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EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

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Rapid variation with remainder and rates of convergence

by

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Eindhoven, Netherlands

February 1988

Rapid variation with remainder and rates of convergence

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ABSTRACT. The remainder term of the class Γ of rapidly varying functions is considered. Some probabilistic applications to limit laws of extreme value theory and to the estimation of the indexparameter of a regularly varying tail are considered.

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1. INTRODUCTION

Let $U:\mathbb{R}\to\mathbb{R}^+$ be a measurable function such that

$$\lim_{x\to\infty} U(tx)/U(x) = t^{\alpha} \quad \text{for every } t>0.$$

Then U is called *regularly varying with index* α ($U\in R_\alpha$). If $\alpha=0$ we say that U is *slowly varying*, while if $\alpha=\infty$ U is called *rapidly varying*.
An important class of rapidly varying functions is the so-called class Γ, introduced by de Haan (1970):

let $f:\mathbb{R}\to\mathbb{R}^+$ be a measurable function, then $f\in\Gamma$ iff there exists a measurable function ϕ: $\mathbb{R}\to\mathbb{R}^+$ such that

$$(1.1) \qquad \lim_{x\to\infty} f(x+u\phi(x))/f(x) = exp(u)$$

locally uniformly (l.u.) in $u\in\mathbb{R}$.

If (1.1) holds, we call ϕ an auxiliary function of f (notation $f\in\Gamma(\phi)$) and it is known that in this case ϕ is *self-neglecting* (see de Haan (1970)):

$$(1.2) \qquad \lim_{x\to\infty} \phi(x+u\phi(x))/\phi(x) = 1$$

l.u. in $u\in\mathbb{R}$.
At this point, notice that our definition of Γ is somewhat more general than the one given by de Haan (1970) as he restricts the class Γ to monotone functions which satisfy (1.1) pointwise in $u\in\mathbb{R}$.
By far the most important probabilistic application of Γ is the characterization of the domain of attraction of the double exponential law in the maximum-scheme: let $X_{1:n}\le X_{2:n}\le\ldots\le X_{n:n}$ denote the order statistics of a sample of size n from a distribution function (df) F. We denote $\bar{F}=1-F$. Then one can find normalizing constants $a_n>0$ and b_n such that for all $x\in\mathbb{R}$,

$$P(X_{n:n}-b_n\le a_n x) \to exp(-exp(-x)) =: \Lambda(x), \quad n\to\infty$$

$$iff$$

$$1/\bar{F} \in \Gamma$$

Another characterizing property of Γ concerns the Hill estimator (Hill(1975), Beirlant and Teugels(1987)): if $F\in C:=\{F|F(0)=0$, F continuous and eventually strictly increasing$\}$, then Hill's estimate

$$H_{m,n}:=m^{-1}\sum_{i=1}^{m} \log X_{n-i+1:n} - \log X_{n-m:n}$$

is attracted as $n\to\infty$ to the gamma law of $m^{-1}\sum_{i=1}^{m} E_i$ ($E_i, i=1,\dots,m$, iid exponential random variables with mean one) iff $1/F\circ exp\in\Gamma$.

Both examples suggest that that we can obtain second order theorems if we could specify (1.1) up to a remainder term. We therefore consider the following asymptotic relations:

let r be a measurable function from $\mathbb{R}$ to $\mathbb{R}^+$ such that $r(x)\to 0$ as $x\to\infty$. Then

(ΓR_1) $\quad f(x+u\phi(x))/f(x) = e^u(1+O(r(x))) \qquad (x\to\infty)$ l.u. in $u\in\mathbb{R}$

(ΓR_2) $\quad f(x+u\phi(x))/f(x) \sim e^u(1+m(u)r(x)) \qquad (x\to\infty)$ l.u. in $u\in\mathbb{R}$

(ΓR_3) $\quad f(x+u\phi(x))/f(x) = e^u(1+o(r(x))) \qquad (x\to\infty)$ l.u. in $u\in\mathbb{R}$.

If f satisfies one of the relations (ΓR_i) (i=1,2,3), with auxiliary functions ϕ and r, we denote it as $f\in\Gamma R_i(\phi,r)$.

It is well-known that Γ is strongly connected with the class Π of slowly varying functions (de Haan (1970)): if f is non-decreasing, $f\in\Gamma(\phi)$ iff

$$(1.3) \qquad \lim_{x\to\infty} (f^{l}(xt) - f^{l}(x))/\phi(f^{l}(x)) = \log(t) \qquad \text{for every } t>0$$

where f^{l} is the inverse of f. We denote (1.3) as $f^{l}\in\Pi(\phi(f^{l}))$.

Similarly as for Γ, we can define remainder versions of Π-variation (see Omey and Willekens(1987)):

for positive measurable functions a and b, consider

(ΠR_1) $\quad f(xt) - f(x) - a(x)\log(t) = O(b(x)) \quad (x\to\infty)$

(ΠR_2) $\quad f(xt) - f(x) - a(x)\log(t) \sim h(u)b(x) \quad (x\to\infty)$

(ΠR_3) $\quad f(xt) - f(x) - a(x)\log(t) = o(b(x)) \quad (x\to\infty)$.

Similarly as above, we use the notation $f\in\Pi R_i(a,b)$, $i=1,2,3$.

As one might expect and as was shown by de Haan and Dekkers (1987), the stated relationship between Γ and Π (see (1.3)) maintains (under appropriate conditions) for the remainder versions, i.e.

$$f\in\Gamma R_i(\phi,r) \quad iff \quad f^{\leftarrow}\in\Pi R_i(\phi(f^{\leftarrow}),\phi(f^{\leftarrow})r(f^{\leftarrow})),\quad i=1,2,3.$$

In the next section we define a transform which also relates the classes ΓR_i and ΠR_i, but which is also valid for non-monotone functions. The analytic results of section 2 are then applied in section 3 to establish rates of convergence in the previously mentioned examples. Before starting with section 2, we notice that ΠR_i is closely related to the concept of slow variation with remainder (SR_i) as defined in Goldie and Smith (1987). Indeed, if $b(x)\to\infty$ $(x\to\infty)$, we have for any function f that $f\in\Pi R_i(0,b)$ iff $\exp f\in SR_i(b)$.

2. SOME ANALYTIC RESULTS

As in Goldie and Smith (1987) and Omey and Willekens (1987) it will be appropriate to impose some conditions on the remainder term r in ΓR_i $(i=1,2,3)$. Unless otherwise stated, we will assume that

(2.1) $$\lim_{x\to\infty} r(x+u\phi(x))/r(x) = \exp(\gamma u) \quad \text{for every } u\in\mathbb{R} \text{ and some } \gamma\le 0.$$

Clearly the limit in (2.1) can only be of the stated form. In the proof of our theorems we will frequently use the following proposition, due to Bingham and Goldie (1983).

Proposition. *Let ϕ be self-neglecting, g satisfy*

$$(2.2) \qquad (g(x+u\phi(x)) - g(x))/z(x) \to 0 \qquad (x\to\infty)$$

with z a measurable function satisfying

$$z(x+u\phi(x))/z(x) \to exp(\gamma u) \qquad (x\to\infty)\ ,\gamma\leq 0,\ u\in\mathbb{R}.$$

Then (2.2) holds uniformly on compact u-sets.

We now define the transform which will be considered in the forthcoming theorem: suppose ϕ is bounded away from zero on any finite interval, and let

$$(2.3) \qquad \Phi(x) := \int_0^x dt/\phi(t), \qquad x\in\mathbb{R}.$$

Then Φ is a strictly increasing continuous function whose inverse is well-defined. Define for any $f\in\Gamma(c\phi), c\in\mathbb{R}_0$,

$$A: f \to A_f := f \circ \Phi^{I} \circ log.$$

It follows from de Haan (1973) that any function f in $\Gamma(\phi)$ can be represented as

$$(2.4) \qquad f(x) = U(exp\Phi(x)) \qquad \text{with } U\in R_I.$$

Clearly with the definition of A, (2.4) implies that $A_f=U$ whence $logA_f \in \Pi(1)$. So the operator A provides an obvious relation between Γ and Π, and it is not hard to imagine that we can expect a similar relation between ΓR_1 and ΠR_1.

Before stating the main theorem of this section, we first consider the function Φ somewhat closer.

By local uniformity in (1.2), we have for any $u\in\mathbb{R}$,

$$\begin{aligned}\Phi(x+u\phi(x)) - \Phi(x) &= \int_x^{x+u\phi(x)} dt/\phi(t)\\ &= \int_0^u \phi(x)/\phi(x+v\phi(x))\ dv\\ &= u + o(1) \qquad (x\to\infty)\end{aligned}$$

Conversely, fixing any number $u \in \mathbb{R}$, one can find $t=t(x)$ with $t(x) \to u$ as $x \to \infty$ such that (cfr. Bingham and Goldie (1983))

(2.5) $$\Phi(x+t(x)\phi(x)) = \Phi(x) + u.$$

This relation is very useful and will be used throughout in the sequel of the paper. We now state our main theorem.

Theorem 2.1. *Let ϕ be self-neglecting and let r satisfy (2.1). For any $i \in \{1,2,3\}$ the following assertions are equivalent:*

(i) $f \in \Gamma R_i(\phi,r)$

(ii) $\log A_f \in \Pi R_i(1,A_r)$ and $A_\phi \in SR_i(A_r)$

(iii) $f(x) = \exp\Phi(x)\ V(\exp\Phi(x))$ with $A_\phi \in SR_i(A_r)$ and $V \in SR_i(A_r)$.

Proof. We first prove the theorem for i=2.

(i)$\Rightarrow$(ii). From the definition of A_f we have that $f(x)=A_f(\exp\Phi(x))$. *Therefore,*

$$f \in \Gamma R_2\ (\phi,r)$$

iff

$$A_f(\exp\Phi(x+u\phi(x)))/A_f(\exp\Phi(x)) - \exp(u) \sim e^u m(u)\ r(x) \quad (x \to \infty)$$

Now with $t(x)$ defined as in (2.5), it follows from local uniformity that

(2.6) $$A_f(e^{\Phi(x)+u})/A_f(e^{\Phi(x)}) - e^u - e^u(t(x)-u) \sim e^u m(u) r(x) \quad (x \to \infty).$$

We first determine the order of $t(x) - u$. Defining

(2.7) $$R_u(x) := (\log f(x+u\phi(x)) - \log f(x) - u)/r(x),$$

we have that

$$R_{u+v}(x) = R_{v(x)}(x+u\phi(x))\ r(x+u\phi(x))/r(x) + R_u(x) + v\{-1 + (\phi(x)/\phi(x+u\phi(x)))\}/r(x)$$

with $v(x)=v\phi(x)/\phi(x+u\phi(x))$.
Then by ΓR_2 and (2.1),

(2.8) $$\lim_{x\to\infty} (\phi(x+u\phi(x))-\phi(x))/\phi(x)r(x) \text{ exists.}$$

Denoting the limit in (2.8) as $k(u)$, it is not hard to show that

$$k(u) = a\int_1^{e^u} \theta^{\gamma-1}d\theta.$$

with a a real constant and γ determined by (2.1).
Using Proposition one can show that convergence in (2.8) holds l.u. in $u\in\mathbb{R}$, so that

$$(\Phi(x+t\phi(x))-\Phi(x)-t)/r(x)= (r(x))^{-1}\int_0^t \{(\phi(x)/\phi(x+u\phi(x))) - 1\}\, du$$
$$\to h(t):= -\int_0^t k(u)du\ ,$$

l.u. in $t\in\mathbb{R}$.
This implies that the function $t(x)$ in (2.5) is of the form

$$t(x)=u-h(u)r(x)+o(r(x)) \qquad (x\to\infty).$$

Then clearly from (2.6), after a change of variables $(y=exp\Phi(x), \lambda=e^u)$

$$logA_f(y\lambda) - logA_f(y) - log\lambda \sim (m(log\lambda)-h(log\lambda))\ A_r(y) \quad (y\to\infty)$$

showing that $logA_f \in \Pi R_2(1,A_r)$.

The fact that $A_\phi \in SR_2(A_r)$ follows immediately from (2.8), local uniformity and the definition of $t(x)$.

(ii)$\Rightarrow$(iii). Obviously $logA_f\in\Pi R_2(1,A_r)$ iff $V(x):=log(A_f(x))/x) \in SR_2(A_r)$.
The representation theorem follows then immediately.

(iii)$\Rightarrow$(i). Immediate.

The proof of the theorem for i=1 or 3 follows exactly the same lines.

Only the limit relations have to be changed in O- or o-versions. □

Remarks.

1. It follows from (2.5) that r satisfies (2.1) iff $A_r \in R_\gamma$.
Clearly for proving Theorem 2.1 if i=1 (i=3), the assumption on r in (2.1) can be relaxed to $r(x+u\phi(x))=O(r(x))$ $(o(r(x)))$ as $x\to\infty$. This then implies that A_r is O-regularly varying (see Goldie and Smith (1987)).

2. Theorem 2.1 implies that if $\gamma<0$ in (2.1), any function f satisfying ΓR_1 is essentially an exponential function. Indeed, if we consider ΓR_2 it follows from $V\in SR_2(A_r)$ and Seneta(1976) (pp. 73-74) that there exists constants c and $d\neq 0$ such that

$$(2.9) \qquad V(\exp\Phi(x)) = d + cr(x) + o(r(x)) \qquad (x\to\infty).$$

For the same reason, there exists constants $c_0\neq 0$ and c_1 such that

$$c_0/\phi(x) = \exp(c_1 r(x) + o(r(x))) \qquad (x\to\infty),$$

from which

$$(2.10) \qquad \Phi(x) = c_2 + x\,c_0^{-1} + c_1 c_0^{-1}\int_{c_3}^{x} r(u)(1+o(1))du.$$

Combining (2.9) and (2.10) we have from Theorem (2.1) that

$$f(x) = (d + cr(x) + o(r(x)))\,\exp(c_2 + xc_0^{-1} + c_1 c_0^{-1}\int_{c_3}^{x} r(u)(1+o(1))du)$$

$$(x\to\infty).$$

3. The proof of Theorem 2.1 shows that from ΓR_2 and (2.1)

$$m(u+v) = m(v)\exp(\gamma u) + m(u) - av\int_1^{e^u} \theta^{\gamma-1}d\theta \qquad (a\in\mathbb{R}).$$

Hence

$$(2.11)\qquad m(u) = \begin{cases} d\gamma^{-1}(\exp(\gamma u) - 1) - a\gamma^{-1}\int_0^u (\exp(\gamma v)-1)dv \quad (d\in\mathbb{R}) & \text{if } \gamma\neq 0 \\ cu - au^2/2 & \text{if } \gamma=0. \end{cases}$$

3. APPLICATIONS IN EXTREME VALUE THEORY

a. Rate of convergence for maxima in domain of attraction of the double exponential distribution.

Let $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$ be an ordered sample from a df F with $1/(-\log F)\in\Gamma(\phi)$. Take $-\log F(b_n)\leq n^{-1}\leq -\log F(b_n-)$ and let $a_n=\phi(b_n)$.

Then it is well-known that (see de Haan (1970))

$$(3.1)\qquad P(X_{n:n}\leq a_n u+b_n)\to\Lambda(u):=\exp(-e^{-u}),\qquad u\in\mathbb{R}.$$

As was mentioned in the introduction, strenghtening the condition $1/(-\log F)\in\Gamma(\phi)$ to $1/(-\log F)\in\Gamma R_i(\phi,r)$ for some $i\in\{1,2,3\}$ and $r(x)\to 0$, will allow us to to study the rate of convergence in (3.1). Indeed, it is easily seen that if $f\in\Gamma R_1(\phi,r)$,

$$(3.2)\qquad \Lambda_n(u) := P(X_{n:n}\leq a_n u+b_n) - \Lambda(u) = O(r(b_n))\qquad (n\to\infty)$$

while if $f\in\Gamma R_2(\phi,r)$,

$$(3.3)\qquad \Lambda_n(u) = \Lambda'(u)m(u)r(b_n) + o(r(b_n))\qquad (n\to\infty)\ .$$

Whereas (3.2) and (3.3) give pointwise rates of convergence, the main problem is to show that they hold uniformly in $u\in\mathbb{R}$.
Although many papers have been devoted to the uniform rate of convergence in (3.1), (see e.g. Anderson (1971), Cohen (1982), Omey and Rachev (1987), Resnick (1986)), it is still an open problem to give the most general conditions that imply the right rate; nearly all of the existing results work by

specifying the Von Mises conditions.
The contrary is true when attraction to the Frechet or Weibull law is concerned. Indeed in these cases Smith (1982) formulated best possible conditions in terms of slow variation with remainder.
In the theorem below, we prove that (3.2) ((3.3)) holds uniformly in $u \in \mathbb{R}$ under ΓR_1 (ΓR_2). Using the theory of section 2, we show that ΓR can be reduced to slow variation with remainder so that we end up exactly with the same problem as was tackled by Smith (1982).
We believe that the present way of proof is properly motivated from the concept of Γ-variation with remainder and that it generalizes the approaches used in the references mentioned above.
The only minor drawback is the following assumption which will be used in the theorem:

$$(3.4) \qquad F^n(b_n - \Phi(b_n)\phi(b_n)) = o(r(b_n)) \qquad (n \to \infty) .$$

Condition (3.4) holds in most instances and may not be satisfied if ϕ is slowly varying with a specified remainder term. The following lemma ensures this statement.

Lemma 3.1. *If any subsequence* $(b'_n)_n$ *of* $(b_n)_n$ *for which*

$$(i) \qquad b'_n - \Phi(b'_n)\phi(b'_n) \to \infty$$

and

$$(ii) \qquad \Phi(b'_n)\phi(b'_n)/b'_n \to 1 \qquad (n \to \infty)$$

satisfies $\quad F^n(b'_n - \Phi(b'_n)\phi(b'_n)) = o(r(b'_n)) \quad (n \to \infty)$

then (3.4) *holds in case* F *is concentrated on an interval of the form* $[z, +\infty)$.

Proof. In case F is concentrated on intervals of the specified type the only subsequences we have to consider are those for which (i) holds.
If furthermore $\limsup_{n \to \infty} \Phi(b'_n)\phi(b'_n)/b'_n < 1$ (3.4) follows from Davis and Resnick (1986) and the fact that A_r is O-regularly varying. In case the *limsup* equals *1* (3.4) follows from the assumptions. □

The theorem reads as follows.

Theorem 3.1. *Suppose ϕ is self-neglecting*

a. *If $1/(-\log F) \in \Gamma R_1(\phi,r)$ and (3.4) is satisfied, and if there exist constants x_0,θ,b,c all positive such that*

(3.5) $\quad bx^{-\theta} \le A_r(xt)/A_r(x) \le c \qquad$ *for all* $x \ge x_0, t \ge 1,$

then

$$\sup_{u\in\mathbb{R}} |\Lambda_n(u)| = O(r(b_n)) \qquad (n\to\infty).$$

b. *If $1/(-\log F) \in \Gamma R(\phi,r)$ and (3.4) is satisfied and if $A_r \in R_\gamma, \gamma\le 0$, then*

$$\Lambda_n(u) = \Lambda'(u)m(u)r(b_n) + o(r(b_n)) \qquad (n\to\infty)$$

uniformly in $u\in\mathbb{R}$. Here $m(u)$ is given as in (2.11).

Proof. First notice that we may assume that F is suported on $[z,\infty)$ for some $z\in\mathbb{R}$ Indeed, putting $Y_i:=max(z,X_i)$, $i=1,\dots,n$, it is clear that for for n large enough

$$\sup_{u\in\mathbb{R}} |P(Y_{n:n}\le a_n u+b_n) - P(X_{n:n}\le a_n u+b_n)| = P(X_{n:n}\le z) = o(r(b_n))$$
$$(n\to\infty)$$

where the last inequality follows from the definition of b_n and (3.5). We now estimate

$$\Delta_n(x) := |\ -\log(\ -n\log F(a_n\log x+b_n)) - \log x\ | \quad \text{for some} x>0.$$

Denoting $-\log F:=f$ and $\exp\Phi(b_n)=:\nu_n$, we have from (2.5) and the definition of b_n that

$$\begin{aligned}\Delta_n(x) &= |\ -\log(\ f(a_n\log x+b_n)) + \log f(b_n) - \log x + o(r(b_n))| \\ &= |\ -\log A_f(\ x\exp(\Phi(b_n)+t(b_n))\ ) + \log A_f(\exp\Phi(b_n)) - \log x \\ &\qquad + o(r(b_n))| \\ &\le |-\log(A_f(\nu_n x)/\nu_n x) + \log(A_f(\nu_n)/\nu_n) + (t(b_n)-\log x) + o(r(b_n))|\end{aligned}$$

Now by Theorem 2.1, $L(x):=A_f(x)/x \in SR_i(A_r)$ with i=1 or 2 depending on whether a. or b. is satisfied . Using the estimation in (3.6), we can copy the proofs of Theorems 1 and 2 of Smith (1982), implying uniform convergence in (3.2) and (3.3) over the region $u=\log x \geq -\log\nu_n + c$, where c is some constant. Hence the proof is finished if we can show that both $\Lambda(-\log\nu_n)$ and $F^n(-a\log\nu_n + b_n)$ are $o(r(b_n))$ as $n\to\infty$. Under the conditions of the theorem, A_r is O-regularly varying such that $\Lambda(-\log\nu_n)=\exp(-\nu_n)=o(A_r(\nu_n))=o(r(b_n))$ $(n\to\infty)$. As to $F^n(-a_n\log\nu_n+b_n)$, notice that $-a_n\log\nu_n+b_n= b_n-\Phi(b_n)\phi(b_n)$. Lemma 3.1 applies now. □

b. Rate of convergence of Hill's estimate.

Beirlant and Teugels (1987) showed that if $1/(1-F)\circ\exp$ belongs to Γ, and F is continuous and strictly increasing in a neigborhood of ∞, Hill's estimate $H_{m,n}$ given in (1.2) is attracted as $n\to\infty$ to the distribution of $m^{-1}\sum_{i=1}^{m} E_i =: E_m$, E_i being iid exponential rv's with mean one.

Let now ϕ be the auxiliary function corresponding to $1/F\circ\exp$,

$l(u) = \phi(\log F^{\leftarrow}(1-u^{-1}))$, $u \in (0,1)$; $q_n=m/n$, and $p_n^2=m(n-m-1)/(n-1)^3$.

Then it was also shown that $n\to\infty, m_n\to\infty, m_n=o(n)$, and

$$\text{(3.7a)} \quad (m_n)^{1/2}\Big(-1 + \int_0^{\infty} (q_n+p_nz)^{-1} \big(1-F(e^{ul(n/m)}F^{\leftarrow}(1-q_n-p_nz))\big)\, du \Big) \to 0 \quad \text{l.u. in } z>0$$

entail that

$$(m_n)^{1/2} \big(H_{m,n}/l(n/m) - 1 \big) \xrightarrow{d} N(0,1) .$$

If we assume that $1/F\circ\exp \in \Gamma R_1(\phi,r)$ then it is clear that condition (3.7a) can be replaced by the more attractive condition

(3.7b) $(m_n)^{1/2} r(\log F^{l}(1-q_n-p_n z)) \to 0$ as $n\to\infty, m_n\to\infty, m_n=o(n)$

l.u. in $z>0$.

With the help of the Berry-Esseen theorem

$$\sup_x \mid P((m_n)^{1/2}(H_{m,n}/l(n/m) - 1) \le x) - \Phi(x) \mid$$

$$\le \tau_n + Cm^{-1/2}$$

where $\tau_n = \sup_x \mid P((m_n)^{1/2}(H_{m,n}/l(n/m) - 1) \le x) - P((m_h)^{1/2}(E_m - 1) \le x) \mid$.

To bound τ_n we apply the well-known smoothing inequality:

$$\tau_n \le \pi^{-1}\int_{-T}^{T} t^{-1} \mid \psi_{m,n}(t) - \kappa_m(t) \mid dt + KT^{-1},$$

where $\psi_{m,n}$, resp. κ_m, denote the characteristic functions of the standardized versions of $H_{m,n}$, resp. E_m, given in Beirlant and Teugels (1987).

We get by choosing $T=m^{1/2}$ that

$$\tau_n \le Km^{-1/2} + A(n,m)\, \pi^{-1}(m!)^{-1} \int_{-m_n^{1/2}}^{m_n^{1/2}} t^{-1} dt \int_0^n (1-v/n)^{n-m-1} v^m \Big| K_n^m(v, m^{\frac{1}{2}}/t\, l(n/m)) - \Big(1+i\int_0^\infty e^{iw-(wm^{\frac{1}{2}}/t)} dw\Big)^m \Big| dv$$

where $A(n,m) = O(1)$ as $n\to\infty$, $m=o(n)$ and

$$K_n(v,u) = 1 + i\int_0^\infty e^{iw}(\bar{F}(e^{w/u}\bar{F}^{i}(1-v/n)))/(\bar{F}(\bar{F}^{i}(1-v/n)))\, dw.$$

First

$$\max\{|K_n(v,u/l(n/m))|,|(1 + i\int_0^\infty e^{iw-w/u}dw)|\} \le 1$$

and if $1/\bar{F}oexp$ satisfies (ΓR_1) we get as $n\to\infty$

$$|K_n(v,u/l(n/m))-(1+i\int_0^\infty e^{iw-w/u}dw| \le e^{-w/u}\, O(r(\log\bar{F}^{i}(1-v/n)))$$

so that by substituting $t/m^{1/2}$ by u we find

$$\tau_n \le Km^{-1/2} + Lm\int_{-1}^{1} du\int_0^{n} (m!)^{-1}(1-v/n)^{n-m-1}v^{m}$$

$$\times|O(r(\log\bar{F}^{i}(1-v/n)))|\, dv$$

for a certain constant L.

So we have derived the following theorem.

Theorem 3.2. *Suppose* $\bar{F}$ *is continuous and strictly increasing in a neighborhood of* ∞. *Moreover assume that* $1/\bar{F}oexp \in \Gamma R_1(\phi,r)$ *and that* (3.1b) *holds.*

Then there exists a positive constant C *such that*

$$\sup_{x\in\mathbb{R}}|P(m^{1/2}(H_{m,n}/l(n/m) - 1) \le x) - \Phi(x)|$$

$$\le C(m^{-1/2} + m\, E(\psi_{m,n,F}(mn^{-1}E_{m+1})))$$

as $n\to\infty, m_n\to\infty$, $m_n=o(n)$, *where* $\psi_{m,n,F}(x) = O(r\ o\ \log\bar{F}^{i}(1-x^{-1}))$ *as* $x\to\infty$.

The above result generalizes results of Falk (1985), who derives rates of convergence for $H_{m,n}$ in more specific models.

References.

Anderson C.W. (1971). Contributions to the asymptotic theory of extreme values. Ph. D; Thesis, University of London .

Beirlant J. and Teugels J.L. (1987). Asymptotics of Hill's estimator. *Th. Probability Appl. 31*, 463-469.

Bingham N.H. and Goldie C.M. (1983). On one-sided Tauberian theorems. *Analysis* 3,159-188.

Bingham N.H., Goldie C.M. and Teugels J.L. (1987). *Regular Variation.* Encyclopedia of Math. and its Applic. 27, Cambridge University Press

Cohen J.P. (1982). Convergence rates for the ultimate and penultimate approximations in extreme-value theory. *Adv. Appl. Prob. 14*, 833-854.

Davis R. and Resnick S.I. (1986). Extremes of moving averages of random variables from the double exponential distribution. Technical Report, Colorado State University.

De Haan L. (1970). On regular variation and its application to the weak convergence of sample extremes. *Math. Centre Tract 32*, Amsterdam.

De Haan L. (1974). Equivalence classes of regularly varying functions. *Stoch. Proc. Appl. 2*, 243-259.

De Haan L. and Dekkers A. (1988). On a consistent estimate to the index of an extreme value distribution. To appear Ann. Probability.

Falk M. (1985). Uniform convergence of extreme order statistics. Habilitationsthesis, University of Siegen.

Goldie C.M. and Smith R.L. (1987). Slow variation with remainder: theory and applications. *Quarterly J. Math. Oxford 38*,45-71.

Hill B.M. (1975). A simple general approach to inference about the tail of a a distribution. *Ann. Statist.* 3,1163-1174.

Omey E. and Willekens E. (1987). π-variation with remainder. To appear *Journal London Math. Soc.*.

Omey E. and Rachev S. (1987). On the rate of convergence in extreme value theory. To appear Th. Prob. Appl.

Reiss R.D. (1987). Estimating the tail index of the claim size distribution. *Blatter DGVM 1*,21-25.

Resnick S.I. (1986). Uniform rates of convergence to extreme value distributions. J. Srivastava (ed.) Probability and Statistics: Essays in honor of F.A. Graybill, N. Holland, Amsterdam.

Seneta E. (1974). Regularly varying functions. *Lecture Notes in Mathematics 508*. Springer Verlag, Berlin.

Smith R.L. (1982). Uniform rates of convergence in extreme-value theory. *Adv. Appl. Prob. 14*, 600-622.