k+\gamma}{n+\delta}$, $0 \leq \gamma \leq \delta$. We now use (16) to compute the second factorial moment. The Pincherle derivative of G is $G' = \frac{1}{a}(-bE^{-b} + (a+b)E^{-a-b}) = E^{-a-b} \left(I - \frac{b}{a} E^a \nabla_a \right)$. Because $(1-xt)^{-2} = \sum_{k=0}^{\infty} (k+1)x^k t^k$, the Isomorphism Theorem yields

$$(G')^{-2} = E^{2a+2b} \left(I - \frac{b}{a} E^a \nabla_a \right)^{-2} = E^{2a+2b} \sum_{k=0}^{\infty} (k+1) b^k E^{ak} \left(\frac{\nabla_a}{a} \right)^k.$$

Since $E^{-bk} \left(\frac{\nabla_a}{a} \right)^k q_n(x) = q_{n-k}(x)$ we have $\left(\frac{\nabla_a}{a} \right)^k q_{n-2}(x) = E^{bk} q_{n-k-2}(x)$. Hence,

$$\mu_{(2)}^Q = \alpha \left[\frac{n(n-1)}{\alpha + \beta} - \beta \frac{\sum_{k=0}^{n-2} b^k (k+1) q_{n-k-2}(\alpha + \beta + (a+b)(k+2))}{q_n(\alpha + \beta)} \right].$$

For $a = 0$ this reduces to the second factorial moment for the probability distribution corresponding to Abel sequence (quasi-binomial II distribution), while for $b = 0$ one obtains the second factorial moment of the Markov-Pólya distribution.

5.2 Appell case

If $Q = D$ and S is an invertible shift invariant operator then $q_k(x) = x^k/k!$ and $s_k(x) = A_k(x) = S^{-1}x^k/k!$ is an Appell sequence. The corresponding approximation operator of the form

$$(L_n^{D,S} f)(x) = \frac{1}{A_n(1)} \sum_{k=0}^n \frac{x^k}{k!} A_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

was introduced and investigated by Manole in [32], [34].

The corresponding Appell sequence for $S = (I + D)^{-1}$ is $A_k(x) = (x^k + kx^{k-1})/k!$ and the approximation operator is given by

$$(L_n^{D,(I+D)^{-1}} f)(x) = \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k-1} (n-k+1-x) f\left(\frac{k}{n}\right).$$

More details are in [32]. The corresponding probability distribution seems not to be studied. The first and second factorial moments can be obtained directly from (22) and (23) and are given by

$$\begin{aligned}\mu_1^{D,S} &= \frac{n\alpha(\alpha + \beta + n - 1)}{(\alpha + \beta)(\alpha + \beta + n)} \\ \mu_{(2)}^{D,S} &= \frac{n(n-1)\alpha^2(\alpha + \beta + n - 2)}{(\alpha + \beta)^2(\alpha + \beta + n)}.\end{aligned}$$

5.3 Sheffer case

- If we take $Q = A = E^{-b}D$ and $S = E^bQ' = I - bD$ then $q_k(x) = x(x + ba)^{k-1}/k!$ is the basic sequence for Q and $s_k(x) = (x + kb)^k/k!$ a Sheffer sequence for Q . In this case $P_{n,\alpha,\beta}^{A,S}$ is the quasi-binomial I distribution (see [30]) and the corresponding operator is the first operator introduced by Cheney and Sharma in [31]:

$$(L_n^{A,I^{-b}D}f)(x) = \frac{1}{(1+nb)^n} \sum_{k=0}^n \binom{n}{k} x(x+kb)^{k-1} (1-x+(n-k)b)^{n-k} f\left(\frac{k}{n}\right)$$

- For the Gould delta operator $Q = \frac{1}{a}E^{-b}\nabla_a = \frac{1}{a}(E^{-b} - E^{-a-b})$ and $S = E^{a+b}Q' = \frac{1}{a}((a+b)I - bE^a)$ we have $q_k(x) = x(x+a+kb)^{[k-1,-a]}/k!$ and $s_k(x) = (x+kb)^{[k,-a]}/k!$. The corresponding probability distribution for the special case $a = -1$ is known as the quasi-hypergeometric distribution I or quasi-Pólya I distribution (see [35]), depending on the signs of α , β and b :

$$P_{n,\alpha,\beta}^{Q,S}\{k\} = \binom{n}{k} \frac{\alpha(\alpha+kb-1)^{[k-1,1]}(\beta+(n-k)b)^{[n-k,1]}}{(\alpha+\beta+nb)^{[n,1]}}.$$

The corresponding approximation operator for general a is given by

$$(L_n^{[a,b]}f)(x) = \frac{\sum_{k=0}^n \binom{n}{k} x(x+a+kb)^{[k-1,-a]} (1-x+(n-k)b)^{[n-k,-a]} f\left(\frac{k}{n}\right)}{(1+nb)^{[n,-a]}}$$

If we replace x with $s(x)$ we obtain an operator which has been studied by Moldovan in [36,37]. The first two moments were obtained after a long computation in [36] without using Umbral Calculus. We now present a computation using our methods, which improves upon similar computations in [1] and [38] both with respect to ease of computation and form of the final result. The first moment equals

$$\mu_1^{G,S} \frac{\alpha \left[E^\alpha (G')^{-1} s_{n-1} \right] (\beta)}{s_n(\alpha + \beta)} = \frac{\alpha \sum_{k=0}^{n-1} b^k s_{n-1-k} (\alpha + \beta + (a+b)(k+1))}{s_n(\alpha + \beta)}.$$

For computing the second factorial moment, we need the second Pincherle derivative of G . Using the expression for G' which has already been computed for the Gould polynomials, we obtain

$$G'' = \frac{1}{a}(b^2 E^{-b} - (a+b)^2 E^{-a-b}) = b^2 Q - (a+2b) E^{-a-b}.$$

We have $(G')^{-3} = E^{3a+3b} \left(I - \frac{b}{a} E^a \nabla_a \right)^{-3}$ but we need a form involving positive powers. Using the identity $(1 - xt)^{-3} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} x^k t^k$ and applying the Isomorphism Theorem we obtain

$$(G')^{-3} = \frac{1}{2} E^{3a+3b} \sum_{k=0}^{\infty} (k+1)(k+2) b^k E^{ak} \left(\frac{\nabla_a}{a} \right)^k.$$

We now apply (20) and obtain

$$\begin{aligned} \mu_{(2)}^{G,S} &= \frac{\alpha E^\alpha}{s_n(\alpha + \beta)} \left[\alpha E^{2(a+b)} \sum_{k=0}^{n-2} b^k (k+1) E^{k(a+b)} Q^k s_{n-2}(x) \right. \\ &\quad - \frac{1}{2} E^{3(a+b)b^2} \sum_{k=0}^{n-3} b^k (k+1)(k+2) E^{k(a+b)} Q^{k+1} s_{n-2}(x) \\ &\quad \left. + \frac{(a+2b)}{2} E^{2(a+b)} \sum_{k=0}^{n-2} b^k (k+1)(k+2) E^{k(a+b)} Q^k s_{n-2}(x) \right]_{x=\beta}. \end{aligned}$$

Combining the first and the third terms and changing the summation index in the second term, we obtain

$$\begin{aligned} \mu_{(2)}^{G,S} &= \frac{\alpha}{s_n(\alpha + \beta)} \left[\sum_{k=0}^{n-2} b^k (k+1) \left(\alpha + \frac{(k+2)(a+2b)}{2} \right) s_{n-k-2}(\alpha + \beta + (k+2)(a+b)) \right. \\ &\quad \left. - \frac{1}{2} \sum_{k=1}^{n-2} b^{k+1} k(k+1) s_{n-k-2}(\alpha + \beta + (k+2)(a+b)) \right]. \end{aligned}$$

Finally we arrive at

$$\begin{aligned} \mu_{(2)}^{G,S} &= \frac{\alpha(\alpha + 2b + a) s_{n-2}(\alpha + \beta + 2(a+b))}{s_n(\alpha + \beta)} \\ &\quad + \frac{\alpha}{s_n(\alpha + \beta)} \sum_{k=1}^{n-2} b^k \frac{(k+1)}{2} [2\alpha + (k+2)a + (k+4)b] s_{n-k-2}(\alpha + \beta + (k+2)(a+b)). \end{aligned}$$

If we take $a = 0$ in the above formula we obtain the second factorial moment for the quasi-binomial I distribution.

6 Generalized umbral calculi

In this section we show that our method can be extended to a larger class of polynomials by considering generalizations of the standard Umbral Calculus. In particular, we are able to treat some q -generalizations of the Bernstein operator.

The proof of the general formula (13) basically requires three facts: the fact that Q is a lowering linear operator, *i.e.*, $Qq_n = q_{n-1}$, and formal generating function (2) and $D^\ell x^n = n^{[\ell]} x^{n-\ell}$. In fact, the proof also works for expressions of the form $\sum_{k=0}^n k^{[\ell]} r_k(\alpha) q_{n-k}$ provided that there is a closed form for the formal generating function $\sum_{k=0}^{\infty} r_k(\alpha) t^k$. In order to have associated approximation operators and probability distributions, it is only necessary to have a proper normalization quantity, *i.e.* a closed form for $\sum_{k=0}^n r_k(\alpha) q_{n-k}(\beta)$. The Sheffer identity (3) is an example, but we will see later in this section that there are other interesting examples.

If we consider all operators commuting with a given lowering operator Q , then we obtain a new umbral calculus (see [39]) with analogues of the shift and differentiation operators, expansion theorems and Sheffer sequences. However, to obtain explicit expressions we now restrict ourselves to a class of generalized umbral calculi with a simple form of differentiation operator.

We briefly review here the most important definitions and results. Most of the results of Section 2 continue to hold with minor modifications (see [40, Chapter 6] for full details). For every sequence of nonzero constants $(c_n)_{n \in \mathbb{N}}$ the generalized derivative D_c is the linear operator defined by

$$D_c x^n = \frac{c_n}{c_{n-1}} x^{n-1}, \text{ for } n > 0 \text{ and } D_c x^0 = 0. \quad (29)$$

Without loss of generality we will always assume that $c_0 = 1$. The generalized shift operator is defined as $E_c^a = \Phi(aD_c)$, where $\Phi(t) = \sum_{n=0}^{\infty} \frac{t^n}{c_n}$. We have the useful property $D_c \Phi(at) = a \Phi(at)$. Note that in general $(E_c^a)^{-1} \neq E_c^{-a}$ and $E_c^a E_c^b \neq E_c^{a+b}$. However, the notions of basic sequence, sequence of convolution type, Sheffer sequence, shift-invariant operators and delta operators remain essentially the same by properly applying the operator E_c^y . *E.g.*, Sheffer sequences are defined by the following identity

$$E_c^y s_n(x) = \sum_{k=0}^n q_k(x) s_{n-k}(y). \quad (30)$$

The Binomial Theorem in this case is given by $s_k(x) = q_k(x) = x^k/c_k$. The

analogue of (4) becomes

$$\sum_{n=0}^{\infty} s_n(x) t^n = \frac{\Phi(xg(t))}{s(g(t))}, \quad (31)$$

where $s_n = s^{-1}(D_c)q_n$. The expansion formulas (5) and (6) remain true without any changes. In particular, linear operators that commute with the generalized shifts E_c^a can be expanded into power series in D_c .

It will be convenient to define $[k]_c := c_k/c_{k-1}$ for $k > 0$ and $[0]_c := 0$. Now we define the analogues of factorials, lower factorials and binomial coefficients by $[k]_c! := [1]_c [2]_c \dots [k]_c = c_k$, $[x]_c^{[\ell]} := [x]_c ([x]_c - [1]_c) \dots ([x]_c - [\ell - 1]_c)$, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_c := \frac{[n]_c!}{[k]_c! [n-k]_c!} = \frac{c_n}{c_k c_{n-k}}.$$

The standard Umbral Calculus is included in this setup as the special case $c_n = n!$, for which $[k]_c = k$ and the generalized factorials and binomial coefficients reduce to the usual ones. Well-known other choices include $c_n = 1$ (the divided differences Umbral Calculus), $c_n = (2n)!$ (hyperbolic Umbral Calculus, see [41]) and $c_n = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n}$ (a q -Umbral Calculus, see Section 6.1).

From now we will assume that $0 < [k]_c < [k+1]_c$ for all $k > 0$, which holds for most common choices of $(c_n)_{n \in \mathbb{N}}$. Let $(s_n)_{n \in \mathbb{N}}$ be a Sheffer sequence for a delta operator Q with basic sequence $(q_n)_{n \in \mathbb{N}}$ in a generalized Umbral Calculus. We throughout assume that $E_c^\alpha s_n(\beta) \neq 0$ and $(q_k(\alpha) s_{n-k}(\beta))/E_c^\alpha s_n(\beta) \geq 0$. We then define a probability distribution on the set $\{[0]_c, [1]_c, \dots, [n]_c\}$ by

$$P \{X_{n,\alpha,\beta}^{Q,S} = [k]_c\} = \frac{q_k(\alpha) s_{n-k}(\beta)}{E_c^\alpha s_n(\beta)}, \quad (32)$$

where $Ss_n = q_n$. We define the corresponding approximation operator by

$$(L_{n,c}^{Q,S} f)(x) = \sum_{k=0}^n \frac{q_k(u(x)) s_{n-k}(v(x))}{E_c^{u(x)} s_n(v(x))} f\left(\frac{[k]_c}{[n]_c}\right), \quad (33)$$

where u and v are real-valued functions on $[0, 1]$. The use of u and v allows us to include some recently introduced q -generalizations of the Bernstein operator. For $c_n = n!$, $s_n = p_n$, $u(x) = x$ and $v(x) = 1 - x$ the operator defined above reduces to (10). It follows from (30) that $L_{n,c}^{Q,S} e_0 = e_0$. As in Section 4, we have

$$\sum_{k=\ell}^{\infty} \frac{c_k}{c_{k-\ell}} r_k(\alpha) Q^k = Q^\ell D_c^\ell \left(\sum_{k=0}^{\infty} r_k(\alpha) t^k \right) \Big|_{t=Q} = Q^\ell D_c^\ell (\Psi(\alpha, t)) \Big|_{t=Q}$$

for some formal power series Ψ and thus

$$\sum_{k=\ell}^n \frac{c_k}{c_{k-\ell}} r_k(\alpha) s_{n-k}(\beta) = Q^\ell D_c^\ell \Psi(\alpha, t) |_{t=Q} s_n(\beta) = D_c^\ell \Psi(\alpha, t) |_{t=Q} s_{n-\ell}(\beta). \quad (34)$$

Further specializations for small values of ℓ are not possible because in general the operators D_c do not obey the chain rule. For Appell sequences (*i.e.*, Sheffer sequences for $Q = D_c$) Formula (34) reduces to

$$\sum_{k=\ell}^n \frac{c_k}{c_{k-\ell}} \frac{q_k(\alpha) A_{n-k}(\beta)}{E_c^\alpha A_n(\beta)} = \alpha^\ell \frac{E_c^\alpha A_{n-\ell}(\beta)}{E_c^\alpha A_n(\beta)}. \quad (35)$$

Formula (34) allows us to compute $\sum_{k=0}^n \frac{c_k}{c_{k-\ell}} P \{X_{n,\alpha,\beta}^{Q,S} = [k]_c\}$, but in general this cannot be expressed in terms of the generalized factorial moment $\sum_{k=0}^n [k]_c^{[\ell]} P \{X_{n,\alpha,\beta}^{Q,S} = [k]_c\}$. An important case in which this is possible is the q -Umbral Calculus of Subsection 6.1. The ordinary moments of $X_{n,\alpha,\beta}^{Q,S}$ can always be expressed in terms of the generalized factorial moments by a generalization of the Stirling of the second kind, and consequently we also have expressions for the action of $L_{n,c}^{Q,S}$ on monomials. Another case where computations are feasible is the hyperbolic Umbral Calculus, where we can directly compute ordinary moments.

6.1 q -Umbral Calculus

If we take $c_n = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n}$, then we obtain a q -Umbral Calculus (cf. [40,42]). In particular, we have $[n]_q = \frac{c_n}{c_{n-1}} = 1 + q + q^2 + \dots + q^{n-1}$, $[n]_q! = \prod_{j=1}^n [j]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$. The action of the q -derivative on a polynomial is given by $D_q p(x) = \frac{p(x) - p(qx)}{x - qx}$. We mention that the q -derivative satisfies a q -Leibniz formula

$$D_q^n (f(t)g(t)) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-k(n-k)} D_q^k f(t) D_q^{n-k} g(q^k t). \quad (36)$$

Define $h_0 = 1$ and $h_\ell(x) = \frac{x(x-[1]_c)\dots(x-[l-1]_c)}{q^{\frac{\ell(\ell-1)}{2}}}$ for $\ell > 0$. It is easy to see that $h_\ell([k]_q) = 0$ for $0 \leq k < \ell$ and $h_\ell([k]_q) = c_k/c_{k-\ell}$ for $k \geq \ell \geq 0$. Hence, (34) is equivalent to generalized factorial moments (*i.e.*, for $\nu_{(n)}^{Q,S} := E h_\ell(X_{n,\alpha,\beta}^{Q,S})$). These factorial moments are directly related to the ordinary moments, since the monomials x^i and the q -factorials are linked through q -Stirling numbers

of the second kind $S_q(i, j)$,

$$x^j = \sum_{i=0}^j S_q(i, j) h_i(x) q^{\frac{i(i-1)}{2}}. \quad (37)$$

The numbers $S_q(i, j)$ satisfy the relations

$$S_q(i+1, j) = S_q(i, j-1) + [j]_q S_q(i, j); \quad S_q(i, i) = 1.$$

$$S_q(i, j) = \frac{1}{[j]!_q} \sum_{r=0}^j (-1)^r q^{\frac{r(r-1)}{2}} \binom{j}{r}_q [j-r]^i.$$

6.1.1 Lupaş operator

Lupaş introduced in [11] the following q -analogue of the Bernstein operator:

$$(B_n^q f)(x) = \sum_{k=0}^n \binom{n}{k}_q \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{\prod_{k=1}^n (1-x+q^{k-1})} f\left(\frac{[k]_q}{[n]_q}\right). \quad (38)$$

In the q -Umbral Calculus at hand, the sequence

$$A_n(x; a) = \frac{1}{c_n} (x-a)(x-qa) \dots (x-q^{n-1}a)$$

is the Appell sequence for the invertible operator E_q^a since $\frac{x^n}{c_n} = E_q^a A_n(x; a)$. The operator (38) may be written as

$$(B_n^q f)(x) = \sum_{k=0}^n \frac{A_k(0; -x) q_{n-k}(1-x)}{A_n(1-x; -x)} f\left(\frac{[k]_q}{[n]_q}\right),$$

where $q_n(x) = \frac{x^n}{c_n}$ is the basic sequence of D_c . Using (31), we see that the generating function for the sequence $(A_n)_{n \in \mathbb{N}}$ is given by

$$\sum_{n=0}^{\infty} A_n(x; a) t^n = \frac{\Phi(xt)}{\Phi(at)}. \quad (39)$$

Hence, $s(g(t)) = \Phi(at)$ for these polynomials. A similar calculation as for (34) yields

$$\sum_{k=\ell}^n \frac{c_k}{c_{k-\ell}} s_k(\alpha) Q^k q_n = Q^\ell \left(D_q^\ell \sum_{k=0}^{\infty} s_k(\alpha) t^k \right) \Big|_{t=Q} q_n = D_q^\ell \frac{\Phi(\alpha g(t))}{s(g(t))} \Big|_{t=Q} q_{n-\ell},$$

where D_q^ℓ is the ℓ -th q -derivative with respect to the variable t . Applying this to $s_k(\alpha) := A_n(\alpha; a)$, using (39) and substituting $\alpha = 0$, we obtain

$$\sum_{k=\ell}^n \frac{c_k}{c_{k-\ell}} A_k(0; a) q_{n-k} = D_q^\ell \frac{1}{\Phi(at)} \Big|_{t=Q} q_{n-\ell}.$$

Applying the definition of D_q with respect to the variable t , we obtain

$$D_q \left[\frac{1}{\Phi(at)} \right] = \frac{\frac{1}{\Phi(at)} - \frac{1}{\Phi(qat)}}{t(1-q)} = -\frac{D_q \Phi(at)}{\Phi(at)\Phi(qat)} = -\frac{a\Phi(at)}{\Phi(at)\Phi(qat)} = -\frac{a}{\Phi(qat)}$$

and $D_q^2 \frac{1}{\Phi(at)} = \frac{a^2 q}{\Phi(q^2 at)}$. So, $\sum_{k=0}^n [k]_q A_k(0; a) q_{n-k}(\beta) = -a(\Phi(qaD_q))^{-1} q_{n-1}(\beta) = -aA_{n-1}(\beta; aq)$ and $\sum_{k=0}^n [k]_q [k-1]_q A_k(0; a) q_{n-k}(\beta) = a^2 q (\Phi(q^2 a D_q))^{-1} q_{n-2}(\beta) = a^2 q A_{n-2}(\beta; aq^2)$. Replacing a with $-x$ and β with $1-x$, we obtain $(B_n^q e_1)(x) = \frac{x[n]_q}{[n]_q} = x$ and

$$(B_n^q e_2)(x) = x^2 \frac{q^2 [n-1]_q}{[n]_q (1-x+qx)} + \frac{x}{[n]_q} = \frac{x^2 q ([n]_q - 1)}{[n]_q (1-x+qx)} + \frac{x}{[n]_q}.$$

The conditions for the uniform convergence of $B_n^q f$ to f are the following: q must depend on n , $q = q(n)$, $0 < q(n) < 1$ and $\lim_{n \rightarrow \infty} q(n) = 1$. These quantities can also be put in terms of moments of probability distribution similar to (32).

6.1.2 Phillips' operator

The q -analogue of Bernstein operator introduced by Phillips ([43]) is given by

$$(\tilde{B}_n^q f)(x) = \sum_{k=0}^n \binom{n}{k}_q x^k \prod_{j=0}^{n-k-1} (1 - q^j x) f\left(\frac{[k]_q}{[n]_q}\right).$$

It can be rewritten in the following form

$$(\tilde{B}_n^q f)(x) = \sum_{k=0}^n \frac{q_k(x) A_{n-k}(1; x)}{E_c^x A_n(1; x)} f\left(\frac{[k]_q}{[n]_q}\right)$$

with q_n and A_n as defined in Section 6.1.1. Here $E_c^x A_n(1; x) = q_n(1)$. The values for the test functions for Phillips operator can be computed using the relations (35) and (37) as follows:

$$(\tilde{B}_n^q e_1)(x) = x \frac{E_c^x A_{n-1}(1; x)}{[n]_q q_n(1)} = x \frac{q_{n-1}(1)}{[n]_q q_n(1)} = x \frac{c_n}{c_{n-1} [n]_q} = x.$$

$$\begin{aligned}
(\tilde{B}_n^q e_2)(x) &= x^2 \frac{q E_c^x A_{n-2}(1; x)}{[n]_q^2 q_n(1)} + \frac{(L_n e_1)(x)}{[n]_q} \\
&= x^2 q \frac{q_{n-2}(1)}{[n]_q^2 q_n(1)} + \frac{x}{[n]_q} = x^2 + \frac{x(1-x)}{[n]_q}.
\end{aligned}$$

The operator B_n^q converges to the given function f under the same conditions as the Lupaş operator L_n^q studied in the previous subsection.

6.1.3 Lewanowicz and Woźny operator

Lewanowicz and Woźny recently introduced a variant of the Phillips operator (see [10]) which can be written in the following form

$$(B_n^{\omega, q} f)(x) = \frac{1}{A_n(1; \omega)} \sum_{k=0}^n A_k(x; \omega) A_{n-k}(1; x) f\left(\frac{[k]_q}{[n]_q}\right),$$

where q and ω are real parameters such that $q \neq 1$ and $\omega \neq 1, q^{-1}, \dots, q^{1-n}$, and $seqA$ is as in Subsection 6.1.1. We now set out to compute the first two generalized factorial moments for the probability distribution $P\{X_{n, \alpha, \beta} = [k]_q\} = \frac{A_k(\alpha; \omega) A_{n-k}(\beta; \alpha)}{A_n(\beta; \omega)}$. For $\beta = 1$ and $\alpha = x$ the moments of this probability distribution correspond to the values of the operator $B_n^{\omega, q}$ on the test functions. Let

D_q^ℓ act on the variable β . First note that

$$\begin{aligned}
\sum_{k=\ell}^n \frac{c_k}{c_{k-\ell}} A_k(\alpha; \omega) A_{n-k}(\beta; \alpha) &= D_q^\ell \left(\sum_{k=\ell}^{\infty} \frac{c_k}{c_{k-\ell}} A_k(\alpha; \omega) t^k \right) \Big|_{t=D_q} A_n(\beta; \alpha) \\
&= D_q^\ell \frac{\Phi(\alpha t)}{\Phi(\omega t)} \Big|_{t=D_q} A_{n-\ell}(\beta; \alpha).
\end{aligned}$$

For $\ell = 1$, it follows from (36) that $D_q \frac{\Phi(\alpha t)}{\Phi(\omega t)} \Big|_{t=D_q} = \left[-\frac{\omega \Phi(\alpha t)}{\Phi(\omega q t)} + \frac{\alpha \Phi(\alpha t)}{\Phi(\omega q t)} \right] \Big|_{t=D_q} = (\alpha - \omega) \left(E_q^{\omega q} \right)^{-1} E_q^\alpha$. So we obtain

$$\mu_1 = \frac{(\alpha - \omega) \left(E_q^{\omega q} \right)^{-1} E_q^\alpha A_{n-1}(\beta; \alpha)}{A_n(\beta; \omega)} = \frac{(\alpha - \omega) \left(E_q^{\omega q} \right)^{-1} q_{n-1}(\beta)}{A_n(\beta; \omega)} = [n]_q \frac{\alpha - \omega}{\beta - \omega},$$

where $q)n(x) = \frac{x^n}{c_n}$. Hence, $(B_n^{\omega, q} e_1)(x) = \frac{x-\omega}{1-\omega}$. Because

$$\begin{aligned}
D_q^2 \frac{\Phi(\alpha t)}{\Phi(\omega t)} \Big|_{t=D_q} &= D_q \left((\alpha - \omega) \frac{\Phi(\alpha t)}{\Phi(\omega q t)} \right) \Big|_{t=D_q} = (\alpha - \omega) (\alpha - \omega q) \frac{\Phi(\alpha t)}{\Phi(\omega q^2 t)} \Big|_{t=D_q} \\
&= (\alpha - \omega) (\alpha - \omega q) \left(E_q^{\omega q^2} \right)^{-1} E_q^\alpha,
\end{aligned}$$

we have

$$\begin{aligned}
\mu_{(2)} &= \frac{(\alpha - \omega)(\alpha - \omega q) \left(E_q^{\omega q}\right)^{-1} E_q^\alpha A_{n-2}(\beta; \alpha)}{A_n(\beta; \omega)} \\
&= \frac{(\alpha - \omega)(\alpha - \omega q) \left(E_q^{\omega q}\right)^{-1} q_{n-2}(\beta; \alpha)}{A_n(\beta; \omega)} \\
&= \frac{(\alpha - \omega)(\alpha - \omega q) A_{n-2}(\beta; \omega q^2)}{A_n(\beta; \omega)} = \frac{c_n}{c_{n-2}} \frac{(\alpha - \omega)(\alpha - \omega q)}{(\beta - \omega)(\beta - \omega q)}.
\end{aligned}$$

It follows that $(B_n^{\omega, q} e_2)(x) = \frac{(x-\omega)}{(1-\omega)(1-\omega q)} \left(x - \omega q + \frac{1-x}{[n]_q}\right)$.

6.2 Hyperbolic Umbral calculus

In [41] Di Bucchianico and Loeb considered a hyperbolic Umbral calculus with $c_n = (2n)!$. The hyperbolic derivative is defined by $D_c x^n = \frac{(2n)!}{(2n-2)!} x^{n-1} = 2n(2n-1)x^{n-1}$. In terms of ordinary derivative the hyperbolic derivative may be written as $D_c = 2(2xD^2 + D)$. In this case $\Phi(t) = \cosh(\sqrt{t})$ so the hyperbolic shift is given by $E_c^y = \cosh(\sqrt{yD_c}) = \sum_{n=0}^{\infty} \frac{y^n D_c^n}{(2n)!}$. If we take as a delta operator the hyperbolic backward difference operator $Q = \frac{1}{a}(I - \Phi(-aD_c)) = \frac{1}{a}(I - \cosh(i\sqrt{aD_c}))$, then the basic sequence is $q_n(x) = \frac{2^n}{(2n)!} \prod_{k=0}^{n-1} (x + ak^2)$. The corresponding approximation operator has the following form

$$(H_n f)(x) = \sum_{k=0}^n \binom{2n}{2k} \frac{\prod_{j=0}^{k-1} (u(x) + aj^2) \prod_{m=0}^{n-k-1} (v(x) + am^2)}{\cosh(\sqrt{v(x)D_c} \left(\prod_{j=0}^{n-1} (u(x) + aj^2)\right))} f\left(\frac{k(2k-1)}{n(2n-1)}\right).$$

This Umbral Calculus leads to new approximation operators and deserves further study.

7 Conclusions

We introduced a general framework for computing the action of approximation operators which extends the class of Popoviciu operators and the related moments of probability distribution operators. In fact there appear also some probability distributions which generalize quasi-binomial and quasi-hypergeometric distributions. Our method is general, but when applied to specific polynomial sequences, the use of a few basic facts from Umbral Calculus leads to quick proofs of known formulas. We also applied our method

to recently developed q -analogues of Bernstein operators. Finally, we showed that our framework leads to new classes of approximation operators that merit further study.

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