## Hankel transformations and spaces of type S

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# HANKEL TRANSFORMATIONS AND SPACES OF TYPE S 

by
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#### Abstract

Summary There exist growth estimates on even functions $\phi \in L_{1}(\mathbb{R})$ and on their Hankel transforms $\mathbb{H}_{v} \phi$ which are necessary and sufficient for $\phi$ to belong to the even subspaces of Gelfand-Shilov's $\mathbf{S}_{\alpha}^{\beta}$ spaces. Consequently, $H_{v}\left(S_{\alpha, \text { even }}^{\beta}\right)=S_{\beta, \text { even. }}^{\alpha}$. Further, $S_{\alpha, \text { even }}^{\beta}$ remains invariant under the fractional differentiation/integration operators $\left(\frac{1}{x} \frac{d}{d x}\right)^{\mu}$.


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## (1) The Hankel transformation $\mathbb{H}_{v}$

The Hankel transformation $H_{v}, v \geq-\frac{1}{2}$, is defined by

$$
\left(H_{v} \phi\right)(x)=\int_{0}^{\infty}(x y)^{-v} J_{v}(x y) \phi(y) y^{2 v+1} d y .
$$

Here $J_{v}$ denotes the Bessel function of the first kind and the order v ,

$$
J_{\mathrm{v}}(t)=\sum_{m=0}^{\infty} \frac{(-1)^{m}\left(\frac{1}{2} t\right)^{v+2 m}}{m!\Gamma(v+m+1)}
$$

Since $J_{\mathrm{v}}(t)=O\left(t^{v}\right)$ as $t \downarrow 0$ and $O\left(t^{-\frac{1}{2}}\right)$ as $t \rightarrow \infty$ the integral expression defining $H_{v} \phi$ converges absolutely for each $\phi \in L_{1}\left((0, \infty), y^{v+1 / 2} d y\right)$ and $H_{v} \phi$ is an even continuous function on $\mathbb{R}$. In fact, the transformation $\mathbb{H}_{v}$ can be extended to a unitary transformation on the Hilbert space $L_{2}\left((0, \infty), y^{2 v+1} d y\right)$ satisfying $\mathbb{H}_{v}^{2}=I$. If we take $v=-\frac{1}{2}$ we obtain the Fourier cosine transformation

$$
\left(\mathbb{H}_{-\frac{1}{2}} \phi\right)(x)=\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos (x y) \phi(y) d y
$$

(1.1) Lemma

Let $\phi \in L_{1}((0, \infty))$ with the property that

$$
\forall_{k \in \mathbb{N}_{0}}: \sup _{x \geq 0} x^{k}|\phi(x)|<\infty .
$$

Then for each $v \geq-\frac{1}{2}$, the Hankel transform $H_{v} \phi$ of $\phi$ is pointwise defined and $H_{v} \phi$ is an even continuous function on $\mathbb{R}$.

In the following lemma we present a comparison between the various Hankel transforms.
(1.2) Lemma

Let $-\frac{1}{2} \leq \mu<v$ and let $\phi \in L_{1}(0, \infty)$ satisfy

$$
\sup _{x \geq 0}\left|x^{k} \phi(x)\right|<\infty \text { and } \sup _{x \geq 0}\left|x^{l}\left(\mathbb{H}_{v} \phi\right)(x)\right|<\infty
$$

for all $k, l \in \mathbb{N}_{0}$. Then for all $x \geq 0$,

$$
\left(\mathbb{H}_{\mu}\left(\mathbb{H}_{v} \phi\right)\right)(x)=\frac{2^{\mu-v+1}}{\Gamma(v-\mu)} \int_{x}^{\infty}\left(\xi^{2}-x^{2}\right)^{v-\mu-1} \phi(\xi) \xi d \xi .
$$

## Proof

The proof is a consequence of the following integral formula

$$
\int_{0}^{\infty} t^{\mu-v+1} J_{\mu}(x t) J_{v}(\xi t) d t=\left\{\begin{array}{l}
\frac{2^{\mu-v+1}}{\Gamma(v-\mu)} x^{\mu} \xi^{-v}\left(\xi^{2}-x^{2}\right)^{v-\mu-1}, \quad \xi>x \\
0, \quad 0<\xi \leq x
\end{array}\right.
$$

cf. [7], p. 100.

## (2) The Schwartz space S

Let $\mathbf{S}$ denote the space of all rapidly decreasing $C^{\infty}$-functions, viz all $C^{\infty}$-functions $\phi$ with

$$
\sup _{x \in \mathbb{R}}\left|x^{k} \phi^{(l)}(x)\right|<\infty, \quad k, l \in \mathbb{N}_{0}
$$

The space $\mathbf{S}$ admits the following characterization.
(2.1) A function $\phi \in L_{1}(\mathbb{R})$ with Fourier transform $\mathbb{F} \phi$ belongs to $S$ if and only if for all $k \in \mathbb{N}_{0}$

$$
\sup _{x \in \mathbb{R}}\left|x^{k} \phi(x)\right|<\infty \text { and } \sup _{x \in \mathbb{R}}\left|x^{k}(\mathbb{F} \phi)(x)\right|<\infty .
$$

This characterization can be obtained by applying standard techniques, cf. [5].
Here we are interested in the space $\mathrm{S}_{\text {even }}$ of all even functions belonging to $\mathbf{S}$. It readily follows that an even function $\phi \in L_{1}(\mathbb{R})$ belongs to $\mathrm{S}_{\text {even }}$ if and only if for all $k \in \mathbb{N}_{0}$

$$
\sup _{x \geq 0}\left|x^{k} \phi(x)\right|<\infty \text { and } \sup _{x \geq 0}\left|x^{k}\left(H_{-\frac{1}{2}} \phi\right)(x)\right|<\infty .
$$

We want to replace the transformation $\mathbb{H}_{-\frac{1}{2}}$ by a Hankel transformation $\mathbb{H}_{v}$ of arbitrary order $v \geq-\frac{1}{2}$. Therefore, we use the result in [2] that the space $S_{\text {even }}$ is Hankel invariant. So $H_{v}\left(\mathrm{~S}_{\text {even }}\right)=\mathrm{S}_{\text {even }}$ for each $v \geq-1 / 2$.

## (2.2) Theorem

Let $v \geq-1 / 2$. An even function $\phi$ in $L_{1}(\mathbb{R})$ belongs to $\mathrm{S}_{\text {even }}$ if and only if for all $k, l \in \mathbb{N}_{0}$

$$
\sup _{x \geq 0}\left|x^{k} \phi(x)\right|<\infty \text { and } \sup _{x \geq 0}\left|x^{l}\left(\mathbb{H}_{v} \phi\right)(x)\right|<\infty .
$$

## Proof

Let $\phi \in \mathbf{S}_{\text {even }}$. Then $\sup _{x \geq 0}\left|x^{k} \phi(x)\right|<\infty$ for all $k \in \mathbb{N}_{0}$. Since also $\mathbb{H}_{v} \phi \in \mathbf{S}_{\text {even }}$, one side of the equivalence is settled.

Conversely, suppose for $\phi \in L_{1}(\mathbb{R}), \sup _{x \geq 0}\left|x^{k} \phi(x)\right|<\infty$ and $\sup _{x \geq 0}\left|x^{l}\left(\mathbb{H}_{v} \phi\right)(x)\right|<\infty$, $k, l \in \mathbb{N}_{0}$. We have to prove that for all $n \in \mathbb{N}_{0}$,

$$
\sup _{x \geq 0}\left|x^{n}\left(\mathbb{H}_{-\frac{1}{2}} \phi\right)(x)\right|<\infty .
$$

First observe that $H_{-\frac{1}{2}} \phi$ is continuous, so that for all $n \in \mathbb{N}_{0}$

$$
\sup _{0 \leq x \leq 1}\left|x^{n}\left(I H_{-\frac{1}{2}} \phi\right)(x)\right|<\infty
$$

We may assume $v>-\frac{1}{2}$. Then $\mathbb{H}_{-\frac{1}{2}} \phi=\left(\mathbb{I} H_{-\frac{1}{2}} I H_{v}\right)\left(\mathbb{I} H_{v} \phi\right)$ and by Lemma (1.2) we can write

$$
\left(H_{-\frac{1}{2}} \phi\right)(x)=C_{v} \int_{x}^{\infty}\left(\xi^{2}-x^{2}\right)^{v-\frac{1}{2}} \xi\left(H_{v} \phi\right)(\xi) d \xi
$$

with $C_{v}=\frac{2^{-v+1 / 2}}{\Gamma\left(v+\frac{1}{2}\right)}$. So for $x \geq 1$ we obtain the estimation

$$
\begin{aligned}
\left|x^{n}\left(H_{-\frac{1}{2}} \phi\right)(x)\right| & \leq C_{v} x^{2 v+n+1} \int_{1}^{\infty}\left(t^{2}-1\right)^{v-1 / 2} t\left|\left(H_{v} \phi\right)(x t)\right| d t \\
& \leq C_{v} \sup _{y \geq 1}\left|y^{2 v+n+2}\left(H_{v} \phi\right)(y)\right| \int_{1}^{\infty} \frac{\left(t^{2}-1\right)^{v-1 / 2}}{t^{2 v+1}} d t
\end{aligned}
$$

and the result follows.

In the proof of the above theorem we used a relation between $\mathbb{H}_{-\frac{1}{2}} \phi$ and $H_{v} \phi$ in order to deduce a growth estimate for $H_{-\frac{1}{2}} \phi$ from the growth estimate satisfied by $I H_{v} \phi$. The following lemma yields a generalization of this result; it will be applied in the next section.
(2.3) Lemma

Let $v>-\frac{1}{2}$ and let $W$ denote a nonnegative function such that the function $x^{-2 v-2} W(x)$ is nondecreasing on $[a, \infty)$ for some $a>0$.
Suppose $\phi \in \mathbf{S}_{\text {even }}$ satisfies the following growth estimate

$$
\sup _{x \geq 0} W(x)\left|\left(\mathbb{H}_{v} \phi\right)(x)\right|<\infty
$$

Then for all $\mu$ with $-\frac{1}{2} \leq \mu<v$ and all $\varepsilon>0$

$$
\sup _{x \geq 0}\left(1+x^{2}\right)^{\mu-v-\varepsilon} W(x)\left|\left(\not H_{v} \phi\right)(x)\right|<\infty .
$$

Proof
We use the same technique as in the proof of Theorem (2.2). Let $-\frac{1}{2} \leq \mu<\nu$ and let $\varepsilon>0$. Since $\mathbb{H}_{\mu} \phi \in \mathrm{S}_{\text {even }}$ we only have to consider $x \geq \max \{a, 1\}$. We write $\mathbb{H}_{\mu} \phi=\left(\mathbb{H}_{\mu} \mathbb{H}_{v}\right)\left(H_{\nu} \phi\right)$ and so by Lemma (1.2)

$$
\left(\mathbb{H}_{\mu} \phi\right)(x)=\frac{2^{\mu-v+1}}{\Gamma(v-\mu)} \int_{x}^{\infty}\left(\xi^{2}-x^{2}\right)^{v-\mu-1} \xi\left(\mathbb{H}_{v} \phi\right)(\xi) d \xi .
$$

Now let $\delta=\min \{\varepsilon, \mu+1\}$ and let $x \geq \max \{a, 1\}$. Then with a straightforward estimation

$$
\begin{array}{rl}
\left(1+x^{2}\right)^{\mu \nu-v} & W(x)\left|\left(\mathbb{H}_{\mu} \phi\right)(x)\right| \leq \\
& \leq C_{v, \mu} \int_{x}^{\infty} \frac{\left(\xi^{2}-x^{2}\right)^{v-\mu-1} \xi}{\left(\xi^{2}+x^{2}\right)^{v-\mu+\delta}} W(\xi)\left|\left(\mathbb{H}_{v} \phi\right)(\xi)\right| d \xi \\
& \left.\leq C_{v, \mu} \sup _{y \geq 1} W(y)\left|\left(\mathbb{H}_{v} \phi\right)(y)\right|\right) \int_{1}^{\infty} \frac{\left(t^{2}-1\right)^{v-\mu-1}}{\left(t^{2}+1\right)^{v-\mu+\delta}} t d t
\end{array}
$$

where $C_{v, \mu}=\frac{2^{\mu+2}}{\Gamma(v-\mu)}$. Hence the result.

## (3) The Gelfand-Shilov spaces $S_{\alpha}^{\beta}$

In the second volume [4] of their celebrated treatise on generalized functions Gelfand and Shilov introduce the following subspaces of the Schwartz space $S$. Let $\alpha \geq 0$ and $\beta \geq 0$.

$$
\begin{aligned}
& \mathbf{S}_{\alpha}:=\left\{\phi \in \mathbf{S} \mid \exists_{A>0} \forall_{l \in \mathbb{N}_{0}} \exists_{B_{l}>0} \forall_{k \in \mathbb{N}_{0}}:\right. \\
& \left.\sup _{x \in \mathbb{R}}\left|x^{k} \phi^{(l)}(x)\right| \leq A^{k} B_{l}(k!)^{\alpha}\right\} \\
& \mathbf{S}^{\beta}:=\left\{\phi \in \mathbf{S} \mid \exists_{B>0} \forall_{k \in \mathbb{N}_{0}} \exists_{A_{k}>0} \forall_{l \in \mathbb{N}_{0}}:\right. \\
& \left.\sup _{x \in \mathbb{R}}\left|x^{k} \phi^{(l)}(x)\right| \leq B^{l} A_{k}(l!)^{\beta}\right\} \\
& \mathbf{S}_{\alpha}^{\beta}:=\left\{\phi \in \mathbf{S} \mid \exists_{A>0, B>0, C>0} \forall_{k \in \mathbb{N}_{0}, l \in \mathbb{N}_{0}}:\right. \\
& \left.\sup _{x \in \mathbb{R}}\left|x^{k} \phi^{(l)}(x)\right| \leq C A^{k} B^{l}(k!)^{\alpha}(l!)^{\beta}\right\} .
\end{aligned}
$$

Only for $\alpha \geq 0, \beta \geq 0$ with $\alpha+\beta<1$ the space $S_{\alpha}^{\beta}$ is trivial, cf. [4], $\S$ IV. 8 .

As a consequence of Sobolev's lemma the supremum norm in the above definitions can be replaced by the $L_{2}(\mathbb{R})$-norm. In [1], Theorem 4.6, the elements $\phi$ of $S_{\alpha}^{\beta}, \alpha>0, \beta>0$, have been characterized in terms of the decay properties of $\phi$ and of its Fourier transform $\mathbb{F} \phi$ :

$$
\begin{align*}
\phi \in \mathbf{S}_{\alpha} \Longleftrightarrow & \exists_{t>0}: \sup _{x \in \mathbb{R}} \exp \left(t|x|^{1 / \alpha}\right)|\phi(x)|<\infty \quad \text { and }  \tag{3.1}\\
& \forall_{k \in \mathbb{N}_{0}}: \sup \left|x^{k}(\mathbb{F} \phi)(x)\right|<\infty \\
\phi \in \mathbf{S}^{\beta} \Leftrightarrow & \forall_{k \in \mathbb{N}_{0}}: \sup _{x \in \mathbb{R}}\left|x^{k} \phi(x)\right|<\infty \text { and } \\
& \exists_{t>0}: \sup _{x \in \mathbb{R}} \exp \left(t|x|^{1 / \beta}\right)|(\mathbb{F} \phi)(x)|<\infty \\
\phi \in \mathbf{S}_{\alpha}^{\beta} \Leftrightarrow & \exists_{t>0}: \sup _{x \in \mathbb{R}} \exp \left(t|x|^{1 / \alpha}\right)|\phi(x)|<\infty \text { and } \\
& \exists_{t>0}: \sup _{x \in \mathbb{R}} \exp \left(t|x|^{1 / \beta}\right)|(\mathbb{F} \phi)(x)|<\infty
\end{align*}
$$

In this section, we derive similar characterizations for the even subspaces $\mathbf{S}_{\alpha, \text { even }}, \mathbf{S}_{\text {even }}^{\beta}$ and $\mathbf{S}_{\alpha, \text { even }}^{\beta}$ in terms of decay estimates for an even function $\phi$ in $L_{1}(\mathbb{R})$ and its Hankel transform $\mathbb{H}_{v} \phi$. First, we present an important auxiliary result.

## (3.2) Lemma

Let $\mathbf{R}$ denote one of the spaces $\mathbf{S}_{\text {even }}, \mathbf{S}_{\alpha, \text { even }}, \mathbf{S}_{\text {even }}^{\beta}$ and $\mathbf{S}_{\alpha, \text { even }}^{\beta}, \alpha>0, \beta>0$. Then the differential operator $\frac{1}{x} \frac{d}{d x}$ maps $\mathbf{R}$ into $\mathbf{R}$.

## Proof

A simple application of Borel's theorem shows that for each $\phi \in \mathbf{S}_{\text {even }}$ there exists $\psi \in \mathbf{S}$ such that $\phi(x)=\psi\left(x^{2}\right), x \in \mathbb{R}$. So the operator $\frac{1}{x} \frac{d}{d x}$ maps $\mathrm{S}_{\text {even }}$ into $\mathrm{S}_{\text {even }}$.
Let $\phi \in \mathbf{S}_{\alpha, \text { even }}$. Then $\frac{1}{x} \phi^{\prime} \in \mathbf{S}_{\text {even }}$ and $\phi^{\prime} \in \mathbf{S}_{\alpha, \text { odd }}$.
So for all $k \in \mathbb{N} N_{0}, \sup _{x \in \mathbb{R}}\left|x^{k}\left(\mathbb{F}\left(\frac{1}{x} \phi^{\prime}\right)\right)(x)\right|<\infty$, and there exists $t>0$ such that $\sup _{x \in \mathbb{R}} \exp \left(t|x|^{1 / \alpha}\right)\left|\frac{1}{x} \phi^{\prime}(x)\right|<\infty$. It follows from (3.1) that $\frac{1}{x} \phi^{\prime} \in \mathrm{S}_{\alpha, \text { even }}$.
Next, let $\phi \in \mathbf{S}_{\text {even }}^{\beta}$. Then, as mentioned,

$$
\exists_{B>0} \forall_{k \in N_{0}} \exists_{A_{k}>0} \forall_{l \in N_{0}}:\left(\int_{-\infty}^{\infty}\left|\xi^{k} \phi^{(l)}(\xi)\right|^{2} d \xi\right)^{1 / 2} \leq A_{k} B^{l}(l!)^{\beta} .
$$

We have

$$
\phi(x)=\phi(0)+\frac{1}{2} x^{2} \phi^{(2)}(0)+\frac{1}{2} x^{3} \int_{0}^{1}(1-t)^{2} \phi^{(3)}(x t) d t .
$$

So for all $n \geq 2$ fixed, we have

$$
\begin{aligned}
\left(\frac{d}{d x}\right)^{n}\left(\frac{1}{x} \phi^{\prime}(x)\right) & =\frac{1}{2} x^{2} \int_{0}^{1}(1-t)^{2} t^{n+1} \phi^{(n+4)}(x t) d t+ \\
& +(n+3 / 2) x \int_{0}^{1}(1-t)^{2} t^{n} \phi^{(n+3)}(x t) d t+ \\
& +\frac{1}{2} n(n+2) \int_{0}^{1}(1-t)^{2} t^{n-1} \phi^{(n+2)}(x t) d t .
\end{aligned}
$$

First, we show that there exist $\bar{B}>0, \tilde{C}>0$ such that

$$
\begin{equation*}
\left(\int_{-\infty}^{\infty}\left|\left(\frac{d}{d x}\right)^{n}\left(\frac{1}{x} \phi^{\prime}\right)(x)\right|^{2} d x\right)^{1 / 2} \leq \bar{C} \bar{B}^{n}(n!)^{\beta} . \tag{*}
\end{equation*}
$$

For $j=0,1,2$ we have

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\left|\int_{0}^{1}(1-t)^{2} t^{n-1}(x t)^{j} \phi^{(n+2+j)}(x t) d t\right|^{2} d x \\
& \leq \int_{-\infty}^{\infty}\left\{\int_{0}^{1} t\left|(x t)^{j} \phi^{(n+2+j)}(x t)\right|^{2} d t d x\right. \\
& \leq\left\{A_{j} B^{n+2+j}(n+2+j)!^{\beta}\right\}^{2} .
\end{aligned}
$$

The result follows since there exist $\tilde{C}, \tilde{B}>0$ such that

$$
\begin{aligned}
& \frac{1}{2} A_{2} B^{n+4}(n+4)!^{\beta}+(n+3 / 2) A_{1} B^{n+3}(n+3)!^{\beta}+\frac{1}{2} n(n+2) A_{0} B^{n+2}(n+2)!^{\beta} \\
& \leq \tilde{C} \tilde{B}^{n}(n!)^{\beta} .
\end{aligned}
$$

Further, we observe that $\frac{1}{x} \phi^{\prime} \in \mathbf{S}_{\text {even }}$ and so for all $k \in \mathbb{N}_{0}$

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left|x^{k} \frac{1}{x} \phi^{\prime}(x)\right|^{2} d x<\infty \tag{}
\end{equation*}
$$

Now the result $\frac{1}{x} \phi^{\prime} \in \mathbf{S}_{\text {even }}^{\beta}$ follows from $\left(^{*}\right)$ and $\left({ }^{* *)}\right.$ by applying Theorem 4.5 in [1].
Finally, let $\phi \in \mathbf{S}_{\alpha, \text { even }}^{\beta}$. Then $\frac{1}{x} \phi^{\prime} \in \mathbf{S}_{\alpha, \text { even }}$ and $\frac{1}{x} \phi^{\prime} \in \mathbf{S}_{\text {even }}^{\beta}$, whence $\frac{1}{x} \phi^{\prime} \in \mathbf{S}_{\alpha, \text { even }} \cap \mathbf{S}_{\text {even }}^{\beta}=$ $\mathbf{S}_{\alpha, \text { even }}^{\beta}$.
(3.3) Lemma

Let $\mathbf{R}$ denote one of the spaces $\mathbf{S}_{\alpha, \text { even }}, \mathbf{S}_{\text {even }}^{\beta}$ and $\mathbf{S}_{\alpha, \text { even }}^{\beta}, \alpha>0, \beta>0$. Then for all $k \in \mathbb{N}_{0}, \mathbb{H}_{-\frac{1}{2}+k}\left(\mathbf{S}_{\alpha, \text { even }}\right)=\mathbf{S}_{\text {even }}^{\alpha}, \mathbb{H}_{-\frac{1}{2}+k}\left(\mathbf{S}_{\text {even }}^{\beta}\right)=\mathbf{S}_{\beta, \text { even }}$ and $\mathbb{H}_{-\frac{1}{2}+k}\left(\mathbf{S}_{\alpha, \text { even }}^{\beta}\right)=\mathbf{S}_{\beta, \text { even }}^{\alpha}$.

## Proof

The recurrence relations (cf. [7], p. 67)

$$
\left(\frac{1}{z} \frac{d}{d z}\right)^{k}\left(z^{-\mu} J_{\mu}(z)\right)=(-1)^{k} z^{-\mu-k} J_{\mu+k}(z), \quad k \in \mathbb{N}_{0}, \mu, z \in \mathbb{C}
$$

imply that for all $\phi \in \mathbf{S}_{\text {even }}$ and all $k \in \mathbb{N}_{0}$,

$$
\left(\frac{1}{x} \frac{d}{d x}\right)^{k} \mathbb{H}_{-\frac{1}{2}} \phi=(-1)^{k} \mathbb{H}_{-\frac{1}{2}+k} \phi
$$

Since $\mathbb{H}_{-\frac{1}{2}+k}\left(\mathbf{S}_{\alpha, \text { even }}\right) \subset S_{\text {even }}^{\alpha}$ ànd $\left(\frac{1}{x} \frac{d}{d x}\right)^{k}\left(S_{\alpha, \text { even }}\right) \subset S_{\alpha, \text { even }}$ we obtain $H_{-\frac{1}{2}+k}\left(S_{\alpha, \text { even }}\right) \subset$ $\mathbf{S}_{\text {even }}^{\alpha}$. Similarly, we obtain $\mathbb{H}_{-\frac{1}{2}+k}\left(\mathbf{S}_{\text {even }}^{\beta}\right) \subset \mathbf{S}_{\beta \text {,even }}$ and $\mathbb{H}_{-\frac{1}{2}+k}\left(\mathbf{S}_{\alpha, \text { even }}^{\beta}\right) \subset \mathbf{S}_{\beta, \text { even }}^{\alpha}$. Now the statements follow since $\left(\mathbb{H}_{-\frac{1}{2}+k}\right)^{2} \phi=\phi, \phi \in \mathbf{S}_{\text {even }}$.
(3.4) Corollary

The operator $\frac{1}{x} \frac{d}{d x}$ is invertible on $\mathbf{S}_{\text {even }}, \mathbf{S}_{\alpha, \text { even }}, S_{\text {even }}^{\beta}$ and $\mathbf{S}_{\alpha, \text { even }}^{\beta}$ with $\left(\frac{1}{x} \frac{d}{d x}\right)^{-1} \phi=-\int_{x}^{\infty} t \phi(t) d t, \phi \in \mathbf{S}_{\text {even }}$.

We arrive at the main theorem of this section.

## (3.5) Theorem

Let $\alpha, \beta>0, \quad v \geq-\frac{1}{2}$ and let $\phi \in L_{1}(\mathbb{R})$ be even.
I. $\quad \phi \in \mathbf{S}_{\alpha, \text { even }}$ iff $\exists_{t>0}: \sup _{x \geq 0} \exp \left(t x^{1 / \alpha}\right)|\phi(x)|<\infty \quad$ and

$$
\forall_{j \in \mathbb{N}_{0}}: \sup _{x \geq 0}\left|x^{j}\left(\mathbb{H}_{v} \phi\right)(x)\right|<\infty
$$

II. $\quad \phi \in S_{\text {even }}^{\beta}$ iff $\forall_{j \in \mathbb{N}}: \sup _{x \geq 0}\left|x^{j} \phi(x)\right|<\infty \quad$ and
$\exists_{t>0}: \quad \sup _{x \geq 0} \exp \left(t x^{1 / \beta}\right)\left|\left(\mathbb{H}_{v} \phi\right)(x)\right|<\infty$
III. $\quad \phi \in \mathbf{S}_{\alpha, \text { even }}^{\beta}$ iff $\exists_{t>0}: \quad \sup _{x \geq 0} \exp \left(t x^{1 / \alpha}\right)|\phi(x)|<\infty \quad$ and
$\exists_{t>0}: \quad \sup _{x \geq 0} \exp \left(t x^{1 / \beta}\right)\left|\left(H_{v} \phi\right)(x)\right|<\infty$.

Proof
For $v=-\frac{1}{2}$ the results are stated in (3.1). So we take $v>-\frac{1}{2}$ in the sequel.
I. Let $\phi \in \mathbf{S}_{\alpha, \text { even }}$. Then $\phi \in \mathbf{S}_{\text {even }}$ and by (3.1) there exists $t>0$ such that $\sup _{x \geq 0} \exp \left(t x^{1 / \alpha}\right)|\phi(x)|<\infty$. Since $\phi \in \mathrm{S}_{\text {even }}$, by Theorem (2.2), $\sup _{x \geq 0}\left|x^{j}\left(H_{v} \phi\right)(x)\right|<\infty$. Conversely, suppose $\phi$ satisfies the stated conditions. Then by Theorem (2.2), $\phi \in \mathbf{S}_{\text {even }}$. It follows that for all $k \in N_{0}, \sup _{x \geq 0}\left|x^{k}\left(\mathbb{H}_{-\frac{1}{2}} \phi\right)(x)\right|<\infty$ and so by $(3.1), \phi \in \mathbf{S}_{\alpha, \text { even }}$.
II. Let $\phi \in S_{\text {even }}^{\beta}$. Then by (3.1) for all $j \in N_{0}, \sup _{x \geq 0} x^{j}|\phi(x)|<\infty$ and, also, for all $l \in \mathbb{N}_{0}, \mathbb{H}_{-\frac{1}{2}+l} \phi \in \mathbf{S}_{\beta \text {,even }}$. So there exists $\tau>0$ such that

$$
\sup _{x \geq 0} \exp \left(\tau x^{1 / \beta}\right)\left|\left(\mathbb{H}_{-\frac{1}{2}+l} \phi\right)(x)\right|<\infty .
$$

Taking a fixed $l>v+1 / 2$ we obtain from Lemma (2.3)

$$
\sup _{x \geq 0} \exp \left(\tau x^{1 / \beta}\right)\left(1+x^{2}\right)^{v-l}\left|\left(I H_{v} \phi\right)(x)\right|<\infty .
$$

Hence for all $t, 0<t<\tau$

$$
\sup _{x \geq 0} \exp \left(t x^{1 / \beta}\right)\left|\left(\mathbb{H}_{v} \phi\right)(x)\right|<\infty .
$$

Conversely, if $\phi$ satisfies the stated conditions, then for all $j \in \mathbb{N}{ }_{0}, \sup _{x \geq 0}\left|x^{j} \phi(x)\right|<\infty$ and by Lemma (2.3)

$$
\sup _{x \geq 0} \exp \left(t x^{1 / \beta}\right)\left(1+x^{2}\right)^{-v-3 / 2}\left|\left(\mathbb{H}_{-\frac{1}{2}} \phi\right)(x)\right|<\infty .
$$

So $\phi \in \mathbf{S}_{\text {even }}^{\beta}$.
III. We observe that $\mathbf{S}_{\alpha, \text { even }}^{\beta}=S_{\alpha, \text { even }} \cap \mathbf{S}_{\text {even }}^{\beta}$.
(3.6) Corollary

Let $v \geq-\frac{1}{2}$ and let $\alpha>0, \beta>0$.
Then $\mathscr{H}_{v}\left(S_{\alpha, \text { even }}\right)=S_{\text {even }}^{\alpha}, \mathscr{H}_{v}\left(S_{\text {even }}^{\beta}\right)=S_{\beta, \text { even }}$, and $\mathscr{H}_{v}\left(S_{\alpha, \text { even }}^{\beta}\right)=S_{\beta, \text { even }}^{\alpha}$.

Proof.
These statements are consequences of the characterizations presented in Theorem (3.5), and the fact that $H_{v}\left(H_{v} \phi\right)=\phi$ for all $\phi \in \mathbf{S}_{\text {even }}$.

## Remark

Partly the results stated in the above corollary are known. In [3] it is proved that $H_{v}\left(\mathbf{S}_{\alpha, \text { even }}^{\alpha}\right)=S_{\alpha, \text { even }}^{\alpha}, \frac{1}{2} \leq \alpha \leq 1$, using properties of the Laguerre polynomials. In [8], the result $H_{v}\left(\mathbf{S}_{\alpha, \text { even }}^{\beta}\right)=\mathbf{S}_{\beta, \text { even }}^{\alpha}$ is stated in case $0<\alpha, \beta<1$. However, a number of proofs in [8] is incorrect. We refer to [9] for correct versions.

As we have seen $\left(\frac{1}{x} \frac{d}{d x}\right)^{l} \phi=\mathscr{H}_{-\frac{1}{2}+l}\left(\mathbb{H}_{-\frac{1}{2}} \phi\right), \phi \in \mathrm{S}_{\text {even }}$. Therefore we define the fractional differentiation operators

$$
\left(\frac{1}{x} \frac{d}{d x}\right)^{v}=H_{-\frac{1}{2}+v} \circ H_{-\frac{1}{2}}, \quad v \geq 0
$$

and the fractional integration operators

$$
\left(\frac{1}{x} \frac{d}{d x}\right)^{-v}=\mathbb{H}_{-\frac{1}{2}} \circ \mathbb{H}_{-\frac{1}{2}+v}, \quad v \geq 0 .
$$

The collection of operators $\left\{\left(\frac{1}{x} \frac{d}{d x}\right)^{v}\right\}_{v \in \mathbb{R}}$ establishes a one-parameter group on $S_{\text {even }}, S_{\alpha, \text { even }}, S_{\text {even }}^{\beta}$ and $S_{\alpha, \text { even }}^{\beta}$. Cf. [6], Ch. 5 for related results.

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