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Abstract. We present a stochastic process algebra including immediate actions, deadlock and termination, and explicit stochastic delays, in the setting of weak choice between immediate actions and passage of time. The operational semantics is a spent time semantics, avoiding explicit clocks. We discuss the embedding of weak-choice real-time process theories and analyze the behavior of parallel composition in the weak choice framework.
Keywords. Stochastic delay, weak choice, race condition, real-time and stochastic process algebra.

1 Introduction

Traditionally, *process algebras* (PAs) like ACP, CCS and CSP are used for qualitative description and verification of processes. In this setting, process behaviour is reflected by the order of actions. However, untimed description of processes is frequently not sufficiently expressive. (See, e.g., [1].) Thus, several timed extensions of traditional PAs emerged. (A detailed overview can be found in [2].) Also, probabilistic behavior of processes was included in PAs supporting probabilistic analysis. (Cf. [3], for example.) Combined efforts, like [4], considering timing aspects and probability, are reported as well.

Often, real-world processes require stochastic behaviour to be incorporated in their description. Early PAs doing so, employed exponentially distributed stochastic delays. Modeling with exponential distributions greatly simplifies the treatment of parallel composition, because of the memoryless property. Prominent Markovian PAs include EMPA, PEPA and Algebra of IMC [5–7]. The first two associate exponential rates with actions, whereas the latter clearly distinguishes between actions and rates.

Although much success has been reported, an abundance of processes cannot be dealt with exponentially. Consequently, several stochastic PAs with general distributions are proposed like SPADES, IGSMP and NMSPA [8–10]. SPADES introduces clocks to record the residual lifetime of stochastic delays. Each clock initialization is governed by a general distribution. Actions are only enabled after all clocks from a particular set have expired. Semantics for SPADES is given in terms of stochastic automata [11]. IGSMP uses clocks to record spent lifetimes. The clocks have an associated expiration time distribution. When a clock

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expires other clocks are redistributed according to the time that has passed. IGSMP semantics is given using generalized semi-Markov processes extended with actions. An interesting feature is the definition of the alternative composition modeled as a probabilistic choice between differently distributed clocks. NMSPA exploits random variables for the distribution of stochastic delays of actions. Also here, expiration of a stochastic delay induces redistribution of other variables according to the time that has passed. The semantics is given in terms of transition systems. The alternative composition is defined over an arbitrary number of summands in order to achieve maximal progress for internal actions. In NMSPA alternative composition of discrete stochastic delays followed by an internal action represents an inherent probabilistic choice. Other stochastic PAs that we mention here are the extension of LOTOS for performance analysis of distributed systems, the stochastic π-calculus and TIPP [12–14]. More details can be found in the overview papers [15, 16].

The main goal of our paper is to deal with standard real-time in stochastic PAs with an semantics that exploits spent-time and avoids explicit clocks. Our aims is to report on preliminary research on the conservative extension of real-time process algebra where delays are governed by probabilistic distributions. To this end, we consider a stochastic PA with immediate actions, deadlock and termination. We model stochastic delays as timed delays guided by discrete random variables, as we wish to distinguish between actions and stochastic delays, similar to IMC [7]. The alternative composition implements weak choice between immediate actions and passage of time similar to real-time PAs in the style of [1]. Here, we give the semantics in terms of stochastic transition systems. In comparison to other stochastic PA our approach is closest to NMSPA. Unlike NMSPA, we define alternative composition on two processes rather than on arbitrary sums and, in our setting, the alternative composition makes no choice in case both summands can delay together as in the real-time PAs. We propose an appropriate version of stochastic bisimulation for our setting, which is a congruence. α-conversion is introduced to pave the way for a treatment of the parallel operator. However, as we show, no expansion law is available in this set-up. We justify, via an embedding of transition systems, the proposed stochastic process algebra being called an extension of real-time process algebra. In our present work, we consider only discrete stochastic delays, mainly because they almost effortlessly model real-time delays as degenerated discrete random variables. Also, as a technical convenience, they allow two different delays to have the same duration, a property not shared by continuous distributions.

Related work Surprisingly, there is not much work on embedding real-time into stochastic time PAs. Markovian PAs cannot embed real-time because they employ exponential distributions only. The extension of LOTOS for performance analysis is an extension of timed LOTOS with stochastic timers, but there are strong syntax restrictions and no embedding is given. We remind the reader that the semantics of SPADES [8] is given in terms of stochastic automata [11]. A structural translation from stochastic automata to timed automata with deadlines is given in [17]. The translation is shown to preserve timed traces, so

SPADES can imitate real-time behaviour. There is a translation from IGSMP into pure real-time models termed Interactive Timed Automata (ITA) [9].

The rest of this paper is organized as follows. Section 2 gives the mathematical background for stochastic delays and the race condition. Section 3 introduces a basic stochastic process algebra with alternative composition and stochastic delay prefix. Section 4 provides the transition system and a notion of stochastic bisimulation, for which congruence properties are given. We define in Section 5 a variant of α-conversion to support the operational semantics. Sections 6 and 7 discuss the parallel operator and the embedding of real-time process theories. Section 8 wraps up with concluding remarks.

2 Preliminaries

We denote the set of discrete random variables by $\mathcal{V}$. For $S \subseteq \mathcal{V}$, $y \in \mathbb{R}$ and $\diamond$ either $<$, $>$, $=$, we write $S \diamond y$ for $X \diamond y$, $X \in S$. We use X, Y and Z for random variables and $F_X(t)$, $F_Y(t)$ and $F_Z(t)$, for $t \geq 0$, for their distribution functions, unless stated otherwise. For durations of a stochastic delay we have $F_X(t) = 0$ for $t < 0$ and we denote the set of such discrete distribution functions by $\mathcal{F}_d$. The support set of random variable X, denoted by $\text{supp}(X)$ contains the values for which $P(X = t) > 0$. By $\overline{F}_X(t)$ we denote the residual distribution function $1 - F_X(t)$. We extend the notion of support set to a set S of random variables by $\text{supp}(S) = \bigcap_{X \in S} \text{supp}(X)$.

A *stochastic delay* is a time delay which duration is guided by a random variable. It is discrete if the random variable is discrete. The notions of stochastic delay and random variable are used interchangeably depending on the context. We give an example of a discrete stochastic delay.

Example 1 (Stochastic delay). If the random variable is X distributed in the following fashion:

t	3.5	7
$P(X = t)$	0.4	0.6

then the stochastic delay, will be observed as having a duration of 3.5 with probability 0.4 and duration of 7 with probability 0.6.

We observe simultaneous passage of time for a number of stochastic delays until at least one of their duration passes. This phenomenon is referred to as the *race condition*. In general, simultaneous multiple stochastic delays can be observed as being the shortest; the shortest duration itself can be different and provided by different delays in different observations. Before we describe the race condition in terms of probability, we illustrate the race condition in the following example.

Example 2 (Race condition). Consider the random variables X and Y distributed in the following fashion:

t	2	5
$P(X = t)$	0.3	0.7

t	3	5	6
$P(Y = t)$	0.3	0.2	0.5

Let us observe the race between two stochastic delays named x and y and guided by X and Y, respectively. One comes to the following conclusions:

1. One observes x having a duration of 2 with probability 0.3 and x wins the race with the same probability since y cannot delay with duration less than 2.
2. Similarly, y has a duration of 3 with probability 0.3, but in order to win the race, x has to exhibit a duration of no less than 3 time units. Thus, the probability of 0.3 that y exhibits duration 3 has to be multiplied by the probability that x will do no less than 3 time units and that is 0.7. Consequently, y can win the race exhibiting a duration of 3 time units with probability $0.3 \cdot 0.7 = 0.21$.
3. Both x and y exhibit a delay of 5 time units. The race will be finished with duration of 5 time units with probability $0.7 \cdot 0.2 + 0.7 \cdot 0.5 = 0.14 + 0.35 = 0.49$, where the first summand gives the probability that both delays finish with duration 5 and the second summand gives the probability that x will win the race. The stochastic delay y cannot win the race alone with any duration greater or equal to 5, since x cannot do a delay which last longer than 5 time units.

The stochastic delay which is observed from this race is distributed as $\min(X, Y)$, i.e.

t	2	3	5
$P(\min(X, Y) = t)$	0.3	0.21	0.49

The probability that x wins the race is the probability that y delays longer for every duration exhibited by x. Thus, $P(\text{“}x \text{ is the winner”})$ can be calculated as

$$\begin{aligned} P(X < Y) &= \sum_{t \in \text{supp}(X)} P(X = t, Y > t) \\ &= P(X = 2, Y > 2) + P(X = 5, Y > 5) \\ &= 0.3 \cdot 1 + 0.7 \cdot 0.5 = 0.65 \end{aligned}$$

Similarly, $P(\text{“}y \text{ is the winner”})$ is obtained as

$$P(Y < X) = \sum_{t \in \text{supp}(Y)} P(Y = t, X > t) = 0.3 \cdot 0.7 + 0.2 \cdot 0 + 0.5 \cdot 0 = 0.21$$

The probability that both x and y finish at the same time is $P(X = Y) = 0.2 \cdot 0.7 = 0.14$.

Observing several stochastic delays we call a *race*. The stochastic delay or delays that have the shortest duration are called ‘winners’. The other ones are called ‘losers’ of the race.

In general, if one observes a race of a set of random variables $V \subseteq \mathcal{V}$, the resulting delay of the race will be distributed as the minimum $\min(V)$ of these random variables with a distribution function $F_{\min(V)}(t) = 1 - \prod_{X \in V} \overline{F}_X(t)$. The probability that the winners are in the set $W \subseteq V$ is

$$P(W = \min(V)) = \sum_{t \in \text{supp}(W)} P(W = t, (V \setminus W) > t).$$

The stochastic delay performed by the winners, is distributed as

$$P(\langle\, X \mid W = \min(V) \,\rangle = t) = \frac{P(W = t, (V \setminus W) > t)}{P(W = \min(V))},$$

for any $X \in W$. We use angle brackets to denote conditional random variables.

Because of associativity and commutativity of the minimum of random variables, it holds that simultaneous observation of all delays amounts to the same as iterated observation of disjoint sets.

Before we finish the section we give special case examples.

Example 3 (Stochastic delay with unique value 0*).* In case a stochastic delay is guided by a random variable X, such that $P(X = 0) = 1$ the result of the race will always be a delay with duration 0.

It might happen that the probability that some process wins the race is 1. Consider the following example.

Example 4 (Only one winner). Let $P(X = 1) = 1$ and $P(Y \leq 1) = 0$. We observe the race between two stochastic delays guided by X and Y, respectively. Obviously, the latter never wins the race, because the probability $P(X > Y) = 0$. Consequently, $\langle Y \mid Y = \min(X, Y)\rangle$ cannot exist as a random variable. Similarly, $\langle Y \mid X = Y\rangle$ is not a properly defined random variable.

In the following section we give a simple PA with discrete stochastic time.

3 Basic Processes with Discrete Stochastic Time

In this section we introduce $\text{BSP}^{\text{dst}}(\mathcal{A}, \mathcal{V})$, a stochastic PA with immediate actions, termination and deadlock, that implements weak choice between actions and time. We refer to BSP^{dst} as Basic Process Theory with Discrete Stochastic Time. The terminology is adopted from [18] and we build on the untimed version $\text{BSP}(\mathcal{A})$. Here, $\mathcal{A}$ is the set of actions and $\mathcal{V}$ is the set of random variables. A new unary operator scheme $\sigma_X.\_$ for $X \in \mathcal{V}$ represents stochastic delays.

3.1 Discrete Stochastic Delays

The process $\sigma_X.p$ executes a stochastic delay guided by the random variable X and continues behaving as p. Because of the race condition, one cannot observe the execution of a stochastic delay in isolation. Informally, an example of a transition system that corresponds to a race between two discrete stochastic delays is depicted in Fig. 1.

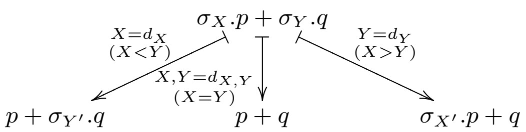

Fig. 1. Race condition

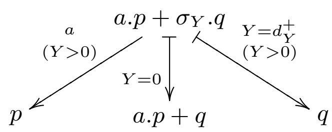

Fig. 2. Weak choice

The relations in the brackets give the condition that enables the transition. Each $\mapsto$ transition represents a stochastic delay. The label shows the winners of the race and their observed duration. The duration is determined by the support set of the winning delay. For clarity, we represent all the transitions by a single transition scheme. For example, the transitions of the stochastic delay guided by X in Fig. 1 are represented by one transition scheme labeled by X and d_X. The observed winning duration d_X takes its values from $\mathrm{supp}(\langle\, X \mid X < Y \,\rangle)$. Thus, the transition scheme replaces $|\,\mathrm{supp}(\langle\, X \mid X < Y \,\rangle)\,|$ different transitions, each executed with its own probability.

When considering the interaction of action transitions and termination versus stochastic delay, we employ weak choice, i.e. a non-deterministic choice between immediate actions, termination and passage of time. The alternative composition depicted in Fig. 2 allows execution of the stochastic delay in the rightmost transition even though the choice is made between an immediate action and passage of time. As a consequence, the losers of the race become dependent on the amount of time that has passed for the winners as in Fig. 1. Thus, the random variables of the remaining stochastic delays do not retain their initial distributions. Another issue we consider is the interaction between immediate actions and zero duration delays. Similar to the timed process theories [1, 2] we take zero duration not to disable immediate actions, as depicted by the middle transition in Fig. 2. Note that the immediate action is enabled only if $F_Y(0) \neq 1$. In order to distinguish between zero and non-zero transitions, we use the notation d_X^+ to denote only positive durations.

In an alternative composition of two stochastic delays, we obtain three transitions. In case the winner is the first summand, one obtains the leftmost transition. The rightmost transition is obtained when the winner is the second summand. The middle transition shows that both delays win the race together with non-zero probability. In this case, the race cannot determine one winner and passage of time does not determine a choice similar as for the real-time setting.

In Fig. 1, the altered probability distributions of X and Y are denoted by X' and Y', respectively. They are termed 'aged' probability distributions of X and Y by the duration d_Y and d_X, respectively. The probability distribution of X' is the aged probability distribution of X by d_Y given by

$$F_{X'}(t) = P(X \leq t \mid X > Y, Y = d_Y) = \frac{F_X(t + d_Y) - F_X(d_Y)}{1 - F_X(d_Y)}.$$

Note that $F_{X'}$ is defined if $P(X > d_Y) > 0$, which is ensured by the condition that $P(X < Y) > 0$. Similarly, for the distribution of Y', one obtains

$$F_{Y'}(t) = \frac{F_Y(t + d_X) - F_Y(d_X)}{1 - F_Y(d_X)}.$$

In order to calculate the actual distribution functions in each state, we require the original distribution function and its age. In order to keep track of the ages of the stochastic delays we introduce an environment to the transition system. The basic idea underlying the environments is that they store the actual distribution function of the random variables. The following definition and property of aging justify the use of environments.

Definition 5. *A distribution function F can be 'aged' by a time duration $d \geq 0$ if $F(d) < 1$. The resulting distribution $F|d$ is $(F|d)(t) = \frac{F(t+d)-F(d)}{1-F(d)}$.*

If the conditions of Definition 5 are fulfilled, then $F|d$ is again a distribution function. We have that iterative application of the aging function is the same as aging the function once by the sum of the time durations as stated by the following lemma.

Lemma 6. *If $(\ldots(F|d_1)\ldots)|d_n$ is defined for $d_1, \ldots d_n \in \mathbb{R}_0^+$, for $n \in \mathbb{N}$ then $(\ldots(F|d_1)\ldots)|d_n = F|\left(\sum_{i=1}^{n} d_i\right)$.*

Proof. By induction on the number of applications of $|$. The case when $n = 1$ is trivial. Assume that the proposition holds for $n = k$, $k \in \mathbb{N}$. We denote by S the sum $S = \sum\limits_{i=1}^{n} d_i$. We prove that the proposition holds for $k + 1$ applications of $|$. One obtains the following derivation:

$$\begin{aligned}
(\ldots(F|d_1)\ldots|d_k)|d &= (F|S)|d \\
&= \frac{(F|S)(t + d) - (F|S)(d)}{1 - (F|S)(d)} \\
&= \frac{\frac{F(t+S+d)-F(S)}{1-F(S)} - \frac{F(S+d)-F(S)}{1-F(S)}}{1 - \frac{F(S+d)-F(S)}{1-F(S)}} \\
&= \frac{F(t + (S + d)) - F(S + d)}{1 - F(S + d)} \\
&= F|(S + d).
\end{aligned}$$

Since the lemma holds for $n = k + 1$ the proof is complete. □

Using this property one easily calculates the age of the losers after each stochastic delay transition by adding the duration for the winners to the existing ages.

The environment is implemented using two injective functions: $\Phi\colon \mathcal{V} \to \mathcal{F}_d$ for the distribution functions and $\Delta\colon \mathcal{V} \to \mathbb{R}_0^+ \cup \{\perp\}$ for the age of the stochastic delays. We add the special symbol $\perp$ to denote that no time has passed for the stochastic delay, i.e. the delay has not participated in a race yet. Note that this is not the same as saying that the delay is of age zero. Having age zero means that the variable has lost a race with a zero duration and, ultimately, that disabled its possible zero duration transitions. Thus, we have to extend the domain of $|$ to $| \colon \mathcal{F}_d \times (\mathbb{R}^+_0 \cup \{\perp\}) \to \mathcal{F}_d$. We put $F|\perp = F$, $x + \perp = x$, for $x \in \mathbb{R}^+_0$, and we write $\mathbb{R}^+_\perp$ instead of $\mathbb{R}^+_0 \cup \{\perp\}$. We consider a well-defined environment to be a pair of two injective functions $(\Phi, \Delta) \in \mathcal{F}_d^{\mathcal{V}} \times {\mathbb{R}^+_\perp}^{\mathcal{V}}$ such that for all $X \in \mathcal{V}$ the probability distribution function $\Phi(X) \mid \Delta(X)$ is defined. The set of well-defined environments is denoted by Env. Next, we introduce the signature of BSP$^{\mathrm{dst}}$ and describe its constants and operators.

3.2 Signature of BSP$^{\mathrm{dst}}$

Definition 7. *The signature of* BSP$^{\mathrm{dst}}$ *contains the two constants* δ *and* ϵ*, the two unary operator schemes* $a.\_$*, for* $a \in A$ *and* $\sigma_X.\_$*, for* $X \in \mathcal{V}$ *and the binary operator* $\_ + \_$*. The syntax of* BSP$^{\mathrm{dst}}$ *is given by*

$$P ::= \delta \mid \epsilon \mid a.P \mid \sigma_X.P \mid P + P,$$

with $a \in A$ *and* $X \in \mathcal{V}$*. The set of closed terms over the signature of* BSP$^{\mathrm{dst}}$ *is denoted by* $\mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ *and it is ranged over by* p*,* q *and* r*.*

We adopt the signature from BSP($\mathcal{A}$) [18] where immediate constants and actions are denoted by $\tilde{\tilde{\delta}}$, $\tilde{\tilde{\epsilon}}$ and $\tilde{\tilde{a}}$. However, here, we do not use the $\approx$-notation. The constant δ represents an immediate deadlock which does not allow passage of time. Immediate termination ϵ terminates without allowing any time to pass. The unary operator scheme $a.p$, for $a \in A$, comprises processes that execute the action a without consuming any time and continue behaving as p. The unary operator scheme $\sigma_X.p$ provides processes that execute a stochastic delay guided by the random variable X and afterwards continue behaving as p. The alternative composition behaves differently depending on three different contexts. It makes a non-deterministic choice between actions, a weak choice between actions, successful termination and stochastic delays, and imposes a race condition on stochastic delays.

In the following subsection we provide operational semantics for BSP$^{\mathrm{dst}}$.

3.3 Structural Operational Semantics

First, we define a *stochastic transition system* (STS) that deals with aging of distributions as informally discussed in the example of Fig. 1. The transitions of the STS are performed in an environment that keeps track of the up-to-date distribution functions of the racing stochastic delays. It contains the distribution functions for the random variables and the age of the delays.

Definition 8. *STS is a structure* $STS = (\mathcal{S}, (\Phi, \Delta), \to, \mapsto, \downarrow)$ *where*

- $\mathcal{S}$ *is a set of states labeled by closed* BSP$^{\mathrm{dst}}$*-terms;*

– $(\Phi, \Delta) \in \mathrm{Env}$ *is a well-defined environment;*
– $\rightarrow \subseteq \mathcal{S} \times \mathrm{Env} \times \mathcal{A} \times \mathcal{S} \times \mathrm{Env}$ *is a labeled transition relation;*
– $\mapsto \subseteq \mathcal{S} \times \mathrm{Env} \times 2^{\mathcal{V}} \times \mathrm{I\!R}_0^+ \times \mathcal{S} \times \mathrm{Env}$ *is a stochastic delay (probabilistic) transition relation;*
– $\downarrow \subseteq \mathcal{S}$ *is an immediate termination predicate.*

For $\rightarrow$ and $\mapsto$ we will use infix notation. By $\langle p, (\Phi, \Delta)\rangle \xrightarrow{a} \langle p', (\Phi, \Delta')\rangle$ we denote that a process term p in the environment (Φ, Δ) does an action transition with the label a to the term p' and changes the environment to (Φ, Δ'). By $\langle p, (\Phi, \Delta)\rangle \overset{S}{\mapsto}_{d_S} \langle p', (\Phi, \Delta')\rangle$ we denote that a term p in the environment (Φ, Δ) exhibits a passage of time of duration d_S, transforms to p' and changes the environment to (Φ, Δ'). The observed time is a result of a race won by the set of stochastic delays that are guided by the set of random variables S. The possible durations of the winners are determined by $d_S \in \mathrm{supp}(\langle\ X \mid S = \min(\mathrm{rd}(p))\ \rangle)$, where $\mathrm{rd}(p)$ (we define this function later) is the set of racing delays of p and $X \in S$. In case there is a separation between zero duration and non-zero duration, we denote the non-zero durations by $d_S^+ > 0$. The random variables $X \in \mathcal{V}$ obtain their probability distributions as $F_X = \Phi(X) \mid \Delta(X)$. The race changes the age binding function Δ by setting age $\perp$ to every winning stochastic delay and increasing the ages of the losing delays by d_S.

Since all transitions only change the 'age parameter' Δ that assigns the ages, we suppress Φ and use the shorthand Δ for the environment (Φ, Δ). The STS represents a scheme because we leave implicit the conditions that enable the transitions and we parameterize multiple delay transitions by their support set. Also, we write X for $\{X\}$ and d_X for $d_{\{X\}}$ in the transition labels. We introduce the the set of all age parameters as $\mathrm{Del} = {\mathrm{I\!R}_\perp^+}^{\mathcal{V}}$. In case we wish to give a transition system for a specific term $p \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ we write $STS(p, (\Phi, \Delta_0))$, where (Φ, Δ_0) is the initial environment. We denote the set of STS's as $\mathcal{STS}$.

The sets of winning stochastic delays are given as labels of the probabilistic transitions. However, not all stochastic delays participate in a race at the same time. So, we have to identify only the racing stochastic delays, i.e. the ones that participate in the race. A function named $\mathrm{rd}\colon \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}) \rightarrow 2^{\mathcal{V}}$ extracts the random variables that guide the racing delays of a process term. They are identified as all stochastic delays that are directly connected by alternative composition.

$$\mathrm{rd}(\epsilon) = \emptyset \quad \mathrm{rd}(a.p) = \emptyset \quad \mathrm{rd}(\delta) = \emptyset \quad \mathrm{rd}(\sigma_X.p) = \{X\} \quad \mathrm{rd}(p + q) = \mathrm{rd}(p) \cup \mathrm{rd}(q)$$

In order to provide a concise presentation of the operational semantics, we define two functions res and age which alter the age parameter Δ of the environment. The function res resets the images of the winners to $\perp$, whereas age ages the losers by the duration observed for the winners.

Definition 9. *For an environment Δ, a set of winners $W \subseteq \mathcal{V}$ and a set of losers $L \subseteq \mathcal{V}$ of a race of duration d, the functions* $\mathrm{res}\colon \mathrm{Del} \times 2^{\mathcal{V}} \rightarrow \mathrm{Del}$ *and* age: Del $\times 2^{\mathcal{V}} \times \mathbb{R}_0^+ \to$ Del *are defined as*

$$\mathrm{res}(\Delta, W) = \begin{cases} \Delta(X) & \textit{if } X \notin W \\ \bot & \textit{if } X \in W \end{cases} \qquad \mathrm{age}(\Delta, L, d) = \begin{cases} \Delta(X) & \textit{if } X \notin L \\ \Delta(X) + d & \textit{if } X \in L. \end{cases}$$

Next, we give the structural operational semantics for $\mathrm{BSP}^{\mathrm{dst}}$.

The structural operation semantics for $\mathrm{BSP}^{\mathrm{dst}}$ are given in Table 1.

Table 1. Structural operational semantics for $\mathrm{BSP}^{\mathrm{dst}}$.

$$1\ \langle \epsilon, \Delta\rangle \downarrow \quad 2\ \frac{\langle p, \Delta\rangle \downarrow}{\langle p+q, \Delta\rangle \downarrow} \quad 3\ \frac{\langle q, \Delta\rangle \downarrow}{\langle p+q, \Delta\rangle \downarrow}$$

$$4\ \langle a.p, \Delta\rangle \xrightarrow{a} \langle p, \Delta\rangle \qquad 5\ \langle \sigma_X.p, \Delta\rangle \overset{X}{\mapsto}_{d_X} \langle p, \mathrm{res}(\Delta, \{X\})\rangle$$

$$6\ \frac{\langle p, \Delta\rangle \xrightarrow{a} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \not\mapsto}{\langle p+q, \Delta\rangle \xrightarrow{a} \langle p', \Delta'\rangle} \qquad 7\ \frac{\langle p, \Delta\rangle \xrightarrow{a} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T^+} \langle q', \Delta''\rangle}{\langle p+q, \Delta\rangle \xrightarrow{a} \langle p', \mathrm{res}(\Delta', \mathrm{rd}(q))\rangle}$$

$$8\ \frac{\langle p, \Delta\rangle \not\mapsto,\ \langle q, \Delta\rangle \xrightarrow{b} \langle q', \Delta'\rangle}{\langle p+q, \Delta\rangle \xrightarrow{b} \langle q', \Delta'\rangle} \qquad 9\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S^+} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \xrightarrow{b} \langle q', \Delta''\rangle}{\langle p+q, \Delta\rangle \xrightarrow{b} \langle q', \mathrm{res}(\Delta'', \mathrm{rd}(p))\rangle}$$

$$10\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_0 \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \not\mapsto}{\langle p+q, \Delta\rangle \overset{S}{\mapsto}_0 \langle p'+q, \Delta'\rangle} \qquad 11\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S^+} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \not\mapsto}{\langle p+q, \Delta\rangle \overset{S}{\mapsto}_{d_S^+} \langle p', \Delta'\rangle}$$

$$12\ \frac{\langle p, \Delta\rangle \not\mapsto,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_0 \langle q', \Delta'\rangle}{\langle p+q, \Delta\rangle \overset{T}{\mapsto}_0 \langle p+q', \Delta'\rangle} \qquad 13\ \frac{\langle p, \Delta\rangle \not\mapsto,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T^+} \langle q', \Delta'\rangle}{\langle p+q, \Delta\rangle \overset{T}{\mapsto}_{d_T^+} \langle q', \Delta'\rangle}$$

$$14\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle q', \Delta''\rangle,\ d_S < d_T}{\langle p+q, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p'+q, \Delta'''\rangle},$$
$$\text{where } \Delta''' = \mathrm{age}(\Delta', \mathrm{rd}(q), d_S)$$

$$15\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle q', \Delta''\rangle,\ d_S > d_T}{\langle p+q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle p+q', \Delta'''\rangle},$$
$$\text{where } \Delta''' = \mathrm{age}(\Delta'', \mathrm{rd}(p), d_T)$$

$$16\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle q', \Delta''\rangle,\ d_S = d_T}{\langle p+q, \Delta\rangle \overset{S\cup T}{\mapsto}_{d_{S\cup T}} \langle p'+q', \Delta'''\rangle},$$
$$\text{where } \Delta''' = \mathrm{res}(\mathrm{age}(\Delta, \mathrm{rd}(p+q), d_{S\cup T}), S\cup T)$$

Rules 1, 2, and 4 are the standard rules for termination and action prefix. Rule 5 states that stochastic delays $\sigma_X.p$ allow passage of time sampling from $\Phi(X)|\Delta(X)$. The non-deterministic choice made by action transitions from the first summand is shown by Rule 6 when the second summand cannot do a stochastic delay and by Rule 7 when it can do a stochastic delay with non-zero duration. Rule 10 states that zero delay of p does not enforce a choice, still allowing action transitions from q. In case p does perform a non-zero delay as in Rule 11 weak choice is enabled between action transitions and passage of time, where passage of time disables the action transitions of q. Rule 14 describes the race in case when the first summand wins the race. The winners given by the set S perform a stochastic delay transition with duration d_S. The racing delays of the losing summand ($\mathrm{rd}(q)$) are aged by d_S using the function age and the environment of the winner Δ' (in which the losers of the first summand are already aged). Note that since the second summand can perform a stochastic delay $d_T > d_S$, the aging of its racing delays is allowed. Rule 16 states that if both summands have stochastic delays that can win with the same duration, the joint race enabled by the alternative composition can be won by the union of the winners of the both summands. The new environment is obtained by aging all racing delays of both summands in the original environment and resetting the winners. The rules 3, 8, 9, 12, 13 and 15 are analogous to 2, 6, 7, 10, 11 and 14 for the second summand.

Next we define a notion of strong bisimulation for closed $\mathrm{BSP}^{\mathrm{dst}}$-terms.

3.4 Stochastic Bisimulation

Before we define a bisimulation relation, we discuss several simple examples to provide intuition what the bisimulation relation captures. We consider bisimulation on stochastic transition systems and process terms.

Example 10 (Bisimilar processes with stochastic time). The processes corresponding to the terms $\sigma_X.\epsilon$ and $\sigma_Y.\epsilon$ should be related, if the probability distribution functions of X and Y are the same. In addition, stochastic delays that lead to bisimilar states should be taken into account. For example, $\sigma_X.a.\epsilon+\sigma_Y.(a.\epsilon+a.\epsilon)$ and $\sigma_X.a.\epsilon + \sigma_Y.a.\epsilon$ should be related.

The example indicates that we should consider two terms to be bisimilar if their action parts are strongly bisimilar, and if they can do time delays with the same duration and probability to the same class of processes. However, in order to compare the stochastic transitions we have to compare the processes on the basis of their actual distributions. Since, we do not carry that information in the process theory, we first define strong stochastic bisimulation of transition systems and afterwards, based on this we define a strong stochastic bisimulation for $\mathrm{BSP}^{\mathrm{dst}}$-terms. The following example gives an example of two bisimilar and stochastically different transition systems.

Example 11 (Bisimilar stochastic transition systems). Next, we give a more elaborated example. Let $p \equiv \sigma_X.\epsilon + \sigma_Y.\epsilon + \sigma_Z.\epsilon$ and $q \equiv \sigma_U.\epsilon + \sigma_V.\epsilon$. Let the environment (Φ, Δ) contain the following distributions for X, Y, Z, U, V:

t	t_1	t_2
$P(X = t) = P(Y = t) = P(Z = t)$	$\frac{1}{3}$	$\frac{2}{3}$

t	t_1	t_2
$P(U = t)$	$\frac{10-\sqrt{73}}{27}$	$\frac{17+\sqrt{73}}{27}$

and

t	t_1	t_2
$P(V = t)$	$\frac{10+\sqrt{73}}{27}$	$\frac{17-\sqrt{73}}{27}$

and $\Delta_0(X) = \bot$, for all $X \in \mathcal{V}$.

Because of the all stochastic delays can have only two possible durations (t_1 or t_2), one concludes that there are only two possibilities for the winners: (1) to perform a stochastic delay of length t_1 or (2) to perform a stochastic delay of length t_2.

In case a time delay of duration t_1 is performed, there are two possibilities: (1.1) all stochastic delays win the race together or (1.2) there are some delays that remain to be executed. If (1.1) then probability that p executes a time delay of length t_1 by all stochastic delays and terminates is the same as for q. Note that $(10 + \sqrt{73}) \cdot (10 - \sqrt{73}) = 27$. Now, one obtains:

$$P(X = t_1, Y = t_1, Z = t_1) = \frac{1}{3^3} = \frac{1}{27} = P(U = t_1, V = t_1).$$

If (1.2) then the losers must execute together a stochastic delay with duration $t_2 - t_1$ and probability 1. There are only 6 different possibilities of executing the delays in such manner for p and 2 for q, so it is straightforward to verify that the probability that a time delay t_1 is executed and it remains to execute a time delay of length $t_2 - t_1$ is the same for both processes. Note that this probability can also be obtained as $1 - P(X = t_1, Y = t_1, Z = t_1) - P(X = t_2, Y = t_2, Z = t_2)$ for p and $1 - P(U = t_1, V = t_1) - P(U = t_2, V = t_2)$ for q.

In case a time delay of duration t_2 is chosen, both processes delay together with all active stochastic delays. This also happens with the same probability of executing a time delay of length t_2 because

$$P(X = t_2, Y = t_2, Z = t_2) = \left(\frac{2}{3}\right)^3 = \frac{8}{27} = P(U = t_1, V = t_1).$$

Thus, an observer has no way of distinguishing between the two processes and they should be considered bisimilar.

Next, we define when two STS's are bisimilar. Intuitively, two STS should be bisimilar if related states (1) do the same action transitions, (2) have the same termination options and (3) go to another class of states with the same accumulative probability of performing a stochastic delay with the same duration. The following definition defines the accumulative probability of (3).

Definition 12. *Let R be an equivalence relation on $\mathcal{S} \times \mathrm{Env}$, $C \in \big(\mathcal{S} \times \mathrm{Env}\big)/R$ an arbitrary class and $(\Phi, \Delta) \in \mathrm{Env}$, where $\mathcal{S} \subseteq \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$. By $\mathrm{ws}(p, \Delta, C, d)$ we define the set of sets of winning stochastic delays that p can do in time d and afterwards transform into a process that belongs to the class C, i.e.*

$$\mathrm{ws}(p, \Delta, C, d) = \textstyle\bigcup_{\langle p', \Delta' \rangle \in C} \{\, S \subseteq \mathrm{rd}(p) \mid \langle p, \Delta \rangle \overset{S}{\mapsto}_d \langle p', \Delta' \rangle \,\}.$$

The accumulative probability of doing a transition from a term to an equivalence class in time d is given as

$$\mathrm{ap}(p, \Delta, C, d) = \begin{cases} 0 & \mathrm{ws}(p, \Delta, C, d) = \emptyset \\ \sum_{S \in \mathrm{ws}(p,\Delta,C,d)} P(S = \min(\mathrm{rd}(p)), S = d) & \mathrm{ws}(p, \Delta, C, d) \neq \emptyset. \end{cases}$$

Next, we define strong bisimulation on STS's.

Definition 13. *A strong bisimulation on STS* $= (\mathcal{S}, (\Phi, \Delta), \to, \mapsto, \downarrow)$ *is an equivalence relation R on* $\mathcal{S} \times \mathrm{Env}$ *such that the following conditions hold:*

1. *if* $\langle p, \Delta\rangle \xrightarrow{a} \langle p', \Delta'\rangle$*, then* $\langle q, \Delta\rangle \xrightarrow{a} \langle q', \Delta'\rangle$*, such that* $(\langle p', \Delta'\rangle, \langle q', \Delta'\rangle) \in R$*,*
2. *for all* $d \geq 0$*, it holds that* $\mathrm{ap}(p, \Delta, C, d) = \mathrm{ap}(q, \Delta, C, d)$*,*
3. *if* $\langle p, \Delta\rangle \downarrow$ *then* $\langle q, \Delta\rangle \downarrow$*,*

for all $p, p', q, q' \in \mathcal{S}$ *and all* $C \in (\mathcal{S} \times \mathrm{Env})_{/R}$ *such that* $(\langle p, \Delta\rangle, \langle q, \Delta\rangle) \in R$*.*

Note that the second transfer condition implies that after doing a stochastic delay, both terms must result again in bisimilar terms. If $\langle p, \Delta\rangle$ and $\langle q, \Delta\rangle$ are related by a strong bisimulation we write $\langle p, \Delta\rangle \leftrightarrow \langle q, \Delta\rangle$. We also note, that if we consider the time duration as a constant in the transition system, we obtain the probabilistic bisimulation given in [19].

Now, we extend to strong bisimulation the term algebra of $\mathrm{BSP}^{\mathrm{dst}}$.

Definition 14. *The term algebra of* $\mathrm{BSP}^{\mathrm{dst}}$ *is*

$$\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}}) = (\mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}), \delta, \epsilon, a.\_ \text{ for } a \in \mathcal{A}, \sigma_X.\_ \text{ for } X \in \mathcal{V}, \_ + \_).$$

We abstract from the environment by considering two terms to be bisimilar if their transitions systems are bisimilar for any initial environment.

Definition 15. *Strong stochastic bisimulation for* $\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}})$ *is an equivalence relation R on* $\mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ *such that for all* $p, q \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ *it holds:*

$$pRq \Longrightarrow \langle p, \Delta\rangle \leftrightarrow \langle q, \Delta\rangle,$$

for all well-defined environments (Φ, Δ)*.*

We overload the relation $\leftrightarrow$ by $p \leftrightarrow q$, which denotes the fact that two closed $\mathrm{BSP}^{\mathrm{dst}}$-terms are strongly bisimilar.

The following lemma gives a important property of strongly bisimilar transition systems states. If two states are bisimilar, then they are bisimilar for any eligible aging of the original state. This property of the bisimilar states is used to prove that $\leftrightarrow$ is a congruence for the alternative composition.

Lemma 16 (Aging of bisimilar states of transition systems). *Suppose* $\langle p, \Delta\rangle \leftrightarrow \langle q, \Delta\rangle$ *are bisimilar states in a well-defined environment* (Φ, Δ)*. Then*

$$\langle p, \mathrm{age}(\Delta, \mathrm{rd}(p), d)\rangle \leftrightarrow \langle q, \mathrm{age}(\Delta, \mathrm{rd}(q), d)\rangle,$$

for any $d \geq 0$ *for which* $\mathrm{age}(\Delta, \mathrm{rd}(p), d)$ *and* $\mathrm{age}(\Delta, \mathrm{rd}(q), d)$ *are defined.*

Proof. The function age does not affect the terms. Thus, the action transitions and the immediate termination options remain unchanged, so we conclude that the first and third transfer conditions are trivially satisfied. It remains to be shown that the second transfer condition is satisfied.

Suppose that ws is defined as in the preliminaries for Definition 13, i.e.

$$\mathrm{ws}(p, \mathit{\Delta}, C, d) = \bigcup_{\langle p', \mathit{\Delta}'\rangle \in C} \{S \subseteq \mathrm{rd}(p) \mid \langle p, \mathit{\Delta}\rangle \overset{S}{\mapsto}_d \langle p', \mathit{\Delta}'\rangle\},$$

for $C \in (\mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}) \times \mathrm{Del})_{/\Leftrightarrow}$.

We observe the sets $\mathrm{ws}(p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d), C, t)$ and $\mathrm{ws}(p, \mathit{\Delta}, C, t{+}d)$, for some arbitrary class C. From Definition 5 of the aging function, Definition 9 of the function age, Lemma 6 for the properties of iterative aging and the assumption that the aging $\mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d)$ is eligible we derive the following:

$$\begin{aligned}
\mathrm{ws}(p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d), C, t) &= \bigcup_{\langle p', \mathit{\Delta}'\rangle \in C} \{S \subseteq \mathrm{rd}(p) \mid \langle p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d)\rangle \overset{S}{\mapsto}_t \langle p', \mathit{\Delta}'\rangle\} \\
&= \bigcup_{\langle p', \mathit{\Delta}'\rangle \in C} \{S \subseteq \mathrm{rd}(p) \mid \langle p, \mathit{\Delta}\rangle \overset{S}{\mapsto}_{t+d} \langle p', \mathit{\Delta}'\rangle\} \\
&= \mathrm{ws}(p, \mathit{\Delta}, C, t + d).
\end{aligned}$$

Now, one calculates the probabilities $\mathrm{ap}(p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d), D, t)$ and $\mathrm{ap}(q, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(q), d), D, t)$ as defined in the preliminaries of Definition 13. There are two possible cases. First, if $\mathrm{ws}(p, \mathrm{age}\mathit{\Delta}, \mathrm{rd}(p), d, C, t) = \emptyset$ then $\mathrm{ws}(p, \mathit{\Delta}, C, t + d) = \emptyset$ and because $p \Leftrightarrow q$, it must be that $\mathrm{ws}(q, \mathit{\Delta}, C, t + d) = \emptyset$, so $\mathrm{ws}(q, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(q), d), C, t) = \emptyset$. Thus, in the first case $\mathrm{ap}(p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d), C, t) = 0 = \mathrm{ap}(q, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(q), d), C, t)$.

In the second case, suppose that $\mathrm{ws}(p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d), D, t) \neq \emptyset$. For the sake of clarity we replace $\mathit{\Phi}(X)|\mathit{\Delta}(X)$ by F_X where appropriate. Now, one obtains the following derivation:

$$\begin{aligned}
&\mathrm{ap}(p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d), D, t) = \\
&= \sum_{S \in \mathrm{ws}(p, \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d), D, t)} \mathrm{ap}(S = \min(\mathrm{rd}(p)), S = t) \\
&= \sum_S P(S = t, \mathrm{rd}(p) \setminus S > t) \\
&= \sum_S \prod_{X \in S} P(X = t) \prod_{X \in \mathrm{rd}(p) \setminus S} (\overline{\mathit{\Phi}(X) | \mathrm{age}(\mathit{\Delta}, \mathrm{rd}(p), d)})(t) \\
&= \sum_{S \in \mathrm{ws}(p, \mathit{\Delta}, C, t+d)} \prod_{X \in S} \tfrac{P(X = d+t)}{1 - (\mathit{\Phi}(X)|\mathit{\Delta}(X))(d)} \prod_{X \in \mathrm{rd}(p) \setminus S} (\overline{\mathit{\Phi}(X) | (\mathit{\Delta}(X) + d)})(t) \\
&= \sum_S \prod_{X \in S} \tfrac{P(X = d+t)}{1 - F_X(d)} \prod_{X \in \mathrm{rd}(p) \setminus S} (\overline{F_X | d})(t) \\
&= \sum_S \prod_{X \in S} \tfrac{P(X = d+t)}{1 - F_X(d)} \prod_{X \in \mathrm{rd}(p) \setminus S} 1 - \tfrac{F_X(d+t) - F_X(d)}{1 - F_X(d)}
\end{aligned}$$

$$= \sum_S \prod_{X \in S} \tfrac{P(X=d+t)}{1-F_X(d)} \prod_{X \in \mathrm{rd}(p) \setminus S} \tfrac{1-F_X(d+t)}{1-F_X(d)}$$

$$= \frac{1}{\prod\limits_{X \in \mathrm{rd}(p)} (1-F_X(d))} \sum_S \prod_{X \in S} P(X = d+t) \prod_{X \in \mathrm{rd}(p) \setminus S} \overline{F_X}(d+t)$$

$$= \frac{1}{\prod\limits_{X \in \mathrm{rd}(p)} \overline{F_X}(d)} \cdot \mathrm{ap}(p, \Delta, D, d+t).$$

Note that if $\langle p, \Delta \rangle \leftrightarrows \langle q, \Delta \rangle$ then the probability that $\langle p, \Delta \rangle$ does a stochastic delay transition of a given duration d is the same as the probability that $\langle q, \Delta \rangle$ does a stochastic delay with the same duration d. This follows immediately from the second transfer condition, since the probability of doing a stochastic delay with the same duration to every class is the same. In addition, the expression $\prod_{X \in \mathrm{rd}(p)} \overline{F_X}(d)$ gives the probability that $\langle p, \Delta \rangle$ does a stochastic delay with a duration longer than d. One easily concludes that $\prod_{X \in \mathrm{rd}(p)} \overline{F_X}(d) = \prod_{Y \in \mathrm{rd}(q)} \overline{F_Y}(d)$. Since one obtains an expression for $\mathrm{ap}(q, \mathrm{age}(\Delta, \mathrm{rd}(q), d), D, d)$ analogously as for $\mathrm{ap}(p, \mathrm{age}(\Delta, \mathrm{rd}(p), d), D, d{+}t)$ and $\mathrm{ap}(p, \Delta, D, d+t) = \mathrm{ap}(q, \Delta, D, d+t)$ the proof is complete. □

Next, we prove the congruence property for $\leftrightarrows$.

Theorem 17. *The bisimulation relation $\leftrightarrows$ is a congruence.*

Proof. First, we show for each operator that the results of the operations are bisimilar when applied to bisimilar states of a stochastic transition system in the same environment (Φ, Δ). We assume that $(\langle p, \Delta \rangle, \langle q, \Delta \rangle) \in R$ and $(\langle p', \Delta \rangle, \langle q', \Delta \rangle) \in R'$ for $p, p', q, q' \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ and Δ is well-defined, where R, R' are strong bisimulation relations. For every operator we define a relation R'' that contains the results of the operator and we prove that its reflexive and transitive closure is again a bisimulation relation.

$a.\_ \ , a \in A$ Define $R'' = R \cup \{(\langle a.p, \Delta \rangle, \langle a.q, \Delta \rangle)\}$. We have to show that the reflexive and transitive closure of R'' is a bisimulation. Since R satisfies the transfer conditions, we have to show that the pair $(\langle a.p, \Delta \rangle, \langle a.q, \Delta \rangle)$ satisfies the transfer conditions.

1. There is only one applicable rule, so $\langle a.p, \Delta \rangle \xrightarrow{a} \langle p, \Delta \rangle$ is the only possible transition. Also, $\langle a.q, \Delta \rangle \xrightarrow{a} \langle q, \Delta \rangle$ and $(\langle p, \Delta \rangle, \langle q, \Delta \rangle) \in R''$.
2. The transfer condition is trivially satisfied since both $a.p$ and $a.q$ cannot do a stochastic delay.
3. The transfer condition is trivially satisfied since both $a.p$ and $a.q$ cannot terminate successfully.

$\sigma_X.\_ \ , X \in \mathcal{V}$ Define $R'' = R \cup \{(\langle \sigma_X.p, \Delta \rangle, \langle \sigma_X.q, \Delta \rangle)\}$.

1. The transfer condition is trivially satisfied since both $\sigma_X.p$ and $\sigma_X.q$ cannot do an action transition.
2. There is only one applicable rule, so $\langle \sigma_X.p, \Delta \rangle \overset{X}{\mapsto}_{d_X} \langle p, \Delta' \rangle$, where $d_X \in \mathrm{supp}(\Phi(X) | \Delta(X))$ and $\Delta' = \mathrm{res}(\Delta, \{X\})$. Similarly $\langle \sigma_X.q, \Delta \rangle \overset{X}{\mapsto}_{d_X} \langle q, \Delta' \rangle$. It is obvious that the transfer condition is satisfied because there is only one active stochastic delay with the same distribution and $(\langle p, \Delta' \rangle, \langle q, \Delta' \rangle) \in R''$.

3. The transfer condition is trivially satisfied since both $\sigma_X.p$ and $\sigma_X.q$ cannot terminate successfully.

– + – Suppose that $\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle \hat{p}, \Delta'\rangle$ and $\langle p', \Delta\rangle \overset{S'}{\mapsto}_{d_{S'}} \langle \hat{p}', \Delta''\rangle$. Then, by performing one stochastic delay transition step (whenever possible) one obtains the transitions depicted in Fig. 3.

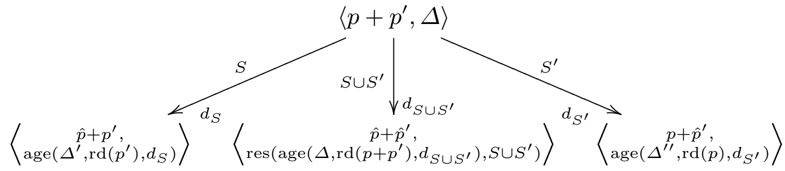

Fig. 3. One stochastic delay transition step of $\langle p + p', \Delta_p \cup \Delta_{p'}\rangle$

Note that there are two type of stochastic delay transitions dependent on the winner of the race (1) only one summand wins the race and (2) both summands win the race. In the first case, the resulting term is a sum of the resulting term of the winning summand and the original term of the losing summand. In the resulting environment of the winning summand the distributions of the stochastic delays of the losing summand are aged by the duration that passed for the winners. Note that the aging is allowed because the losing summand can do a stochastic delay with longer duration then the one observed in the transition. In the second case, the resulting term is a sum of the resulting terms of both summands and the resulting environment contains the aged distributions of the random variables that guide the losing delays from both terms. Note that if one of the summands makes an action transition instead of stochastic delay, then the alternative composition makes a non-deterministic choice and the transition system continues behaving as the resulting transition system of one of the summands.
Based on the previous observation and Lemma 16 we build the bisimulation relation by assuming all possible eligible stochastic delay transitions for all bisimilar transition system states. In order to find all possible transitions we have to find all stochastic delay durations which are eligible for the losing term. The duration of the winning stochastic delay transition must be shorter than the maximal duration that the losing delays must perform. More precisely, it must be shorter than the minimum of all durations with maximal length of every stochastic delay that the losing term can perform. This duration depends on distributions of the active stochastic delays and the process term. We denote it by $d_{max}(p, \Delta)$ and we formally define it as:

$$d_{\max}(p, \Delta) = \min\{\max(\text{supp}(\Phi(X)|\Delta(X))) \mid X \in \text{rd}(p)\}.$$

Now, we define a relation R'' that relates the alternative composition of every possible aging for every pair in R and R'. Note that R'' must also contain the original bisimilar pairs in case an action transition has to be executed.

$$
\begin{aligned}
R'' = {} & R \cup R' \cup \\
& \{(\langle p+p', \Delta\rangle, \langle q+q', \Delta\rangle) \mid (\langle p, \Delta\rangle, \langle q, \Delta\rangle) \in R, (\langle p', \Delta\rangle, \langle q', \Delta\rangle) \in R'\} \cup \\
& \{(\langle p+p', \mathrm{age}(\Delta, \mathrm{rd}(p'), d)\rangle, \langle q+q', \mathrm{age}(\Delta, \mathrm{rd}(q'), d)\rangle) \mid \\
& \quad (\langle p, \Delta\rangle, \langle q, \Delta\rangle) \in R, (\langle p', \Delta\rangle, \langle q', \Delta\rangle) \in R', d < d_{\max}(p', \Delta)\} \cup \\
& \{(\langle p+p', \mathrm{age}(\Delta, \mathrm{rd}(p), d')\rangle, \langle q+q', \mathrm{age}(\Delta, \mathrm{rd}(q), d')\rangle) \mid \\
& \quad (\langle p, \Delta\rangle, \langle q, \Delta\rangle) \in R, (\langle p', \Delta\rangle, \langle q', \Delta\rangle) \in R', d' < d_{\max}(p, \Delta)\} \cup \\
& \{(\langle p+p', \mathrm{age}(\mathrm{age}(\Delta, \mathrm{rd}(p), d'), \mathrm{rd}(p'), d)\rangle, \langle q+q', \mathrm{age}(\mathrm{age}(\Delta, \mathrm{rd}(q), d'), \mathrm{rd}(q'), d)\rangle) \\
& \quad \mid (\langle p, \Delta\rangle, \langle q, \Delta\rangle) \in R, (\langle p', \Delta\rangle, \langle q', \Delta\rangle) \in R', d < d_{\max}(p', \Delta), d' < d_{\max}(p, \Delta)\}
\end{aligned}
$$

1. Suppose $\langle p+p', \Delta\rangle \xrightarrow{a} \langle r, \Delta\rangle$ for some $a \in A$ and $r \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$. According to the deduction rules, either (1) $\langle p, \Delta\rangle \xrightarrow{a} \langle r, \Delta\rangle$ and p' does not have to do a zero delay or (2) $\langle p', \Delta\rangle \xrightarrow{a} \langle r, \Delta\rangle$ and p does not have to do a zero delay. If (1) then $\langle q, \Delta\rangle \xrightarrow{a} \langle s, \Delta'\rangle$ for some $s \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ because $\langle p', \Delta\rangle \leftrightarroweq \langle q', \Delta\rangle$, so $\langle q', \Delta\rangle$ does not have to do a zero delay. Then $\langle q+q', \Delta\rangle \xrightarrow{a} \langle s, \Delta'\rangle$ and $(r, s) \in R''$ because $(\langle p, \Delta\rangle, \langle q, \Delta\rangle) \in R$. Analogously for (2).
2. Suppose $(\langle p+p', \Delta\rangle, \langle q+q', \Delta\rangle) \in R''$. There are two possible cases: (1) one of the summands won the race or (2) both summands won the race. First, we consider the case when one of the summands wins the race. Suppose that $\langle p, \Delta\rangle$ can win the race in time d. Then, for every class $C \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})_{/R}$ it holds that $\mathrm{ap}(p, \Delta, C, d) = \mathrm{ap}(q, \Delta, C, d)$. The probability that the winning stochastic delays of $\langle p+p', \Delta\rangle$ came from $\langle p, \Delta\rangle$ is equal to the probability that $\langle p, \Delta\rangle$ won the race in time d and $\langle p', \Delta\rangle$ does a stochastic delay with duration longer than d. However, the probability that $\langle p', \Delta\rangle$ does a stochastic delay longer than some duration d is the same as the probability that $\langle q', \Delta\rangle$ does a stochastic delay with duration longer than d since $\langle p', \Delta\rangle \leftrightarroweq \langle q', \Delta\rangle$ (see the proof of Lemma 16). Note that by construction alternative compositions of bisimilar terms of R and bisimilar terms of R' belong to the same class of R''. Now, we have that the probability of going to a part of the class induced by R'' and by performing a stochastic delay from $\langle p, \Delta\rangle$ and $\langle q, \Delta\rangle$, respectively, is the same for both $\langle p+p', \Delta\rangle$ and $\langle q+q', \Delta\rangle$. Thus, we conclude that $\mathrm{ap}(p+p', \Delta, C'', d) = \mathrm{ap}(q+q', \Delta, C'', d)$, for $C'' \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})_{/R''}$. The argument is analogous for the case when $\langle p', \Delta\rangle$ wins the race.
   In the second case, both summands perform a stochastic delay transition with the same duration. The probability of going to some equivalence class is simply the product of the individual probabilities. Since the individual probabilities are the same we finish the proof by argument similar to the first case.
3. If $\langle p+p', \Delta\rangle \downarrow$ then (1) $\langle p, \Delta\rangle \downarrow$, (2) $\langle p', \Delta\rangle \downarrow$ or (3) both (1) and (2) hold. If (1) then $\langle q, \Delta\rangle \downarrow$, so $\langle q+q', \Delta\rangle \downarrow$. Analogously for (2). If (3) then the transfer condition immediately follows from (1) or (2).

Now, let us consider the strong bisimulation on $\mathrm{BSP}^{\mathrm{dst}}$-terms. By definition, $p \leftrightarrows q$ if $\langle p, \Delta\rangle \leftrightarrows \langle q, \Delta\rangle$, for all well-defined environments (Φ, Δ). Since we did the congruence proof for all operators over $\langle p, \Delta\rangle \leftrightarrows \langle q, \Delta\rangle$ for an arbitrary environment (Φ, Δ), the proof is complete. □

In the following section we identify a conflicting behaviour that can be exhibited by the STS's and propose a solution by adding α-conversion in the structural operational semantics.

4 α-conversion

We proceed by analyzing a conflicting behaviour of the STSs defined so far that occurs when two racing delays are guided by the same random variable. Consider the following example.

Example 18. Suppose $p \equiv \sigma_X.\epsilon$. We observe $STS(p + p, (\Phi, \Delta))$. Consider the transition $\langle \sigma_X.\epsilon + \sigma_X.\epsilon, \Delta\rangle \overset{X}{\mapsto}_{d_X} \langle \epsilon + \sigma_X.\epsilon, \mathrm{res}(\mathrm{age}(\Delta, \{X\}, d_X), X)\rangle$. In the resulting environment, X is a random variable that guides both the winning and the losing stochastic delay. Such behavior leads to conflict because $\Delta(X)$ should contain both, $\perp$, because X won the race and d_X, because X lost the race. On the other hand, the term $p+p$ is not bisimilar to $\sigma_X.(\epsilon+\epsilon)$ because, in general, the distribution functions of X and $\min(X, X)$ are not equal. Therefore, we wish to express that the left and the right summand have equally distributed stochastic delays and the distribution function is provided by the random variable X.

We resolve the conflict by renaming one of the variables and ensuring that the original and the replacement have the same distribution. So, $\sigma_X.\epsilon + \sigma_X.\epsilon$ and $\sigma_X.\epsilon + \sigma_Y.\epsilon$ behave the same under the assumption that $F_X = F_Y$, because the behavior of the STS does not depend on the name of the variable, but on its distribution function. However, the second term has proper semantics, since there is no conflicting behavior in its STS. For an technical underpinning of this, we define the relation $\simeq_\alpha$ on $\mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}) \times \mathrm{Env}$ as the least relation such that

$$\langle \delta, \Delta\rangle \simeq_\alpha \langle \delta, \Delta\rangle \qquad \langle \epsilon, \Delta\rangle \simeq_\alpha \langle \epsilon, \Delta\rangle \qquad \frac{\langle p, \Delta\rangle \simeq_\alpha \langle q, \Delta\rangle}{\langle a.p, \Delta\rangle \simeq_\alpha \langle a.q, \Delta\rangle}$$

$$\frac{\langle p, \Delta\rangle \simeq_\alpha \langle q, \Delta\rangle, \Phi(X) = \Phi(Y), \Delta(X) = \Delta(Y)}{\langle \sigma_X.p, \Delta\rangle \simeq_\alpha \langle \sigma_Y.q, \Delta\rangle} \qquad \frac{\langle p, \Delta\rangle \simeq_\alpha \langle q, \Delta\rangle, \langle p', \Delta\rangle \simeq_\alpha \langle q', \Delta\rangle}{\langle p + p', \Delta\rangle \simeq_\alpha \langle q + q', \Delta\rangle}$$

Clearly, $\simeq_\alpha$ is a congruence. In the literature, a relation as $\simeq_\alpha$ is referred to as α-congruence or α-conversion [13, 8].

We define a function $\mathrm{cv}\colon \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}) \to 2^{\mathcal{V}}$ to identify conflicting random variables that guide multiple stochastic delays in the same race. The function cv is defined using structural induction.

$$\mathrm{cv}(\epsilon) = \emptyset \qquad \mathrm{cv}(\delta) = \emptyset \qquad \mathrm{cv}(a.p) = \emptyset \qquad \mathrm{cv}(\sigma_X.p) = \emptyset$$
$$\mathrm{cv}(p + q) = \mathrm{cv}(p) \cup (\mathrm{rd}(p) \cap \mathrm{rd}(q)) \cup \mathrm{cv}(q).$$

If a term does not contain conflicting variables, we say that it is conflict-free. We characterize such terms using a predicate cf that checks whether the set of conflict variables in the current step is empty. Given a process term p, $\mathrm{cf}(p)$ is true if and only if $\mathrm{cv}(p) = \emptyset$.

We add an α-conversion rule to the structural operational semantics, viz.

$$(\alpha) \quad \frac{\langle q, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle,\ \mathrm{cf}(q),\ \mathrm{cf}(p'),\ p \simeq_\alpha q}{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \mathrm{res}(\Delta', \mathrm{rd}(p') \setminus \mathrm{rd}(q))\rangle}.$$

This rule guarantees that the stochastic delay transitions are performed as a result of a race which does not lead to conflicting behavior. This is achieved by finding an α-converted term that is conflict-free and performing the race with it. Note that the non-racing terms of p' can get an age in the process of α-converting p to q, so we have to reset them in the resulting environment. For example, α-converting $\sigma_Y.\sigma_X.\epsilon + \sigma_X.\epsilon$ to $\sigma_Y.\sigma_U.\epsilon + \sigma_X.\epsilon$, where $\Delta(X) \neq \bot$ results in $\Delta(U) \neq \bot$, but U has not participated in any race. In order to exclude conflicting behavior, we use the predicate cf.

This means that we have to adapt the operational semantics by adding an extra conflict-freeness condition for every state that has the option to perform a stochastic delay. For example, the adapted version of Rule 11 is:

$$11\alpha \quad \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S^+} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \not\mapsto,\ \mathrm{cf}(p+q)}{\langle p+q, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle}.$$

The obtained theory is denoted as $\mathrm{BSP}_\alpha^{\mathrm{dst}}$. In the following we give an example of a STS in order to illustrate the operational semantics rules.

Example 19. In Fig. 4 we give the $STS(\sigma_X.\sigma_X.\epsilon + \sigma_X.a.\epsilon, (\Phi, \Delta_0))$, where initially $\Phi = \{X \mapsto F, Y \mapsto F, Z \mapsto F\}$ and $\Delta_0 = \{X \mapsto \bot, Y \mapsto \bot, Z \mapsto \bot\}$. Note that we give possible α-conversions in brackets for clarification, but it is not a part of the transition system.

In the following section we present the equational theory of $\mathrm{BSP}^{\mathrm{dst}}$.

4.1 Equational Theory

First, we define a term model of $\mathrm{BSP}_\alpha^{\mathrm{dst}}$ up to α-congruence. Note that $\mathbb{P}(\mathrm{BSP}_\alpha^{\mathrm{dst}}) = \mathbb{P}(\mathrm{BSP}^{\mathrm{dst}})$.

Definition 20 (Term model of $\mathrm{BSP}_\alpha^{\mathrm{dst}}$). *The term model of* $\mathrm{BSP}_\alpha^{\mathrm{dst}}$ *is the quotient algebra* $\mathbb{P}(\mathrm{BSP}_\alpha^{\mathrm{dst}})_{/\leftrightarroweq}$.

In the following section we present the process theory $\mathrm{BSP}_\alpha^{\mathrm{dst}}$ and show that it is a sound and complete axiomatization for $\mathbb{P}(\mathrm{BSP}_\alpha^{\mathrm{dst}})_{/\leftrightarroweq}$.

Before we present the equational theory of $\mathrm{BSP}_\alpha^{\mathrm{dst}}$ we briefly review the standard axioms of the un-timed theory $\mathrm{BSP}(\mathcal{A})$ [1]. The commutative and

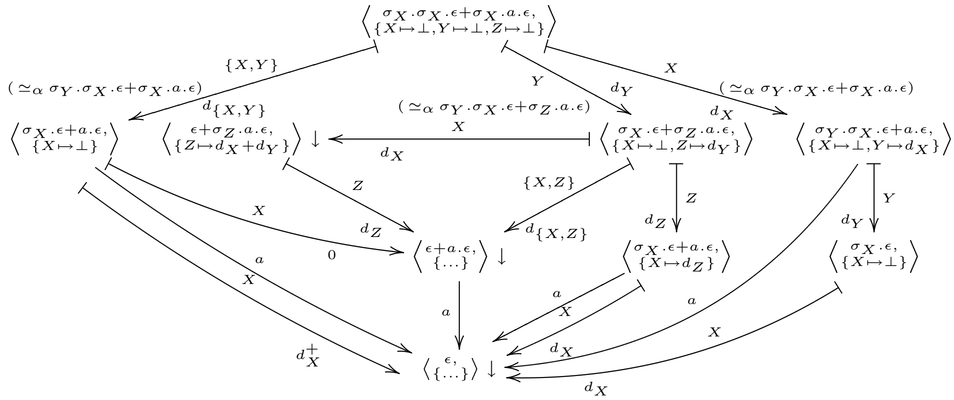

Fig. 4. Stochastic transition system of $\sigma_X.\sigma_X.\epsilon + \sigma_X.a.\epsilon$

associative property of the alternative composition should still hold since the race condition depends on the minimum of distributions of the stochastic delays which is commutative and associative. Alternative composition will not remain an idempotent operator because of the race condition. However, alternative composition is idempotent when it presents a non-deterministic choice. The neutral element of the alternative composition remains the immediate deadlock δ. The equational theory of $\mathrm{BSP}^{\mathrm{dst}}_\alpha$ is given in the following table.

$p + q = q + p$	$A1$	$(p+q)+r = p+(q+r)$	$A2$
$\epsilon + \epsilon = \epsilon$	$A3\epsilon$	$a.p + a.p = a.p$	$A3a$
$p + \delta = p$	$A6$	$\sigma_X.p = \sigma_Y.p$ if $F_X = F_Y$	α

We discuss the new axioms. The axioms $A3\epsilon$ and $A3a$ define the idempotent behavior of the alternative composition when performed on ϵ or on action prefixed processes, respectively. Note that $A6$ covers the idempotent behavior of the alternative composition for the δ. The axiom α characterizes the α-conversion and introduces side conditions for identically distributed stochastic delays.

The process theory $\mathrm{BSP}^{\mathrm{dst}}_\alpha$ is a sound and ground-complete axiomatization for the proposed term model as stated in the following theorems.

Theorem 21. *The process theory* $\mathrm{BSP}^{\mathrm{dst}}_\alpha$ *is sound axiomatization of* $\mathbb{P}(\mathrm{BSP}^{\mathrm{dst}}_\alpha)_{/\leftrightarrows}$.

Proof. It suffices to give a bisimulation relation R for every $l, r \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$, such that l presents the left side and r presents the right side of the axioms of $\mathrm{BSP}^{\mathrm{dst}}_\alpha$. In the following we will always assume that the proposed relation R for the basis of the bisimulation is extended to equivalence, if required.

$A1$ Let $R = \{(p+q, q+p) \mid p, q \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})\}$.

1. Suppose $p + q \xrightarrow{a} r$ for some $a \in A$ and $r \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$. Then, either $p \xrightarrow{a} r$ and q does not have to do a zero delay or $q \xrightarrow{a} r$ and p does not have to do a zero delay. In both cases $q + p \xrightarrow{a} r$ and vice versa.

2. The racing delays of $p+q$ and $q+p$ are the same, i.e. $\mathrm{rd}(p+q) = \mathrm{rd}(q+p)$. Thus, the winning sets and the losing sets of $p+q$ and $q+p$ are the same, so the accumulative probability is the same. We conclude that $p+q \leftrightarrow q+p$.
3. If $p+q\downarrow$ then either $p\downarrow$ or $q\downarrow$. In either case, $q+p\downarrow$. Analogously for $q+p$.

A2 Let $R = \{((p+q)+r, p+(q+r)) \mid p,q,r \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})\}$.

1. Suppose $(p+q)+r \xrightarrow{a} s$ for some $a \in A$ and $s \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}_\alpha)$. Then, either $p+q \xrightarrow{a} s$ and r does not have to do a zero delay or $r \xrightarrow{a} s$ and $p+q$ does not have to do a zero delay. If $p+q \xrightarrow{a} s$ then either $p \xrightarrow{a} s$ and q does not have to do a zero delay or $q \xrightarrow{a} s$ and p does not have to do a zero delay. In all three cases $p+(q+r) \xrightarrow{a} s$ and $p+(q+r)$ has no additional transitions.
2. Analogous to *A1*, we obtain that $\mathrm{rd}((p+q)+r) = \mathrm{rd}(p+(q+r))$, so $(p+q)+r \leftrightarrow p+(q+r)$.
3. If $(p+q)+r\downarrow$ then either $p+q\downarrow$ or $r\downarrow$. If $p+q\downarrow$ then $p\downarrow$ or $q\downarrow$. In all three cases $p+(q+r)\downarrow$. Analogously for $p+(q+r)$.

A3ϵ Let $R = \{(\epsilon+\epsilon, \epsilon)\}$.

1. Trivially satisfied since neither $\epsilon+\epsilon$, nor ϵ can do an action transition.
2. Trivially satisfied since neither $\epsilon+\epsilon$, nor ϵ can do a stochastic delay.
3. According to the deduction rules $\epsilon+\epsilon\downarrow$ and $\epsilon\downarrow$.

A3a Let $R = \{(a.p + a.p, a.p) \mid p \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})\}$.

1. Obviously, $a.p + a.p \xrightarrow{a} p$ for some $a \in A$ and $a.p \xrightarrow{a} p$ are the only transitions this terms can do.
2. Trivially satisfied since neither $a.p + a.p$, nor $a.p$ can do a stochastic delay.
3. Trivially satisfied since neither $a.p + a.p$, nor $a.p$ can terminate successfully.

A6 Let $R = \{(p+\delta, p) \mid p \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})\}$.

1. Suppose $p+\delta \xrightarrow{a} p'$ for some $a \in A$ and $p' \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}_\alpha)$. Since, δ cannot do any transitions, then $p \xrightarrow{a} p'$.
2. Analogous to *A1* since $\mathrm{rd}(p+\delta) = \mathrm{rd}(p)$.
3. Suppose $p+\delta\downarrow$. Since δ cannot terminate then $p\downarrow$.

α Let $R = \{(\sigma_X.p, \sigma_Y.p) \mid X, Y \in \mathcal{V}, p \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}}), F_X = F_Y\}$.

1. Trivially satisfied since neither $\sigma_X.p$, nor $\sigma_Y.p$ can do action transitions.
2. Both $\sigma_X.p$ and $\sigma_Y.p$ can do only one stochastic delay to $[p]_{/R}$. Since $F_X = F_Y$ it follows that $P(\sigma_X, \varDelta, C, d) = P(\sigma_Y, \varDelta, C, d)$ for every eligible $\varDelta$ and arbitrary equivalence class C and $d \geq 0$, so we conclude $\sigma_X.p \leftrightarrow \sigma_Y.p$
3. Trivially satisfied since neither $\sigma_X.p$, nor $\sigma_Y.p$ can successfully terminate.

□

Corollary 22 ((Normal) form of closed $\mathrm{BSP}^{\mathrm{dst}}_\alpha$-terms). *Using the axioms each closed $\mathrm{BSP}^{\mathrm{dst}}_\alpha$-term can be written in the following (normal) form:*

$$p = \sum_{i=1}^{l'} a_i.p'_i + \sum_{i=1}^{l''} \sigma_{X_i}.p''_i [+\epsilon]$$

where $\sum_{i=1}^{n}$ *denotes the alternative composition of* n *closed* $\mathrm{BSP}_{\alpha}^{\mathrm{dst}}$*-terms if* $n > 0$*, or* δ *if* $n = 0$*. The square brackets around* ϵ *denote that it is an optional summand and* $l', l'' \in \mathbb{N}$*,* $a_i \in \mathcal{A}$*,* $X_j \in \mathcal{V}$*,* $p'_i, p''_j \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ *for* $1 \leq i \leq l', 1 \leq j \leq l''$*.*

Theorem 23 (Ground-completeness of $\mathrm{BSP}_{\alpha}^{\mathrm{dst}}$**).** *The process theory* $\mathrm{BSP}_{\alpha}^{\mathrm{dst}}$ *is a axiomatization of* $\mathbb{P}(\mathrm{BSP}_{\alpha}^{\mathrm{dst}})_{/\Leftrightarrow}$*, i.e.* $\mathbb{P}(\mathrm{BSP}_{\alpha}^{\mathrm{dst}})_{/\Leftrightarrow} \models p = q$ *implies* $\mathrm{BSP}_{\alpha}^{\mathrm{dst}} \vdash p = q$*.*

Proof. We will prove the theorem by natural induction on the total number of symbols in p and q.

The base case is when p and q are either δ or ϵ. Trivially $\delta \Leftrightarrow \delta$ implies $\delta = \delta$. Analogous for ϵ. Suppose that the total number of symbols is s and $p \Leftrightarrow q$. Using Corollary 22 we transform the terms p and q in the following form:

$$p = \sum_{i=1}^{l'} a_i.p'_i + \sum_{i=1}^{l''} \sigma_{X_i}.p''_i[+\epsilon]$$

$$q = \sum_{j=1}^{m'} b_j.q'_j + \sum_{j=1}^{m''} \sigma_{Y_j}.q''_j[+\epsilon]$$

where $p'_i, q'_j \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ and $a_i, b_j \in A$ for $1 \leq i \leq l', 1 \leq j \leq m'$ and $p''_i, q''_j \in \mathcal{C}(\mathrm{BSP}^{\mathrm{dst}})$ and $X_i, Y_j \in \mathcal{V}$ for $1 \leq i \leq l'', 1 \leq j \leq m''$. All terms p, q have an optional ϵ summand denoted with square brackets.

If $p \downarrow$ then it must be that $q \downarrow$, so if p contains an ϵ summand then also q contains an ϵ summand and vice versa.

Now we separate action prefixed and stochastic time prefixed summands. The sum of all action prefixed summands of a process p is denoted by p_a and the sum of all stochastic time prefixed summands is denoted by p_s. Thus, we write $p = p_a + p_s[+\epsilon]$ and $q = q_a + q_s[+\epsilon]$.

If $p \Leftrightarrow q$ then $p_a \Leftrightarrow q_a$ and $p_s \Leftrightarrow q_s$. The statement holds because the processes p_a and q_a cannot do any stochastic delays and cannot terminate and vice versa, p_s and q_s cannot do action transitions and they cannot terminate successfully.

We can easily reuse the proof from [1], by example, to show that if $\mathbb{P}(\mathrm{BSP}_{\alpha}^{\mathrm{dst}})_{/\Leftrightarrow} \models p_a = q_a$ then $\mathrm{BSP}_{\alpha}^{\mathrm{dst}} \vdash p_a = q_a$. It remains to be proven that if $\mathbb{P}(\mathrm{BSP}_{\alpha}^{\mathrm{dst}})_{/\Leftrightarrow} \models p_s = q_s$ then $\mathrm{BSP}_{\alpha}^{\mathrm{dst}} \vdash p_s = q_s$.

First, we show that if $p_s \Leftrightarrow q_s$ then $l'' = m''$, where l'' and m'' are the limits of the sums for stochastic time prefix summands in the normal form representation. The proof is done by contradiction. More precisely, if $l'' \neq m''$, then either $l'' > m''$ or $l'' < m''$. Let us assume that $l'' > m''$. We choose some distribution F to assign to all the variables such that $|\mathrm{supp}(F)| = l''$. Then, p_s can do l'' different time steps with non-zero probability, whereas q_s cannot do the same time steps with non-zero probability, which is a contradiction. The case when $m'' > l''$ is proven analogously.

Next we show that p_s and q_s can be rewritten in a form such that $F_{X_i} = F_{Y_i}$, for $1 \leq i \leq l'' = m''$. We assume that p_s and q_s cannot be rewritten in the required form. It follows that there is at least one index $1 \leq j \leq l''$, such that $F_{X_j} \neq F_{Y_k}$, where Y_k is not related with any other variable. We assign a distribution function that can do some delay $t \geq 0$ with probability 1 to F_{X_j} and a distribution function that does a delay $t+1 \geq 0$ to every other variable. Then, p_s can do a delay of duration t with probability 1, where q_s cannot, which is a contradiction.

Now using the inductional hypothesis and Axiom α, we easily obtain that $p_s = q_s$. By using the congruence property of the equational theory we obtain that $p_a + p_s = q_a + q_s$. Finally, using Corollary 22, we obtain that $p = q$, which completes the proof. □

Next, we investigate the behavior of the parallel composition in the current setting.

5 Parallel Composition

We add an ACP-style parallel composition to the theory $\mathrm{BSP}_\alpha^{\mathrm{dst}}$ and obtain the theory of Basic Communication Processes with Discrete Stochastic Time and α-conversion $\mathrm{BCP}_\alpha^{\mathrm{dst}}(\mathcal{A}, \mathcal{V}, \gamma)$, where γ is the ACP-style communication function. As the parallel composition allows both interleaving and communication of immediate actions, in the present setting it should also cater for interleaving and synchronization of stochastic delays. Similarly to real-time PA's, we merge the delays in case the processes perform stochastic delays of different duration. We synchronize the processes in case their delays are of the same duration. Immediate actions always take precedence over time in the parallel composition, except when performing zero duration delays. It is important to perform all possible zero delays and afterwards the immediate actions because otherwise we may lose communication options. For example, $\sigma_X.a.\epsilon \parallel b.\epsilon$ should allow a and b to communicate if $F_X(0) \neq 0$.

The definitions of rd, cv and $\simeq_\alpha$ are extended straightforwardly to apply to a parallel process $p \parallel q$. We give the operational semantics of the parallel composition in the following table:

We briefly discuss the new rules. Rule 17 states when the parallel composition has the termination option. Rule 18 enables zero delays before immediate actions similar to the alternative composition. Rules 20 and 22 enable interleaving of actions, by allowing the left operand to perform an immediate action if the right one cannot delay or it can delay with positive duration, in which case the zero durations are disabled by aging of 0 in Rule 22. Rule 24 states that synchronization of actions can occur, only if their communication is defined by the communication function γ. Rule 25 enables the race condition, similar to the Rule 14 for the alternative composition. Rule 27 enables simultaneous passage of time for the left and right operand which allows synchronization of stochastic delays that exhibit the same duration. Rules 19, 21, 23 and 26 are analogous to the rules 18, 20, 22 and 25 for the righthand operand.

It is easily observed that the parallel operator is both commutative and associative. The proof for the action transitions is standard. Regarding stochastic

Table 2. Structural operational semantics for $\mathrm{BCP}_{\alpha}^{\mathrm{dst}}$.

$$17\ \frac{\langle p, \Delta\rangle \downarrow, \langle q, \Delta\rangle \downarrow}{\langle p \parallel q, \Delta\rangle \downarrow} \quad 18\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_0 \langle p', \Delta'\rangle,\ q \not\mapsto}{\langle p \parallel q, \Delta\rangle \overset{S}{\mapsto}_0 \langle p' \parallel q, \Delta'\rangle} \quad 19\ \frac{\langle p, \Delta\rangle \not\mapsto,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_0 \langle q', \Delta'\rangle}{\langle p \parallel q, \Delta\rangle \overset{T}{\mapsto}_0 \langle p \parallel q', \Delta'\rangle}$$

$$20\ \frac{\langle p, \Delta\rangle \xrightarrow{a} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \not\mapsto}{\langle p \parallel q, \Delta\rangle \xrightarrow{a} \langle p' \parallel q, \Delta'\rangle} \quad 22\ \frac{\langle p, \Delta\rangle \xrightarrow{a} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T^+} \langle q', \Delta''\rangle}{\langle p \parallel q, \Delta\rangle \xrightarrow{a} \langle p' \parallel q, \mathrm{age}(\Delta', \mathrm{rd}(q), 0)\rangle}$$

$$21\ \frac{\langle p, \Delta\rangle \not\mapsto,\ \langle q, \Delta\rangle \xrightarrow{b} \langle q', \Delta'\rangle}{\langle p \parallel q, \Delta\rangle \xrightarrow{b} \langle p \parallel q', \Delta'\rangle} \quad 23\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S^+} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \xrightarrow{b} \langle q', \Delta'\rangle}{\langle p \parallel q, \Delta\rangle \xrightarrow{a} \langle p \parallel q', \mathrm{age}(\Delta', \mathrm{rd}(p), 0)\rangle}$$

$$24\ \frac{\langle p, \Delta\rangle \xrightarrow{a} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \xrightarrow{b} \langle q', \Delta'\rangle,\ \gamma(a, b) = c}{\langle p \parallel q, \Delta\rangle \xrightarrow{c} \langle p' \parallel q', \Delta'\rangle}$$

$$25\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle q', \Delta''\rangle,\ d_S < d_T}{\langle p \parallel q, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p' \parallel q, \Delta'''\rangle}, \quad \text{where } \Delta''' = \mathrm{age}(\Delta', \mathrm{rd}(q), d_S)$$

$$26\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle q', \Delta''\rangle,\ d_S > d_T}{\langle p \parallel q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle p \parallel q', \Delta'''\rangle}, \quad \text{where } \Delta''' = \mathrm{age}(\Delta'', \mathrm{rd}(p), d_T)$$

$$27\ \frac{\langle p, \Delta\rangle \overset{S}{\mapsto}_{d_S} \langle p', \Delta'\rangle,\ \langle q, \Delta\rangle \overset{T}{\mapsto}_{d_T} \langle q', \Delta''\rangle,\ d_S = d_T}{\langle p \parallel q, \Delta\rangle \overset{S \cup T}{\mapsto}_{d_{S \cup T}} \langle p' \parallel q', \Delta'''\rangle}, \quad \text{where } \Delta''' = \mathrm{res}(\mathrm{age}(\Delta, \mathrm{rd}(p \parallel q), d_{S \cup T}), S \cup T).$$

delays, the properties follow immediately from the structural operational semantics. Note that the race imposed by the parallel operator is the same as for the alternative composition. In the following example we illustrate some problems introduced by the weak choice and the α-conversion for the parallel operator, ultimately leading to absence of a standard expansion law.

Example 24 (No expansion law for $\mathrm{BCP}_{\alpha}^{\mathrm{dst}}$*).* Let $p \equiv \sigma_X.\epsilon$ and $q \equiv \sigma_Y.\epsilon$. We observe their parallel composition $p \parallel q$ and $p \mathbin{⌊\!⌊} q + q \mathbin{⌊\!⌊} p + p \mid q$ as its standard expansion. Note that $p \mathbin{⌊\!⌊} q$ can perform a delay guided by X if $P(X < Y) > 0$. Same holds for $q \mathbin{⌊\!⌊} p$, whereas $p \mid q$ performs a delay if $P(X = Y) > 0$. Suppose (Φ, Δ) is the environment. Then $\langle \sigma_X.\epsilon, \Delta\rangle \overset{X}{\mapsto}_{d_X} \langle \epsilon, \Delta'\rangle$ and $\langle \sigma_Y.\epsilon, \Delta\rangle \overset{Y}{\mapsto}_{d_Y} \langle \epsilon, \Delta'\rangle$. Let us assume that $d_X < d_Y$. Then one obtains the transition $\langle \sigma_X.\epsilon \parallel \sigma_Y.\epsilon, \Delta\rangle \overset{X}{\mapsto}_{d_X} \langle \epsilon \parallel \sigma_Y.\epsilon, \Delta''\rangle$, where $\Delta''(Y) = d_X$ and the transition system deadlocks.

Next, let us observe the process term obtained by the standard expansion law $\sigma_X.\epsilon \mathbin{⌊\!⌊} \sigma_Y.\epsilon + \sigma_Y.\epsilon \mathbin{⌊\!⌊} \sigma_X.\epsilon + \sigma_X.\epsilon \mid \sigma_Y.\epsilon$. This term has semantics only if it is first α-converted to $\sigma_X.\epsilon \mathbin{⌊\!⌊} \sigma_Y.\epsilon + \sigma_{Y'}.\epsilon \mathbin{⌊\!⌊} \sigma_{X'}.\epsilon + \sigma_{X''}.\epsilon \mid \sigma_{Y''}.\epsilon$, where $F_X = F_{X'} = F_{X''}$ and $F_Y = F_{Y'} = F_{Y''}$. Now, it is straightforward to observe that the parallel composition and its standard expansion do not have the same transition systems. For example, due to the weak choice the standard expansion term can do a stochastic delay guided by X, followed by a stochastic delay guided by Y' and aged by d_X and afterwards it finally deadlocks.

Based on the previous observations we conclude that the lack of total order on the durations of the stochastic delays and the presence of weak choice and α-conversion made it difficult to obtain a standard expansion law. However, because we retained the weak choice we are able to embed real-time in the STS's, which is presented in the following section.

6 Embedding Real Time in Stochastic Time

We consider the embedding of $\mathrm{BCP}^{\mathrm{srt}}$ into $\mathrm{BCP}_{\alpha}^{\mathrm{dst}}$. $\mathrm{BCP}^{\mathrm{srt}}(\mathcal{A}, \gamma)$ is a real-time extension of $\mathrm{BSP}(\mathcal{A})$ with parallel composition that allows synchronization of time delays with the same duration. It is a variant of the process algebra $\mathrm{TCP}_{\mathrm{srt}}(\mathcal{A}, \gamma)$ of [18] without sequential composition. Its semantics is given in terms of *timed transition systems* (TTS's).

Definition 25. *TTS is a structure* $TTS = (\mathcal{S}, \rightarrow, \mapsto, \downarrow)$ *where*

- $\mathcal{S}$ *is a set of states labeled by closed* $\mathrm{BCP}^{\mathrm{srt}}$*-terms;*
- $\rightarrow \subseteq \mathcal{S} \times \mathcal{A} \times \mathcal{S}$ *is a labeled transition relation;*
- $\mapsto \subseteq \mathcal{S} \times \mathbb{R}_0^+ \times \mathcal{S}$ *is a timed transition relation;*
- $\downarrow \subseteq \mathcal{S}$ *is an immediate termination predicate.*

Similarly to STSs, we use infix notation for $\rightarrow$ and $\mapsto$. By $\overset{t}{\mapsto}$ we denote that time $t \geq 0$ has passed. The TTS of a term p is denoted by $TTS(p)$. We denote the set of TTSs by $\mathcal{TTS}$.

The embedding of TTSs into STSs is given by an embedding of $\mathrm{BCP}^{\mathrm{srt}}$-terms in $\mathrm{BCP}_{\alpha}^{\mathrm{dst}}$-terms that will effectively replace each timed delay of duration d by a stochastic delay guided by a degenerated random variable X_{d}, such that $P(X_{\mathrm{d}} = d) = 1$. The restrictions to degenerated random variables are denoted by a subscript deg. The embedding is given by the mapping $\xi\colon \mathcal{TTS} \to \mathcal{STS}$:

$$\xi(TTS(p)) = STS(\varepsilon(p), (\Phi_{\mathrm{deg}}, \Delta_{\perp})),$$

where Φ_{deg} is restricted to degenerated distributions, $\Delta_{\perp}(X) = \perp$, for all $X \in \mathcal{V}_{\mathrm{deg}}$ and the mapping $\varepsilon\colon \mathcal{C}(\mathrm{BCP}^{\mathrm{srt}}) \to \mathcal{C}(\mathrm{BCP}_{\alpha}^{\mathrm{dst}})$ is given by:

$$\begin{array}{lll} \varepsilon(\epsilon) = \epsilon & \varepsilon(\delta) = \delta & \varepsilon(a.p) = a.\varepsilon(p) \\ \varepsilon(\sigma^{\,t}.p) = \sigma_{X_t}.\varepsilon(p) & \varepsilon(p + q) = \varepsilon(p) + \varepsilon(q) & \varepsilon(p \parallel q) = \varepsilon(p) \parallel \varepsilon(q). \end{array}$$

Note that because of the degenerated distributions the stochastic transition system only deals with the probabilities 0 and 1. Therefore, in that setting our bisimulation coincides with strong timed bisimulation of [18], were only the durations of delays are required to match. We observe that only one of the operational rules 12, 13 and 14 is applicable at the same time and the stochastic delay with the shortest duration wins. Moreover, we realize that in this setting there is no need for α-conversion, since all stochastic delays guided by the same random variable either win the race together or age the same duration of time together. The behavior of the zero delay is captured by the rules 8 and 10 and the weak choice by the rules 9 and 11. The time interpolation of the real-time PA's is embedded by aging the racing delays by the interpolation time.

Taking all together we have the following theorem.

Theorem 26. *The mapping $\xi\colon \mathcal{TTS} \to \mathcal{STS}$ is an embedding.*

Proof. The embedding does not change the actions and because $P(X_0 = 0) = 1$ and that is the only degenerate delay that can have zero duration, all actions that are enabled in the TTS are enabled in the STS. Thus, both processes perform the same action transitions. When considering only degenerate random variables, the result from the race condition is always the delay with the shortest duration. When considering timed delays, all timed delays first execute a timed delay with the shortest duration enabled by time interpolation, i.e. $\sigma^{t+s}.p + \sigma^{t}.q \overset{t}{\mapsto}_{\sigma} {}^{s}.p + q$. However, in the STS this behavior is mimicked using the age parameter of the environment, i.e. $\langle \sigma_{X_{t+s}}.p + \sigma_{X_t}.q, \Delta\rangle \overset{X_t}{\mapsto}_t \langle \sigma_{X_{t+s}}.p + q, \Delta'\rangle$, where $\Delta'(X_{t+s}) = t$. Thus, after the aging X_{t+s} has remaining duration of s. The fact that ages are set to $\perp$ in the initial states of STS makes sure that the timed and the stochastic delays have the same duration. □

Example 27. In Fig. 5, we have for the term $p \equiv \sigma^{\,t+s}.a.\epsilon \parallel \sigma^{\,t}.(\sigma^{\,s}.b.\epsilon + a.\epsilon)$, for $s, t > 0$ and $\gamma(a, b) = c$ the original $TTS(p)$ on the left, and its embedding, the $STS(\varepsilon(p), (\Phi_{\mathrm{deg}}, \Delta_{\perp}))$ on the right, where $\varepsilon(p) = \sigma_{X_{t+s}}.a.\epsilon \parallel \sigma_{X_t}.(\sigma_{X_s}.b.\epsilon + a.\epsilon)$. We represent only the important part of the environment.

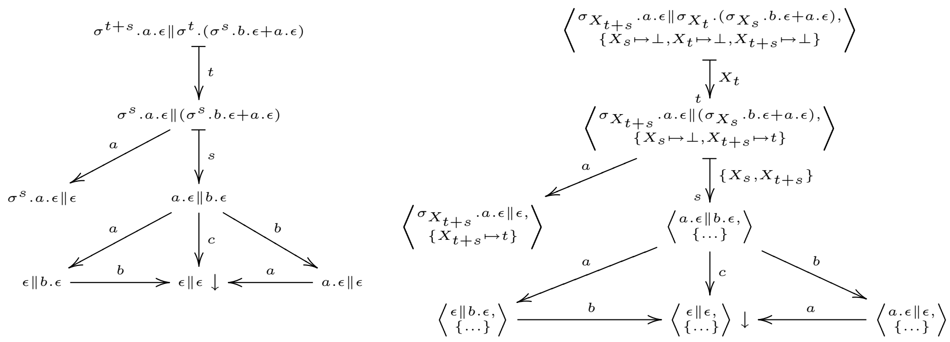

Fig. 5. Example embedding

7 Conclusion and Future Work

We have proposed a stochastic process algebra with immediate actions, termination and deadlock, and discrete distributions as an extension of un-timed process algebra. We introduced a notion of a stochastic transition system and gave a definition of strong bisimulation in that setting that conforms to the probabilistic bisimulation when considering the time as a constant and it corresponds to strong timed bisimulation when only considering probabilities of 0 and 1. We have argued that the bisimulation is a congruence. We showed conflicting behavior of the STS's and introduced α-conversion in order to deal with stochastic delays that are guided by conflicting variables.

We considered extending the algebra with parallel composition. However, expansion of the parallel operator using the alternative composition with weak choice turned out to be problematic. We identified the lack of total ordering on the durations observed by the stochastic delays as the main reason for failure of the standard expansion law when considering alternative composition with weak choice and α-conversion. However, because we retained the weak choice, we were able to propose an intuitive embedding of TTS into stochastic ones by restricting to discrete degenerated stochastic delays.

As future work we schedule an alternative way to obtain an expansion law for the parallel composition, as part of the identification of an axiomatic theory that conservatively extends the underlying real-time theory. Because of the semantical basis, we do not expect major difficulties when incorporating recursion. Also, we plan to extend the current setting with continuous stochastic time. Afterwards, we will consider case studies, especially in protocol verification (e.g. sliding window protocols), since successful modeling of real-time delays paves the way for an easy specification of time-outs.

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