

On a pairing heuristic in binpacking

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EINDHOVEN UNIVERSITY OF TECHNOLOGY

Faculty of Mathematics and Computing Science

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On a pairing heuristic
in binpacking

by

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ON A PAIRING HEURISTIC IN BINPACKING

ABSTRACT

For the analysis of a pairing heuristic in binpacking an important result is used without proof in [1] and [2].

In this note we discuss this result and give a detailed proof of it.

Introduction

Let $n \in \mathbb{N}$ be given and suppose (X_1, \dots, X_n) is a n -dimensional stochastic vector with joint density $f(x_1, \dots, x_n)$

Moreover assume

- (i) $0 \leq X_i \leq 1 \quad i = 1, \dots, n$
- (ii) The stochastic vector $(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)})$ is distributed as (X_1, X_2, \dots, X_n) for every permutation σ on $\{1, \dots, n\}$.
- (iii) $f(x_1, x_2, \dots, x_n) = f(1-x_1, \dots, x_n)$

Remark

Condition (ii) states that we are dealing with a finite sequence of so-called exchangeable random variables (cf. [3]), while condition (iii) is a symmetry condition.

Note that by (ii) the symmetry in (iii) holds in every component.

Before stating the main result introduce the following notations

$$1_A := \begin{cases} 1 & \text{if the event } A \text{ happens} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_i := (1 - X_i)1_{\{X_i > \frac{1}{2}\}} + X_i 1_{\{X_i \leq \frac{1}{2}\}} \quad i = 1, \dots, n$$

$$(i) := \begin{cases} +1 & \text{if } X_i > \frac{1}{2} \\ -1 & \text{if } X_i \leq \frac{1}{2} \end{cases} \quad i = 1, \dots, n$$

If we order the random variables Y_i in non-decreasing order, say $Y_{i_1} \leq Y_{i_2} \leq \dots \leq Y_{i_n}$, we denote by (i_k) the label of the k -order statistic of the sequence $\{Y_i\}_{i=1}^n$.

Now the main result reads as follows.

Theorem 1

Suppose the random variables $\{X_i\}_{i=1}^n$ satisfy the conditions (i), (ii) and (iii).

Then the following results hold

a) $\{Y_i, i \in A\}$ and $\{\underline{I}_i, i \in A\}$ are independent for every subset $A \subset \{1, 2, \dots, n\}$

$$\begin{aligned} \text{b) } P\{(\underline{I}_k) = \pi(i_k), k \in A\} &= \prod_{k \in A} P\{(\underline{I}_k) = \pi(i_k)\} \\ &= 2^{-|A|} \end{aligned}$$

for every subset $A \subset \{1, 2, \dots, n\}$ and

for every function $\pi: \{1, 2, \dots, n\} \rightarrow \{-1, 1\}$.

Proof For every sequence $\{y_i\}_{i=1}^n$ with $y_i \in (0, \frac{1}{2})$ and σ some permutation on $\{1, \dots, n\}$ we obtain

$$\begin{aligned} P\{Y_{\sigma(i)} \leq y_{\sigma(i)}, (\underline{\sigma}(i)) = \pi(\sigma(i)), i=1, \dots, k\} &= \\ = P\{1 - X_{\sigma(i)} \leq y_{\sigma(i)} (i \in C) \wedge X_{\sigma(i)} \leq y_{\sigma(i)} (i \in \{1, \dots, k\} - C)\} \end{aligned}$$

where $1 \leq k \leq n$ and $C := \{j: 1 \leq j \leq k \ \& \ \pi(\sigma(j)) = 1\}$

By (ii) and (iii) it follows easily

$$P\{Y_{\sigma(i)} \leq y_{\sigma(i)}, (\underline{\sigma}(i)) = \pi(\sigma(i)) \ i = 1, \dots, k\} =$$

$$(1) \quad P\{X_{\sigma(i)} \leq y_{\sigma(i)}; i = 1, \dots, k\} = P\{X_i \leq y_{\sigma(i)}; i = 1, \dots, k\}$$

and this implies

$$\begin{aligned} P\{Y_{\sigma(i)} \leq y_{\sigma(i)}; i = 1, \dots, k\} &= \\ = \sum_{\tau \in D} P\{Y_{\sigma(i)} \leq y_{\sigma(i)}, (\underline{\sigma}(i)) = \tau(\sigma(i)); i = 1, \dots, k\} &= \end{aligned}$$

$$(2) \quad = \sum_{\tau \in D} P\{X_i \leq y_{\sigma(i)}; i = 1, \dots, k\} = 2^k P\{X_i \leq y_{\sigma(i)}; i = 1, \dots, k\}$$

where D is the set of functions $\tau: \{1, 2, \dots, n\} \rightarrow \{-1, +1\}$ which are different on $\{\sigma(1), \dots, \sigma(k)\}$.

Moreover by (1)

$$\begin{aligned}
 & \mathbb{P} \{(\underline{\sigma(i)}) = \pi(\sigma(i)); i = 1, \dots, k\} = \\
 & = \mathbb{P} \{Y_{\sigma(i)} \leq \frac{1}{2}, (\underline{\sigma(i)}) = \pi(\sigma(i)), i = 1, \dots, k\} = \\
 (3) \quad & = \mathbb{P} \{X_i \leq \frac{1}{2}; i = 1, \dots, k\} .
 \end{aligned}$$

Since the density $f(x_1, \dots, x_n)$ is symmetric it is easy to prove that for every $1 \leq l \leq n-1$

$$\mathbb{P} \{X_1 \leq \frac{1}{2}, \dots, X_l \leq \frac{1}{2}\} = 2 \mathbb{P} \{X_1 \leq \frac{1}{2}, \dots, X_{l+1} \leq \frac{1}{2}\}$$

and this implies using $\mathbb{P} \{X_1 \leq \frac{1}{2}\} = \frac{1}{2}$ that

$$(4) \quad \mathbb{P} \{X_1 \leq \frac{1}{2}, \dots, X_l \leq \frac{1}{2}\} = 2^{-l}$$

Now by the relations (1), (2), (3) and (4)

$$\begin{aligned}
 & \mathbb{P} \{Y_{\sigma(i)} \leq y_{\sigma(i)}, (\underline{\sigma(i)}) = \pi(\sigma(i)); i=1, \dots, k\} = \\
 & \mathbb{P} \{X_i \leq y_{\sigma(i)}; i = 1, \dots, k\} = \\
 & 2^{-k} \cdot 2^k \mathbb{P} \{X_i \leq y_{\sigma(i)}; i = 1, \dots, k\} = \\
 & \mathbb{P} \{(\underline{\sigma(i)}) = \pi(\sigma(i)); i=1, \dots, k\} \cdot \mathbb{P} \{Y_{\sigma(i)} \leq y_{\sigma(i)}; i=1, \dots, k\}
 \end{aligned}$$

and so we have proved the result in (a)

In order to prove the result in b) we note that for every subset $A \subset \{1, 2, \dots, n\}$ and every function $\pi: \{1, 2, \dots, n\} \rightarrow \{-1, +1\}$

$$\begin{aligned}
 & \mathbb{P} \{(\underline{i_k}) = \pi(i_k); k \in A\} = \\
 & = \sum_{\sigma} \mathbb{P} \{Y_{\sigma(i)} \leq \frac{1}{2}, Y_{\sigma(1)} \leq Y_{\sigma(2)} \leq \dots \leq Y_{\sigma(n)}, (\underline{\sigma(k)}) = \pi(\sigma(k)); k \in A\} \\
 & = \sum_{\sigma} \mathbb{P} \{Y_{\sigma(i)} \leq \frac{1}{2}, Y_{\sigma(1)} \leq \dots \leq Y_{\sigma(n)}\} \mathbb{P} \{(\underline{\sigma(k)}) = \pi(\sigma(k)); k \in A\}
 \end{aligned}$$

where we have used (a) to obtain the last equality.

Hence

$$\mathbb{P} \{(\underline{i_k}) = \pi(i_k); k \in A\} =$$

$$= 2^{-|A|} \sum_{\sigma} P \{ Y_{\sigma(1)} \leq \dots \leq Y_{\sigma(n)}, Y_{\sigma(i)} \leq \frac{1}{2}; i = 1, \dots, n \}$$

$$= 2^{-|A|} = \prod_{k \in A} P \{ (i_k) = \pi(i_k) \}$$

□

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