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ON A PAIRING HEURISTIC IN BINPACKING

ABSTRACT

For the analysis of a pairing heuristic in binpacking an important result is used without proof in [1] and [2].

In this note we discuss this result and give a detailed proof of it.

Introduction

Let $n \in \mathbb{N}$ be given and suppose $(\underline{X}_1, \cdots, \underline{X}_n)$ is a n-dimensional stochastic vector with joint density $f(x_1, \cdots, x_n)$

Moreover assume

(i) $0 \le \underline{X}_i \le 1 \qquad i = 1, \cdots, n$

(ii) The stochastic vector $(\underline{X}_{\sigma(1)}, \underline{X}_{\sigma(2)}, \cdots, \underline{X}_{\sigma(n)})$ is distributed as $(\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_n)$ for every permutation σ on $\{1, \cdots, n\}$.

(iii) $f(x_1, x_2, \cdots, x_n) = f(1-x_1, \cdots, x_n)$

Remark

Condition (ii) states that we are dealing with a finite sequence of so-called exchangeable random variables (cf. [3]), while condition (iii) is a symmetry condition.

Note that by (ii) the symmetry in (iii) holds in every component.

Before stating the main result introduce the following notations

$$1_A := \begin{cases} 1 & \text{if the event } A \text{ happens} \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{Y}_i := (1-\underline{X}_i) 1_{\{\underline{X}_i > \frac{1}{2}\}} + \underline{X}_i \, 1_{\{\underline{X}_i \le \frac{1}{2}\}} \quad . \quad i = 1, \cdots, n$$

$$(\underline{i}) := \begin{cases} +1 & \text{if } \underline{X}_i > \frac{1}{2} \\ -1 & \text{if } \underline{X}_i \le \frac{1}{2} \end{cases} \quad i = 1, \cdots, n$$

If we order the random variables $\underline{Y}_i$ in non-decreasing order, say $\underline{Y}_{i_1} \le \underline{Y}_{i_2} \le \cdots \le \underline{Y}_{i_n}$, we denote by $(\underline{i_k})$ the label of the k-order statistic of the sequence $\{\underline{Y}_i\}_{i=1}^n$.

Now the main result reads as follows.

Theorem 1

Suppose the random variables $\{\underline{X}_i\}_{i=1}^n$ satisfy the conditions (i), (ii) and (iii).
Then the following results hold

a) $\{\underline{Y}_i, i \in A\}$ and $\{(\underline{i}), i \in A\}$ are independent for every subset $A \subset \{1,2,\cdots,n\}$

b) $$\mathbb{P}\ \{(\underline{i_k}) = \pi(i_k),\, k \in A\} = \prod_{k \in A} \mathbb{P}\,\{(\underline{i_k}) = \pi(i_k)\}$$

$$= 2^{-|A|}$$

for every subset $A \subset \{1,2,\cdots,n\}$ and

for every function $\pi\colon \{1,2,\cdots,n\} \to \{-1,1\}$.

Proof For every sequence $\{y_i\}_{i=1}^n$ with $y_i \in (0,\frac{1}{2})$ and σ some permutation on $\{1,\cdots,n\}$ we obtain

$$\mathbb{P}\,\{\underline{Y}_{\sigma(i)} \le y_{\sigma(i)},\, (\underline{\sigma(i)}) = \pi(\sigma(i)),\, i=1,\cdots,k\} =$$

$$= \mathbb{P}\,\{1-\underline{X}_{\sigma(i)} \le y_{\sigma(i)}\ (i \in C)\ \wedge\ \underline{X}_{\sigma(i)} \le y_{\sigma(i)}\ \ (i \in \{1,\cdots,k\}-C)\}$$

where $1 \le k \le n$ and $C := \{j\colon 1 \le j \le k\ \&\ \pi(\sigma(j)) = 1\}$
By (ii) and (iii) it follows easily

$$\mathbb{P}\,\{\underline{Y}_{\sigma(i)} \le y_{\sigma(i)}, (\underline{\sigma(i)}) = \pi(\sigma(i))\ i = 1,\cdots,k\} =$$

(1) $$\mathbb{P}\,\{\underline{X}_{\sigma(i)} \le y_{\sigma(i)};\ i = 1,\cdots,k\} = \mathbb{P}\,\{\underline{X}_i \le y_{\sigma(i)}; i = 1,\cdots,k\}$$

and this implies

$$\mathbb{P}\,\{\underline{Y}_{\sigma(i)} \le y_{\sigma(i)};\ i = 1,\cdots,k\} =$$

$$= \sum_{\tau \in D} \mathbb{P}\,\{\underline{Y}_{\sigma(i)} \le y_{\sigma(i)}, (\underline{\sigma(i)}) = \tau(\sigma(i)); i = 1,\cdots,k\} =$$

(2) $$= \sum_{\tau \in D} \mathbb{P}\,\{\underline{X}_i \le y_{\sigma(i)};\ i = 1,\cdots,k\} = 2^k\, \mathbb{P}\,\{\underline{X}_i \le y_{\sigma(i)};\ i = 1,\cdots,k\}$$

where D is the set of functions $\tau\colon \{1,2,\cdots,n\} \to \{-1,+1\}$ which are different on $\{\sigma(1),\cdots,\sigma(k)\}$.
Moreover by (1)

$$\mathbb{P}\{(\underline{\sigma(i)}) = \pi(\sigma(i));\ i = 1, \cdots, k\} =$$

$$= \mathbb{P}\{\underline{Y}_{\sigma(i)} \le \tfrac{1}{2}, (\underline{\sigma(i)}) = \pi(\sigma(i)),\ i = 1, \cdots, k\} =$$

(3) $$= \mathbb{P}\{\underline{X}_i \le \tfrac{1}{2};\ i = 1, \cdots, k\} \ .$$

Since the density $f(x_1, \cdots, x_n)$ is symmetric it is easy to prove that for every $1 \le l \le n-1$

$$\mathbb{P}\{\underline{X}_1 \le \tfrac{1}{2}, \cdots, \underline{X}_l \le \tfrac{1}{2}\} = 2\mathbb{P}\{\underline{X}_1 \le \tfrac{1}{2}, \cdots, \underline{X}_{l+1} \le \tfrac{1}{2}\}$$

and this implies using $\mathbb{P}\{\underline{X}_1 \le \frac{1}{2}\} = \frac{1}{2}$ that

(4) $$\mathbb{P}\{\underline{X}_1 \le \tfrac{1}{2}, \cdots \underline{X}_l \le \tfrac{1}{2}\} = 2^{-l}$$

Now by the relations (1), (2), (3) and (4)

$$\mathbb{P}\{\underline{Y}_{\sigma(i)} \le y_{\sigma(i)}, (\underline{\sigma(i)}) = \pi(\sigma(i));\ i=1, \cdots, k\} =$$

$$\mathbb{P}\{\underline{X}_i \le y_{\sigma(i)};\ i = 1, \cdots, k\} =$$

$$2^{-k}.2^k\ \mathbb{P}\{\underline{X}_i \le y_{\sigma(i)};\ i = 1, \cdots, k\} =$$

$$\mathbb{P}\{(\underline{\sigma(i)}) = \pi(\sigma(i));\ i=1, \cdots, k\}.\mathbb{P}\{\underline{Y}_{\sigma(i)} \le y_{\sigma(i)};\ i=1, \cdots, k\}$$

and so we have proved the result in (a)

In order to prove the result in b) we note that for every subset $A \subset \{1, 2, \cdots, n\}$ and every function $\pi\colon \{1, 2, \cdots, n\} \to \{-1, +1\}$

$$\mathbb{P}\{(\underline{i_k}) = \pi(i_k);\ k \in A\} =$$

$$= \sum_{\sigma} \mathbb{P}\{\underline{Y}_{\sigma(i)} \le \tfrac{1}{2}, \underline{Y}_{\sigma(1)} \le \underline{Y}_{\sigma(2)} \le \cdots \le \underline{Y}_{\sigma(n)}, (\underline{\sigma(k)}) = \pi(\sigma(k));\ k \in A\}$$

$$= \sum_{\sigma} \mathbb{P}\{\underline{Y}_{\sigma(i)} \le \tfrac{1}{2}, \underline{Y}_{\sigma(1)} \le \cdots \le \underline{Y}_{\sigma(n)}\}\ \mathbb{P}\{(\underline{\sigma(k)}) = \pi(\sigma(k));\ k \in A\}$$

where we have used (a) to obtain the last equality.

Hence

$$\mathbb{P}\{(\underline{i_k}) = \pi(i_k);\ k \in A\} =$$

$$= 2^{-|A|} \sum_{\sigma} \mathbb{P}\,\{\underline{Y}_{\sigma(1)} \leq \cdots \leq \underline{Y}_{\sigma(n)}, \underline{Y}_{\sigma(i)} \leq \tfrac{1}{2}\,;\, i = 1, \cdots, n\}$$

$$= 2^{-|A|} = \prod_{k \in A} \mathbb{P}\,\{(\underline{i_k}) = \pi(i_k)\}$$

□

References

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[3] Feller, W., An Introduction to Probability Theory and its Applications, vol II, Wiley, 1971.