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Eindhoven University of Technology
Department of Mathematics and Computing Science

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S.J.L. van Eijndhoven

and

C.A.M. van Berkel

Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

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Summary

There exist growth estimates on even functions $\phi \in L_1(\mathbb{R})$ and on their Hankel transforms $\mathbb{H}_\nu \phi$ which are necessary and sufficient for ϕ to belong to the even subspaces of Gelfand-Shilov's S_α^β-spaces. Consequently, $\mathbb{H}_\nu(S_{\alpha,\text{even}}^\beta) = S_{\beta,\text{even}}^\alpha$. Further, $S_{\alpha,\text{even}}^\beta$ remains invariant under the fractional differentiation/integration operators $(\frac{1}{x}\frac{d}{dx})^\mu$.

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(1) The Hankel transformation $\mathbb{H}_\nu$

The Hankel transformation $\mathbb{H}_\nu$, $\nu \geq -\frac{1}{2}$, is defined by

$$(\mathbb{H}_\nu \phi)(x) = \int_0^\infty (xy)^{-\nu} J_\nu(xy)\, \phi(y)\, y^{2\nu+1}\, dy.$$

Here J_ν denotes the Bessel function of the first kind and the order ν,

$$J_\nu(t) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}t)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}.$$

Since $J_\nu(t) = O(t^\nu)$ as $t \downarrow 0$ and $O(t^{-\frac{1}{2}})$ as $t \to \infty$ the integral expression defining $\mathbb{H}_\nu \phi$ converges absolutely for each $\phi \in L_1((0,\infty), y^{\nu+\frac{1}{2}} dy)$ and $\mathbb{H}_\nu \phi$ is an even continuous function on $\mathbb{R}$. In fact, the transformation $\mathbb{H}_\nu$ can be extended to a unitary transformation on the Hilbert space $L_2((0,\infty), y^{2\nu+1} dy)$ satisfying $\mathbb{H}_\nu^2 = I$. If we take $\nu = -\frac{1}{2}$ we obtain the Fourier cosine transformation

$$(\mathbb{H}_{-\frac{1}{2}} \phi)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xy)\, \phi(y)\, dy.$$

(1.1) Lemma

Let $\phi \in L_1((0,\infty))$ with the property that

$$\forall_{k \in \mathbb{N}_0} : \sup_{x \geq 0} x^k \,|\phi(x)| < \infty.$$

Then for each $\nu \geq -\frac{1}{2}$, the Hankel transform $\mathbb{H}_\nu \phi$ of ϕ is pointwise defined and $\mathbb{H}_\nu \phi$ is an even continuous function on $\mathbb{R}$. □

In the following lemma we present a comparison between the various Hankel transforms.

(1.2) Lemma

Let $-\frac{1}{2} \leq \mu < \nu$ and let $\phi \in L_1(0,\infty)$ satisfy

$$\sup_{x \geq 0} |x^k \phi(x)| < \infty \quad \text{and} \quad \sup_{x \geq 0} |x^l (\mathbb{H}_\nu \phi)(x)| < \infty$$

for all $k, l \in \mathbb{N}_0$. Then for all $x \geq 0$,

$$(\mathbb{H}_\mu(\mathbb{H}_\nu \phi))(x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} \int_x^\infty (\xi^2 - x^2)^{\nu-\mu-1}\, \phi(\xi)\, \xi\, d\xi.$$

Proof

The proof is a consequence of the following integral formula

$$\int_0^\infty t^{\mu-\nu+1} J_\mu(xt) J_\nu(\xi t)\, dt = \begin{cases} \dfrac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} x^\mu \xi^{-\nu} (\xi^2 - x^2)^{\nu-\mu-1}, & \xi > x, \\ 0, & 0 < \xi \le x \end{cases}$$

cf. [7], p. 100. □

(2) The Schwartz space S

Let S denote the space of all rapidly decreasing C^∞-functions, viz all C^∞-functions ϕ with

$$\sup_{x\in \mathbb{R}} | x^k \phi^{(l)}(x) | < \infty, \quad k, l \in \mathbb{N}_0.$$

The space S admits the following characterization.

(2.1) A function $\phi \in L_1(\mathbb{R})$ with Fourier transform $\mathbb{F}\phi$ belongs to S if and only if for all $k \in \mathbb{N}_0$

$$\sup_{x\in \mathbb{R}} | x^k \phi(x) | < \infty \text{ and } \sup_{x\in \mathbb{R}} | x^k(\mathbb{F}\phi)(x) | < \infty.$$

This characterization can be obtained by applying standard techniques, cf. [5].
Here we are interested in the space S_{even} of all even functions belonging to S. It readily follows that an even function $\phi \in L_1(\mathbb{R})$ belongs to S_{even} if and only if for all $k \in \mathbb{N}_0$

$$\sup_{x\geq 0} | x^k \phi(x) | < \infty \text{ and } \sup_{x\geq 0} | x^k(\mathbb{H}_{-\frac{1}{2}}\phi)(x) | < \infty.$$

We want to replace the transformation $\mathbb{H}_{-\frac{1}{2}}$ by a Hankel transformation $\mathbb{H}_\nu$ of arbitrary order $\nu \geq -\frac{1}{2}$. Therefore, we use the result in [2] that the space S_{even} is Hankel invariant. So $\mathbb{H}_\nu(S_{even}) = S_{even}$ for each $\nu \geq -½$.

(2.2) Theorem

Let $\nu \geq -½$. An even function ϕ in $L_1(\mathbb{R})$ belongs to S_{even} if and only if for all $k, l \in \mathbb{N}_0$

$$\sup_{x\geq 0} | x^k \phi(x) | < \infty \text{ and } \sup_{x\geq 0} | x^l(\mathbb{H}_\nu\phi)(x) | < \infty.$$

Proof

Let $\phi \in S_{even}$. Then $\sup_{x\geq 0} | x^k \phi(x) | < \infty$ for all $k \in \mathbb{N}_0$. Since also $\mathbb{H}_\nu \phi \in S_{even}$, one side of the equivalence is settled.

Conversely, suppose for $\phi \in L_1(\mathbb{R})$, $\sup_{x\geq 0} |x^k \phi(x)| < \infty$ and $\sup_{x\geq 0} |x^l (\mathbb{H}_\nu \phi)(x)| < \infty$, $k, l \in \mathbb{N}_0$. We have to prove that for all $n \in \mathbb{N}_0$,

$$\sup_{x\geq 0} |x^n (\mathbb{H}_{-\frac{1}{2}} \phi)(x)| < \infty.$$

First observe that $\mathbb{H}_{-\frac{1}{2}} \phi$ is continuous, so that for all $n \in \mathbb{N}_0$

$$\sup_{0\leq x\leq 1} |x^n (\mathbb{H}_{-\frac{1}{2}} \phi)(x)| < \infty.$$

We may assume $\nu > -\frac{1}{2}$. Then $\mathbb{H}_{-\frac{1}{2}} \phi = (\mathbb{H}_{-\frac{1}{2}} \mathbb{H}_\nu)(\mathbb{H}_\nu \phi)$ and by Lemma (1.2) we can write

$$(\mathbb{H}_{-\frac{1}{2}} \phi)(x) = C_\nu \int_x^\infty (\xi^2 - x^2)^{\nu - \frac{1}{2}} \xi (\mathbb{H}_\nu \phi)(\xi)\, d\xi$$

with $C_\nu = \dfrac{2^{-\nu+½}}{\Gamma(\nu+\frac{1}{2})}$. So for $x \geq 1$ we obtain the estimation

$$|x^n (\mathbb{H}_{-\frac{1}{2}} \phi)(x)| \leq C_\nu x^{2\nu+n+1} \int_1^\infty (t^2-1)^{\nu-½}\, t\, |(\mathbb{H}_\nu \phi)(xt)|\, dt$$

$$\leq C_\nu \sup_{y\geq 1} |y^{2\nu+n+2} (\mathbb{H}_\nu \phi)(y)| \int_1^\infty \frac{(t^2-1)^{\nu-½}}{t^{2\nu+1}}\, dt$$

and the result follows. □

In the proof of the above theorem we used a relation between $\mathbb{H}_{-\frac{1}{2}} \phi$ and $\mathbb{H}_\nu \phi$ in order to deduce a growth estimate for $\mathbb{H}_{-\frac{1}{2}} \phi$ from the growth estimate satisfied by $\mathbb{H}_\nu \phi$. The following lemma yields a generalization of this result; it will be applied in the next section.

(2.3) <u>Lemma</u>
Let $\nu > -\frac{1}{2}$ and let W denote a nonnegative function such that the function $x^{-2\nu-2} W(x)$ is nondecreasing on $[a, \infty)$ for some $a > 0$.
Suppose $\phi \in S_{\text{even}}$ satisfies the following growth estimate

$$\sup_{x\geq 0} W(x) |(\mathbb{H}_\nu \phi)(x)| < \infty.$$

Then for all μ with $-\frac{1}{2} \leq \mu < \nu$ and all $\varepsilon > 0$

$$\sup_{x\geq 0} (1+x^2)^{\mu-\nu-\varepsilon} W(x) |(\mathbb{H}_\nu \phi)(x)| < \infty.$$

Proof

We use the same technique as in the proof of Theorem (2.2). Let $-\frac{1}{2} \le \mu < \nu$ and let $\varepsilon > 0$. Since $I\!H_\mu \phi \in S_{even}$ we only have to consider $x \ge \max\{a, 1\}$. We write $I\!H_\mu \phi = (I\!H_\mu\, I\!H_\nu)(I\!H_\nu \phi)$ and so by Lemma (1.2)

$$(I\!H_\mu \phi)(x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} \int_x^\infty (\xi^2 - x^2)^{\nu-\mu-1}\, \xi (I\!H_\nu \phi)(\xi)\, d\xi.$$

Now let $\delta = \min\{\varepsilon, \mu+1\}$ and let $x \ge \max\{a, 1\}$. Then with a straightforward estimation

$$(1+x^2)^{\mu-\nu-\varepsilon}\, W(x)\, |(I\!H_\mu \phi)(x)| \le$$

$$\le C_{\nu,\mu} \int_x^\infty \frac{(\xi^2-x^2)^{\nu-\mu-1}\,\xi}{(\xi^2+x^2)^{\nu-\mu+\delta}}\, W(\xi)\, |(I\!H_\nu \phi)(\xi)|\, d\xi$$

$$\le C_{\nu,\mu} \left(\sup_{y\ge 1} W(y)\, |(I\!H_\nu \phi)(y)|\right) \int_1^\infty \frac{(t^2-1)^{\nu-\mu-1}}{(t^2+1)^{\nu-\mu+\delta}}\, t\, dt$$

where $C_{\nu,\mu} = \dfrac{2^{\mu+2}}{\Gamma(\nu-\mu)}$. Hence the result. ▯

(3) The Gelfand-Shilov spaces S_α^β

In the second volume [4] of their celebrated treatise on generalized functions Gelfand and Shilov introduce the following subspaces of the Schwartz space S. Let $\alpha \ge 0$ and $\beta \ge 0$.

$$S_\alpha := \{\phi \in S \mid \exists_{A>0}\, \forall_{l \in I\!N_0}\, \exists_{B_l>0}\, \forall_{k \in I\!N_0} :$$

$$\sup_{x \in I\!R} |x^k \phi^{(l)}(x)| \le A^k B_l (k!)^\alpha\}$$

$$S^\beta := \{\phi \in S \mid \exists_{B>0}\, \forall_{k \in I\!N_0}\, \exists_{A_k>0}\, \forall_{l \in I\!N_0} :$$

$$\sup_{x \in I\!R} |x^k \phi^{(l)}(x)| \le B^l A_k (l!)^\beta\}$$

$$S_\alpha^\beta := \{\phi \in S \mid \exists_{A>0, B>0, C>0}\, \forall_{k \in I\!N_0, l \in I\!N_0} :$$

$$\sup_{x \in I\!R} |x^k \phi^{(l)}(x)| \le C A^k B^l (k!)^\alpha (l!)^\beta\}.$$

Only for $\alpha \ge 0$, $\beta \ge 0$ with $\alpha + \beta < 1$ the space S_α^β is trivial, cf. [4], § IV. 8.

As a consequence of Sobolev's lemma the supremum norm in the above definitions can be replaced by the $L_2(I\!R)$-norm. In [1], Theorem 4.6, the elements ϕ of S_α^β, $\alpha > 0$, $\beta > 0$, have been characterized in terms of the decay properties of ϕ and of its Fourier transform $I\!F \phi$:

$$(3.1)\qquad \phi \in S_\alpha \iff \exists_{t>0} : \sup_{x\in \mathbb{R}} \exp(t\,|x|^{1/\alpha})\,|\phi(x)| < \infty \quad \text{and}$$

$$\forall_{k\in \mathbb{N}_0} : \sup |x^k (\mathbb{F}\phi)(x)| < \infty$$

$$\phi \in S^\beta \iff \forall_{k\in \mathbb{N}_0} : \sup_{x\in \mathbb{R}} |x^k \phi(x)| < \infty \quad \text{and}$$

$$\exists_{t>0} : \sup_{x\in \mathbb{R}} \exp(t\,|x|^{1/\beta})\,|(\mathbb{F}\phi)(x)| < \infty$$

$$\phi \in S^\beta_\alpha \iff \exists_{t>0} : \sup_{x\in \mathbb{R}} \exp(t\,|x|^{1/\alpha})\,|\phi(x)| < \infty \quad \text{and}$$

$$\exists_{t>0} : \sup_{x\in \mathbb{R}} \exp(t\,|x|^{1/\beta})\,|(\mathbb{F}\phi)(x)| < \infty$$

In this section, we derive similar characterizations for the even subspaces $S_{\alpha,\text{even}}$, S^β_{even} and $S^\beta_{\alpha,\text{even}}$ in terms of decay estimates for an even function ϕ in $L_1(\mathbb{R})$ and its Hankel transform $\mathbb{H}_\nu \phi$. First, we present an important auxiliary result.

(3.2) Lemma

Let **R** denote one of the spaces S_{even}, $S_{\alpha,\text{even}}$, S^β_{even} and $S^\beta_{\alpha,\text{even}}$, $\alpha > 0$, $\beta > 0$. Then the differential operator $\frac{1}{x}\frac{d}{dx}$ maps **R** into **R**.

Proof

A simple application of Borel's theorem shows that for each $\phi \in S_{\text{even}}$ there exists $\psi \in S$ such that $\phi(x) = \psi(x^2)$, $x \in \mathbb{R}$. So the operator $\frac{1}{x}\frac{d}{dx}$ maps S_{even} into S_{even}.

Let $\phi \in S_{\alpha,\text{even}}$. Then $\frac{1}{x}\phi' \in S_{\text{even}}$ and $\phi' \in S_{\alpha,\text{odd}}$.

So for all $k \in \mathbb{N}_0$, $\sup_{x\in \mathbb{R}} |x^k(\mathbb{F}(\frac{1}{x}\phi'))(x)| < \infty$, and there exists $t > 0$ such that $\sup_{x\in \mathbb{R}} \exp(t\,|x|^{1/\alpha})\,|\frac{1}{x}\phi'(x)| < \infty$. It follows from (3.1) that $\frac{1}{x}\phi' \in S_{\alpha,\text{even}}$.

Next, let $\phi \in S^\beta_{\text{even}}$. Then, as mentioned,

$$\exists_{B>0}\,\forall_{k\in \mathbb{N}_0}\,\exists_{A_k>0}\,\forall_{l\in \mathbb{N}_0} : \Big(\int_{-\infty}^{\infty} |\xi^k \phi^{(l)}(\xi)|^2\, d\xi\Big)^{1/2} \le A_k B^l (l!)^\beta .$$

We have

$$\phi(x) = \phi(0) + \tfrac{1}{2}\,x^2\,\phi^{(2)}(0) + \tfrac{1}{2}\,x^3 \int_0^1 (1-t)^2\,\phi^{(3)}(xt)\,dt.$$

So for all $n \ge 2$ fixed, we have

$$(\frac{d}{dx})^n (\frac{1}{x} \phi'(x)) = \frac{1}{2} x^2 \int_0^1 (1-t)^2 t^{n+1} \phi^{(n+4)}(xt)\, dt +$$

$$+ (n+3/2)\, x \int_0^1 (1-t)^2 t^n \phi^{(n+3)}(xt)\, dt +$$

$$+ \frac{1}{2} n (n+2) \int_0^1 (1-t)^2 t^{n-1} \phi^{(n+2)}(xt)\, dt.$$

First, we show that there exist $\tilde{B} > 0$, $\tilde{C} > 0$ such that

$$(*) \qquad (\int_{-\infty}^{\infty} |(\frac{d}{dx})^n (\frac{1}{x} \phi')(x)|^2 dx)^{1/2} \leq \tilde{C} \tilde{B}^n (n!)^\beta .$$

For $j = 0, 1, 2$ we have

$$\int_{-\infty}^{\infty} | \int_0^1 (1-t)^2 t^{n-1} (xt)^j \phi^{(n+2+j)}(xt)\, dt |^2 dx$$

$$\leq \int_{-\infty}^{\infty} \{ \int_0^1 t \, |(xt)^j \phi^{(n+2+j)}(xt)|^2 dt \;\; dx$$

$$\leq \{ A_j B^{n+2+j} (n+2+j)!^\beta \}^2 .$$

The result follows since there exist $\tilde{C}$, $\tilde{B} > 0$ such that

$$\frac{1}{2} A_2 B^{n+4} (n+4)!^\beta + (n+3/2) A_1 B^{n+3} (n+3)!^\beta + \frac{1}{2} n(n+2) A_0 B^{n+2} (n+2)!^\beta$$

$$\leq \tilde{C} \tilde{B}^n (n!)^\beta .$$

Further, we observe that $\frac{1}{x} \phi' \in S_{even}$ and so for all $k \in \mathbb{N}_0$

$$(**) \qquad \int_{-\infty}^{\infty} | x^k \frac{1}{x} \phi'(x) |^2 dx < \infty .$$

Now the result $\frac{1}{x} \phi' \in S^\beta_{even}$ follows from (*) and (**) by applying Theorem 4.5 in [1].

Finally, let $\phi \in S^\beta_{\alpha,even}$. Then $\frac{1}{x} \phi' \in S_{\alpha,even}$ and $\frac{1}{x} \phi' \in S^\beta_{even}$, whence $\frac{1}{x} \phi' \in S_{\alpha,even} \cap S^\beta_{even} = S^\beta_{\alpha,even}$. □

(3.3) Lemma

Let $\mathbf{R}$ denote one of the spaces $S_{\alpha,even}$, S^β_{even} and $S^\beta_{\alpha,even}$, $\alpha > 0$, $\beta > 0$. Then for all $k \in \mathbb{N}_0$, $\mathbb{H}_{-\frac{1}{2}+k}(S_{\alpha,even}) = S^\alpha_{even}$, $\mathbb{H}_{-\frac{1}{2}+k}(S^\beta_{even}) = S_{\beta,even}$ and $\mathbb{H}_{-\frac{1}{2}+k}(S^\beta_{\alpha,even}) = S^\alpha_{\beta,even}$.

Proof

The recurrence relations (cf. [7], p. 67)

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k (z^{-\mu} J_\mu(z)) = (-1)^k z^{-\mu-k} J_{\mu+k}(z), \quad k \in \mathbb{N}_0, \mu, z \in \mathbb{C}$$

imply that for all $\phi \in S_{even}$ and all $k \in \mathbb{N}_0$,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k \mathbb{H}_{-\frac{1}{2}} \phi = (-1)^k \mathbb{H}_{-\frac{1}{2}+k} \phi.$$

Since $\mathbb{H}_{-\frac{1}{2}+k}(S_{\alpha,even}) \subset S^{\alpha}_{even}$ and $\left(\frac{1}{x}\frac{d}{dx}\right)^k (S_{\alpha,even}) \subset S_{\alpha,even}$ we obtain $\mathbb{H}_{-\frac{1}{2}+k}(S_{\alpha,even}) \subset S^{\alpha}_{even}$. Similarly, we obtain $\mathbb{H}_{-\frac{1}{2}+k}(S^{\beta}_{even}) \subset S_{\beta,even}$ and $\mathbb{H}_{-\frac{1}{2}+k}(S^{\beta}_{\alpha,even}) \subset S^{\alpha}_{\beta,even}$. Now the statements follow since $(\mathbb{H}_{-\frac{1}{2}+k})^2 \phi = \phi$, $\phi \in S_{even}$. □

(3.4) Corollary

The operator $\frac{1}{x}\frac{d}{dx}$ is invertible on S_{even}, $S_{\alpha,even}$, S^{β}_{even} and $S^{\beta}_{\alpha,even}$ with $\left(\frac{1}{x}\frac{d}{dx}\right)^{-1} \phi = -\int_x^\infty t\,\phi(t)\,dt$, $\phi \in S_{even}$. □

We arrive at the main theorem of this section.

(3.5) Theorem

Let $\alpha, \beta > 0$, $\nu \geq -\frac{1}{2}$ and let $\phi \in L_1(\mathbb{R})$ be even.

I. $\phi \in S_{\alpha,even}$ iff $\exists_{t>0}$: $\sup_{x\geq 0} \exp(t x^{1/\alpha}) \, |\phi(x)| < \infty$ and

$\forall_{j \in \mathbb{N}_0}$: $\sup_{x\geq 0} |x^j (\mathbb{H}_\nu \phi)(x)| < \infty$

II. $\phi \in S^{\beta}_{even}$ iff $\forall_{j \in \mathbb{N}_0}$: $\sup_{x\geq 0} |x^j \phi(x)| < \infty$ and

$\exists_{t>0}$: $\sup_{x\geq 0} \exp(t x^{1/\beta}) \, |(\mathbb{H}_\nu \phi)(x)| < \infty$

III. $\phi \in S^{\beta}_{\alpha,even}$ iff $\exists_{t>0}$: $\sup_{x\geq 0} \exp(t x^{1/\alpha}) \, |\phi(x)| < \infty$ and

$\exists_{t>0}$: $\sup_{x\geq 0} \exp(t x^{1/\beta}) \, |(\mathbb{H}_\nu \phi)(x)| < \infty$.

Proof

For $\nu = -\frac{1}{2}$ the results are stated in (3.1). So we take $\nu > -\frac{1}{2}$ in the sequel.

I. Let $\phi \in S_{\alpha,\text{even}}$. Then $\phi \in S_{\text{even}}$ and by (3.1) there exists $t > 0$ such that $\sup_{x\geq 0} \exp(t x^{1/\alpha}) \mid \phi(x) \mid < \infty$. Since $\phi \in S_{\text{even}}$, by Theorem (2.2), $\sup_{x\geq 0} \mid x^j (I\!H_\nu \phi)(x) \mid < \infty$.
Conversely, suppose ϕ satisfies the stated conditions. Then by Theorem (2.2), $\phi \in S_{\text{even}}$. It follows that for all $k \in I\!N_0$, $\sup_{x\geq 0} \mid x^k (I\!H_{-\frac{1}{2}} \phi)(x) \mid < \infty$ and so by (3.1), $\phi \in S_{\alpha,\text{even}}$.

II. Let $\phi \in S^{\beta}_{\text{even}}$. Then by (3.1) for all $j \in I\!N_0$, $\sup_{x\geq 0} x^j \mid \phi(x) \mid < \infty$ and, also, for all $l \in I\!N_0$, $I\!H_{-\frac{1}{2}+l}\, \phi \in S_{\beta,\text{even}}$. So there exists $\tau > 0$ such that

$$\sup_{x\geq 0} \exp(\tau x^{1/\beta}) \mid (I\!H_{-\frac{1}{2}+l}\, \phi)(x) \mid < \infty.$$

Taking a fixed $l > \nu + ½$ we obtain from Lemma (2.3)

$$\sup_{x\geq 0} \exp(\tau x^{1/\beta}) (1+x^2)^{\nu-l} \mid (I\!H_\nu \phi)(x) \mid < \infty.$$

Hence for all t, $0 < t < \tau$

$$\sup_{x\geq 0} \exp(t x^{1/\beta}) \mid (I\!H_\nu \phi)(x) \mid < \infty.$$

Conversely, if ϕ satisfies the stated conditions, then for all $j \in I\!N_0$, $\sup_{x\geq 0} \mid x^j \phi(x) \mid < \infty$ and by Lemma (2.3)

$$\sup_{x\geq 0} \exp(t x^{1/\beta}) (1+x^2)^{-\nu-3/2} \mid (I\!H_{-\frac{1}{2}} \phi)(x) \mid < \infty.$$

So $\phi \in S^{\beta}_{\text{even}}$.

III. We observe that $S^{\beta}_{\alpha,\text{even}} = S_{\alpha,\text{even}} \cap S^{\beta}_{\text{even}}$. □

(3.6) Corollary
Let $\nu \geq -\frac{1}{2}$ and let $\alpha > 0$, $\beta > 0$.
Then $I\!H_\nu(S_{\alpha,\text{even}}) = S^{\alpha}_{\text{even}}$, $I\!H_\nu(S^{\beta}_{\text{even}}) = S_{\beta,\text{even}}$, and $I\!H_\nu(S^{\beta}_{\alpha,\text{even}}) = S^{\alpha}_{\beta,\text{even}}$.

Proof.
These statements are consequences of the characterizations presented in Theorem (3.5), and the fact that $I\!H_\nu(I\!H_\nu \phi) = \phi$ for all $\phi \in S_{\text{even}}$. □

Remark
Partly the results stated in the above corollary are known. In [3] it is proved that $I\!H_\nu(S^{\alpha}_{\alpha,\text{even}}) = S^{\alpha}_{\alpha,\text{even}}$, $\frac{1}{2} \leq \alpha \leq 1$, using properties of the Laguerre polynomials. In [8], the result $I\!H_\nu(S^{\beta}_{\alpha,\text{even}}) = S^{\alpha}_{\beta,\text{even}}$ is stated in case $0 < \alpha, \beta < 1$. However, a number of proofs in [8] is incorrect. We refer to [9] for correct versions.

As we have seen $(\frac{1}{x}\frac{d}{dx})^l\,\phi = \mathbb{H}_{-\frac{1}{2}+l}\,(\mathbb{H}_{-\frac{1}{2}}\,\phi)$, $\phi \in S_{\text{even}}$. Therefore we define the fractional differentiation operators

$$(\frac{1}{x}\frac{d}{dx})^{\nu} = \mathbb{H}_{-\frac{1}{2}+\nu} \circ \mathbb{H}_{-\frac{1}{2}}\,, \quad \nu \geq 0,$$

and the fractional integration operators

$$(\frac{1}{x}\frac{d}{dx})^{-\nu} = \mathbb{H}_{-\frac{1}{2}} \circ \mathbb{H}_{-\frac{1}{2}+\nu}\,, \quad \nu \geq 0.$$

The collection of operators $\{(\frac{1}{x}\frac{d}{dx})^{\nu}\}_{\nu\in\mathbb{R}}$ establishes a one-parameter group on S_{even}, $S_{\alpha,\text{even}}$, S^{β}_{even} and $S^{\beta}_{\alpha,\text{even}}$. Cf. [6], Ch. 5 for related results.

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