

Hankel transformations and spaces of type S

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HANKEL TRANSFORMATIONS AND SPACES OF TYPE S

by

S.J.L. van Eijndhoven and C.A.M. van Berkel

Summary

There exist growth estimates on even functions $\phi \in L_1(\mathbb{R})$ and on their Hankel transforms $\mathbb{H}_v \phi$ which are necessary and sufficient for ϕ to belong to the even subspaces of Gelfand-Shilov's S_{α}^{β} -spaces. Consequently, $\mathbb{H}_v(S_{\alpha,\text{even}}^{\beta}) = S_{\beta,\text{even}}^{\alpha}$. Further, $S_{\alpha,\text{even}}^{\beta}$ remains invariant under the fractional differentiation/integration operators $(\frac{1}{x} \frac{d}{dx})^{\mu}$.

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(1) The Hankel transformation IH_{v}

The Hankel transformation \mathbb{H}_{ν} , $\nu \ge -\frac{1}{2}$, is defined by

$$(I\!H_{\nu}\phi)(x) = \int_{0}^{\infty} (xy)^{-\nu} J_{\nu}(xy)\phi(y) y^{2\nu+1} dy.$$

Here J_{v} denotes the Bessel function of the first kind and the order v,

$$J_{\nu}(t) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2} t\right)^{\nu+2m}}{m! \, \Gamma(\nu+m+1)}.$$

Since $J_{\nu}(t) = O(t^{\nu})$ as $t \downarrow 0$ and $O(t^{-\frac{1}{2}})$ as $t \to \infty$ the integral expression defining $I\!H_{\nu}\phi$ converges absolutely for each $\phi \in L_1((0, \infty), y^{\nu+\frac{1}{2}} dy)$ and $I\!H_{\nu}\phi$ is an even continuous function on $I\!R$. In fact, the transformation $I\!H_{\nu}$ can be extended to a unitary transformation on the Hilbert space $L_2((0, \infty), y^{2\nu+1} dy)$ satisfying $I\!H_{\nu}^2 = I$. If we take $\nu = -\frac{1}{2}$ we obtain the Fourier cosine transformation

$$(IH_{-\frac{1}{2}}\phi)(x) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\cos(xy)\phi(y)\,dy.$$

(1.1) Lemma Let $\phi \in L_1((0, \infty))$ with the property that

$$\forall_{k\in IN_{\mathfrak{o}}}: \sup_{x\geq 0} x^{k} \mid \phi(x) \mid <\infty.$$

Then for each $v \ge -\frac{1}{2}$, the Hankel transform $I\!H_v \phi$ of ϕ is pointwise defined and $I\!H_v \phi$ is an even continuous function on $I\!R$.

In the following lemma we present a comparison between the various Hankel transforms.

(1.2) <u>Lemma</u> Let $-\frac{1}{2} \le \mu < \nu$ and let $\phi \in L_1(0, \infty)$ satisfy

$$\sup_{x \ge 0} |x^k \phi(x)| < \infty \text{ and } \sup_{x \ge 0} |x^l(IH_v \phi)(x)| < \infty$$

for all $k, l \in \mathbb{N}_0$. Then for all $x \ge 0$,

$$(I\!H_{\mu}(I\!H_{\nu}\phi))(x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} \int_{x}^{\infty} (\xi^{2} - x^{2})^{\nu-\mu-1} \phi(\xi) \xi d\xi.$$

Proof

The proof is a consequence of the following integral formula

$$\int_{0}^{\infty} t^{\mu-\nu+1} J_{\mu}(xt) J_{\nu}(\xi t) dt = \begin{cases} \frac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} x^{\mu} \xi^{-\nu} (\xi^{2}-x^{2})^{\nu-\mu-1}, & \xi > x, \\ 0, & 0 < \xi \le x \end{cases}$$

cf. [7], p. 100.

(2) The Schwartz space S

Let S denote the space of all rapidly decreasing C^{∞} -functions, viz all C^{∞} -functions ϕ with

$$\sup_{x \in \mathbb{R}} |x^k \phi^{(l)}(x)| < \infty, \quad k, l \in \mathbb{N}_0.$$

The space S admits the following characterization.

(2.1) A function $\phi \in L_1(\mathbb{R})$ with Fourier transform $\mathbb{I} F \phi$ belongs to S if and only if for all $k \in \mathbb{N}_0$

$$\sup_{x \in I\!\!R} |x^k \phi(x)| < \infty \text{ and } \sup_{x \in I\!\!R} |x^k(I\!\!F \phi)(x)| < \infty.$$

This characterization can be obtained by applying standard techniques, cf. [5].

Here we are interested in the space S_{even} of all even functions belonging to S. It readily follows that an even function $\phi \in L_1(\mathbb{R})$ belongs to S_{even} if and only if for all $k \in \mathbb{N}_0$

$$\sup_{x \ge 0} |x^k \phi(x)| < \infty \text{ and } \sup_{x \ge 0} |x^k (IH_{-\frac{1}{2}} \phi)(x)| < \infty.$$

We want to replace the transformation $I\!H_{-\frac{1}{2}}$ by a Hankel transformation $I\!H_{\nu}$ of arbitrary order $\nu \ge -\frac{1}{2}$. Therefore, we use the result in [2] that the space S_{even} is Hankel invariant. So $I\!H_{\nu}(S_{even}) = S_{even}$ for each $\nu \ge -\frac{1}{2}$.

(2.2) <u>Theorem</u> Let $v \ge -\frac{1}{2}$. An even function ϕ in $L_1(\mathbb{R})$ belongs to S_{even} if and only if for all $k, l \in \mathbb{N}_0$

$$\sup_{x\geq 0} |x^k \phi(x)| < \infty \text{ and } \sup_{x\geq 0} |x^l(IH_v \phi)(x)| < \infty.$$

Proof

Let $\phi \in S_{even}$. Then $\sup_{x \ge 0} |x^k \phi(x)| < \infty$ for all $k \in \mathbb{N}_0$. Since also $\mathbb{H}_v \phi \in S_{even}$, one side of the equivalence is settled.

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Conversely, suppose for $\phi \in L_1(\mathbb{R})$, $\sup_{x \ge 0} |x^k \phi(x)| < \infty$ and $\sup_{x \ge 0} |x^l(\mathbb{H}_v \phi)(x)| < \infty$, $k, l \in \mathbb{N}_0$. We have to prove that for all $n \in \mathbb{N}_0$,

$$\sup_{x\geq 0} |x^n(H_{-\frac{1}{2}}\phi)(x)| < \infty.$$

First observe that $H_{-\frac{1}{2}} \phi$ is continuous, so that for all $n \in \mathbb{N}_0$

$$\sup_{0\leq x\leq 1} |x^n(IH_{-\frac{1}{2}}\phi)(x)| < \infty.$$

We may assume $v > -\frac{1}{2}$. Then $H_{-\frac{1}{2}} \phi = (H_{-\frac{1}{2}} H_v) (H_v \phi)$ and by Lemma (1.2) we can write

$$(I\!H_{-\frac{1}{2}}\phi)(x) = C_{\nu} \int_{x}^{\infty} (\xi^{2} - x^{2})^{\nu - \frac{1}{2}} \xi(I\!H_{\nu}\phi)(\xi) d\xi$$

with $C_v = \frac{2^{-v+1/2}}{\Gamma(v+\frac{1}{2})}$. So for $x \ge 1$ we obtain the estimation

$$|x^{n}(IH_{-\frac{1}{2}}\phi)(x)| \leq C_{v} x^{2v+n+1} \int_{1}^{\infty} (t^{2}-1)^{v-\frac{1}{2}} t | (IH_{v}\phi)(xt)| dt$$
$$\leq C_{v} \sup_{y\geq 1} |y^{2v+n+2}(IH_{v}\phi)(y)| \int_{1}^{\infty} \frac{(t^{2}-1)^{v-\frac{1}{2}}}{t^{2v+1}} dt$$

and the result follows.

In the proof of the above theorem we used a relation between $I\!H_{-\frac{1}{2}}\phi$ and $I\!H_v\phi$ in order to deduce a growth estimate for $I\!H_{-\frac{1}{2}}\phi$ from the growth estimate satisfied by $I\!H_{\nu}\phi$. The following lemma yields a generalization of this result; it will be applied in the next section.

(2.3) Lemma

Let $v > -\frac{1}{2}$ and let W denote a nonnegative function such that the function $x^{-2v-2} W(x)$ is nondecreasing on $[a, \infty)$ for some a > 0. Suppose $\phi \in \mathbf{S}_{even}$ satisfies the following growth estimate

Suppose
$$\phi \in S_{even}$$
 satisfies the following growth estimation of $\phi \in S_{even}$

$$\sup_{x\geq 0} W(x) \mid (I\!H_{v}\phi)(x) \mid <\infty.$$

Then for all μ with $-\frac{1}{2} \le \mu < \nu$ and all $\epsilon > 0$

$$\sup_{x\geq 0} (1+x^2)^{\mu-\nu-\epsilon} W(x) \mid (I\!\!H_{\nu}\phi)(x) \mid <\infty.$$

Proof

We use the same technique as in the proof of Theorem (2.2). Let $-\frac{1}{2} \le \mu < \nu$ and let $\varepsilon > 0$. Since $I\!H_{\mu}\phi \in S_{even}$ we only have to consider $x \ge \max\{a, 1\}$. We write $I\!H_{\mu}\phi = (I\!H_{\mu} I\!H_{\nu})(I\!H_{\nu}\phi)$ and so by Lemma (1.2)

$$(I\!H_{\mu}\phi)(x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} \int_{x}^{\infty} (\xi^2 - x^2)^{\nu-\mu-1} \xi(I\!H_{\nu}\phi)(\xi) d\xi.$$

Now let $\delta = \min{\{\varepsilon, \mu + 1\}}$ and let $x \ge \max{\{a, 1\}}$. Then with a straightforward estimation

$$(1+x^{2})^{\mu-\nu-\epsilon} W(x) | (IH_{\mu} \phi) (x) | \leq \\ \leq C_{\nu,\mu} \int_{x}^{\infty} \frac{(\xi^{2}-x^{2})^{\nu-\mu-1} \xi}{(\xi^{2}+x^{2})^{\nu-\mu+\delta}} W(\xi) | (IH_{\nu} \phi) (\xi) | d\xi \\ \leq C_{\nu,\mu} (\sup_{y\geq 1} W(y) | (IH_{\nu} \phi) (y) |) \int_{1}^{\infty} \frac{(t^{2}-1)^{\nu-\mu-1}}{(t^{2}+1)^{\nu-\mu+\delta}} t dt$$

where $C_{\nu,\mu} = \frac{2^{\mu+2}}{\Gamma(\nu-\mu)}$. Hence the result.

(3) The Gelfand-Shilov spaces S^{β}_{α}

In the second volume [4] of their celebrated treatise on generalized functions Gelfand and Shilov introduce the following subspaces of the Schwartz space S. Let $\alpha \ge 0$ and $\beta \ge 0$.

$$S_{\alpha} := \{ \phi \in S \mid \exists_{A>0} \forall_{l \in IN_{0}} \exists_{B_{l}>0} \forall_{k \in IN_{0}} :$$

$$\sup_{x \in IR} \mid x^{k} \phi^{(l)}(x) \mid \leq A^{k} B_{l}(k!)^{\alpha} \}$$

$$S^{\beta} := \{ \phi \in S \mid \exists_{B>0} \forall_{k \in IN_{0}} \exists_{A_{k}>0} \forall_{l \in IN_{0}} :$$

$$\sup_{x \in IR} \mid x^{k} \phi^{(l)}(x) \mid \leq B^{l} A_{k}(l!)^{\beta} \}$$

$$S^{\beta}_{\alpha} := \{ \phi \in S \mid \exists_{A>0, B>0, C>0} \forall_{k \in IN_{0}, l \in IN_{0}} :$$

$$\sup_{x \in IR} \mid x^{k} \phi^{(l)}(x) \mid \leq C A^{k} B^{l}(k!)^{\alpha} (l!)^{\beta} \}.$$

Only for $\alpha \ge 0$, $\beta \ge 0$ with $\alpha + \beta < 1$ the space S_{α}^{β} is trivial, cf. [4], § IV. 8.

As a consequence of Sobolev's lemma the supremum norm in the above definitions can be replaced by the $L_2(\mathbb{R})$ -norm. In [1], Theorem 4.6, the elements ϕ of S_{α}^{β} , $\alpha > 0$, $\beta > 0$, have been characterized in terms of the decay properties of ϕ and of its Fourier transform \mathbb{R} ϕ :

$$(3.1) \qquad \varphi \in S_{\alpha} \iff \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\alpha}) | \phi(x) | < \infty \quad \text{and} \\ \forall_{k \in IN_{0}} : \sup_{x \in IR} |x^{k} (IF \phi) (x)| < \infty \\ \varphi \in S^{\beta} \iff \forall_{k \in IN_{0}} : \sup_{x \in IR} |x^{k} \phi(x)| < \infty \quad \text{and} \\ \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \\ \varphi \in S_{\alpha}^{\beta} \iff \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | \phi(x)| < \infty \quad \text{and} \\ \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \\ \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \text{and} \quad \exists_{t>0} : \sup_{x \in IR} \exp(t |x|^{1/\beta}) | (IF \phi) (x)| < \infty \quad \exists_{t>0} : \bigcup_{x \in IR} \oplus t \mid \forall_{t>0} : \bigcup_{x \in IR} \oplus t \mid \forall_{x \in IR} \oplus t \mid \forall_{x$$

In this section, we derive similar characterizations for the even subspaces $S_{\alpha,\text{even}}$, S_{even}^{β} and $S_{\alpha,\text{even}}^{\beta}$ in terms of decay estimates for an even function ϕ in $L_1(\mathbb{R})$ and its Hankel transform $\mathbb{H}_{\gamma}\phi$. First, we present an important auxiliary result.

(3.2) Lemma

Let **R** denote one of the spaces S_{even} , $S_{\alpha,even}$, S_{even}^{β} and $S_{\alpha,even}^{\beta}$, $\alpha > 0$, $\beta > 0$. Then the differential operator $\frac{1}{x} \frac{d}{dx}$ maps **R** into **R**.

Proof

A simple application of Borel's theorem shows that for each $\phi \in S_{even}$ there exists $\psi \in S$ such that $\phi(x) = \psi(x^2)$, $x \in \mathbb{R}$. So the operator $\frac{1}{x} \frac{d}{dx}$ maps S_{even} into S_{even} . Let $\phi \in S_{\alpha, even}$. Then $\frac{1}{x} \phi' \in S_{even}$ and $\phi' \in S_{\alpha, odd}$. So for all $k \in \mathbb{N}_0$, $\sup_{x \in \mathbb{R}} |x^k(\mathbb{F}(\frac{1}{x}\phi'))(x)| < \infty$, and there exists t > 0 such that $\sup_{x \in \mathbb{R}} \exp(t |x|^{1/\alpha}) |\frac{1}{x} \phi'(x)| < \infty$. It follows from (3.1) that $\frac{1}{x} \phi' \in S_{\alpha, even}$. Next, let $\phi \in S_{even}^{\beta}$. Then, as mentioned,

$$\exists_{B>0} \forall_{k \in IN_0} \exists_{A_k>0} \forall_{l \in IN_0} : (\int_{-\infty}^{\infty} |\xi^k \phi^{(l)}(\xi)|^2 d\xi)^{\frac{1}{2}} \leq A_k B^l (l!)^{\beta}.$$

We have

$$\phi(x) = \phi(0) + \frac{1}{2} x^2 \phi^{(2)}(0) + \frac{1}{2} x^3 \int_0^1 (1-t)^2 \phi^{(3)}(xt) dt.$$

So for all $n \ge 2$ fixed, we have

$$\left(\frac{d}{dx}\right)^{n} \left(\frac{1}{x} \phi'(x)\right) = \frac{1}{2} x^{2} \int_{0}^{1} (1-t)^{2} t^{n+1} \phi^{(n+4)}(xt) dt + + (n+3/2) x \int_{0}^{1} (1-t)^{2} t^{n} \phi^{(n+3)}(xt) dt + + \frac{1}{2} n (n+2) \int_{0}^{1} (1-t)^{2} t^{n-1} \phi^{(n+2)}(xt) dt$$

First, we show that there exist $\tilde{B} > 0$, $\tilde{C} > 0$ such that

(*)
$$(\int_{-\infty}^{\infty} |(\frac{d}{dx})^n (\frac{1}{x} \phi')(x)|^2 dx)^{\frac{1}{2}} \leq \tilde{C} \tilde{B}^n (n!)^{\beta}.$$

For j = 0, 1, 2 we have

$$\int_{-\infty}^{\infty} \int_{0}^{1} \int_{0}^{1} (1-t)^{2} t^{n-1} (xt)^{j} \phi^{(n+2+j)} (xt) dt |^{2} dx$$

$$\leq \int_{-\infty}^{\infty} \{ \int_{0}^{1} t | (xt)^{j} \phi^{(n+2+j)} (xt) |^{2} dt dx$$

$$\leq \{ A_{j} B^{n+2+j} (n+2+j) !^{\beta} \}^{2}.$$

The result follows since there exist \tilde{C} , $\tilde{B} > 0$ such that

$$\begin{split} & \frac{1}{2} A_2 B^{n+4} (n+4) !^{\beta} + (n+3/2) A_1 B^{n+3} (n+3) !^{\beta} + \frac{1}{2} n(n+2) A_0 B^{n+2} (n+2) !^{\beta} \\ & \leq \tilde{C} \tilde{B}^n (n!)^{\beta}. \end{split}$$

Further, we observe that $\frac{1}{x} \phi' \in S_{even}$ and so for all $k \in \mathbb{N}_0$

$$(**) \qquad \int_{-\infty}^{\infty} |x^k \frac{1}{x} \phi'(x)|^2 dx < \infty.$$

Now the result $\frac{1}{x} \phi' \in S_{even}^{\beta}$ follows from (*) and (**) by applying Theorem 4.5 in [1].

Finally, let $\phi \in S_{\alpha, \text{even}}^{\beta}$. Then $\frac{1}{x} \phi' \in S_{\alpha, \text{even}}$ and $\frac{1}{x} \phi' \in S_{\text{even}}^{\beta}$, whence $\frac{1}{x} \phi' \in S_{\alpha, \text{even}} \cap S_{\text{even}}^{\beta} = S_{\alpha, \text{even}}^{\beta}$.

(3.3) Lemma

Let **R** denote one of the spaces $S_{\alpha,\text{even}}$, S_{even}^{β} and $S_{\alpha,\text{even}}^{\beta}$, $\alpha > 0$, $\beta > 0$. Then for all $k \in \mathbb{N}_0$, $\mathbb{H}_{-\frac{1}{2}+k}(S_{\alpha,\text{even}}) = S_{\text{even}}^{\alpha}$, $\mathbb{H}_{-\frac{1}{2}+k}(S_{\text{even}}^{\beta}) = S_{\beta,\text{even}}$ and $\mathbb{H}_{-\frac{1}{2}+k}(S_{\alpha,\text{even}}^{\beta}) = S_{\beta,\text{even}}^{\alpha}$.

<u>Proof</u>

The recurrence relations (cf. [7], p. 67)

$$\left(\frac{1}{z} \frac{d}{dz}\right)^{k} (z^{-\mu} J_{\mu}(z)) = (-1)^{k} z^{-\mu-k} J_{\mu+k}(z), \quad k \in \mathbb{N}_{0}, \, \mu, \, z \in \mathbb{C}$$

imply that for all $\phi \in S_{even}$ and all $k \in \mathbb{N}_0$,

$$\left(\frac{1}{x} \frac{d}{dx}\right)^{k} IH_{-\frac{1}{2}} \phi = (-1)^{k} IH_{-\frac{1}{2}+k} \phi.$$

Since $I\!H_{-\frac{1}{2}+k}(S_{\alpha,even}) \subset S_{even}^{\alpha}$ and $(\frac{1}{x} \frac{d}{dx})^k (S_{\alpha,even}) \subset S_{\alpha,even}$ we obtain $I\!H_{-\frac{1}{2}+k}(S_{\alpha,even}) \subset S_{\alpha,even}$. S_{even}^{α} . Similarly, we obtain $I\!H_{-\frac{1}{2}+k}(S_{even}^{\beta}) \subset S_{\beta,even}$ and $I\!H_{-\frac{1}{2}+k}(S_{\alpha,even}^{\beta}) \subset S_{\beta,even}^{\alpha}$. Now the statements follow since $(I\!H_{-\frac{1}{2}+k})^2 \phi = \phi, \phi \in S_{even}$.

(3.4) Corollary

The operator
$$\frac{1}{x} \frac{d}{dx}$$
 is invertible on S_{even} , $S_{\alpha,even}$, S_{even}^{β} and $S_{\alpha,even}^{\beta}$ with $(\frac{1}{x} \frac{d}{dx})^{-1} \phi = -\int_{x}^{\infty} t \phi(t) dt, \phi \in S_{even}$.

We arrive at the main theorem of this section.

 $(3.5) \underline{\text{Theorem}}$ Let α , $\beta > 0$, $\nu \ge -\frac{1}{2}$ and let $\phi \in L_1(\mathbb{R})$ be even. I. $\phi \in S_{\alpha, \text{even}}$ iff $\exists_{t>0}$: $\sup_{x\ge 0} \exp(t x^{1/\alpha}) | \phi(x) | < \infty$ and $\forall_{j\in \mathbb{I}N_0}$: $\sup_{x\ge 0} |x^j(\mathbb{H}_{\nu}\phi)(x)| < \infty$ II. $\phi \in S_{\text{even}}^{\beta}$ iff $\forall_{j\in \mathbb{I}N_0}$: $\sup_{x\ge 0} |x^j\phi(x)| < \infty$ and $\exists_{t>0}$: $\sup_{x\ge 0} \exp(t x^{1/\beta}) | (\mathbb{H}_{\nu}\phi)(x)| < \infty$ III. $\phi \in S_{\alpha, \text{even}}^{\beta}$ iff $\exists_{t>0}$: $\sup_{x\ge 0} \exp(t x^{1/\beta}) | (\mathbb{H}_{\nu}\phi)(x)| < \infty$ and $\exists_{t>0}$: $\sup_{x\ge 0} \exp(t x^{1/\beta}) | \phi(x)| < \infty$ and $\exists_{t>0}$: $\sup_{x\ge 0} \exp(t x^{1/\beta}) | (\mathbb{H}_{\nu}\phi)(x)| < \infty$

Proof

For $v = -\frac{1}{2}$ the results are stated in (3.1). So we take $v > -\frac{1}{2}$ in the sequel.

- I. Let $\phi \in S_{\alpha, even}$. Then $\phi \in S_{even}$ and by (3.1) there exists t > 0 such that $\sup_{x \ge 0} \exp(t x^{1/\alpha}) | \phi(x) | < \infty$. Since $\phi \in S_{even}$, by Theorem (2.2), $\sup_{x \ge 0} | x^j (H_v \phi) (x) | < \infty$. Conversely, suppose ϕ satisfies the stated conditions. Then by Theorem (2.2), $\phi \in S_{even}$. It follows that for all $k \in IN_0$, $\sup_{x \ge 0} | x^k (IH_{-\frac{1}{2}} \phi) (x) | < \infty$ and so by (3.1), $\phi \in S_{\alpha, even}$.
- II. Let $\phi \in S_{\text{even}}^{\beta}$. Then by (3.1) for all $j \in \mathbb{N}_0$, $\sup_{x \ge 0} x^j |\phi(x)| < \infty$ and, also, for all $l \in \mathbb{N}_0$, $\mathbb{H}_{-\frac{1}{2}+l} \phi \in S_{\beta,\text{even}}$. So there exists $\tau > 0$ such that

$$\sup_{x>0} \exp\left(\tau x^{1/\beta}\right) \mid \left(I\!H_{-\frac{1}{2}+l}\phi\right)(x) \mid <\infty.$$

Taking a fixed $l > v + \frac{1}{2}$ we obtain from Lemma (2.3)

$$\sup_{x \ge 0} \exp\left(\tau x^{1/\beta}\right) (1+x^2)^{\nu-l} \mid (I\!H_{\nu}\phi)(x) \mid < \infty.$$

Hence for all t, $0 < t < \tau$

$$\sup_{x\geq 0} \exp\left(t\,x^{1/\beta}\right) \mid (I\!H_{\nu}\,\phi)(x) \mid <\infty.$$

Conversely, if ϕ satisfies the stated conditions, then for all $j \in \mathbb{N}_0$, $\sup_{x \ge 0} |x^j \phi(x)| < \infty$ and by Lemma (2.3)

$$\sup_{x \ge 0} \exp(t x^{1/\beta}) (1 + x^2)^{-\nu - 3/2} \mid (IH_{-\frac{1}{2}} \phi)(x) \mid < \infty.$$

So $\phi \in S_{even}^{\beta}$.

III. We observe that $S^{\beta}_{\alpha,even} = S_{\alpha,even} \cap S^{\beta}_{even}$.

(3.6) <u>Corollary</u> Let $v \ge -\frac{1}{2}$ and let $\alpha > 0$, $\beta > 0$. Then $I\!H_v(\mathbf{S}_{\alpha,\text{even}}) = \mathbf{S}_{\text{even}}^{\alpha}$, $I\!H_v(\mathbf{S}_{\text{even}}^{\beta}) = \mathbf{S}_{\beta,\text{even}}$, and $I\!H_v(\mathbf{S}_{\alpha,\text{even}}^{\beta}) = \mathbf{S}_{\beta,\text{even}}^{\alpha}$.

Proof.

These statements are consequences of the characterizations presented in Theorem (3.5), and the fact that $I\!H_v(I\!H_v\phi) = \phi$ for all $\phi \in S_{even}$.

Remark

Partly the results stated in the above corollary are known. In [3] it is proved that $I\!H_{\nu}(S^{\alpha}_{\alpha,\text{even}}) = S^{\alpha}_{\alpha,\text{even}}, \frac{1}{2} \le \alpha \le 1$, using properties of the Laguerre polynomials. In [8], the result $I\!H_{\nu}(S^{\beta}_{\alpha,\text{even}}) = S^{\alpha}_{\beta,\text{even}}$ is stated in case $0 < \alpha, \beta < 1$. However, a number of proofs in [8] is incorrect. We refer to [9] for correct versions.

As we have seen $(\frac{1}{x} \frac{d}{dx})^l \phi = I\!\!H_{-\frac{1}{2}+l} (I\!\!H_{-\frac{1}{2}} \phi), \phi \in S_{even}$. Therefore we define the fractional differentiation operators

$$(\frac{1}{x} \frac{d}{dx})^{v} = I H_{-\frac{1}{2}+v} \circ I H_{-\frac{1}{2}}, \quad v \ge 0,$$

and the fractional integration operators

$$\left(\frac{1}{x} \frac{d}{dx}\right)^{-\nu} = IH_{-\frac{1}{2}} \circ IH_{-\frac{1}{2}+\nu}, \quad \nu \ge 0.$$

The collection of operators $\{(\frac{1}{x}, \frac{d}{dx})^{\nu}\}_{\nu \in \mathbb{R}}$ establishes a one-parameter group on S_{even} , $S_{\alpha,even}$, S_{β}^{β} and $S_{\alpha,even}^{\beta}$. Cf. [6], Ch. 5 for related results.

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