Tactical design of production-distribution networks : safety stocks, shipment consolidation and production planning

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van
de Rector Magnificus, prof.dr. R.A. van Santen,
voor een commissie aangewezen door het College
voor Promoties in het openbaar te verdedigen
op woensdag 21 mei 2003 om 16.00 uur

door

Sanne Smits

geboren te Huizen

Dit proefschrift is goedgekeurd door de promotoren:

prof.dr. A.G. de Kok
en
prof.dr.ir. W.M.P. van der Aalst

Copromotor:
dr. G.P. Kiesmüller

Voorwoord

Het onderzoek dat ten grondslag ligt van dit proefschrift is februari 1999 gestart. Van augustus 1998 tot januari 1999 heb ik mijn afstudeerderopdracht Econometrie bij CQM b.v. uitgevoerd. Deze opdracht bestond uit het ontwikkelen van een generiek model om in een distributienetwerk de optimale locaties van magazijnen te bepalen. Tijdens mijn stage viel het me op dat de bestaande strategische modellen een aantal lacunes hadden. Jan van Doremalen, mijn stage begeleider bij CQM, heeft mij toen attent gemaakt op een voorgesteld promotieonderzoek van de onderzoek school Beta dat precies op deze punten inging. Kort daarna ben ik door Peter van Laarhoven en Ton de Kok aangenomen voor dit project.

Veel dank ben ik verschuldigd aan mijn begeleiders en aan CQM. Peter van Laarhoven, die aan het begin van dit promotieproject mijn eerste promotor was, is helaas na ruim 1 jaar vertrokken naar TPG. Hij heeft me geholpen met het structureren van het onderzoek en het overzicht over het project te houden. Dank ook aan Ton de Kok die aan het begin van dit promotieproject mijn tweede promotor was is na het vertrek van Peter mijn eerste promotor geworden. Ton heeft me veel inspiratie gegeven en ideen aan de hand gedaan. Hij heeft me geholpen wanneer ik weer ergens in het onderzoek vast zat en veel geduld getoond bij het lezen van mijn teksten en van dit proefschrift. Ook heeft hij het mogelijk gemaakt dat ik heb kunnen samenwerken met Michael Wagner van de universiteit van Augsburg. Wil van der Aalst is aan het einde van dit promotieproject mijn tweede promotor geworden. Wil was een belangrijke en kritische lezer van dit proefschrift en zijn opmerkingen over het proefschrift waren nuttig. Naast de promotoren heb ik nog

een aantal dagelijkse begeleiders gehad. In het begin was dit Werner Rutten, die helaas kort na het begin van dit project ergens anders is gaan werken. Daarna is Bart Vos mijn begeleider geweest totdat ook hij na 1 jaar vertrokken is naar de universiteit van Tilburg. Het laatste anderhalve jaar van mijn project werd Gudrun Kiesmüller mijn dagelijkse begeleider. Ze heeft me veel geholpen met het duidelijker uitleggen van sommige onderdelen in dit proefschrift. Vanuit CQM waren Fred Janssen, Jan van Doremalen en Mijnt Zijlstra mijn contactpersonen. Verder heeft CQM dit onderzoek gedeeltelijk gesponseerd.

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Sanne Smits

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Chapter 1

Introduction

1.1 Objective and motivation

In the 1990's supply chain management emerged as a new management concept. It concentrates on interactions at the various levels in a supply chain. A supply chain, which is also referred to as a logistics network (cf. Simchi-Levi et al. (2000)), consists of suppliers, manufacturing centers, warehouses, distribution centers, and retail outlets. According to The Council of Logistic Management, logistics or supply chain management is: "The process of planning, implementing and controlling the efficient, costs effective flow and storage of raw materials, in-process inventory, finished goods, and related information from the point-of-origin to the point-of-consumption in order to meet customers' requirements." In this thesis we concentrate on a part of supply chain management, namely the tuning of the production process to the distribution process and the management of the distribution process itself. Two important elements in this are the evaluation of the costs and the customer's requirements.

In European companies the distribution costs represents on average 15 % of the selling price. However, this can vary depending on the type and the nature of the product. For example it is 30 % for light bulbs and 7 % for television sets and video players. The main elements of the distribution costs are the transportation costs with 32 %, the inventory costs with 31 % and the facility costs (warehouses, handling and internal transport) with 28 % (cf. van Damme (2000)). The previous figures illustrate that the distribution costs are significant and therefore it is necessary to analyse them carefully.

Besides costs another important element of logistic management are customers' requirements. A product is of little value if it is not available to customers at the time and place they wish to consume or use it. We distinguish between the required, desired and perceived customer service level. The first one is defined by the company, the second one is desired by the customers and the third one is the service level, which is observed by the customers. In practice we notice a gap between these three service levels. Let us illustrate this with the following example from Gourdin (2001): "PC manufacturing routinely announce 10 day order lead times (but the customers want the order in 3 days), 90 % of the orders are delivered directly from stock (the customers think it should be 95 % or 99 %). Yet when asking the customers what they actually got, you hear of order lead times between 20 and 30 days and 50 to 65 % of the orders which are delivered directly from stock". In other industries this mismatch also occurs. On average the companies announce to deliver 95 % of the orders on time and the customers only perceive 80 % (cf. van Goor et al. (1999)). This illustrates the importance of accurately evaluating the customers' service level.

Further, because the environment is constantly changing, it is necessary to evaluate the distribution costs and the customer service levels of the logistic network regularly. For example economic developments, like recessions, may affect the demand. Technological advances, like information systems, make it possible to speed up the check for the availability of materials. Regulations of the European Union encourage companies to increase the transport efficiency and the following will probably affect the transportation policies and costs in the near future.

From the previously mentioned facts we conclude that it is important to consider carefully distribution costs and customer service levels. To illustrate which decisions in the logistic network affect the costs and service levels, we introduce the following example. Let us consider a large industrial company, which produces and sells in

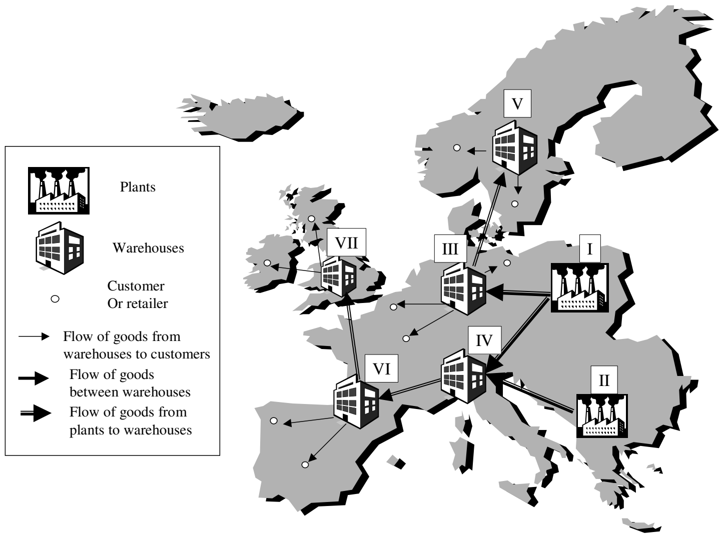


Figure 1.1: Logistic network.

Europe four electronic consumer goods which we name A, B, C and D (for example television sets, light bulbs, coffee machines and video players). The logistic network considered consists of two factories, five warehouses and nine retailers or customers. For the flow of goods we refer to Figure 1.1.

From benchmark research the company has observed that their distribution costs are 10 % higher than the industry average. Therefore, this company wants to reconsider its logistics network. When reconsidering the logistics network Ballou (1992) distinguishes between three different levels of decision-making depending on the time horizon, namely, strategic, tactical, and operational. The strategic level considers time horizons of more than one year. The operational level involves short-term decisions, often less than an hour or a day. The tactical level falls in between those extremes.

- Strategic level

  Typical strategic decisions are concerned with the opening or closing of a warehouse or a factory or the outsourcing of transportation. For example, where should new warehouses be located and what should be their capacity? Should the company have its own fleet of trucks or should it outsource the transportation? Another strategic decision is the allocation of production to a factory. For example, should the company produce the new item Z at factory II or I? Certain strategic decisions could also be assigned to the tactical level depending on the flexibility of the distribution or production process. For example, when the transportation and warehousing are outsourced, the flexibility of the distribution process increases because it is then possible to increase the space used at the warehouse or the amount of trucks used within a year.

- Tactical level

  Typical tactical decisions are related to the control of the operational processes, i.e. inventory, transportation, warehousing and production control. Inventory management addresses the following basic questions: what, when and from where should we order? Should the company order periodically or when the inventories are below a threshold value? Should we use joint replenishments or order goods independently? Transportation management addresses the following questions: When should goods be shipped and what mode should be used? Warehouse management addresses the following questions: where and how should the items be stored, what packaging unit should be used or which storage and equipment techniques should be chosen? Should the company use fully mechanized equipment at warehouse IV or not? Should the packaging unit be pallets or boxes? Should the company change from pallet to box at warehouses III and IV or at warehouses V, VI? Finally, production management addresses the following questions. What should the production lot be, how should the production be scheduled and are there scheduling priorities to be taken into account?

- Operational level

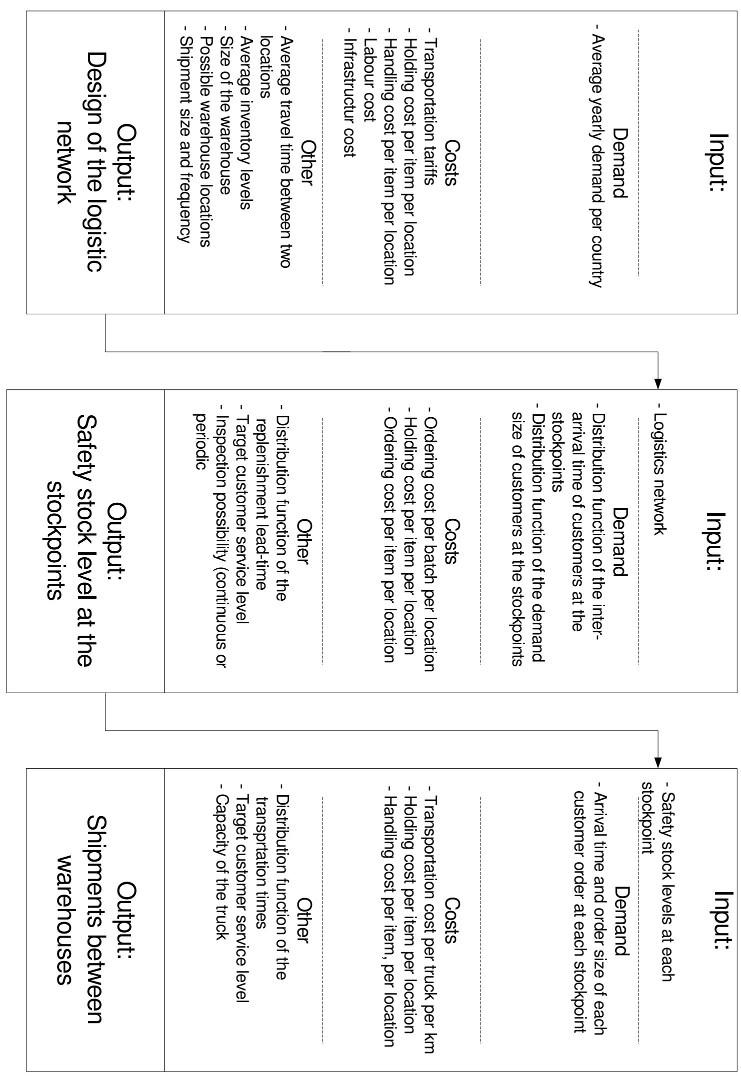
  Operational decisions are related to the operational processes, i.e. inventory, transportation, warehousing and production processes. The inventory process addresses for example the following questions. Should IV order some extra product because a large demand is expected in the near future or should the company tranship some products from III to IV? A question at the transportation process could be for example which route should be taken? Should the company first deliver the customer in France and then the one in Belgium or the other way round? A question at the production process is for example when should the company expedite the order of product Z?

A distinction between these three levels is made because they use different information. This will be illustrated with an example: Figure 1.2 indicates which information is necessary to tackle the strategic decision "What should be the design of the network ?", the tactical decision "What should be the level of safety stocks at each stockpoint ?" and the operational decision "What should be shipped between warehouses ?".

To be able to tackle these decisions planning information is required related to future periods up to some time horizon, which differs for the three levels of decision-making (strategic, tactical and operational). Unfortunately, most information is not known with certainty. For example what will be the yearly demand in France three years from now? What will be the demand for product A at warehouse III in two months and what will be the transportation time between the customer in France and the one in Belgium tomorrow? To overcome this problem, forecasts are made of the unknown information.

When the company has to decide between different possible solutions for each of these decision-making levels it is important that the required information is reliable. However, there can be a high degree of uncertainty in these forecasts. A technique used to increase the reliability of the information is the aggregation of data. For example, when we want to know the demand in two years, we do not forecast the demand for A, B, C and D separately, but we forecast the aggregated demand over A, B, C and D. Besides product aggregation there is also time aggregation and

Figure 1.2: Information necessary to tackle a strategic, tactical and operational decision



customer aggregation. An example of time aggregation is not to forecast the weekly demand but the yearly demand.

Generally speaking, the reliability of information about the behaviour of a system in some future decreases. The further this period is from the present point on time the less reliable the information is. In order to guarantee some degree of reliability one needs to aggregate information about the system behaviour for periods further out into the future. This implies that the aggregation level is different for the three levels of decision-making. This implies for example that the strategic decisions are based on average yearly demand per country, tactical decisions are based on the distribution functions of the interarrival time and demand size per item per customer and the operational decisions are based on forecasts and actual data about each customer order.

For the three levels of decision-making also different kinds of models are used. The models dealing with strategic decisions are suited to compare a large number of different possible logistic networks. To be able to find a solution within reasonable time often linear models are used. If the model is linear, efficient techniques like linear programming can be used to quickly find the optimal solution of the model under consideration. The tactical models generally assume more detail and include non-linearity in the model. Finally, the operational models again use often linear models because the time horizon is short and the time available to find a solution is restricted.

As we have explained previously the strategic models use aggregated forecasts and linear models. What is the consequence of using these approximations? Previously we mentioned that it is essential to estimate correctly the customer service level. To satisfy the customer service level, it is necessary to keep stocks at the warehouses. We will illustrate this with the following example. When warehouse VII places an order at VI, the replenishment lead-time is on average 7 days. The replenishment lead-time is the time elapsed between the placement and the receipt of an order. When should we place at VII an order such that the required service level is satisfied? The idea is to place the order early enough, such that the number of items demanded during the replenishment lead-time will not result in a stock out too often. Since the demand and replenishment lead-time are unknown in advance, we estimate them by means of forecasts. To take into account the uncertainties in the forecasts of the demand and replenishment lead-time some extra stocks are kept at the warehouse, which we call safety stocks. Since the strategic models only forecast the average replenishment lead-time and average yearly demand, these safety stocks are neglected or assumed to be exogenous to the model. However, these safety stocks are in fact endogenous to the model. An exogenous variable is a variable, which is not influenced by the model, and an endogenous variable is one, which is influenced by it. When the logistics network changes due to a decision at the strategic level, for example warehouse VII is closed and the customers in England and Ireland are re-allocated to III, not only the average demand at warehouse III changes but also the uncertainty in the demand changes and therefore the safety stocks are in fact endogenous to the model. Another variable, which is endogenous in reality and exogenous in most strategic models, is the replenishment lead-time. For example if IV is closed and the replenishment orders of VI are re-allocated to I and II or if the amount of safety stocks at IV is reduced then the replenishment lead-time of the products at VI will probably increase. In the following section we will explain how different strategic and tactical decisions influence the replenishment lead-time.

1.2 Replenishment lead-time

Before we explain which strategic and tactical decisions affect the replenishment lead-time, we will describe the processes at the factories and warehouses in more detail. In the example, we have four different items A, B, C, and D. They are produced at the production facility at factory I and the items C and D at factory II, see Figure 1.3. There are stockpoints and a consolidation dock at each factory. The stockpoints are necessary to satisfy demand coming from warehouse III and IV on time and the consolidation dock is needed to consolidate the orders for warehouse III and IV to increase the transportation efficiency, i.e. minimization of the transportation costs using economies of scale. Note that at the consolidation dock two

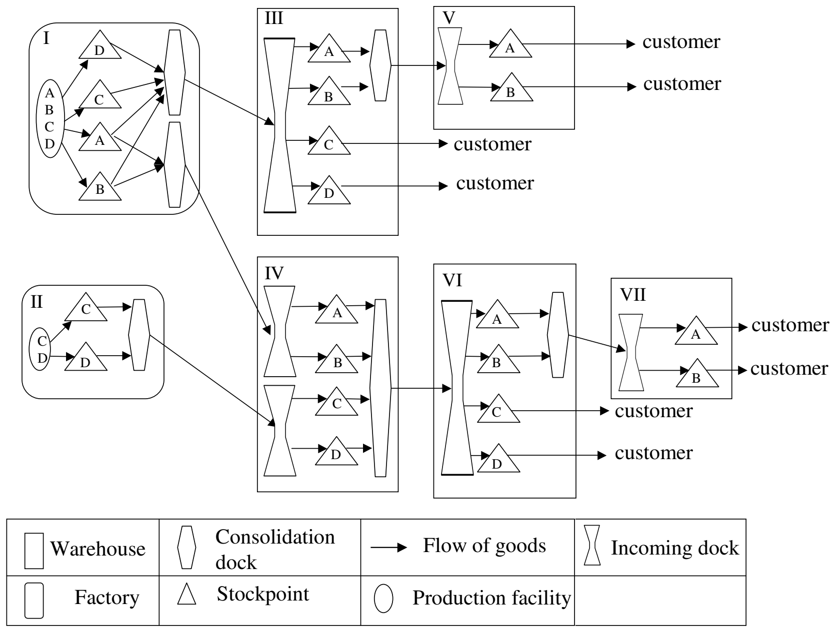


Figure 1.3: Schematic representation of the logistic network.

separate consolidation processes for the two warehouses take place. At warehouses III and IV the orders are arriving at an incoming dock. From there they are internally shipped to the stockpoints where they are available for demand coming from V and customers for III and from VI for IV. Further at V, VI and VII similarly to III and IV there are an arrival dock and stockpoints and at III, IV and VI there are also consolidation docks. We consider a divergent logistic network. A divergent network is a network where each stockpoint has a single predecessor.

At the warehouses III, V, VI and VII customer demand for the different items arrives. This can be described by an interarrival time distribution and a demand size distribution for each item. For example on average one order of product A is placed during a week at warehouse V and the average order size is 50 products. For both interarrival times and order sizes, the averages and variances are forecasted. To satisfy the customers' requirements, enough stock should be kept at each end stockpoint. An end-stockpoint is a stockpoint where customer demand takes place. Therefore, a target customer service level at the end-stockpoints is defined, for example a target percentage of orders that should be delivered directly from stock. We assume that orders are backordered if the warehouse is out of stock, which means that they wait until the stock is replenished.

To manage the inventory process such that this target customer service level is met, we need to know the replenishment lead-time. As explained previously, in most strategic models the replenishment lead-time is assumed to be an exogenous variable. This means that the replenishment lead-time of a stockpoint is an input parameter of the model. In the following we will demonstrate that the lead-time is influenced in many ways by strategic and tactical decisions. Therefore, the replenishment lead-time should be considered as an endogenous variable. Its four elements are: an exogenous delay, the waiting time due to lack of stock at the preceding warehouse, the waiting time due to shipment consolidation and the waiting time due to production.

The first element of the lead-time we consider is the exogenous delay, which is denoted by L^d. This element can be split up in three parts: the time needed to administrate incoming order ($L^{d(1)}$), the time needed to handle the order in the warehouse ($L^{d(2)}$) and the time needed for external transport from the warehouse to the delivery point ($L^{d(3)}$).

A second element of the replenishment lead-time is the waiting time due to a lack of stock at the preceding warehouse, which we will denote as W^s. Recall that in the example the inventories were located at all warehouses and factories. This means that the inventories at warehouses VII, V and VI are respectively backed up by inventories at warehouses VI, III and IV. In turn, inventories at III and IV are backed up by inventories at the two factories II and I. If substantial amounts of inventories are maintained at VI, IV and III, then the probability of being out of stock at these locations is small and orders for the warehouses V, VI VII do not have to wait very often due to a stock out at their preceding warehouse. The waiting time due to a lack of stock at the preceding warehouse is dependent on the structure of the logistic network and on the amount of safety stocks at the preceding stockpoint. In the example, the waiting time of VII due to a lack of stock at its preceding warehouse changes if for example the company decides to deliver VII from III instead of VI, if the safety stocks at VI change, if VII places different orders, and if the lead-time towards VI changes.

A third element of the replenishment lead-time is the waiting time due to shipment consolidation, which we will denote W^c. It depends on the structure of the logistic network and on the chosen consolidation policy. In the example we have consolidation docks at the factories and warehouses. Suppose we ship between I and III when a full truckload is consolidated. If the distribution network changes and VII is re-allocated to warehouse III, it will take less time until a full truckload at I is consolidated because the demand intensity is larger. On the other hand it will take more time to consolidate a full truck from I to IV, because the demand intensity at VI has decreased. Further, W^c changes also when the shipment consolidation policy changes, for example the company decides to ship every three days instead of waiting until the truck is full.

The last element of the replenishment lead-time is the waiting time due to production, W^p. It is influenced by the structure of the logistic network, by the production schedule and by a change in the demand at the production facility. Suppose we re-allocate the production of item C from the production facility at factory II to the production facility at factory I, the utilization degree of the production facility at factory I will increase and thus the waiting time due to production will increase. Note that only the beginning stockpoints experience a waiting time due to production. The beginning stockpoints are the stockpoints which are replenished by a production facility.

In Figure 1.4, we recapitulate all the different elements of the replenishment lead-time.

Besides that the replenishment lead-time is necessary in determining the level of safety stocks such that the customer service level is satisfied, it is also essential in evaluating accurately the costs of the following decisions: where should the company

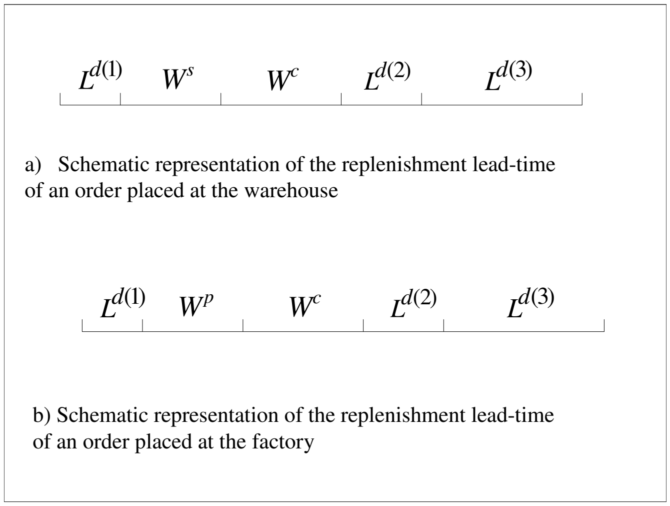


Figure 1.4: Schematic representation of the replenishment lead-time.

place the safety stocks, what transportation frequency should be used and what should be the production schedule?

A tactical decision in the logistic network is the choice of where to place the safety stocks. For example how much safety stocks should be kept at VI and VII? If less safety stocks are kept at VI, then the waiting time due to a lack of stock at VI and the replenishment lead-time towards VII increase. Since the replenishment lead-time increases, more safety stocks are needed at VII to keep the same customer service level. So the inventory costs at VI decrease and at VII the inventory costs increase. Therefore, to decide where to keep the safety stocks, it is essential to include the waiting time due to a lack of stock at the preceding stockpoint in the replenishment lead-time.

Deciding on the frequency of transportation between two locations also belongs to the tactical level. For example should the company transport twice or once a week between warehouse IV and VI? If we ship once a week instead of twice, then the waiting time due to shipment consolidation and the replenishment lead-time decreases. Further, if we ship once a week instead of twice the replenishment order size increases. Both, the replenishment lead-time and the replenishment order size, influence the amount of safety stocks necessary to keep the same customer service level and hence the inventory costs. The transportation costs increase because off more frequent shipments.

To model the trade-off between the inventory and transportation costs it is essential to include the waiting time due to shipment consolidation in the replenishment lead-time.

Finally, other tactical decisions in the logistic network are the decisions on production lot sizes and production sequence. At the production facility different products can be produced. When the production changes from one product to another, the production facility must be set-up and this incurs costs, which is called set-up costs. If the production schedule changes and product A is produced every week instead of every four weeks, then the waiting time due to production, the replenishment lead-time and the inventory costs change and the set-up and production costs increase due to more frequent set-ups. Therefore, to model the change in inventory and production costs due to a change in the production schedule it is again essential to include the waiting time due to production time into the replenishment lead-time.

In summary, we advocate that in models dealing with strategic and tactical decisions in the logistic network, the replenishment lead-time should be an endogenous variable in order to make proper trade-offs between customer service levels, inventory holding costs, transportation costs and manufacturing costs.

From above follows that the replenishment lead-time (L) can be formulated as

$$L = L^d + W^s + W^c + W^p \tag{1.1}$$

In this thesis we provide insight in the following questions:

- How is the replenishment lead time influenced by safety stocks at the preceding stockpoint?

- How is the replenishment lead time influenced by shipment consolidation policies?

- How is the replenishment lead-time influenced by the production schedule?

Before handling in detail these questions, we describe the used methodology in this thesis.

1.3 Methodology

In the literature only a few models have been studied in which the replenishment lead-time is considered to be endogenous. In these models all the variables, like for example the interarrival time, the replenishment lead-time, the demand and the batchsizes are often assumed to be continuous variables. Whereas in practice above mentioned variables are often discrete variables. This approximation is necessary to reduce the complexity of the problem. It is commonly assumed that the demand is pure Poisson or Compound poisson. The interarrival times are in both (the pure Poisson and compound Poisson) exponentially distributed and the demand sizes are in pure Poisson unitary and in compound Poisson arbitrary distributed random variables. Besides that the analysis restricts to single or two-echelon models. Each echelon is a level between the production facility and the customer where stock is kept. The example in Figure 1.3 is a four-echelon network. However, the two assumptions (pure Poisson demand or Compound Poisson demand and single or two-echelons models) do not always hold, in logistic networks the interarrival times are not always exponentially distributed and the number of echelons is usually more than two. The contribution of this thesis to literature is that we assume compound renewal demand and multiple echelons. In compound renewal demand, the interarrival time as well as the order size have an continuous arbitrary distribution. Relaxing the compound Poisson demand and the two-echelon network increases the complexity of the problem and therefore it is difficult, if not impossible, to find exact expressions for the average and variance of the replenishment lead-time. Since it is difficult to find exact expressions, we provide approximations for the average and variance of the replenishment lead-time based on asymptotic results from renewal theory. The validity of these approximations is tested by discrete event simulations. A large number of different cases is simulated to measure the sensitivity of the approximations to certain input parameters.

1.4 Outline of the thesis

This thesis is concerned with the evaluation of multi-product, multi-echelon logistic networks. The development of fast and accurate approximations for the first two moments of the replenishment lead-time is the key to such an evaluation.

In the first part of Chapter 2, we review the literature on strategic and tactical decisions for logistic networks. We distinguish between five different model classes. The first class consists of models supporting the tactical decision level with exogenous lead-time. The second class consists of tactical models where W^s is included in L. The third class are the tactical models where W^c is included in L. The fourth class consider W^p as a element of L and the fifth class consists of models supporting the strategic decision level. For each model we describe the network structure, the assumptions used and how the customer service level and the trade-offs in costs are modeled. In the second part of Chapter 2 key results in renewal theory are summarized. These results are used to derive the approximations for the average and variance of the replenishment lead-time.

In Chapter 3, we investigate the influence of the safety stocks at the preceding stockpoint on the replenishment lead-time. We consider a single product multi-echelon distribution system, for example see Figure 1.5. The customer demand arrives at the end-stockpoint according to a compound renewal process. The customer service level under consideration is expressed as a target fill rate, which is defined as the percentage of orders that are delivered directly from stock. In multi-echelon models the demand process at non end-stockpoints and the replenishment lead-time at non-starting stockpoints are unknown. Therefore, in Chapter 3 an algorithm is developed to determine the demand process at each stockpoint by a compound renewal demand process. Further, approximations are derived for the average and variance of the waiting time due to a lack of stock at the preceding stockpoint. Finally, in the last part of Chapter 3 extensive discrete event simulations validate the approximations.

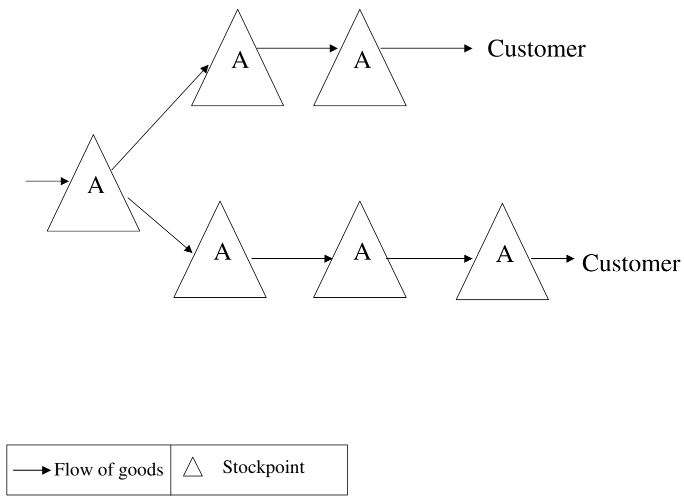


Figure 1.5: Multi-echelon inventory problem.

In Chapter 4, we investigate the influence of shipment consolidation policies on the replenishment lead-time in multi-product, multi-echelon distribution networks. We consider a multi-echelon distribution network with multiple products and shipment consolidation between the warehouses to increase transportation efficiency, see Figure 1.6. Similar to Chapter 3, we assume compound renewal customer demand and target fill rates at the end-stockpoints. We distinguish between two different shipment consolidation policies: the time policy and the quantity policy. The time policy dispatches the orders when a target shipping date has expired. The quantity policy dispatches the orders when a target quantity is consolidated. Further, when a stockpoint is out of stock, the arriving orders wait first until the stockpoint is replenished and then the orders are shipped to the consolidation dock and wait until the next scheduled shipment departs. For the time policy we derive exact

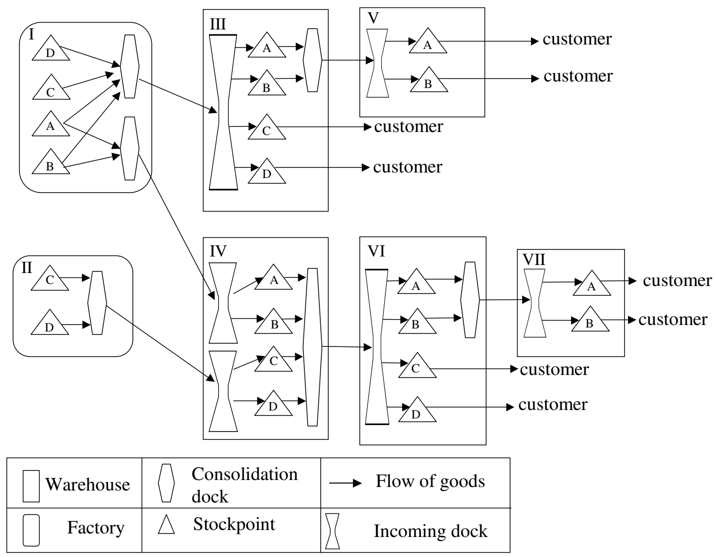


Figure 1.6: Inventory and transportation problem.

expressions for the distribution function of the waiting time due to shipment consolidation (W^c) and we prove that W^c and W^s are independent of each other. For the quantity policy we derive approximations for the average and variance of W^c under the assumption that W^c and W^s are independent of each other. The validity of the approximations and the independence assumptions are tested via discrete event simulation.

In Chapter 5, we investigate the influence of the production schedule on the replenishment lead-time in a multi-product, single echelon logistic network. We consider a factory with one production facility, which produces the products, and multiple stockpoints, one for each product. A schematic representation of the model is given in Figure 1.7. Similar to Chapter 3 and 4 we assume compound renewal demand and target fill rates at the stockpoints. The replenishment orders arrive at the production facility and are produced in a First In First Out (FIFO) order.

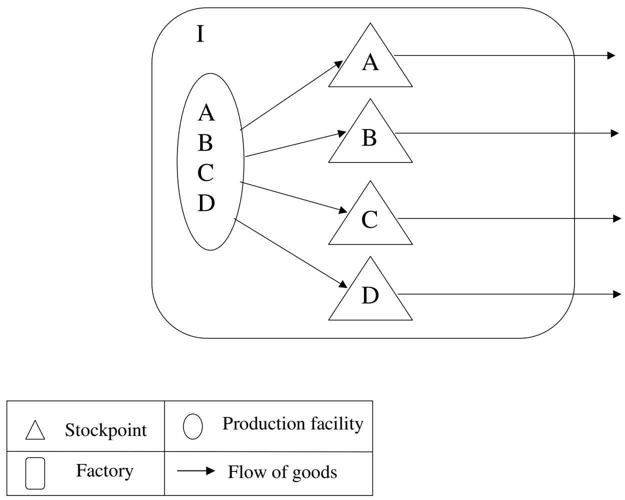


Figure 1.7: Inventory and production problem.

The production facility needs a fixed time to produce one unit. Further, the production facility is set-up between the production of two orders. The replenishment lead-time of a stockpoint consists of waiting time before the machine is available for production, set-up time and time necessary to produce the entire order. We derive approximations for the average and variance of the waiting time and the total production time and the validity of these approximations is tested via discrete event simulation. In the last part of Chapter 5 we develop a local search heuristic to determine close to optimal production schedules.

Finally, in Chapter 6, a summary of the main results is given, and some topics for future results are proposed.

Chapter 2

Literature review and mathematical background

2.1 Literature review

This thesis focuses on the evaluation of multi-product, multi-echelon logistic networks. In Chapter 1, we have explained that the replenishment lead-time should be an endogenous variable in models dealing with strategic and tactical decisions in the logistic network. This in order to determine accurately the costs and the levels of safety stocks such that required customer service levels are achieved.

In the first part of this chapter, we will review the literature dealing with strategic and tactical decisions in logistic networks and describe how in previous research the level of safety stocks and the costs are determined. We distinguish five model classes as mentioned in Chapter 1. The first class consists of tactical models considering a simple single item stockpoint. The second class are the tactical models considering a single item, multi-echelon logistic system with multiple stockpoints. The third class are the tactical models considering a multi-item, single echelon model with shipment consolidation. The fourth class consists of models considering the tactical decisions on production lot sizes and production sequence in a multi-item logistic network.

The last class consists of the models supporting the strategic decision level. In each of these classes an extensive literature exists. Therefore, we refer the reader to recent reviews and discuss the most important contributions. For each contribution, we describe briefly the considered logistic network, the different control concepts, the operational decisions to be taken, the modeling of the costs, the replenishment lead-time, the customer service level, and the considered objective.

2.1.1 Tactical models

Single item inventory models

Let us start with the simplest logistic model: a single item logistic model with one stockpoint. The two fundamental tactical decisions considered in this system are when and how much to order?

When deciding when to order, the literature distinguishes between two ways of monitoring the inventories: continuous and periodic review. In continuous review policies an order is placed as soon as the inventory position drops below a threshold value. The inventory position is equal to the physical inventory level plus the amount of items on order minus the backorders. We denote the inventory position by Y. The threshold value is called the reorder level and is denoted by s. In periodic review policies an order is placed when at the review moment Y is below s. We denote the review period by R.

When deciding which quantity to order, the literature distinguish between: order-up-to and order quantity policies. In order-up-to policies an order is placed to raise the inventory position up to a target level, which we denote by S. In order quantity policies a multiple of a fixed quantity is ordered. This fixed quantity is called the batchsize and is denoted by Q. Upon ordering, the amount ordered is such that the inventory position is raised to a value between s and $s+Q$. The decisions, when to order and which quantity to order, give rise to four type of policies. The first type is the (s,nQ)-policy. The (s,nQ)-policy is a control policy where Y is reviewed continuously and if Y drops below s, nQ is ordered such that Y is raised between s and $s+Q$. Note that in the literature often (s,Q)-policies are assumed, which is a policy where the replenishment order size is always equal to Q, but this is only true when the probability that the demand is larger than the batchsize is negligible. The second type is the (R,s,nQ)-policy. The (R,s,nQ)-policy is the same as the (s,nQ)-policy, but instead that the inventory position is reviewed continuously, it is reviewed each R units of time. The third type is the (s,S)-policy. The (s,S)-policy is a control policy where Y is reviewed continuously and if Y drops below s, an order is placed as to raise the inventory position up to the order-up-to level S. The fourth type is the (R,s,S)-policy. The (R,s,S)-policy is the same as the (s,S)-policy but instead that the inventory position is reviewed continuously, it is reviewed each R units of time. Note that a (R,S)-policy is a policy, where at each review period the inventory is inspected and an order is placed as to raise the inventory position up to S. In chapters 3 and 4 of this thesis, we consider (s,nQ)-inventory policies and in Chapter 5 (R,S)-inventory policies.

Excellent reviews of papers considering the single item inventory model are given in Lee and Nahmias (1993), Silver et al. (1998) and Scarf (2002). The most important contributions in this area are summarized below.

Harris (1913) is the first one who has analyzed this kind of model. He considers stationary deterministic demand per period. The inventories are controlled by (s,Q) policies and the replenishment lead-time is assumed to be zero. The customer service level is expressed as a constraint. It is assumed that the demand is always satisfied directly from stock. The reorder level s is always equal to zero, since the replenishment lead-time is zero and the demand is deterministic and must always be satisfied directly from stock. The author derives the optimal batchsize Q in terms of cost. The considered costs are the fixed ordering, the unit ordering and the holding costs. This optimal Q is denoted as the Economic Order Quantity (EOQ), which is quite robust and can also be used for much broader applications.

Wagner and Whitin (1958) consider the same model but with deterministic dynamic demand per period. They develop a dynamic optimization model to determine the optimal batchsizes in terms of costs.

Scarf and Karlin (1958) assume stationary stochastic demand per period. They assume that the product lifetime is only one planning period. This is the case, for example, for newspapers (hence: 'news vendor problem'). They consider holding, ordering and penalty costs and optimize the order quantity. In this model the required customer service level is not expressed as a constraint but is implicitly taken into account by considering penalty costs.

Hadley and Whitin (1963) give a rigorous treatment of an (s,nQ)-model. The demand per time unit is assumed to be randomly distributed. Further, Hadley and Whitin (1963) assume that the reorder level, the batchsizes and the demand are continuous variables. They consider ordering, holding and penalty costs and determine the optimal s and Q in terms of cost. The essential result obtained is that the inventory position immediately after an arrival of a customer is homogenously distributed on the interval $(s,s+Q)$.

Tijms and Groeneveld (1984) present simple approximations for the reorder point in periodic and continuous review (s,S) inventory systems with service level constraints. The considered costs are the holding and ordering costs and with a given S, the optimal reorder level s is determined. Further, they assume compound Poisson customer demand and stochastic replenishment lead-times. In compound Poisson demand, the interarrival times of customers at the stockpoint are exponentially distributed and the demand sizes have an arbitrary distribution. They introduce a technique to fit mixed-Erlang distributions, which will also be used throughout this monograph.

De Kok (1991) presents approximations derived from asymptotic results for renewal processes to determine the customer service level given the inventory control parameters (i.e. s and Q for (s,nQ) policies). The author considers four different control policies: (s,nQ), (s,S), (R,s,nQ) and (R,s,S). Further, he assumes compound renewal customer demand and stochastic lead times. In compound renewal demand, the interarrival time as well as the order size have an arbitrary distribution.

Single item, multi-echelon inventory models

The tactical decisions considered in a single item, multi-echelon logistic system are when and how much to order at each stockpoint in the system. The difference between the multi and single echelon models is that in the multi-echelon models the replenishment lead-time is an endogenous variable instead of being an exogenous variable. The endogenous component of the replenishment lead-time in multi-echelon, single item models is the waiting time due to a lack of stock at the preceding stockpoint (W^s). For reviews of this model we refer to Federgruen (1993), Axsäter (1993a), Diks et al. (1996), and Houtum et al. (1996).

The literature considering single item, multi-echelon logistic networks, distinguishes between two different control concepts: centralized and decentralized control. Under decentralized control each stockpoint is controlled separately. This means that the ordering decision is, similar to the one stockpoint logistic system, based on the inventory position, Y, of the stockpoint itself. In contrast to this, the stockpoints in a logistic system can also be controlled centrally, which means that the ordering decisions depend on the state of the system as a whole. A specific centralized control concept is based on the echelon inventory position. The echelon inventory position of a stockpoint equals its echelon stock plus all material in transit to that stockpoint. The echelon stock of a stockpoint is the sum of its physical stock plus the transit to or on hand at its downstream stockpoints minus possible backorders at its end-stockpoints.

The literature distinguishes 4 types of single item, multiple echelon logistic networks. The first type is the serial system, which is a system where each stockpoint has a single predecessor and successor. The second type is the divergent system, which is a system where each stockpoint has a single predecessor and multiple successors. The third type is the convergent system, this system is characterized by multiple predecessors and a single successor and the last type is the general networks with multiple successors and predecessors.

Most important contributions considering the centralized control concept are the following articles.

Scarf and Clark (1960) were the first ones to define the concept of echelon stock. They assume that the inventories are controlled by periodic (R,S)-policies with stationary stochastic demand per period. The considered costs are holding, ordering and penalty costs. For a serial system, they derive exact values for the control parameters (i.e. R and S) of the inventory policies.

Chen and Zheng (1994) consider (s,nQ) policies in serial systems and develop a recursive procedure to derive exact steady-state results for the echelon inventory positions given the inventory control parameters (i.e. s and Q). These exact results can be used to evaluate the long-run average holding and penalty costs, as well as the customer service level.

Diks and de Kok (1999) present fast, accurate and easy-to-implement algorithms that generate close-to-optimal echelon periodic order-up-to-policies for divergent networks in terms of cost. They assume stochastic demand per period and consider penalty and holding costs.

De Kok and Visschers (1999) generalize the model of Scarf and Clark (1960) for assembling systems. An assembly system is a logistic system where each item may be built from multiple items. With each item a stockpoint is associated, implying that a stockpoint may have multiple predecessors.

Most important contributions considering the decentralized control concept are the following articles.

Deuermeyer and Schwartz (1981) analyze (s,nQ) policies in a divergent two-echelon system. They assume a distribution system with one warehouse and several identical retailers allocated to it. The customer demand at the retailers is modeled by a pure Poisson process. In a pure Poisson process, the interarrival times are exponentially distributed and the demand sizes are unitary. The authors determine the customer service level for given inventory control parameters (i.e. s and Q are given). Their replenishment lead-time is composed of a delay (L^d) and a retard (waiting time due to a lack of stock at the higher echelon stockpoint). The delay is fixed and the mean retard is evaluated. From this they estimate the average replenishment lead-time and approximate the lead-time distribution by a normal distribution with the same mean.

Svoronos and Zipkin (1988) propose several refinements to improve the model proposed by Deuermeyer and Schwartz (1981). The most notable one is that they estimate both the mean and the variance of the replenishment lead-time and approximate the lead-time distribution by a mixture of two shifted Poisson distributions with the same mean and variance. This leads to more accurate approximations since also the variance of the replenishment lead-time is taken into consideration.

Axsäter (1993b) presents a two-echelon inventory system with decentralized controlled (s,Q) policies, pure Poisson customer demand and fixed delay. Andersson et al. (1998) consider the same model but assume compound Poisson customer demand. In both models it is assumed that all Q's are identical and that the reorder point and the initial inventory position are multiples of Q. Both models assume that Q is fixed and use exact formulae to determine the waiting time due to lack of stock at the preceding stockpoint and to optimize the reorder levels in terms of costs. The costs evaluation includes inventory holding costs at both echelons and shortage costs.

Axsäter (2001b) considers a two-echelon model with stationary compound Poisson demand. The inventories are controlled by decentralized (s,nQ) policies. A technique is developed to derive the waiting time due to a lack of stock and to optimize the reorder levels s for given batchsizes Q in terms of costs. The considered costs are the holding and penalty costs.

In Chapter 3 of this thesis, we consider a multi-echelon divergent system with compound renewal customer demand. The replenishment lead-time consists of delay and waiting time due to a lack of stock. The delay is an exogenous variable and we assume that it has an arbitrary distribution. The waiting time due to a lack of stock is an endogenous variable. The inventories are controlled by decentralized (s,nQ) policies. We derive approximations for the average and variance of the waiting time due to a lack of stock at the preceding stockpoint and for the average and variance of the replenishment lead-time. Further, given the batchsizes Q, we determine the reorder levels s such that the target customer service levels are satisfied.

In the first column of Table 2.1 the most important contributions are summarized and the third column indicates how the replenishment lead-time (L) is modeled. When $L = L^d$ then the replenishment lead-time consists only of a exogenous delay (L^d) and when $L = L^d + W^s$ then it includes also the endogenous waiting time due to lack of stock at the preceding stockpoint (W^s).

Table 2.1: Tactical single item inventory models

Assumptions	Tactical models	Replenishment lead-time
Single echelon model with stationary deterministic demand per period	Harris (1913)	$L = L^d$
Single echelon model with stationary stochastic demand per period	Scarf and Karlin (1958) Hadley and Whitin (1963)	$L = L^d$ $L = L^d$
Single echelon model with stationary compound Poisson demand	Tijms and Groeneveld (1984)	$L = L^d$
Single echelon model with stationary compound renewal demand	De Kok (1991)	$L = L^d$
Single echelon model with dynamic deterministic demand	Wagner and Whitin (1958)	$L = L^d$
N-echelon, serial model with stationary stochastic demand per period	Scarf and Clark (1960) Chen and Zheng (1994)	$L = L^d + W^s$ $L = L^d + W^s$
N-echelon, assembly model with stationary stochastic demand per period	De Kok and Visschers (1999)	$L = L^d + W^s$
N-echelon, divergent model with stationary stochastic demand per period	Diks and de Kok (1999)	$L = L^d + W^s$
2-echelon,divergent model with stationary pure Poisson customer demand	Deuermeyer and Schwartz (1981) Svoronos and Zipkin (1988) Axsäter (1993b)	$L = L^d + W^s$ $L = L^d + W^s$ $L = L^d + W^s$
2-echelon, divergent model with stationary compound Poisson demand	Andersson et al. (1998) Axsäter (2001a)	$L = L^d + W^s$ $L = L^d + W^s$
N-echelon, divergent model with stationary compound renewal demand	Chapter 3 of this thesis	$L = L^d + W^s$

Multi-item inventory models with shipment consolidation

Up to now, we have only considered models dealing with single item logistic networks. The models mentioned in the preceding two sections assume that when the replenishment order is available at the preceding stockpoint, it is shipped immediately without any consolidation. Shipment consolidation is the logistic process of combining two or more orders of different products for the same intermediary warehouse to increase transport efficiency. In the following, we assume a multi-item logistic network where the replenishment orders of the different items at a warehouse are consolidated to increase transport efficiency.

For a review of the literature on deterministic and stochastic models in this area, see Chapter 11 of Silver et al. (1998), Goyal and Satir (1989), van Eijs (1993), and Simpson and Erenguc (1995).

The literature considering multi-item logistic networks with shipment consolidation distinguishes between two classes of models: joint replenishment models and models that explicitly consider the shipment consolidation process. In joint replenishment models, the replenishment moments of the different items at a warehouse coincide and the replenishment lead-time is exogenous. In the models, that explicitly consider the shipment consolidation process, the replenishment moments do not coincide but the consolidation is realized by letting the replenishment orders wait for a certain time or until a certain quantity is consolidated at the preceding warehouse. In these models the replenishment lead-time is an endogenous variable and the endogenous component of the replenishment lead-time is the waiting time due to shipment consolidation (W^c).

Below we summarize the most important contributions to the analysis of joint replenishment models.

Silver (1974) assumes a one echelon multi-item inventory system with continuous can-order inventory policies and pure Poisson customer demand. Further, the replenishment lead-times are assumed to be equal to zero. In the continuous can-order policy, (s_i,c_i,S_i) where $s_i \leq c_i < S_i$, an order is triggered for item i when its inventory position falls to or below the reorder level s_i. Any other item j for which the inventory position is at or below its can order level c_j is included in this order and the inventory position of each item j included in the order is replenished up to S_j. The author provides an iterative method to compute suboptimal (s_i,c_i,S_i) policies. This heuristic method decomposes the coordinated control problem into single item problems. Each single item problem has "normal" replenishment opportunities with major setup costs, occurring at the demand epochs for this item; in addition there are special replenishment opportunities, with reduced setup costs, at epochs generated by a Poisson process which is an approximation of the superposition of the ordering processes triggered by the other items.

Federgruen et al. (1984) assume a one echelon multi-item inventory system with continuous can-order inventory policies, compound Poisson customer demand and stochastic replenishment lead-times at the stockpoints. For each item i close-to-optimal values for s_i, c_i and S_i are derived in terms of cost. The considered costs are major ordering cost (in Silver (1974) called normal cost), minor ordering cost (in Silver (1974) called reduced setup cost), inventory cost and penalty cost. An algorithm is presented to minimize the long run average costs. The algorithm uses a heuristic decomposition procedure to transform the multi-item problem into single-item subproblems, which are solved using a specialized policy iteration technique.

Atkins and Iyogun (1988) present a comparison between periodic and can-order policies for coordinated multi-item, single echelon inventory systems. In the periodic review policies, the review periods of the different items are multiples of some base period. The main results of their paper is that a simple lower bound is available for the evaluation of heuristics, where periodic replenishment policies can out-perform can-order policies for coordinated replenishment inventory systems. An important advantage of coordinated periodic review policies is that the determination of close-to-optimal policies is relatively easy.

Van Eijs (1994) considers a periodic review policy for the coordinated multi-item, single echelon inventory system with the possibility to enlarge the order such that a target quantity is reached. He assumes pure Poisson customer demand and proposes a heuristic which decides whether to enlarge the initial order or not at a review time. This decision is based on a comparison of the expected extra holding cost and the expected extra saved shipping cost from an extra order. The author performs numerous simulations, which show that the total cost can be substantially decreased (up to 20 %) in case the shipment is enlarged.

As stated above, the joint replenishment models coordinate the replenishment moments, thereby ensuring economics in transportation. However, such economies can also be achieved in situations without such coordination of replenishment moments. In that case the consolidation of shipments must be explicitly modeled.

Higginson and Bookbinder (1994) distinguish between three types of consolidation policies: the time policy, the quantity policy and the time-quantity policy. The time policy dispatches the orders when a shipping date is expired. The quantity policy dispatches the orders when a target quantity is consolidated and the time-quantity policy dispatches the orders when one of both (shipping date or target quantity) is reached. They assume a single echelon network with stationary stochastic demand per period. Numerous simulations are performed for a large range of order-arrival rates and these policies are compared on basis of costs per load and average order delay. From these results, the author defines a rule, which indicates when to use the different policies. This rule is based on the percentage of minimum cost per unit weight accumulated in holding time. When this percentage is between 0 and 0.73 there is no clear choice of which policy to use. When this percentage is between 0.73 and 1.4 the time policy should be used and when this percentage is larger than 1.4 the quantity or the time-quantity policy should be used.

Higginson and Bookbinder (1995a) give some normative approaches to set the shipping date in single echelon networks with stationary stochastic demand per period.

Higginson and Bookbinder (1995b) consider a single echelon network with a time-quantity based shipment consolidation policy and compound Poisson customer demand. The author proposes a discrete-time Markovian decision process approach for determining when to release consolidated loads. Whenever a customer places an order, a choice must be made between dispatching this order plus all other waiting immediately or continuing to consolidate until the next arrival. The cost considered are the transportation and holding costs. The author proposes two kind of transportation cost function: the private fleet and the common carriage.

Çetinkaya and Lee (2000) assume an analytical model for coordinating inventory and transportation decisions in single echelon networks. They consider a time-based consolidation policy and compound Poisson customer demand. The stockpoints at the warehouse (vendor) are controlled by (s,S) inventory policy and the replenishment lead-time of the stockpoint is assumed to be negligible and therefore $s = 0$. The authors compute the optimal replenishment quantity ($Q = S - s$) and the dispatch frequency simultaneously in terms of costs. The considered costs are procurement, holding, customer waiting, and transportation costs.

Table 2.2: Multi-item logistic networks with shipment consolidation

Assumptions	Tactical models	Replenishment lead-time
Single echelon model with stationary stochastic demand per period	Higginson and Bookbinder (1994) Higginson and Bookbinder (1995a)	$L = L^d + W^c$ $L = L^d + W^c$
Single echelon model with stationary pure Poisson demand	Silver (1974) van Eijs (1994)	$L = L^d$ $L = L^d + W^c$
Single echelon network with stationary compound Poisson demand	Federgruen et al. (1984) Atkins and Iyogun (1988) Higginson and Bookbinder (1995b) Çetinkaya and Lee (2000) Axsäter (2001a)	$L = L^d$ $L = L^d$ $L = L^d + W^c$ $L = L^d + W^c$ $L = L^d + W^c$
N-echelon, divergent model with stationary compound renewal demand	Chapter 4 of this thesis	$L = L^d + W^s + W^c$

Axsäter (2001a) shows that the model of Çetinkaya and Lee (2000) is optimized by an approximate technique and he suggests a new approximation and an adjustment that can be used to improve both the original and the new heuristic.

In Chapter 4 of this thesis, we integrate the shipment consolidation decision into the multi-echelon distribution network. Thus, we consider a multi-item, multi-echelon logistic network with shipment consolidation and compound renewal customer demand. The replenishment lead-time consists of an exogenous delay, an endogenous waiting time due to lack of stock at the preceding stockpoint and an endogenous waiting time due to shipment consolidation. The inventories are controlled by decentralized (s,nQ) policies. For the time policy we derive exact results for the distribution of the waiting time due to shipment consolidation and we prove that the waiting time due to shipment consolidation and the waiting time due to a lack of stock are independent of each other. For the quantity policy approximations are derived for the average and variance of the waiting time due to shipment consolidation and the replenishment lead-time. Further, given the shipment consolidation policy and the batchsizes Q, we determine the reorder levels s such that the target customer service levels are met. Table 2.2 summarizes the literature on multi-item logistic networks with consolidated shipments.

Multi-item inventory models and production scheduling

Besides the need to consider the influence of shipment consolidation on the replenishment lead-time, it is also necessary to consider the influence of production scheduling on the replenishment lead-time. In multi-item tactical models supporting production lot-sizes and production sequence decisions, an endogenous component of the replenishment lead-time is the waiting time due to production (W^p). The operational decisions considered are at which moment and how much to order and when and how much to produce? For a general literature review, we refer to Chapter 11 in Silver et al. (1998), for a review on models with deterministic demand we refer to Lawler et al. (1993). For models considering stochastic lot scheduling we refer to Graves (1980), Qiu and Loulou (1995), Vergin and Lee (1978), Leachman et al. (1994, 1991), and Sox et al. (1999). The stochastic lot scheduling model deals with the scheduling of multiple items with random demand on a single facility and significant setups.

The literature on stochastic lot scheduling distinguishes between three different control concepts: independent stochastic control, joint deterministic control and joint stochastic control concept. In the independent stochastic control concept, single item inventory control policies (i.e. (s,S) or (s,Q)) are used to generate the quantity and release time. The replenishment order release process determines the production schedule with the possibility of priority rules. In the joint deterministic control approach, the schedule and the inventory control policy are constructed simultaneously based on the deterministic demand. Afterwards safety stocks and recovery policies (like overtime or machine disruption) are integrated to guarantee a certain service level. In the joint stochastic control concept, the schedule and inventory control policy are constructed based on the stochastic demand.

The most important contribution considering the independent stochastic control is the following article.

Lambrecht and Vandaele (1996) assume an inventory-production system with one production facility and multiple stockpoints, one for each item. The replenishment orders of all the items are produced with an exponential rate on the production facility with significant setup times. At the stockpoints, which are controlled by (s,Q) inventory policies, the customer demand arrives according to a pure Poisson process. When the production of a replenishment order is finished, the production facility is setup to produce the next replenishment order. The orders are handled in a FIFO order. An approximation is developed to determine the queueing delays and the lead-time.

The articles considering the joint deterministic control are the following articles.

Gallego (1990) consider a production-inventory system with a joint deterministic control and pure renewal customer demand. The author determines from the expected demand per period the close-to-optimal cyclic production schedule in terms of costs with a power-of-two algorithm (see Khouja et al. (1998)). The costs considered are setup, and holding costs. This implemented deterministic schedule may be disrupted when the inventory position of an item is too low. The author formulates the disruption problem as a control problem and obtains a linear recovery policy. Given the recovery policy, the level of safety stocks is determined via simulations such that the customer service level is satisfied.

Bourland and Yano (1994) consider a production-inventory system with joint deterministic control and pure renewal customer demand. The stockpoints are controlled by continuous (s,Q) inventory policies. From the expected demand a pure rotation cycle is constructed and instead of disrupting the machine like Gallego

(1990) overtime is used in shortage situations.

Anupindi and Tayur (1998) develop a simulation-based heuristic to optimize the (s,S)-policy for a given cyclic schedule with compound renewal customer demand. Three different types of service levels are assumed: expected costs based on response times, service levels based on quoted lead times and expected fraction of orders that are satisfied immediately from stock.

Markowitz et al. (2000) propose a class of dynamic policies. The pure rotation cycle is constructed from the expected demand. At each moment in time the production facility has three options: produce a unit of the product that is set-up, change over to the next product in the cycle or remain idle. The authors consider compound renewal demand and propose a production rule, based on heavy traffic approximations. The production rule consists of a dynamic lot-sizing policy and a server idling threshold value based on the system wide vector of inventories.

Most important contributions considering the joint stochastic control concept are the following papers:

Federgruen and Katalan (1994) consider an inventory-production system with pure Poisson customer demand. The stockpoints are controlled by (s,S) policies, where $s = S - 1$, and at the production facility there is a queue for each item. The production facility visits the queues in a cyclic order, produces a given number of orders and then goes to the next queue. The replenishment lead-time consists of waiting time in the queue, setup time, and production time. This model is a so-called polling model. The authors derive for this polling model approximations for the waiting time distribution for the gated and exhaustive policy given the production cycle and the inventory control parameters (i.e. given S). In the exhaustive policy all orders in the queue are produced before the production facility switches to the production of the next item. In the gated case only the orders present at its arrival are produced.

Further, Federgruen and Katalan (1996) determine the inventory control parameters for a given periodic production sequence and Federgruen and Katalan (1998) complement their paper of (1996) and determine the optimal production sequence and inventory control parameters simultaneously in terms of cost. The costs considered are holding, penalty and setup costs.

In Chapter 5 of this thesis, we consider a joint stochastic control concept. The inventories are controlled by (R,S) policies and compound renewal customer demand is assumed. In this model, the review periods R determine the periodic production sequence. The replenishment orders from the release process are produced on a first in, first out order. The replenishment lead time consists of waiting time before the production starts, setup time and production time. In the first part of Chapter 5, given the production sequence, approximations for the average and variance of the waiting time and the replenishment lead-time are derived and the order-up-to levels S are evaluated such that the target customer service levels are satisfied. In the second part of Chapter 5, a local search algorithm is developed to optimize simultaneously the inventory control parameters (i.e. R and S) and the production sequence in terms of cost under the condition of satisfying a target customer service level. The costs considered are the holding and setup costs.

Table 2.3 summarizes the literature in this area.

Table 2.3: Multi-item logistic networks with production scheduling

assumptions	Tactical models	Replenishment lead-time
Independent stochastic control with pure Poisson customer demand	Lambrecht and Vandaele (1996)	$L = W^p$
Joint deterministic control with pure renewal customer demand	Gallego (1990) Bourland and Yano (1994) Markowitz et al. (2000)	$L = W^p$ $L = W^p$ $L = W^p$
Joint deterministic control with compound renewal demand.	Anupindi and Tayur (1998)	$L = W^p$
Joint stochastic control with pure Poisson customer demand	Federgruen and Katalan (1994) Federgruen and Katalan (1996) Federgruen and Katalan (1998)	$L = W^p$ $L = W^p$ $L = W^p$
Joint stochastic control with compound renewal customer demand	Chapter 5 of this thesis	$L = W^p$

2.1.2 Strategic models

In spite that this thesis focusses on tactical decisions, for sake of completeness, we consider also the recent literature supporting strategic decisions in logistic networks since these models consider implicitly tactical decisions. For example when different designs are compared in terms of costs, one of the costs components is the inventory costs. To determine these inventory costs the inventory control parameters should be known.

Aikens (1985) presents an early review of the literature from 1963 until 1978. Vidal and Goetschalckx (1997) review the literature dealing with large scale production-distribution networks. Owen and Daskin (1998) consider the literature on stochastic and dynamic customer demand and Dasci and Verter (2001) review the literature considering continuous facility location, which means that the costs are represented by spacial distribution of the costs and customer demand.

Balachandran and Jain (1976) assume a single echelon model. The logistic network in this model consists of factories, warehouses and customers. Only at the warehouse inventories are kept and they are controlled by (R,S) policies. A stationary stochastic demand per period is given. The considered decisions are the location of the factories and the allocation of the warehouses to the factories. Both the location of the factories and allocation of warehouses to the factories are expressed by binary variables. The costs consist of shipping, facility, holding and penalty costs. The objective is to minimize the costs and close-to-optimal solutions are found for this model by means of heuristics.

Gross et al. (1981) consider a serial multi-echelon distribution network with stochastic customer demand per period. The model considers a number of possible network structures for which the inventory and penalty costs are evaluated with the model of Scarf and Clark (1960). The inventories are controlled by (R,S) echelon stock policies. These inventory and penalty costs per network structure are input for a linear optimization model where also transportation and facility costs are considered.

Fleischmann (1993) assumes a logistic network with factories, central warehouses and transhipment points or regional warehouses. The operational decisions to be taken are the number, location and required capacity of the warehouses and the replenishment frequency. The costs considered are the warehousing, inventory and transportation costs. The warehousing costs are modeled as the fixed costs to open a facility at a location. The inventories are decomposed into working stock and safety stock. The working stock is a half of the average replenishment quantity, but at least the daily throughput. The safety stock is assumed to be proportional to $\sqrt{x}$, where x is the demand during the lead-time at a warehouse. The following approximation is derived by Eppen (1979) and holds under strong assumptions. For the transportation costs, a transport tariff for a single shipment of quantity q from i to j is given. From the demand per period and the replenishment frequency the average replenishment quantity is estimated. From this average replenishment quantity, a discrete distribution of the replenishment quantity is derived by multiplying it with a given weight function. The model is optimized by an iterative local linearization of the costs.

Goetschalckx et al. (1995) consider a logistic network with factories, warehouses and customers. The stationary deterministic demand per period per customer per product is assumed to be given. The decision variables are the number and location of warehouses and factories, the connections between the factories, warehouses and customers and the transportation modes. The considered costs are the inventory, transportation, facility and pipeline costs. Similar to Fleischmann (1993), the model distinguishes between working stock and safety stock. However, in this model the safety stocks are exogenous to the model. Further, different transport modes (carriers) are considered in this model. Each transportation mode has a different replenishment frequency, capacity of the carrier and transportation costs of one carrier from i to j. For the evaluation of the transportation costs, the volume and weight of the average shipped quantity is evaluated with help of the demand per period and the replenishment frequency. From the volume and weight of the average shipped quantity, the number of used carriers and transportation costs are determined. The facility costs are expressed as fixed costs to open or close a facility. The stock in the pipeline is evaluated in a similar manner as the working stock, thus equal to one half of the average replenishment quantity. The customer service level is expressed by the exogenous safety stocks and a maximum distance between the warehouse and the customer. The resulting model is linear and can be solved to optimality with known techniques from linear programming.

Cole (1995) considers a multi-echelon serial logistic network with stationary stochastic demand per period. The considered costs are similar to Goetschalckx et al. (1995) and he uses the model of Scarf and Clark (1960) to evaluate the level of safety stocks. The objective function of the model is linearized to find a solution within reasonable time with a local search algorithm.

Vidal and Goetschalckx (2000) assume a model with stochastic lead times. The model shows how uncertainties in the lead-time affect the design of the production-distribution network.

Table 2.4: The modeling of inventories in tactical and strategic models considering single item inventory models

Assumptions	Tactical models	Replenishment lead-time
Single echelon model with stationary deterministic demand per period	Fleischmann (1993) Goetschalckx et al. (1995)	$L = L^d$ $L = L^d$
Single echelon model with stationary stochastic demand per period	Balachandran and Jain (1976) Vidal and Goetschalckx (2000)	$L = L^d$ $L = L^d$
Single echelon model with dynamic deterministic demand per period	Arntzen et al. (1995) Dogan and Goetschalckx (1999)	$L = L^d$ $L = L^d$
N-echelon, serial model with stationary stochastic demand per period	Gross et al. (1981) Cole (1995)	$L = L^d + W^s$ $L = L^d + W^s$

The literature supporting strategic decisions in the logistic network compare different logistic networks in terms of cost. One important cost component is the inventory cost. To evaluate the inventory levels and cost, the strategic models use two types of tactical models mentioned previously. The first type is the single-item, single echelon models with exogenous replenishment lead-times and the second type is the single item, serial N-echelon models with endogenous replenishment lead-times. For references, we refer to Table 2.4. An interesting question would be to consider multi-item, multi-echelon networks and to investigate the effect of shipment consolidation, production lot sizes and production sequence decisions on the replenishment lead-time in those networks, which will be the subject of this thesis.

2.1.3 Concluding remarks

In the literature on the evaluation and optimization of logistics networks an important discriminator between the various contributions is whether the replenishment of an order is assumed to be exogenous or endogenous. In Chapter 1, we concluded that in multi-item, multi-echelon networks with shipment consolidation, production lot sizes, and production scheduling decisions it is important to determine accurately the costs and the customer service level. One way to realize this is to model the replenishment lead-time as an endogenous variable. An advantage of this method is that it permits to decompose these complex networks into simple single stockpoint models, which can be solved independently of each other.

Table 2.5 summarizes how the replenishment lead-time is modeled in the different papers. The components of the replenishment lead-time L are an exogenous delay (L^d), the waiting time due to a lack of stock at the preceding stockpoint (W^s), the waiting time due to shipment consolidation (W^c) and the waiting time due to production (W^p).

In the following, we summarize the main contributions of this thesis to the literature.

- In Chapter 3, we derive analytical approximations for the average and variance of the waiting time due to lack of stock at the preceding stockpoint in divergent multi-echelon, single item networks with compound renewal customer demand and no restriction on the demand sizes and batchsizes.

- In Chapter 4, we derive analytical approximations for the average and variance of the waiting time due to shipment consolidation for the time and the quantity policy in divergent multi-echelon, multi-item networks with compound renewal customer demand.

Table 2.5: Literature considering endogenous replenishment lead-times

Replenishment lead-time	Literature
$L = L^d$	Harris (1913), Scarf and Karlin (1958) Hadley and Whitin (1963), Tijms and Groeneveld (1984) De Kok (1991), Wagner and Whitin (1958), Silver (1974) Federgruen et al. (1984), Atkins and Iyogun (1988) Fleischmann (1993), Goetschalckx et al. (1995) Vidal and Goetschalckx (2000), Dogan and Goetschalckx (1999)
$L = L^d + W^s$	Scarf and Clark (1960), Gross et al. (1981) Deuermeyer and Schwartz (1981), Svoronos and Zipkin (1988) Chen and Zheng (1994), Cole (1995), De Kok and Visschers (1999) Axsäter (1993b), Andersson et al. (1998), Diks and de Kok (1999) Axsäter (2001a), Chapter 3 of this thesis
$L = L^d + W^c$	van Eijs (1994), Higginson and Bookbinder (1994, 1995a,b) Çetinkaya and Lee (2000), Axsäter (2001a)
$L = L^d + W^s + W^c$	Chapter 4 of this thesis
$L = W^p$	Gallego (1990, 1994), Bourland and Yano (1994) Federgruen and Katalan (1996, 1998) Lambrecht and Vandaele (1996), Anupindi and Tayur (1998) Markowitz et al. (2000), Chapter 5 of this thesis

- In Chapter 5, we derive analytical approximations for the average and variance of the waiting time due to production, and we develop a local search algorithm which determines simultaneously the optimal production schedule and inventory control parameters, such that the target customer levels are satisfied, in terms of cost.

Besides that the approximations presented in this thesis contribute to literature they can also be a valuable contribution to industry practice. Because the logistic network of numerous companies consist of multiple items and multiple echelons. Furthermore, throughout this thesis compound renewal demand is assumed and this demand process suits better to the real customer demand process than demand processes where only the interarrival times are stochastic (pure renewal demand) or only the demand size is stochastic. Finally, the approximations presented are quite easy to implement and do not need excessive computation time.

2.2 Mathematical background

Many of the approximations applied in this thesis are derived from asymptotic results from renewal processes. For example to determine the level of safety stocks such that a target customer service level is satisfied, the distribution function of the demand during the replenishment lead-time should be known. The demand during the replenishment lead-time depends on the amount of customers arriving at the stockpoint during the replenishment lead-time. This counting process is called a renewal process. In Section 2.2.1, we give a general introduction to renewal theory and present asymptotic approximations for the first two moments of the counting process and for the first two moments of the quantity collected on a time interval.

For certain random variables, like the demand during the replenishment lead-time, the exact distribution function is intractable, because off the complexity of the problem. If this is the case, we evaluate the first two moments of the random variable and we approximate the distribution of the random variable by a mixed-Erlang distribution with the same first two moments. In Section 2.2.2, we define first mixed-Erlang distributions and then we present this fitting technique.

To be able to evaluate the level of safety stocks, the arrival process at a stockpoint should be known. What is the total (superposed) interarrival process of multiple interarrival processes at a stockpoint and what is the order size process of multiple order size processes at a stockpoint? In Section 2.2.3, asymptotic approximations are derived to evaluate the first two moments of the superposed interarrival process and in Section 2.2.4, asymptotic approximations are derived to evaluate the first two moments of the superposed order size process.

The approximations derived in this thesis are validated by discrete event simulations. In Section 2.2.5, a standard notation for error measurement is presented. Before we continue, let us first introduce the functions and operators used.

Used functions and operators.

$E[X]$	Expectation of the random variable X
$\sigma^2(X)$	Variance of the random variable X
$E[X^2]$	Second moment of the random variable X
c_X	Coefficient of variation of X, $c_X = \frac{\sigma(X)}{E[X]}$
$(X)^+$	Maximum of the random variable X and zero
ρ_X	Auto-correlation coefficient of stochastic process $\{X_n\}_{n=1}^{\infty}$
$F_X(x)$	Cumulative distribution function of X, $F_X(x) = P\{X \leq x\}$
$f_X(x)$	Probability density function of X
$P\{A\}$	Probability of event A
$F_X^{(n)*}(x)$	n-fold convolution of $F_X(x)$
$M_X(x)$	Renewal function, $M_X(x) := \sum_{n=0}^{\infty} F_X^{n*}(x)$
$\lceil A \rceil$	Largest integer smaller than or equal to A
$\lfloor A \rfloor$	Smallest integer larger than or equal to A

A list of notations used throughout this thesis is given at the end of this thesis.

2.2.1 Renewal theory

A renewal process is a counting process where the times between successive events are independently distributed with an arbitrary distribution function. Renewal theory began with the study of problems related to the failure and replacements of components, such as electric light bulbs. Since then it has been applied to a wide range of practical problems, including inventory management and queuing problems. Let X_i denote the time between the $(i-1)$st and ist renewal, $i \geq 1$ and $X_i \geq 0$. X_i is a continuous random variable. We define S_n as the time until the n-th renewal

$$S_0 := 0 \qquad \textit{and} \qquad S_n := \sum_{i=1}^{n} X_i \qquad n \geq 1 \tag{2.1}$$

We define $N(t)$ as the number of renewals up to t. Then $\{N(t), t > 0\}$ is a counting process defined by

$$N(t) := \max\{n : S_n \leq t\} \tag{2.2}$$

Given S_n and $N(t)$, Ross (1993) defines a renewal process as follow

Definition 2.1 *If the sequence of non-negative random variables $X_1, X_2, \ldots$ is independent and identically distributed, then the counting process $\{N(t), t > 0\}$ is a renewal process.*

Distribution of $\mathbf{N(t)}$

Note first that $N(t) \geq n \Longleftrightarrow S_n \leq t$. From this we obtain

$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geq n\} - P\{N(t) \geq n+1\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} \end{aligned} \tag{2.3}$$

Since the random variables X_i, $i \geq 1$, are independent and have a common distribution $F_X(x)$, it follows that $S_n = \sum_{i=1}^{n} X_i$ is distributed as $F_X^{n*}(x)$, the n-fold convolution of $F_X(x)$. Therefore,

$$P\{N(t) = n\} = F_X^{n*}(t) - F_X^{(n+1)*}(t) \tag{2.4}$$

The renewal function $M_X(t)$, defined as $E[N(t)] + 1$, is given by

$$\begin{aligned} M_X(t) &= E[N(t)] + 1 \\ &= \sum_{n=0}^{\infty} P\{N(t) \geq n\} \\ &= \sum_{n=0}^{\infty} F_X^{n*}(t) \end{aligned} \tag{2.5}$$

Key renewal theorem

Suppose $a(t)$, $t \geq 0$, is a given, integrable function that is bounded on finite intervals. Let the function $Z(t)$, $t \geq 0$, be defined by the integral equation

$$Z(t) = a(t) + \int_0^t Z(t-x) f_X(x) dx \qquad t \geq 0 \tag{2.6}$$

where $f_X(x)$ is the density of the inter-renewal time X_t. Then

$$\lim_{t \to \infty} Z(t) = \frac{1}{E[X_t]} \int_0^\infty a(x) dx \tag{2.7}$$

Equation (2.6) has an unique solution that is bounded on finite intervals. Given the renewal function $M_X(t)$, the renewal equation (2.6) has an unique solution

$$Z(t) = a(t) + \int_0^t a(t-x) M_X(x) dx \qquad t \geq 0 \tag{2.8}$$

For a proof we refer to Feller (1970).

The residual lifetime

We assume a renewal process with X_i, S_n and $N(t)$ as defined in the previous section. The residual lifetime, $U(t)$, or the forward recurrence-time is defined as the time until the next renewal at time t.

$$U(t) = S_{N(t)+1} - t \tag{2.9}$$

By conditioning on the time the first renewal and noting that after each renewal the renewal process probabilistically starts over and under $t \geq 0$, it follows that

$$
\begin{aligned}
P\{U(t) > u\} &= \int_0^t P\{U(t-x) > u\} f_X(x)dx + \int_{t+u}^{\infty} f_X(x)dx \\
&= \sum_{n=0}^{\infty} \int_0^t (1 - F_X(t - x + u)) f_X^{n*}(x)dx + \int_{t+u}^{\infty} f_X(x)dx \\
&= \int_0^t (1 - F_X(t + u - x)) M_X(x)dx + (1 - F_X(t + u)) \qquad (2.10)
\end{aligned}
$$

We can apply the key renewal theorem to (2.10) and obtain the asymptotic residual lifetime (U). $U := \lim_{t\to\infty} U(t)$

$$
P\{U > u\} = \frac{1}{E[X]} \int_0^u (1 - F_X(x))dx \qquad (2.11)
$$

From (2.11) we can compute the moments of U

$$
E[U^k] = \frac{E[X^{k+1}]}{(k+1)E[X]} \qquad (2.12)
$$

Tijms (1994) has shown by numerical investigations that the asymptotic approximations for the distribution function of the residual lifetime yield accurate results for

$$
t \geq \begin{cases} \frac{3}{2} c_X^2 E[X] & if \quad c_X^2 > 1 \\ E[X] & if \quad 0.2 < c_X^2 \leq 1 \\ \frac{1}{2c_X^2} E[X] & if \quad 0 < c_X^2 \leq 0.2 \end{cases} \qquad (2.13)
$$

Finally, we define the following lemma, which can be very useful for example in the derivation of the long run average inventory level.

Lemma 2.2 *Given the continuous random variable X, with probability distribution function $F_X(x)$ and renewal function $M_X(x)$ and let $F_U(x)$ be the associated asymptotic residual lifetime time distribution, then*

$$M_X(x) * F_U(x) = \frac{x}{E[X]} \tag{2.14}$$

The proof of this lemma can be found in De Kok (1991) and Janssen (1998), which presents a direct proof from Laplace transform.

Asymptotic approximations for the first two moments of $N(t)$

In this section asymptotic approximations for the first two moments of $N(t)$ are derived using standard renewal theory (e.g. Cox (1962) or De Kok (1991)). At time epoch zero the renewal process is assumed to be stationary. A distinction is made between two situations:

1. At time epoch zero an arrival occurred and the counting process starts after this arrival. In this case

$$E[N(t)] \simeq \frac{t}{E[X]} + \frac{E[X^2]}{2E[X]^2} - 1 \tag{2.15}$$

$$E[N(t)^2] \simeq \frac{t^2}{E[X]^2} + t(\frac{2E[X^2]}{E[X^3]} - \frac{3}{E[X]}) + \frac{3E[X^2]^2}{2E[X]^4} - \frac{2E[X^3]}{3E[X]^3} - \frac{3E[X^2]}{2E[X]^2} + 1 \tag{2.16}$$

This case applies for example for the number of customers arriving at a stock-point during the replenishment lead-time in a (s,nQ)-inventory policy. In this policy the counting process starts after the arrival, which causes the inventory position to drop below s.

2. Time epoch zero is an arbitrary point in time, i.e. the time between the starting of the counting process and the arrival of the first renewal is distributed according to the residual lifetime. In this case

$$E[N(t)] \simeq \frac{t}{E[X]} \tag{2.17}$$

$$E[N(t)^2] \simeq \frac{t^2}{E[X]^2} + t(\frac{E[X^2]}{E[X]^3} - \frac{1}{E[X]}) + \frac{E[X^2]^2}{2E[X]^4} - \frac{E[X^3]}{3E[X]^3} \qquad (2.18)$$

This case applies for example for the number of customers arriving at a stock-point during the replenishment lead-time in a (R,S)-inventory policy with stochastic demand. In the periodic review policy every R units of time the inventory is inspected. This inspection period is independent of the arrival process.

Compound renewal processes

A compound renewal process is not only characterized by the continuous random interarrival time X but also by a continuous random quantity Y. The sequence Y_i is assumed to be independently identically distributed. An example of a random quantity is in inventory management the demand size. The cumulative quantity on the interval $(0, t]$ is evaluated as follows:

$$Y(t) = \sum_{i=1}^{N(t)} Y_i \qquad (2.19)$$

The first two moments are:

$$E[Y(t)] = E[N(t)]E[Y] \qquad (2.20)$$

$$E[Y(t)^2] = E[N(t)]\sigma^2(Y) + E[N(t)^2]E[Y]^2 \qquad (2.21)$$

2.2.2 Mixed-Erlang distributions

In this thesis, we use for the fitting technique and in the simulations mixed-Erlang distributions. In the analysis, we assume compound renewal customer demand and arbitrarily distributed exogenous delay. In the numerical analysis we translate this by using mixed-Erlang distributions for the interarrival times and the demand sizes of the customer demand and for the exogenous delays. A mixed-Erlang distribution is defined as the mixture of two-Erlang distributions. A random variable X has

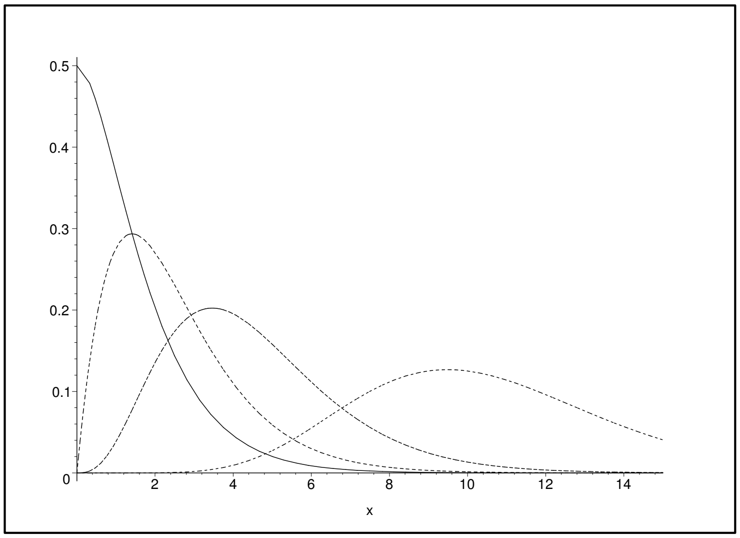


Figure 2.1: The density of various mixed-Erlang distributions

a mixed-Erlang distribution, $\mathbf{E}_{k_1,k_2}((\mu_1,\mu_2),(p_1,p_2))$, if X is with probability p_1 (resp. $p_2 = p_1 - 1$) the sum of $k_1 - 1$ (resp. $k_2 - 1$) independent exponentials with respectively mean $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$. The density of an $\mathbf{E}_{k_1,k_2}((\mu_1,\mu_2),(p_1,p_2))$ distribution is given by

$$f_X(x) := \sum_{i=1}^{2} p_i \mu_i^{k_i} \frac{x^{k_i-1}}{(k_i-1)!} e^{-\mu_i x} \qquad x > 0 \tag{2.22}$$

In Figure 2.1 we display from left to right the densities of $E_{1,2}((1,1),(0.5,0.5))$ (solid line), $E_{2,3}((1,1),(0.5,0.5))$ (dotted line), $E_{4,5}((1,1),(0.5,0.5))$ (dashed line) and $E_{10,11}((1,1),(0.5,0.5))$ (dash-dotted line). This distribution function is useful for fitting a distribution if the first two moments of a random variable are known, since for each possible combination of first two moments an unique mixture of Erlang distributions can be found with the same first two moments.

Fitting of mixed-Erlang distributions based on the first two moments

When an exact expression for the distribution function of a random variable is intractable we can resort to the following approximation. We evaluate the first two moments of the random variable and we approximate the distribution of the random variable by a mixed-Erlang distributions with the same first two moments. The parameters, in Equation (2.22), p_i, k_i and μ_i for i=1,2 can be evaluated as follows

- If $c_X^2 < 1$, then one fits a mixture of two Erlang distributions with the same scale parameter. Hence,

  $k_1 = \lfloor \frac{1}{c_X^2} \rfloor$

  $k_2 = k_1 + 1$

  $p_1 = \frac{1}{1+c_X^2} \left(k_2 c_X^2 - \sqrt{k_2(1 + c_X^2) - k_2^2 c_X^2} \right)$

  $p_2 = 1 - p_1$

  $\mu_1 = \frac{k_2 - p_1}{E[X]}$

  $\mu_2 = \mu_1$

- If $c_X^2 \geq 1$, then one fits a mixture of two exponential distributions with balanced means.

  $k_1 = 1$

  $k_2 = 1$

  $\mu_1 = \frac{2}{E[X]} \left(1 + \sqrt{\frac{c_X^2 - \frac{1}{2}}{c_X^2 + 1}} \right)$

  $\mu_2 = \frac{4}{E[X]} - \mu_1$

  $p_1 = \frac{\mu_1(\mu_2 E[X] - 1)}{\mu_2 - \mu_1}$

  $p_2 = (1 - p_2)$

The fitting technique is common, see for example Tijms (1994), de Kok (1985) or Federgruen and Katalan (1996).

Evaluation of frequently used functions

In this thesis, quantities like $E[(X-z)^+]$ where X is a mixed-Erlang distributed random variable with the same parameters as Equation (2.22) and z is constant, are frequently used, therefore we will compute them in this section. We will first solve some auxiliary integrals

$$\int_z^\infty x^n e^{-\mu x} dx = \sum_{i=0}^{n} z^i \frac{n!}{i!} \mu^{i-n-1} e^{-\mu z} \tag{2.23}$$

$$\int_0^z x^n e^{-\mu x} dx = \frac{n!}{\mu^{n+1}} - \sum_{i=0}^{n} z^i \frac{n!}{i!} \mu^{i-n-1} e^{-\mu z} \tag{2.24}$$

$$\int_0^\infty x^n e^{-\mu x} dx = \frac{n!}{\mu^{n+1}} \tag{2.25}$$

We can use these expressions to compute $E[((X-z)^+)^m]$.

$$\begin{aligned}
E[((X-z)^+)^m] &= \int_z^\infty (x-z)^m dF_X(x) \\
&= \sum_{j=0}^{m} \binom{m}{m-j} (-z)^j \int_z^\infty x^{m-j} dF_X(x) \\
&= \sum_{j=0}^{m} \binom{m}{m-j} (-z)^j \int_z^\infty x^{m-j} \Big(\sum_{i=1}^{2} p_i \mu_i^{k_i} \frac{x^{k_i-1}}{(k_i-1)!} e^{-\mu_i x}\Big) dx \\
&= \sum_{j=0}^{m} \binom{m}{m-j} (-z)^j \left(\sum_{i=1}^{2} p_i \frac{\mu_i^{k_i}}{(k_i-1)!} \int_z^\infty x^{m-j+k_i-1} e^{-\mu_i x} \right) dx \\
&= \sum_{j=0}^{m} \binom{m}{m-j} (-z)^j \left(\sum_{i=1}^{2} p_i \frac{\mu_i^{k_i}}{(k_i-1)!} \sum_{l=0}^{m-j+k_i-1} \frac{z^l (m-j+k_i-1)!}{l! \mu_i^{m-j-l+k_i}} e^{-\mu_i z} \right)
\end{aligned}$$

Similarly we can compute $E[((z-X)^+)^m]$:

$$E[((z-X)^+)^m] = \int_0^z (z-x)^m dF_X(x)$$
$$= \sum_{j=0}^{m} \binom{m}{m-j} (z)^{m-j} (-1)^j \sum_{i=1}^{2} p_i \frac{\mu_i^{k_i}}{(k_i-1)!}$$
$$\left(\frac{(j+k_i-1)!}{\mu_i^{j+k_i}} - \sum_{l=0}^{j+k_i-1} \frac{z^l (j+k_i-1)!}{l! \mu^{j-l+k_i}} e^{-\mu_i z} \right)$$

Further, we can also compute $E[((X-Z)^+)^m]$ where X is distributed according to a mixed-Erlang distribution with parameters p_1, p_2, k_1, k_2, μ_1 and μ_2 and Z is also distributed according to a mixed-Erlang distribution but with parameters q_1, q_2, l_1, l_2, λ_1 and λ_2.

$$E[((X-Z)^+)^m] = \sum_{j=0}^{m} \binom{m}{m-j} (-1)^j \int_0^\infty z^j \int_z^\infty x^{m-j} dF(x) dG(z)$$
$$= \sum_{j=0}^{m} \binom{m}{m-j} (-1)^j \sum_{i=1}^{2} \frac{p_i}{(k_i-1)!} \sum_{n=0}^{m-j+k_i-1} \frac{(m-j+k_i-1)!}{n!} \mu_i^{n-m+j}$$
$$\left(\sum_{r=1}^{2} q_r \lambda_r^{l_r} \frac{(l_r-1+j+n)!}{(\lambda_r+\mu_i)^{l_r+j+n} (l_r-1)!} \right)$$

2.2.3 Analytical approximations for the first two moments of a superposed interarrival process

In this section, we derive approximations for the first two moments of the superposed interarrival process. Suppose we have N independent interarrival processes at a server (for example a stockpoint). What is the superposed arrival at the server? We define X_k with $k = 1, ..., N$ as the interarrival time for each process k and we assume X_k ($k = 1, ..., N$) to be independently identically distributed. We define

X_0 as the superposed interarrival process. To find approximations for the first two moments of X_0, we apply the stationary interval method developed by Whitt (1982), to superpose renewal processes. The stationary-interval method equates the moments of the renewal interval with the moments of the stationary interval in the point process to be approximated. Whitt (1982) assumes that the point process to be approximated satisfies the central limit theorem. Instead of superposing hyper-exponential and shifted exponential distributions we superpose mixtures of Erlang distributions. In the superposition procedure it is assumed that the superposed process is a renewal process, which is not true. If we superpose N renewal processes, the first renewal time of the superposed process should be the minimum of the first renewal times of the N individual renewal processes. Tijms (1994) shows that under the condition of the central limit theorem, the superposition converges to a Poisson process when N tends to infinity. The first two moments of X_0 are as follows: (we refer to Whitt (1982) for a derivation)

$$E[X_0] = \frac{1}{\sum\limits_{k=1}^{N} \frac{1}{E[X_k]}} \tag{2.26}$$

$$E[X_0^2] \simeq 2E[X_0] \int\limits_0^\infty \left(\prod_{k=1}^{N} \frac{1}{E[X_k]} \right) \left(\prod_{k=1}^{N} \int\limits_z^\infty (1 - F_{X_k}(y))dy \right) dz \tag{2.27}$$

If N is 2 then we can easily compute the right hand side of (2.27) numerically but if N is large then it becomes quite complicated and time consuming to compute the right hand side of (2.27). Therefore, following Whitt (1982), we can simplify the computation by superposing two processes at a time and fitting a mixed-Erlang distribution to the first two moments of the superposed process. The resulting iteration scheme is presented below.

1. Order the N interarrival processes from the largest to the smallest first moment.

2. Compute

$$E[X_0^{(1)}] := \frac{1}{\sum_{i=1}^{2} E[X_i]} \tag{2.28}$$

$$E[(X_0^{(1)})^2] := 2E[X_0^{(1)}] \int_0^\infty \left(\prod_{i=1}^{2} \frac{1}{E[X_i]} \right) \left(\prod_{i=1}^{2} \int_z^\infty (1 - F_{X_k}(y)) dy \right) dz \tag{2.29}$$

if X_1 is distributed according to a mixed-Erlang distribution with parameters p_i, k_i, μ_i with $i = 1, 2$ and X_2 is also distributed according to a mixed-Erlang distribution with parameters q_i, l_i, λ_i with $i = 1, 2$ then

$$\begin{aligned} E[(X_0^{(1)})^2] &:= 2\frac{E[X_0^{(1)}]}{E[X_1]E[X_2]} \int_0^\infty \left(\int_z^\infty \sum_{j=1}^{2} p_j \sum_{n=0}^{k_j-1} \frac{(\mu_j y)^n}{n!} e^{-\mu_j y} dy \right) \\ &\left(\int_z^\infty \sum_{i=1}^{2} q_i \sum_{s=0}^{l_i-1} \frac{(\lambda_i y)^s}{s!} e^{-\lambda_i y} dy \right) dz \\ E[(X_0^{(1)})^2] &:= 2\frac{E[X_0^{(1)}]}{E[X_1]E[X_2]} \sum_{j=1}^{2} \sum_{i=1}^{2} p_j q_i \sum_{n=0}^{k_j-1} \sum_{s=0}^{l_i-1} (k_j - n)(l_j - 1) \\ &\binom{n+s}{n} \frac{\mu_j^{n-1} \lambda^{s-1}}{(\mu_j + \lambda_i)^{n+s+1}} \end{aligned} \tag{2.30}$$

3. Fit a mixed-Erlang distribution to the first two moments of $X_0^{(1)}$.

4. Initially set n:=2 and i:=3.

5. Compute

$$E[X_0^{(n)}] := \frac{1}{E[X_0^{(n-1)}] + E[X_i]} \tag{2.31}$$

$$E[(X_0^{(n)})^2] := 2E[X_0^{(n)}] \int_0^\infty \frac{1}{E[X_0^{(n-1)}]E[X_i]}$$
$$\left(\int_z^\infty (1 - F_{X_0^{(n-1)}}(y))dy \int_z^\infty (1 - F_{X_i}(y))dy \right) dz \tag{2.32}$$

Fit a mixed-Erlang distribution to the first two moments of $X_0^{(n)}$

6. If $n < N$ then $n := n + 1$, $i := i + 1$ and go to step 5 else $E[X_0] := E[X_0^{(N)}]$ and $E[X_0^2] := E[(X_0^{(N)})^2]$

2.2.4 Analytical expressions for the aggregated compound renewal process

As we mentioned previously the compound renewal process is not only characterized by the random interarrival times X but also by a random quantity Y. Suppose we have N independent compound renewal processes at the server. What is the demand size and interarrival process at the server? We define X_k with $k = 1, ..., N$ as the interarrival time for each process k and Y_k with $k = 1, ..., N$ as the demand size for each process k. We assume X_k and Y_k $(k = 1, ..., N)$ to be independently identically distributed. We define X_0 as the interarrival process and Y_0 as the demand size process at the server. The expressions derived in Section 2.2.3 can be used to determine the interarrival time. The demand process at the server Y_0 can be calculated straightforwardly by taking the weighted sum of the individual demand sizes:

$$E[Y_0] = E[X_0] \sum_{i=1}^{N} \frac{E[Y_i]}{E[X_i]} \tag{2.33}$$

$$E[Y_0^2] = E[X_0] \sum_{i=1}^{N} \frac{E[Y_i^2]}{E[X_i]} \tag{2.34}$$

2.2.5 Error measurements for analytical approximations

In this thesis we derive several analytical approximations, which are tested by means of discrete event simulations. As a performance measure for a parameter Z, we define $\delta_i^{(Z)}$ the percentage error and $\Delta_i^{(Z)}$ the absolute error of Z_i for simulation i ($i = 1, ..., \mathrm{I}$).

$$\delta_i^{(Z)} = \frac{|Z_i^{simu} - Z_i|}{Z_i^{simu}} * 100 \qquad i = 1, ..., \mathrm{I} \tag{2.35}$$

$$\Delta_i^{(Z)} = |Z_i^{simu} - Z_i| * 100 \qquad i = 1, ..., \mathrm{I} \tag{2.36}$$

where Z is the approximated one and Z^{simu} is the simulated one and we define $\bar{\delta}^{(Z)}$ and $\max(\delta^{(Z)})$ as

$$\bar{\delta}^{(Z)} = \frac{\sum_{i=1}^{\mathrm{I}} \delta_i^{(Z)}}{\mathrm{I}} \tag{2.37}$$

$$\max(\delta^{(Z)}) = \max_{i=1,...,\mathrm{I}} \quad \delta_i^{(Z)} \tag{2.38}$$

Similarly we can define $\bar{\Delta}^{(Z)}$ and $\max(\Delta^{(Z)})$.

In the previous section we proposed an approximative method to evaluate the first two moments of the superposed process X_0. To test the approximation for the superposed process, we performed 73 discrete event simulations. We simulate 1×10^6 arrivals at the server. We performed each simulation for 10 different seeds if the results where significant different from each other we increased the number of arrivals. N is varied between 2, 4, 8, 16, 32, 64 and 128. The N interarrival processes are identical. $E[X_k]$ for $k = 1, ..., N$ is chosen such that $E[X_0] = 0.5$ and $c^2_{X_k}$ for $k = 1, ..., N$ is varied between 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8 and 2. We assume that X_k for $k = 1, ..., N$ are mixed-Erlang distributions. The results are presented in Table 2.6. Between parentheses the 95 % confidence interval is indicated as derived by Law and Kelton (1991), for more details see Law and Kelton (1991) p 533. This means that for example for the fourth situation ($N = 2$ and $C^2_{X_k} = 0.8$) we can claim with 95 % confidence that $\bar{\delta}^{(c^2_{X_0})}$ is contained in the interval 0.11 ± 0.12, i.e. [-0.01,0.23].

We define ρ_{X_0} as the auto-correlation coefficient of X_0.

$$\rho_{X_0} = \frac{E[(X_0^{(n)} - E[X_0])(X_0^{(n+1)} - E[X_0])]}{\sigma^2(X_0)} \tag{2.39}$$

In Table 2.6, we observe that when N increases $c^2_{X_0}$ tends to 1 and ρ_{X_0} to 0, this is in line with the results found in Whitt (1982). Further we see that the process X_0 is strongly correlated, when $c^2_{X_0} < 1$ then $\rho_{X_0} < 0$ and when $c^2_{X_0} > 1$ then $\rho_{X_0} > 0$. Finally, we remark that for $N = 2$ the simulated $c^2_{X_0}$ is close to the calculated $c^2_{X_0}$, but for $N > 2$ the simulated $c^2_{X_0}$ is different from the calculated $c^2_{X_0}$.

Table 2.6: Results of the approximation for superposing interarrival time distributions.

N	$c^2_{X_k}$	$\bar{\delta}^{(c^2_{X_0})}$	ρ_{X_0}	N	$c^2_{X_k}$	$\bar{\delta}^{(c^2_{X_0})}$	ρ_{X_0}
2	0.2	0.00 (±0.00)	-0.366 (±0.000)	16	1.2	1.55 (±0.13)	0.013 (±0.000)
2	0.4	0.00 (±0.00)	-0.156 (±0.000)	16	1.4	3.11 (±0.13)	0.024 (±0.000)
2	0.6	0.00 (±0.00)	-0.070 (±0.000)	16	1.6	4.60 (±0.16)	0.033 (±0.000)
2	0.8	0.11 (±0.12)	-0.028 (±0.000)	16	1.8	6.41 (±0.13)	0.041 (±0.000)
2	1.0	0.00 (±0.00)	0.000 (±0.000)	16	2	7.88 (±0.16)	0.048 (±0.000)
2	1.2	0.09 (±0.10)	0.015 (±0.000)	32	0.2	2.77 (±0.00)	-0.032 (±0.000)
2	1.4	0.00 (±0.00)	0.024 (±0.000)	32	0.4	1.38 (±0.00)	-0.028 (±0.000)
2	1.6	0.00 (±0.00)	0.030 (±0.000)	32	0.6	0.52 (±0.00)	-0.017 (±0.000)
2	1.8	0.00 (±0.00)	0.034 (±0.000)	32	0.8	0.20 (±0.07)	-0.006 (±0.000)
2	2	0.00 (±0.00)	0.035 (±0.000)	32	1	0.00 (±0.00)	0.000 (±0.000)
4	0.2	6.37 (±0.00)	-0.236 (±0.000)	32	1.2	1.27 (±0.10)	0.008 (±0.000)
4	0.4	2.70 (±0.00)	-0.132 (±0.000)	32	1.4	2.33 (±0.12)	0.014 (±0.000)
4	0.6	0.87 (±0.00)	-0.068 (±0.000)	32	1.6	3.65 (±0.12)	0.020 (±0.000)
4	0.8	0.22 (±0.00)	-0.027 (±0.000)	32	1.8	4.55 (±0.09)	0.024 (±0.000)
4	1.0	0.00 (±0.00)	0.000 (±0.000)	32	2	5.83 (±0.10)	0.029 (±0.000)
4	1.2	1.09 (±0.07)	0.022 (±0.000)	64	0.2	1.24 (±0.07)	-0.016 (±0.000)
4	1.4	1.92 (±0.12)	0.037 (±0.000)	64	0.4	0.82 (±0.12)	-0.014 (±0.000)
4	1.6	2.55 (±0.08)	0.050 (±0.000)	64	0.6	0.41 (±0.00)	-0.009 (±0.000)
4	1.8	3.46 (±0.12)	0.059 (±0.000)	64	0.8	0.10 (±0.13)	-0.003 (±0.000)
4	2	4.19 (±0.09)	0.066 (±0.000)	64	1	0.00 (±0.00)	0.000 (±0.000)
8	0.2	6.89 (±0.00)	-0.128 (±0.000)	64	1.2	0.69 (±0.07)	0.004 (±0.000)
8	0.4	2.94 (±0.14)	-0.088 (±0.000)	64	1.4	1.78 (±0.00)	0.007 (±0.000)
8	0.6	1.02 (±0.13)	-0.049 (±0.000)	64	1.6	2.35 (±0.10)	0.010 (±0.000)
8	0.8	0.21 (±0.08)	-0.019 (±0.000)	64	1.8	2.92 (±0.12)	0.013 (±0.000)
8	1.0	0.00 (±0.00)	0.000 (±0.000)	64	2	3.49 (±0.11)	0.015 (±0.000)
8	1.2	1.51 (±0.12)	0.018 (±0.000)	128	0.2	0.41 (±0.07)	-0.008 (±0.000)
8	1.4	3.14 (±0.11)	0.035 (±0.000)	128	0.4	0.21 (±0.07)	-0.007 (±0.000)
8	1.6	4.63 (±0.00)	0.048 (±0.000)	128	0.6	0.20 (±0.12)	-0.005 (±0.000)
8	1.8	6.17 (±0.10)	0.059 (±0.000)	128	0.8	0.10 (±0.07)	-0.002 (±0.000)
8	2	7.86 (±0.11)	0.069 (±0.000)	128	1	0.00 (±0.00)	0.000 (±0.000)
16	0.2	4.53 (±0.00)	-0.065 (±0.000)	128	1.2	0.50 (±0.07)	0.002 (±0.000)
16	0.4	2.23 (±0.08)	-0.051 (±0.000)	128	1.4	1.09 (±0.12)	0.004 (±0.000)
16	0.6	0.75 (±0.00)	-0.031 (±0.000)	128	1.6	1.39 (±0.10)	0.006 (±0.000)
16	0.8	0.31 (±0.00)	-0.011 (±0.000)	128	1.8	1.78 (±0.12)	0.006 (±0.000)
16	1.0	0.00 (±0.00)	0.000 (±0.000)	128	2	2.10 (±0.00)	0.008 (±0.000)

Chapter 3

A single item, multi-echelon inventory model

The content of this chapter is joint work with A.G. de Kok and G.P. Kiesmüller and has appeared in Smits et al. (2002)

3.1 Introduction

The objective of this chapter is to derive expressions for the waiting time of a replenishment order due to a lack of stock at the preceding stockpoint. Consequently, we consider a single item, multi-echelon logistic system with no shipment consolidation and production scheduling. We assume that the stockpoints in the system are controlled by (s,nQ)-installation stock policies. The (s,nQ)-installation stock inventory policy operates as follows: as soon as the inventory position, which is defined as the physical inventory plus the stock on order minus the backorders, drops below s an amount nQ is ordered such that the inventory position is raised to a value between s and $s + Q$. Q is called the batch size and s is called the reorder level. The replenishment lead-time consists of an exogenous delay and a waiting time due to a lack of stock at the preceding stockpoint. A more detailed description of the model will follow in Section 3.2.

In single item, single echelon networks, the demand process and the replenishment lead-time at the stockpoints are known. In contrast to this, in single item, multi-echelon networks only the demand process at the end-stockpoints and the replenishment lead-time at the beginning stockpoints is known. Therefore, in Section 3.3 approximations are derived to evaluate the replenishment lead-time and the demand process at each stockpoint in the system. Further, in this section we show also how to evaluate different performance measures for given reorder levels s and batchsizes Q. In Section 3.4, we introduce a numerical example. The validity of the derived approximations is tested by discrete event simulations in Section 3.5 and the chapter concludes with a summary and an outlook for future research.

3.2 Problem formulation

3.2.1 The model

A divergent multi-echelon distribution system is considered, through which a single item flows from an external supplier, through intermediate stockpoints to end stockpoints that deliver to customers. An example system is depicted in Figure 3.1.

The stockpoints in the single item system are uniquely numbered. Let $\mathcal{M}$ denote the set of stockpoints.

$\mathcal{S}_k$ is defined as the set of all immediate successors of $k \in \mathcal{M}$, where $j \in \mathcal{M}$ is a successor of k if and only if replenishment orders of j are part of the demand process of k. As an example see Figure 3.1 where $\mathcal{S}_1 = \{2, 3\}$ and $\mathcal{S}_4 = \emptyset$.

$\mathcal{P}_k$ is the immediate predecessor of $k \in \mathcal{M}$, i.e. replenishment orders of k are part of the demand process of $j \in \mathcal{P}_k$. As an example see Figure 3.1 where $\mathcal{P}_5 = \{3\}$ and $\mathcal{P}_1 = \emptyset$.

Further, we denote the set of end-stockpoints by $\mathcal{E}$, i.e. $\mathcal{E} = \{k|\mathcal{S}_k = \emptyset, k \in \mathcal{M}\}$. Denote the root notes of the system by $\mathcal{B}$, i.e. $\mathcal{B} = \{k|\mathcal{P}_k = \emptyset, k \in \mathcal{M}\}$ and we define the intermediate stockpoints as $\mathcal{I}$, i.e. $\mathcal{I} = \{k| \in \mathcal{M}\backslash(\mathcal{B} \bigcup \mathcal{E})\}$. Note that since the system is divergent, $|\mathcal{P}_k| = 1$ for all $k \in \mathcal{M}\backslash\mathcal{B}$ and $|\mathcal{P}_k| = 0$ for $k \in \mathcal{B}$.

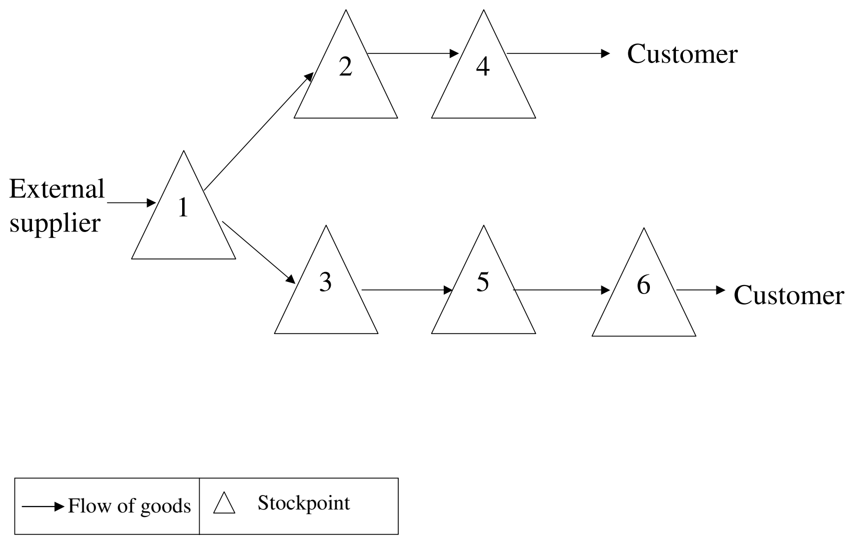


Figure 3.1: Multi-echelon inventory system.

We assume demand arrives at stockpoint $k \in \mathcal{E}$ according to a continuous compound renewal process. Let A_k, $k \in \mathcal{E}$, denote the length of an arbitrary interarrival time and D_k, $k \in \mathcal{E}$, denote the demand size of an arbitrary customer at stockpoint $k \in \mathcal{M}$. Note that A_k and D_k are continuous stochastic variables. All stockpoints are controlled by (s,nQ)-installation stock policies. The (s_k,nQ_k)-installation stock policy operates as follows: as soon as the inventory position is below s_k an amount nQ_k is ordered such that the inventory position is raised to a value between s_k and $s_k + Q_k$. The replenishment order size is denoted by O_k. Further, $O_k = nQ_k$ where O_k is assumed to be continuous stochastic variable, n is a stochastic integer and Q_k is a given constant. The time between subsequent replenishments is denoted by R_k.

The replenishment process (R_k,O_k) is independent of the reorder level s_k. Suppose the inventory position is equal to s_k+Z, just before an order arrives causing the inventory position to drop below s_k. This order arrives at epoch τ_1 and its demand size is $D_k^1 \geq Z$. When the order arrives a replenishment order of size $\lceil \frac{D_k^1-Z}{Q_k} \rceil Q_k$ is placed at $j \in \mathcal{P}_k$. The size of this replenishment order is independent of the reorder level s_k. Note that $D_k^1 - Z$ is called the undershoot. The undershoot (U_k) is defined as the difference between s_k and the inventory position just before the placement of a replenishment order. The independency between R_k and s_k can be derived as follows. From renewal theory follows that

$$R_k = \sum_{i=1}^{N(R_k)} A_k^i \quad \forall k \in \mathcal{M}. \tag{3.1}$$

where $N(R_k)$ is defined as the number of customer arrivals during a replenishment cycle at stockpoint $k \in \mathcal{M}$ and A_k^i as the i^{th} interarrival time during a replenishment cycle at $k \in \mathcal{M}$. For more detail see Section 2.2.1. But $N(R_k)$ can also be expressed as follows

$$E[O_k] = E[\sum_{i=1}^{N(R_k)} D_k^i] \quad \forall k \in \mathcal{M}. \tag{3.2}$$

From equations (3.1) and (3.2), we observe that R_k is dependent on O_k, D_k and A_k and therefore independent of s_k

As mentioned earlier the replenishment lead-time L_k for $k \in \mathcal{M}$ is the sum of the waiting time due to a lack of stock at $j \in \mathcal{P}_k$ and an exogenous delay L_k^d from $j \in \mathcal{P}_k$ to k.

$$L_k = L_k^d + W_k^s. \tag{3.3}$$

In Chapter 1, we mentioned that the exogenous delay is composed of three elements: time needed for the administration of the incoming order, time needed to handle the order in the warehouse and time needed for external transportation from the warehouse to the delivery point. We assume that these three elements are not influenced by the system, thus they are exogenous variables and we assume that they are independent of W_k^s. Further, we assume that the waiting time at $k \in \mathcal{B}$ due to a lack of stock is zero and that for all $k \in \mathcal{M}$ the exogenous delay L_k^d has a known probability density function.

To evaluate the quality of the control of the inventory system, we consider the performance measures as defined in Silver et al. (1998) for the stockpoint $k \in \mathcal{M}$. For the notation we refer to Schneider (1981) because it is frequently used.

1. α_k: the long run non-stockout probability per replenishment cycle.

   The replenishment cycle is defined as the time between two replenishment order placements. Equivalently, α_k is the fraction of cycles in which a stockout does not occur. A stockout is defined as an occasion when the physical stock drops to the zero level.

2. β_k: the long run fraction of demand delivered directly from stock.

   β_k is also called the fill rate. It is the fraction of customer demand that is met routinely; that is, without backorders or lost sales. This service measure has considerable appeal to practitioners.

3. γ_k: the long run fraction of time the net stock is positive.

   γ_k is also called the ready rate. The ready rate is the fraction of time during which the net stock is positive; that is, there is some stock on the shelf. The ready rate finds common application in the case of equipment used for emergency purposes.

4. $P\{W_j^s > 0\}$: the long run probability that an arbitrary customer j has to wait due to a lack of stock.

   The waiting time of a customer is measured from the time of arrival until the time at which the demand is completely satisfied.

5. $E[b_k]$: the long run expected backlog level.

6. $E[I_k]$: the long run expected physical inventory level.

Our analysis enables us to find approximations for the performance measures for given values of (s_k,Q_k) for $k \in \mathcal{M}$. Given these approximations and given a target performance measure and the batchsize for $k \in \mathcal{M}$, the reorder level s_k can be determined, which involves a one-dimensional search according to e.g. the bisection rule. We assume in this thesis that the fill rates at all stockpoints are given. However in practice only the fill rates at the end-stockpoints are known. Further research is necessary to develop a technique to find the optimal fill rates at $k \in \mathcal{M}\backslash\mathcal{E}$. Additionally, we consider the following four assumptions:

1. Subsequent orders are not allowed to overtake each other.

2. Only complete deliveries between stockpoints are allowed.

   In the deterministic demand case, algorithms are developed to be able to determine the optimal batchsizes in divergent networks, see for example Roundy and Sun (1994). These algorithms consider fixed order cost and inventory cost and assume complete deliveries. When partial deliveries are allowed between stockpoints the fixed order cost are higher than in the complete case and these algorithms are not applicable anymore.

3. Partial deliveries between end-stockpoints and customers are allowed

4. Shortages are backordered

Below the notation and the assumptions are summarized.
The following variables are the control parameters of the inventory policy.

Q_k	Batchsize at $k \in \mathcal{M}$
s_k	Reorder level at $k \in \mathcal{M}$

The following random variables are supposed to have a known probability density function.

D_k	Demand size for $k \in \mathcal{E}$
A_k	Time between two subsequent arrivals of orders $k \in \mathcal{E}$
L_k^d	Exogenous delay to $k \in \mathcal{M}$

The following parameter is assumed to be zero.

W_k^s	Waiting time at stockpoint $k \in \mathcal{B}$ due to a lack of stock at the external supplier.

The distribution function of the following random variables is unknown.

O_k	Replenishment order size of $k \in \mathcal{M}$
R_k	Time between two subsequent replenishments $k \in \mathcal{M}$
L_k	Replenishment lead-time at $k \in \mathcal{M}$ ($L_k = L_k^d + W_k^s$)
D_k	Order size of the demand process at $k \in \mathcal{M}\backslash\mathcal{E}$
A_k	Time between two subsequent order arrivals of the demand process at $k \in \mathcal{M}\backslash\mathcal{E}$
U_k	Undershoot at $k \in \mathcal{M}$, which is defined as the difference between s_k and the inventory position just before the placement of an replenishment order
$D_k(L_k)$	Demand at $k \in \mathcal{M}$ during the replenishment lead-time L_k
$N(R_k)$	Number of customers arriving during an arbitrary replenishment cycle at $k \in \mathcal{M}$. (The replenishment cycle is defined as the time between two subsequent replenishments.)
$\tilde{N}(R_k)$	Number of customers arriving during an arbitrary replenishment cycle at $k \in \mathcal{M}$ under the condition that Q_k is large.
W_k^s	Waiting time of $k \in \mathcal{M}\backslash\mathcal{B}$ due to a lack of stock at $j \in \mathcal{P}_k$
H_k	Residual lifetime distribution of D_k, for more details we refer to Section 2.2.1.

Exact expressions for the distribution function of the random variables D_k, A_k ($k \in \mathcal{M}\backslash\mathcal{E}$) and O_k, R_k, L_k, $D_k(L_k)$ ($k \in \mathcal{M}$) are in general intractable. The distribution function of these variables is approximated by a mixed-Erlang distribution based on (approximate) expressions for the first two moments, as indicated in Section 2.2.2.

3.3 Analytical approximations

3.3.1 Outline of the algorithm to evaluate the demand and waiting time due to lack of stock at all stockpoint $k \in \mathcal{M}$

In single echelon inventory models the replenishment lead-time and the demand at the stockpoint are known. Therefore, it is straightforward to calculate the inventory control parameters. In contrast to this in multi-echelon inventory models the replenishment lead-times at stockpoints $k \in \mathcal{M} \backslash \mathcal{B}$ is unknown because the waiting time due to a lack of stock are unknown and only the demand process at the end-stockpoints is known.

In this chapter, approximations for the first two moments of the waiting time due to a lack of stock and the demand processes at all stockpoints in a general multi-echelon distribution system are provided.

Since the replenishment process is independent of the reorder levels, the demand processes in the entire system can be determined iteratively independently of the reorder levels. To evaluate the waiting time we need the demand process at the stockpoints. Therefore, in a first step the demand processes (A_k,D_k) at all $k \in \mathcal{M} \backslash \mathcal{E}$ are determined and in a second step the waiting times of $k \in \mathcal{M} \backslash \mathcal{B}$ due to a lack of stock at the preceding stockpoint are determined.

Iteration scheme to evaluate the demand process at all $k \in \mathcal{M} \backslash \mathcal{E}$

- For all $k \in \mathcal{E}$, we translate the demand process (A_k,D_k) into a compound renewal approximation of the replenishment process (R_k,O_k).
- For all $\{j \in \mathcal{M} | \mathcal{S}_j \subset \mathcal{E}\}$ we can determine the demand process (A_j,D_j) by aggregating the replenishment processes (R_k,O_k) of stockpoints $k \in \mathcal{S}_j$.
- After that, we can translate the demand process (A_j,D_j) for $\{j \in \mathcal{M} | \mathcal{S}_j \subset \mathcal{E}\}$ into a replenishment process (R_j,O_j).
- These two previous steps can be repeated by iterating echelon by echelon until the demand process (A_k,D_k) for $k \in \mathcal{B}$ is determined.

Iteration scheme to evaluate the waiting time due to a lack of stock at all $k \in \mathcal{M} \backslash \mathcal{B}$

- For $k \in \mathcal{B}$, the demand process (A_k,D_k) and the replenishment lead-time L_k is known, since $W_k^s = 0$. Given the demand process, the replenishment lead-time, the batchsize and the reorder level at $k \in \mathcal{B}$, the waiting times W_j^s ($j \in \mathcal{S}_k$) can be determined. Alternatively, we can specify a target performance measure and calculate the corresponding s_k and W_j^s ($j \in \mathcal{S}_k$).

- Having expressions for W_j^s ($j \in \mathcal{S}_k$), the replenishment lead-times for all $j \in \mathcal{S}_k$ can be evaluated.

- These two previous steps can be repeated by iterating echelon by echelon until the replenishment lead-times at all $k \in \mathcal{M}$ are determined.

3.3.2 The aggregate demand process (A_k, D_k)

In this section, we derive expressions for the first two moments of the interarrival time A_k and the demand size D_k of the aggregate demand process at $k \in \mathcal{M} \backslash \mathcal{E}$. From the analysis in Section 2.2.3 of the superposed process of a finite number of renewal processes we obtain the first two moments of A_k for all $k \in \mathcal{M} \backslash \mathcal{E}$.

$$E[A_k] = \frac{1}{\sum\limits_{j \in \mathcal{S}_k} \frac{1}{E[R_j]}} \tag{3.4}$$

$$E[A_k^2] \simeq 2E[A_k] \left(\prod_{j \in \mathcal{S}_k} \frac{1}{E[R_j]} \right) \int\limits_0^\infty \left(\prod_{j \in \mathcal{S}_k} \int\limits_x^\infty (1 - F_{R_j}(y))dy \right) dx. \tag{3.5}$$

The first two moments of the aggregate demand size, D_k ($k \in \mathcal{M} \backslash \mathcal{E}$) are straightforward to calculate by taking the weighted sum of the individual demand sizes:

$$E[D_k] = E[A_k] \sum_{j \in \mathcal{S}_k} \frac{E[O_j]}{E[R_j]} \quad \forall k \in \mathcal{M} \backslash \mathcal{E} \tag{3.6}$$

$$E[D_k^2] = E[A_k] \sum_{j \in \mathcal{S}_k} \frac{E[O_j^2]}{E[R_j]} \quad \forall k \in \mathcal{M} \backslash \mathcal{E}. \tag{3.7}$$

3.3.3 The replenishment process (O_k,R_k)

Derivation of the first two moments of the order size O_k

In this subsection we present a procedure that translates the demand process at $k \in \mathcal{M}$ to a replenishment process at $j \in \mathcal{P}_k$. As we mentioned in Chapter 2, in the literature it is often assumed that the average replenishment order size in an (s_k,nQ_k) model equals Q_k, but this is only true when $Q_k \gg E[D_k]$. As explained in Section 3.2.1, the amount of Q_k ordered, n, depends on the size of the undershoot, U_k. It is easy to see that the following relation holds:

$$n = i \Leftrightarrow (i-1)Q_k < U_k \leq iQ_k \qquad for \qquad i = 1, 2, ... \qquad and \qquad k \in \mathcal{M} \tag{3.8}$$

For the computation of the first two moments of O_k the distribution function of U_k is needed. Pyke et al. (1996) derive the following theorem:

Theorem 3.1 *For a given batchsize Q_k and given a continuous demand distribution F_{D_k} the distribution function of the undershoot is given as follows*

$$P\{U_k \geq u\} = \frac{\int_0^{Q_k} (1 - F_{D_k}(z+u))dz}{\int_0^{Q_k} (1 - F_{D_k}(z))dz} \quad \forall k \in \mathcal{M}. \tag{3.9}$$

For a proof of Theorem 3.1 we refer to Appendix 3. Using Theorem 3.1 and Relation (3.8) exact expressions for the first two moments of O_k can be derived as follows.

Corollary 3.2 *The first moment of the order size is given as:*

$$E[O_k] = \frac{Q_k E[D_k]}{\int_0^{Q_k} P\{D_k \geq x\}dx} \quad \forall k \in \mathcal{M} \tag{3.10}$$

Proof.

The following proof comes from Pyke et al. (1996).

$$\begin{aligned}
E[O_k] &= \sum_{i=1}^{\infty} Q_k i[P\{U \geq (i-1)Q_k\} - P\{U \geq iQ_k\}] \\
&= \sum_{i=0}^{\infty} Q_k((i+1)-i)P\{U \geq iQ_k\} \\
&= Q_k \sum_{i=0}^{\infty} \frac{\int_0^{Q_k}(1-F_{D_k}(y+u))dy}{\int_0^{Q_k}(1-F_{D_k}(y))dy} \\
&= \frac{Q_k}{\int_0^{Q_k}(1-F_{D_k}(y))dy} \sum_{i=0}^{\infty} \int_{iQ_k}^{(i+1)Q_k} (1-F_{D_k}(y))dy \\
&= \frac{Q_k}{\int_0^{Q_k}(1-F_{D_k}(y))dy} \int_0^{\infty}(1-F_{D_k}(y))dy \\
&= \frac{Q_k}{\int_0^{Q_k}(1-F_{D_k}(y))dy} E[D_k] \\
&= \frac{Q_k E[D_k]}{\int_0^{Q_k} P\{D_k \geq x\}dx}
\end{aligned}$$

Corollary 3.3 *The second moment of the order size is given as:*

$$E[O_k^2] = Q_k^2 \sum_{i=0}^{\infty}(2i+1)P\{U_k \geq iQ_k\} \quad \forall k \in \mathcal{M} \tag{3.11}$$

Proof.

The following proof comes from de Pyke et al. (1996).

$$
\begin{aligned}
E[O_k^2] &= \sum_{i=1}^{\infty}(Q_k i)^2[P\{U \geq (i-1)Q_k\} - P\{U \geq iQ_k\}] \\
&= \sum_{i=0}^{\infty} Q_k^2((i+1)^2 - i^2)P\{U \geq iQ_k\} \\
&= \sum_{i=0}^{\infty} Q_k^2(2i+1)P\{U \geq iQ_k\}
\end{aligned}
$$

Derivation of the first two moments of the time between replenishments R_k

To derive an expression for the first two moments of R_k, the time between two replenishments from $k \in \mathcal{M}$, we use the following expression for R_k,

$$R_k = \sum_{i=1}^{N(R_k)} A_k^i \quad \forall k \in \mathcal{M} \tag{3.12}$$

where $N(R_k)$ is defined as the number of customer arrivals during a replenishment cycle at stockpoint $k \in \mathcal{M}$ and A_k^i as the i^{th} inter-arrival time during a replenishment cycle at $k \in \mathcal{M}$. Since $N(R_k)$ and $\{A_k^i\}$ are independent random variables and A_k^i is identically distributed, the following equations hold (see Section 2.2.1)

$$E[R_k] = E[N(R_k)]E[A_k] \quad \forall k \in \mathcal{M} \tag{3.13}$$

$$E[R_k^2] = E[N(R_k)]\sigma^2(A_k) + E[N(R_k)^2]E^2[A_k] \quad \forall k \in \mathcal{M}. \tag{3.14}$$

The first two moments of A_k are already known, therefore it remains to determine the moments of $N(R_k)$. $N(R_k)$ equals also to the number of renewals constituted by D_k^i in an interval of length O_k, therefore we can write

$$E[O_k] = E[\sum_{i=1}^{N(R_k)} D_k^i] \quad \forall k \in \mathcal{M}. \tag{3.15}$$

Since $N(R_k)$ is a stopping time, it follows that

$$E[N(R_k)] = \frac{E[O_k]}{E[D_k]} \quad \forall k \in \mathcal{M} \tag{3.16}$$

An exact expression for the second moment of $N(R_k)$ is in general intractable. Hence we resort to approximations. First note that for Q_k sufficiently large we order only one Q_k. This implies that a cycle starts with inventory position $s + Q - U$. We assume as an approximation that U is distributed according to the residual life time distribution associated with the renewal process constituted by D_k^i, the renewal process is stationary at the start of the renewal.

This yields:

$$E[\tilde{N}(R_k)^2] \simeq \frac{Q_k^2}{E[D_k]^2} + Q_k \frac{c_{D_k}^2}{E[D_k]} + \frac{E[D_k^2]^2}{2E[D_k]^4} - \frac{E[D_k^3]}{3E[D_k]^3} \quad \forall k \in \mathcal{M} \tag{3.17}$$

Note that the third moment of D_k can be calculated by fitting a mixed-Erlang distribution to the first two moments of D_k, for more details we refer to Section 2.2.2. Further, for the cases that Q_k is not sufficiently large but $\frac{Q_k}{E[D_k]} > 1$, we found, after performing numerous simulations (for more detail see De Kok (1993)), a correction factor which leads to the following formula:

$$E[N(R_k)^2] \simeq (E[\tilde{N}(R_k)^2]) \frac{E[O_k]}{Q_k} \quad \forall k \in \mathcal{M} \tag{3.18}$$

Note that if Q_k is large then $\frac{E[O_k]}{Q_k} = 1$ and expressions (3.17) and (3.18) are identical.

3.3.4 Waiting time due to a lack of stock W_k^s

Approximations for the first two moments of the waiting time W_k^s

The waiting time, W_k^s ($k \in \mathcal{M} \backslash \mathcal{B}$), due to a lack of stock at $j \in \mathcal{P}_k$ is the time elapsed between the moment the order O_k is placed and the moment until the entire order O_k is received at j.

The waiting time W_k^s of an order O_k received at installation $k \in \mathcal{M} \backslash \mathcal{B}$ can be expressed as a function of s_j, A_j, D_j, Q_j, O_k and L_j ($j \in \mathcal{P}_k$). These variables or their distributions are given. Further, we assume that $s_j \geq 0$ and we know from standard probability theory that the following equations hold

$$E[W_k^s] = \int_0^\infty P\{W_k^s > x\}dx \qquad \forall k \in \mathcal{M}\backslash\mathcal{B} \tag{3.19}$$

$$E[(W_k^s)^2] = 2\int_0^\infty xP\{W_k^s > x\}dx \qquad \forall k \in \mathcal{M}\backslash\mathcal{B} \tag{3.20}$$

To compute these two moments we need an expression for $P\{W_k^s > x\}$. In Janssen (1998) and De Kok (1993) the following proposition is proven.

Proposition 3.4 *The conditional distribution function of W_k^s ($k \in \mathcal{M}\backslash\mathcal{B}$) under the condition that the replenishment lead-time L_j is known can be evaluated for $0 \leq x \leq l$ and $j \in \mathcal{P}_k$ as follows*

$$P\{W_k^s \leq x | L_j = l\} = 1 - \frac{1}{Q_j}\bigg(E[(D_j(l-x) + O_k - s_j)^+] \\ - E[(D_j(l-x) + O_k - (s_j + Q_j))^+]\bigg) \tag{3.21}$$

For the proof we refer to Appendix 3.

To get an expression for $E[W_k^s]$, we condition on $L_j = l$ in the expression (3.19), and fill in expression (3.21).

Proposition 3.5 *The first moment of the waiting time due to a lack of stock is given for all $k \in \mathcal{M}$ as*

$$E[W_k^s] = E[L_j]\frac{E[(D_j(\hat{L}_j) + O_k - s_j)^+] - E[(D_j(\hat{L}_j) + O_k - (s_j + Q_j))^+]}{Q_j} \tag{3.22}$$

where $\hat{L}_j$ ($j \in \mathcal{P}_k$) is a random variable with the following distribution

$$F_{\hat{L}_j}(y) = \frac{1}{E[L_j]} \int_0^y (1 - F_{L_j}(z))dz \qquad (3.23)$$

Note that $\hat{L}_j$ is the residual lifetime of L_j.

For a proof we refer to Appendix 3.

Proposition 3.6 *The second moment of the waiting time due to a lack of stock is given for $k \in \mathcal{M}$ as follows*

$$E[(W_k^s)^2] = E[L_j^2] \frac{E[(D_j(\tilde{L}_j) + O_k - s_j)^+] - E[(D_j(\tilde{L}_j) + O_k - (s_j + Q_j))^+]}{Q_j} \qquad (3.24)$$

where $\tilde{L}_j$ $(j \in \mathcal{P}_k)$ is a random variable with the following distribution function

$$F_{\tilde{L}_j}(y) = \frac{2}{E[L_j^2]} \int_0^y \int_x^\infty (z - x) dF_{L_j}(z) dx \qquad (3.25)$$

For the proof we refer to Appendix 3.

Notice that Propositions 3.5 and 3.6 show that the random variables W_k^s, are different for different $k \in \mathcal{S}_j$, i.e. the waiting time of a replenishment order at a stockpoint depends partly on the successor that generates the order. Because if the replenishment order is large then the probability that this replenishment order has to wait is large.

Computation of the first two moments of the waiting time due to a lack of stock W_k^s

Having derived expressions for the first two moments of $E[W_k^s]$ and $E[(W_k^s)^2]$, we now explain how they can be computed.

1. We compute the first two moments of $\hat{L}_j$ and $\tilde{L}_j$ from formula's (3.23) and (3.25) resulting in

$$E[\hat{L}_j] = \frac{E[L_j^2]}{2E[L_j]} \qquad j \in \mathcal{P}_k \tag{3.26}$$

$$E[\hat{L}_j^2] = \frac{E[L_j^3]}{3E[L_j]} \qquad j \in \mathcal{P}_k \tag{3.27}$$

$$E[\tilde{L}_j] = \frac{E[L_j^3]}{3E[L_j^2]} \qquad j \in \mathcal{P}_k \tag{3.28}$$

$$E[\tilde{L}_j^2] = \frac{E[L_j^4]}{6E[L_j^2]} \qquad j \in \mathcal{P}_k \tag{3.29}$$

The third and the fourth moments of L_j can be calculated straightforwardly by using our distribution assumption, that all random variables are mixed-Erlang distributed random variables, see Section 2.2.2.

2. We need expressions for the first two moments of $D_j(\hat{L}_j)$ to compute $E[W_k^s]$ and these can be found in Section 2.2.1 equations (2.20) and (2.21). This gives

$$E[D_j(\hat{L}_j)] = E[N(\hat{L}_j)]E[D_j] \qquad j \in \mathcal{P}_k \tag{3.30}$$

$$E[D_j^2(\hat{L}_j)] = E[N(\hat{L}_j)]\sigma^2(D_j) + E[N(\hat{L}_j)^2]E[D_j]^2 \qquad j \in \mathcal{P}_k. \tag{3.31}$$

where $N(\hat{L}_j)$ denotes the number of orders arriving at j during $\hat{L}_j$. $N(\hat{L}_j)$ is a counting process. Based on asymptotic results from renewal theory we can derive expressions for the first two moments of $N(\hat{L}_j)$. For the derivations of the first two moments of $N(\hat{L}_j)$ for all $j \in \mathcal{P}_k$ we refer to Section 2.2.1

$$E[N(\hat{L}_j)] \simeq \frac{E[\hat{L}_j]}{E[A_j]} + \frac{E[A_j^2]}{2E[A_j]^2} - 1 \tag{3.32}$$

$$
\begin{aligned}
E[N(\hat{L}_j)^2] \quad \simeq \quad & \frac{E[\hat{L}_j^2]}{E[A_j]^2} + E[\hat{L}_j]\left(\frac{2E[A_j^2]}{E[A_j]^3} - \frac{3}{E[A_j]}\right) \\
& + \frac{3E[A_j^2]^2}{2E[A_j]^4} - \frac{2E[A_j^3]}{3E[A_j]^3} - \frac{3E[A_j^2]}{2E[A_j]^2} + 1
\end{aligned} \tag{3.33}
$$

Further, the first two moments of $D_j(\tilde{L}_j)$ can be computed in the same manner as the first two moments of $D_j(\hat{L}_j)$.

3. For the computation of $E[(D_j(\hat{L}_j)+O_k-s_j)^+]$, $E[(D_j(\hat{L}_j)+O_k-(s_j+Q_j))^+]$, $E[(D_j(\tilde{L}_j)+O_k-s_j)^+]$ and $E[(D_j(\tilde{L}_j)+O_k-s_j+Q_j)^+]$ we refer to Section 2.2.2 After that we compute $E[W_k^s]$ and $E[(W_k^s)^2]$ using formulas (3.22) and (3.24).

3.3.5 Analytical calculation of the performance measures

Given the interarrival process A_k, the demand size D_k, the batchsize Q_k, the reorder level s_k and the replenishment lead-time L_k at $k \in \mathcal{M}$, the different performance measures can be analytically evaluated.

$$
\alpha_k = P\{s_k - U_k - D_k(L_k) > 0\} \tag{3.34}
$$

$$
\beta_k = 1 - \frac{E[(D_k(L_k)+U_k-s_k)^+] - E[(D_k(L_k)+U_k-s_k-Q_k)^+]}{Q_k} \tag{3.35}
$$

$$
\gamma_k \simeq \frac{E[(s_k+Q_k-D_k(L_k))^+] - E[(s_k-D_k(L_k))^+]}{Q} \tag{3.36}
$$

$$
P\{W_k^s > 0\} \simeq 1 - \frac{E[(D_j(L_j)+O_k-s_j)^+] - E[(D_j(L_j)+O_k-s_j-Q_j)^+]}{Q_j} \tag{3.37}
$$

$$
E[b_k] \simeq \frac{1}{2Q_k}\left(E[\left((D_k(L_k)-s_k)^+\right)^2] - E[\left((D_k(L_k)-(s_k+Q_k))^+\right)^2]\right) \tag{3.38}
$$

$$E[I_k] \simeq \frac{1}{2Q_k}\left(E[\left((s_k + Q_k - D_k(L_k))^+\right)^2] - E[\left((s_k - D_k(L_k))^+\right)^2]\right) \tag{3.39}$$

The equations (3.35) and (3.34) hold for all $k \in \mathcal{E}$, for a proof we refer to Silver et al. (1998). Note that in equation (3.35) in contrast to the standard expression given in Silver et al. (1998) we also take into account the expected shortage at the beginning of the replenishment cycle. For a proof of equation (3.39) we refer to Appendix 3. Equations (3.36), (3.38) can be derived in a similar manner, for more details see Janssen (1998). Finally, for a proof of equation (3.37) we refer to Section 3.3.4.

Equations (3.34), (3.35), (3.36), (3.38) and (3.39) assume that if there is not enough stock a partial delivery takes place. But we assumed that between stockpoints only complete deliveries are allowed. Therefore, we propose expressions for the fill rate β_k^c and the long average inventory level $E[I_k^c]$ in case of complete deliveries between the stockpoints.

$$\beta_k^c = \beta_k - \frac{(P\{D_k(L_k) + U_k > s_k\} - P\{D_k(L_k) + U_k > s_k + Q_k\})E[H_k]}{Q_k} \tag{3.40}$$

$$E[I_k^c] = E[I_k] + (1 - \gamma_k)E[H_k] \tag{3.41}$$

where H_k is the residual lifetime distribution of D_k, see Section 2.2.1. For an explanation of the both equations we refer to Appendix 3.

In some of the equations above we need an expression for the first two moments of the undershoot U_k. U_k can be evaluated using Theorem 3.1. Another approximation to evaluate the first two moments of U_k is to assume that U_k is distributed according to the stationary residual lifetime associated with the renewal process, see Section 2.2.1. For equations (3.34), (3.35) and (3.40) we evaluate the first two moments of $U_k + D_k(L_k)$, for equations (3.36), for equations (3.38), (3.39) and (3.41) the first two moments of $D_k(L_k)$ and for equation (3.37) the first two moments of $D_j(L_j) + O_k$. After that we fit a mixed-Erlang distribution to these first two moments and use the fitted distribution function to compute the performance measure.

Previously mentioned equations for the performance measures can also be used to evaluate the reorder level s_k such that a target performance measure is met. To do this a bisection method can be used, for more details see Winston (1993).

3.4 Numerical Example

In this section we illustrate the analytical approximations derived in Section 3.3 by a numerical example. We consider a two-echelon distribution system, with one beginning stockpoint $k \in \mathcal{B}$ denoted by 5 and 4 end-stockpoints $k \in \mathcal{E}$ denoted by $\{1, 2, 3, 4\}$. A schematic representation of the system is given in Figure 3.2.

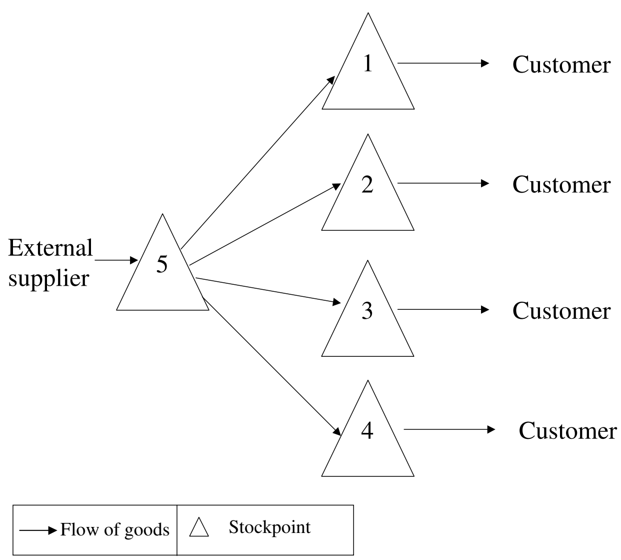


Figure 3.2: Schematic representation of the logistic system

Similarly to Silver et al. (1998), we assume that the batchsizes (Q_k) and target fill rates β_k at $k \in \mathcal{M}$ are given. With the algorithm described in Section 3.3.1 we compute the demand processes, the replenishment lead-times and the reorder levels s_k at all stockpoints $k \in \mathcal{M}$ such that these target fill rates are met. In strategic and tactical models different alternatives are compared by means of costs. An important element in these costs is the inventory cost, which is computed from the long-run average inventory level. Therefore as a second performance measure we also compute the long-run average physical inventory levels ($E[I_k]$).

For the interarrival time and order size process we assume exponential distributions. The average interarrival time is 1. In practice, we can observe that the demand size at only a few end-stockpoints is large, which we will denote as large accounts and at many end-stockpoints it is small, which we will denote as small accounts. The ratio between small and large accounts is approximately 75:25. (i.e. for $|\mathcal{E}| = 4$ the number of large accounts is 1 which is stockpoint 4). The average demand size ($E[D_k]$) at end-stockpoints for the small accounts is uniformly drawn on the interval (10,40) and for the large accounts on the interval (50,80). Further, we assume the exogenous delay L_k^d to be constant.

The input parameters of the numerical example are presented in Table 3.1.

Table 3.1: Input parameters of the numerical example.

k	1	2	3	4	5
$E[D_k]$	34	16	22	64	
$\sigma^2(D_k)$	1183	270	490	4120	
$E[A_k]$	1	1	1	1	
$\sigma^2(A_k)$	1	1	1	1	
L_k^d	2	2	2	2	8
Q_k	69	33	44	128	1027
β_k	95 %	95 %	95 %	95 %	80 %

In a first step, we evaluate, with the approximations presented in Section 3.3.3, the first two moments of the time between replenishments (R_k) and of the replenishment order size (O_k) of the end-stockpoints towards the beginning stockpoint. The results are presented in Table 3.2.

In a second step, we evaluate, with the approximations presented in Section 3.3.2, the first two moments of the interarrival time (A_k) and of the demand size

Table 3.2: The replenishment processes of the end-stockpoints towards the beginning stockpoint.

k	1	2	3	4
$E[O_k]$	80	38	51	148
$E[O_k^2]$	7188	1640	2977	25028
$E[R_k]$	2.31	2.31	2.31	2.31
$E[R_k^2]$	9.26	9.26	9.26	9.26

(D_k) at the beginning stockpoint. This gives

$E[D_5] = 79$

$\sigma^2(D_5) = 2918$

$E[A_5] = 0.578$

$\sigma^2(A_5) = 0.294$

In a third step, we evaluate via a bisection rule the reorder level at the beginning stockpoint and the long-run average inventory level. This results in $s_k = 1071$ and $E[I_k] = 535$ for $k \in \mathcal{B}$.

In a fourth step, we compute, with the approximations presented in Section 3.3.4, the first two moments of the waiting times (W_k^s) of the end-stockpoints. These results are presented in Table 3.3.

Table 3.3: The waiting time of the end-stockpoints due to a lack at the beginning stockpoint.

k	1	2	3	4
$E[W_k^s]$	0.31	0.28	0.29	0.39
$E[(W_k^s)^2]$	0.83	0.73	0.76	1.01

Finally, we compute via a bisection rule the reorder level at the end-stockpoints and the long-run average inventory level. These results are presented in Table 3.4.

In the following, we test the performance of the approximations for this numerical example by using discrete event simulations. The simulation runs until 3×10^5 customers have arrived at one of the stockpoints $k \in \mathcal{E}$ and we repeat this for 10

Table 3.4: The reorder level and the long-run average inventory levels at the end-stockpoints.

k	1	2	3	4	5
s_k	251	118	160	480	1071
$E[I_k]$	216	102	138	412	535

different seeds. The simulations length was determined such that accurate results were obtained for the performance characteristics. In Table 3.5 we recapitulate the input values and the values obtained for the different variables with the analytical approximations and we present the results of these values obtained from the simulation. Similarly to Chapter 2, the value between parentheses is the 95% confidence interval of the corresponding error, for more details we refer to Section 2.2.5 p 54.

To evaluate these variables, numerous approximations are made. In the following, we recapitulate the different approximations.

1. To evaluate $E[A_k^2]$, we approximate the distributions of R_j $j \in \mathcal{S}_k$ by mixed-Erlang distributions with the same first two moments.

2. To evaluate the first two moments of O_k, we approximate the distribution of D_k by a mixed-Erlang distribution with the same first two moments.

3. To evaluate $E[N(R_k^2)]$, we use an asymptotic approximation from renewal theory. Tijms (1994) shows that this approximation perform correctly under condition (2.13) p 44. Further, we introduce a correction factor for the case that Q_k is not sufficiently large.

4. To evaluate the first two moments of $\hat{L}$ and $\tilde{L}$, we approximate the distribution of L_k by a mixed-Erlang distribution with the same first two moments with the same first two moments.

5. To evaluate the first two moments of $N(\hat{L})$ and $N(\tilde{L})$, we use an asymptotic approximation from renewal theory. Tijms (1994) shows that this approximation perform correctly under condition (2.13) p 44.

Table 3.5: Results of the numerical example for the analytical approximations and simulations.

Input parameters and analytical approximations					
k	1	2	3	4	5
$E[D_k]$	34	16	22	64	79
$\sigma^2(D_k)$	1184	270	490	4120	2918
$E[A_k]$	1	1	1	1	0.57
$\sigma^2(A_k)$	1	1	1	1	0.29
ρ_{A_k}	0	0	0	0	0
Q_k	69	33	44	128	1027
L_k^d	2	2	2	2	8
$E[O_k]$	80	38	51	148	–
$E[O_k^2]$	7188	1640	2977	25028	–
$E[R_k]$	2.31	2.31	2.31	2.31	–
$E[R_k^2]$	9.25	9.25	9.25	9.252	–
$E[W_k^s]$	0.31	0.28	0.29	0.39	0
$E[(W_k^s)^2]$	0.83	0.73	0.76	1.01	0
β_k^{target}	0.95	0.95	0.95	0.95	0.80
s_k	251	118	160	480	1071
$E[I_k]$	216	102	138	412	535
Simulations					
k	1	2	3	4	5
$E[D_k]$	34 (±0)	16 (±0)	22 (±0)	64 (±0)	79 (±0)
$\sigma^2(D_k)$	1182 (±5)	270 (±0)	490 (±2)	4121 (±21)	2914 (±13)
$E[A_k]$	1.00 (±0.01)	1.00 (±0.01)	1.00 (±0.01)	1.00 (±0.001)	0.56 (±0.01)
$\sigma^2(A_k)$	1.00 (±0.01)	1.00 (±0.01)	1.00 (±0.01)	1.00 (±0.01)	0.27 (±0.00)
ρ_{A_k}	0.00 (±0.00)	0.00 (±0.00)	0.00 (±0.00)	0.00 (±0.00)	-0.04 (±0.00)
$E[O_k]$	80 (±0)	38 (±0)	51 (±0)	149 (±0)	–
$E[O_k^2]$	7192 (±15)	1641 (±2)	2981 (±5)	25052 (±77)	–
$E[R_k]$	2.31 (±0.01)	2.31 (±0.01)	2.31 (±0.01)	2.31 (±0.01)	–
$E[R_k^2]$	9.25 (±0.04)	9.23 (±0.03)	9.26 (±0.03)	9.25 (±0.03)	–
$E[W_k^s]$	0.29 (±0.00)	0.25 (±0.00)	0.28 (±0.00)	0.37 (±0.00)	0.00 (±0.00)
$E[(W_k^s)^2]$	0.76 (±0.01)	0.64 (±0.01)	0.68 (±0.01)	0.98 (±0.01)	0.00 (±0.00)
β_k	0.958 (±0.000)	0.958 (±0.000)	0.957 (±0.000)	0.957 (±0.000)	0.806 (±0.000)
$E[I_k]$	223 (±0)	108 (±0)	143 (±0)	432 (±0)	525 (±1)

6. To evaluate $E[(D_j(\hat{L}_j) + O_k - s_j)^+]$ and $E[(D_j(\hat{L}_j) + O_k - (s_j + Q_j))^+]$, we approximate the distribution of $D_j(\hat{L}_j) + O_k$ by a mixed-Erlang distribution with the same first two moments.

7. To evaluate $E[(D_j(\tilde{L}_j) + O_k - s_j)^+]$ and $E[(D_j(\tilde{L}_j) + O_k - s_j + Q_j)^+]$, we approximate the distribution of $D_j(\hat{L}_j) + O_k$ by a mixed-Erlang distribution with the same first two moments.

8. To evaluate the performance measures, we approximate respectively the distributions of $D_k(L_k)$, $D_k(L_k) + U_k$ and $D_j(L_j) + O_k$ by mixed-Erlang distributions with the same first two moments.

Since (A_k,D_k) ($k \in \mathcal{E}$) are exponential distributed, the evaluation of the first moment of R_k and the first two moments of O_k are exact. The approximations used to evaluate the second moment of R_k perform correctly, since from Table 3.5 we observe that the difference between the simulated and the evaluated R_k is small. As expected $E[A_5]$ is correctly evaluated, since $E[R_k]$ $k \in \mathcal{E}$ is correctly evaluated and we have an exact formula to derive $E[A_5]$ from $E[R_k]$ $k \in \mathcal{E}$. Also as expected there is a small error in $\sigma^2(A_5)$ and $\rho_{A_5} < 0$, this is caused by the superposition method (for more details we refer to Section 2.2.5). These errors will decrease when $|\mathcal{E}|$ increases. The formulae to evaluate the first two moments of D_5 from the first two moments O_k, $E[R_k]$ and $E[A_5]$ ($k \in \mathcal{E}$) are exact therefore the error in $\sigma^2(D_5)$ comes from an error in the input ($E[O_k]$,$E[O_k^2]$,$E[R_k]$ and $E[A_5]$). The errors in the first two moments of W_k^s can be caused by approximations 1-7. The errors in β_k and $E[I_k]$ can be caused by all approximations. Therefore it is difficult to determine for these four variables, what the effect is of each approximation. For this numerical example, we observe that the errors are small. But to be able to generalize these results, we need to test the quality of the approximations for numerous other cases.

3.5 Numerical analysis

In this section we further investigate the analytical approximations derived in Section 3.3. This is realized by performing discrete event simulations. Similarly to 3.4, we assume given target fill rates β_k for all $k \in \mathcal{M}$ and with the algorithm described in Section 3.3.1 we compute the demand processes, the replenishment lead-times and the reorder levels s_k at all stockpoints $k \in \mathcal{M}$ such that the target fill rate is met. Further, we also assume as a second performance measure the long-run average physical inventory levels. After that we run a simulation of the distribution system with the given customer demands, exogenous delays, batchsizes and the calculated reorder levels. The simulation runs until 3×10^5 customers have arrived at one of the stockpoints $k \in \mathcal{E}$ and we repeat this for 10 different seeds, to have accurate point estimates for all relevant performance characteristics. We compare the resulting fill rate with the target fill rate and the expected physical inventory level obtained with the simulation and with the analytical approximations. Additionally, we test the approximations for the first two moments of the waiting time. Since we observe in Section 3.4 that these four variables are most critical.

For every simulation we calculate the absolute percentage error in the fill rate, the percentage error in the waiting time due to a lack of stock and the percentage error in the long-run expected physical inventory level. For the definitions of the errors we refer to Section 2.2.5.

To be able to draw meaningful conclusions, we define acceptable margins for the $\delta_i^{(E[W_k^s])}$, $\delta_i^{(E[(W_k^s)^2])}$ $\Delta_i^{(\beta_k)}$ values and the $\delta_i^{(E[I_k])}$ values. To construct a realistic margin, we look at the error in the probability of having backlog $(1-\beta_k^{target})$. In Table 3.6 good and acceptable values are defined for the fill rate. The good and the acceptable margins for the $\delta_i^{(E[W_k^s])}$, $\delta_i^{(E[(W_k^s)^2])}$ and $\delta_i^{(E[I_k])}$ are respectively 5 and 10 %.

Due to the numerous layers of approximations invoked in order to analyze this general system, it is difficult to understand the impact of the individual approximations. However by varying the following six input parameters, we can investigate the influence of the approximations in general.

1. Coefficient of variation of the inter-arrival times at $k \in \mathcal{E}$ $(c^2_{A_k})$.
2. Coefficient of variation of the demand sizes at $k \in \mathcal{E}$ $(c^2_{D_k})$.
3. Target fill rate at the end-stockpoints $(\beta_k^{target}, k \in \mathcal{E})$.

Table 3.6: Good and acceptable fill rates ($k \in \mathcal{M}$).

β_k^{target}	Good fill rates			Acceptable fill rates		
	Min	Max	$\Delta_i^{(\beta_k)}$	Min	Max	$\Delta_i^{(\beta_k)}$
60 %	56 %	64 %	≤ 4	52 %	68 %	$4 < \Delta_i^{(\beta_k)} \leq 8$
80 %	78 %	82 %	≤ 2	76 %	84 %	$2 < \Delta_i^{(\beta_k)} \leq 4$
90 %	89 %	91 %	≤ 1	88 %	92 %	$1 < \Delta_i^{(\beta_k)} \leq 2$
95 %	94.5 %	95.5 %	≤ 0.5	94 %	96 %	$0.5 < \Delta_i^{(\beta_k)} \leq 1$
99 %	98.9 %	99.1 %	≤ 0.1	98.8 %	99.2 %	$0.1 < \Delta_i^{(\beta_k)} \leq 0.2$

4. Number of first echelon stockpoints ($|\mathcal{E}|$).

5. Batchsizes (Q_k, $k \in \mathcal{M}$).

6. Target fill rate at the beginning stockpoints (β_k^{target}, $k \in \mathcal{B}$).

In the following, we will investigate the sensitivity of above mentioned input parameters on the performance of the approximations. In Section 3.5.1 we report the results for a 2-echelon distribution networks and in Section 3.5.2 for a 3-echelon distribution networks.

3.5.1 A 2-echelon distribution networks

In the following section we consider a two-echelon distribution system consisting of a single beginning stockpoint and multiple end-stockpoints. Note a two echelon system has no intermediate stockpoints. For the simulations we use the following input. We vary the number of first echelon stockpoints ($|\mathcal{E}|$) between 4, 8 and 16. We assume that the interarrival time and the order size at the end-stockpoints are mixed-Erlang distributed. The average interarrival time ($E[A_k]$) is 1 for all $k \in \mathcal{E}$ and the squared coefficient of variation ($c^2_{A_k}$) is varied between 0.4, 1 and 1.8. Further in practice, we can observe that the demand size at only a few $k \in \mathcal{E}$ is large, which we will denote as large accounts and at many $k \in \mathcal{E}$ it is small, which we will denote as small accounts. The ratio between small and large accounts is approximately 75:25. (e.i. for $|\mathcal{E}| = 4$ the number of large accounts is 1, for $|\mathcal{E}| = 8$ it is 2 and for $|\mathcal{E}| = 16$ it is 4). The average demand size ($E[D_k]$) at $k \in \mathcal{E}$ for the small accounts is uniformly drawn on the interval (10,40) and for the large accounts on the interval (50,80). The squared coefficient of variation $c^2_{D_k}$ at $k \in \mathcal{E}$ is varied over 0.4, 1 and 1.8. The exogenous delay ($E[L^d_k]$) is assumed to be mixed-Erlang distributed with average 2 for $k \in \mathcal{E}$ and 8 for $k \in \mathcal{B}$ and $c^2_{L^d_k} = 0.1$ for $k \in \mathcal{M}$. The batchsizes Q_k varies between $2E[D_k]$ and $5E[D_k]$ for $k \in \mathcal{E}$ and Q_j varied $4\max\limits_{k\in\mathcal{E}}(Q_k)$ and $8\max\limits_{k\in\mathcal{E}}(Q_k)$ for $j \in \mathcal{B}$. The target fill rate at $k \in \mathcal{B}$ is varied between 0.6, 0.8 and 0.9 and $k \in \mathcal{E}$ between 0.90, 0.95 and 0.99. The results are given in Tables 3.7, 3.8, 3.9, 3.10, 3.11, 3.12 and 3.13. For the calculations of the errors, we refer to Equation (2.37). Similar to Section 3.4, the value between parentheses is the 95% confidence interval of the corresponding error. This means that for example for the first situation ($\beta^{target}_k = 0.9$, $c^2_{A_k} = 0.4$ and $c^2_{D_k} = 0.4$), we can claim with approximately 95 % that $\bar{\delta}^{E[W_k]}$ is contained in the interval 8.61 ± 2.48, i.e. [6.13,11.09]. In the simulations we have observed that the errors for the large and small accounts reveal approximately the same behaviour, therefore no information is lost considering the average of all accounts.

We now investigate the errors in $E[W^s_k]$, $E[(W^s_k)^2]$, β_k and $E[I_k]$ ($k \in \mathcal{M}$). The errors in the first two moments of W^s_k are all within acceptable margins. The error in the second moment of W^s_k is generally higher than the error in the first moment. An error in β_j ($j \in \mathcal{B}$) can lead to an error in the waiting times due to a lack of stock experienced by the $k \in \mathcal{S}_j$ and this can lead to an error in β_k. But we see that the propagation of the error between $\bar{\Delta}^{(\beta_j)}$ and $\bar{\Delta}^{(\beta_k)}$ ($k \in \mathcal{E}$ and $j \in \mathcal{B}$) is not so large. For example in Table 3.7, when $c^2_{A_k} = 0.4$ and $c^2_{D_k} = 1$ the error in β_j ($j \in \mathcal{B}$) is relatively high, but this results not in a large increase in the error in β_k $k \in \mathcal{E}$. This is important, because in practice, an error in β_k ($k \in \mathcal{E}$) has more serious consequences than an error in β_j ($j \in \mathcal{B}$), because β_k denotes the fill rate the customers will receive. If we compare the $\bar{\Delta}^{(\beta_k)}$ ($k \in \mathcal{E}$) for the different simulations with the margins we can conclude that the fill rate at the first echelon stockpoints are correctly estimated. The same is true for the average physical inventory level at the end-stockpoints.

Table 3.7: Results for the first two moments of the waiting time for a 2-echelon distribution system with $|\mathcal{E}| = 4$, $\beta_j^{target} = 0.80$, $Q_k = 5E[D_k]$ and $Q_j = 8 \max_{k \in \mathcal{E}} Q_k$ where $k \in \mathcal{E}$ and $j \in \mathcal{B}$. Between parentheses the 95% confidence interval of the corresponding error is indicated

Problem data			$k \in \mathcal{E}$	
β_k^{target}	$c^2_{A_k}$	$c^2_{D_k}$	$\bar{\delta}^{(E[W_k^s])}$	$\bar{\delta}^{(E[(W_k^s)^2])}$
0.9	0.4	0.4	8.61 ($\pm$2.48)	8.79 ($\pm$3.82)
0.9	0.4	1	7.36 ($\pm$3.44)	9.36 ($\pm$4.92)
0.9	0.4	1.8	7.02 ($\pm$2.53)	7.84 ($\pm$2.90)
0.9	1	0.4	7.97 ($\pm$1.73)	8.11 ($\pm$2.33)
0.9	1	1	7.86 ($\pm$1.69)	8.35 ($\pm$2.37)
0.9	1	1.8	5.54 ($\pm$2.15)	10.22 ($\pm$3.43)
0.9	1.8	0.4	6.77 ($\pm$1.87)	10.67 ($\pm$3.04)
0.9	1.8	1	6.37 ($\pm$2.00)	6.45 ($\pm$2.40)
0.9	1.8	1.8	6.45 ($\pm$1.51)	7.66 ($\pm$2.69)
0.95	0.4	0.4	8.61 ($\pm$2.48)	8.79 ($\pm$3.82)
0.95	0.4	1	7.36 ($\pm$3.44)	9.36 ($\pm$4.92)
0.95	0.4	1.8	7.02 ($\pm$2.53)	7.84 ($\pm$2.90)
0.95	1	0.4	7.97 ($\pm$1.73)	8.11 ($\pm$2.33)
0.95	1	1	7.86 ($\pm$1.69)	8.35 ($\pm$2.37)
0.95	1	1.8	5.54 ($\pm$2.15)	10.22 ($\pm$3.43)
0.95	1.8	0.4	6.77 ($\pm$1.87)	10.67 ($\pm$3.04)
0.95	1.8	1	6.37 ($\pm$2.00)	6.45 ($\pm$2.40)
0.95	1.8	1.8	6.45 ($\pm$1.51)	7.66 ($\pm$2.69)
0.99	0.4	0.4	8.61 ($\pm$2.48)	8.79 ($\pm$3.82)
0.99	0.4	1	7.36 ($\pm$3.44)	9.36 ($\pm$4.92)
0.99	0.4	1.8	7.02 ($\pm$2.53)	7.84 ($\pm$2.90)
0.99	1	0.4	7.97 ($\pm$1.73)	8.11 ($\pm$2.33)
0.99	1	1	7.86 ($\pm$1.69)	8.35 ($\pm$2.37)
0.99	1	1.8	5.54 ($\pm$2.15)	10.22 ($\pm$3.43)
0.99	1.8	0.4	6.77 ($\pm$1.87)	10.67 ($\pm$3.04)
0.99	1.8	1	6.37 ($\pm$2.00)	6.45 ($\pm$2.40)
0.99	1.8	1.8	6.45 ($\pm$1.51)	7.66 ($\pm$2.69)

Table 3.8: Results for the fill rate and the long-run average inventory level for a 2-echelon distribution system with $|\mathcal{E}| = 4$, $\beta_j^{target} = 0.80$, $Q_k = 5E[D_k]$ and $Q_j = 8 \max_{k \in \mathcal{E}} Q_k$ where $k \in \mathcal{E}$ and $j \in \mathcal{B}$. Between parentheses the 95% confidence interval of the corresponding error is indicated

Problem data			$k \in \mathcal{E}$		$j \in \mathcal{B}$	
β_k^{target}	$c^2_{A_k}$	$c^2_{D_k}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$	$\bar{\Delta}^{(\beta_j)}$	$\bar{\delta}^{(E[I_j])}$
0.9	0.4	0.4	0.70 (±0.17)	2.64 (±0.34)	1.66 (±0.31)	2.93 (±1.13)
0.9	0.4	1	0.82 (±0.23)	5.47 (±0.36)	2.76 (±0.54)	3.39 (±0.59)
0.9	0.4	1.8	0.33 (±0.14)	5.18 (±0.29)	0.74 (±0.26)	1.66 (±0.49)
0.9	1	0.4	0.42 (±01.2)	6.83 (±0.27)	0.83 (±0.17)	3.03 (±0.48)
0.9	1	1	0.43 (±0.17)	6.62 (±0.33)	2.40 (±0.25)	2.87 (±0.52)
0.9	1	1.8	0.41 (±0.19)	4.68 (±0.32)	0.45 (±0.16)	1.36 (±0.39)
0.9	1.8	0.4	0.40 (±0.15)	5.80 (±0.28)	0.45 (±0.27)	1.68 (±0.54)
0.9	1.8	1	0.35 (±0.15)	8.51 (±0.26)	1.59 (±0.24)	1.88 (±0.57)
0.9	1.8	1.8	0.72 (±0.21)	7.14 (±0.18)	0.47 (±0.22)	0.99 (±0.56)
0.95	0.4	0.4	0.27 (±0.10)	2.80 (±0.29)	1.66 (±0.31)	2.93 (±1.13)
0.95	0.4	1	0.39 (±0.16)	1.82 (±0.28)	2.76 (±0.54)	3.39 (±0.59)
0.95	0.4	1.8	0.22 (±0.12)	1.54 (±0.24)	0.74 (±0.26)	1.66 (±0.49)
0.95	1	0.4	0.28 (±0.10)	7.56 (±0.22)	0.83 (±0.17)	3.03 (±0.48)
0.95	1	1	0.26 (±0.10)	6.71 (±0.28)	2.40 (±0.25)	2.87 (±0.52)
0.95	1	1.8	0.38 (±0.18)	5.77 (±0.25)	0.45 (±0.16)	1.36 (±0.39)
0.95	1.8	0.4	0.34 (±0.12)	6.45 (±0.23)	0.45 (±0.27)	1.68 (±0.54)
0.95	1.8	1	0.32 (±0.12)	6.33 (±0.22)	1.59 (±0.24)	1.88 (±0.57)
0.95	1.8	1.8	0.70 (±0.18)	9.27 (±0.14)	0.47 (±0.22)	0.99 (±0.56)
0.99	0.4	0.4	0.30 (±0.08)	3.00 (±0.21)	1.66 (±0.31)	2.93 (±1.13)
0.99	0.4	1	0.13 (±0.07)	0.37 (±0.16)	2.76 (±0.54)	3.39 (±0.59)
0.99	0.4	1.8	0.21 (±0.08)	0.32 (±0.14)	0.74 (±0.26)	1.66 (±0.49)
0.99	1	0.4	0.20 (±0.09)	6.17 (±0.15)	0.83 (±0.17)	3.03 (±0.48)
0.99	1	1	0.18 (±0.08)	3.39 (±0.19)	2.40 (±0.25)	2.87 (±0.52)
0.99	1	1.8	0.18 (±0.07)	2.57 (±0.17)	0.45 (±0.16)	1.36 (±0.39)
0.99	1.8	0.4	0.19 (±0.08)	9.33 (±0.15)	0.45 (±0.27)	1.68 (±0.54)
0.99	1.8	1	0.19 (±0.09)	6.03 (±0.14)	1.59 (±0.24)	1.88 (±0.57)
0.99	1.8	1.8	0.23 (±0.09)	4.95 (±0.11)	0.47 (±0.22)	0.99 (±0.56)

Table 3.9: Results for the first two moments of the waiting time for a 2-echelon distribution system with $|\mathcal{E}| = 4$, $\beta_k^{target} = 0.95$ and $\beta_j^{target} = 0.80$ where $x = 4 \max Q_k$, $y = 8 \max Q_k$, $k \in \mathcal{E}$ and $j \in \mathcal{B}$. Between parentheses the 95% confidence interval of the corresponding error is given

Problem data				$k \in \mathcal{E}$	
Q_k	Q_j	$c^2_{A_k}$	$c^2_{D_k}$	$\bar{\delta}^{(E[W^s_k])}$	$\bar{\delta}^{(E[(W^s_k)^2])}$
$5E[D_k]$	x	0.4	0.4	3.77 (±1.53)	9.59 (±2.19)
$5E[D_k]$	x	0.4	1	7.38 (±2.81)	8.62 (±4.15)
$5E[D_k]$	x	0.4	1.8	8.77 (±3.07)	11.14 (±4.06)
$5E[D_k]$	x	1	0.4	5.90 (±2.27)	8.37 (±2.70)
$5E[D_k]$	x	1	1	5.78 (±2.10)	9.92 (±3.60)
$5E[D_k]$	x	1	1.8	6.02 (±1.93)	8.85 (±3.48)
$5E[D_k]$	x	1.8	0.4	6.76 (±2.59)	7.98 (±3.49)
$5E[D_k]$	x	1.8	1	5.29 (±2.16)	6.82 (±2.59)
$5E[D_k]$	x	1.8	1.8	8.11 (±2.29)	10.25 (±3.46)
$2E[D_k]$	y	0.4	0.4	6.89 (±1.77)	11.27 (±2.56)
$2E[D_k]$	y	0.4	1	6.47 (±1.95)	6.88 (±2.53)
$2E[D_k]$	y	0.4	1.8	7.76 (±2.21)	11.01 (±2.83)
$2E[D_k]$	y	1	0.4	6.49 (±2.37)	6.55 (±3.31)
$2E[D_k]$	y	1	1	4.89 (±2.34)	7.70 (±3.17)
$2E[D_k]$	y	1	1.8	5.51 (±2.10)	9.48 (±3.17)
$2E[D_k]$	y	1.8	0.4	3.57 (±1.38)	6.72 (±1.77)
$2E[D_k]$	y	1.8	1	8.38 (±2.07)	10.47 (±2.61)
$2E[D_k]$	y	1.8	1.8	4.62 (±1.56)	8.82 (±2.18)

Table 3.10: Results for the fill rate and the long-run average inventory level for a 2-echelon distribution system with $|\mathcal{E}| = 4$, $\beta_k^{target} = 0.95$ and $\beta_j^{target} = 0.80$ where $x = 4 \max Q_k$, $y = 8 \max Q_k$, $k \in \mathcal{E}$ and $j \in \mathcal{B}$. Between parentheses the 95% confidence interval of the corresponding error is given

Problem data				$k \in \mathcal{E}$		$j \in \mathcal{B}$	
Q_k	Q_j	$c^2_{A_k}$	$c^2_{D_k}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$	$\bar{\Delta}^{(\beta_j)}$	$\bar{\delta}^{(E[I_j])}$
$5E[D_k]$	x	0.4	0.4	0.31 (±0.10)	3.32 (±0.17)	1.87 (±0.21)	5.28 (±1.78)
$5E[D_k]$	x	0.4	1	0.49 (±0.16)	1.83 (±0.28)	2.58 (±0.43)	4.46 (±0.82)
$5E[D_k]$	x	0.4	1.8	0.30 (±0.11)	1.32 (±0.28)	0.72 (±0.25)	2.04 (±0.65)
$5E[D_k]$	x	1	0.4	0.22 (±0.08)	8.27 (±0.25)	1.07 (±0.26)	3.44 (±0.37)
$5E[D_k]$	x	1	1	0.22 (±0.12)	6.74 (±0.20)	2.74 (±0.22)	4.07 (±0.51)
$5E[D_k]$	x	1	1.8	0.48 (±0.18)	5.50 (±0.23)	0.53 (±0.28)	1.15 (±0.32)
$5E[D_k]$	x	1.8	0.4	0.39 (±0.17)	4.96 (±0.24)	0.39 (±0.12)	2.08 (±0.68)
$5E[D_k]$	x	1.8	1	0.35 (±0.10)	10.82 (±0.16)	1.89 (±0.19)	2.55 (±0.42)
$5E[D_k]$	x	1.8	1.8	0.84 (±0.27)	9.16 (±0.24)	1.23 (±0.48)	0.84 (±0.40)
$2E[D_k]$	y	0.4	0.4	0.86 (±0.28)	4.88 (±0.26)	1.16 (±0.35)	2.67 (±0.70)
$2E[D_k]$	y	0.4	1	0.61 (±0.15)	3.77 (±0.23)	1.35 (±0.16)	0.60 (±0.32)
$2E[D_k]$	y	0.4	1.8	0.94 (±0.23)	2.22 (±0.26)	1.46 (±0.15)	1.37 (±0.52)
$2E[D_k]$	y	1	0.4	0.60 (±0.19)	5.68 (±0.32)	1.59 (±0.17)	4.08 (±0.57)
$2E[D_k]$	y	1	1	0.58 (±0.26)	5.03 (±0.36)	1.36 (±0.21)	1.10 (±0.21)
$2E[D_k]$	y	1	1.8	0.47 (±0.16)	3.88 (±0.31)	1.59 (±0.13)	2.64 (±0.82)
$2E[D_k]$	y	1.8	0.4	0.78 (±0.33)	0.57 (±0.26)	1.39 (±0.21)	5.13 (±0.65)
$2E[D_k]$	y	1.8	1	0.77 (±0.22)	0.49 (±0.20)	1.16 (±0.23)	2.28 (±0.70)
$2E[D_k]$	y	1.8	1.8	0.70 (±0.31)	0.46 (±0.16)	1.41 (±1.11)	4.00 (±0.47)

Table 3.11: Results for the first two moments of the waiting time for a 2-echelon distribution system with $Q_k = 5E[D_k]$, $Q_j = 8 \max_{k \in \mathcal{E}} Q_k$, $\beta_k^{target} = 0.95$ and $|\mathcal{E}| = 4$ where $k \in \mathcal{E}$ and $j \in \mathcal{B}$. Between parentheses the 95% confidence interval of the corresponding error is given.

Problem data			$k \in \mathcal{E}$	
β_j^{target}	$c^2_{A_k}$	$c^2_{D_k}$	$\bar{\delta}^{(E[W_k^s])}$	$\bar{\delta}^{(E[(W_k^s)^2])}$
0.60	0.4	0.4	2.42 (±0.76)	6.46 (±1.38)
0.60	0.4	1	3.60 (±1.11)	4.77 (±1.30)
0.60	0.4	1.8	4.88 (±1.09)	7.40 (±1.93)
0.60	1	0.4	5.24 (±1.53)	5.72 (±1.69)
0.60	1	1	3.86 (±1.16)	6.15 (±1.61)
0.60	1	1.8	4.08 (±1.43)	4.93 (±1.77)
0.60	1.8	0.4	3.91 (±1.05)	8.16 (±1.73)
0.60	1.8	1	3.35 (±1.62)	6.45 (±2.45)
0.60	1.8	1.8	4.48 (±1.05)	6.00 (±1.80)
0.90	0.4	0.4	9.80 (±2.75)	10.33 (±4.02)
0.90	0.4	1	6.44 (±2.98)	9.96 (±3.45)
0.90	0.4	1.8	7.01 (±2.83)	8.16 (±3.30)
0.90	1	0.4	7.17 (±2.59)	9.82 (±3.35)
0.90	1	1	7.13 (±3.45)	9.14 (±4.04)
0.90	1	1.8	5.31 (±2.09)	7.12 (±3.95)
0.90	1.8	0.4	8.24 (±3.06)	12.09 (±3.96)
0.90	1.8	1	5.26 (±2.29)	10.41 (±2.29)
0.90	1.8	1.8	6.92 (±3.70)	10.87 (±5.94)

Table 3.12: Results for the fill rate and the long-run average inventory level for a 2-echelon distribution system with $Q_k = 5E[D_k]$, $Q_j = 8 \max_{k \in \mathcal{E}} Q_k$, $\beta_k^{target} = 0.95$ and $|\mathcal{E}| = 4$ where $k \in \mathcal{E}$ and $j \in \mathcal{B}$. Between parentheses the 95% confidence interval of the corresponding error is given.

Problem data			$k \in \mathcal{E}$		$j \in \mathcal{B}$	
β_j^{target}	$c_{A_k}^2$	$c_{D_k}^2$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$	$\bar{\Delta}^{(\beta_j)}$	$\bar{\delta}^{(E[I_j])}$
0.60	0.4	0.4	0.94 (±0.14)	1.63 (±0.22)	2.22 (±0.16)	3.52 (±0.42)
0.60	0.4	1	0.52 (±0.17)	1.46 (±0.27)	3.31 (±0.27)	5.67 (±0.70)
0.60	0.4	1.8	0.53 (±0.16)	1.64 (±0.27)	0.50 (±0.31)	1.84 (±0.47)
0.60	1	0.4	0.62 (±0.18)	6.63 (±0.35)	1.92 (±0.36)	2.21 (±0.53)
0.60	1	1	0.45 (±0.16)	6.34 (±0.25)	3.21 (±0.27)	3.92 (±0.64)
0.60	1	1.8	0.54 (±0.18)	5.73 (±0.28)	1.60 (±0.42)	0.99 (±0.31)
0.60	1.8	0.4	0.56 (±0.16)	4.10 (±0.29)	1.48 (±0.17)	1.29 (±0.40)
0.60	1.8	1	0.49 (±0.19)	3.97 (±0.39)	2.71 (±0.31)	2.63 (±0.50)
0.60	1.8	1.8	0.67 (±0.24)	3.37 (±0.31)	0.85 (±0.26)	0.83 (±0.42)
0.90	0.4	0.4	0.42 (±0.11)	3.37 (±0.21)	0.79 (±0.20)	3.17 (±0.55)
0.90	0.4	1	0.55 (±0.13)	1.45 (±0.13)	2.19 (±0.12)	2.70 (±0.47)
0.90	0.4	1.8	0.22 (±0.10)	1.41 (±0.19)	0.31 (±0.22)	1.82 (±0.73)
0.90	1	0.4	0.20 (±0.09)	8.57 (±0.24)	0.39 (±0.19)	2.27 (±0.61)
0.90	1	1	0.26 (±0.09)	6.64 (±0.18)	1.52 (±0.24)	2.62 (±0.53)
0.90	1	1.8	0.29 (±0.13)	5.67 (±0.17)	0.38 (±0.12)	1.25 (±0.45)
0.90	1.8	0.4	0.27 (±0.10)	4.54 (±0.25)	0.35 (±0.16)	1.75 (±0.45)
0.90	1.8	1	0.34 (±0.11)	5.64 (±0.19)	1.03 (±0.24)	1.75 (±1.07)
0.90	1.8	1.8	0.62 (±0.21)	2.91 (±0.24)	0.67 (±0.23)	1.20 (±0.76)

Table 3.13: Results 2-echelon distribution system with $\beta_k^{target} = 0.95$, $\beta_k^{target} = 0.80$, $Q_k = 5E[D_k]$ and $Q_j = 8 \max_{k \in \mathcal{E}} Q_k$ where $k \in \mathcal{E}$ and $j \in \mathcal{B}$. Between parentheses the 95% confidence interval of the corresponding error is indicated.

Problem data			$k \in \mathcal{E}$		$j \in \mathcal{B}$	
$\|\mathcal{E}\|$	$c^2_{A_k}$	$c^2_{D_k}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$	$\bar{\Delta}^{(\beta_j)}$	$\bar{\delta}^{(E[I_j])}$
8	0.4	0.4	0.54 (±0.07)	6.06 (±0.15)	0.64 (±0.45)	1.95 (±0.96)
8	0.4	1	0.53 (±0.07)	2.49 (±0.11)	1.85 (±0.25)	1.10 (±0.53)
8	0.4	1.8	0.39 (±0.07)	2.80 (±0.09)	1.16 (±0.28)	0.35 (±0.21)
8	1	0.4	0.38 (±0.08)	5.63 (±0.72)	0.38 (±0.23)	0.42 (±0.32)
8	1	1	0.26 (±0.06)	7.87 (±0.11)	0.87 (±0.31)	0.66 (±0.44)
8	1	1.8	0.26 (±0.06)	7.13 (±0.08)	1.01 (±0.15)	0.61 (±0.50)
8	1.8	0.4	0.24 (±0.05)	7.90 (±0.76)	0.24 (±0.15)	0.40 (±0.34)
8	1.8	1	0.39 (±0.07)	6.76 (±0.69)	0.96 (±0.38)	0.35 (±0.33)
8	1.8	1.8	0.62 (±0.10)	7.11 (±0.62)	1.15 (±0.27)	1.19 (±0.72)
16	0.4	0.4	0.53 (±0.04)	6.21 (±0.07)	0.26 (±0.14)	1.09 (±0.77)
16	0.4	1	0.47 (±0.04)	2.36 (±0.06)	0.30 (±0.14)	1.61 (±0.67)
16	0.4	1.8	0.36 (±0.04)	2.73 (±0.05)	1.03 (±0.18)	2.43 (±0.42)
16	1	0.4	0.28 (±0.04)	5.40 (±0.51)	0.56 (±0.21)	2.14 (±0.59)
16	1	1	0.21 (±0.04)	7.73 (±0.07)	0.50 (±0.39)	2.19 (±0.28)
16	1	1.8	0.23 (±0.04)	7.02 (±0.06)	1.17 (±0.06)	3.06 (±0.59)
16	1.8	0.4	0.23 (±0.04)	6.51 (±0.04)	1.26 (±0.39)	2.69 (±0.87)
16	1.8	1	0.37 (±0.04)	6.26 (±0.54)	0.34 (±0.25)	2.38 (±0.59)
16	1.8	1.8	0.60 (±0.07)	6.44 (±0.50)	0.46 (±0.24)	3.50 (±0.63)

In the simulations we varied six different input parameters: $c^2_{A_k}$ $(k \in \mathcal{E})$, $c^2_{D_k}$ $(k \in \mathcal{E})$, β_k^{target} $(k \in \mathcal{E})$, $|\mathcal{E}|$, Q_k $(k \in \mathcal{M})$ and β_j^{target} $(j \in \mathcal{B})$. In the following we will investigate the effects of previously mentioned input parameters.

When both $c^2_{A_k}$ and $c^2_{D_k}$ are high (1.8) or low (0.4) then the largest errors in β_k and $E[I_k]$ $(k \in \mathcal{E})$ are observed. This is in line with the expectations because some approximations, like the asymptotic approximations of the residual lifetime, perform best when the variable is exponentially distributed which means that c^2 is close to 1.

When the target β_k increases then the performance of the approximations decrease in comparison with the defined ranges. This can be observed in tables 3.7, 3.8 and Figure 3.3.

When $Q_j = 8 \max\limits_{k \in \mathcal{E}} Q_k$ and $Q_k = 2E[D_k]$ then the errors in β_j, β_k, $E[I_k]$ and $E[I_j]$ $(k \in \mathcal{E}$ and $j \in \mathcal{B})$ are the highest. For more details see tables 3.9, 3.10 and Figure 3.4.

When β_j^{target} $(j \in \mathcal{B})$ increases then the errors decrease. The following can be observed in tables 3.11 and 3.12. When β_j is larger then the first two moments of W_k^s are smaller in comparison to the first two moments of the replenishment lead-time. Therefore the influence of the waiting time due to a lack of stock on the replenishment lead-time is smaller.

When $|\mathcal{E}|$ increases from 4 to 8 then the errors in β_k for all $k \in \mathcal{M}$ seem to increase and when $|\mathcal{E}|$ increases from 8 to 16 then the errors in β_k for all $k \in \mathcal{M}$ seem to decrease. This can be observed in Table 3.13 and Figure 3.5. The same pattern is observed, in Section 2.2.5, for the errors in the second moment of the superposed process.

To conclude, nearly all results are within the acceptable margins and therefore the approximations are of sufficient quality for practical purposes.

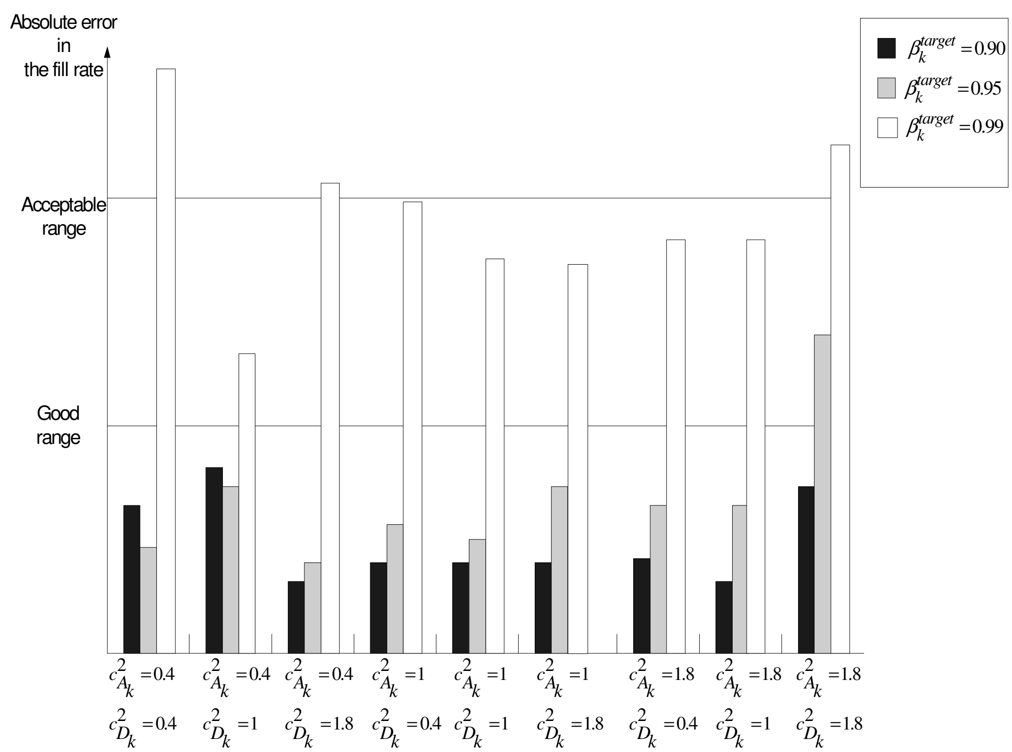


Figure 3.3: Absolute error in β_k ($k \in \mathcal{E}$) for different values of β_k^{target}, $c^2_{A_k}$ and $c^2_{D_k}$. Given that $|\mathcal{E}| = 4$, $\beta_j^{target} = 0.80$, $Q_k = 5 * E[D_k]$ and $Q_j = 8 \max_{k \in \mathcal{E}} Q_j$ ($k \in \mathcal{E}$ and $j \in \mathcal{B}$).

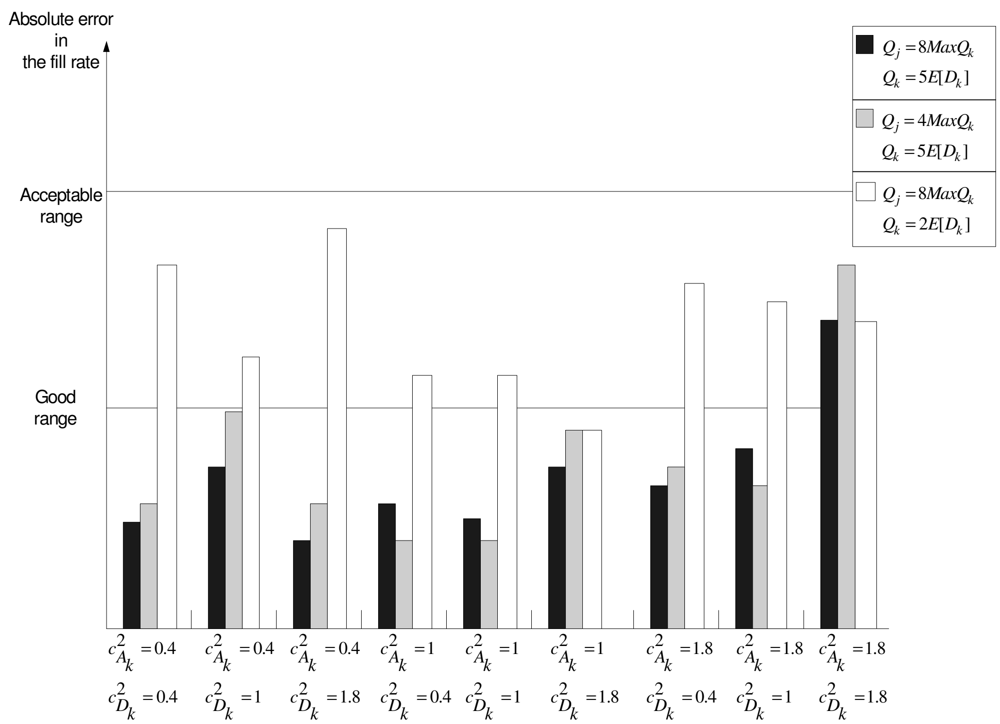


Figure 3.4: Absolute error in β_k ($k \in \mathcal{E}$) for different values of Q_k, Q_j, $c^2_{A_k}$ and $c^2_{D_k}$. Given that $|\mathcal{E}| = 4$, $\beta_j^{target} = 0.80$ and $\beta_k^{target} = 0.95$ ($k \in \mathcal{E}$ and $j \in \mathcal{B}$).

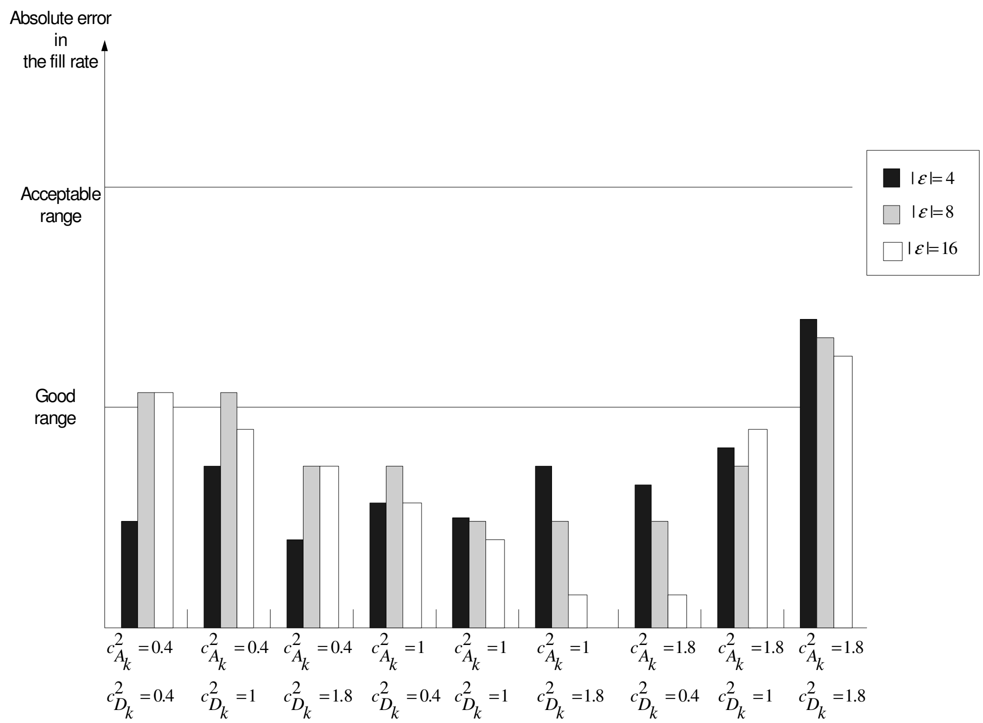


Figure 3.5: Absolute error in β_k ($k \in \mathcal{E}$) for different values of $|\mathcal{E}|$, $c^2_{A_k}$ and $c^2_{D_k}$. Given that $\beta_j^{target} = 0.80$, $\beta_k^{target} = 0.95$, $Q_k = 5 * E[D_k]$ and $Q_j = 8 \max_{k \in \mathcal{E}} Q_j$ ($k \in \mathcal{E}$ and $j \in \mathcal{B}$).

3.5.2 3-echelon case

In this paragraph, we test the analytical approximations in a 3-echelon system. The distribution system is constituted of one third echelon stockpoint $i \in \mathcal{B}$, a number of second echelon stockpoints $j \in \mathcal{I}$ allocated to k and a number of first echelon stockpoints $k \in \mathcal{E}$ allocated to j. We vary $|\mathcal{E}|$ between 32 and 16 and set $|\mathcal{I}| = 4$. The batchsizes Q_k is $5E[D_k]$ for $k \in \mathcal{E}$ and $Q_j = 8 \max_{j \in \mathcal{S}_k}(Q_k)$ for $j \in \mathcal{M} \backslash \mathcal{E}$. $E[L_k^d]$ is 4 for $k \in \mathcal{I}$ and 6 for $k \in \mathcal{B}$. The rest of the input is the same as in the 2-echelon case.

The values of $\bar{\Delta}^{(\beta_k)}$ and $\bar{\delta}^{(\beta_k)}$ $k \in \mathcal{M}$ are higher than in the 2-echelon case. The analytical approximations perform less good, which seems reasonable due to the accumulations of more approximations. The results are presented in Table 3.14.

Similar to the two echelon distribution system the magnitude $\bar{\Delta}^{(\beta_k)}$ for $k \in \mathcal{E}$ is for most cases inside the indicated range, which is important because this is the service level the customers experience and is not related with the error in $\bar{\Delta}^{(\beta_j)}$ $(j \in \mathcal{M} \backslash \mathcal{E})$.

The errors in the fill rate and the average physical inventory level in the 3-echelon case increased compared to the 2-echelon case, but the results are still of sufficient quality for practical purposes.

Table 3.14: Results 3-echelon distribution system.

Problem data			$k \in \mathcal{E}$	
$\lvert\mathcal{E}\rvert$	$c^2_{D_k}$	$c^2_{A_k}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$
16	0.4	1	0.33 (±0.09)	2.01 (±0.20)
16	0.4	1.8	1.10 (±0.07)	7.00 (±0.26)
16	1	1	0.39 (±0.10)	7.24 (±0.47)
16	1.8	1	0.22 (±0.05)	0.74 (±0.09)
16	1.8	1.8	0.74 (±0.09)	7.21 (±0.97)
32	0.4	1	0.62 (±0.07)	2.80 (±0.11)
32	0.4	1.8	0.60 (±0.20)	3.59 (±0.57)
32	1	1	0.25 (±0.04)	7.91 (±0.08)
32	1.8	1	0.37 (±0.06)	8.14 (±0.17)
32	1.8	1.8	0.76 (±0.14)	8.01 (±0.35)
Problem data			$j \in \mathcal{I}$	
$\lvert\mathcal{E}\rvert$	$c^2_{D_k}$	$c^2_{A_k}$	$\bar{\Delta}^{(\beta_j)}$	$\bar{\delta}^{(E[I_j])}$
16	0.4	1	1.30 (±0.17)	2.82 (±0.45)
16	0.4	1.8	2.93 (±0.25)	3.30 (±0.33)
16	1	1	0.76 (±0.22)	1.70 (±0.57)
16	1.8	1	0.46 (±0.18)	1.58 (±0.81)
16	1.8	1.8	0.43 (±0.20)	0.57 (±0.32)
32	0.4	1	2.34 (±0.55)	8.62 (±0.60)
32	0.4	1.8	0.33 (±0.16)	7.42 (±0.62)
32	1	1	1.18 (±0.48)	8.19 (±0.46)
32	1.8	1	0.66 (±0.41)	7.01 (±0.80)
32	1.8	1.8	2.02 (±0.34)	5.34 (±0.56)
Problem data			$I \in \mathcal{B}$	
$\lvert\mathcal{E}\rvert$	$c^2_{D_k}$	$c^2_{A_k}$	$\bar{\Delta}^{(\beta_i)}$	$\bar{\delta}^{(E[I_i])}$
16	0.4	1	1.87 (±0.37)	7.51 (±0.44)
16	0.4	1.8	2.14 (±0.36)	7.65 (±0.18)
16	1	1	1.54 (±0.32)	6.15 (±1.57)
16	1.8	1	1.44 (±0.29)	7.35 (±0.76)
16	1.8	1.8	1.31 (±0.89)	5.74 (±0.35)
32	0.4	1	1.76 (±0.70)	6.62 (±1.66)
32	0.4	1.8	1.98 (±0.98)	6.24 (±0.61)
32	1	1	2.31 (±0.30)	5.92 (±0.01)
32	1.8	1	1.85 (±0.80)	5.46 (±0.87)
32	1.8	1.8	1.60 (±0.16)	4.32 (±0.13)

3.6 Summary and outlook to Chapter 4

In this chapter we provided analytical approximations to determine the waiting time due to a lack of stock at the preceding stockpoint, the replenishment lead-time and different service measures in a divergent single item, multi-echelon distribution system.

The model under consideration is based on ten assumptions: The first assumption implies that subsequent orders are not allowed to overtake each other. The second one implies that only complete deliveries between stockpoints are allowed. The third one implies that partial deliveries between stockpoints and customers are allowed. The fourth one implies that W_k^s is zero for $k \in \mathcal{B}$. The fifth one implies that all stockpoints are controlled by (s,nQ)-installation stock policies. The sixth one implies a divergent distribution system. The seventh one implies that the batchsizes and the reorder levels or target fill rates are given. The eighth one implies that shortages are backordered. The ninth assumption implies that the exogenous delay is independent of the waiting time due to a lack of stock and last one implies stationary compound renewal customer demand. Assumptions 2 and 3 can easily be relaxed with the formulae given in Section 3.3.5. Assumptions 4 can easily be extended to a given distribution function for W_k^s for $k \in \mathcal{B}$. Further research is necessary to be able to relax the other assumptions.

The analytical approximations derived in this chapter are based on asymptotic results, which may lead to errors when applying them. We refer to Section 3.4 for a recapitulation of all the used approximations. However, the numerical analysis shows that the analytical approximations are accurate in 2-echelon and 3-echelon distribution networks. Moreover, the approximations work better in heterogeneous distribution structures, i.e. non-identical demand streams and batchsizes and when the number of stockpoints per echelon is large. The performance of the approximations decreases when the number of echelons increase, but the results are still satisfactory for 3-echelon networks. An advantage of these analytical approximations is that they can be used for any number of echelons and that they are easy to implement.

In this chapter we assumed that when the replenishment order is available at the preceding stockpoint, it is shipped immediately without any consolidation. However, to increase transport efficiency different items for the same intermediary warehouse could be combined. In the following chapter, we will investigate how different shipment consolidation policies influence the replenishment lead-time.

Appendix 3

Proof of Theorem 3.1

We define U_k as the undershoot and D_k as the demand size at $k \in \mathcal{M}$ which is assumed to have a continuous distribution function. The inventory position of k at an arbitrary demand epoch is uniformly distributed on $(s_k, s_k + Q_k)$ (Hadley and Whitin (1963)). Suppose that z is the difference between the inventory position and s_k just before an order arrives. The probability that an undershoot occurs equals $(1 - F_{D_k}(z))$. The probability that an undershoot exceeding u occurs equals $(1 - F_{D_k}(z+u))$. Hence the probability that an arbitrary demand causes undershoot equals $\frac{1}{Q_k} \int_0^{Q_k} (1 - F_{D_k}(z))dz$. The probability that an arbitrary demand causes an undershoot exceeding u equals $\frac{1}{Q_k} \int_0^{Q_k} (1 - F_{D_k}(z+u))dz$. This lead to

$$P\{U_k \geq u\} = \frac{\int_0^{Q_k} (1 - F_{D_k}(z+u))dz}{\int_0^{Q_k} (1 - F_{D_k}(z))dz}$$

Proof of Proposition 3.4

We assume that $s_j \geq 0$ $(j \in \mathcal{M})$. Let $Y_j(t)$ denote the inventory position of stockpoint j at time t and $(W_k^s)^t$ the waiting time of $k \in \mathcal{S}_j$ when a demand arrives at time epoch t and has size $O_k(t)$. $D_j(t+x-l,t)$ is the aggregate demand at j during the interval $(t+x-l,t]$. If $L_j = l$ with $0 \leq x \leq l$ then $Y_j(t+x-l)$ minus $D_j(t+x-l,t)$ is larger than $O_k(t)$ if and only if k has to wait less than x which implies $Y_j(t) \geq O_k(t)$. Hence,

$$P\{(W_k^s)^t \leq x | L_j = l\} = P\{Y_j(t+x-l) - D_j(t+x-l,t) \geq O_k(t)\}$$

Conditioning on $Y_j(t+x-l)$, and using that the stockpoint is controlled by a (s, nQ) and that for an (s, nQ) policy $Y_j(t+x-l)$ is uniformly distributed on $(s, s+Q)$ therefore we find

$$
\begin{aligned}
P\{W_k^s \le x | L_j = l\} &= \frac{1}{Q_j} \int_{s_j}^{s_j+Q_j} P\{D_j(0, l-x) + O_k \le u\} du \\
&= 1 - \frac{1}{Q_j} \int_{s_j}^{s_j+Q_j} P\{D_j(0, l-x) + O_k \ge u\} du \\
&= 1 - \frac{1}{Q_j} \left(\int_0^\infty P\{D_j(0, l-x) + O_k - s_j \ge t\} dt - \right. \\
&\quad \left. \int_0^\infty P\{D_j(0, l-x) + O_k - s_j - Q_j \ge t\} dt \right) \\
&= 1 - \frac{1}{Q_j} \left(E[(D_j(l-x) + O_k - s_j)^+] \right. \\
&\quad \left. - E[(D_j(l-x) + O_k - (s_j + Q_j))^+] \right) \qquad (3.44)
\end{aligned}
$$

with $D_j(l-x) = D_j(0, l-x)$

Proof of Proposition 3.5

Using formula (3.44) by conditioning on $L_j = l$ we get

$$
\begin{aligned}
E[W_k^s] &= \int_0^\infty P\{W_k^s > x\} dx \\
&= \int_0^\infty \int_0^\infty P\{W_k^s > x | L_j = l\} dF_{L_j}(l) dx
\end{aligned}
$$

Changing the order of the integration we get

$$
\begin{aligned}
E[W_k^s] &= \int_0^\infty \int_0^l P\{W_k^s > x | L_j = l\} dx dF_{L_j}(l) \\
&= \int_0^\infty \int_0^l \tfrac{1}{Q_j} (E[(D_j(l-x) + O_k - s_j)^+] \\
&\quad - E[(D_j(l-x) + O_k - (s_j + Q_j))^+]) dx dF_{L_j}(l)
\end{aligned}
$$

We introduce the variable $z = l - x$

$$\begin{aligned}
E[W_k^s] &= \int_0^\infty \int_0^l \tfrac{E[(D_j(z)+O_k-s_j)^+]-E[(D_j(z)+O_k-(s_j+Q_j))^+]}{Q_j} dz dF_{L_j}(l) \\
&= \int_0^\infty \int_z^\infty \tfrac{E[(D_j(z)+O_k-s_j)^+]-E[(D_j(z)+O_k-(s_j+Q_j))^+]}{Q_j} dF_{L_j}(l) dz \\
&= \int_0^\infty \tfrac{E[(D_j(z)+O_k-s_j)^+]-E[(D_j(z)+O_k-(s_j+Q_j))^+]}{Q_j} (1-F_{L_j}(z)) dz
\end{aligned}$$

In the next step we substitute $F_{\hat{L}_j}(z) = \frac{1}{E[L_j]} \int_0^z (1-F_{L_j}(l))dl$ in the equation above and we get

$$\begin{aligned}
E[W_k^s] &= \tfrac{E[L_j]}{Q_j} \int_0^\infty \big(E[(D_j(z)+O_k-s_j)^+] \\
&\quad -E[(D_j(z)+O_k-(s_j+Q_j))^+]\big) dF_{\hat{L}_j}(z) \\
&= \tfrac{E[L_j]}{Q_j} \big(E[(D_j(\hat{L}_j)+O_k-s_j)^+] \\
&\quad -E[(D_j(\hat{L}_j)+O_k-(s_j+Q_j))^+]\big)
\end{aligned}$$

Proof of Proposition 3.6

In Proposition 3.6 we gave an expression for $E[(W_k^s)^2]$, $(k \in \mathcal{M} \backslash \mathcal{B})$. Using formula (3.20) by conditioning on $L_j = l$ we get

$$\begin{aligned}
E[(W_k^s)^2] &= \int_0^\infty 2xP\{W_k^s > x\} dx \\
&= \int_0^\infty \int_0^\infty 2xP\{W_k^s > x | L_j = l\} dF_{L_j}(l) dx
\end{aligned}$$

Changing the order of the integration we get

$$\begin{aligned}
E[(W_k^s)^2] &= \int_0^\infty \int_0^l 2xP\{W_k^s > x | L_j = l\} dx dF_{L_j}(l) \\
&= \int_0^\infty \int_0^l 2x \tfrac{1}{Q_j} (E[(D_j(l-x)+O_k-s_j)^+] \\
&\quad -E[(D_j(l-x)+O_k-(s_j+Q_j))^+]) dx dF_{L_j}(l)
\end{aligned}$$

We introduce the variable $y = l - x$

$$\begin{aligned}
E[(W_k^s)^2] &= \int_0^\infty \int_0^l 2(l-y) \tfrac{1}{Q_j} (E[(D_j(y)+O_k-s_j)^+] \\
&\quad -E[(D_j(y)+O_k-(s_j+Q_j))^+]) dy dF_{L_j}(l) \\
&= \int_0^\infty \int_y^\infty 2(l-y) \tfrac{1}{Q_j} (E[(D_j(y)+O_k-s_j)^+] \\
&\quad -E[(D_j(y)+O_k-(s_j+Q_j))^+]) dF_{L_j}(l) dy
\end{aligned}$$

After that we substitute $F_{\tilde{L}_j}(y) = \frac{2}{E[L_j^2]} \int_0^y \int_x^\infty (l-x) dF_{L_j}(l) dx$ in the equation above and get

$$\begin{aligned} E[(W_k^s)^2] &= \tfrac{E[L_j^2]}{Q_j}\int_0^\infty (E[(D_j(y)+O_k-s_j)^+] \\ &\quad -E[(D_j(y)+O_k-(s_j+Q_j))^+])dF_{\tilde{L}_j}(y) \\ &= \tfrac{E[L_j^2]}{Q_j}\big(E[(D_j(\tilde{L}_j)+O_k-s_j)^+] \\ &\quad -E[(D_j(\tilde{L}_j)+O_k-(s_j+Q_j))^+]\big) \end{aligned}$$

Explanation of Equation (3.39)

In Equation (3.39) we derived an expression for the expected long run inventory level. We will first define the notation used. D_k is the random demand process ($k \in \mathcal{M}$), we denote M as the renewal function of D_k and U as its stationary residual lifetime. Janssen (1998) gives the following prove for the long run average inventory level.

Define $H(x)$ as the expected area between the physical inventory level and the zero level, given that the physical inventory level on epoch zero equals x, and there are no replenishments. Then conditioning on the demand in the next period results in

$$H(x) = x + \int_0^x H(x-y)dF_{D_k}(y) \tag{3.45}$$

Repeated substitution yields

$$H(x) = \int_0^x (x-y)dM(y) \tag{3.46}$$

The expected physical inventory level at the beginning of the replenishment cycle (just after the replenishment arrived) is equal to $E[(s_k+Q_k-U_k-D_k(L_k))^+]$. The expected physical inventory level at the end of the replenishment cycle (just before the replenishment arrives) is equal to $E[(s_k-U_k-D_k(L_k))^+]$. Then the total expected area between the physical inventory level is equal to $E[H(s_k+Q_k-U_k-D_k(L_k))]-E[H(s_k-U_k-D_k(L_k))]$. The expected duration of a replenishment cycle is given by $Q_k/E[D_k]$. Conditioning on $s_k+Q_k-U_k-D_k(L_k)$ and $s_k-U_k-D_k(L_k)$ using (3.46) and Lemma 2.2 we find

$$
\begin{aligned}
E[I_k] &= \tfrac{E[H(s_k+Q_k-U_k-D_k(L_k))]-E[H(s_k-U_k-D_k(L_k))]}{Q_k/E[D_k]} \\
&= \tfrac{E[D_k]}{Q_k}\Bigg(\int_0^{s_k+Q_k} H(s_k+Q_k-x)d(F_{D_k(L_k)}*U)(x) \\
&\quad -\int_0^{s_k} H(s_k-x)d(F_{D_k(L_k)}*U)(x)\Bigg) \\
&= \tfrac{E[D_k]}{Q_k}\Bigg(\int_0^{s_k+Q_k}\int_0^{s_k+Q_k-x}(s_k+Q_k-x-y)dM(y)d(F_{D_k(L_k)}*U)(x) \\
&\quad -\int_0^{s_k}\int_0^{s_k-x}(s_k-x-y)dM(y)d(F_{D_k(L_k)}*U)(x)\Bigg) \\
&= \tfrac{E[D_k]}{Q_k}\Bigg(\int_0^{s_k+Q_k}\int_0^{s_k+Q_k-x}(s_k+Q_k-x-y)d(M*U)(y)dF_{D_k(L_k)}(x) \\
&\quad -\int_0^{s_k}\int_0^{s_k-x}(s_k-x-y)d(M*U)(y)dF_{D_k(L_k)}(x)\Bigg) \\
&\simeq \tfrac{E[D_k]}{Q_k}\Bigg(\int_0^{s_k+Q_k}\int_0^{s_k+Q_k-x}\tfrac{(s_k+Q_k-x-y)}{E[D_k]}dydF_{D_k(L_k)}(x) \\
&\quad -\int_0^{s_k}\int_0^{s_k-x}\tfrac{(s_k-x-y)}{E[D_k]}dydF_{D_k(L_k)}(x)\Bigg) \\
&= \int_0^{s_k+Q_k}\tfrac{(s_k+Q_k-x)^2}{2Q_k}dF_{D_k(L_k)}(x)-\int_0^{s_k}\tfrac{(s_k-x)^2}{2Q_k}dF_{D_k(L_k)}(x) \\
&= \tfrac{E[(s_k+Q_k-D_k(L_k))^{+2}]-E[(s_k-D_k(L_k))^{+2}]}{2Q_k}
\end{aligned}
$$

Equations (3.36) and (3.38) can be derived in a similar manner. For more detail we refer to Janssen (1998)

Explanation of Equation (3.40)

In case of complete delivery, when an order arrives and there is not enough stock the entire order waits until the stock is replenished. In the figures 3.6 and 3.7 a schematic representation of the inventory position is given for the case $I_b > 0$ and the case $I_b < 0$.

The fill rate is the fraction of demand that is not directly delivered from stock. It can be written as

$$\beta = 1 - \frac{E[X]}{Q} \tag{3.47}$$

where X is the demand, which is not delivered directly from stock and Q is the average batchsize. Further, we define:

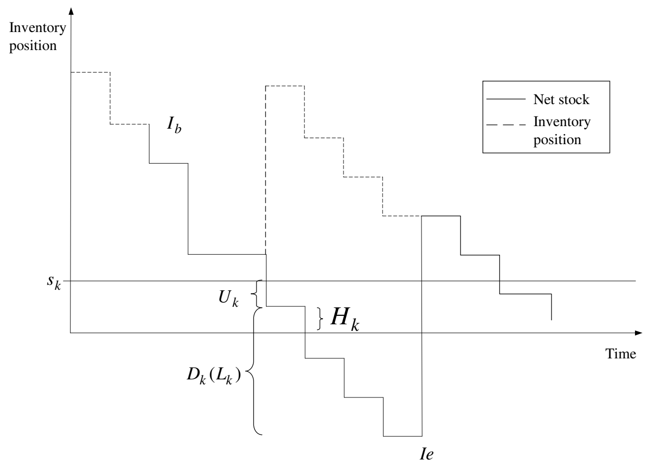


Figure 3.6: Schematic representation of the inventory position for the case $I_b > 0$.

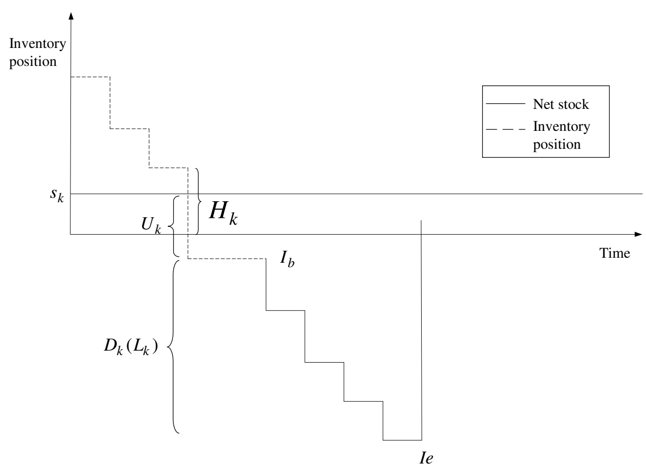


Figure 3.7: Schematic representation of the inventory position for the case $I_b < 0$.

- X_p as the demand, which is not delivered directly from stock in the partial delivery case

- X_c as the demand, which is not delivered directly from stock in the complete delivery case

- I_e as the net stock at the end of the replenishment cycle, just before the replenishment order arrives.

- I_b as the net stock at the beginning of the replenishment cycle, just after the replenishment order arrived.

In the partial delivery case, the first moment of X_p is
$E[X_p] = P\{I_e < 0, I_b > 0\}E[-I_e|I_e < 0, I_b > 0] + P\{I_e < 0, I_b < 0\}$
$E[-I_e + I_b|I_e < 0, I_b < 0]$
In the complete delivery case the first moment of X_c is
$E[X_c] = P\{I_e < 0, I_b > 0\}E[-I_e + H_k|I_e < 0, I_b > 0] + P\{I_e < 0, I_b < 0\}$
$E[-I_e + H_k + I_b - H_k|I_e < 0, I_b > 0]$
$E[X_c] = E[X_p] + P\{I_e < 0, I_b > 0\}E[H_k]$
$E[X_c] = E[X_p] + (P\{I_e < 0\} - P\{I_e < 0, I_b < 0\})E[H_k]$
$E[X_c] = E[X_p] + (P\{I_e < 0\} - P\{I_b < 0\})E[H_k]$

After that we can substitute I_e by $s_k - D_k(L_k) - U_k$ and I_b by $Q_k + s_k - D_k(L_k) - U_k$, resulting in

$$E[X_c] = E[X_p] + (P\{D_k(L_k) + U_k > s_k\} - P\{D_k(L_k) + U_k > s_k + Q_k\})E[H_k] \quad (3.48)$$

Substituting equation (3.48) in equation (3.47) results in
$\beta_k^c = 1 - \frac{X_p + (P\{D_k(L_k)+U_k>s_k\} - P\{D_k(L_k)+U_k>s_k+Q_k\})E[H_k]}{Q_k}$
$\beta_k^c = 1 - \frac{X_p}{Q_k} - \frac{(P\{D_k(L_k)+U_k>s_k\} - P\{D_k(L_k)+U_k>s_k+Q_k\})E[H_k]}{Q_k}$
$\beta_k^c = \beta_k - \frac{(P\{D_k(L_k)+U_k>s_k\} - P\{D_k(L_k)+U_k>s_k+Q_k\})E[H_k]}{Q_k}$

Explanation of Equation (3.41)

In figures 3.8 and 3.9, a schematic representation of the physical stock in the partial and the complete delivery case is given.

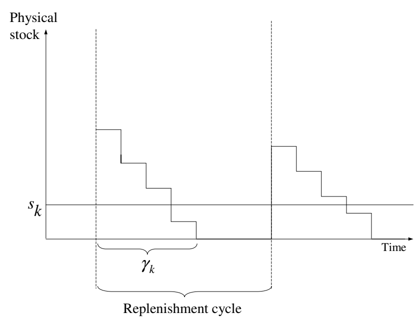


Figure 3.8: Physical stock in the partial delivery case.

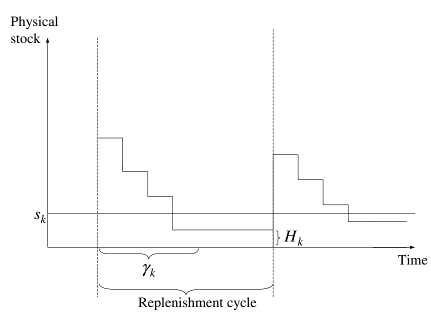


Figure 3.9: Physical stock in the complete delivery case.

In the partial delivery case, we distinguish between two cases: when the net stock is positive and when the net stock is zero. Given γ_k, which is the fraction of time that the net physical stock is positive, the long-run expected inventory level can be written as

$$\begin{aligned} E[I_k] &= \gamma_k E[I_k|I_k > 0] + (1-\gamma_k)E[I_k|I_k \leq 0] \\ E[I_k] &= \gamma_k E[I_k|I_k > 0] + (1-\gamma_k)0 \end{aligned} \tag{3.49}$$

In the complete case if a lack of stock occurs the entire order waits until the stock is replenished. Therefore during fraction of $1-\gamma_k$ a part of an order stays in inventory (H_k). From renewal theory we can derive that H_k is equal to the residual lifetime of D_k. Thus, for the complete delivery case, the expected long-run average inventory level is equal to

$$E[I_k^c] = \gamma_k E[I_k^c|I_k > 0] + (1-\gamma_k)E[H_k] \tag{3.50}$$

To be able to evaluate (3.50), an expression for $E[I_k^c|I_k > 0]$ is needed. Note that the net stock in the partial and complete case is the same when $I_k > 0$, therefore

$$E[I_k|I_k > 0] = E[I_k^c|I_k > 0] \tag{3.51}$$

Substituting equation (3.51) into (3.49) and (3.49) into (3.50) gives

$$E[I_k^c] = E[I_k] + (1-\gamma_k)E[H_k] \tag{3.52}$$

Chapter 4

A multi-item, multi-echelon inventory model with shipment consolidation

The content of this chapter is joint work with A.G. de Kok and G.P. Kiesmüller, and has appeared in Smits and de Kok (2002) and Smits et al. (2002). Further, I would like to thanks A. van Harten, who proposed an improvement to the approximation used to evaluate $E[C_{k,m}^2]$

4.1 Introduction

In Chapter 3, we assumed that replenishment orders satisfied at the preceding stockpoint are shipped immediately, without any shipment consolidation. Considering a multi-item inventory model with shipment consolidation can reduce the total logistics costs, due to an increase in transport efficiency. Therefore we extend our model and consider multiple items for which the replenishment orders towards the same intermediary warehouse are consolidated. As mentioned in Chapter 2, the literature considering multi-item logistic networks with shipment consolidation distinguishes between two classes of models: joint replenishment models and models that explicitly consider the shipment consolidation process. In joint replenishment models, the replenishment moments of the different items at a warehouse coincide and the replenishment lead-time is exogenous. In the models, that explicitly consider the shipment consolidation process, the replenishment moments do not coincide but the consolidation is realized by letting the replenishment orders wait for a certain time or until a certain quantity is consolidated at the preceding warehouse. In these models the replenishment lead-time is an endogenous variable and the endogenous component of the replenishment lead-time is the waiting time due to shipment consolidation. In this thesis we consider the case that the shipment consolidation process is explicitly modeled. Thus, in addition to the exogenous delay and the waiting time due to a lack of stock at the preceding stockpoint, the replenishment lead-time includes the waiting time due to shipment consolidation. The objective of this chapter is to derive expressions for the replenishment lead-time in this model.

When there is a lack of stock at a stockpoint the replenishment order first waits until the stock is replenished and then the order goes to the consolidation dock and leaves with the next planned truck. A more detailed description of the model and additional notation is given in Section 4.2. Similar to Higginson and Bookbinder (1994) we consider two types of order consolidation policies: the time-based policy and the quantity-based policy. In Section 4.3, we present, for the time-based policy, exact results for the distribution of the waiting time due to shipment consolidation and we show that the waiting time due to shipment consolidation is independent of the waiting time due to a lack of stock at the preceding stockpoint. In Section 4.4, we derive approximations for the first two moments of the waiting time due to shipment consolidation for the quantity policy. In Section 4.5, we present a numerical example. The performance of the approximations for the quantity policy is tested through extensive computer simulations in Section 4.6 and the chapter concludes with a summary and an outlook to the next chapter.

4.2 Model description

In this section, we describe the model in detail and introduce additional notation. We extend the model in Chapter 3 by considering multiple-items for which replenishment orders for the same warehouse are consolidated and shipped together.

The warehouses are uniquely numbered with roman numbers and the stockpoints with arabic numbers. The stockpoint numbering indicates a combination of an item and a location. In the example we can derive that stockpoint 1 is the stockpoint of item A at location I, stockpoint 2 is the stockpoint of item B at location I and so on, see Figure 4.1 and 4.2. The set of warehouses is denoted by $\mathcal{W}$ and the set of stockpoints located at warehouse m is denoted by $\mathcal{L}_m$. In the example in Figure 4.2 we get $\mathcal{M} = \{1, \ldots, 22\}$, $\mathcal{W} = \{I, II, III, IV, V, VI, VII\}$, $\mathcal{L}_I = \{1, 2, 3, 4\}$, $\mathcal{L}_{II} = \{5, 6\}$, $\mathcal{L}_{III} = \{7, 8, 9, 10\}$, $\mathcal{L}_{IV} = \{11, 12, 13, 14\}$, $\mathcal{L}_V = \{15, 16\}$, $\mathcal{L}_{VI} = \{17, 18, 19, 20\}$ and $\mathcal{L}_{VII} = \{21, 22\}$.

As we have demonstrated in Chapter 3, given batchsizes Q_k the aggregated demand processes (A_k, D_k) can be determined independently of the reorder levels using a compound renewal approximation of the replenishment process (R_k, O_k) of the successors (see Section 3.3). In the following we distinguish between two types of arrival processes (see Figure 4.3): the arrival process of replenishment orders of stockpoint k at stockpoint $j \in \mathcal{P}_k$ and the arrival process of replenishment orders at the consolidation dock. We assume that stockpoint k is located at warehouse m. $C_{k,m}$ is defined as the stationary inter-arrival time of the latter process and $B_{k,m}$ as the corresponding demand size, which is expressed, for example, in number of pallets, weight or volume. In this chapter, we assume that $B_{k,m}$ is expressed in volume. If there are enough items on stock then both replenishment processes are equivalent. Otherwise the replenishment orders at the consolidation dock are delayed and also clustering of the orders is possible, which means that more than one replenishment order for the same stockpoint can arrive simultaneously at the consolidation dock.

The arrival process of replenishment orders of an arbitrary order from warehouse m at the consolidation dock is denoted with (C_m^*, B_m^*). Without loss of generality we assume that all stockpoints at warehouse $m \in \mathcal{W}$ are replenished by the same

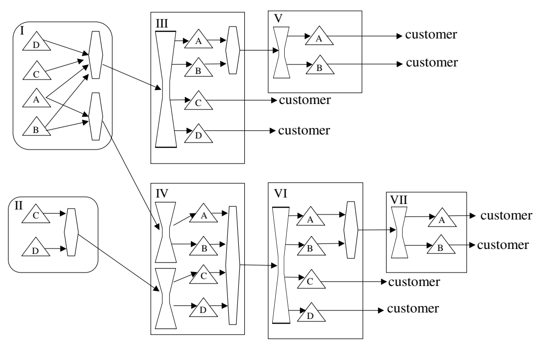


Figure 4.1: Multi-echelon, multi-item network with shipment consolidation.

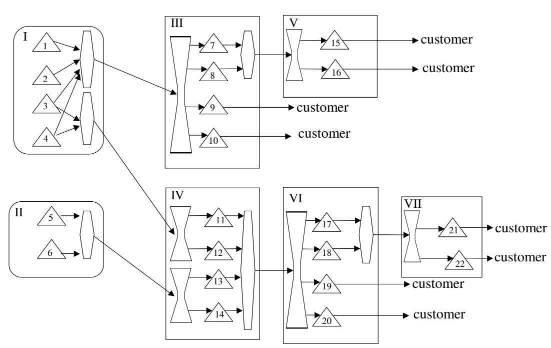


Figure 4.2: Multi-echelon, multi-item network with shipment consolidation.

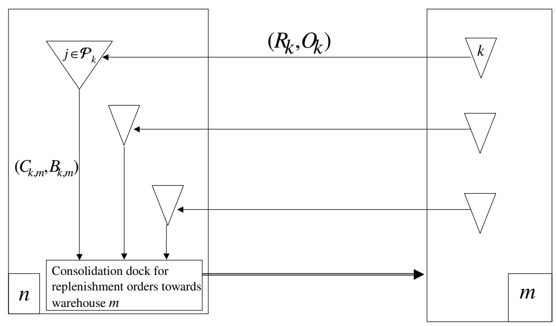


Figure 4.3: The replenishment process

warehouse $n \in \mathcal{W}$ to facilitate the notation. In case not all stockpoints at warehouse $m \in \mathcal{W}$ are replenished by the same warehouse $n \in \mathcal{W}$, the warehouse $m \in \mathcal{W}$ can be split in a number of pseudo-warehouses such that the stockpoints of each pseudo-warehouse are replenished by a single warehouse. In the example in Figure 4.2, warehouse IV can be split in two pseudo-warehouses: warehouse IV' with stockpoints 11 and 12 replenished by warehouse I and warehouse IV" with stockpoints 13 and 14 replenished by warehouse II.

In this chapter two different shipment consolidation policies are investigated: a time-based policy and a quantity-based policy. In the time-based policy the trucks depart from n to m ($n, m \in \mathcal{W}$) at fixed time intervals T_m (for example, every week). We assume in this case that the truck capacity is unlimited. In the quantity-based policy the orders are dispatched when a target quantity Q_m^c is consolidated. This target quantity is expressed in the same unit as the replenishment orders at the consolidation dock B_m^*. We assume that $B_m^* < Q_m^c$, which means that all the replenishment orders are smaller than the target quantity. It is possible that the consolidated quantity is larger than the target quantity. In this situation, we distinguish between three different cases:

1. The last order is split such that the consolidated quantity is equal to the target quantity. In this situation a part of the order leaves directly and has no waiting time due to shipment consolidation. The remaining part V_m has to wait until the next truck departs. Suppose O_m^l is the last order than $O_m^l - V_m$ has waiting time 0 and V_m has to wait until the next truck departure, which we define as W^0. The replenishment lead-time of order $O_m^l = \frac{O_m^l - V_m}{O_m^l} 0 + \frac{V_m}{O_m^l} W^0$

2. The last order leaves with the next truck. Thus, the shipped quantity is smaller than the target quantity and the last order has to wait until the next truck departs.

3. The last order leaves with the current truck. Thus, the shipped quantity is larger than the target quantity and the last order has no waiting time due to shipment consolidation.

We present the analysis for the case that the last order just before the truck leaves is split (i.e. case 1). For the two other cases the waiting time due to shipment consolidation can be derived in a similar manner, consequently we only present the final results for the first two moments of the waiting time due to shipment consolidation in the appendix. Note that the waiting time due to shipment consolidation in the three cases converge to each other when the average replenishment order size decreases in comparison to the target consolidated quantity.

Obviously, the shipment consolidation policy influences the replenishment lead-time L_k ($k \in \mathcal{M}$). Moreover, when stockpoint k ($k \in \mathcal{M}$) places a replenishment order two different situations can be observed at $j \in \mathcal{P}_k$:

1. There is enough stock at $j \in \mathcal{P}_k$ to fulfill the entire order.
   Then the order is directly available for consolidation.

2. There is not enough stock at $j \in \mathcal{P}_k$ to fulfill the entire order.
   Then the order has to wait until $j \in \mathcal{P}_k$ is replenished, which is W_k^s, and afterwards the order becomes available for consolidation. If more than one

order is waiting due to a lack of stock, a FIFO (first in, first out) rule is applied.

We assume that the probability distribution function of the exogenous delay L_k^d is given and that subsequent orders cannot overtake, which implies that subsequent replenishment lead-times are independent. From the above definitions it follows that

$$L_k = L_k^d + W_k^s + W_{k,m}^c. \tag{4.1}$$

Additionally, we assume that the waiting time due to shipment consolidation $W_{k,m}^c$ and the waiting time due to a lack of stock W_k^s at $k \in \mathcal{B}$ are zero. Finally, we assume that L_k^d is independent of W_k^s and $W_{k,m}^c$.
Below we summarize the additionally notation used throughout this chapter. For the remaining notation we refer to Chapter 3.

We assume that the following variable is zero.

$W_{k,m}^c$	Waiting time due to consolidation for $k \in \mathcal{B}$, $m \in \mathcal{W}$

We assume that the following variables are deterministic and given.

Q_k	Batchsize at $k \in \mathcal{M}$
T_m	Time between two truck departures towards $m \in \mathcal{W}$ (time-based policy)
Q_m^c	Predetermined consolidation quantity towards $m \in \mathcal{W}$ (quantity-based policy)

The distribution functions of the following random variables are unknown.

$W_{k,m}^c$	Waiting time due to shipment consolidation towards $k \in \mathcal{M}$, $m \in \mathcal{W}$
V_m	Remaining part of the split order for consolidation towards $m \in \mathcal{W}$
$X_{k,m}$	Time between the last truck departure towards $m \in \mathcal{W}$ and the arrival of an arbitrary order from $k \in \mathcal{L}_m$
$Y_{k,m}$	Amount consolidated between the last truck departure towards $m \in \mathcal{W}$ and the arrival of an arbitrary order from $k \in \mathcal{L}_m$
$N(Q_m^c - Y_m)$	Number of arrivals of replenishment orders from m between the placement of replenishment order at the consolidation dock from $k \in \mathcal{L}_m$ and the departure of the truck towards $m \in \mathcal{W}$.

$C_{k,m}$	Inter-arrival time of replenishment orders from stockpoint k at the consolidation dock for $k \in \mathcal{M}$, $m \in \mathcal{W}$
C_m^*	Inter-arrival time of an arbitrary replenishment order from warehouse m at the consolidation dock for $m \in \mathcal{W}$
$B_{k,m}$	Order size of replenishment orders in volume at the consolidation dock for $k \in \mathcal{M}$, $m \in \mathcal{W}$
B_m^*	Order size of replenishment orders in volume at the consolidation dock for $m \in \mathcal{W}$

Similar to Chapter 3, when the distribution function of a random variable is intractable, we evaluate the first two moments of the random variable and we approximate the distribution of it by a mixed-Erlang distribution based on (approximate) expressions for the first two moments, as indicated in Section 2.2.2.

4.3 The time-based consolidation policy

In this section we determine the first two moments for the waiting time due to shipment consolidation $W_{k,m}^c$ for all $k \in \mathcal{M}\backslash\mathcal{B}$ and $m \in \mathcal{W}$ in case of a time-based policy. If a time-based policy is used then the trucks at warehouse n depart to warehouse m ($n, m \in \mathcal{W}$) at fixed time intervals T_m (for example, every week). We assume that a truck travels from n to m at time 0 and that the next one leaves from n to m at T_m. Further, we assume that this truck has an unlimited capacity in order to reduce the complexity of the problem. During the interval $(0, T_m]$ all replenishment orders at the consolidation dock from the stockpoints k ($k \in \mathcal{L}_m$) are collected and then shipped together at T_m. Assuming that all processes are stationary the truck departure process can be modeled as a renewal process with deterministic inter-renewal times.

To determine the first two moments of $W_{k,m}^c$, we introduce the random variable $X_{k,m}$, which is the time between the last truck departure at n towards m and the arrival of an order k from m at n with $k \in \mathcal{L}_m$ and $m \in \mathcal{W}$. Then we can write: (see Figure 4.4)

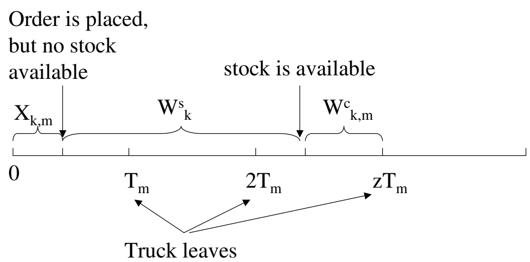


Figure 4.4: Schematic representation of the time-based consolidation policy

$$X_{k,m} + W_k^s + W_{k,m}^c = zT_m \qquad z \in \mathbb{N}, \forall k \in \mathcal{L}_m, m \in \mathcal{W} \tag{4.2}$$

where z is stochastic integer. Using equation (4.2) the following theorem can be proven (see Appendix 4)

Theorem 4.1 *Given that $X_{k,m}$, the time between the last truck departure from n to m and the arrival of a replenishment order from warehouse m, is uniformly distributed on (0,T_m] and W_k^s is independent of $X_{k,m}$ ($k \in \mathcal{L}_m$ and $m \in \mathcal{W}$), $W_{k,m}^c$ is uniformly distributed on the interval (0,T_m] and the following formulae hold:*

$$E[W_{k,m}^c] = \frac{T_m}{2} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W} \tag{4.3}$$

and

$$E[(W_{k,m}^c)^2] = \frac{T_m^2}{3} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W} \tag{4.4}$$

Further, we have to prove that $X_{k,m}$ ($k \in \mathcal{L}_m$ and $m \in \mathcal{W}$) is uniformly distributed on the interval $(0,T_m]$.

Theorem 4.2 *$X_{k,m}$, which is the time between the last truck departure from n to m and the arrival of an order from stockpoint k, is uniformly distributed on the interval $(0,T_m)$, where $k \in \mathcal{L}_m$ and $m \in \mathcal{W}$.*

The proof is given in Appendix 4.

Since W_k^s is independent of the truck departure process, the W_k^s is independent of $X_{k,m}$. Further, we have to prove that $W_{k,m}^c$ and W_k^s are independent of each other.The proof of the following theorem is also given in Appendix 4.

Theorem 4.3 *$W_{k,m}^c$ and W_k^s are independent of each other, $k \in \mathcal{M}$ and $m \in \mathcal{W}$.*

By substituting the results from theorems 4.1, 4.2 and 4.3 into Formula (4.1) we find expressions for the first two moments of L_k for all $k \in \mathcal{M}$. The analysis of Chapter 3, yielded good approximations for the replenishment lead-time and the different performance measures, like for example the fill rate. In this chapter we derived exact results for $W_{k,m}^c$, therefore without loss of generality we can consider $W_{k,m}^c$ as part of the exogenous delay in Chapter 3, which will also yield good results for the replenishment lead-time and the different performance measures.

4.4 The quantity-based consolidation policy

In contrast to the time-based policy the time between truck departures in the quantity-based policy is a random variable, because it depends on the arrival process of replenishment orders at the consolidation dock. Replenishment orders at the consolidation dock are collected until the amount is equal or larger than a predetermined quantity Q_m^c and the last order O_m^l is split such that the shipped quantity is equal to Q_m^c ($m \in \mathcal{W}$). Then the consolidation process starts all over again with the remaining part V_m. So one part of the last replenishment order leaves directly and the other part V_m with the next truck which is illustrated in Figure 4.5. V_m has to wait has to wait until the next truck departure which is W^0. The replenishment lead-time of order O_m^l is equal to $\frac{O_m^l - V_m}{O_m^l}0 + \frac{V_m}{O_m^l}W^0$.

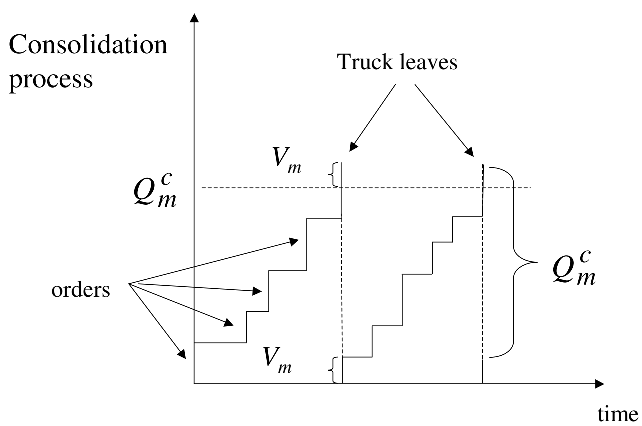


Figure 4.5: Evolution of the consolidation policy

4.4.1 The arrival process of replenishment orders at the consolidation dock

Since the arrival process of replenishment orders at the consolidation dock has a big impact on the truck departure process and therefore also on the waiting time due to shipment consolidation we investigate the arrival process at the consolidation dock in more detail and we derive approximate formulae for the first two moments of the interarrival time C_k, m of an individual replenishment process at the consolidation dock.

As already mentioned in Section 4.2 the interarrival times $C_{k,m}$ of the replenishment orders from stockpoint k warehouse m at the consolidation dock are different from the interarrival times R_k of the replenishment processes at the stockpoint $j \in \mathcal{P}_k$ due to the delay and clustering of arrivals at the consolidation dock. By clustering we mean that more than one replenishment order can arrive simultaneously at the consolidation dock in out-of-stock situations. See for an illustration Figure 4.6.

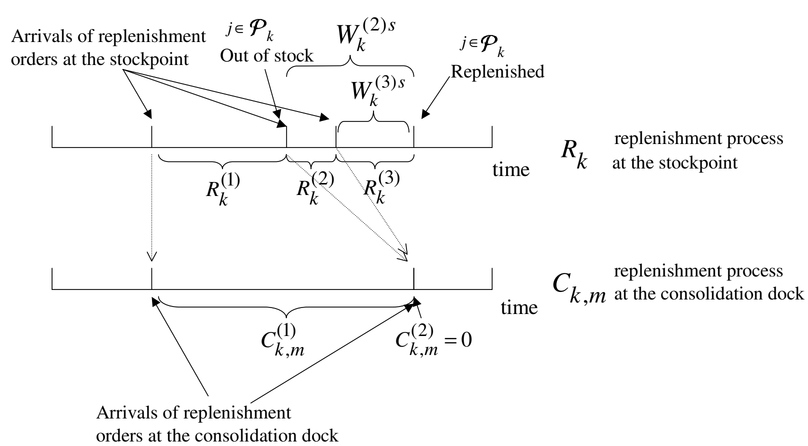


Figure 4.6: Demonstration of the clustering of arrivals at the consolidation dock

Denoting the interarrival time between the n-th and $(n+1)$-th order of stockpoint k at the consolidation dock by $C_{k,m}^{(n)}$, the following equation holds,

$$C_{k,m}^{(n)} = R_k^{(n)} - W_{k,m}^{(n)s} + W_{k,m}^{(n+1)s} \qquad \forall k \in \mathcal{M} \tag{4.5}$$

Letting $n \to \infty$, i.e. considering the stationary situation, results in

$$E[C_{k,m}] = E[R_k] \qquad \forall k \in \mathcal{M} \qquad m \in \mathcal{W}. \tag{4.6}$$

The second moment of C_k is

$$E[(C_{k,m}^{(n)})^2] = E[(R_k^{(n)} - W_{k,m}^{(n)s} + W_k^{(n+1)s})^2] \qquad \forall k \in \mathcal{M} \qquad m \in \mathcal{W} \tag{4.7}$$

Equation (4.7) is difficult to compute because $W_{k,m}^{(n)s}$ and $W_{k,m}^{(n+1)s}$ are dependent as illustrated in Figure 4.6. If the time between replenishments of $j \in \mathcal{S}_k$ is large in comparison to the replenishment lead-time L_k and β_k is large then the probability that the arrival of replenishment order at the consolidation dock is delayed because of a stockout occasion is small and thus the probability that replenishment orders at the consolidation dock for the same stockpoint are delayed and clustered is small. Therefore we assume that the delay and clustering effect of the replenishment orders for stockpoint k at the consolidation dock can be neglected. Based on this assumption the following formula can be obtained:

$$\sigma^2(C_{k,m}) \simeq \sigma^2(R_k) \qquad \forall k \in \mathcal{M} \tag{4.8}$$

Notice that we are not only interested in the arrival process of replenishment orders of one stockpoint k at the consolidation dock but even more in the superposed arrival process of arbitrary orders from warehouse m at the consolidation dock. To evaluate the superposed process, we used the superposition method presented in 2.2.3 In the superposition method, the superposed variable converges to an exponential distribution when the number of superposed processes goes to infinity. So, both C_m^* and R_m^* will converge to an exponential distribution.

Finally, it is straightforward to see that the replenishment order size process, O_k, at the stockpoint $j \in \mathcal{P}_k$ is equivalent to the replenishment order size process, $B_{k,m}$, at the consolidation dock. Only the dimension can differ, since O_k is generally given in units whereas $B_{k,m}$ is expressed in volumes.

4.4.2 The waiting time due to shipment consolidation

To calculate the waiting time $W^c_{k,m}$ due to shipment consolidation for stockpoint $k \in \mathcal{L}_m$ and $m \in \mathcal{W}$, we distinguish between two situations, depending on $Q^c_m - Y_{k,m}$ where $Y_{k,m}$ is defined as the amount collected immediately after a replenishment order of stockpoint k has arrived at the consolidation dock:

1. $\mathbf{Q^c_m - Y_{k,m} \geq 0}$

   In this case either the replenishment order from stockpoint k is not the last order, O^l_m, or it is the last order and $V_m = 0$. In this situation, the replenishment order is not split and therefore we can write the waiting time as follows:

$$W^c_{k,m} = \sum_{i=1}^{N(Q^c_m - Y_{k,m})} (C^*_m)^{(i)} \tag{4.9}$$

   where $(C^*_m)^{(i)}$ is the interarrival time between the i-th and $i+1$-th order at the consolidation dock and $N(Q^c_m - Y_{k,m})$ is the number of arrivals between a replenishment order of stockpoint k at the consolidation dock and the first departure of a truck.

   Assuming that, $N(Q^c_m - Y_{k,m})$ and $(C^*_m)^{(i)}$ are independent random variables, and $(C^*_m)^{(i)}$ is identically distributed for all i, it follows from (4.9) that

$$E[W^c_{k,m}] \simeq E[N(Q^c_m - Y_{k,m})]E[C^*_m] \tag{4.10}$$

$$E[(W^c_{k,m})^2] \simeq E[N(Q^c_m - Y_{k,m})]\sigma^2(C^*_m) + E[N(Q^c_m - Y_{k,m})^2]E^2[C^*_m] \tag{4.11}$$

   If the order size of the replenishment orders at the consolidation dock are identical and deterministic $B_{k,m} = \tilde{B}$ for all $k \in \mathcal{L}_m$ and the volume of the truck is a multiple of $\tilde{B}$ then $Q^c_m - Y_{k,m}$ is always larger or equal to 0. For this situation we can derive exact expressions for $N(Q^c_m - Y_{k,m})$. We observe that in practice this occurs for example when the sizes of the replenishment orders are tuned to a pallet size and the truck size is also tuned to a pallet size.

2. $\mathbf{Q_m^c - Y_{k,m} < 0}$
   If $Q_m^c - Y_{k,m} < 0$ then the order of stockpoint k is the last order and $V_m > 0$. In this case $Y_{k,m} - V_m$ has waiting time 0 and V_m has to wait until the next truck departure. An explicit expression for the waiting time is difficult to obtain. Therefore we will provide an approximation for the first two moments of the waiting time.

Identical and deterministic replenishment order sizes

In this subsection we derive exact expressions for the first two moments of $N(Q_m^c - Y_{k,m})$ for the case that $B_{k,m} = \tilde{B}$ for all $k \in \mathcal{M}$, $\tilde{B}$ is deterministic and additionally Q_m^c is a multiple of $\tilde{B}$. Since Q_m^c is a multiple of $\tilde{B}$, the remaining part V_m of the last order, O_m^l, is equal to zero. Moreover, in the steady state the truck consolidation process is uniformly distributed over $\{0, \tilde{B}, 2\tilde{B}, ..., (\frac{Q_m^c}{\tilde{B}} - 1)\tilde{B}\}$ with the different possibilities having the same probability namely $\frac{\tilde{B}}{Q_m^c}$ and it easily follows that

$$E[N(Q_m^c - Y_{k,m})] = \frac{\tilde{B}}{Q_m^c} \sum_{s=0}^{(\frac{Q_m^c}{\tilde{B}}-1)} s = \frac{1}{2}(\frac{Q_m^c}{\tilde{B}} - 1) \tag{4.12}$$

$$E[N(Q_m^c - Y_{k,m})^2] = \frac{\tilde{B}}{Q_m^c} \sum_{s=0}^{(\frac{Q_m^c}{\tilde{B}}-1)} s^2 = \frac{1}{3}(\frac{Q_m^c}{\tilde{B}} - 1)(\frac{Q_m^c}{\tilde{B}} - \frac{1}{2}) \tag{4.13}$$

We can substitute these expressions for the first two moments of $N(Q_m^c - Y_{k,m})$ together with the expressions for the first two moments of C_m^* in equations (4.10) and (4.11) to get approximations for the first two moments of the waiting time due to shipment consolidation.

Non-identical and stochastic replenishment order sizes

In this section we provide a heuristic for the computation of the first two moments of the waiting time due to shipment consolidation. The approximation of the average waiting time is based on the following equation:

$$E[W^c_{k,m}] = \sum_{n_k=0}^{\infty}\sum_{z=0}^{\infty} E[W^c_{k,m}|n_m^{\neq k} = z, n_k]P\{n_m^{\neq k} = z, n_k\} \qquad (4.14)$$

Here n_k denotes the number of replenishment orders at the consolidation dock from stockpoint k between two truck departures and $n_m^{\neq k}$ the number of replenishment orders at the consolidation dock from stockpoints j with $j \in \mathcal{L}_m\backslash\{k\}$ between two truck departures

To evaluate the righthand sight of (4.14) we assume that there is at maximum only one replenishment order from stockpoint $k \in \mathcal{L}_m$ in a truck, i.e. $P\{n_k = 1\} = 1$. This assumption is based on the observation that in general it is not efficient to have a high probability of having more than two replenishment orders of the same stockpoint in one truck. When there is high probability of having two replenishments orders of the same stockpoint in one truck, we can increase the batchsize, Q_k, without increasing the inventory level which leads to the same inventory costs but may lead to lower handling costs.

1. Expressions for $\mathbf{P\{n_m^{\neq k} = z\}}$

Using the assumption $P\{n_k = 1\} = 1$ we get:

$$P\{n_m^{\neq k} = z\} = P\{n_m^{\neq k} \geq z\} - P\{n_m^{\neq k} \geq z+1\} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W} \quad (4.15)$$

It can easily be seen that the number of orders between two subsequent truck departures depends on the volume of the replenishment orders at the consolidation dock. Under the assumption mentioned above the number of orders from stockpoint k arriving at the consolidation dock between an arrival of an order from stockpoint k and the departure of the truck is zero. So only replenishment orders $j \neq k$, $\forall j \in \mathcal{L}_m$ have to be considered. We introduce $(B_m^{\neq k})$ as the size of an arbitrary replenishment order at the consolidation dock of all stockpoints $j \in \mathcal{L}_m\backslash\{k\}$ (see Figure 4.7).

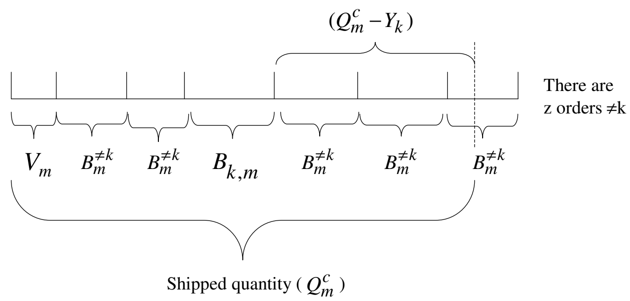


Figure 4.7: Schematic representation of the quantity-based consolidation policy

We can rewrite (4.15) as follows:

$$P\{n_m^{\neq k} = z\} \simeq P\{\sum_{j=1}^{z} B_m^{\neq k(j)} + V_m + B_{k,m} \leq Q_m^c\} \\ - P\{\sum_{j=1}^{z+1} B_m^{\neq k(j)} + V_m + B_{k,m} \leq Q_m^c\} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W} \quad (4.16)$$

where we assume that each $B_m^{\neq k(j)}$ $j = 1, 2, ...$ is identically independently distributed. Exact formulae for the probabilities at the righthand sight in (4.16) are difficult to obtain. Therefore, we determine the first two moments of $\sum_{j=1}^{z} B_m^{\neq k(j)} + V_m + B_{k,m}$ and $\sum_{j=1}^{z+1} B_m^{\neq k(j)} + V_m + B_{k,m}$ and fit a mixed Erlang distribution, see Section 2.2.2.

Moments for V_m can easily be obtained using Lemma 4.4 which implies that V_m is equivalent to the undershoot process in an $(0, nQ_m^c)$ inventory model.

Lemma 4.4 *The consolidation process of a quantity policy with partial shipments under a compound renewal demand process is equivalent to the inventory position process under an $(0, nQ_m^c)$ control policy and compound renewal customer demand.*

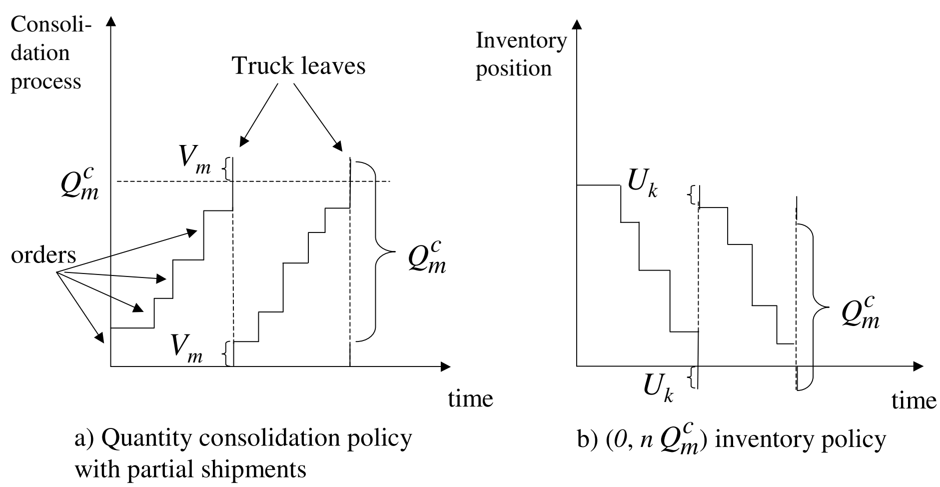


Figure 4.8: Evolution of the consolidated quantity and inventory position

The proof follows from comparison of the sample paths of the two processes (see Figure 4.8 compare a) and b)). Applying Lemma 4.4 and using asymptotic results for the residual life distribution we obtain an approximation for the distribution function of V_m

$$P\{V_m < v\} \simeq \frac{1}{E[B_m^*]} \int_0^v (1 - F_{B_m^*}(x))dx \qquad \forall m \in \mathcal{W} \tag{4.17}$$

Therefore, the first two moments of V_m are given as

$$E[V_m] \simeq \frac{E[(B_m^*)^2]}{2E[B_m^*]} \qquad \forall m \in \mathcal{W} \tag{4.18}$$

$$E[V_m^2] \simeq \frac{E[(B_m^*)^3]}{3E[B_m^*]} \qquad \forall m \in \mathcal{W} \tag{4.19}$$

For the calculation of the first two moments of B_m^* we refer to equations (3.6) and (3.7) in Section 3.3.2. To evaluate $E[(B_m^*)^3]$ we fit a mixed-Erlang distribution on the first two moments B_m^*, for references see Section 2.2.

Additionally, we need for the evaluation of (4.16) the first two moments of $\sum_{j=1}^{z} B_m^{\neq k(j)}$ and $B_m^{\neq k}$. The first two moments of $B_m^{\neq k}$ are straightforward to calculate by taking the weighted sum of the individual order sizes, for references see Section 3.3.2. Further by assuming that $B_m^{\neq k(j)}$ $j = 1, 2, 3, ...$ are independent and identically distributed, we can straightforwardly compute the first two moments of $\sum_{j=1}^{z} B_m^{\neq k(j)}$.

2. $\mathbf{E[W_k^c | n_m^{\neq k} = z, n_k = 1]}$

If the order from stockpoint k is the first order at the consolidation dock, then $Y_{k,m} = V_m + B_{k,m}$ and $W_k^c = \sum_{i=1}^{z} (C_m^*)^i$ under the condition $n_m^{\neq k} = z$ and $n_k = 1$. If it is the second one then $Y_{k,m} = V_m + B_{k,m} + B_m^{\neq k}$ and $W_{k,m}^c = \sum_{i=1}^{z-1} (C_m^*)^i$ under the condition $n_m^{\neq k} = z, n_k = 1$ and so on. For the case that the order from stockpoint k is the last order, $Q_m^c - Y_{k,m} < 0$, and the order has to be split. We assume that $Y_{k,m} - V_m$ of the order has waiting time 0 and V_m has to wait until the next truck departure $\sum_{i=1}^{z+1} (C_m^*)^i$.

Further, we assume, similarly to the identical replenishment case, that all the possibilities above are equally likely. Therefore, the probability that the replenishment order of stockpoint k being the first one at the consolidation dock, is equal to the probability that it is the second one, the third or the last one. Moreover, this probability is given as $\frac{1}{z+1}$, which leads for all $k \in \mathcal{L}_m$ and $m \in \mathcal{W}$ to

$$E[W^c_{k,m}|n^{\neq k}_m = z, n_k = 1] \simeq \frac{1}{z+1}(\sum_{s=1}^{z} sE[C^*_m] + \frac{E[V^*_m]}{E[B^*_m]}\frac{E[B^*_m]}{Q^c_m}E[C^*_m]) \tag{4.20}$$

We substitute this in equation (4.14) and get

$$E[W^c_{k,m}] \simeq \sum_{z=0}^{\infty} P\{n^{\neq k}_m = z, n_k = 1\}\left(\frac{1}{z+1}(\sum_{s=1}^{z} sE[C^*_m] + \frac{E[V^*_m]}{E[B^*_m]}\frac{E[B^*_m]}{Q^c_m}E[C^*_m])\right) \tag{4.21}$$

Similarly, we can derive an expression for the second moment of $E[(W^c_{k,m})^2]$

$$E[(W^c_{k,m})^2] \simeq \sum_{z=0}^{\infty} P\{n^{\neq k}_m = z, n_k = 1\}\Bigg(\frac{1}{z+1}(\sum_{s=1}^{z} s^2 E[(C^*_m)^2] + (\frac{E[V^*_m]}{E[B^*_m]}\frac{E[B^*_m]}{Q^c_m})^2 E[(C^*_m)^2])\Bigg) \tag{4.22}$$

Finally, we assume that the waiting time due to a lack of stock is independent of the waiting time due to shipment consolidation which allows the computation of the first two moments of the replenishment lead-time as given in Formula 4.1.

4.5 Numerical example

In this section we illustrate the analytical approximations derived in Section 4.4 with a numerical example. We consider a two-echelon distribution network, with 5 warehouses and 40 stockpoints (i.e. 8 stockpoints at each warehouse). The central warehouse with stockpoints $k \in \mathcal{B}$ is denoted by V. The decentral warehouses with stockpoints $k \in \mathcal{E}$ are denoted by $\{I, II, III, IV\}$.

Similarly to Section 3.4, we assume that the batchsizes (Q_k) and target fill rates (β_k^{target}) at $k \in \mathcal{M}$ are given and as a second performance measure we consider the long-run average physical inventory level. Further, we consider the identical and deterministic replenishment order sizes case and we assume that the volume of each product is 1.

For the interarrival times and the order size processes at the end-stockpoints we assume exponential distributions and the exogenous delay L_k^d to be constant.

The input parameters of the numerical example are presented in Table 4.1.

Table 4.1: Input parameters of the numerical example.

Parameter	Value
$E[D_k]$ $(k \in \mathcal{E})$	50
$\sigma^2(D_k)$ $(k \in \mathcal{E})$	2500
$E[A_k]$ $(k \in \mathcal{E})$	1
$\sigma^2(A_k)$ $(k \in \mathcal{E})$	1
L_k^d $(k \in \mathcal{E})$	2
L_k^d $(k \in \mathcal{B})$	4
Q_k $(k \in \mathcal{E})$	500
Q_k $(k \in \mathcal{B})$	4000
β_k^{target} $(k \in \mathcal{E})$	0.95
β_k^{target} $(k \in \mathcal{B})$	0.90
Q_m^c $(m \in \{I, II, III, IV\})$	2000

The procedure to evaluate all the variables in the logistic system is as follows:

In a first step we evaluate (A_k,D_k) for $k \in \mathcal{B}$, (R_k,O_k) for $k \in \mathcal{E}$, s_k for $k \in \mathcal{B}$, $E[I_k]$ for $k \in \mathcal{B}$ and W_k^s for $k \in \mathcal{E}$. These variables can be evaluated with the analytical approximations presented in Chapter 3. The results are given in Table 4.2.

Table 4.2: Results Part 1.

Variable	Value
$E[R_k]$ $(k \in \mathcal{E})$	10
$E[R_k^2]$ $(k \in \mathcal{E})$	120
$E[O_k]$ $(k \in \mathcal{E})$	500
$E[O_k^2]$ $(k \in \mathcal{E})$	250 000
$E[A_k]$ $(k \in \mathcal{B})$	2.5
$\sigma^2(A_k)$ $(k \in \mathcal{B})$	3.50
$E[D_k]$ $(k \in \mathcal{B})$	500
$\sigma^2(D_k)$ $(k \in \mathcal{B})$	0
s_k $(k \in \mathcal{B})$	789
$E[I_k]$ $(k \in \mathcal{B})$	2119
$E[W_k^s]$ $(k \in \mathcal{E})$	0.15
$E[(W_k^s)^2]$ $(k \in \mathcal{E})$	0.32

In a second step, with the analytical approximations in Section 4.4.1 , we evaluate the replenishment processes at the consolidation dock ($C_{k,m}$, $B_{k,m}$) for $k \in \mathcal{E}$ and we use the superposition method in Section 3.3.2 to evaluate (C_m^*,B_m^*) for $\{I, II, III, IV\}$. These results are presented in Table 4.3.

Table 4.3: Results Part 2.

Variable	Value
$E[C_{k,m}]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	10
$E[C_{k,m}^2]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	120
$E[B_{k,m}]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	500
$E[B_{k,m}^2]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	250 000
$E[C_m^*]$ $(m \in \{I, II, III, IV\})$	1.25
$E[(C_m^*)^2]$ $(m \in \{I, II, III, IV\})$	2.70
$E[B_m^*]$ $(m \in \{I, II, III, IV\})$	500
$E[(B_m^*)^2]$ $(m \in \{I, II, III, IV\})$	250 000

In a third step, we use the analytical approximations presented in Section 4.4.2 to evaluate the waiting time due to shipment consolidation for $k \in \mathcal{E}$. These results are presented in Table 4.4.

Table 4.4: Results Part 3.

Variable	Value
$E[W^c_{k,m}]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	1.87
$E[(W^c_{k,m})^2]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	7.16

Finally, with the analytical approximations presented in Chapter 3, we evaluate the replenishment lead-times for $k \in \mathcal{E}$, the reorder levels for $k \in \mathcal{M}$ and the long-run average physical inventory level for $k \in \mathcal{M}$. These results are presented in Table 4.5.

Table 4.5: Results Part 4.

Variable	Value
$E[L_k]$ $(k \in \mathcal{E})$	4.02
$\sigma^2(L_k)$ $(k \in \mathcal{E})$	3.96
s_k $(k \in \mathcal{E})$	411
$E[I_k]$ $(k \in \mathcal{E})$	472

We tested the performance of the approximations for this numerical example by using discrete event simulations. The simulation runs until 3×10^5 customers have arrived at one of the stockpoints $k \in \mathcal{E}$ and we repeat this for 10 different seeds. The simulations length was determined such that accurate results were obtained for the performance characteristics. In Table 4.6 we recapitulate the input values and the values obtained for the different variables with the analytical approximations and we present the results of these values obtained from the simulation. Similarly to chapter 2 and 3, the value between parentheses is the 95% confidence interval of the corresponding error.

To evaluate the variables in Table 4.6, several approximations are made. For the approximations used to evaluate the first two moments of (A_k,D_k) $(k \in \mathcal{B})$, (R_k,O_k) $(k \in \mathcal{E})$ and W^s_k $(k \in \mathcal{E})$ and s_k $(k \in \mathcal{M})$ and $E[I_k]$ $(k \in \mathcal{M})$, we refer to Section 3.4 p 79. Further, in case of identical and deterministic replenishment order

Table 4.6: Results of the numerical example for the analytical approximation and the simulation.

Variable	Input parameters and analytical approximation	Simulation
$E[D_k]$ $(k \in \mathcal{E})$	50	50.02 ($\pm$0.26)
$\sigma^2(D_k)$ $(k \in \mathcal{E})$	2500	2503.20 ($\pm$36.04)
$E[D_k]$ $(k \in \mathcal{B})$	500	500.02 ($\pm$0.02)
$E[D_k^2]$ $(k \in \mathcal{B})$	250000	250011.71 ($\pm$11.81)
$E[A_k]$ $(k \in \mathcal{E})$	1	1 ($\pm$0.00)
$\sigma^2(A_k)$ $(k \in \mathcal{E})$	1	0.99 ($\pm$0.02)
$E[A_k]$ $(k \in \mathcal{B})$	2.5	2.49 ($\pm$0.01)
$\sigma^2(A_k)$ $(k \in \mathcal{B})$	3.50	3.98 ($\pm$0.05)
ρ_{A_k} $(k \in \mathcal{B})$	0	-0.22 ($\pm$0.00)
$E[O_k]$ $(k \in \mathcal{E})$	500	500 ($\pm$0.01)
$E[O_k^2]$ $(k \in \mathcal{E})$	250000	250033.21 ($\pm$17.10)
$E[R_k]$ $(k \in \mathcal{E})$	10	10.03 ($\pm$0.10)
$E[R_k^2]$ $(k \in \mathcal{E})$	120	120.16 ($\pm$1.23)
$E[W_k^s]$ $(k \in \mathcal{E})$	0.15	0.16 ($\pm$0.01)
$E[(W_k^s)^2]$ $(k \in \mathcal{E})$	0.32	0.28 ($\pm$0.02)
$E[B_{k,m}]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	500	500 ($\pm$0.00)
$E[B_{k,m}^2]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	250000	250033.21 ($\pm$17.10)
$E[C_{k,m}]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	10	9.99 ($\pm$0.07)
$E[C_{k,m}^2]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	120	121.54 ($\pm$2.86)
$E[B_m^*]$ $(m \in \{I, II, III, IV\})$	500	500 ($\pm$0.00)
$E[(B_m^*)^2]$ $(m \in \{I, II, III, IV\})$	250000	250033.21($\pm$17.10)
$E[C_m^*]$ $(m \in \{I, II, III, IV\})$	1.25	1.25 ($\pm$0.01)
$E[(C_m^*)^2]$ $(m \in \{I, II, III, IV\})$	2.70	2.79 ($\pm$0.01)
β_k $(k \in \mathcal{E})$	0.95	0.953 ($\pm$0.002)
β_k $(k \in \mathcal{B})$	0.90	0.93 ($\pm$0.001)
$E[W_{k,m}^c]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	1.87	1.91 ($\pm$0.03)
$E[(W_{k,m}^c)^2]$ $(k \in \mathcal{E},\, m \in \{I, II, III, IV\})$	7.16	7.20 ($\pm$0.18)
$E[I_k]$ $(k \in \mathcal{E})$	472	504.63 ($\pm$2.23)
$E[I_k]$ $(k \in \mathcal{B})$	2119	2254.20 ($\pm$5.93)

sizes, three additional approximations are made to evaluate the first two moments of ($C_{k,m}$,$B_{k,m}$) ($k \in \mathcal{E}$, $m \in \{I, II, III, IV\}$), (C^*_m,B^*_m) ($m \in \{I, II, III, IV\}$), and $W^c_{k,m}$.

1. To evaluate the $E[C^2_{k,m}]$, we neglect the delay and the clustering effect.
2. To evaluate $E[(C^*_m)^2]$, we approximate the distributions of $C_{k,m}$ ($k \in \mathcal{L}_m$) by mixed-Erlang distributions with the same first two moments.
3. To evaluate the first two moments of the replenishment lead-time, we assume that the waiting time due to a lack of stock is independent of the waiting time due to shipment consolidation

In the numerical example β_k^{target} ($k \in \mathcal{B}$) is high and $E[R_j] > E[L_k]$ ($j \in \mathcal{E}$ and $k \in \mathcal{B}$), therefore the clustering effect is small and as expected the error in $E[C^2_j]$ ($j \in \mathcal{E}$) is small. Further, we observe that also the errors in $E[C^*_m]$ and the first two moments of W^c_k are small.

The errors in β_k and $E[I_k]$ ($k \in \mathcal{B}$) are mainly caused by Approximation 8 in Section 3.4 p 80. Approximation 8 implies that to evaluate the performance measure, we approximate the distribution of $D_k(L_k)$ by a mixed-Erlang distribution with the same first two moments. However, $D_k(L_k)$ ($k \in \mathcal{B}$) is nearly discrete.

We observe that for this numerical example the results for the first two moments of $W^c_{k,m}$ are of sufficient quality for practical purposes. To be able to generalize these results, we need to test the quality of the analytical approximations for numerous other cases.

4.6 Numerical analysis of the quantity-based shipment consolidation policy

In this section, we report on the performance of the approximations for the first two moments of the waiting time due to shipment consolidation for the quantity-based policy and the consequence of an error in the waiting time on the fill rate and the physical average inventory level. We consider the fill rate and the average physical inventory level because they are commonly used in practice, see Silver et al. (1998). When the first two moments of the replenishment lead-time are given we can calculate the reorder level and average physical inventory level such that the target fill rate is met with the approximations given in Section 3.3.5. The performance is tested using discrete event simulations.

In Section 4.4, we described a method to evaluate the first two moments of the waiting time due to shipment consolidation for the quantity-based policy. In the case of $Q^c_m - Y_{k,m} \geq 0$, we can apply equations (4.10) and (4.11). In this situation, the first two moments of $W^c_{k,m}$ depends only on the first two moments of C^*_m and $N(Q^c_m - Y_{k,m})$. In Section 4.6.1, we analyze the approximation used for the first two moments of $C_{k,m}$ and the effects on C^*_m. In Section 4.6.2 the errors in the waiting time due to shipment consolidation for the identical $B_{k,m}$ case, $Q^c_m - Y_{k,m} \leq 0$, are investigated. After that, in Section 4.6.3, we handle the non-identical $B_{k,m}$ case for which we derived approximations for the first two moments of $W^c_{k,m}$.

4.6.1 Approximations for the second moment of C^*_m

We assume 4 warehouses, one central warehouse denoted I and three decentral warehouses II, III and IV. At $k \in \mathcal{E}$ demand arrives according to a compound renewal process. The interarrival time and the order size of $k \in \mathcal{E}$ are mixed-Erlang distributed with known first two moments. $|\mathcal{L}_m|$ ($m \in \{II, III, IV\}$) is varied between 4 and 16 to analyze the errors in the individual and the aggregated arrival processes of replenishment orders at the consolidation dock.

We consider 5 different cases. For each case we evaluate $E[C^2_{k,m}]$ and $E[(C^*_m)^2]$ with the approximations given in Section 4.4.1. We simulate the cases and compare the results. The simulations stop after 1×10^5 arrivals of item orders and we performed the simulations for 10 different seeds. Table 4.7 indicates the parameter values used for the 5 cases.

We first report on the quality of the approximation made for $E[C^2_{k,m}]$ for all $k \in \mathcal{E}$ and then for $E[(C^*_m)^2]$ for $m \in \{II, III, IV\}$ for distribution networks with $|\mathcal{L}_m|$=4 and 16 ($m \in \{II, III, IV\}$). The results of the approximations for the arrival processes at the consolidation dock are presented in Table 4.8. Similarly

Table 4.7: Parameter values

Parameters	1	2	3	4	5
$E[D_k]$ $(k \in \mathcal{E})$	100	100	100	100	100
$c^2_{D_k}$ $(k \in \mathcal{E})$	1	1	1	1	1
$E[A_k]$ $(k \in \mathcal{E})$	1	1	1	1	1
$c^2_{A_k}$ $(k \in \mathcal{E})$	1	1	1	1	1
Q_k $(k \in \mathcal{E})$	100	100	100	100	100
Q_k $(k \in \mathcal{B})$	2000	2000	2000	2000	2000
L^d_k $(k \in \mathcal{E})$	2	2	2	2	2
L_k $(k \in \mathcal{B})$	16	16	16	16	8
β^{target}_k $(k \in \mathcal{B})$	0.60	0.60	0.60	0.60	0.80
Q^c_m $(m \in \{II, III, IV\})$	2000	2000	2000	2000	2000

to Section 4.5, the value between parentheses is the 95% confidence interval of the corresponding error.

Table 4.8: Errors in the second moment of the interarrival process at the consolidation dock, $\forall k \in \mathcal{E}_1$ and $m \in \{II, III, IV\}$.

cases	$\lvert\mathcal{L}_m\rvert = 4$		$\lvert\mathcal{L}_m\rvert = 16$	
	$\bar{\delta}^{(E[C^2_{k,m}])}$	$\bar{\delta}^{(E[(C^*_m)^2])}$	$\bar{\delta}^{(E[C^2_{k,m}])}$	$\bar{\delta}^{(E[(C^*_m)^2])}$
1	28.02 (±0.82)	19.36 (±1.30)	29.28 (±0.81)	10.06 (±0.85)
2	20.32 (±0.98)	13.27 (±3.51)	19.72 (±1.01)	8.93 (±0.41)
3	17.01 (±1.18)	14.85 (±1.17)	17.26 (±1.13)	4.58 (±0.70)
4	12.28 (±1.88)	10.59 (±1.28)	10.23 (±1.61)	4.62 (±3.22)
5	0.49 (±0.21)	0.26 (±0.17)	0.61 (±0.25)	0.14 (±0.15)

For the evaluation of $E[C^2_{k,m}]$ we make use of the approximation that the delay and the clustering effect are negligible. We noticed in Section 4.4.1 that when $E[R_k] >> E[L_j]$ and β^{target}_j ($k \in \mathcal{E}$ and $j \in \mathcal{P}_k$) delay and clustering can occur. In the first four generated cases $E[R_k] >> E[L_j]$ and β^{target}_j is small, therefore we observe in Table 4.8 that the errors in $E[C^2_{k,m}]$ are large and thus delays and clusterings take place. Note that the first four cases are extreme ones, which are not frequent in practice. Case 5 is the more common in practice.

For the evaluation of $E[(C_m^*)^2]$, we use the superposition method which approximates the distributions of $C_{k,m}$ ($k \in \mathcal{M}$) by mixed-Erlang distributions with the same first two moments. In the superposition method, the superposed variable converges to an exponential distribution when the number of superposed processes goes to infinity. So, both the evaluated and simulated C_m^* will converge to an exponential distribution.

As expected, we observe that the decrease in error between $E[C_{k,m}^2]$ and $E[(C_m^*)^2]$ is large. However, the error in $E[(C_m^*)^2]$ for the first four cases and $|\mathcal{L}_m| = 4$ is still large. Therefore, we present this improvement for the approximation used to evaluate $E[C_{k,m}^2]$. This improved is proposed by A. van Harten. This approximation considers only the delay effect and not the clustering effect. We introduce p_k, which is the probability that W_k^s is larger than zero and w_k, which is waiting time due to a lack of stock given that W_k^s is larger than zero. From definition follows that $f_{\mathrm{w}_k}(x) = \frac{1}{p} f_{W_k^s}(x)$ for $x > 0$.

Since we assume no clustering, if $W_k^{(n)s} > 0$ then $W_k^{(n+1)s} = 0$ and $W_k^{(n-1)s} = 0$. We can distinguish between three different type of events.

1. When $W_k^{(n)s} = 0$ and $W_k^{(n+1)s} > 0$, then the arrival at the consolidation dock is delayed by $W_k^{(n+1)s}$. The interarrival time at the consolidation dock is $R_k^{(n)} + \mathrm{w}_k^{(n+1)}$. The probability that this event takes place is p, since
$P\{W_k^{(n)s} = 0, W_k^{(n+1)s} > 0\} = P\{W_k^{(n+1)s} > 0\} P\{W_k^{(n)s} = 0 | W_k^{(n+1)s} > 0\}$
$P\{W_k^{(n)s} = 0, W_k^{(n+1)s} > 0\} = p$

2. When $W_k^{(n)s} > 0$ and $W_k^{(n+1)s} = 0$, then the interarrival time at the consolidation dock is $R_k^{(n)} - \mathrm{w}_k^{(n)}$. The probability that this event takes place is p, since
$P\{W_k^{(n)s} > 0, W_k^{(n+1)s} = 0\} = P\{W_k^{(n)s} > 0\} P\{W_k^{(n+1)s} = 0 | W_k^{(n)s} > 0\}$
$P\{W_k^{(n)s} > 0, W_k^{(n+1)s} = 0\} = p$

3. When $W_k^{(n)s} = 0$ and $W_k^{(n+1)s} = 0$, then the interarrival time at the consolidation dock is $R_k^{(n)}$. The probability that this event takes place is $1 - 2p$, since the probabilities should sum up to one.

In Figure 4.9, we give a schematic representation of the arrival process at the stockpoint and at the consolidation dock. In Figure 4.9 $C_{k,m}^{(1)}$ is of type 1, $C_{k,m}^{(2)}$ is of type 2 and $C_{k,m}^{(3)}$ and $C_{k,m}^{(4)}$ are of type 3. This results in the following approximation for $E[C_{k,m}^2]$,

$$
\begin{aligned}
E[C_{k,m}^2] &\simeq (1-2p)E[R_k^2] + pE[(R_k + \mathrm{w}_k)^2] + pE[(R_k - \mathrm{w}_k)^2] \\
E[C_{k,m}^2] &\simeq (1-2p)E[R_k^2] + pE[(R_k^2 + 2R_k\mathrm{w}_k + \mathrm{w}_k^2)] + pE[(R_k^2 - 2R_k\mathrm{w}_k + \mathrm{w}_k^2)] \\
E[C_{k,m}^2] &\simeq E[R_k^2] + 2pE[\mathrm{w}_k^2] \\
E[C_{k,m}^2] &\simeq E[R_k^2] + 2E[(W_k^s)^2] \qquad (4.23)
\end{aligned}
$$

The results of this new approximation for the same five cases are presented in Table 4.9. The value between parentheses is the 95% confidence interval of the corresponding error, for more detail we refer to Section 2.2.5.

Table 4.9: Errors in the variance of the consolidation process for the improved approximation, $\forall k \in \mathcal{E}_1$ and $m \in \{II, III, IV\}$.

cases	$\lvert\mathcal{L}_m\rvert = 4$		$\lvert\mathcal{L}_m\rvert = 16$	
	$\bar{\delta}^{(E[C_k^2])}$	$\bar{\delta}^{(E[(C_m^*)^2])}$	$\bar{\delta}^{(E[C_k^2])}$	$\bar{\delta}^{(E[(C_m^*)^2)]}$
1	1.75 (±1.61)	1.65 (±1.28)	1.87 (±1.57)	1.21 (±0.80)
2	5.26 (±1.02)	3.72 (±1.42)	4.73 (±1.35)	1.04 (±0.90)
3	5.18 (±1.27)	4.65 (±1.52)	4.76 (±1.39)	1.52 (±0.24)
4	3.93 (±1.73)	3.41 (±1.86)	3.98 (±1.30)	1.36 (±0.78)
5	0.25 (±0.18)	0.02 (±0.08)	0.37 (±0.15)	0.06 (±0.07)

We observe that the performance of this new approximation is excellent even for the first four cases, which are extreme situations. Further, we conclude from the simulation results that even in the extreme situations the delay effect is more important than the clustering effect since this new approximation perform correctly and only the delay is taken into account.

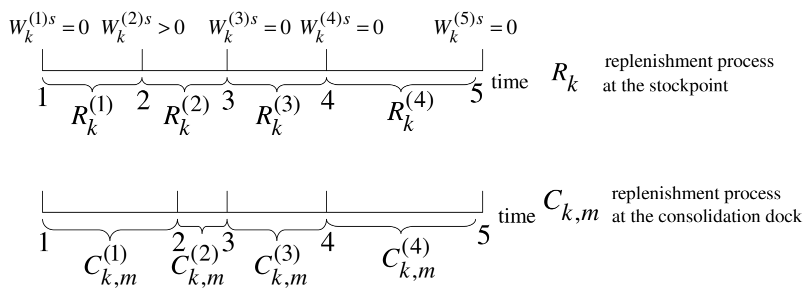


Figure 4.9: Schematic representation of the arrival process at the stockpoint and at the consolidation dock

4.6.2 Identical and deterministic replenishment order sizes

In this section, we test the performance of the approximations for the first two moments of the waiting time due to shipment consolidation, the fill rate and the average inventory level for the identical and deterministic $B_{k,m}$ case. The first two moments of the waiting time depend on the first two moments of C_m^* and of $N(Q_m^c - Y_{k,m})$. In Section 4.4 we derived exact results for the first two moments of $N(Q_m^c - Y_{k,m})$, therefore we test in this section the effect of the error in C_m^* on the waiting time due to shipment consolidation, the fill rate and the average inventory level.

We assume 5 warehouses, one central warehouse denoted I and four decentral warehouses II, III, IV and V. At $k \in \mathcal{E}$ demand arrives according to a compound renewal process. The interarrival time and the order size of $k \in \mathcal{E}$ are mixed-Erlang distributed with known first two moments. Similarly to Section 4.5 the volume of each product 1. We choose $E[D_k] << Q_k$, $E[D_k] = U(10, 50)$ and $Q_k = 500$ for $k \in \mathcal{M}$ such that the $B_{k,m}$ are identical and deterministic. Further, we assume that $c^2_{D_k} = 1$, $E[A_k] = U(1, 5)$, $c^2_{A_k} = 1$, L^d_k=4 for $k \in \mathcal{E}$, L^d_k for $k \in \mathcal{B}$ is varied between 4, 8 and 16, β^{target}_k for $k \in \mathcal{E}$ is 95% and β^{target}_k for $k \in \mathcal{B}$ is 80%. The $|\mathcal{L}_m|$ ($m \in \{II, III, IV, V\}$) is varied between 4, 8 and 16. The input parameters of the 18 different simulations are given in Table 4.10. The results of the 18 simulations are given in Table 4.11. The value between parentheses is the 95% confidence interval of the corresponding error, for more detail we refer to Section 2.2.5.

Table 4.10: Parameter values

case	$\lvert\mathcal{L}_m\rvert$	L^d_k $(k \in \mathcal{B})$	Q^c_m $(m \in \{II, III, IV, V\})$
1	4	4	2000
2	4	8	2000
3	4	16	2000
4	4	4	4000
5	4	8	4000
6	4	16	4000
7	8	4	2000
8	8	8	2000
9	8	16	2000
10	8	4	4000
11	8	8	4000
12	8	16	4000
13	16	4	2000
14	16	8	2000
15	16	16	2000
16	16	4	4000
17	16	8	4000
18	16	16	4000

To evaluate the first two moments of $W^c_{k,m}$, we make use of the following approximations:

1. To evaluate the second moment of R_k we use approximation 3 of the approximations enumerated in Section 3.4, which is an asymptotic approximation

Table 4.11: Performance of the approximations.

Case	$\bar{\delta}^{(E[W^c_{k,m}])}$	$\bar{\delta}^{(E[(W^c_{k,m})^2)]}$	$\bar{\Delta}^{\beta_k}$	$\bar{\delta}^{E[I_k]}$
1	3.87 (±2.45)	4.60 (±3.94)	1.58 (±0.28)	3.33 (±0.80)
2	4.77 (±1.00)	3.66 (±1.87)	1.26 (±0.29)	2.45 (±0.87)
3	3.59 (±2.23)	4.49 (±4.71)	1.59 (±0.31)	2.91 (±0.98)
4	0.77 (±1.97)	3.59 (±2.07)	1.67 (±0.25)	1.97 (±0.93)
5	1.21 (±1.98)	2.62 (±1.95)	1.44 (±0.29)	1.69 (±0.94)
6	1.03 (±2.07)	3.07 (±2.34)	1.60 (±0.30)	2.06 (±0.96)
7	3.33 (±2.18)	4.40 (±2.60)	0.75 (±0.26)	5.92 (±0.62)
8	4.58 (±2.45)	5.50 (±2.47)	0.66 (±0.31)	4.04 (±0.69)
9	3.52 (±2.18)	4.28 (±3.47)	1.36 (±0.29)	5.63 (±0.61)
10	1.77 (±1.71)	2.94 (±2.98)	1.44 (±0.30)	4.80 (±0.79)
11	2.19 (±1.83)	1.96 (±1.13)	1.07 (±0.31)	3.31 (±0.84)
12	2.38 (±1.81)	3.77 (±1.79)	1.37 (±0.27)	4.89 (±0.77)
13	6.11 (±2.30)	7.59 (±1.79)	0.75 (±0.24)	6.33 (±0.51)
14	6.18 (±2.63)	8.84 (±2.41)	1.10 (±0.29)	5.79 (±0.55)
15	6.17 (±2.14)	8.41 (±3.96)	1.48 (±0.29)	5.01 (±0.53)
16	3.57 (±1.81)	4.41 (±2.21)	0.74 (±0.26)	6.36 (±0.58)
17	2.97 (±1.78)	5.97 (±2.52)	1.04 (±0.29)	5.97 (±0.63)
18	3.09 (±1.81)	6.68 (±2.78)	1.22 (±0.28)	5.56 (±0.62)

from renewal theory.

2. To evaluate $E[C^2_{k,m}]$, we neglect the delay and clustering effect.
3. To evaluate $E[(C^*_m)^2]$, we use the superposition technique and approximate the distributions of C_k by mixed-Erlang distributions with the same first two moments.

To evaluate the fill rates and the long-run physical average inventory levels, we make use of the approximations recapitulated in Section 3.4, we assume that the waiting time due to a lack of stock is independent of the waiting time due to shipment consolidation and the approximations used to evaluate the first two moments of $W^c_{k,m}$.

In Table 4.11, we observe that the errors in $E[W^c_{k,m}]$ and $E[(W^c_{k,m})^2]$ are within acceptable margins. Further from Table 4.11 we observe that there is positive relation between the target consolidation quantity and the errors in $E[W^c_{k,m}]$ and $E[(W^c_{k,m})^2]$.

In Table 4.11, we observe that the errors in β_k ($k \in \mathcal{E}$) are large. This is mainly caused by approximation 8 in Section 3.4. Approximation 8 implies that to evaluate the performance measure, we approximate the distribution of $D_k(L_k)$ by a mixed-Erlang distribution with the same first two moments. However, $D_k(L_k)$ ($k \in \mathcal{B}$) is nearly discrete. In Table 4.11 we observe that the errors in $E[I_k]$ ($k \in \mathcal{E}$) are within acceptable margins. We conclude that the simulations are still of sufficient quality for practical purposes

4.6.3 Non-identical and stochastic replenishment order size

In this section, we test the performance of the approximations for the first two moments of the waiting time due to shipment consolidation, the fill rate and the average inventory level. Similar to Section 4.6.2, we consider the same distribution network with one central warehouse and four decentral warehouses. At $k \in \mathcal{E}$ demand arrives according to a compound renewal process. The inter-arrival time and the order size of $k \in \mathcal{E}$ are mixed-Erlang distributed with known first two moments. We assume that the last order before the truck departs is split.

In Section 4.4.2 we derived expressions for the first two moments of $W^c_{k,m}$. To derive these expressions we make use of the following approximations.

1. We neglect the clustering effect to evaluate the second moment of $C_{k,m}$.
2. To evaluate C^*_m, we approximate the distributions of $C_{k,m}$ $(k \in \mathcal{M})$ by mixed-Erlang distributions with the same first two moments,
3. We approximate the distribution of $\sum_{i=1}^{z} B^{\neq k}_{m,i} + V_m + B_m$, $z \in \mathbb{N}$ by a mixed-Erlang distribution with the same first two moments.
4. We neglect the probability that more than 1 replenishment order from stockpoint k are in one truck. $(P\{n_k = 1\} = 1)$
5. The probability that replenishment order k is the first one, the second one or the last one is the same.

The effect of approximations 1 and 2 were investigated in the previous two sections. Therefore, we concentrate on the effect of approximations 3 to 5. To do this we consider 11 different cases.

In cases 1 to 4, we investigate the effects of approximation 3 by increasing the average number of different orders in a truck, $\frac{Q^c_m}{\sum_{k \in \mathcal{E}} Q_k/|\mathcal{L}_m|}$. We increase also the number of stockpoints at warehouse m, $|\mathcal{L}_m|$, to keep the same probability of having more than one orders of the same stockpoint in one truck.

In cases 5 to 7, we investigate the effects of approximation 4 by increasing the average number of orders in a truck, $\frac{Q^c_m}{\sum_{k \in \mathcal{E}} Q_k/|\mathcal{E}|}$, but we keep $|\mathcal{L}_m| = 4$. So $P\{n_k > 1\}$ increases gradually, when the number of orders in a truck is raised from 5 to 10 to 20.

In cases 8 to 11, we test the effect of approximation 5 by considering two types of stockpoints at warehouse m, the ones with small $B_{k,m}$ and the ones with large $B_{k,m}$. The set of stockpoints with a small $B_{k,m}$ is denoted $\mathcal{R}$ and with a large $B_{k,m}$ is denoted $\mathcal{D}$. At the warehouse there are 8 stockpoints, 4 with a small $B_{k,m}$ and 4 with a large $B_{k,m}$. We assumed in Section 4.4.2 that the probability that order k is the first one, the second one or the last one is the same. When an order is large the probability that it is the last one before the truck leaves is larger. Therefore we increase the difference between the $B_{k,m}$'s gradually.

Similarly to Section 4.5 the volume of each product is 1. We assume that the interarrival times and the order sizes are mixed-Erlang distributed. The average order size and interarrival time at stockpoints ($k \in \mathcal{E}$) are randomly generated on respectively an interval $U(10, 50)$ and $U(1, 5)$. The coefficient of variations of the order size and the interarrival time at stockpoints ($k \in \mathcal{E}$) are 1. The batchsizes at stockpoints $k \in \mathcal{B}$ are 4000. The exogenous delay is assumed to be deterministic. L_k^d ($k \in \mathcal{E}$) is equal to 4 and L_k^d ($k \in \mathcal{B}$) is equal to 16. The target fill rate is, at stockpoints $k \in \mathcal{E}$, 95 % and at $k \in \mathcal{B}$ 90 %.

The remaining input parameters of the 11 different cases are presented in Table 4.12. The results of the simulations are presented in Table 4.13 .

Table 4.12: Parameter values

case	$\lvert\mathcal{L}_m\rvert$ ($m \in \{II, III, IV, V\}$)	Q_k ($k \in \mathcal{E}$)	Q_m^c ($m \in \{II, III, IV, V\}$)
1	4	$U(600, 1000)$	2000
2	8	$U(600, 1000)$	4000
3	16	$U(600, 1000)$	8000
4	32	$U(600, 1000)$	16000
5	4	$U(600, 1000)$	4000
6	4	$U(600, 1000)$	8000
7	4	$U(600, 1000)$	16000
8	8	$U(500, 600)$ and $U(700, 800)$	4000
9	8	$U(400, 500)$ and $U(800, 900)$	4000
10	8	$U(300, 400)$ and $U(900, 1000)$	4000
11	8	$U(200, 300)$ and $U(1000, 1100)$	4000

If we compare the results for the first two moments of the waiting time due to shipment consolidation between the identical and non-identical $B_{k,m}$, as expected the errors in the non-identical case are larger, but the approximations for the first two moments of $W_{k,m}^c$ are in the non-identical case still good. If we compare the errors in the fill rates and average inventory level then the non-identical $B_{k,m}$ seems to work better than the identical $B_{k,m}$, this is due to approximation 8 in Section 3.4 because in the identical $B_{k,m}$ case $D(L_k)$ is nearly discrete.

Table 4.13: Performance of the approximations.

Case	$\bar{\delta}^{(E[W_k^c])}$	$\bar{\delta}^{(E[(W_k^c)^2)]}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$
1	3.28 (±1.82)	3.03 (±2.64)	1.15 (±0.16)	2.80 (±0.32)
2	2.48 (±1.14)	5.16 (±1.49)	0.56 (±0.19)	3.17 (±0.31)
3	1.26 (±0.88)	2.67 (±1.31)	0.36 (±0.20)	4.86 (±0.32)
4	1.60 (±0.85)	2.97 (±1.73)	0.40 (±0.22)	5.84 (±0.34)
5	4.09 (±1.01)	6.27 (±1.78)	1.90 (±0.17)	1.09 (±0.31)
6	6.88 (±1.49)	7.38 (±0.96)	2.40 (±0.16)	1.98 (±0.43)
7	8.38 (±1.74)	11.42 (±0.71)	2.75 (±0.15)	2.57 (±0.47)
8 $k \in \mathcal{R}$	3.63 (±1.98)	4.83 (±1.81)	1.29 (±0.55)	4.24 (±1.26)
8 $k \in \mathcal{D}$	3.23 (±1.25)	4.69 (±2.12)	1.29 (±0.55)	3.70 (±1.27)
9 $k \in \mathcal{R}$	3.27 (±1.67)	4.70 (±2.37)	1.30 (±0.54)	4.57 (±1.13)
9 $k \in \mathcal{D}$	3.15 (±1.65)	4.37 (±2.25)	1.25 (±0.60)	3.52 (±1.27)
10 $k \in \mathcal{R}$	6.45 (±3.18)	8.66 (±2.57)	1.29 (±0.58)	4.82 (±1.13)
10 $k \in \mathcal{D}$	2.36 (±1.36)	4.44 (±2.44)	1.23 (±0.56)	3.55 (±1.42)
11 $k \in \mathcal{R}$	8.27 (±3.27)	10.98 (±2.24)	1.41 (±0.62)	3.81 (±1.35)
11 $k \in \mathcal{D}$	4.77 (±1.41)	7.12 (±2.21)	0.92 (±0.56)	3.40 (±1.05)

Further, from Table 4.13 we can make the following observations:

1. The effect of varying the number of different orders in one truck on the errors in the first two moments of $W^c_{k,m}$ is only noticeable from cases 1 to 2.

2. The effect of varying the number of different orders in one truck on the errors in the first two moments of $W^c_{k,m}$, β_k and $E[I_k]$ is small.

3. The effect of varying the average number of orders in one truck on the errors in the first two moments of $W^c_{k,m}$, β_k and $E[I_k]$ is large.

4. The effect of considering two type of customers on the errors in the first two moments of $W^c_{k,m}$, β_k and $E[I_k]$ is moderate.

To conclude when $n_k \geq 2$ and we use the analytical approximations to evaluate $W^c_{k,m}$ the errors are large. But we observed in Section 4.4.2 that it is not cost efficient to have $n_k \geq 2$, because we could increase Q_k without increasing the inventory level, which leads to the same inventory costs but may lead to lower handling costs.

Until now we assumed that the last order before the truck leaves is split. In the appendix of this chapter, we also derived approximations for the first two moments of the waiting time due to shipment consolidation for the cases that the last order leaves with the current truck and that the last order leaves with the next truck. These approximations perform less as the split order case but they are still of sufficient quality for practical purposes, cf. see Smits and de Kok (2002).

4.7 Summary and outlook to Chapter 5

Transportation consolidation policies directly influence the replenishment lead-times towards stockpoints whose orders are consolidated. Since the replenishment lead-time is a major input for the analysis of an inventory control policy, it is clear that issues related to transportation consolidation and inventory control should be addressed simultaneously. As we observed in Chapter 1, the replenishment lead-time is important when considering tactical and strategic decisions in the logistic network. Therefore it is important to observe the integrated model (transportation policies as well as inventory policies).

The model under consideration in this chapter is based on twelve assumptions. In the time-based policy, we assume that the truck capacity is unlimited. Further, we assume that W_k^c is zero for $k \in \mathcal{B}$, the shipment consolidation policy is given and the nine assumptions enumerated in Section 3.6. Similar to Chapter 3, the partial and complete delivery assumptions can easily be relaxed with the formulae given in Section 3.3.5. The assumptions that W_k^c and W_k^s are zero for $k \in \mathcal{B}$ can also easily be extended to a given distribution function. Further research is necessary to be able to relax the other assumptions.

In this chapter, we derived exact expressions for the waiting due to shipment consolidation for time-based consolidation policies and approximations for quantity-based consolidation policies. For a recapitulation of the different approximations, we refer to Section 4.6. An extensive numerical study was conducted to understand and investigate the quality of these approximations. The study has revealed that the approximations performed good for the quantity-based policy with deterministic and identical replenishment order sizes and well for the quantity-based policy with stochastic replenishment order sizes. In the stochastic replenishment order size case, the approximations perform less when there are less well than two replenishment orders of each stockpoint in a truck. This condition is realistic, since we observed in Section 4.4.2 that it is not cost efficient to have more than 1 replenishment order of each stockpoint in a truck, because we could increase the batchsize Q_k without increasing the inventory level, which leads to the same inventory costs but may lead to lower handling costs.

Besides transportation consolidation policies also production scheduling policies influence directly the replenishment lead-times. Therefore in the following chapter we will evaluate expressions for the first two moments of the waiting time due to production.

Appendix 4

Theorem 4.1 *Given that $X_{k,m}$, the time between the last truck departure from n to m and the arrival of a replenishment order from stockpoint k, is uniformly distributed on (0,T_m] and W_k^s is independent of $X_{k,m}$ ($k \in \mathcal{L}_m$ and $m \in \mathcal{W}$), $W_{k,m}^c$ is uniformly distributed on the interval (0,T_m] and the following formulae hold:*

$$E[W_{k,m}^c] = \frac{T_m}{2} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W} \tag{4.24}$$

and

$$E[(W_{k,m}^c)^2] = \frac{T_m^2}{3} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W} \tag{4.25}$$

proof $X_{k,m}$ is the time between the last truck departure from n to m $(n, m \in \mathcal{W})$ and the arrival of an arbitrary replenishment order from $k \in \mathcal{L}_m$. W_k^s is the waiting time due to a lack of stock (see Figure 4.4) and $W_{k,m}^c$ is the waiting time due to shipment consolidation, $\forall k \in \mathcal{L}_m$. We define $X = X_{k,m}$, $U = W_k^s$, $Y = W_{k,m}^c$ and $T = T_m$ to make the proof easier to see. Given that $Y \in [0, T]$

$$\begin{aligned}
P\{Y \leq y\} &= P\{X + U \in (zT - y, zT), z \in \mathbb{N}\} \\
&= \sum_{z=1}^{\infty} \frac{1}{T} \int_0^T P\{U \in (zT - y - x, zT - x)\} dx \\
&= \sum_{z=1}^{\infty} \frac{1}{T} \int_0^T \left(\int_{zT-y-x}^{\infty} dF_U(u) - \int_{zT-x}^{\infty} dF_U(u) \right) dx \\
&= \frac{1}{T} \left(\int_0^{T-y} \int_{T-y-x}^{\infty} dF_U(u) dx + \int_{T-y}^{T} \int_0^{\infty} dF_U(u) dx - \int_0^T \int_{T-x}^{\infty} dF_U(u) dx \right) \\
&\quad + \sum_{z=2}^{\infty} \frac{1}{T} \left(\int_{zT-y}^{\infty} \int_0^T dx dF_U(u) + \int_{((z-1)T-y)}^{zT-y} \int_{(zT-y-u)}^{T} dx dF_U(u) \right. \\
&\quad \left. - \int_{zT}^{\infty} \int_0^T dx dF_U(u) - \int_{((z-1)T)}^{zT} \int_{(zT-u)}^{T} dx dF_U(u) \right)
\end{aligned}$$

$$\begin{aligned}
&= \tfrac{1}{T}\Bigg(\int\limits_0^{T-y}\int\limits_{T-y-u}^{T-y} dxdF_U(u) + \int\limits_{T-y}^{\infty}\int\limits_0^{T-y} dxdF_U(u) - \int\limits_T^{\infty}\int\limits_0^{T} dxdF_U(u) \\
&\quad + y - \int\limits_0^{T}\int\limits_{T-u}^{T} dxdF_U(u)\Bigg) + \sum_{z=2}^{\infty}\Bigg(\int\limits_{(zT-y)}^{\infty} dF_U(u) - \int\limits_{zT}^{\infty} dF_U(u) \\
&\quad + \int\limits_{(z-1)T-y}^{zT-y} (\tfrac{u-y-(z-1)T}{T})dF_U(u) - \int\limits_{(z-1)T}^{zT} \tfrac{u-(z-1)T}{T}dF_U(u)\Bigg) \\
&= \tfrac{y}{T} + \int\limits_{T-y}^{\infty} \tfrac{T-y}{T}dF_U(u) - \int\limits_{T-y}^{T} \tfrac{u}{T}dF_U(u) - \int\limits_T^{\infty} dF_U(u) \\
&\quad + \sum_{z=2}^{\infty}\Bigg(\int\limits_{(z-1)T-y}^{zT-y} \tfrac{y}{T}dF_U(u) + \int\limits_{(z-1)T-y}^{(z-1)T} \tfrac{u-(z-1)T}{T}dF_U(u) \\
&\quad - \int\limits_{zT-y}^{zT} \tfrac{u-zT}{T}dF_U(u)\Bigg) \\
&= \tfrac{y}{T} + \int\limits_{T-y}^{\infty} \tfrac{T-y}{T}dF_U(u) - \int\limits_{T-y}^{T} \tfrac{u}{T}dF_U(u) - \int\limits_T^{\infty} dF_U(u) \\
&\quad + \int\limits_{T-y}^{\infty} \tfrac{y}{T}dF_U(u) + \int\limits_{T-y}^{T} \tfrac{u-T}{T}dF_U(u) \\
&= \tfrac{y}{T}
\end{aligned}$$

Theorem 4.2 *$X_{k,m}$, which is the time between the last truck departure from n to m and the arrival of an order from stockpoint k, is uniformly distributed on the interval (0,T_m), where $k \in \mathcal{L}_m$ and $m \in \mathcal{W}$.*

proof $X_{k,m}$ is the time between the last truck departure from n to m ($n, m \in \mathcal{W}$) and the arrival of an arbitrary replenishment from k in $\mathcal{L}_m$. We define $Y_{k,m}$ as the time between the arrival of an arbitrary replenishment from k and departure of a truck from n to m. If $Y_{k,m}$ is uniformly distributed over the interval (0,T_m) then $X_{k,m}$ is uniformly distributed over the interval (0,T_m), since $T_m - Y_{k,m} = X_{k,m}$. We define $\tilde{R}^*_m$ as the residual life time of the inter-arrival time of an arbitrary replenishment from m to n at an arbitrary moment in time. We define $\tilde{T}_m$ as the residual lifetime of the truck arrival process at time t from n to m. Since $\tilde{T}_m$ is the time between an arbitrary moment in time and the departure of the truck, $\tilde{T}_m$ is uniformly distributed over $(0, T_m)$, for references see Doob (1953).
$\tilde{R}^*_m + Y_{k,m} = \tilde{T}_m + zT_m$,
$\tilde{R}^*_m + Y_{k,m} + T_m - \tilde{T}_m = (z+1)T_m$, $z \in N$

We define $U = T_m - \tilde{T}_m$, it is easy to see that U is uniform distributed between $(0, T_m)$ and $\tilde{z} = z + 1$.

$$\begin{aligned} P\{Y_{k,m} \leq y_{k,m}\} &= P\{U + \tilde{R}^*_m \in (\tilde{z}T_m - y_{k,m}, \tilde{z}T_m), \tilde{z} \in N\} \\ &= \tfrac{y_{k,m}}{T_m} \end{aligned}$$

See proof of Theorem 4.1. Since $Y_{k,m}$ is uniformly distributed over $(0, T_m)$, $X_{k,m}$ is uniformly distributed over $(0, T_m)$.

Theorem 4.3 *$W^c_{k,m}$ and W^s_k are independent of each other, $k \in \mathcal{M}$.*

proof

$P\{W^c_{k,m} \leq w^c_{k,m} | W^s_k = w^s_k\} = P\{ZT_m - w^s_k - X_{k,m} \leq w^c_{k,m}\}$

The value of Z depends on the value of $w^s_k + X_{k,m}$.

We define $r(l) = \lceil \frac{w^s_k}{T_m} \rceil T_m - w^s_k$ and $z_y = \lceil \frac{w^s_k}{T_m} \rceil$.

It is easy to see that $\frac{w^s_k + X_{k,m} + w^c_{k,m}}{T_m} = \lceil \frac{w^s_k + X_{k,m}}{T_m} \rceil = Z$, $0 \leq X_{k,m} \leq T_m$ and $0 \leq W^c_{k,m} \leq T_m$.

Further,

$$r(l) = \lceil \tfrac{w^s_k}{T_m} \rceil T_m - w^s_k$$

$$r(l) \geq \tfrac{w^s_k}{T_m} T_m - w^s_k$$

$$r(l) \geq 0.$$

$$P\{W^c_{k,m} \leq w^c_{k,m} | W^s_k = w^s_k\} = \int_0^{T_m} P\{ZT_m - w^s_k - x_{k,m} \leq W^c_{k,m}\} \tfrac{1}{T_m} dx_{k,m}$$

We distinguish between two situations:

1) $x_{k,m} < r(l)$

   if $x_{k,m} < r(l)$ then the following conditions hold:

   a) $\lceil \frac{w^s_k + X_{k,m}}{T_m} \rceil = \lceil \frac{w^s_k}{T_m} \rceil$

   b) $Z = z_y$

c) $W^c_{k,m} \leq r(l)$
$\frac{w^s_k + X_{k,m} + W^c_{k,m}}{T_m} = \lceil \frac{w^s_k + X_{k,m}}{T_m} \rceil = \lceil \frac{w^s_k}{T_m} \rceil$ multiplying by T_m and subtracting w^s_k we obtain $X_{k,m} + W^c_{k,m} = r(l)$ and since $X_{k,m} \geq 0$, $W^c_{k,m}$ cannot be larger than $r(l)$

2) $x_{k,m} \geq r(l)$
if $x_{k,m} \geq r(l)$ then the following conditions hold:

a) $\lceil \frac{w^s_k + X_{k,m}}{T_m} \rceil = \lceil \frac{w^s_k}{T_m} \rceil + 1$

b) $Z = z_y + 1$

$$P\{W^c_{k,m} \leq w^c_{k,m} | W^s_k = w^s_k\} = \frac{1}{T_m}\Big(\int_0^{r(l)} P\{z_y T_m - x_{k,m} - w^s_k \leq w^c_{k,m}\} dx_{k,m} + \int_{r(l)}^{T_m} P\{(z_y+1)T_m - x_{k,m} - w^s_k \leq w^c_{k,m}\} dx_{k,m}\Big)$$

$$P\{W^c_{k,m} \leq w^c_{k,m} | W^s_k = w^s_k\} = \frac{1}{T_m}\Big(\int_0^{r(l)} P\{r(l) - x_{k,m} \leq w^c_{k,m}\} dx_{k,m} + \int_{r(l)}^{T_m} P\{r(l) + T_m - x_{k,m} \leq w^c_{k,m}\} dx_{k,m}\Big) \quad (4.26)$$

We again distinguish between two situations

i) $r(l) \geq w^c_{k,m}$
The first term of formula (4.26), $P\{r(l) - x_{k,m} \leq w^c_{k,m}\}$ is true if and only if $r(l) - w^c_{k,m} \leq x_{k,m}$.
The second term of formula (4.26), $P\{r(l) + T_m - x_{k,m} \leq w^c_{k,m}\}$ is true if and only if $x_{k,m} \geq T_m$
$0 < r(l) - w^c_{k,m} \leq T_m - x_{k,m} \Rightarrow x_{k,m} > T_m$
This gives:

$$
\begin{aligned}
P\{W^c_{k,m} \le w^c_{k,m} | W^s_k = w^s_k\} &= \frac{1}{T_m} \int\limits_{r(l)-w^c_{k,m}}^{r(l)} P\{r(l) - x_{k,m} \le w^c_{k,m}\} dx_{k,m} + \\
&\quad \frac{1}{T_m} \int\limits_{T_m}^{T_m} P\{r(l) + T_m - x_{k,m} \le w^c_{k,m}\} dx_{k,m} \\
&= \frac{w^c_{k,m}}{T_m} \\
&= P\{W^c_{k,m} \le w^c_{k,m}\}
\end{aligned}
$$

ii) $r(l) < w^c_{k,m}$

The first term of formula (4.26), $P\{r(l) - x_{k,m} \le w^c_{k,m}\}$ can never hold because of condition 1)c. The second term of formula (4.26), $P\{r(l) + T_m - x_{k,m} \le w^c_{k,m}\}$ is true if and only if $T_m - w^c_{k,m} \le x_{k,m}$

$r(l) + T_m - x_{k,m} \le w^c_{k,m} \Longleftrightarrow r(l) \le w^c_{k,m} - T_m + x_{k,m}$, we know $0 \le r(l)$ therefore

$T_m - w^c_{k,m} \le x_{k,m}$

$$
\begin{aligned}
P\{W^c_{k,m} \le w^c_{k,m} | W^s_k = w^s_k\} &= \frac{1}{T_m} \int\limits_{T_m - w^c_{k,m}}^{T_m} P\{r(l) + T_m - x_{k,m} \le w^c_{k,m}\} \\
&= \frac{w^c_{k,m}}{T_m} \\
&= P\{W^c_{k,m} \le w^c_{k,m}\}
\end{aligned}
$$

First two moments for the waiting time due to shipment consolidation for the cases that the last replenishment order is not split.

In the analysis above we assumed that the last order before the truck departs is split. In a similar manner we can also derive approximations for the first two moments of the waiting due to shipment consolidation for the case that the entire last order leaves directly and for the case that the entire last order leaves with the current truck.

1. Last order leaves directly.

   In this case Equation (4.16) becomes

$$P\{n_m^{\neq k} = z, n_k = 1\} \simeq P\{\sum_{j=1}^{z} B_m^{\neq k(j)} + B_{k,m} \leq Q_m^c\}$$
$$- P\{\sum_{j=1}^{z+1} B_m^{\neq k(j)} + B_{k,m} \leq Q_m^c\} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W}. \quad (4.27)$$

The first two moments of the waiting time due to consolidation are

$$E[W_{k,m}^c] \simeq \sum_{z=0}^{\infty} P\{n_m^{\neq k} = z, n_k = 1\}(\frac{1}{z+1}(\sum_{s=1}^{z+1} sE[C_m^*]) \qquad (4.28)$$

$$E[(W_{k,m}^c)^2] \simeq \sum_{z=0}^{\infty} P\{n_m^{\neq k} = z, n_k = 1\}(\frac{1}{z+1}(\sum_{s=1}^{z+1} s^2 E[(C_m^*)^2])). \qquad (4.29)$$

2. Last order leaves with next truck.
   In this case we get for equation (4.16)

$$P\{n_m^{\neq k} = z, n_k = 1\} \simeq P\{\sum_{j=1}^{z} B_m^{\neq k(j)} + B_{k,m} + B_m^* \leq Q_m^c\}$$
$$- P\{\sum_{j=1}^{z+1} B_m^{\neq k(j)} + B_{k,m} + B_m^* \leq Q_m^c\} \qquad \forall k \in \mathcal{L}_m, m \in \mathcal{W} \quad (4.30)$$

The first two moments of the waiting time due to consolidation are

$$E[W_{k,m}^c] \simeq \sum_{z=0}^{\infty} P\{n_m^{\neq k} = z, n_k = 1\}(\frac{1}{z+1}(\sum_{s=1}^{z} sE[C_m^*] + (\frac{E[B_m^*]}{Q_m^c})E[C_m^*])) \qquad (4.31)$$

$$E[(W_{k,m}^c)^2] \simeq \sum_{z=0}^{\infty} P\{n_m^{\neq k} = z, n_k = 1\}(\frac{1}{z+1}(\sum_{s=1}^{z} s^2 E[(C_m^*)^2] + (\frac{E[B_m^*]}{Q_m^c})^2 E[(C_m^*)^2])). \qquad (4.32)$$

Chapter 5

A multi-item inventory model with production scheduling

The content of this chapter is joint work with M. Wagner and A.G. de Kok and has appeared in Smits, Wagner and de Kok (2002), Wagner and Smits (2002) and Smits, Adan and de Kok (2002)

5.1 Introduction

In Chapter 1, we observed that the three endogenous elements of the replenishment lead-time are: the waiting time due to a lack of stock at the preceding stockpoint, the waiting time due to shipment consolidation and the waiting time due to production. In Chapter 3, we derived expressions for the first two moments of the waiting time due to a lack of stock in a single item, multi-echelon distribution system. In Chapter 4, we derived expressions for the waiting time due to shipment consolidation in a multi-item, multi-echelon distribution system with shipment consolidation.

The objective of this chapter is to derive expressions for the waiting time due to production.

In the literature and in practice different production processes exist. The literature distinguishes among four broad classes of processes: job shop, batch flow, assembly line and continuous process industry. Job shops typically manufacture customized products such as circuit boards. Batch flow processes manufacture mass products. Assembly line products are for example automobiles and continuous process industry include chemicals. Products produced by job shops and assembly lines are typical make-to-order products, whereas products produced by batch flow processes and continuous process industry are typical make-to-stock products. Further the literature distinguishes between multi- and single stage production processes. The production process considered in this chapter is a single stage batch flow process.

In Chapter 2, we observed that the existing literature in this area is limited. Therefore, we will first consider a multi-item, single echelon model with production and without shipment consolidation. The considered logistic system consists of multiple stockpoints, one for each item, and a single production facility, which produces with limited production capacity and (significant) setup times the replenishment orders released by the stockpoints. In contrast to chapters 3 and 4, we assume that the stockpoints are controlled by periodic (R,S) installation stock policies. The periodic review policies are in production situations preferred to continuous policies because it allows a better control of the workload at the production facility. At the stockpoints the customer demand arrives according to a compound renewal process. The replenishment orders released by the stockpoints are produced according to FIFO (First In, First Out) discipline. The production sequence is cyclic and follows from the review period of each stockpoint. The replenishment lead-time consists of an exogenous delay and a waiting time due to production. The waiting time due to production can be split up in three parts: waiting time in the queue, setup time and production time.

In Section 5.2 we give a more detailed description of the model and we derive approximations for the first two moments of the replenishment lead-time and the performance measures of the stockpoints. To be able to determine the first two moments of the waiting time in the queue, the production process is modeled as a queueing model. Since the arrival process is deterministic and cyclic, the queueing model is a $D/G/1$ queue with cyclic arrivals. Therefore, in Section 5.2 we present an algorithm to determine the first two moments of the waiting time in a $D/G/1$ queue with cyclic arrivals. In Section 5.4 we present a numerical example. In Section 5.3 we develop a heuristic to find close-to-optimal solutions for R and the production schedule in terms of costs. In Section 5.5 we present a numerical analysis. In this numerical analysis we first test the approximations for the first two moments of the waiting time in queue with discrete event simulation and then we test the performance of the heuristic by comparing it with the optimal solution.

5.2 Model description

In this section a method is presented to evaluate for a given production schedule the replenishment lead-time and performance measures. Before presenting the method, we will describe the model in detail and introduce the notation used.

5.2.1 The model

We consider a system with a stockpoint for each of the M items and a single production facility, which produces all the items. At the stockpoints demand arrives according to a compound renewal process. The inter-arrival time at stockpoint k is denoted as A_k and the demand size is denoted as D_k ($k \in \mathcal{M}$). In Figure 5.1 a schematic representation of the model is given for M=4.

The stockpoints are controlled by periodic order-up-to policies, where R_k is review period of stockpoint $k \in \mathcal{M}$ and S_k is the order-up-to level of stockpoint $k \in \mathcal{M}$. The periodic review policies are preferred to continuous policies in production situations because it allows a better control of the workload at the production facility. In this section, we assume that the review periods and the release schedule are given. The release schedule indicates at which epoch the production orders of each item are placed at the production facility. Similarly to Chapter 3, we assume

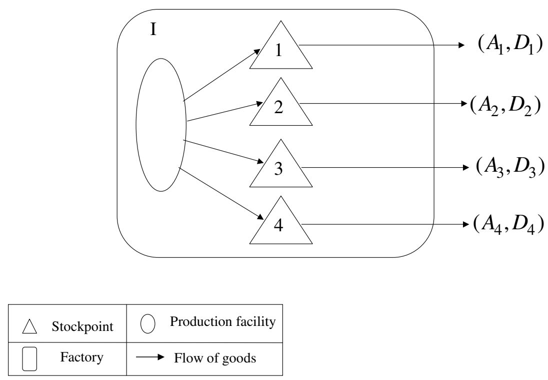


Figure 5.1: Schematic representation of the model.

that shortages are backordered and partial deliveries to the customers are allowed. Later in Section 5.3 we present a local search heuristic to optimize the review periods and the release schedule. The release schedule is determined by t_k^0 and R_k $(k \in \mathcal{M})$. t_k^0 is defined as the first time item k is setup. R_k is assumed to be equal to $m_k T^{BP}$ where T^{BP} is called the basic period and m_k is an integer. Given T^{BP} and m_k $(k \in \mathcal{M})$, the release schedule has a fixed cyclic sequence with a total cycle of duration $T = T^{BP} m_c$, where m_c is the least common multiple of m_k $(m_c = \max_{k \in \mathcal{M}} m_k)$. In Figure 5.2 a schematic representation of a schedule is given for four different stockpoints.

The number of orders placed at the production facility during a total cycle is $N = m_c$. The different orders are denoted with index i, $i = 1, ..., N$. We define $v(i)$ as the kind of item released by order i. In the example in Figure 5.2, we get $v(1) = 2$, $v(2) = 1$, $v(3) = 2$, $v(4) = 3$ and $v(5) = 4$. During the cycle, the stockpoint of item k places $J^k = \frac{N}{m_k}$ orders at the production facility during the time period T. In the example in Figure 5.2, we get $J^1 = 1$, $J^2 = 2$ and $J^3 = 1$ and $J^4 = 1$. Further, we define $f(i)$ as a function indicating which order of an item is placed at i. In the

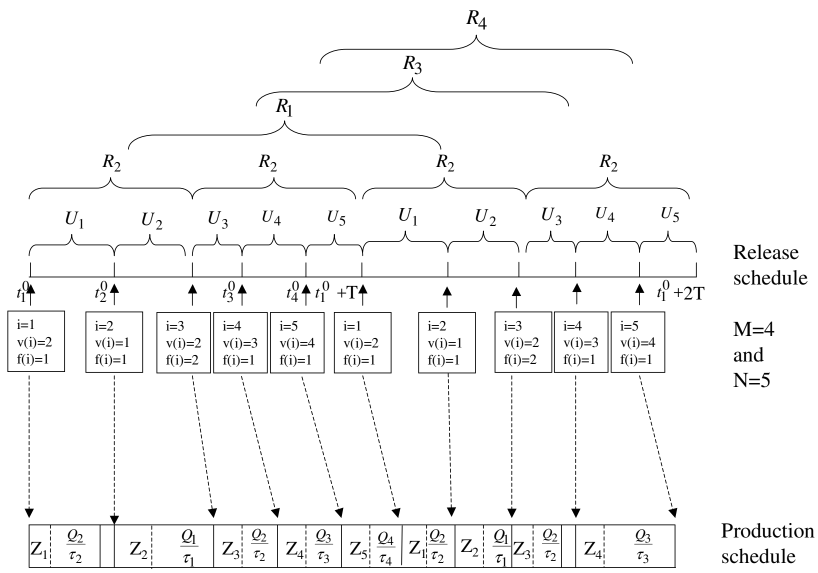


Figure 5.2: Schematic representation of the production schedule.

example in Figure 5.2, we get $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 1$ and $f(5) = 1$.

We define U_i as the time between the placement of order i and $i+1$ and U_N between N and order 1. U_i is deterministic and can be derived from t_k^0 and R_k ($k \in \mathcal{M}$). Note that $\sum_{i=1}^{N} U_i = T$.

The size of the order i is denoted by $Q_{v(i)}$, which is a stochastic variable. We assume that the probability that $Q_{v(i)}$ is zero is zero. If an order of item $v(i)$ is released at t, the size of the released order is equal to the demand arriving at the stockpoint between $t - R_{v(i)}$ and t, i.e. $Q_{v(i)} = D(t - R_{v(i)}, t]$. The released order, is produced on the production facility with a continuous rate, $\tau_{v(i)}$. Further, we assume batch availability at the end of the production run.

If the production of order i is finished before order $i+1$ is placed then the production facility is idle until order $i+1$ arrives. As soon as order $i+1$ is placed the production facility can be setup to produce order $i+1$. The setup time of order i is denoted as Z_i. Note that the setup time is sequence dependent and stochastic. Since, we assumed that $P\{Q_{v(i)} = 0\} = 0$, the probability that there is no demand during an interval R_k is also zero. However, in practice this can occur, in this case a (R_k, s_k, S_k)-inventory policy should be used. Unfortunately, the analysis derived in this chapter is not applicable anymore because U_i is no longer deterministic.

However, if the production facility is producing when order $i+1$ is placed then it has to wait until the production facility is idle. As soon as the production facility has finished with the production of all earlier orders, it is setup to produce order $i+1$. The time order i has to wait is called the waiting time in the queue and is denoted as W_i^q. Approximations for the first two moments of W_i^q are derived in Section 5.2.2.

The replenishment lead-time includes an exogenous delay $L_{v(i)}^d$, the waiting time in the queue W_i^q, the setup time Z_i and the production time $W_{v(i)}^f = \frac{Q_{v(i)}}{\tau_{v(i)}}$. The exogenous delay includes the time needed to administrate incoming orders, the time needed to handle the order in the warehouse and the time needed for external transport from the warehouse to the delivery point. We assume that the exogenous delay is independent of the waiting time in the queue, the setup time and the production time.

$$L_{v(i),f(i)} = L_{v(i)}^d + W_i^q + Z_i + W_{v(i)}^f \tag{5.1}$$

Since the waiting time in the queue is order dependent, the replenishment lead-times ($L_{v(i),f(i)}$) are also order dependent.

In Section 5.2.3, we derive expressions for $\beta_{v(i)}$ and $E[I_{v(i)}]$. Similarly to Chapter 3 these expressions can be used to evaluate the order-up-to level for a given β_k^{target} or $E[I_k]$ ($i = 1, ..., N$). Below, we summarize the notation used.

Inventory control parameters.

R_k	review period of item k
S_k	Order-up-to level of item k

The following deterministic variables and parameters are assumed to be given.

N	number of orders placed during a total cycle
M	number of items
$v(i)$	function indicating which item is ordered at i, $(i = 1, ..., N)$
J^k	number of orders for item $k \in \mathcal{M}$ placed during a total cycle
$f(i)$	function indicating which order of item k is ordered at i, $(i = 1, ..., N)$
τ_k	production rate of the item of stockpoint $k \in \mathcal{M}$
SC_k	setup costs for the production of the item of stockpoint k, $k \in \mathcal{M}$
HC_k	holding costs of the item of stockpoint $k \in \mathcal{M}$
U_i	time between the placement of order i and $i+1$, $i = 1, ..., N-1$
U_N	time between the placement of order N and 1
t_k^0	epoch at which the first order of stockpoint k is setup
T^{BP}	basic period
T	length of the total cycle
ϵ	a small positive value
φ	a large positive value

The following stochastic variables are assumed to be given.

D_k	demand size at stockpoint k, $k \in \mathcal{M}$
A_k	interarrival time between item orders at the stockpoint k, $k \in \mathcal{M}$
L_k^d	Exogenous delay of replenishment orders at stockpoint k, $k \in \mathcal{M}$
Z_i	setup time of production order i , $i = 1, ..., N$

The following variables are unknown.

$E[I_k]$	long-run average physical inventory of stockpoint $k \in \mathcal{M}$
ρ	traffic intensity or utilization degree of the production facility

The following random variables are unknown.

Q_i	size of the production order i, $i = 1, ..., N$
W_i^q	waiting time of order i in the queue, $i = 1, ..., N$
W_k^f	production time of item order $k = 1, ..., M$
$L_{v(i),f(i)}$	replenishment lead-time of order $f(i)$ for item $v(i)$, $i = 1, ..., N$

$I^b_{v(i),f(i)}$	net stock of item $v(i)$ just after production order $f(i)$ arrives, $i = 1, ..., N$
$I^e_{v(i),f(i)}$	net stock of item $v(i)$ just before production order $f(i) + 1$ arrives, $i = 1, ..., N$
B_i	service time of order i, $i = 1, ..., N$, which is defined as the production time plus the setup time. $B_i = Z_i + W^f_{v(i)}$

5.2.2 Determination of the replenishment lead-time

In this section we derive expressions for the first two moments of the replenishment lead-time. This is done as follows. In a first step we evaluate the first two moments of the service time, B_i, of order i ($i = 1, ..., N$), which is the sum of the waiting time due to production and the setup time and in a second step an algorithm is described to evaluate the first two moments of the waiting time in the queue, (W^q_i ($i = 1, ..., N$)).

1. Expressions for the first two moments of B_i ($i = 1, ..., N$)

   The service time of order i is equal to $\frac{Q_{v(i)}}{\tau_{v(i)}} + Z_i$ where $Q_{v(i)}$ is equal to the demand during a review period $R_{v(i)}$ denoted with $D(R_{v(i)})$.

   The average and variance of $Q_{v(i)} = D(R_{v(i)})$ can be derived from asymptotic relations in renewal theory, since the inventory inspection moments are independent of the interarrival process at the stockpoint. The inspection of the inventory and the counting process starts at an arbitrary point in time. This means that the demand process is stationary and the time until the next customer arrival is distributed according to the residual lifetime, see Section 2.2.1. We get the following formulae for the first two moments of the demand during the review period.

$$E[D(R_{v(i)})] \simeq \left(\frac{R_{v(i)}}{E[A_{v(i)}]}\right) E[D_{v(i)}] \qquad i = 1, .., N \tag{5.2}$$

   and

$$E[D(R_{v(i)})^2] \simeq \left(\frac{R_{v(i)}}{E[A_{v(i)}]}\right)\sigma^2(D_{v(i)}) + \left[\frac{R^2_{v(i)}}{E[A^2_{v(i)}]} + R_{v(i)}\left(\frac{E[A^2_{v(i)}]}{E[A_{v(i)}]^3} - \frac{1}{E[A_{v(i)}]}\right) + \frac{E[A^2_{v(i)}]^2}{2E[A_{v(i)}]^4} - \frac{E[A^3_{v(i)}]}{3E[A_{v(i)}]^3}\right] E[D_{v(i)}]^2 \qquad i = 1, .., N \tag{5.3}$$

The average and variance of service time B_i is:

$$E[B_i] = \frac{E[D(R_{v(i)})]}{\tau_{v(i)}} + E[Z_i] \qquad i = 1, .., N \tag{5.4}$$

$$\sigma^2(B_i) = \frac{\sigma^2(D(R_{v(i)}))}{\tau^2_{v(i)}} + \sigma^2(Z_i) \qquad i = 1, .., N \tag{5.5}$$

2. Algorithm to evaluate the first two moments of W_i^q

   The production process can be modeled as a queueing model. The arrival process is defined as the placement of orders to be produced at the production facility. The time between the placement of production lot i and $i+1$ is U_i, which is deterministic. The service time of order i is B_i. Since the arrival time is deterministic and cyclic the queueing model is a cyclic $D/G/1$ system.

   For continuous-time GI/G/1 queueing systems, de Kok (1989) presents a simple and accurate algorithm, the moment-iteration method, for the determination of the waiting time characteristics by using Lindley's equation. In the situation with one order ($N = 1$), we have a GI/G/1 queue and we can apply this algorithm as follows. If j denotes the time between the arrival of customer j and $j+1$, the following recurrence relation can be derived

$$W^q_{1,j+1} = (W^q_{1,j} + B_{1,j} - A_{1,j})^+ \tag{5.6}$$

   with $W^q_{1,j}$ the waiting time of the order in the j-th cycle ($j = 1, 2, 3, ...,$). The service time of the order in the j-th cycle, $B_{1,j}$ ($j = 1, 2, 3, ...,$) as well as the interarrival times of the order in the j-th cycle, $A_{1,j}$ $(j = 1, 2, 3, ...,)$ are identically and independently distributed with known averages and variances. The algorithm calculates the waiting time characteristics of an arbitrary customer.

When $N > 1$ then each order $i = 1, ..., N$ has its own service time B_i and interarrival time U_i. Therefore the waiting time of each order $i = 1, ..., N$ during a total cycle is different. Using the recurrence relation in this situation, we can write the following set of recursive equations.

$$W^q_{i+1,j} = (W^q_{i,j} + B_{i,j} - U_i)^+ \qquad \forall i = 1, .., N-1 \tag{5.7}$$

and

$$W^q_{1,j+1} = (W_{N,j} + B_{N,j} - U_N)^+ \tag{5.8}$$

where i and j refer to the i-th order in the j-th cycle, respectively. We assume that for each $i = 1, .., N$, $B_{i,j}$ $(j = 1, 2, 3...)$ is identically and independently distributed with known average $E[B_i]$ and variance $\sigma^2(B_i)$. U_i is deterministic. Under the assumptions that the queue is in a stationary state, these recursive equations can be translated into Lindley's integral equation. The algorithm of de Kok (1989) with cyclical arrivals will approximate the following performance characteristics of the D/G/1 queue. First we consider the probability that the waiting time is positive for $j \to \infty$.

$$\pi_i := \lim_{j\to\infty} P\{W^q_{i,j} > 0\} \tag{5.9}$$

Second, we consider the n-th moment of the waiting time in the queue for $j \to \infty$.

$$E[(W^q_i)^n] := \lim_{j\to\infty} E[(W^q_{i,j})^n] \tag{5.10}$$

These limits exist if the total traffic intensity, ρ, is smaller than 1.

$$\rho = \frac{\sum_{i=1}^{N} E[B_i]}{\sum_{i=1}^{N} E[U_i]} < 1 \tag{5.11}$$

Equation (5.6) relates the waiting time in the queue of an order to the waiting time in the queue of the previous order. From this equation, we get the following expressions for n-th moment of $W^q_{i,j}$.

$$E[(W^q_{i+1,j})^n] = \int_{U_i}^{\infty} (W^q_{i,j} + B_{i,j} - U_i)^n dF_X(x) \qquad i = 1, ..., N-1, j = 1, 2, .. \tag{5.12}$$

$$E[(W^q_{1,j+1})^n] = \int_{U_N}^{\infty} (W^q_{N,j} + B_{N,j} - U_N)^n dF_X(x) \qquad i = N, j = 1, 2, .. \tag{5.13}$$

Here we concentrate on the first two moments (so n=1,2). If the first two moments $W^q_{i,j} + B_{i,j}$ of the previous order are known and we fit a mixed-Erlang distribution to these first two moments, then the above expression with the fitted distributions can be used to compute the first two moments of the waiting time in the queue of the present order. This procedure is then repeated for the next order and so on. The resulting iteration scheme is presented below.

Iteration scheme

(a) Initially set ϵ to a small positive number, $j := 1$, $E[W^q_{i,j}] := 0$ and $E[(W^q_{i,j})^2] := 0$ for $i = 1, ..., N$ and $u := N$.

(b) Fit a mixed-Erlang distribution to $E[W^q_{u,j} + B_{u,j}]$ and $E[(W^q_{u,j} + B_{u,j})^2]$, see Section 2.2.2.

(c) If $u + 1 > N$ then $u := 0$. Compute $E[W^q_{u+1,j}]$ and $E[(W^q_{u+1,j})^2]$ from equations (5.12) and (5.13).

(d) If $u+1<N$ then $u:=0$ and we go to Step (b)

If $u+1=N$ then

begin

If $\sum_{s=1}^{N}|E[W^q_{s,j-1}]-E[W^q_{s,j}]|<\epsilon$

and $\sum_{s=1}^{N}|E[(W^q_{s,j-1})^2]-E[(W^q_{s,j})^2]|<\epsilon$

then we stop and have approximations for $E[W^q_i]$ and $E[(W^q_i)^2]$ for $i=1,...,N$ else $j:=j+1$ and we go to Step (b).

end.

The performance of this algorithm is tested in Section 5.5.1, in Smits, Wagner and de Kok (2002) and Smits, Adan and de Kok (2002). In Smits, Adan and de Kok (2002) also exact expressions for the first two moments of the waiting time in the queue for Erlang distributed service and interarrival times are derived and the results of the algorithm are also compared with exact results.

By substituting the results for the first two moments of the waiting time due to production into Formula (5.1) we find expressions for the first two moments of L_k for all $k \in \mathcal{M}$.

5.2.3 Determination of the fill rates and the physical inventory levels

In the previous section expressions for the first two moments of the replenishment lead-time were derived. Given the replenishment lead-time, the demand process, the review period and the order-up-to level at stockpoint $v(i)$, we can derive expressions for the fill rate and the long-run physical average inventory level. Periodically the stockpoint is inspected and an order is placed to raise the inventory position to the order-up-to level. Figure 5.3, gives a schematic representation of the evolution of the inventory position and the net stock.

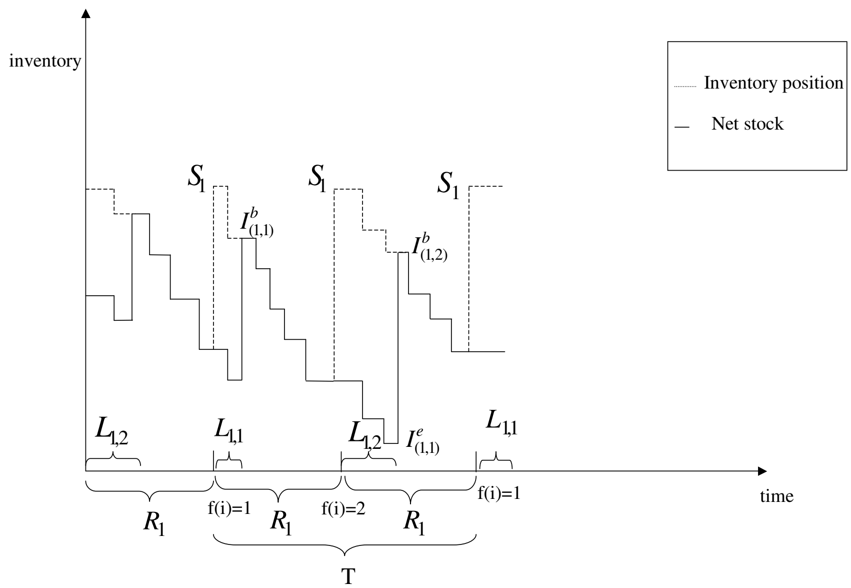


Figure 5.3: Evolution of the inventory position and the net stock

In Section 5.2, we defined $S_{v(i)}$ as the order-up-to level of item $v(i)$. $I^b_{v(i),f(i)}$ is the net stock at stockpoint $v(i)$ just after production order $f(i)$ arrives. $I^e_{v(i),f(i)}$ is the net stock at stockpoint $v(i)$ just before production order $(f(i)modN)+1$ arrives.

$$I^b_{(v(i),f(i))} = S_{v(i)} - D(L_{v(i),f(i)}) \tag{5.14}$$

$$I^e_{(v(i),f(i))} = S_{v(i)} - D(R_{v(i)} + L_{v(i),f(i)+1}) \tag{5.15}$$

Since the demand process and the replenishment lead-time at each stockpoint are known, we can derive the following expression for the fill rate of item $v(i)$.

$$\beta_{v(i)} = \frac{\sum_{j=1}^{J^{v(i)}} R_{v(i)}\left(1 - \frac{E[(-I^{e}_{(v(i),j)})^{+}]-E[(-I^{b}_{(v(i),j)})^{+}]}{E[D(R_{v(i)})]}\right)}{T} \tag{5.16}$$

For a proof see Chapter 3. To compute (5.16), we need expressions for the first two moments of $D(R_{v(i)} + L_{v(i),f(i)+1})$ and $D(L_{v(i),f(i)})$. Since for the demand during $L_{v(i),f(i)}$ the counting starts at an arbitrary point in time which is independent of the arrival process at the stockpoint, the average and variance of $D(L_{v(i),f(i)})$ can be derived from asymptotic relations in renewal theory, see Section 2.2.1 The asymptotic relation can be used under the condition that $E[L_{v(i),f(i)}] \gg E[A_{v(i)}]$.

$$E[D(L_{v(i),f(i)})] \simeq \left(\frac{E[L_{v(i),f(i)}]}{E[A_{v(i)}]}\right) E[D_{v(i)}] \tag{5.17}$$

$$\begin{aligned} E[D(L_{v(i),f(i)})^2] \simeq & \left(\frac{E[L_{v(i),f(i)}]}{E[A_{v(i)}]}\right)\sigma^2(D_{v(i)}) + \Bigg(\frac{E[L^2_{v(i),f(i)}]}{E[A^2_{v(i)}]} + E[L_{v(i),f(i)}] \\ & \left(\frac{E[A^2_{v(i)}]}{E[A_{v(i)}]^3} - \frac{1}{E[A_{v(i)}]}\right) + \frac{E[A^2_{v(i)}]^2}{2E[A_{v(i)}]^4} - \frac{E[A^3_{v(i)}]}{3E[A_{v(i)}]^3}\Bigg)E[D_{v(i)}]^2 \end{aligned} \tag{5.18}$$

Since $D(L_{v(i),f(i)}) = D(W^q_{v(i),f(i)+1} + Z_i + \frac{D(R_{v(i)})}{\tau_{v(i)}})$, $D(R_{v(i)})$ and $D(L_{v(i),f(i)+1})$ are dependent. Therefore, the first two moments of $D(R_{v(i)} + L_{v(i),f(i)+1})$ are more difficult to derive.

$$\begin{aligned} E[D(R_{v(i)} + L_{v(i),f(i)+1})] = E[D(R_{v(i)})] + E[D(W^q_{v(i),f(i)+1} + Z_i \\ + \frac{E[D(R_{v(i)})]}{\tau_{v(i)}})] \end{aligned} \tag{5.19}$$

$E[D(W^q_{v(i),f(i)+1} + Z_i)]$ can be calculated in a similar manner as $E[D(L_{v(i),f(i)+1})]$. The second moment of $D(R_{v(i)} + L_{v(i),f(i)+1})$ is as follows:

$$
\begin{aligned}
E[D(R_{v(i)} + L_{v(i),f(i)+1})^2] &= E[D(R_{v(i)})^2 + D(L_{v(i),f(i)+1})^2 \\
&\quad + 2D(R_{v(i)})D(L_{v(i),f(i)+1})] \\
&= E[D(R_{v(i)})^2] + E[D(L_{v(i),f(i)+1})^2] + \\
&\quad 2E[D(R_{v(i)})D(W^q_{v(i),f(i)+1} + Z_i + \frac{D(R_{v(i)})}{\tau_{v(i)}})] \qquad (5.20)
\end{aligned}
$$

To evaluate (5.20) we apply the following Lemma 5.1 where $X = D(R_{v(i)})$ and $Y = W^q_{v(i),f(i)+1} + Z_i$

Lemma 5.1 *Given the demand process $(A_{v(i)}, D_{v(i)})$, the constant production rate $\tau_{v(i)}$ and the random variable X and Y then*

$$
E[XD(Y + \frac{X}{\tau_{v(i)}})] \simeq \left(\frac{E[X]E[Y]E[D_{v(i)}]}{E[A_{v(i)}]} + \frac{E[X^2]E[D_{v(i)}]}{\tau_{v(i)}E[A_{v(i)}]} \right). \qquad (5.21)
$$

We define $D(Y + \frac{X}{\tau_{v(i)}})$ as the demand arriving during $Y + \frac{X}{\tau_{v(i)}}$,

Proof.

$$
\begin{aligned}
&E[XD(Y + \tfrac{X}{\tau_{v(i)}})] \\
&= \int_0^\infty x \left(E[D(Y)] + D(\tfrac{x}{\tau_{v(i)}}) \right) dF_X(x) \\
&\simeq \int_0^\infty x \left(E[D(Y)] + \tfrac{x}{\tau_{v(i)}} \tfrac{E[D_{v(i)}]}{E[A_{v(i)}]} \right) dF_X(x) \\
&\simeq \left(\tfrac{E[X]E[Y]E[D_{v(i)}]}{E[A_{v(i)}]} + \tfrac{E[X^2]E[D_{v(i)}]}{\tau_{v(i)}E[A_{v(i)}]} \right)
\end{aligned}
$$

For the long-run physical average inventory level at stockpoint $v(i)$, we use an approximation derived in Silver et al. (1998).

$$
E[I_{v(i),f(i)}] \simeq \frac{E[(I^b_{(v(i),f(i))})^+] + E[(I^e_{(v(i),f(i))})^+]}{2} \qquad (5.22)
$$

$$E[I_{v(i)}] \simeq \frac{\sum_{j=1}^{J^{v(i)}} R_{v(i)} E[I_{v(i),j}]}{T} \tag{5.23}$$

For the computation of $E[(I^b_{(v(i),f(i))})^+]$ and $E[(I^e_{(v(i),f(i))})^+]$, we refer to Section 2.2.2.

Previously mentioned equations for the long run fill rate and long run average inventory level can also be used to evaluate the order-up-to level corresponding to a target performance measure, see Chapter 3.

5.3 Local Search algorithm

The objective of the stochastic economic lot scheduling problem (SELSP) is to minimize the sum of the long-run expected setup and inventory costs while satisfying customer service requirements defined by a target fill rate. The decision variables are R_k and t_k^0 ($k \in \mathcal{M}$).
The optimization problem of the SELSP is given by:

$$\begin{aligned}
&\min_{R_1,\ldots,R_M;t_1^0,\ldots,t_M^0} \sum_{k=1}^{M} \left(\frac{SC_k}{R_k} + HC_k E[I_k] \right) \qquad (5.24)\\
&Subject \quad to\\
&\beta_k = \beta_k^{target} \qquad k \in \mathcal{M}\\
&R_k = m_k T^{BP} \qquad k \in \mathcal{M} \qquad and \qquad m_k \in \mathbb{N}\\
&T^{BP} \max_{k \in \mathcal{M}} m_k = T
\end{aligned}$$

We define TC as the total costs, this results in

$$TC = \sum_{k=1}^{M} \left(\frac{SC_k}{R_k} + HC_k E[I_k] \right) \tag{5.25}$$

The first part of (5.24) calculates the setup costs per period (SC_k are the fixed setup costs for item k ($k \in \mathcal{M}$)) and the second part the costs for holding inventory

(HC_k denotes the holding costs per time unit for item k ($k \in \mathcal{M}$)). The average inventory $E[I_k]$ can be computed from equation (5.23).

The stochastic economic lot scheduling problem is NP-hard (Sox et al. (1999)), which makes it unlikely that optimal solutions can be found in a reasonable amount of time. Therefore, we assume from now that the setups are sequence independent. Further, we develop a heuristic to find a close to optimal solution within a reasonable computation time.

To facilitate the construction of a schedule, we restrict the range of values for m_k to power-of-two values, $m_k \in \{2^0, 2^1, 2^2, ...\}$. This constraint is introduced by Haessler (1979) to facilitate the scheduling problem (but it doesn't guarantee the feasibility of the scheduling problem).

The heuristic uses a local search algorithm to find close to optimal solutions for m_k ($k \in \mathcal{M}$). For more details on local search algorithms we refer to Aarts and Lenstra (1997). The local search algorithm starts with an initial solution which is generated by Algorithm 1 from a relaxed formulation of the problem. Given this initial solution for m_k, we determine T^{BP} with Algorithm 2 by using a modified golden rule search. Given m_k and T^{BP}, we determine U_i with Algorithm 3 which is an algorithm proposed by Tunasar and Rajgopal (1996). Finally, given m_k, T^{BP} and U_i, we can determine the total costs of this solution by using the formulae in Section 5.2. The local search heuristic then searches for a better solution in the neighborhood of the current solution with Algorithm 4. If a better solution is found, it is accepted as the new current solution. This process is continued until no better solution is found, which terminates the search process. Thus, the local search heuristic uses four different algorithms.

Before describing the four algorithms, we define a lower bound for T^{BP}.

Lemma 5.2 *The lower bound of T^{BP} ensures, that all products can be produced according to their specific cycle. Therefore, the condition is a necessary but not sufficient constraint for feasibility.*

$$T^{BP} > \sum_{k \in \mathcal{M}} \frac{E[A_k]\tau_k m_k}{E[A_k]\tau_k m_k - E[D_k]} \sum_{k \in \mathcal{M}} \frac{E[Z_k]}{m_k} \tag{5.26}$$

Proof. According to the traffic intensity, ρ should be larger than 1. This results in,

$$\frac{\sum_{i=1}^{N} E[B_i]}{\sum_{i=1}^{N} E[U_i]} < 1$$

$$\sum_{i=1}^{N} E[B_i] < \sum_{i=1}^{N} E[U_i]$$

Per definition $\sum_{i=1}^{N} E[U_i] = T$ and $T = m_c T^{BP}$

$$\sum_{i=1}^{N} E[B_i] < m_c T^{BP}$$

We substitute equation 5.4 for $E[B_i]$

$$\sum_{k \in \mathcal{M}} \frac{m_c}{m_k}\left(\frac{E[D_k(T^{BP})]}{\tau_k} + E[Z_k]\right) < m_c T^{BP}$$

We rearrange the terms and obtain

$$T^{BP} > \sum_{k \in \mathcal{M}} \frac{1}{m_k}\left(\frac{E[D_k(T^{BP})]}{\tau_k} + E[Z_k]\right)$$

From renewal theory we know that $E[D_k(T^{BP})] = \frac{T^{BP}}{E[A_k]} E[D_k]$.

$$\left(1 - \sum_{k \in \mathcal{M}} \frac{E[D_k]}{E[A_k] m_k \tau_k}\right) T^{BP} > \sum_{k \in \mathcal{M}} \frac{1}{m_k} E[Z_k]$$

$$T^{BP} > \left(\sum_{k \in \mathcal{M}} \frac{E[A_k]\tau_k m_k}{E[A_k]\tau_k m_k - E[D_k]}\right) \sum_{k \in \mathcal{M}} \frac{E[Z_k]}{m_k}$$

In the following we will first describe the four algorithms and then we present the iteration scheme of the local search heuristic.

1. Algorithm 1 generates the initial multipliers m_k $(k \in \mathcal{M})$

   The initial multiples m_k $(k \in \mathcal{M})$ are generated from a relaxed formulation of the problem. In the relaxed problem W_i^q $(i = 1, ..., N)$ is equal to zero. Therefore this relaxed problem is independent of the schedule U_i $(i = 1, ..., N)$. Given this relaxed problem close to optimal values are found for R_k $(k \in \mathcal{M})$ with a modified golden rule search. For more details see Winston (1993). From numerous test cases we observe that TC is convex in R_k but due to the numerous approximations it is difficult or even impossible to prove the convexity analytically. Therefore, we apply a modified golden rule search. The modified golden rule search consist in first applying the standard golden rule search. We choose as starting values for the standard golden rule search

$R_k^{(1)} := E[A_k]$ and $R_k^{(2)} := \varphi E[A_k]$, where φ is a large value. For more details about the standard golden rule search we refer to Winston (1993). If during the search, we find that TC is not convex, we stop the search and determine the R_k by searching over the whole range by varying R_k by small steps. Given R_k $(k \in \mathcal{M})$, we evaluate $\frac{R_k}{\min_{j \in \mathcal{M}} R_j}$ for each $k \in \mathcal{M}$ and round-off to the closest power-of-two multiple to find m_k.

2. Algorithm 2 generates a close to optimal solution for T^{BP}

   Given m_k $(k \in \mathcal{M})$, this algorithm generates close to optimal solution for T^{BP}. Similarly to Algorithm 1, we use the modified golden rule search, because here also we observe from numerous test cases that TC is convex in T^{BP} and due to the numerous approximations it is difficult to determine the convexity analytically. However the start values are different to Algorithm 1. For the lower bound of T^{BP} we use Equation (5.26). Note that for each value of T^{BP} we evaluate U_i $(i = 1, ..., N)$ with Algorithm 3.

3. Algorithm 3 generates feasible values for U_i $(i = 1, ..., N)$

   Tunasar and Rajgopal (1996) propose an algorithm to determine a feasible schedule in deterministic economic lot scheduling problems under the assumption that m_k and T^{BP} are known and that m_k are power-of-two multiples The algorithm defines time buckets and assigns production orders of items to these buckets in ascending order of their multiples m_k $(k \in \mathcal{M})$. This is reasonable because the flexibility for scheduling is increasing for items which have to be produced less often in the total cycle. Similarly, the flexibility for items with low service times is higher and thus, items with the same multiple are sorted according to the service times. When assigning an item to a time bucket, the first production order of the item is assigned to the time bucket with most capacity remaining, in contrast to Tunasar and Rajgopal (1996), who assign the item to the first feasible time bucket found by the algorithm. The resulting iteration scheme is presented below.

(a) Calculate the least common multiple of m_k, $m_c = \max_{k \in \mathcal{M}} m_k$ and the complete rotation cycle, $T = T^{BP} m_c$.

(b) Define time buckets w_j $j = 1, ..., m_c$ of length T^{BP}

(c) Sort all items k in ascending order of their multiples m_k $(k \in \mathcal{M})$

(d) Sort all items k with equal multiples in ascending order of $\frac{\sum_{i \in \{j|v(j)=k\}} E[B_i]}{J^k}$ $k \in \mathcal{M}$

(e) Initially set $k := 1$

(f) Select item k and the bucket with most capacity remaining $w_{max} = \max_j w_j$ $j = 1, ..., m_c$

(g) Assign item k to this bucket, update the capacity usage $w_{max} := w_{max} - E[B_i]$ with $i \in \{j|v(j) = k, f(j) = 1\}$ and fix the first setup t_k^0 $(k \in \mathcal{M})$ for item k. If $w_{max} < 0$ then the algorithm stops and no feasible schedule is found.

(h) If $J^k > 1$ then item k is setup more than once during the total cycle. Therefore we need also to update the capacity for the other orders of item k. This results in assigning item k to $\frac{m_c}{m_k} - 1$ subsequent time buckets with intervening gaps of $m_k - 1$. After that we update the capacity usage and fix the setup time for item k. If $w_j < 0$ $(j = 0, ..., m_c)$ then the algorithm stops and no feasible schedule is found.

(i) k:=k+1. If $k < N$ then go to step (f) else the algorithm stops and a feasible schedule is found.

U_i can then be determined from the setup times t_i $(i = 1, ..., N)$. Our tests in Section 5 showed, that in most cases the procedure finds a feasible schedule if one exists. Nevertheless, it should be noted that the heuristic cuts off some solutions as "infeasible" which are feasible.

4. Algorithm 4 generates a new candidate for m_k $(k \in \mathcal{M})$.

We remember all the candidates for m_k $(k \in \mathcal{M})$. We assume that we have already generated n candidates for m_k $(k \in \mathcal{M})$ and we want to draw candidate

$n+1$. We define $m_k^{(l)}$ $l = 1,2,3,...,n$ $(k \in \mathcal{M})$ as the m_k of candidate l and i as the last modified item. The following iteration scheme is used to generate the new candidate $m_k^{(n+1)}$ $(k \in \mathcal{M})$.

(a) Initially set $j := i$

(b) $m_j^{(n+1)} := 2m_j^{(n)}$.

(c) If $(m_1^{(n+1)},...,m_M^{(n+1)}) \neq (m_1^{(l)},...,m_M^{(l)})$ $(l = 1,...,n)$
   then $m_k^{(n+1)}$ $(k \in \mathcal{M})$ is the new candidate and the algorithm stops
   else begin
      $m_j^{(n+1)} := \frac{1}{2}m_j^{(n)}$.
      If $(m_1^{(n+1)},...,m_M^{(n+1)}) \neq (m_1^{(l)},...,m_M^{(l)})$ $(l = 1,...,n)$
      then $m_k^{(n+1)}$ $(k \in \mathcal{M})$ is the new candidate and the algorithm stops
      else $j := j+1$.
   end.

(d) If $j > M$ then $j := 1$.

(e) If $j \neq i$
   then go to step (b)
   else the algorithm stops and no new candidate is found.

Iteration scheme of the Local search heuristic

1. Generate an initial solution for $m_k^{(1)}$ $(k \in \mathcal{M})$ with Algorithm 1.

2. Determine T^{BP} with Algorithm 2, U_i $(i = 1,...,N)$ with Algorithm 3 and $TC^{(1)}$.

3. Generate a new candidate $m_k^{(2)}$ $(k \in \mathcal{M})$ with Algorithm 4.

4. Determine T^{BP} with Algorithm 2, U_i $(i = 1,...,N)$ with Algorithm 3 and $TC^{(2)}$.

5. If $TC^{(1)} > TC^{(2)}$ then $TC^{(1)} := TC^{(2)}$, $m_k^{(1)} := m_k^{(2)}$ $(k \in \mathcal{M})$ and generate a new candidate $m_k^{(2)}$ $(k \in \mathcal{M})$ with Algorithm 4. If there are no new candidates stop the heuristic else go to Step 4.

   If $TC^{(1)} < TC^{(2)}$ then generate a new candidate $m_k^{(2)}$ $(k \in \mathcal{M})$ with Algorithm 4. If there are no new candidates stop the heuristic else go to Step 4.

In local search heuristics there is a trade-off between the quality of solution and the computation time. The presented heuristic is a so-called conventional local search heuristic which is fast. However, a shortcoming of this heuristic is that it terminates at the very first local optimum. Another shortcoming is that the final solution strongly depends on the choice of the initial solution. Other algorithms like simulated annealing, tabu search, genetic algorithms or variable depth search have a solution of better quality but have also a longer computation time. In Section 5.5.2, we test the quality of this local search heuristic and the computation time.

5.4 Numerical Example

In this section we illustrate the analytical approximations derived in Section 5.2 by a numerical example. We consider a logistic system, with four stockpoints denoted by $\{1, 2, 3, 4\}$ and one production facility. A schematic representation of the system is given in Figure 5.4.

Similarly to chapters 3 and 4 we assume that target fill rates β_k at $k \in \mathcal{M}$ are given. Further, we assume that the review periods (R_k) at $k \in \mathcal{M}$ and the release schedule (U_i) for $i = 1, ..., N$ are given. With the expressions given in Section 5.2 we compute the production order processes (Q_k), the waiting due in the queue W_i^q, replenishment lead-times and the order-up-to levels S_k at all stockpoints $k \in \mathcal{M}$ such that these target fill rates are met. In strategic and tactical models different alternatives are compared by means of costs. An important element in these costs is the inventory cost, which is computed from the long-run average inventory level. Therefore as a second performance measure we also compute the long-run average physical inventory levels $(E[I_k])$.

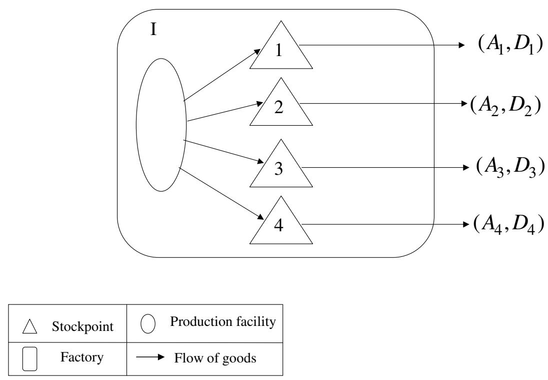


Figure 5.4: Schematic representation of the logistic system.

For the interarrival process and the order size process at the end-stockpoints we assume exponential distributions. Further, we assume that each product is produced once during a total cycle. This implies that for $k \in \mathcal{M}$ the review period (R_k) is equal to the length of the total cycle T. The length of the total cycle T is determined from $\sum_i^N U_i$. The production speed is chosen such that the traffic intensity is less than one. The input parameters of the numerical example are presented in Table 5.1.

The procedure to evaluate all the variables in the logistic system is as follows:

In a first step, we evaluate the first two moments of B_i $(i = 1, ..., N)$ with the analytical approximations presented in Section 5.2.2. In a second step, we compute the first two moments of W_i^q $(i = 1, ..., N)$. Given the first two moments of W_i^q we can determine the first two moments of the replenishment lead-times. In a third step, we evaluate the order-up-to levels such that the fill rates are met with help of a bisection rule and finally, we compute the average physical inventory levels. The results of the computations are presented in table 5.2.

In the following, we test the performance of the approximations for this numerical

Table 5.1: Input parameters of the numerical example.

k	1	2	3	4
$E[D_k]$	93	55	26	99
$\sigma^2(D_k)$	8628	3069	687	9786
$E[A_k]$	7.61	5.69	2.12	2.10
$\sigma^2(A_k)$	57.93	32.33	4.51	4.43
L_k^d	0	0	0	0
U_k	21.06	27.63	29.76	26.88
$E[Z_i]$	1	1	1	1
$\sigma^2(Z_i)$	1	1	1	1
τ_k	70.48	42.40	49.73	210.34
β_k	95 %	95 %	95 %	95 %

Table 5.2: Results of the numerical example.

k	1	2	3	4
$E[B_i]$	19.24	25.20	27.15	24.52
$\sigma^2(B_k)$	49.68	64.26	28.56	23.13
$E[W_k^q]$	5.42	5.76	6.11	5.81
$E[(W_k^q)^2]$	77.30	92.75	108.25	91.34
S_k	2222.44	1752.42	1916.48	712.06
$E[I_k]$	1310.79	963.24	855.59	3212.55

example by using discrete event simulations. The simulation runs until 1×10^6 customers have arrived at one of the stockpoints $k \in \mathcal{E}$ and we repeat this for 10 different seeds. The simulations length was determined such that accurate results were obtained for the performance characteristics. In Table 5.3 we present the results of these values obtained from the simulation. Similar to chapters 2, 3 and 4, the value between parentheses is the 95% confidence interval of the corresponding error.

Table 5.3: Results of the numerical example and simulations.

k	1	2	3	4
Numerical example				
$E[B_i]$	19.24	25.20	27.15	24.52
$\sigma^2(B_k)$	49.68	64.26	28.56	23.13
$E[W_k^q]$	5.42	5.76	6.11	5.81
$E[(W_k^q)^2]$	77.30	92.75	108.25	91.34
S_k	2222.44	1752.42	1916.48	712.06
$E[I_k]$	1310.79	963.24	855.59	3212.55
Simulations				
$E[B_i]$	19.18 (±0.02)	25.25 (±0.05)	27.12 (±0.08)	24.50 (±0.05)
$\sigma^2(B_k)$	47.81 (±0.27)	63.61 (±0.71)	27.32 (±0.89)	21.88 (±0.43)
$E[W_k^q]$	5.47 (±0.06)	5.89 (±0.04)	6.25 (±0.09)	5.93 (±0.03)
$E[(W_k^q)^2]$	87.61 (±1.87)	119.39 (±2.22)	124.55 (±4.69)	110.05 (±3.93)
β_k	0.950 (±0.001)	0.946 (±0.003)	0.944 (±0.003)	0.945 (±0.001)
$E[I_k]$	1268.92 (±4.11)	909.31 (±1.04)	840.77 (±5.34)	3169.80 (±16.63)

To evaluate the variables in Table 5.3 several approximations are made. In the following we recapitulate the different approximations.

1. To evaluate the first two moments of B_i $(i = 1, ..., N)$, we use an asymptotic approximation from renewal theory. Tijms (1994) shows that this approximation performs correctly under condition 2.13 p 44.

2. To evaluate the first two moments of W_i^q $(i = 1, ..., N)$, we approximate the distribution of $B_i - U_i$ $(i = 1, ..., N)$ by a mixed-Erlang distribution with the same first two moments.

3. To evaluate β_k and $E[I_k]$ $(k \in \mathcal{M})$, we approximate the distributions of

$D(L_{v(i),f(i)})$ and $D(R_{v(i)} + L_{v(i),f(i)})$ $(i = 1, ..., N)$ by mixed-Erlang distributions with the same first two moments.

When comparing all the results in tables 5.2 and 5.3, we observe that for this numerical example the errors are small. To be able to generalize this result, we need to test for numerous other cases the quality of the approximations.

5.5 Numerical analysis

5.5.1 Simulations

In this section we further investigate the analytical approximations derived in Section 5.2. This is realized by performing discrete event simulations. Similarly to Section 5.4, we assume given target fill rates and review period for all $k \in \mathcal{M}$ and a given release schedule. We compute the service times, the waiting times due to production, the replenishment lead-times and the order-up-to levels (S_k) at all stockpoints $k \in \mathcal{M}$ such that the target fill rate is met. Further, we also assume as a second performance measure the long-run average physical inventory level. After that we run a simulation of the distribution system with the given customer demands, exogenous delays, batchsizes and the calculated order-up-to levels.

We assume one production facility and M products which have to be produced on this production facility. The cyclic release schedule for the production of the different items on the production facility is given.

First, we assume a simple schedule where each item is produced once during the total cycle. After that we assume schedules where the items can be produced more than once during the total cycle. For the simple schedules, the number of items and orders is varied between 2, 5 and 25. For the schedules where items can be produced more than once during a common cycle, we assume a schedule with three items and four orders, where $v(1) = 1$, $v(2) = 2$, $v(3) = 1$ and $v(4) = 3$. U_3 is $U(10, U_1 + U_2)$ and U_4 is chosen such that $U_1 + U_2 - U_3 = U_4$, we will denote these simulations with 4^*.

In Section 5.4, the different approximations used to determine the waiting time due to production and the order-up-to level are recapitulated. By varying the following four input parameters, we can investigate the influence of the approximations in general.

1. Coefficient of variation of the inter-arrival times at $k \in \mathcal{E}$ ($c^2_{A_k}$).

2. Coefficient of variation of the demand sizes at $k \in \mathcal{E}$ ($c^2_{D_k}$).

3. The release schedule U_i ($i = 1, ..., N$).

4. The production rate τ_k ($k \in \mathcal{M}$)

We consider 15 different cases. For each case we perform 10 simulations with different seeds and the simulations stop after the arrival of 6×10^6 item orders at one of the stockpoints.

The item orders arrive at the stockpoint according to a compound renewal demand process. The interarrival time and the order size of the item orders are mixed Erlang distributed with known first two moments. The expected order size is randomly generated on the interval $(10, 100)$ and the expected interarrival time on the interval $(1, 10)$. The target fill rate at the stockpoints ($k \in \mathcal{M}$) is 95 % and the exogenous delay is 0. The average setup is 1 and the coefficient of variation of the setup time is 1. The production speed is chosen such that the traffic intensity is less than 1. The production rate is varied by the coefficient x where $\tau_k = \frac{E[D_{R_k}]}{U_k - E[Z_i]} x$. The values of the other parameters for the 15 different cases is presented in Table 5.4.

For each case, we calculated the error in the first two moments of the waiting time in the queue, the absolute error in the fill rate and the percentage error in long run average inventory level, since we observed in Section 5.4 that those are most critical (for references see Section 2.2.5). Similar to Section 5.4, the value between parentheses is the 95% confidence interval of the corresponding error.

The results of the simulations are summarized in table 5.5, 5.6 and 5.7 . In Table 5.8, 4* denotes the simulations with schedules where item 1 is produced twice during the total cycle.

Table 5.4: Parameter values, where $\tau_k = \frac{E[D_{R_k}]}{U_k - E[Z_i]} x$

case	U_i $(i = 1, .., N)$	x	$c^2_{D_k}$ $(k \in \mathcal{M})$	$c^2_{A_k}$ $(k \in \mathcal{M})$
1	$U(40, 50)$	1.1	1	1
2	$U(40, 50)$	1.1	0.2	1
3	$U(40, 50)$	1.1	2	1
4	$U(40, 50)$	1.1	0.2	0.2
5	$U(40, 50)$	1.1	1	0.2
6	$U(40, 50)$	1.1	2	0.2
7	$U(40, 50)$	1.1	0.2	2
8	$U(40, 50)$	1.1	1	2
9	$U(40, 50)$	1.1	2	2
10	$U(30, 40)$	1.1	1	1
11	$U(50, 60)$	1.1	1	1
12	$U(60, 70)$	1.1	1	1
13	$U(70, 80)$	1.1	1	1
14	$U(40, 50)$	1.2	1	1
15	$U(40, 50)$	1.3	1	1

Table 5.5: Results for $N = 2$.

Case	$\bar{\delta}^{(E[W_i^q])}$	$\bar{\delta}^{(E[(W_i^q)^2])}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$
1	3.68 (±1.38)	8.87 (±1.03)	1.43 (±0.08)	8.61 (±0.78)
2	2.17 (±2.09)	12.33 (±2.05)	1.21 (±0.11)	6.86 (±0.48)
3	3.83 (±1.92)	8.47 (±1.60)	1.52 (±0.10)	8.33 (±0.73)
4	4.54 (±1.72)	7.87 (±3.73)	1.59 (±0.12)	11.43 (±0.72)
5	8.48 (±1.50)	12.29 (±5.25)	1.89 (±0.52)	12.97 (±1.08)
6	8.49 (±2.66)	10.79 (±5.82)	1.57 (±0.07)	10.61 (±0.79)
7	7.66 (±2.81)	14.12 (±2.68)	0.23 (±0.11)	1.43 (±0.48)
8	5.02 (±1.59)	12.54 (±4.70)	1.13 (±0.15)	3.83 (±0.12)
9	11.45 (±3.25)	18.56 (±10.16)	1.26 (±0.31)	3.14 (±0.79)
10	3.90 (±3.22)	8.99 (±4.74)	1.42 (±0.16)	7.21 (±1.22)
11	2.85 (±2.34)	7.47 (±1.37)	1.57 (±0.09)	8.24 (±0.77)
12	5.36 (±2.68)	16.05 (±1.19)	0.81 (±0.14)	2.56 (±0.44)
13	4.18 (±2.70)	14.44 (±3.12)	0.73 (±0.13)	4.16 (±0.49)
14	2.88 (±1.61)	13.48 (±2.90)	0.50 (±0.07)	4.76 (±0.26)
15	0.57 (±0.86)	4.23 (±4.09)	0.39 (±0.09)	7.19 (±0.27)

Table 5.6: Results for $N = 5$.

Case	$\bar{\delta}^{(E[W_i^q])}$	$\bar{\delta}^{(E[(W_i^q)^2])}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$
1	2.89 (±4.00)	15.80 (±4.25)	0.18 (±0.01)	9.45 (±0.43)
2	6.00 (±3.45)	10.40 (±5.80)	0.20 (±0.16)	5.21 (±0.13)
3	2.96 (±3.35)	11.56 (±3.30)	0.54 (±0.18)	5.01 (±0.52)
4	7.03 (±0.59)	9.77 (±1.49)	0.18 (±0.07)	1.64 (±0.03)
5	5.58 (±4.08)	15.37 (±7.31)	0.32 (±0.16)	16.95 (±0.10)
6	7.68 (±1.04)	14.46 (±8.71)	0.44 (±0.16)	12.90 (±0.20)
7	4.19 (±2.63)	11.97 (±6.95)	0.11 (±0.16)	5.84 (±0.31)
8	5.32 (±3.15)	7.20 (±3.41)	0.12 (±0.13)	11.67 (±0.36)
9	5.51 (±2.48)	6.35 (±6.61)	0.25 (±0.14)	5.01 (±0.36)
10	2.85 (±1.67)	5.69 (±5.13)	0.22 (±0.09)	8.56 (±0.25)
11	3.55 (±3.66)	8.39 (±7.12)	0.13 (±0.06)	5.39 (±0.30)
12	8.86 (±1.05)	8.27 (±4.42)	0.11 (±0.04)	11.13 (±0.09)
13	4.36 (±2.24)	17.83 (±4.63)	0.14 (±0.02)	5.90 (±0.33)
14	7.58 (±1.97)	7.93 (±2.73)	0.09 (±0.10)	4.31 (±0.03)
15	7.45 (±6.78)	12.76 (±6.74)	0.10 (±0.06)	4.84 (±0.08)

Table 5.7: Results for $N = 25$.

Case	$\bar{\delta}^{(E[W_i^q])}$	$\bar{\delta}^{(E[(W_i^q)^2])}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$
1	13.90 (±13.00)	27.66 (±15.28)	0.21 (±0.26)	2.31 (±0.32)
2	15.71 (±11.79)	28.27 (±14.31)	0.12 (±0.23)	2.70 (±0.57)
3	9.17 (±3.22)	24.12 (±14.06)	0.21 (±0.29)	1.16 (±0.20)
4	13.27 (±10.04)	14.54 (±11.17)	0.15 (±0.08)	4.72 (±0.41)
5	19.65 (±14.96)	25.27 (±19.70)	0.26 (±0.25)	3.20 (±0.58)
6	10.58 (±7.22)	26.03 (±14.30)	0.16 (±0.12)	1.26 (±0.10)
7	8.54 (±7.29)	22.80 (±11.80)	0.17 (±0.21)	1.22 (±0.07)
8	9.30 (±5.42)	16.42 (±9.80)	0.20 (±0.32)	1.05 (±0.36)
9	7.72 (±5.54)	25.01 (±8.03)	0.26 (±0.29)	1.37 (±0.69)
10	8.76 (±1.05)	22.45 (±13.23)	0.23 (±0.19)	1.32 (±0.03)
11	18.35 (±8.92)	19.48 (±8.81)	0.20 (±0.31)	2.23 (±0.21)
12	17.69 (±6.57)	20.01 (±6.41)	0.28 (±0.30)	2.06 (±0.25)
13	15.25 (±6.67)	27.71 (±17.43)	0.21 (±0.35)	2.30 (±0.62)
14	16.85 (±8.47)	18.48 (±12.35)	0.23 (±0.27)	1.46 (±0.05)
15	11.87 (±3.00)	24.41 (±9.41)	0.19 (±0.31)	1.73 (±0.01)

Table 5.8: Results for $N = 4^*$.

Case	$\bar{\delta}^{(E[W_i^q])}$	$\bar{\delta}^{(E[(W_i^q)^2])}$	$\bar{\Delta}^{(\beta_k)}$	$\bar{\delta}^{(E[I_k])}$
1	7.19 (±3.92)	13.33 (±2.45)	0.35 (±0.05)	2.88 (±0.19)
2	4.10 (±2.16)	6.68 (±4.80)	2.08 (±0.04)	7.30 (±0.42)
3	5.96 (±1.12)	9.48 (±1.79)	1.00 (±0.04)	8.16 (±0.36)
4	9.31 (±2.60)	9.92 (±6.67)	0.87 (±0.03)	9.04 (±4.77)
5	1.51 (±0.13)	4.65 (±1.88)	0.44 (±0.04)	8.76 (±1.12)
6	6.89 (±1.03)	16.13 (±2.33)	0.43 (±0.06)	5.61 (±2.14)
7	11.64 (±1.52)	11.79 (±5.98)	0.17 (±0.03)	4.04 (±0.29)
8	5.95 (±1.09)	11.19 (±4.14)	1.30 (±0.76)	6.59 (±0.91)
9	2.62 (±1.02)	9.00 (±2.08)	0.60 (±0.08)	5.65 (±0.38)
10	10.29 (±1.34)	4.17 (±1.96)	2.08 (±1.45)	4.88 (±0.67)
11	3.31 (±1.94)	6.02 (±3.11)	0.45 (±0.11)	4.65 (±0.40)
12	8.50 (±2.88)	15.38 (±4.37)	1.68 (±0.00)	6.16 (±0.38)
13	12.30 (±1.99)	11.90 (±3.91)	2.08 (±0.56)	1.59 (±0.58)
14	13.26 (±1.51)	17.63 (±3.47)	1.51 (±0.03)	4.79 (±0.22)
15	8.55 (±1.93)	9.26 (±4.21)	0.95 (±0.01)	7.59 (±0.23)

We now investigate the errors in $E[W_i^q]$, $E[(W_i^q)^2]$, β_k and $E[I_k]$ ($k \in \mathcal{M}$ and $i = 1, ..., N$). For $N = 2$, $N = 5$ and $N = 4^*$, the errors in $E[W_i^q]$ and $E[(W_i^q)^2]$ ($i = 1, ..., N$) are within acceptable margins. For $N = 25$ the errors in $E[W_i^q]$ and $E[(W_i^q)^2]$ ($i = 1, ..., N$) are high, but also the confidence intervals are high. When $N = 25$ then B_k is nearly discrete and this causes $E[W_i^q]$ and $E[(W_i^q)^2]$ ($i = 1, ..., N$) to be close to 0. We observe that there is relation between $E[W_i^q]$ and $E[(W_i^q)^2]$ ($i = 1, ..., N$), the errors in these variables and there confidence intervals. When $E[W_i^q]$ and $E[(W_i^q)^2]$ ($i = 1, ..., N$) are small then the errors in these variables are large but also the confidence intervals are large. Note that this observation holds also for standard $G/G/1$ queues, cf. de Kok (1989). It is therefore difficult to determine the quality of the approximations for the case that the waiting times are small. In Smits, Adan and de Kok (2002), we also derived exact expressions for the first two moments of the waiting time in the queue for Erlang distributed service and interarrival times. This permitted us to determine correctly the quality of the approximations and it performed well for small W_i^q, with an average error for $E[W_i^q]$ of 6.97 % and for $E[(W_i^q)^2]$ of 12.88 % ($i = 1, ..., N$). For more details we refer to Smits, Adan and de Kok (2002). Further, the errors in $E[(W_i^q)^2]$ are higher than the errors in $E[W_i^q]$ $(i = 1, ..., N)$. This is also an observation that holds for standard $G/G/1$ queues, cf. de Kok (1989).

For $N = 2$ the errors in β_k and $E[I_k]$ $(k \in \mathcal{M})$ are relatively high. When $N = 2$ then R_k is small in comparison to $E[W_i^q]$ therefore the errors in $E[W_i^q]$ propagate to errors in β_k and $E[I_k]$. For $N = 5$, $N = 25$ and $N = 4^*$ the errors in β_k and $E[I_k]$ $(k \in \mathcal{M})$ are within acceptable margins.

In cases 1 to 9, we vary the coefficient of variation of $C^2_{D_k}$ and $C^2_{A_k}$, because we expect that the approximations are sensitive to different coefficients of variation. But in cases 1 to 9 in tables 5.5, 5.6, 5.7 and 5.8, we observe that there is not a clear relation between the value of the coefficients of variations of $C^2_{D_k}$ and $C^2_{A_k}$ and the errors in $E[W_i^q]$ and $E[(W_i^q)^2]$, β_k and $E[I_k]$ $(i = 1, ..., N$ and $k \in \mathcal{M})$.

In cases 10 to 13, we vary the length of U_i, because we expect that the approximations are sensitive to different length of U_i. However, we observe in cases 10 to 13 in tables 5.5, 5.6, 5.7 and 5.8 that there is not a clear relation between the length of U_i and the errors in $E[W_i^q]$ and $E[(W_i^q)^2]$, β_k and $E[I_k]$ $(i = 1, ..., N$ and $k \in \mathcal{M})$.

In cases 14 and 15, we vary the production speed τ_k, because we expect that the approximations are sensitive to different length of τ_k. However, we observe in cases 14 and 15 in tables 5.5, 5.6, 5.7 and 5.8 that there is not a clear relation between τ_k and the errors in $E[W_i^q]$ and $E[(W_i^q)^2]$, β_k and $E[I_k]$ $(i = 1, ..., N$ and $k \in \mathcal{M})$.

To conclude, nearly all results are within the acceptable margins and therefore the approximations are of sufficient quality for practical purposes.

5.5.2 Local search

In the section we will test the performance of the local search heuristic to determine the m_k $(k \in \mathcal{M})$. Note that in the analysis we restrict the values of m_k $(k \in \mathcal{M})$ to power-of-two values. We generate 25 random small test cases with 5 items. For both cases, we evaluate the optimal values for m_k $(k \in \mathcal{M})$ and the ones obtained with the local search algorithm. The values of the input parameters of the 25 randomly test cases are presented in 5.9.

The optimal values for m_k $(k \in \mathcal{M})$ are found by calculating for all possible combinations for $m_k \in \{1, 2, 4, ..., 1024\}$ the total costs.

Table 5.9: Parameter values

Parameter	Values
$E[D_k]$ $(k \in \mathcal{M})$	$U(10, 100)$
$c^2_{D_k}$ $(k \in \mathcal{M})$	$U(0.2, 2)$
$E[A_k]$ $(k \in \mathcal{M})$	1
$c^2_{A_k}$ $(k \in \mathcal{M})$	1
$E[Z_k]$ $(k \in \mathcal{M})$	$U(0.1, 1)$
$c^2_{Z_k}$ $(k \in \mathcal{M})$	0
SC_k $(k \in \mathcal{M})$	$U(100, 4000)$
HC_k $(k \in \mathcal{M})$	0.5
β_k^{target} $(k \in \mathcal{M})$	95%
τ_k $(k \in \mathcal{M})$	$\sum_{k \in \mathcal{M}} \frac{E[D_k]}{E[A_k]} 1.4$

In Table 5.10 a comparison is made between the optimal solutions for m_k $(k \in \mathcal{M})$ and the one obtained with the local search heuristic. ξ denotes the computation time in seconds. The local search heuristic is approximately 381 times faster than the algorithm that evaluates the optimal m_k $(k \in \mathcal{M})$. From the results in Table 5.10 we can observe that the local search heuristic performs well for small test cases, the largest error in TC for the 25 test cases is 1.40%. Further, when an error occurs, m_k $(k \in \mathcal{M})$ differs only one power-of-two for multiple items. Therefore a better heuristic for m_k $(k \in \mathcal{M})$ would be the variable depth search. In the variable depth search first 1 item is moved 1 multiple up and down, then two items are moved one multiple up and down, then three and so on until M, for more detail see Aarts and Lenstra (1997). The values of m_k are in this analysis restricted to power-of-two variables, which means that better solutions to the problems could be obtained if this assumption is relaxed. However, if we relax this assumption it is difficult to construct a feasible schedule from m_k and T^{BP}. Further research is necessary to determine the additional costs in the stochastic case.

Table 5.10: Comparison between optimal and local search heuristic.

Case	Optimal				Local search				δ^{TC}
	TC	ξ	T^{BP}	m_k $(k \in \mathcal{M})$	TC	ξ	T^{BP}	m_k $(k \in \mathcal{M})$	
1	2530.44	2411	8.56	{2,2,2,1,2}	2530.44	2	8.56	{2,2,2,1,2}	0
2	3781.41	2358	7.59	{2,2,1,1,1}	3826.08	3	5.77	{2,2,2,2,1}	1.18
3	3555.85	2408	7.11	{2,2,2,1,1}	3555.85	2	7.11	{2,2,2,1,1}	0
4	3635.84	2360	5.79	{2,2,2,2,1}	3635.84	3	5.79	{2,2,2,2,1}	0
5	3002.60	2363	6.89	{1,1,2,2,2}	3002.60	51	6.89	{1,1,2,2,2}	0
6	2963.48	2375	9.85	{1,1,1,1,1}	2965.78	4	6.19	{2,2,2,2,2}	0.08
7	2477.20	2451	7.91	{2,4,4,1,1}	2494.15	5	10.08	{2,2,2,1,1}	0.68
8	2585.23	2375	6.63	{2,1,2,2,2}	2585.23	2	6.63	{2,1,2,2,2}	0
9	3771.86	2381	12.55	{1,1,1,1,2}	3771.86	2	12.55	{1,1,1,1,1}	0
10	2796.93	1410	11.81	{1,2,1,1,1}	2796.93	3	11.81	{1,2,1,1,1}	0
11	3620.00	2378	9.32	{2,2,1,1,1}	3620.00	3	9.32	{2,2,1,1,1}	0
12	2412.23	2425	7.95	{1,4,4,2,1}	2412.23	10	7.95	{1,4,4,2,1}	0
13	3084.50	2380	6.83	{2,2,1,2,2}	3127.58	3	11.10	{1,1,1,1,1}	1.40
14	2317.70	2395	6.64	{1,2,4,2,2}	3198.72	5	10.38	{1,1,2,1,1}	0.66
15	2101.45	2487	6.73	{4,1,1,2,4}	2101.45	10	6.73	{4,1,1,2,4}	0
16	2462.88	2434	7.79	{4,1,2,4,1}	2462.88	7	7.79	{4,1,2,4,1}	0
17	2987.09	2487	8.33	{2,2,2,2,1}	2987.09	4	8.33	{2,2,2,2,1}	0
18	3253.62	2468	12.41	{1,1,1,1,1}	3253.62	4	12.41	{1,1,1,1,1}	0
19	2991.70	2539	6.25	{2,2,4,2,1}	2997.24	4	10.03	{1,1,2,1,1}	0.19
20	2862.99	2461	7.31	{4,1,2,1,1}	2862.99	4	7.31	{4,1,2,1,1}	0
21	2758.06	2456	8.97	{2,1,2,1,2}	2758.06	6	8.97	{2,1,2,1,2}	0
22	2726.19	2468	6.22	{2,2,2,2,1}	2726.19	2	6.22	{2,2,2,2,1}	0
23	3516.20	2421	13.15	{1,1,1,1,1}	3516.20	4	13.15	{1,1,1,1,1}	0
24	3957.22	2376	10.60	{1,1,1,1,1}	3957.22	5	10.60	{1,1,1,1,1}	0
25	2967.67	2439	14.00	{1,2,1,1,1}	2967.67	7	14.00	{1,2,1,1,1}	0

5.6 Summary and outlook to Chapter 6

In the first part of this chapter, we derived expressions to evaluate the first two moments of the waiting time due to production. We considered a single echelon, multi-item logistic system where the stockpoints are controlled by (R, S) installation stock policies and a fixed cyclic production sequence. In the second part of this chapter, we developed a local search heuristic to find close to optimal values for the review period R_k $(k \in \mathcal{M})$ and the release schedule U_i $(i = 1, ..., N)$. By optimal we mean minimization of the setup and inventory holding costs under the condition that the target fill rate is met.

The model under consideration in the first part of this chapter is based on ten assumptions: The first one implies that all stockpoints are controlled by (R,S)-installation stock policies. The second one implies stationary compound renewal customer demand. The third one implies that subsequent orders are not allowed to overtake each other. The fourth one implies that partial deliveries between stockpoints and customers are allowed. The fifth one implies a constant production rate. The sixth one implies a single echelon distribution system. The seventh one implies that the review periods, the release schedule and the order-up-to levels or target fill rates are given. The eighth one implies that shortages are backordered. The ninth assumption implies sequence dependent and stochastic setups and the last one implies a fixed cyclic production schedule.

Assumption 3 is a standard assumptions from inventory theory and holds usually also in practice. Assumption 4 can easily be relaxed with the formulae given in Section 3.3.5. Assumption 5 can easily be extended to stochastic production rates. Assumption 7 is relaxed in the second part of this chapter. If we relax assumption 8, then the lost sales case is considerate and in the lost sales case the waiting time due to production is zero. Assumption 9 is general and holds often in practice. Further research is necessary to be able to relax the other assumptions.

The model under consideration in the second part of this chapter is based on ten assumptions: Assumptions 1, 2, 3, 4, 5, 6 and 8 of part one. The eighth assumption implies a given target fill rate, fixed setup costs and inventory costs and the last one implies sequence dependent setups. Further research is necessary to be able to relax also assumptions 8 and 9.

The presented algorithm to calculate the waiting time of a production order is based on 3 approximations. These approximations can be found in Section 5.4 p 178. The quality of these approximations were investigated by discrete event simulations and they were acceptable. Further, for small values of the first to moments of the waiting time in the queue, the confidence intervals of the simulations were large. Note that this observation holds also for standard $G/G/1$ queues, cf. de Kok (1989). It is therefore difficult to determine the quality of the approximations for the case that the waiting times are small. In Smits, Adan and de Kok (2002), we also derived exact expressions for the first two moments of the waiting time in the queue for Erlang distributed service and interarrival times. This enabled us to determine correctly the quality of the approximations and the quality of the approximations where sufficient for practical purposes. Further, to determine the fill rate and the physical average inventory level we use three types of approximations. The first approximation type concerns the assumption that a random variable is mixed-Erlang distributed, when only the first two moments of this random variable are tractable. The second approximation type is the application of asymptotic results from renewal theory to processes of finite time arrivals. The third approximation type stems from the assumption of a renewal process for a point processes emerging in this model. Numerical analysis indicates that these approximations perform correctly.

In the second part of Chapter 5, we presented a heuristic to find close to optimal values for the review period and the release schedule. Due to the large number of variables and the iterative algorithm to determine the waiting time in the queue, it is hard to find within a reasonable time a solution. Therefore we consider the following approximations. We use a basic period approach and restrict m_k $(k \in \mathcal{M})$ to power-of-two values, for more details see Haessler (1979). We determine close-to-optimal power-of-two values for m_k $(k \in \mathcal{M})$ with a standard local search heuristic. We use an algorithm developed by Tunasar and Rajgopal (1996) to determine a feasible release schedule given m_k $(k \in \mathcal{M})$ and the basic period.

In Section 5.5.2, we analyzed the performance of the local search heuristic. The analysis reveals that the performance of the local search algorithm is excellent on small test cases. Further investigation is necessary to determine its performance on larger test cases. From the errors in the test cases, we derive that a good alternative for the conventional local search would be to apply a variable depth heuristic.

Further investigation is necessary, to determine for stochastic instances if these algorithms give solutions that are close to the optimal solution (without restricting m_k).

In this thesis we derived expressions for the first two moments of three endogenous components of the replenishment lead-time. As we mentioned throughout this thesis, the three models developed are based on numerous assumptions and make use of numerous approximations to determine above mentioned endogenous components of the replenishment lead-time. In Chapter 6, we will summarize the assumptions, the used approximations and the quality of the approximations. Further, we will indicate directions for further research.

Chapter 6

Conclusions and suggestions for future research

6.1 Contributions of this thesis

A typical production-distribution has a large number of different items and customers. For instance, a known soft drink company has between 100000 and 120000 customers and the number of products that flow through the network is in the hundreds (cf. Simchi-Levi et al. (2000)). The demand of the customers at the warehouses can be described by an interarrival time and a demand size. When answering tactical and strategic decisions, we need to know the demand 1 to 5 years from now. Hence, the interarrival times and the demand size are uncertain. To take this into account the interarrival time and the demand size should be described as stochastic variables. In practice it is common to solve these strategic and tactical decisions within a logistic network with computer programs, such as the Supply Chain Designer from CAPS Logistics and CAST DPM from Radical. In these programs demand is modeled by deterministic interarrival times and demand sizes for strategic decisions and by deterministic interarrival times and stochastic demand sizes for the tactical decisions. Similarly to the computer programs also in literature many models considering strategic decisions within the logistic network assume deterministic demand and in some models dealing with tactical decisions within the logistic network, the interarrival times of the customer demand process are assumed to be random and the demand sizes are deterministic (see for example Zipkin (2000)). Others models considering tactical decisions assume the interarrival times to be deterministic and the demand sizes to be random (see for example Svoronos and Zipkin (1988)). Finally, few models assume compound Poisson demand, which means that both the interarrival times and the order sizes are modeled as exponential distributions with a coefficient of variation equal to 1 (see for example Andersson et al. (1998)). In contrast to these models we consider a compound renewal customer demand process, which permits us to model both interarrival and demand size processes with a coefficient of variation different from 1.

Close examination of a large number of companies revealed that the customers are not identical, 20 % of the customers account for 80 % of the total demand, (cf. Silver et al. (1998)). However, in many publications assumptions like identical retailers, identical batchsizes or identical demand streams can be found; see for example Andersson et al. (1998) who assume identical batchsizes, or Cachon (2001) who consider identical demand streams. In contrast to this, the approach in this thesis allows for non-identical end-stockpoint demands and for different batchsizes and demand streams.

In this thesis, the replenishment lead-time consists of endogenous elements and an exogenous delay. The exogenous delay can be split up in three elements: the time needed to administrate the incoming orders, the time needed to handle the orders in the warehouses and the time needed for external transportation from the warehouse to the destination point. In practice these three elements are often uncertain, for example the transportation time is dependent on the traffic situation. In most existing models the exogenous delay is deterministic. In this thesis we assume that the exogenous delay is a random variable.

In some companies the number of echelons is larger than two. For example, several electronic companies manufacturing products in Asia and selling them in Europe, have a three echelon distribution network. The products are first shipped from the different factories in Asia to a central warehouse in Asia, then by boat the products are shipped to a central warehouse in Europe and finally they are shipped to regional warehouses in Europe. The regional warehouses are necessary to satisfy the customer demand on time (cf. Simchi-Levi et al. (2000)). In the literature mostly problems restricted to two echelon distribution systems with one warehouse and several retailers are analyzed. See for example Cachon (2001). In chapters 3 and 4 of this thesis we consider a N-echelon distribution network.

In Chapter 1, we explained that to take into account uncertainties in the forecasts of the demand and replenishment lead-time additional stocks are kept at the warehouse, which are called safety stocks. The level of safety stocks should be an endogenous variable in models dealing with strategic and tactical decisions within the logistic network. For example, in a two echelon distribution network with one central warehouse and a number of decentral warehouses, if we close the central warehouse and allocate the decentral warehouses to the factories then the replenishment lead-time will change and consequently also the level of safety stocks will change. In Supply Chain Designer and CAST DPM, programs that support strategic decision making, the safety stocks are exogenous to the model and a tool is included to evaluate for a given distribution network the level of safety stocks such that the customer service level is satisfied. The tool of the Supply Chain Designer evaluates analytically the safety stock level. The tool assumes a single echelon network, i.i.d. demand per period and uses the formulas described in Silver et al. (1998) to evaluate the inventory control parameters. In the tool of CAST DPM the level of safety stocks is determined by discrete event simulations. This permits to handle multi-echelon networks but it requires a long computation time to evaluate the safety stocks. Hence existing models and software dealing with strategic decisions within the logistic network assume also that the safety stock level is an exogenous variable. In our model approach, the level of safety stock is an endogenous variable.

The rest of this chapter is organized as follows. Section 6.2 provides an overview of the thesis and in Section 6.3 we present directions for further research.

6.2 Overview of the thesis

In this thesis we have highlighted the importance of understanding the relationship between the replenishment lead-time of a stockpoint and the inventory, transportation and production control policies used. The replenishment lead-time impacts the investments in inventory capital to meet customer service requirements. In turn, intermediate customer service requirements, shipment consolidation policies and production planning policies determine the behaviour of the replenishment lead-time towards a stockpoint.

As mentioned in Chapter 1, it is crucial to determine correctly the distribution costs and the customer service level of a distribution network. The distribution costs can be a substantial part of the selling price, for example for light bulbs up to 30 %, and a correct estimation of the customer service level is important because a product is of little value if it is not delivered on time. In general a gap exists between the customer service level announced by the company and the one perceived by the customers. On average the companies announce to deliver 95 % of the orders on time and the customers perceive that only 80 % of the orders is on time (cf. van Goor et al. (1999)). We explained in Chapter 1 that decisions within the logistic network influence the costs and the customer service levels. In models dealing with strategic and tactical decisions within the logistic network the replenishment lead-time is often considered as an exogenous variable, whereas in reality it is an endogenous variable. In this thesis we discuss three decisions that influence the replenishment lead-time:

- What should be the level of safety stocks at intermediary stockpoints?
- What should be the shipment consolidation policy?
- What should be the production schedule?

In the first part of Chapter 2 we reviewed the literature dealing with strategic and tactical decisions in logistic networks to understand how the replenishment lead-time can be modeled. From this literature review we concluded that current models with endogenous replenishment lead-times have restrictions which are not realistic in practice. For example, the models that take into account the level of safety stocks at the intermediary stockpoints, are restricted to two echelon distribution networks, mostly assume identical demand at the end-stockpoints and consider pure Poisson or independent identical distributed demands per period. The models taking into account shipment consolidation are restricted to a single echelon distribution network and consider pure Poisson demand or stochastic demand per period. Finally, in the models, which take production scheduling into account, the replenishment lead-time is mostly determined by simulations. In the second part of Chapter 2 key results in renewal theory have been summarized.

In Chapter 3, we provide insight into the question how the replenishment lead-time is influenced by the level of safety stocks at the preceding stockpoint. The replenishment lead-time consists of a waiting time due to a lack of stock at the preceding stockpoint and an exogenous delay. The exogenous delay can be split up in three elements: the time needed to administrate the incoming order, the time needed to handle the order in the warehouse and the time needed for external transport from the warehouse to the delivery point. In Chapter 3, we derived expressions for the first two moments of the waiting time due to a lack of stock at the preceding stockpoint in a multi-echelon, single item logistic system. We assumed that the demand at the end-stockpoints is compound renewal process, which means that both the interarrival times and the demand sizes are described by general distribution functions. We considered a divergent network and each stockpoint was controlled by an (s,nQ)-installation stock policy. We assumed that the exogenous delay was given and independent of the waiting time due to a lack of stock at the preceding stockpoint. The shortages were backordered and subsequent orders were not allowed to overtake. Only complete deliveries between stockpoints and only partial deliveries between customers and stockpoints were allowed. Further, we also derived expressions for various customer performance measures for given values of the inventory policy. Given these approximations, a target performance measure and the batchsize for each stockpoint we can determine the reorder level. The reorder level can be determined by a one-dimensional search according to the bisection rule. These derivations make use of two types of approximations. The first type of approximation concerns the assumption that a random variable is mixed-Erlang distributed, when only the first two moments of this random variable are tractable. The second type of approximation uses asymptotic results from renewal theory to processes on finite time arrivals. The numerical analysis in Section 3.5 showed that the approximations are accurate in 2 and 3 echelon distribution networks. The approximations perform even better when the distribution network is large, i.e. each stockpoint has many successors, and when the demand at the end-stockpoints is heterogeneous.

In Chapter 4, we provided insight in the question how is the replenishment lead-time influenced by shipment consolidation policies? The replenishment lead-time additionally consisted of the waiting time due to shipment consolidation. We derived expressions for the waiting time due to shipment consolidation in a divergent multi-echelon, multi-item logistic network. Similar to Chapter 3, we assume compound renewal customer demand, (s,nQ)-installation stock policies, subsequent orders are not allowed to overtake, shortages are backordered and only complete shipments are allowed between stockpoints and only partial shipments between stockpoints and customers. We distinguished between two types of shipment consolidation policies, the time-based and the quantity-based policy. For the time-based policy we derived exact results for the distribution function of the waiting time due to shipment consolidation. For the quantity-based policy we derived expressions for the first two moments of the waiting time due to shipment consolidation. These expressions use three additional types of assumptions/approximations. Firstly, we neglected the delay and clustering effect of replenishment orders at the consolidation dock. Secondly, we neglect the probability that there are multiple replenishment orders of a stockpoint in one truck. Thirdly, we approximate the probability that a replenishment order from a stockpoint is the n-th one in the consolidation process by the probability that each position in the series of orders consolidated in the transportation lot are equally likely. An extensive numerical study was conducted in Section 4.6. The study has revealed that the approximations performed well when there are less than two replenishment orders of each stockpoint in a truck. However, we observed in Section 4.4.2 that it is not cost efficient to have more than one replenishment order of each stockpoint in a truck, because we could increase the batchsize without increasing the inventory level, which leads to the same inventory costs but may lead to lower handling costs. This supports the general applicability of the results obtained in this chapter.

In Chapter 5, we provided insight in how the replenishment lead-time is influenced by the production schedule? From the literature review in Chapter 2, we observe that the number of papers dealing with stochastic demand is limited. Therefore, we consider a simple system, with a stockpoint for each item and a single production facility, which produces all the items. We assume that the replenishment lead-time consists of an exogenous delay and waiting time due to production scheduling. In Chapter 5, we derived expressions for the first two moments of the waiting time due to production in a single echelon, multi-item logistic system without shipment consolidation. Similarly to chapters 3 and 4 we assume compound renewal customer demand, shortages are backordered and orders are not allowed to overtake each other. In contrast to chapters 3 and 4, the stockpoints are controlled by (R,S)-installation stock policies, since in production situations periodic review policies are preferred above continuous policies because it allows a better control of the workload at the production facility and enables a schedule such that no time is wasted on setups. We assumed a fixed production rate per item and sequence dependent setups and the exogenous delay is independent of the waiting time due to production. We assume that the production order is larger than zero. The derived expressions for the waiting time due to production use of the same two types of approximations as in Chapter 3. The performance of our approximations was tested using discrete event simulation, which proves the validity of the approximations. Further, we developed an algorithm to determine close-to-optimal solutions for the review periods of the stockpoints and the resulting production schedule. The algorithm was based on the following approximations. We used a basic period approach and restricted the multipliers to power-of-two values. We determined close-to-optimal power-of-two values for the multipliers with a standard local search heuristic. Next we determined a feasible release schedule given the multipliers and the basic period. The performance of the local search heuristic to determine the optimal power-of-two value for multipliers was tested by comparing it with the optimal case for small instances. The maximum error between the optimal total costs (restricting m_k to power-of-two values) and the total costs evaluated with the local search algorithm was 1.18 % for 25 random generated cases.

6.3 Further research

After embedding the contributions of this thesis in existing literature and giving an overview of this thesis, we present some possible directions for further research.

In some cases, the errors in the first two moments of the waiting time were rather large ($> 10\%$). For example in Chapter 5, the error in the second moment of the waiting time in the queue was in certain situation above 20%. A direction of further research could be to improve the approximations. We assumed mixed-Erlang distribution of random variables whose distribution is intractable except for its first two moments. We evaluated the first two moments of the random variable and we approximated the distribution function of the random variable by a mixed-Erlang distribution with the same first two moments. To improve the approximations, we could try to match the first three (or even more) moments of the random variable or to approximate the shape of the random variable. In de Kok (1989) a three moment approximation is described, this method is only applicable for coefficients of variation larger than 1.

Throughout this thesis, we tested the performance of the approximations by discrete event simulation. In these simulations we considered mixed-Erlang distributions for the interarrival times and order sizes at the end-stockpoints and for the exogenous delays. However, in practice the interarrival times and order size could have another distribution function. Therefore, a direction of further research could be to investigate the quality of approximations for different distribution functions, like for example uniform distributed interarrival times.

In Chapter 1, we mentioned that to take the optimal strategic and tactical decisions in terms of costs and service level, it is of interest to determine accurately the costs and service level. In this thesis, we provided approximations, which enabled us to determine more accurately the costs and service level. Therefore, a third possibility for further research could be to tackle the strategic and tactical decisions within the logistic network with these accurate expressions for the costs and service level. Typical tactical decisions that could be tackled are for example the performance measures at the non end-stockpoints, the batchsizes, and the shipment consolidation policies. Typical strategic decisions could be for example the structure of the logistic network. Due to the numerous approximations and the non-linearity of the objective, it is likely that long computation time is required to solve these decisions to optimality with standard optimization methods. However, heuristics could be developed to find close-to-optimal solutions. Possible heuristics could be the ones presented in Chapter 5. However, in certain situations the number of decision variables will be so large that even the heuristics presented in Chapter 5 will require long computation times. In this case, one could resort to apply a design of experiment method or a response surface model. This implies that we evaluate for various cases the costs and that we construct a regression model of problem. For more detail, we refer to Law and Kelton (1991).

As mentioned in Section 6.2 we made several assumptions for each model, for example the type of inventory, shipment consolidation and production policy, the type of demand and network structure. Some of these assumptions are not always valid in practice and an interesting direction for further research could be to relax certain assumptions and considering different policies. Below we handle the most interesting and promising extensions to the current models.

In Section 6.1, we mentioned that numerous companies have distribution networks with 3-echelons. In Chapter 5, we assumed a single echelon model, whereas in chapters 3 and 4 we considered a multi-echelon model. An interesting direction for further research could be to extend the model in Chapter 5 to a multi-echelon network. In Chapter 5, we assumed (R,S) inventory policies and in Chapter 3 we derived expressions for the waiting time due to a lack of stock at the preceding stockpoint for (s,nQ)-policies. Therefore to be able to extend the model in Chapter 5 to a multi-echelon model, we must derive expressions for the waiting time due to a lack of stock at the preceding stockpoint for the (R,S) inventory policy. Note that van der Heijden and de Kok (1992) performed already some preliminary research in this area using compound Poisson demand.

In Chapter 4, we considered two different kinds of shipment consolidation policies: the time-based policy and the quantity-based policy. We mentioned in Chapter 2 that the literature distinguishes between two classes of models: the joint replenishment models and the models that explicitly consider the shipment consolidation policy. The time-based policy and the quantity-based policy belong to the second class. The first class of models are more common in practice, therefore it would be interesting to derive expressions for the replenishment lead-time and performance measures for the multi-echelon models with joint replenishments. To be able to realize this we could make use of the approximations derived by Federgruen et al. (1984) for compound Poisson demand.

In Chapter 5, we considered (R,S) inventory policies and we assumed that the probability that there is no demand during a review period is zero. This is in practice not always a realistic assumption, therefore an interesting direction for further research could be to develop expressions for the waiting time due to production for a (R,s,S) inventory policy. If a (R,s,S) policy is used the inter-arrival times are no longer deterministic and the release order process is no longer constant. Therefore the approximations presented in Chapter 5 are no longer valid. Note that Adan and Kulkarni (2002) did some preliminary research in this area. Adan and Kulkarni (2002) derive approximations for the waiting time for a single-server queue with Markov dependent interarrival and service times. The interarrival times are exponential distributed and the service times are independently identically distributed.

As observed in this last chapter there are still numerous interesting questions that remain unanswered. Nevertheless, we consider this thesis a step forwards to a more comprehensive understanding of the relationship between replenishment lead-times and inventory, transportation and production planning.

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Notations

Below we list the notations used in this thesis.

- Operators and functions

$E[X]$	Expectation of the random variable X
$\sigma^2(X)$	Variance of the random variable X
$E[X^2]$	Second moment of the random variable X
c_X	Coefficient of variation of X, $c_X = \frac{\sigma(X)}{E[X]}$
$(X)^+$	Maximum of the random variable X and zero
ρ_X	Auto-correlation coefficient of stochastic process $\{X_n\}_{n=1}^{\infty}$
$F_X(x)$	Cumulative distribution function of X, $F_X(x) = P\{X \leq x\}$
$f_X(x)$	Probability density function of X
$P\{A\}$	Probability of event A
$F_X^{(n)*}(x)$	n-fold convolution of $F_X(x)$
$M_X(x)$	Renewal function, $M_X(x) := \sum\limits_{n=0}^{\infty} F_X^{n*}(x)$
$\lceil A \rceil$	Largest integer smaller than or equal to A
$\lfloor A \rfloor$	Smallest integer larger than or equal to A

- Error measurement

$\delta_i^{(Z)}$	Percentage error in Z for simulation i

$\Delta_i^{(Z)}$	Absolute error in Z for simulation i
$\bar{\delta}_i^{(Z)}$	Average percentage error in Z
$\bar{\Delta}_i^{(Z)}$	Average absolute error in Z
$\max(\delta_i^{(Z)})$	Maximum percentage error in Z
$\max(\Delta_i^{(Z)})$	Maximum absolute error in Z

- Definition of the distribution network

$\mathcal{W}$	Set of warehouses
$\mathcal{M}$	Set of stockpoints
$\mathcal{L}_m$	Set of stockpoints at location $m \in \mathcal{F} \bigcup \mathcal{W}$
$\mathcal{S}_k$	Set of all immediate succeeding stockpoints of stockpoint $k \in \mathcal{M}$
$\mathcal{P}_k$	Set of all immediate predeceasing stockpoints of stockpoint $k \in \mathcal{M}$
$\mathcal{E}$	Set of all end-stockpoints stockpoints
$\mathcal{I}$	Set of all intermediary stockpoints
$\mathcal{B}$	Set of all beginning stockpoints
M	Number of items
N	Number of production orders

- Inventory control parameters

s_k	Reorder level at $k \in \mathcal{M}$
S_k	Order-up to level at $k \in \mathcal{M}$
Q_k	Batchsize at $k \in \mathcal{M}$
R_k	Review period at $k \in \mathcal{M}$

- Performance measures

α_k	Long run non-stockout probability per replenishment cycle at $k \in \mathcal{M}$, in case of partial deliveries.
β_k	Long run fraction of demand delivered directly from stock at $k \in \mathcal{M}$, in case of partial deliveries.
β_k^c	Long run fraction of demand delivered directly from stock at $k \in \mathcal{M}$, in case of complete deliveries.
γ_k	Long run fraction of time the net stock is positive at $k \in \mathcal{M}$, in case of partial deliveries.
$P\{W_j^s > 0\}$	Long run probability that an arbitrary customer j has to wait due to a lack of stock at $k \in \mathcal{P}_j$, In case of complete deliveries.
$E[b_k]$	Long run expected backlog level at $k \in \mathcal{M}$, in case of partial deliveries.
$E[I_k]$	Long run expected physical inventory level at $k \in \mathcal{M}$, in case of partial deliveries.
$E[I_k^c]$	Long run expected physical inventory level at $k \in \mathcal{M}$, in case of complete deliveries.

- Chapter 3 and 4

The following variable are supposed to be given

D_k	Demand size for $k \in \mathcal{E}$
A_k	Time between two subsequent arrivals of orders $k \in \mathcal{E}$
L_k^d	Transportation time to $k \in \mathcal{M}$
W_k^s	Waiting time of $k \in \mathcal{B}$ due to a lack of stock
Q_k	Batchsize at $k \in \mathcal{M}$
W_k^c	Waiting time due to consolidation for $k \in \mathcal{B}$
T_m	Time between two truck departures towards $m \in \mathcal{W}$ (time policy)

Q^c_m	Predetermined consolidation quantity towards $m \in \mathcal{W}$ (quantity policy)
β^{target}_k	Target fill rate at $k \in \mathcal{M}$

- The following are computed in Chapter 3

O_k	Replenishment order size of $k \in \mathcal{M}$
R_k	Time between two subsequent replenishments $k \in \mathcal{M}$
L_k	Replenishment lead-time at $k \in \mathcal{M}$ ($L_k = L^d_k + W^s_k$)
D_k	Order size of the demand process at $k \in \mathcal{M} \backslash \mathcal{E}$
A_k	Time between two subsequent order arrivals of the demand process at $k \in \mathcal{M} \backslash \mathcal{E}$
U_k	Undershoot at $k \in \mathcal{M}$, which is defined as the difference between s_k and the inventory position just before the placement of an replenishment order
$D_k(L_k)$	Demand at $k \in \mathcal{M}$ during the replenishment lead-time L_k
$N(R_k)$	Number of customers arriving during an arbitrary replenishment cycle at $k \in \mathcal{M}$. (The replenishment cycle is defined as the time between two subsequent replenishments.)
$\tilde{N}(R_k)$	Number of customers arriving during an arbitrary replenishment cycle at $k \in \mathcal{M}$ under the condition that Q_k is large.
W^s_k	Waiting time of $k \in \mathcal{M} \backslash \mathcal{B}$ due to a lack of stock at $j \in \mathcal{P}_k$
H_k	Residual lifetime distribution of D_k, for more details we refer to Section 2.2.1.

- The following are computed in Chapter 4

$W^c_{k,m}$	Waiting time due to shipment consolidation towards $k \in \mathcal{M}$, $m \in \mathcal{W}$

V_m	Remaining part of the split order for consolidation towards $m \in \mathcal{W}$
$X_{k,m}$	Time between the last truck departure towards $m \in \mathcal{W}$ and the arrival of an arbitrary order from $k \in \mathcal{L}_m$
$Y_{k,m}$	Amount consolidated between the last truck departure towards $m \in \mathcal{W}$ and the arrival of an arbitrary order from $k \in \mathcal{L}_m$
$N(Q_m^c - Y_m)$	Number of arrivals of replenishment orders from m between the placement of replenishment order at the consolidation dock from $k \in \mathcal{L}_m$ and the departure of the truck towards $m \in \mathcal{W}$.
$C_{k,m}$	Inter-arrival time of replenishment orders from stockpoint k at the consolidation dock for $k \in \mathcal{M}$, $m \in \mathcal{W}$
C_m^*	Inter-arrival time of an arbitrary replenishment order from warehouse m at the consolidation dock for $m \in \mathcal{W}$
$B_{k,m}$	Order size of replenishment orders in volume at the consolidation dock for $k \in \mathcal{M}$, $m \in \mathcal{W}$
B_m^*	Order size of replenishment orders in volume at the consolidation dock for $m \in \mathcal{W}$

- Chapter 5 The following variable are supposed to be given

N	number of orders placed during a total cycle
M	number of items
$v(i)$	function indicating which item is ordered at i, $(i = 1, ..., N)$
J^k	number of orders for item $k \in \mathcal{M}$ placed during a total cycle
$f(i)$	function indicating which order of item k is ordered at i, $(i = 1, ..., N)$
τ_k	production rate of the item of stockpoint $k \in \mathcal{M}$
SC_k	setup costs for the production of the item of stockpoint k, $k \in \mathcal{M}$
HC_k	holding costs of the item of stockpoint $k \in \mathcal{M}$

U_i	time between the placement of order i and $i+1$, $i = 1, ..., N-1$
U_N	time between the placement of order N and 1
t_k^0	epoch at which the first order of stockpoint k is setup
T^{BP}	basic period
T	length of the total cycle
ϵ	a small positive value
φ	a large positive value
Z_k	setup time of production order i , $i = 1, ..., N$

- The following are computed in Chapter 5

Q_i	size of the production order i, $i = 1, ..., N$
W_i^q	waiting time of order i in the queue, $i = 1, ..., N$
W_k^f	production time of item order $k = 1, ..., M$
$L_{v(i),f(i)}$	replenishment lead-time of order $f(i)$ for item $v(i)$, $i = 1, ..., N$
$I_{v(i),f(i)}^b$	net stock of item $v(i)$ just after production order $f(i)$ arrives, $i = 1, ..., N$
$I_{v(i),f(i)}^e$	net stock of item $v(i)$ just before production order $f(i)+1$ arrives, $i = 1, ..., N$
B_i	service time of order i, $i = 1, ..., N$, which is defined as the production time plus the setup time. $B_i = Z_i + W_{v(i)}^f$
ρ	traffic intensity or utilization degree of the production facility

Samenvatting

Het onderzoek dat in dit proefschrift beschreven wordt concentreert zich op het bepalen van de beleveringstijd van voorraadpunten in logistieke netwerken. Het beschouwde netwerk bestaat uit fabricage-eenheden, magazijnen, overslagpunten en klanten (bijvoorbeeld de detailhandel). De belangrijkste bijdrage van dit proefschrift aan de literatuur is dat algemene distributie netwerken beschouwd worden die rekening houden met het serviceniveau dat de klant wenst. Het logistieke netwerk bestaat uit meerdere echelons en zowel de voorraadpunten bij de fabriek als voorraadpunten in het distributienetwerk worden gemodelleerd. Verder beschouwen we "compound renewal" klantenvraag, dit wil zeggen dat zowel de tussenaankomsttijd als de bestelgrootte stochastische variabelen zijn.

In logistieke netwerken wordt de behoefte van de klant uitgedrukt als een gewenst serviceniveau waaraan voldaan moet worden. Om de hoeveelheid voorraad te bepalen die nodig is om dit gewenste serviceniveau te realiseren moet de beleveringstijd van het voorraadpunt accuraat bepaald worden. De beleveringstijd is de tijd die verloopt tussen het bestellen en het ontvangen van een bevoorradingsorder. In strategische en tactische modellen betreffende het logistieke netwerk wordt de beleveringstijd meestal als een exogene variabele beschouwd. Echter, om de voorraadhoeveelheid te bepalen zodanig dat het gewenste serviceniveau behaald kan worden en om de afwegingen in kosten te bepalen, zou de beleveringstijd gemodelleerd moeten worden als een endogene variabele.

Drie componenten van de beleveringstijd worden beïnvloed door strategische en tactische beslissingen die betrekking hebben op het logistieke netwerk en deze moeten daarom expliciet in het model worden meegenomen.

De eerste component is de wachttijd veroorzaakt door een tekort aan voorraad bij het voorgaande voorraadpunt. Deze variabele is afhankelijk van de hoeveelheid voorraad bij het voorgaande voorraadpunt. Veronderstel een voorraadpunt A in een logistiek netwerk. Wanneer de hoeveelheid voorraad bij het voorgaande voorraadpunt van A afneemt dan neemt de beleveringstijd van het voorraadpunt A toe omdat de wachttijd die veroorzaakt wordt door een tekort aan voorraad bij het voorgaande voorraadpunt toeneemt. Doordat de beleveringstijd van voorraadpunt A toeneemt moet meer voorraad gehouden worden bij A om hetzelfde serviceniveau te bieden. Dit houdt ook in dat de voorraadkosten bij A toenemen en bij de voorganger van A afnemen.

De tweede component van de beleveringstijd is de wachttijd door orderconsolidatie. Deze is afhankelijk van de gekozen transportconsolidatieregel. De transportconsolidatieregel is essentieel in het modelleren van de afwegingen tussen transport- en voorraad-kosten. Bijvoorbeeld, als er twee keer per week in plaats van een keer per week een vrachtwagen rijdt dan neemt de beleveringstijd af, omdat de wachttijd door order consolidatie afneemt.

De derde component is wachttijd veroorzaakt door productie. Deze is grotendeels afhankelijk van de productieplanning. Wanneer de productieplanning verandert dan verandert de wachttijd veroorzaakt door productie en vandaar ook de beleveringstijd.

In dit proefschrift worden wiskundige benaderingen afgeleid voor de eerste twee momenten van deze drie wachttijden. Dit laat ons toe om nauwkeurig de beleveringstijd te bepalen. Hierdoor zijn we in staat om de logistieke kosten te bepalen zodanig dat aan het serviceniveau van de klant wordt voldaan. De correctheid van deze benaderingen zijn getest met behulp van simulaties. De simulaties wijzen uit dat in de meeste gevallen de approximaties goede benaderingen geven.

Curriculum vitae

Sanne Smits was born in 1976 in Huizen, the Netherlands. From 1981-1984 and from 1986-1992 she followed her education in France. In 1994 she received her International Baccalaureate at the Lorentz international school in Arnhem, Holland. She studied Ecometrics at the Rijksuniversiteit Groningen and received her master's degree in February 1999 after a research on optimal location of warehouses in a distribution network. This research was conducted at CQM b.v. and was supervised by prof.dr. Wim Klein-Haneveld en drs. Tibor Huizinga.

Since February 1999 she has been conducting research at the Technische Universiteit Eindhoven, faculty Operations, Planning and Control concerning the tactical design of production-distribution networks. The project was supervised by prof.dr. Ton de Kok. During the research, which was sponsored by CQM b.v., she also worked there part-time.

In May 2002, she visited the university of Augsburg, Germany, to work together with dipl.kfm. Michael Wagner and prof.dr. Bernhard Fleischmann on a project dealing with the coordination between production and inventory control.

Stellingen

behorende bij het proefschrift

Tactical design of production-distribution networks:

safety stocks, shipment consolidation and production planning

van

Sanne Smits

I

We advocate that in models dealing with strategic and tactical decisions in the logistic network, the replenishment lead-time should be modelled as an endogenous variable in order to make the trade-offs between customer service levels, inventory holding costs, transportation costs and manufacturing costs.
For more details we refer to Chapter 1 of this thesis.

II

Jayaraman (1998) notices the importance of simultaneously considering the transportation and inventory policy, since there is a strong interdependence among them. He states that the selection of transport and inventory policies should be made simultaneously based on costs and transit time. Unfortunately the choice of inventory policies in his proposed model is restricted since only the cycle stock, which is determined from the shipping frequency, is considered.
Jayaraman, V. (1998) *Transportation, facility location and inventory issues in the distribution network design* International Journal of Operations and Production Management, vol 18, p 471-494

III

A correct evaluation of the replenishment lead-time is crucial. A product is of little value if it is not available to customers at the time and place they wish to consume or use it. We distinguish between the required, desired and perceived customer service level. In practice we notice a gap between these three service levels. Let us illustrate this with the following example from Gourdin (2001): "PC manufacturing routinely announce 10 day order lead times (but the customers want the order in 3 days), 90 % of the orders are delivered directly from stock (the customers think it should be 95 % or 99 %). Yet when asking the customers what they actually got, you hear of order lead times between 20 and 30 days and 50 to 65 % of the orders which are delivered directly from stock".
Gourdin, K. N. *Global logistics management: a competitive advantage for the new millennium* Blackwell Publishers

IV

Exact expressions for the distribution function of numerous random variables are in this thesis intractable due to the assumption of compound renewal customer demand. The key approximation used throughout this thesis to solve this problem concerns the assumption that a random variable is mixed-Erlang distributed, when only the first two moments of the random variable are known.

V

The proof of Theorem 4.2 is less obvious than it seems to be.
Theorem 4.2: Given a time-based policy where the trucks leaves at fixed time intervals T_m. Further, we assume that the truck capacity is unlimited and that all processes are stationary. In this case $X_{n,m}$, the time between the last truck departure from warehouse n to warehouse m and the arrival of an arbitrary replenishment order from m, is an uniformly distributed random variable on the interval $(0, T_m]$.
For more details we refer to Chapter 4 of this thesis.

VI

The moment-iteration method of de Kok (1989) is also applicable to a much more general class of problems namely cyclic $G/G/1$ queues. In cyclic $G/G/1$ queues the interarrival and service times are cyclic. This model may be used, for example, when the inflow of customers depends on the day of the week, or on the hour of the day.
For more detail, we refer to Chapter 5 of this thesis and **Smits, S.R., Adan, I., and de Kok, A.G. (2003)** *Waiting time characteristics in cyclic queues.*
Kok, A.G. de (1989) *A Moment-iteration method for approximating the waiting-time characteristics of the GI/G/1 queue.* Probability in the Engineering Informational Sciences, vol 3, p 273-287

VII

Alleen een echte monarchist schrijft met een kroontjespen.

VIII

La brouette ou les grandes inventions

Le paon fait la roue
le hasard fait le reste
Dieu s´assoit dedans
et l´homme le pousse

Jacques Prévert

IX

Wanneer je een vreemde hand in een vreemde zak vindt is het of je hand niet, of je zak niet.

X

Car si l´amour n´est pas dans l´air, je préfére rester sur terre. Mais si l´amour n´est pas sur terre, je préfére reprendre l´air.

Paris Combo

XI

Je leert iets pas door het te doen; want ook als je denkt dat je het weet, weet je het nooit zeker, voordat je het probeert.

Sophocles

XII

Mašala