# Optimal control of serial, multi-echelon inventory/production systems with periodic batching 

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# Optimal Control of Serial, Multi-Echelon Inventory/Production Systems with Periodic Batching 

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#### Abstract

We consider a single-item, periodic-review, serial, multi-echelon inventory system, with linear inventory holding and penalty costs. In order to facilitate shipment consolidation and capacity planning, we assume the system has implemented periodic batching: each stage is allowed to order at given equidistant times. Further, for each stage except the most downstream one, the reorder interval is assumed to be an integer multiple of the reorder interval of the next downstream stage. This reflects the fact that the further upstream in a supply chain, the higher setup times and costs tend to be, and thus stronger batching is desired. Our model with periodic batching is a direct generalization of the serial, multi-echelon model of Clark and Scarf (1960). For this generalized model, we prove the optimality of basestock policies, we derive Newsboy-type characterizations for the optimal basestock levels, and we describe an efficient exact solution procedure for the case with mixed Erlang demands. Finally, we present extensions to assembly systems and to systems with a modified fill rate constraint instead of backorder costs.


Subject classification: Inventory/Production: Multi-echelon, stochastic demand, periodic batching, optimal policies.

## 1 Introduction

Two main costs of supply chains consist of capacity costs and material costs; hence decisions affecting each of these should be made taking decisions regarding the other into account. Typically capacity decisions are made for a longer term ( $>5$ years, say) than material decisions; thus capacity decisions are often made first, with materials decisions following afterwards (and being revisited more often). Such material decisions concern batching rules (often used to facilitate capacity partitioning among different products), which may be reviewed annually, and reorder and basestock levels, which may even be adapted on a daily or weekly basis (e.g., when procedures like exponential smoothing are used for demand forecasting). These materials decisions, constrained to accommodate previous capacity choices, are the focus of this paper. Specifically we consider the setting of optimal reorder points and basestock levels in multi-echelon supply chains utilizing periodic batching.

With respect to batching, for a single-item, single-stage situation, with either continuous or discrete time, we distinguish the following forms:
(i) Periodic batching: Think of an $(R, S)$-policy, where $R$ represents the reorder interval and $S$ the basestock or order-up-to level. Under this policy, every $R$ time units (or periods) an order is placed to return the inventory position to $S$;
(ii) Fixed batch sizes: Think of an $(s, Q)$-policy, where $s$ represents the reorder level and $Q$ the fixed batch size. Each time that the inventory position has dropped to or below $s$, one or more batches of size $Q$ are ordered to bring the inventory position up to or above $s$;
(iii) A combination of both: Think of an $(R, s, Q)$-policy with the review interval $R$, fixed batch size $Q$, and reorder level $s$ as decision variables. Under this policy, every $R$ time units one is allowed to place an order, and if at such a time epoch the inventory position has dropped to or below $s$, then one or more batches of size $Q$ are ordered to bring the inventory position up to or above $s$.

When looking at multiple items and/or multiple stages, one also has these three forms, and further many different combinations are possible. Which form of batching is best is a very hard question, because aspects like setups (setup times and costs), capacity constraints, capacity flexibility, and shipment consolidation are
affected. In fact, a clear answer is only known for a periodic-review, single-item, single-stage system with fixed ordering costs and convex inventory holding and penalty costs, and for some variants of this system. Then an $(s, S)$-policy is optimal (cf. Scarf, 1960, Zipkin, 2000, and Porteus, 2002; notice that an $(s, S)$-policy is equivalent to an $(s, Q)$-policy in certain cases, e.g. under continuous review and a Poisson demand process). Further, Rao (2003) compared an optimal $(R, S)$-policy (where both $R$ and $S$ are optimized) to an optimal $(s, Q)$-policy for a single-item, single-stage system with fixed ordering costs and linear inventory holding and penalty costs. Computational results, for instances with Poisson demand processes, show that the $(s, Q)$-policy is superior but the relative difference in costs is limited. In a subsequent study, Feng and Rao (2003) compare nested multi-echelon $(R, S)$-policies to nested multi-echelon $(s, Q)$-policies for a two-echelon serial system with fixed ordering costs at both stages. Based on a series of instances with Poisson demand processes, they find that the optimal multi-echelon $(s, Q)$-policy is better than the optimal multi-echelon $(R, S)$-policy, but the difference is small. And, thus, $(R, S)$-policies easily may become more attractive when other factors than just fixed ordering costs are important. For all other cases, not so much is known. Even for the relatively simple Stochastic Economic Lotsizing Problem (which is multi-item, single-stage), no clear answers are available.

Because of the complexity, for the materials decisions in supply chains, we, as many authors before us, advocate use of a hierarchical approach with two decision levels:
(i) an upper level to decide on the form of batching and the batch sizes and reorder intervals, where a multi-item, multi-echelon view is taken in order to deal with setups, capacity constraints, capacity flexibilities, and shipment consolidation; and
(ii) a lower level to decide on reorder and basestock levels, where the batching rule is given and a singleitem, multi-echelon view can be taken.

This is in line with the separation between batching and safety stock decisions as advocated by Graves (1996), and with comments of several other authors; see e.g. Yano and Carlson (1988), Zipkin (2000, p. 235), and Rao (2003).

Models for the upper level can be of various types. If only setup costs play a role, then lotsizing models with deterministic demand may be appropriate (see e.g. Roundy, 1986). However, for real-life problems,
more sophisticated lotsizing models are often needed. Such models should work in concert with appropriate models for the lower level, which are the focus of this work. For this lower level, models with given fixed batch sizes - multi-echelon $(s, Q)$-policies - are available, see e.g. DeBodt and Graves (1985), Chen (2000), and the many references therein. General multi-echelon systems with periodic batching - multi-echelon $(R, S)$ policies -, or a combination of fixed batch sizes and periodic batching - multi-echelon $(R, s, Q)$-policies -, lack such definitive results however. This is quite surprising, given the fact that periodic batching was already recognized as common practice to facilitate freight consolidations and logistics/production scheduling nearly ten years ago; see e.g. Graves (1996, p. 4) who reports on periodic replenishments in the context of large retail chains such as WalMarts. Graves also comments on the expectation that the replenishment schedule would be "nested", what we will refer to as the integer ratio and synchronization constraint.

We do not wish to imply that periodic batching models have been ignored in the literature, as in fact some results for special cases are known. For two-echelon serial systems, there is recent work by Feng and Rao (2003), as mentioned above. For a two-stage assembly system with periodic batching and normally distributed demand, Yano and Carlson (1988) developed an approximate evaluation and optimization procedure for installation-stock basestock policies. Further, a number of papers has been devoted to two-echelon distribution systems with periodic batching. Partial characterizations of optimal policies have been derived by Jackson (1988), McGavin, Schwartz, and Ward (1993), and Güllü and Erkip (1996). Several computational experiments were executed by these authors and by Jönsson and Silver (1987), Graves (1996), and Van der Heijden (1999), in order to gain further insights into appropriate control rules. In addition, Graves (1996) provides an exact evaluation (but not optimization) for basestock policies in multi-echelon distribution systems with periodic batching and Poisson demand processes. Conspicuously absent from this literature is a characterization of the exact form of the optimal policy; we demonstrate this in this paper.

In this paper, we study a single-item, periodic-review, serial, multi-echelon inventory system with periodic batching. We assume that for each stage a reorder interval has already been determined, and for each stage except the most downstream one the reorder interval is an integer multiple of the reorder interval in the next downstream stage. We call this the integer-ratio constraint. This constraint facilitates synchronization within the chain and reflects the fact that the further upstream we are in a supply chain, the higher setup
times and costs tend to be, and thus stronger batching is desired. Further, we assume the reorder epochs are timed such that arriving materials at one stockpoint can be forwarded immediately to the next stockpoint if desired. This is called the synchronization constraint. (Together, these two constraints formalize the concept of nesting of Graves, 1996.) Our model is a direct generalization of the Clark-and-Scarf model (Clark and Scarf, 1960), and extends many of the known Clark and Scarf results (see next paragraph) to the periodic batching domain. We also complement the work by Chen (2000), who generalized several of the existing results for the Clark-and-Scarf model to the case with a given fixed batch size per stage. As we allow batch sizes to vary, and instead fix reorder intervals, our model is analogous to an $(R, S)$-policy, whereas Chen (2000) is analogous to an $(s, Q)$-policy.

For the general Clark-and-Scarf model, many results are available. Clark and Scarf (1960) proved the optimality of basestock policies in a finite-horizon setting. Federgruen and Zipkin (1984) extended this result to the infinite-horizon case. Alternative proofs were given by Langenhoff and Zijm (1990) and Chen and Zheng (1994). Van Houtum and Zijm (1991, 1997) derived Newsboy-type characterizations for the optimal basestock levels, and proved the optimality of basestock policies under a modified service level constraint (which is equivalent to an average backlog constraint). In addition, they derived an efficient exact solution procedure for the case with mixed Erlang demands, and an even faster approximate solution procedure for the general demand case (see also Van Houtum, Inderfurth, and Zijm, 1996). As stated above, Chen (2000) generalized the main results for the case with given fixed batch sizes. Further, Chen and Song (2001) considered the extension to Markov modulated demand, and Gallego and Özer (2003) to a specific type of advance demand information. In both cases, the optimality of state-dependent basestock policies was derived, together with an efficient algorithm for the determination of the optimal basestock levels. Finally, bounds allowing simple spreadsheet computations were derived by Shang and Song (2003).

The main contributions of this paper are as follows: First, this paper derives, for the first time, the optimal policy for a general multi-echelon system with a given periodic batching rule. Second, we generalize many of the existing results for the Clark-and-Scarf model to the periodic batching domain - we prove the optimality of basestock policies, derive Newsboy-type characterizations for the optimal basestock levels, and describe an efficient exact solution procedure for the case with mixed Erlang demands. Third, for our proofs
we follow the line sketched in Van Houtum et al. (1996), but obtain more clear formulae and a simpler exact solution procedure than has been available so far for the Clark-and-Scarf model. Forth, we discuss extensions to assembly systems, systems with a $\gamma$-service level constraint (modified fill rate constraint), and systems violating the integer-ratio or synchronization constraint.

This paper is organized as follows. In Section 2, we discuss the model. The complete analysis is presented in Section 3, and extensions are discussed in Section 4. Finally, concluding remarks are made in Section 5. Throughout the paper, an illustrative example is used to support the presentation.

## 2 Model

In this section, we describe our model for a serial, multi-echelon inventory system with periodic batching. The assumptions are given in Subsection 2.1, the notation is listed in Subsection 2.2, and the objective and the concept of echelon costs are introduced in Subsection 2.3.

### 2.1 Assumptions

- The inventory/production system consists of a number of stages in series. Inventory may be held in stock at the end of each stage.
- Time is divided into periods of equal length (w.l.o.g., the length of each period is assumed to be equal to 1 ), and the time horizon that we consider is infinitely long. The periods are numbered $0,1, \ldots$
- External demand occurs at the most downstream stage only. The demands per period are i.i.d., strictly continuously distributed random variables on $(0, \infty)$.
- Leadtimes are constant.
- The most upstream stage orders at an external supplier, which can always deliver.
- Demand that cannot be satisfied from stock at the most downstream stockpoint is backlogged and satisfied as soon as possible (in FCFS order).
- For each stage, a fixed reorder interval is given. The reorder intervals are nondecreasing from downstream to upstream in the chain, and the reorder intervals satisfy an integer-ratio constraint, i.e., for each except the most downstream stage, the reorder interval is an integer multiple of the reorder interval of the next downstream stage. Further, the reorder time instants are synchronized, i.e., each stage, excluding the most upstream stage, has its order moments at the time instants that orders of the next upstream stage arrive and at equidistant time instants in between.
- In each period the following events occur: (i) At each stage, an order is placed if this is allowed in that period; (ii) Arrival of orders; (iii) Demand occurs; (iv) Inventory holding and penalty costs are charged. The first two events are assumed to take place at the beginning of the period, and the order of these two events may be interchanged, except for the most downstream stage when its leadtime equals 0 . The last event is assumed to occur at the end of a period. The third event, the demand, may occur anywhere in between.


### 2.2 Notation

We define notation below, illustrating it with an illustrative example described below (see also Figures 1 and $2)$.

## General

- $N$ : Number of stages of the serial system $(N \in \mathbb{N}, N \geq 2)$. The stages (and the corresponding stockpoints) are numbered from $n=1, \ldots, N$, where 1 denotes the most downstream stage and $N$ the most upstream stage.


## Leadtimes

- $l_{n}, n=1, \ldots, N$ : Fixed leadtime for the $n$-th stage $\left(l_{n} \in \mathbb{N}\right.$ for $n=2, \ldots, N$, and $\left.l_{1} \in \mathbb{N}_{0}:=\{0,1, \ldots\}\right)$.
- $L_{n}, n=1, \ldots, N: L_{n}:=\sum_{i=n}^{N} l_{n}$ denotes the cumulative leadtime from the external supplier to the stockpoint at the end of stage $n$. For notational convenience, $L_{N+1}:=0$.


## Demand

- $D_{t_{1}, t_{2}}, t_{1}, t_{2} \in \mathbb{N}_{0}, t_{2} \geq t_{1}$ : Random variable which denotes the cumulative demand over the periods $t_{1}, \ldots, t_{2}$.
- $F$ : Generic distribution function of the demand $D_{t, t}$ in an arbitrary period $t \in \mathbb{N}_{0}$. We assume that $F$ is a continuous distribution on $(0, \infty)$, with $F(0)=0$. (Later, when we consider the same model with a $\gamma$-service level constraint, we in addition assume that $F$ has a compact support, i.e., that $F$ is strictly increasing from 0 to 1 on an interval $\left[a_{n}, b_{n}\right)$ with $0 \leq a_{n}<b_{n} \leq \infty$.)
- $\mu$ : Expected value of $D_{t, t}$; notice that $\mu>0$.


## Reorder Intervals

- $R_{n}, n=1, \ldots, N$ : The given reorder interval for the $n$-th stage (exogenous variable). The reorder intervals are assumed to satisfy the integer-ratio constraint, i.e., we assume that for each $n=1, \ldots, N-$ $1, R_{n+1} / R_{n}=r_{n} \in \mathbb{N}$, or, equivalently $R_{n}=R_{n+1} / r_{n}$, where $r_{n}$ denotes the number of times that stage $n$ can order per order of stage $n+1$. Let $r_{0}:=R_{1}$ (so, $r_{0}$ denotes how often customer demand occurs per order of stage 1) and $r_{N}:=1$. For all $n=1, \ldots, N, R_{n}=\prod_{i=0}^{n-1} r_{i}$. The $r_{n}$ are called the relative reorder frequencies.
- $T_{n}, n=1, \ldots, N$ : The set of periods $t \in \mathbb{N}_{0}$ in which stage $n$ is allowed to place an order. W.l.o.g., we assume that stage $N$ places its first order in period 0 . Then $T_{N}:=\left\{k R_{N} \mid k \in \mathbb{N}_{0}\right\}$, and $T_{n}:=$ $\left\{L_{n+1}+k R_{n} \mid t \in T_{N}, k \in \mathbb{Z}, t+L_{n+1}+k R_{n} \geq 0\right\}$ for all $n=1, \ldots, N-1$.

Note that the reorder epochs $T_{n}$ are offset by $L_{n+1}$ to allow the lower stage to reorder from the upper at the exact moment an order arrives at the upper stage (and equidistant times thereafter). This constraint is called the synchronization constraint.

## Costs

- $H_{n}, n=1, \ldots, N$ : The inventory holding cost parameters. A cost of $H_{n}, n=2, \ldots, N$ is charged for each unit that is on stock in stockpoint $n$ at the end of a period and for each unit in the pipeline from the $n$-th to the $(n-1)$-th stockpoint. A cost of $H_{1}$ is charged for each unit that is on stock in


Figure 1: The 2-stage serial inventory/production system of Example 1.
stockpoint 1 at the end of a period. So, we pretend that in each stage the value of a product changes at the end of the stage only, i.e., when the product enters the stockpoint at the end of the stage. We assume that we have nonnegative added value per stage, i.e., that $H_{1} \geq H_{2} \geq \ldots \geq H_{N} \geq 0$. For notational convenience, $H_{N+1}:=0$.

- $h_{n}, n=1, \ldots, N$ : The additional inventory holding cost parameters; $h_{n}:=H_{n}-H_{n+1}$ for $n=N, \ldots, 1$. Notice that $h_{n} \geq 0$ for all $n=1, \ldots, N$.
- $p$ : Penalty cost parameter. A cost $p$ is charged per unit of backlog at stockpoint 1 at the end of a period. We assume that $p>0$.

Further notation will be introduced during the analysis as needed.

Example 1 During this paper we use a serial system consisting of $N=2$ stages, with leadtimes $l_{1}=l_{2}=1$ and reorder intervals $R_{1}=2$ and $R_{2}=4$, as an illustrative example. This implies that $L_{3}=0, L_{2}=1$, $L_{1}=2, r_{2}=1, r_{1}=2, r_{0}=2, T_{2}=\{0,4,8, \ldots\}, T_{1}=\{1,3,5, \ldots\}$. The other input variables are specified when needed. See Figures 1 and 2 for a visualization of the input variables for our example and the time epochs at which orders are placed by stages 1 and 2 .

### 2.3 Objective Function and Echelon Costs

Let $\Pi$ denote the set of all feasible ordering policies, and let $G(\pi)$ denote the average costs of ordering policy $\pi$ for all $\pi \in \Pi$. The objective is to find an ordering policy under which the average costs per period are


Figure 2: Timing of orders placed by stages 1 and 2 for the system of Example 1.
minimized, or, equivalently, to solve:

$$
\begin{array}{lll}
(P): & \operatorname{Min} & G(\pi) \\
& \text { s.t. } & \pi \in \Pi .
\end{array}
$$

Here, the average costs consist of inventory holding costs and penalty costs. In Section 4 we will show how the optimal ordering policy can also be found when the objective is to minimize the average inventory holding costs subject to a $\gamma$-service level constraint.

Before we start with the analysis we have to introduce the concepts echelon stock and echelon inventory position, as well as some relevant cost functions. These are all standard in multi-echelon inventory theory, so we just summarize them here. For an explanation in greater depth, the reader is referred to Zipkin (2000, p. 120-124).

The portion of our serial supply chain from the most downstream stockpoint 1 up to any other stockpoint is called an echelon. Echelons are numbered according to the highest stockpoint in that echelon. The echelon stock, or echelon inventory level, of a given echelon $n$ denotes all physical stock at stockpoint $n$ plus all materials in transit to or on hand at any stockpoint downstream, minus a possible backlog at stockpoint 1. The echelon inventory position of an echelon $n$ is defined as its echelon stock plus all materials which are in transit to stockpoint $n$. As we have centralized control, we may assume w.l.o.g. that a stockpoint never orders more than what is available at the next upstream stockpoint (cf. Chen and Zheng, 1994). Hence, our definition of echelon inventory position is equivalent to defining the echelon inventory position as the echelon stock plus all materials which are on order. The echelon stock and echelon inventory position of echelon $n$
are also called echelon stock $n$ and echelon inventory position $n$, respectively.
We now define so-called costs attached per echelon. Let $x_{n}$ denote echelon stock $n$ at the end of a period. Notice that, by the above definitions, it holds that $x_{n} \geq x_{n-1}$ for $n=2, \ldots, N$. Therefore we find that the total costs at the end of the period under consideration are equal to

$$
\begin{aligned}
& \sum_{n=2}^{N} H_{n}\left(x_{n}-x_{n-1}\right)+H_{1} x_{1}^{+}+p x_{1}^{-} \\
& =H_{N} x_{N}+\sum_{n=1}^{N-1}\left(H_{n}-H_{n+1}\right) x_{n}+\left(p+H_{1}\right) x_{1}^{-} \\
& =\sum_{n=1}^{N} h_{n} x_{n}+\left(p+H_{1}\right) x_{1}^{-}
\end{aligned}
$$

where $x^{+}=\max \{0, x\}$ and $x^{-}=\max \{0,-x\}=-\min \{0, x\}$ for any $x \in \mathbb{R}$. This formula shows that the costs may be written as a sum of cost terms per echelon. The costs $h_{n} x_{n}$ are the costs attached to echelon $n$ (or the echelon $n$ costs), $n=2, \ldots, N$, and the costs $h_{1} x_{1}+\left(p+H_{1}\right) x_{1}^{-}$are the costs attached to echelon 1. Notice that the terms $h_{n} x_{n}$ always appear, independent of the sign of $x_{n}$.

Notationally, for each $n=1, \ldots, N$ and $t \in \mathbb{N}_{0}$, we let $I L_{t, n}$ and $I P_{t, n}$ denote echelon stock $n$ ( $=$ echelon inventory level $n$ ) and echelon inventory position $n$ at the beginning of period $t$ (just before the demand occurs), and we let $C_{t, n}$ denote the costs attached to echelon $n$ at the end of period $t$.

## 3 Analysis

We first show the relationship between, and direct impact of, ordering decisions of different stages, in Subsection 3.1. This constitutes the basis for the analysis of basestock policies and the derivation of an optimal policy, in Subsections 3.2 and 3.3 , respectively. In Subsection 3.4, we describe additional results that follow directly from the main results. Finally, an exact solution procedure for the case with mixed Erlang demands is described in Subsection 3.5.

### 3.1 Setup of the Analysis

In this subsection, we describe the connection between ordering decisions at different stages and which costs they affect.

Let $t_{0}$ be a period in which stage $N$ may place an order; i.e., $t_{0} \in T_{N}$. By this order, $I P_{t_{0}, N}$ is increased to a certain level $z$. We say that this order starts a whole order cycle in the chain; the ordering decision for stage $N$ in period $t_{0}$ affects a whole tree of decisions. First, the decision directly affects the ordering by stage $N-1$ in the periods $\tau_{k}:=t_{0}+l_{N}+(k-1) R_{N-1}$, with $k=1, \ldots, r_{N-1}$, by which echelon inventory level $I L_{N-1}$ is increased. Next, each of the ordering decisions for stage $N-1$ directly affects the ordering by stage $N-2$ in the periods $\tau_{k}+l_{N-1}+(m-1) R_{N-2}$, with $m=1, \ldots, r_{N-2}$; and so on.

To denote the whole tree of decisions, we introduce the following set of vectors, which has a direct correspondence with the decisions in the order cycle:

$$
\begin{gathered}
A \stackrel{\text { def }}{=}\left\{\left(a_{0}, a_{1}, \ldots, a_{N}\right) \in \mathbb{N}_{0}^{N+1} \mid a_{i}=0 \text { for } 0 \leq i \leq k-1, \text { where } k \in\{0, \ldots, N\},\right. \\
\text { and } \left.a_{i} \in\left\{1, \ldots, r_{i}\right\} \text { for } k \leq i \leq N\right\} .
\end{gathered}
$$

In addition, for each $a \in A$, we define $\operatorname{lev}(a)$ as the index of the first nonzero component of $a$, i.e.,

$$
\operatorname{lev}(a) \stackrel{\text { def }}{=} \min \left\{i \mid a_{i}>0\right\} .
$$

The vector $(0, \ldots, 0,1)$ is the only vector in $A$ with $\operatorname{lev}(a)=N$. This vector is used as the label for the ordering decision for echelon inventory position $N$ at the beginning of period $t_{0}$. Next, the vectors $(0, \ldots, 0, k, 1), k=1, \ldots, r_{N-1}$ (each with level $\left.N-1\right)$ are used to denote the ordering decisions for stage $N-1$ in the periods $t_{0}+l_{N}+(k-1) R_{N-1}$; etcetera. Define

$$
A_{n} \stackrel{\text { def }}{=}\{a \in A \mid \operatorname{lev}(a)=n\}, \quad n=0, \ldots, N
$$

Thus the vectors $a \in A_{n}, n=1, \ldots, N$, denote the decisions at level $n$ in the order cycle, with each vector corresponding to a decision epoch in a specific way. Specifically, vector $a \in A_{n}, n \geq 1$, denotes the ordering decision for stage $n$ at the beginning of period $t_{a} \stackrel{\text { def }}{=} t_{0}+L_{n+1}+\sum_{i=n}^{N}\left(a_{i}-1\right) R_{i}$. The remaining elements of $A$, i.e. the vectors $a \in A_{0}$, are used to label the relevant backlogs as seen by the customers. Vector $a \in A_{0}$ is the label for the backlog at stage 1 at the end of period $t_{a} \stackrel{\text { def }}{=} t_{0}+L_{1}+\sum_{i=0}^{N}\left(a_{i}-1\right) R_{i}$. Finally, for each $a \in A \backslash\{(0, \ldots, 0,1)\}$, we define $\operatorname{par}(a)$ as the parent of $a$, obtained by replacing the first non-zero component of $a$ by a zero. For each $a \in A \backslash A_{0}$, the vectors $\tilde{a} \in A$ for which $\operatorname{par}(\tilde{a})=a$ are called the children of $a$. For all $a \in A_{n}, n \geq 2$, decision $a$ directly affects the decisions $\tilde{a} \in A_{n-1}$ for which $\operatorname{par}(\tilde{a})=a$. For all $a \in A_{1}$, decision $a$ directly affects the backlogs $\tilde{a} \in A_{0}$ for which $\operatorname{par}(\tilde{a})=a$.

Example 1 (continued) For our 2-stage example system, we have:

$$
\begin{aligned}
& A=\{(0,0,1),(0,1,1),(0,2,1),(1,1,1),(2,1,1),(1,2,1),(2,2,1)\} \\
& \operatorname{lev}((0,0,1))=2, \operatorname{lev}((0,1,1))=\operatorname{lev}((0,2,1))=1 \\
& \operatorname{lev}((1,1,1))=\operatorname{lev}((2,1,1))=\operatorname{lev}((1,2,1))=\operatorname{lev}((2,2,1))=0 \\
& A_{2}=\{(0,0,1)\}, A_{1}=\{(0,1,1),(0,2,1)\}, A_{0}=\{(1,1,1),(2,1,1),(1,2,1),(2,2,1)\}, \\
& t_{(0,0,1)}=t_{0}, \quad t_{(0,1,1)}=t_{0}+1, \quad t_{(0,2,1)}=t_{0}+3, \quad t_{(1,1,1)}=t_{0}+2 \\
& t_{(2,1,1)}=t_{0}+3, \quad t_{(1,2,1)}=t_{0}+4, \quad t_{(2,2,1)}=t_{0}+5, \quad \text { where } t_{0} \in T_{2} \\
& \operatorname{par}((0,1,1))=\operatorname{par}((0,2,1))=(0,0,1), \quad \operatorname{par}((1,1,1))=\operatorname{par}((2,1,1))=(0,1,1) \\
& \operatorname{par}((1,2,1))=\operatorname{par}((2,2,1))=(0,2,1) .
\end{aligned}
$$

Here, e.g., the vector $(0,2,1)$ denotes the ordering decision taken at stage 1 at time $t_{0}+3$, where $t_{0} \in T_{2}$. This decision is directly affected by the parent decision $(0,0,1)$. The vector $(2,2,1)$ is the label for the backlog at the end of period $t_{0}+5$. See Figure 3 .

We now describe which costs are directly affected by the decisions $a \in A \backslash A_{0}$, and, while doing that, we also give a more detailed description of the relationship between these decisions $a \in A \backslash A_{0}$,

- Decision $a=(0, \ldots, 0,1)$ : This decision concerns the decision at the beginning of period $t_{0}$ with respect to the order placed by stage $N$ at the external supplier. Suppose that this order is such that the echelon inventory position $I P_{t_{0}, N}$ becomes equal to some level $z_{(0, \ldots, 0,1)}$. First of all, this decision directly affects the echelon $N$ costs at the end of the periods $t_{0}+l_{N}+k, k=0, \ldots, R_{N}-1$. The expected values of these costs are equal to

$$
\begin{aligned}
\mathrm{E}\left\{C_{t_{0}+l_{N}+k, N} \mid I P_{t_{0}, N}=z_{(0, \ldots, 0,1)}\right\} & =\mathrm{E}\left\{h_{N}\left(z_{(0, \ldots, 0,1)}-D_{t_{0}, t_{0}+l_{N}+k}\right)\right\} \\
& =h_{N}\left(z_{(0, \ldots, 0,1)}-\left(l_{N}+k+1\right) \mu\right)
\end{aligned}
$$

and for the sum we find

$$
\begin{align*}
\sum_{k=0}^{R_{N}-1} \mathrm{E}\left\{C_{t_{0}+l_{N}+k, N} \mid I P_{t_{0}, N}=z_{(0, \ldots, 0,1)}\right\}= & R_{N} h_{N} *\left[z_{(0, \ldots, 0,1)}\right. \\
& \left.-\left(l_{N}+\frac{1}{2}\left(R_{N}+1\right)\right) \mu\right] . \tag{1}
\end{align*}
$$

Second, decision $(0, \ldots, 0,1)$ affects the decisions $a \in A_{N-1}$. At the beginning of period $\tau_{k}=t_{0}+$ $l_{N}+(k-1) R_{N-1}, k=1, \ldots, r_{N-1}$, echelon stock $N$ becomes equal to $I L_{\tau_{k}, N}=z_{(0, \ldots, 0,1)}-D_{t_{0}, \tau_{k}-1}$, and this limits the level to which $I P_{\tau_{k}, N-1}$ can be increased at the beginning of that period $\tau_{k}$. These decisions at level $N-1$ are the next decisions to consider.

- Decisions $a \in A_{n}$, for $n=N-1, \ldots, 2$ (this range of values for $n$ is empty when $N=2$ ): Let $n \in\{2, \ldots, N-1\}$ and $a \in A_{n}$. Decision $a$ concerns the decision with respect to the order placed by stage $n$ at the beginning of period $t_{a}=t_{0}+L_{n+1}+\sum_{i=n}^{N}\left(a_{i}-1\right) R_{i}$. Suppose that by this order $I P_{t_{a}, n}$ becomes equal to some level $z_{a}$. First of all, this decision directly affects the echelon $n$ costs at the end of the periods $t_{a}+l_{n}+k, k=0, \ldots, R_{n}-1$. The expected values of these costs are equal to

$$
\mathrm{E}\left\{C_{t_{a}+l_{n}+k, n} \mid I P_{t_{a}, n}=z_{a}\right\}=\mathrm{E}\left\{h_{n}\left(z_{a}-D_{t_{a}, t_{a}+l_{n}+k}\right)\right\}=h_{n}\left(z_{a}-\left(l_{n}+k+1\right) \mu\right)
$$

and for the sum we find

$$
\begin{equation*}
\sum_{k=0}^{R_{n}-1} \mathrm{E}\left\{C_{t_{a}+l_{n}+k, n} \mid I P_{t_{a}, n}=z_{a}\right\}=R_{n} h_{n}\left[z_{a}-\left(l_{n}+\frac{1}{2}\left(R_{n}+1\right)\right) \mu\right] \tag{2}
\end{equation*}
$$

Second, decision $a$ affects the decisions $\tilde{a} \in A_{n-1}$ for which $\operatorname{par}(\tilde{a})=a$. At the beginning of period $\tau_{m}:=t_{a}+l_{n}+(m-1) R_{n-1}, m=1, \ldots, r_{n-1}$, echelon stock $n$ becomes equal to $I L_{\tau_{m}, n}=z_{a}-D_{t_{a}, \tau_{m}-1}$, and this limits the level to which $I P_{\tau_{m}, n-1}$ can be increased at the beginning of period $\tau_{m}$.

- Decisions $a \in A_{1}$ : Let $a \in A_{1}$. Decision $a$ concerns the decision with respect to the order placed by stage 1 at the beginning of period $t_{a}=t_{0}+L_{2}+\sum_{i=1}^{N}\left(a_{i}-1\right) R_{i}$. Suppose that by this order $I P_{t_{a}, 1}$ becomes equal to some level $z_{a}$. This decision directly affects the echelon 1 costs at the end of the periods $t_{a}+l_{1}+k, k=0, \ldots, R_{1}-1$. The expected values of these costs are equal to

$$
\begin{aligned}
\mathrm{E}\left\{C_{t_{a}+l_{1}+k, 1} \mid I P_{t_{a}, 1}=z_{a}\right\}= & \mathrm{E}\left\{h_{1}\left(z_{a}-D_{t_{a}, t_{a}+l_{1}+k}\right)+\right. \\
& \left.\left(p+H_{1}\right)\left(D_{t_{a}, t_{a}+l_{1}+k}-z_{a}\right)^{+}\right\} \\
= & h_{1}\left(z_{a}-\left(l_{1}+k+1\right) \mu\right)+ \\
& \left(p+H_{1}\right) \mathrm{E}\left\{\left(D_{t_{a}, t_{a}+l_{1}+k}-z_{a}\right)^{+}\right\} .
\end{aligned}
$$



Figure 3: The relationship between and the consequences of the decisions $a \in A \backslash A_{0}$ for the system of Example 1.

The sum of these expected costs equals

$$
\begin{align*}
\sum_{k=0}^{R_{1}-1} \mathrm{E}\left\{C_{t_{a}+l_{1}+k, 1} \mid I P_{t_{a}, 1}=z_{a}\right\}= & R_{1} h_{1}\left[z_{a}-\left(l_{1}+\frac{1}{2}\left(R_{1}+1\right)\right) \mu\right] \\
& +\left(p+H_{1}\right) \sum_{k=0}^{R_{1}-1} \mathrm{E}\left\{\left(D_{t_{a}, t_{a}+l_{1}+k}-z_{a}\right)^{+}\right\} \tag{3}
\end{align*}
$$

Figure 3 illustrates the way in which the above decisions affect each other and which costs are determined by them for Example 1.

In the description above, we have explicitly described for each decision $a \in \cup_{n=1}^{N-1} A_{n}$ how the level $z_{a}$ to which $I P_{t_{a}, l e v(a)}$ is increased, is bounded from above. We will need this in the analysis below. Obviously, for each decision $a \in \cup_{n=1}^{N-1} A_{n}$, it also holds that the level $z_{a}$ to which $I P_{t_{a}, l e v(a)}$ is increased, is bounded from below (by the level that one already has for echelon inventory position $\operatorname{lev}(a)$ just before the new order is placed). In the analysis below, this is taken into account too. But, for the analysis the bounding from below will appear to be less important.

The tree of decisions $a \in A \backslash A_{0}$ starts with decision $(0, \ldots, 0,1)$ taken in a period $t_{0} \in T_{N}$. It determines the costs $C_{t_{0}}$ over the corresponding cycle:

$$
C_{t_{0}} \stackrel{\text { def }}{=} \sum_{n=1}^{N} \sum_{k=0}^{R_{N}-1} C_{t_{0}+L_{n}+k, n}
$$

These costs are defined for each period $t_{0} \in T_{N}$, and we call them the total costs attached to cycle $t_{0}$. Notice that $C_{t_{0}}$ contains costs over different shifted time intervals for different echelons. It is easily verified that $C_{t_{0}}$ also may be written as

$$
C_{t_{0}}=\sum_{n=1}^{N} \sum_{a \in A_{n}} \sum_{k=0}^{R_{n}-1} C_{t_{a}+l_{n}+k, n}
$$

Example 1 (continued) Let us continue with the illustrative example in order to explain the expressions for $C_{t_{0}}$. Then for each $t_{0} \in T_{2}$ :

$$
\begin{aligned}
C_{t_{0}}= & \left(C_{t_{0}+1,2}+C_{t_{0}+2,2}+C_{t_{0}+3,2}+C_{t_{0}+4,2}\right) \\
& +\left(C_{t_{0}+2,1}+C_{t_{0}+3,1}\right)+\left(C_{t_{0}+4,1}+C_{t_{0}+5,1}\right) \\
= & \sum_{n=1}^{2} \sum_{a \in A_{n}} \sum_{k=0}^{R_{n}-1} C_{t_{a}+l_{n}+k, n} .
\end{aligned}
$$

For each positive recurrent policy $\pi \in \Pi$, the average costs are equal to the average value of the costs $C_{t_{0}}$ over all cycles $t_{0} \in T_{N}$ divided by the cycle length $R_{N}$ :

$$
\begin{align*}
G(\pi) & =\lim _{T \rightarrow \infty} \frac{1}{T} \mathrm{E}\left\{\sum_{t=0}^{T-1} \sum_{n=1}^{N} C_{t, n}\right\}=\lim _{k \rightarrow \infty} \frac{1}{k R_{N}} \mathrm{E}\left\{\sum_{t=0}^{k R_{N}-1} \sum_{n=1}^{N} C_{t, n}\right\} \\
& =\lim _{k \rightarrow \infty} \frac{1}{k R_{N}} \mathrm{E}\left\{\sum_{j=0}^{k-1} C_{j R_{N}}+\sum_{n=1}^{N} \sum_{t=0}^{L_{n}-1} C_{t, n}-\sum_{n=1}^{N} \sum_{t=k R_{N}}^{k R_{N}+L_{n}-1} C_{t, n}\right\} \\
& =\lim _{k \rightarrow \infty} \frac{1}{k R_{N}} \sum_{j=0}^{k-1} \mathrm{E} C_{j R_{N}} . \tag{4}
\end{align*}
$$

The above expression requires that the expectations exist and be finite. While this need not be true for general inventory policies (in particular those that do not order sufficiently to satisfy demand), any policy that is positive recurrent will meet this requirement. The class we consider below, basestock policies, are well known to be positive recurrent.

### 3.2 Basestock Policies

A relevant class of ordering policies is constituted by the class of basestock policies. A basestock policy is denoted by a tuple $\left(y_{1}, \ldots, y_{N}\right)$, where $y_{n} \in \mathbb{R}$ denotes the desired order-up-to level for the echelon inventory position $n$. Under basestock policy $\left(y_{1}, \ldots, y_{N}\right)$, the ordering decisions are taken as follows: at the beginning of each period $t \in T_{N}$, echelon inventory position $N$ is increased to $y_{N}$. For each $n=N-1, \ldots, 1$, at the beginning of each period $t \in T_{n}$, echelon inventory position $n$ is increased to the minimum of $y_{n}$ and the actual echelon stock of echelon $n+1$ (the start up phenomena, occurring in case the initial echelon inventory positions are larger than the desired levels, are ignored, since the long run average costs are not affected by these). Notice that we do not require that the basestock levels be nondecreasing.

The average costs for a basestock policy $\left(y_{1}, \ldots, y_{N}\right)$ are denoted by $G\left(y_{1}, \ldots, y_{N}\right)$. It is easily seen that

$$
\begin{align*}
G\left(y_{1}, \ldots, y_{N}\right)= & \frac{1}{R_{N}} \mathrm{E} C_{t_{0}} \\
= & \frac{1}{R_{N}} \mathrm{E}\left\{\sum_{n=1}^{N} \sum_{a \in A_{n}} \sum_{k=0}^{R_{n}-1} C_{t_{a}+l_{n}+k, n} \mid z_{(0, \ldots, 0,1)}=y_{N}\right. \\
& \left.z_{a}=\min \left\{I L_{t_{a}, n+1}, y_{n}\right\} \text { for all } n=N-1, \ldots, 1 \text { and } a \in A_{n}\right\}, \tag{5}
\end{align*}
$$

where the tree of decisions $a \in A \backslash A_{0}$ starts with decision $(0, \ldots, 0,1)$ at the beginning of some period $t_{(0, \ldots, 0,1)}=t_{0} \in T_{N}$ (as described in the previous subsection).

We now analyze the sums $\sum_{k=0}^{R_{n}-1} C_{t_{a}+l_{n}+k, n}$ for $n=N, \ldots, 1$ and $a \in A_{n}$, referring to formulae (1)(3). The term for $n=N$ and $a=(0, \ldots, 0,1)$ is the simplest one. The expected value of the costs $\sum_{k=0}^{R_{N}-1} C_{t_{a}+l_{N}+k, N}$ equals (cf. (1))

$$
\sum_{k=0}^{R_{N}-1} \mathrm{E} C_{t_{a}+l_{N}+k, N}=R_{N} h_{N}\left[y_{N}-\left(l_{N}+\frac{1}{2}\left(R_{N}+1\right)\right) \mu\right]
$$

Next, we consider $\sum_{k=0}^{R_{n}-1} C_{t_{a}+l_{n}+k, n}$ for $n=N-1$ and $a \in A_{N-1}$. The expected value of this sum is equal to (cf. (2))

$$
\sum_{k=0}^{R_{N-1}-1} \mathrm{E} C_{t_{a}+l_{N-1}+k, N-1}=R_{N-1} h_{N-1}\left[\mathrm{E}\left\{z_{a}\right\}-\left(l_{N-1}+\frac{1}{2}\left(R_{N-1}+1\right) \mu\right],\right.
$$

with $z_{a}=\min \left\{y_{N}-D_{t_{0}, t_{a}-1}, y_{N-1}\right\}$. This holds if $N \geq 3$; below we describe the formulae for the general case with $N \geq 2$. The level $z_{a}$ denotes the actual level to which $I P_{t_{a}, N-1}$ is increased. The difference between this and the desired level $y_{N-1}$ is called the shortfall, which can also be seen as a backlog at
stage $N$ (it would be the backlog at stage $N$ if stage $N-1$ would order such that $I P_{t_{a}, N-1}$ is increased up to $y_{N-1}$, without taking into account how much is available at stage $N$ ). We denote this shortfall by $B_{a}=y_{N-1}-z_{a}=y_{N-1}-\min \left\{y_{N}-D_{t_{0}, t_{a}-1}, y_{N-1}\right\}=\left(D_{t_{0}, t_{a}-1}-\left(y_{N}-y_{N-1}\right)\right)^{+}$. (Notice that this shortfall is 0 if and only if $y_{N} \geq y_{N-1}$ and $D_{t_{0}, t_{a}-1} \leq y_{N}-y_{N-1}$; if $y_{N}<y_{N-1}$, then this shortfall is positive by definition.) We now find that

$$
\sum_{k=0}^{R_{N-1}-1} \mathrm{E} C_{t_{a}+l_{N-1}+k, N-1}=R_{N-1} h_{N-1}\left[y_{N-1}-\left(l_{N-1}+\frac{1}{2}\left(R_{N-1}+1\right) \mu-\mathrm{E} B_{a}\right] .\right.
$$

A similar expression holds for the sums $\sum_{k=0}^{R_{n}-1} C_{t_{a}+l_{n}+k, n}$ with $n \leq N-2$ and $a \in A_{n}$.
To find the general expressions for the expected values of the sums $\sum_{k=0}^{R_{n}-1} C_{t_{a}+l_{n}+k, n}$, for the general case with $N \geq 2$, we define

$$
\begin{align*}
B_{a} & =0 \text { for } a=(0, \ldots, 0,1)  \tag{6}\\
B_{a} & =\left(B_{\operatorname{par}(a)}+D_{t_{\operatorname{par}(a)}, t_{a}-1}-\left(y_{n+1}-y_{n}\right)\right)^{+} \text {for all } n=N-1, \ldots, 1, a \in A_{n},  \tag{7}\\
B_{a} & =\left(B_{\operatorname{par}(a)}+D_{t_{\operatorname{par}(a)}, t_{a}}-y_{1}\right)^{+} \text {for all } a \in A_{0} . \tag{8}
\end{align*}
$$

For each $a \in A \backslash A_{0}, B_{a}$ denotes the shortfall when decision $a$ is taken, i.e., the shortfall at stage lev $(a)+1$ (read external supplier when $\operatorname{lev}(a)=N)$ at the beginning of period $t_{a}$. For each $a \in A_{0}$, the random variable $B_{a}$ denotes the backlog at stage 1 at the end of period $t_{a}$. Then, using (2) and (3), it can be shown that

$$
\begin{aligned}
\sum_{k=0}^{R_{n}-1} \mathrm{E} C_{t_{a}+l_{n}+k, n}= & R_{n} h_{n}\left[y_{n}-\left(l_{n}+\frac{1}{2}\left(R_{n}+1\right)\right) \mu-\mathrm{E} B_{a}\right], n=N, \ldots, 2, a \in A_{n} \\
\sum_{k=0}^{R_{1}-1} \mathrm{E} C_{t_{a}+l_{1}+k, 1}= & R_{1} h_{1}\left[y_{1}-\left(l_{1}+\frac{1}{2}\left(R_{1}+1\right)\right) \mu-\mathrm{E} B_{a}\right] \\
& +\left(p+H_{1}\right) \sum_{\tilde{a} \in A_{0}, \operatorname{par}(\tilde{a})=a} \mathrm{E} B_{\tilde{a}}, a \in A_{1} .
\end{aligned}
$$

By substitution of these formulae into (5), we obtain the following theorem.

Theorem 2 The average costs of a basestock policy $\left(y_{1}, \ldots, y_{N}\right)$, with $y_{n} \in \mathbb{R}$ for all $n=1, \ldots, N$, are equal to

$$
\begin{aligned}
G\left(y_{1}, \ldots, y_{N}\right)= & \sum_{n=1}^{N} h_{n}\left\{y_{n}-\left(l_{n}+\frac{1}{2}\left(R_{n}+1\right)\right) \mu-\frac{R_{n}}{R_{N}} \sum_{a \in A_{n}} \mathrm{E} B_{a}\right\} \\
& +\left(p+H_{1}\right) \frac{1}{R_{N}} \sum_{a \in A_{0}} \mathrm{E} B_{a},
\end{aligned}
$$

where the random variables $B_{a}$ are given by (6)-(8).

Proof : Starting with the substitution of the expressions given just before the theorem into equation (5), we obtain:

$$
\begin{aligned}
G\left(y_{1}, \ldots, y_{N}\right)= & \frac{1}{R_{N}}\left[\sum_{n=2}^{N} \sum_{a \in A_{n}} R_{n} h_{n}\left[y_{n}-\left(l_{n}+\frac{1}{2}\left(R_{n}+1\right)\right) \mu-\mathrm{E} B_{a}\right]\right. \\
& +\sum_{a \in A_{1}}\left\{R_{1} h_{1}\left[y_{1}-\left(l_{1}+\frac{1}{2}\left(R_{1}+1\right)\right) \mu-\mathrm{E} B_{a}\right]\right. \\
& \left.\left.+\left(p+H_{1}\right) \sum_{\tilde{a} \in A_{0}, \operatorname{par}(\tilde{a})=a} \mathrm{E} B_{\tilde{a}}\right\}\right] \\
= & \sum_{n=1}^{N} \sum_{a \in A_{n}} \frac{R_{n}}{R_{N}} h_{n}\left[y_{n}-\left(l_{n}+\frac{1}{2}\left(R_{n}+1\right)\right) \mu-\mathrm{E} B_{a}\right] \\
& +\left(p+H_{1}\right) \frac{1}{R_{N}} \sum_{a \in A_{1}} \sum_{\tilde{a} \in A_{0}, p a r(\tilde{a})=a} \mathrm{E} B_{\tilde{a}} \\
= & \sum_{n=1}^{N} h_{n}\left\{y_{n}-\left(l_{n}+\frac{1}{2}\left(R_{n}+1\right)\right) \mu-\frac{R_{n}}{R_{N}} \sum_{a \in A_{n}} \mathrm{E} B_{a}\right\} \\
& +\left(p+H_{1}\right) \frac{1}{R_{N}} \sum_{a \in A_{0}} \mathrm{E} B_{a},
\end{aligned}
$$

where in the last step we use that the number of elements of $A_{n}$ is equal to

$$
\left|A_{n}\right|=\prod_{i=n}^{N} r_{i}=\frac{R_{N}}{R_{n}}
$$

Note that as the $\mathrm{E} B_{a}$ depend on $a$, we cannot make this substitution for the remaining sums $\sum_{a \in A_{n}} \mathrm{E} B_{a}$ in the last expression.

Theorem 2 gives a very simple expression for the average costs, subject to the evaluation of the average shortfalls/backlogs $\mathrm{E} B_{a}, a \in A$. This idea will be important in Subsection 3.5

Example 1 (continued) For our illustrative example, we find that the average costs of a basestock policy $\left(y_{1}, y_{2}\right), y_{1}, y_{2} \in \mathbb{R}$, are equal to

$$
\begin{align*}
G\left(y_{1}, y_{2}\right)= & h_{2}\left\{y_{2}-\frac{7}{2} \mu\right\}+h_{1}\left\{y_{1}-\frac{5}{2} \mu-\frac{1}{2}\left(\mathrm{E} B_{(0,1,1)}+\mathrm{E} B_{(0,2,1)}\right)\right\} \\
& +\left(p+H_{1}\right) \cdot \frac{1}{4}\left(\mathrm{E} B_{(1,1,1)}+\mathrm{E} B_{(2,1,1)}+\mathrm{E} B_{(1,2,1)}+\mathrm{E} B_{(2,2,1)}\right) \tag{9}
\end{align*}
$$

where

$$
\begin{aligned}
& B_{(0,1,1)}=\left(D_{t_{0}, t_{0}}-\left(y_{2}-y_{1}\right)\right)^{+}, \quad B_{(0,2,1)}=\left(D_{t_{0}, t_{0}+2}-\left(y_{2}-y_{1}\right)\right)^{+}, \\
& B_{(1,1,1)}=\left(B_{(0,1,1)}+D_{t_{0}+1, t_{0}+2}-y_{1}\right)^{+}, \quad B_{(2,1,1)}=\left(B_{(0,1,1)}+D_{t_{0}+1, t_{0}+3}-y_{1}\right)^{+}, \\
& B_{(1,2,1)}=\left(B_{(0,2,1)}+D_{t_{0}+3, t_{0}+4}-y_{1}\right)^{+}, \quad B_{(2,2,1)}=\left(B_{(0,2,1)}+D_{t_{0}+3, t_{0}+5}-y_{1}\right)^{+} .
\end{aligned}
$$

For the sake of the analysis below, we now introduce cost functions $G_{n}\left(y_{1}, \ldots, y_{n}\right)$, with $n=1, \ldots, N$, and $y_{i} \in \mathbb{R}$ for all $i=1, \ldots, n$. The function $G_{n}\left(y_{1}, \ldots, y_{n}\right)$ is defined as the average costs attached to the echelons $1, \ldots, n$ when each of the stages $1, \ldots, n$ applies a basestock policy with basestock level $y_{i}$ and when stage $n+1$ can always deliver. Obviously, $G_{N}\left(y_{1}, \ldots, y_{N}\right)=G\left(y_{1}, \ldots, y_{N}\right)$. For the functions $G_{n}\left(y_{1}, \ldots, y_{n}\right)$ we can derive expressions similar to $G\left(y_{1}, \ldots, y_{N}\right)$. We first define

$$
\begin{array}{r}
A^{(n)} \stackrel{\text { def }}{=}\left\{\left(a_{0}, a_{1}, \ldots, a_{n}\right) \in \mathbb{N}_{0}^{n+1} \mid a_{i}=0 \text { for } 0 \leq i \leq k-1, \text { where } k \in\{0, \ldots, n\},\right. \\
\left.a_{i} \in\left\{1, \ldots, r_{i}\right\} \text { for } k \leq i \leq n-1, a_{n}=1\right\},
\end{array}
$$

and then define $\operatorname{lev}(a), A_{i}^{(n)}$ and $\operatorname{par}(a)$ respective to $A^{(n)}$ in an analogous manner as we did for the set $A$. Then it is easily verified that (cf. Equation (5))

$$
\begin{align*}
& G_{n}\left(y_{1}, \ldots, y_{n}\right)=\frac{1}{R_{n}} \mathrm{E}\left\{\sum_{i=1}^{n} \sum_{a \in A_{i}^{(n)}} \sum_{k=0}^{R_{i}-1} C_{t_{a}+l_{i}+k, i} \mid z_{(0, \ldots, 0,1)}=y_{n}\right. \\
& z_{a}\left.=\min \left\{I L_{t_{a}, i+1}, y_{i}\right\} \text { for all } 1 \leq i \leq n-1 \text { and } a \in A_{i}^{(n)}\right\} \tag{10}
\end{align*}
$$

where the tree of decisions $a \in A^{(n)} \backslash A_{0}^{(n)}$ starts with decision $(0, \ldots, 0,1)$ at the beginning of some period $t_{(0, \ldots, 0,1)}=t_{0} \in T_{n}$, and

$$
t_{a} \stackrel{\text { def }}{=} t_{0}+\sum_{j=i+1}^{n} l_{j}+\sum_{j=i}^{n}\left(a_{j}-1\right) R_{j}, \quad i=0, \ldots, n, a \in A_{i}^{(n)}
$$

Next, along the same lines as Theorem 2, we find the following result.

Lemma 3 For $n=1, \ldots, N$, and $y_{i} \in \mathbb{R}$ for all $i=1, \ldots, n$,

$$
\begin{aligned}
G_{n}\left(y_{1}, \ldots, y_{n}\right)= & \sum_{i=1}^{n} h_{i}\left\{y_{i}-\left(l_{i}+\frac{1}{2}\left(R_{i}+1\right)\right) \mu-\frac{R_{i}}{R_{n}} \sum_{a \in A_{i}^{(n)}} \mathrm{E} B_{a}^{(n)}\right\} \\
& +\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{E} B_{a}^{(n)}
\end{aligned}
$$

where the random variables $B_{a}^{(n)}$ are defined by

$$
\begin{aligned}
B_{a}^{(n)} & =0 \text { for } a=(0, \ldots, 0,1) \\
B_{a}^{(n)} & =\left(B_{\text {par }(a)}^{(n)}+D_{t_{p a r(a)}, t_{a}-1}-\left(y_{i+1}-y_{i}\right)\right)^{+} \text {for all } 1 \leq i \leq n-1, a \in A_{i}^{(n)}, \\
B_{a}^{(n)} & =\left(B_{p a r(a)}^{(n)}+D_{t_{p a r(a)}, t_{a}}-y_{1}\right)^{+} \text {for all } a \in A_{0}^{(n)} .
\end{aligned}
$$

Notice that $B_{a}^{(N)}=B_{a}$ for all $a \in A^{(N)}=A$. For $n=1, \ldots, N-1$ (the formulation that follows also holds for $n=N)$, the $B_{a}^{(n)}$ are related to the $B_{a}$ as follows. Let $n \in\{1, \ldots, N-1\}$ and let $\tilde{a} \in A_{n}$. Then

$$
B_{a}^{(n)} \stackrel{d}{=}\left(B_{\left(a_{0}, \ldots, a_{n-1}, \tilde{a}_{n}, \ldots, \tilde{a}_{N}\right)} \mid B_{\tilde{a}}=0\right) \quad \text { for all } a \in A^{(n)}
$$

where $\stackrel{d}{=}$ denotes equality in distribution. In other words, $B_{a}^{(n)}$ is equal in distribution to the shortfall of any full $N$-vector on the same decision epoch under the condition that there is no shortfall upstream of stage $n$ in the $N$-vector.

Similarly, the $B_{a}^{(k)}$ are related to the $B_{a}^{(n)}$ for some $k<n$ as follows. Let $k, n \in\{1, \ldots, N\}$ with $k<n$ and let $\tilde{a} \in A_{k}^{(n)}$. Then

$$
\begin{equation*}
B_{a}^{(k)} \stackrel{d}{=}\left(B_{\left(a_{0}, \ldots, a_{k-1}, \tilde{a}_{k}, \ldots, \tilde{a}_{n}\right)}^{(n)} \mid B_{\tilde{a}}^{(n)}=0\right) \quad \text { for all } a \in A^{(k)} \tag{11}
\end{equation*}
$$

For each of the functions $G_{n}\left(y_{1}, \ldots, y_{n}\right)$, we also need the partial derivative with respect to the last component $y_{n}$. Hence, for $n=1, \ldots, N$, we define

$$
g_{n}\left(y_{1}, \ldots, y_{n}\right) \stackrel{\text { def }}{=} \frac{\delta}{\delta y_{n}}\left\{G_{n}\left(y_{1}, \ldots, y_{n}\right)\right\}, \quad y_{i} \in \mathbb{R} \text { for all } i=1, \ldots, n
$$

For the partial derivatives, the following result holds.

Lemma 4 For $n=1, \ldots, N$, and $y_{i} \in \mathbb{R}$ for all $i=1, \ldots, n$,

$$
\begin{aligned}
g_{n}\left(y_{1}, \ldots, y_{n}\right)= & \sum_{i=1}^{n} h_{i}-\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}>0\right\} \\
& -\sum_{i=1}^{n-1} \frac{R_{i}}{R_{n}} \sum_{a \in A_{i}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}=0\right\} g_{i}\left(y_{1}, \ldots, y_{i}\right)
\end{aligned}
$$

(with the convention that the last sum on the righthand side is equal to 0 when $n=1$ ).

The proof of this lemma is given in the appendix.

### 3.3 Derivation of an Optimal Policy

Consider again the tree of decisions $a \in A \backslash A_{0}$ which starts with decision $(0, \ldots, 0,1)$ in some period $t_{0} \in T_{N}$; see the description in Subsection 3.1. We first consider how these decisions can be taken such that the expected total costs attached to cycle $t_{0}\left(=\mathrm{E} C_{t_{0}}\right)$ are minimized. Each decision $a \in A_{n}$, with $n=1, \ldots, N$, is described by the level $z_{a}$, to which echelon inventory position $n$ is increased at the beginning of period $t_{a}$. The choice for the level $z_{a}$ is limited from above by what is available at the next upstream stage and from below by the value of echelon inventory position $n$ just before the order is placed. For the moment, we neglect the bounding from below, and we consider the following relaxed problem $\left(R P\left(t_{0}\right)\right)$ :

$$
\begin{aligned}
& \text { Min } \\
& \text { s.t. } C_{t_{0}}=\sum_{n=1}^{N} \sum_{a \in A_{n}} \sum_{k=0}^{R_{n}-1} \mathrm{E} C_{t_{a}+l_{n}+k, n} \\
& \text { s. } \\
& \sum_{k=0}^{R_{1}-1} \mathrm{E} C_{t_{a}+l_{1}+k, 1}=R_{1} h_{1}\left[z_{a}-\left(l_{1}+\frac{1}{2}\left(R_{1}+1\right)\right) \mu\right] \\
& \quad+\left(p+H_{1}\right) \sum_{k=0}^{R_{1}-1} \mathrm{E}\left\{\left(D_{t_{a}, t_{a}+l_{1}+k}-z_{a}\right)^{+}\right\}, a \in A_{1}, \\
& \\
& \sum_{k=0}^{R_{n}-1} \mathrm{E} C_{t_{a}+l_{n}+k, n}=R_{n} h_{n}\left[z_{a}-\left(l_{n}+\frac{1}{2}\left(R_{n}+1\right)\right) \mu\right], \quad n=2, \ldots, N, a \in A_{n}, \\
& z_{a} \leq I L_{t_{a}, n+1}, \quad n=1, \ldots, N-1, a \in A_{n}, \\
& I L_{t_{a}, n+1}=z_{p a r(a)}-D_{t_{p a r(a)}, t_{a}-1}, \quad n=1, \ldots, N-1, a \in A_{n} .
\end{aligned}
$$

The solution of this $N$-stage stochastic programming problem follows from the following lemma.

Lemma 5 For $n$ equal to successively $1, \ldots, N$ :
(i) If $n=1$, then

$$
g_{1}\left(y_{1}\right)=h_{1}-\left(p+H_{1}\right) \frac{1}{R_{1}} \sum_{a \in A_{0}^{(1)}} \mathrm{P}\left\{B_{a}^{(1)}>0\right\}, \quad y_{1} \in \mathbb{R},
$$

with

$$
B_{a}^{(1)}=\left(D_{t_{p a r(a)}, t_{a}}-y_{1}\right)^{+} \text {for all } a \in A_{0}^{(1)} .
$$

If $n \in\{2, \ldots, N\}$, then

$$
g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)=\sum_{i=1}^{n} h_{i}-\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}>0\right\}, \quad y_{n} \in \mathbb{R},
$$

with

$$
\begin{aligned}
& B_{a}^{(n)}=0 \quad \text { for } a=(0, \ldots, 0,1), \\
& B_{a}^{(n)}=\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{p a r(a)}, t_{a}-1}-\left(y_{n}-S_{n-1}\right)\right)^{+} \text {for all } a \in A_{n-1}^{(n)}, \\
& B_{a}^{(n)}=\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{\operatorname{par}(a)}, t_{a}-1}-\left(S_{i+1}-S_{i}\right)\right)^{+} \text {for all } i=n-2, \ldots, 1, a \in A_{i}^{(n)}, \\
& B_{a}^{(n)}=\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{p a r(a)}, t_{a}}-S_{1}\right)^{+} \text {for all } a \in A_{0}^{(n)} .
\end{aligned}
$$

(if one or more of the $S_{i}$ are equal to infinity, then in these formulae the $S_{i}$ have to be read as if they are equal to a very large finite constant).
(ii) $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$ is continuous and nondecreasing as a function of $y_{n}$. In particular, $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)=$ $-\left(p+H_{n+1}\right)(<0)$ for all $y_{n} \leq 0$ and $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right) \uparrow h_{n}(\geq 0)$ as $y_{n} \rightarrow \infty$.
(iii) $G_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$ is convex as a function of $y_{n}$.
(iv) Let $S_{n}(\in \mathbb{R} \cup\{\infty\})$ be chosen such that

$$
S_{n} \stackrel{\text { def }}{=} \operatorname{argmin}_{y_{n} \in \mathbb{R}} G_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)
$$

Then $S_{n}$ is such that $g_{n}\left(S_{1}, \ldots, S_{n-1}, S_{n}\right)=0$. In particular, $S_{n}$ is positive and finite if $h_{n}>0$; $S_{n}=\infty$ if $h_{n}=0$ and $F$ has infinite support; $S_{n}$ is positive and may be finite as well as infinite if $h_{n}=0$ and $F$ has finite support.
(v) For the problem $\left(R P\left(t_{0}\right)\right)$, it is optimal to choose each of the levels $z_{a}, a \in A_{n}$, equal to $S_{n}$, or as high as possible if this level can not be reached.

The proof of this lemma is given in the appendix.

By, Lemma 5, basestock policy $\left(S_{1}, \ldots, S_{N}\right)$ is optimal for the relaxed problem $\mathrm{RP}\left(t_{0}\right)$. The problem was obtained by neglecting the bounding from below when placing orders. However, the optimality of basestock policy $\left(S_{1}, \ldots, S_{N}\right)$ holds for each cycle $t_{0} \in T_{N}$. If this basestock policy is used in all cycles, then these lower bounds at most constitute a limitation during a transient period (when the echelon inventory positions may be above the $S_{n}$, and nothing should be ordered). Hence, in the long run basestock policy $\left(S_{1}, \ldots, S_{N}\right)$ is also feasible for the un-relaxed version of $\operatorname{RP}\left(t_{0}\right)$, and hence also optimal for this problem. Thus basestock policy $\left(S_{1}, \ldots, S_{N}\right)$ is also optimal for problem (P).

Theorem 6 Basestock policy $\left(S_{1}, \ldots, S_{N}\right)$ with the $S_{n}$ as defined in Lemma 5 is optimal for problem ( $P$ ).

The optimal basestock levels $\left(S_{1}, \ldots, S_{N}\right)$ satisfy the Newsboy-type characterizations listed in the following corollary, which immediately follows from the parts (i)-(iv) of Lemma 5.

Corollary 7 The optimal basestock levels $S_{1}, \ldots, S_{N}$ are such that for each $n=1, \ldots, N$,

$$
\frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}=0\right\}=\frac{p+H_{n+1}}{p+H_{1}}
$$

with

$$
\begin{aligned}
B_{a}^{(n)} & =0 \quad \text { for } a=(0, \ldots, 0,1) \\
B_{a}^{(n)} & =\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{\text {par }(a)}, t_{a}-1}-\left(S_{i+1}-S_{i}\right)\right)^{+} \quad \text { for all } i=n-1, \ldots, 1, a \in A_{i}^{(n)}, \\
B_{a}^{(n)} & =\left(B_{\text {par }(a)}^{(n)}+D_{t_{\operatorname{par}(a)}, t_{a}}-S_{1}\right)^{+} \quad \text { for all } a \in A_{0}^{(n)} .
\end{aligned}
$$

This corollary says that, when $S_{n}$ is determined, it is imagined that stage $n+1$ can always deliver (i.e., the analysis is limited to the chain consisting of the stages $n, \ldots, 1$ ) and the value for $S_{n}$ is chosen such that the average probability for a nonnegative stock at the most downstream stage 1 is equal to $\frac{p+H_{n+1}}{p+H_{1}}$ (notice that the non-stockout probability in any period has a cyclic pattern).

Example 1 (continued) For our illustrative example, an optimal policy $\left(S_{1}, S_{2}\right)$ is as follows. By Corollary 7 and some algebra, we find that basestock level $S_{1}$ satisfies

$$
\begin{equation*}
\frac{1}{2}\left\{\mathrm{P}\left\{D_{t_{0}, t_{0}+1} \leq S_{1}\right\}+\mathrm{P}\left\{D_{t_{0}, t_{0}+2} \leq S_{1}\right\}\right\}=\frac{p+H_{2}}{p+H_{1}} \tag{12}
\end{equation*}
$$

where $t_{0}$ is an arbitrary period in which stage 1 orders (i.e., $t_{0} \in T_{1}$ ). So, under the assumption that stage 1 never experiences a shortage at stage $2, S_{1}$ is such that the average non-stockout probability per cycle of 2 periods is equal to $\frac{p+H_{2}}{p+H_{1}}$. Basestock level $S_{2}$ satisfies

$$
\begin{align*}
& \frac{1}{4}\left\{\mathrm{P}\left\{\left(D_{t_{0}, t_{0}}-\left(S_{2}-S_{1}\right)\right)^{+}+D_{t_{0}+1, t_{0}+2} \leq S_{1}\right\}\right. \\
& \quad+\mathrm{P}\left\{\left(D_{t_{0}, t_{0}}-\left(S_{2}-S_{1}\right)\right)^{+}+D_{t_{0}+1, t_{0}+3} \leq S_{1}\right\} \\
& \quad+\mathrm{P}\left\{\left(D_{t_{0}, t_{0}+2}-\left(S_{2}-S_{1}\right)\right)^{+}+D_{t_{0}+3, t_{0}+4} \leq S_{1}\right\} \\
& \left.\quad+\mathrm{P}\left\{\left(D_{t_{0}, t_{0}+2}-\left(S_{2}-S_{1}\right)\right)^{+}+D_{t_{0}+3, t_{0}+5} \leq S_{1}\right\}\right\}=\frac{p}{p+H_{1}} \tag{13}
\end{align*}
$$

where $t_{0}$ is an arbitrary period in which stage 2 orders (i.e., $t_{0} \in T_{2}$ ). $S_{2}$ is such that the average non-stockout probability per cycle of 4 periods is equal to $\frac{p}{p+H_{1}}$. Note that since the demand distribution $F$ is continuous, $\left(S_{1}, S_{2}\right)$ can be chosen to satisfy these equations.

### 3.4 Additional Results

The following corollary says that it is sufficient to hold stocks at the end of the most downstream stage 1 and in front of stages where value is added to the product; i.e., it is not necessary to hold stock in front of stages $n(<N)$ with $h_{n}=0$.

Corollary 8 There exists an optimal basestock policy under which no safety stocks are held in front of stages where zero value is added.

Proof: Suppose that $h_{n}=0$ for some stage $n=1, \ldots, N-1$. Then, by part (iv) of Lemma $5, S_{n}$ may be chosen equal to $S_{n}=\infty$. This implies that in each period all goods arriving in stockpoint $n+1$ are immediately forwarded to stockpoint $n$. This means that there is never stock present in stockpoint $n+1$ at the end of a period.

For the basestock policies we have not assumed that they have nondecreasing basestock levels. In fact, such an assumption would have complicated our analysis. The following corollary relates a general basestock policy to a basestock policy with nondecreasing basestock levels.

Corollary 9 Let $y_{n} \in \mathbb{R}$ for $n=1, \ldots, N$, and define $\tilde{y}_{n} \stackrel{\text { def }}{=} \min \left\{y_{n}, \ldots, y_{N}\right\}$ for $n=1, \ldots, N$. Then $G\left(\tilde{y}_{1}, \ldots, \tilde{y}_{N}\right)=G\left(y_{1}, \ldots, y_{N}\right)$.

This corollary follows directly from Theorem 2. In fact, it is easily verified that under basestock policy $\left(\tilde{y}_{1}, \ldots, \tilde{y}_{N}\right)$ the orders are identical to the orders generated under basestock policy $\left(y_{1}, \ldots, y_{N}\right)$ (at least in the long run; in the first periods of the horizon there may be differences).

### 3.5 Exact Solution Procedure for Mixed Erlang Demands

In general, an optimal basestock policy $\left(S_{1}, \ldots, S_{N}\right)$ and the corresponding average costs can be computed as follows. First, for $n=1, \ldots, N, S_{n}$ may be computed by the Newsboy-type characterization in Corollary 7, where in each step bisection is applied. Next, the optimal average costs $G\left(S_{1}, \ldots, S_{N}\right)$ follow from Theorem 2. For both, it is required that for a given basestock policy we are able to evaluate the shortfalls/backlogs $B_{a}^{(n)}$ as defined in Corollary 7 and the $B_{a}$ given by (6)-(8). These shortfalls/backlogs may be determined recursively after a sufficiently fine discretization of the one-period demand distribution $F$, although that may be computationally inefficient, in particular as $N$ grows large. We describe an alternative, efficient procedure that is applicable when the one-period demand is a mixture of Erlang distributions with the same scale parameter. This is described below. Modeling demand as a mixture of Erlangs with the same scale parameter is motivated by the fact that this class of distributions is dense in the class of all distributions on $[0, \infty)$. We assume that the one-period demand is simply given as such a distribution. In practice, however, often only the first two moments of the one-period demand are given, and then a two-moment fit may be applied first: a so-called Erlang $(k-1, k)$ distribution can be fitted if the coefficient of variation of the demand is smaller than or equal to 1 , and a so-called $\operatorname{Erlang}(1, k)$ distribution otherwise. More moments may be fit as desired, yielding a larger mixture.

Let us assume that the demand distribution $F$ is a mixture of Erlang distributions with the same scale parameter; i.e., there is a discrete distribution on $\mathbb{N},\left\{q_{k}\right\}_{k \in \mathbb{N}}$, and a $\lambda>0$ such that $F(x)=\sum_{k=1}^{\infty} q_{k} E_{k, \lambda}(x)$, $x \in \mathbb{R}$, where $E_{k, \lambda}$ denotes the Erlang distribution with $k \in \mathbb{N}$ phases and scale parameter $\lambda>0$. We describe the evaluation of the $B_{a}^{(n)}$ as defined in Corollary 7, for $n \geq 3$. The evaluation for all other cases is similar. For $a=(0, \ldots, 0,1)$, which is the only element of $A_{n}^{(n)}$, it is given that $B_{a}^{(n)}=0$ (which may be seen as a mixture of $E_{k, \lambda}$ with all probability mass in 0 . Next, we can determine the $B_{a}^{(n)}$ for each $a \in A_{n-1}^{(n)}$. In this case, $B_{a}^{(n)}=\left(D_{t_{p a r(a)}, t_{a}-1}-\left(S_{n}-S_{n-1}\right)\right)^{+}$. The random variable $D_{t_{p a r(a)}, t_{a}-1}$ is the $\left(t_{a}-t_{p a r(a)}\right)$-fold convolution of the one-period demand, and thus is also a mixture of Erlang distributions with scale parameter $\lambda$. The mixture probabilities are equal to the $\left(t_{a}-t_{\operatorname{par}(a)}\right)$-fold convolution of $\left\{q_{k}\right\}_{k \in \mathbb{N}}$. Next, if $S_{n}>S_{n-1}$, then we shift the distribution of $D_{t_{\operatorname{par}(a)}, t_{a}-1}$ to the left over a distance $S_{n}-S_{n-1}$, while the probability mass that would arrive in the negative range is absorbed in 0 . By that operation, we obtain the distribution
of $\left(D_{t_{\operatorname{par}(a)}, t_{a}-1}-\left(S_{n}-S_{n-1}\right)\right)^{+}$, which is a mixture of $E_{k, \lambda}$ distributions with a positive probability mass in 0 (for the computation of the new mixture probabilities, see Scheller-Wolf et al., 2003). If $S_{n} \leq S_{n-1}$, then we shift the distribution of $D_{t_{\operatorname{par}(a)}, t_{a}-1}$ to the right over a distance $S_{n-1}-S_{n}$. Then, the random $\operatorname{variable}\left(D_{t_{\operatorname{par}(a)}, t_{a}-1}-\left(S_{n}-S_{n-1}\right)\right)^{+}$consists of a deterministic part plus a mixture of $E_{k, \lambda}$ distributions. Next, the $B_{a}^{(n)}=\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{\operatorname{par}(a),}, t_{a}-1}-\left(S_{n-1}-S_{n-2}\right)\right)^{+}$for each $a \in A_{n-2}^{(n)}$ may be determined, and so on. The procedure is the same then, except that each time first the convolution of $D_{t_{\text {par }(a), t_{a}-1}}$ with $B_{\operatorname{par}(a)}^{(n)}$ has to be determined. This again will be a mixture of Erlang distributions with the same scale parameter $\lambda$, possibly plus a deterministic part (see Scheller-Wolf et al., 2003).

The reason that the above procedure works is that the class of Erlang distributions with the same scale parameter, with possibly a positive probability mass in 0 or a deterministic part being added, is closed under the shift operation as described above and when convolutions are taken. The above procedure is simpler to implement than the exact procedure described in Van Houtum and Zijm (1997) for the standard Clark-and-Scarf model.

Finally, the general class of phase-type distributions is likewise closed under the convolution and shift operation. So, an exact procedure can also be derived for phase-type distributions, although the different steps may become more complicated. If one is satisfied with accurate approximations, then one may use the simple approximate procedure based on two-moment fits as described in Van Houtum and Zijm (1991).

Example 1 (continued) For our illustrative example, we have the formulae (9), (12), and (13) available. Assume that $H_{2}=0.5, H_{1}=1, p=20$ and that one-period demand is exponential with mean $\mu=1$. Then, by the exact solution procedure, we find unique optimal basestock levels $S_{1}=6.67$ and $S_{2}=9.90$, and optimal costs $G\left(S_{1}, S_{2}\right)=5.84$. The corresponding computation time on a regular Pentium III PC is negligibly small (only a flash).

## 4 Extensions

### 4.1 Service Level Problem

We now consider our model with a $\gamma$-service level constraint (which is equivalent to an average backlog constraint) instead of penalty costs. The $\gamma$-service level is also known as modified fill rate, and is closely related to the regular fill rate. For high service levels (more precisely, as long as any demand is not backordered for more than one period), both measures are identical (see Van Houtum et al., 1996). Let $\gamma_{0}$ be the target $\gamma$-service level. We assume that the demand distribution $F$ has a compact support. Then an optimal policy is obtained as described below.

First, by Theorem 2, we find that the $\gamma$-service level of a basestock policy $\left(y_{1}, \ldots, u_{N}\right)$ equals

$$
\gamma\left(y_{1}, \ldots, y_{N}\right)=1-\frac{1}{\mu} \frac{1}{R_{N}} \sum_{a \in A_{0}} \mathrm{E} B_{a},
$$

where the random variables $B_{a}$ are given by (6)-(8). Second, if for the penalty cost model with a given penalty cost parameter $p$, the resulting optimal policy $\left(S_{1}, \ldots, S_{N}\right)$ has a $\gamma$-service level $\gamma\left(S_{1}, \ldots, S_{N}\right)=\gamma(p)$ that is precisely equal $\gamma_{0}$, then $\left(S_{1}, \ldots, S_{N}\right)$ is also optimal for the service level problem with target service level $\gamma_{0}$ (see Theorem 1 of Van Houtum and Zijm, 2000). Third, the function $\gamma(p)$ is nondecreasing as a function of $p$ (see Theorem 2 of Van Houtum and Zijm, 2000). Forth, under the assumption that $F$ has compact support, one can show that the optimal basestock levels $S_{1}, \ldots, S_{N}$ are continuous as a function of $p$; thus, also $\gamma(p)$ is continuous as a function of $p$, and further $\gamma(p) \uparrow 1$ as $p \rightarrow \infty$.

The above shows that the service level problem with a given target level $\gamma_{0}<1$ may be solved by solving the penalty cost problem and tuning the penalty cost parameter $p$ such that the $\gamma$-service level $\gamma(p)$ of the optimal policy equals $\gamma_{0}$. Also, this implies that the class of basestock policies is also optimal for the service level problem with a $\gamma$-service level constraint.

### 4.2 Non-Synchronized Reorder Epochs

We have assumed that the synchronization constraint is satisfied for the reorder epochs $T_{n}$ of the stages $n=1, \ldots, N$. If this constraint is not satisfied, then the results still hold but for a system with adapted leadtimes. This is illustrated for the system of Example 1. Assume that for this example system everything
is as previously described but stage 1 is now allowed to order at the beginning of the periods $0,2,4, \ldots$; i.e., $T_{1}=\{0,2,4, \ldots\}$ instead of $T_{1}=\{1,3,5, \ldots\}$. Then each shipment arriving in stockpoint 2 has to wait at least one period before materials may enter stage 1 , as if the leadtime for stage 2 equals $\tilde{l}_{2}=2$ instead of $l_{2}=1$. Everything else is the same as in the original system synchronized reorder epochs. Thus all results apply, where in all formulae $l_{2}$ has to be replaced by $\tilde{l}_{2}$.

### 4.3 Order Frequencies with Non-Integer Ratios

In our model, we also assumed that the reorder intervals satisfy the integer-ratio constraint. If this constraint is not satisfied, then the optimal policy may be a basestock policy with time-dependent basestock levels or possibly even more complicated. Nevertheless, for practical purposes, it may be useful to consider basestock policies only, and to optimize within this class. For the average costs, one then has to look at a cycle with a length equal to the least common multiple of $R_{1}, \ldots, R_{N}$. For the average costs of a basestock policy, similar formulae will be obtained as in Theorem 2. To find optimal basestock levels, we suggest deriving partial derivatives and to apply a standard procedure for the minimization of a multi-dimensional function.

### 4.4 Assembly Systems

Assembly systems without periodic batching can be shown to be equivalent to serial systems; see Rosling (1989). This equivalence is built on the property that for a given component it is never optimal to order more than the amount of companion components that will be available at the next assembly point. This property also applies to assembly systems with periodic batching. Thus, each assembly system with periodic batching is equivalent to a serial system with periodic batching. If this equivalent serial system satisfies the synchronization and integer-ratio constraint, then all results of this paper apply again. Otherwise, see Subsections 4.2 and 4.3.

## 5 Concluding Remarks

We have studied a serial, multi-echelon inventory system with periodic batching. We have been able to generalize many of the results known for the standard Clark-and-Scarf model. In further research, we will
consider multi-echelon systems with a divergent structure and periodic batching, which are relevant for production and distribution environments. There we can build on available results for divergent systems without batching; see e.g. Diks and De Kok (1998). Also, we will work on developing appropriate models for the upper decision level of the hierarchical approach as described in the introduction of this paper.

## Appendix

## Proof of Lemma 4

By Lemma 3, for all $n=1, \ldots, N$ and $y_{i} \in \mathbb{R}$ for $i=1, \ldots, n$,

$$
g_{n}\left(y_{1}, \ldots, y_{n}\right)=h_{n}-\sum_{i=1}^{n-1} h_{i} \frac{R_{i}}{R_{n}} \sum_{a \in A_{i}^{(n)}} \frac{\delta}{\delta y_{n}}\left\{\mathrm{E} B_{a}^{(n)}\right\}+\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \frac{\delta}{\delta y_{n}}\left\{\mathrm{E} B_{a}^{(n)}\right\}
$$

noting that $B_{a}^{(n)}=0$ for $a \in A_{n}^{(n)}$.
For each $a=\left(a_{0}, a_{1}, \ldots, a_{n}\right) \in A^{(n)}$, we define

$$
p_{j}(a) \stackrel{\text { def }}{=}\left(0, \ldots, 0, a_{j}, \ldots, a_{n}\right) \quad \text { for } j=\operatorname{lev}(a), \ldots, n \text {. }
$$

The vector $p_{j}(a)$ is the unique vector at level $j$ on the path from $a$ to $(0, \ldots, 0,1)$ in the tree constituted by the elements of $A^{(n)}$, where the tree is rooted at $(0, \ldots, 0,1)$ and edges correspond to parent-child relationships.

In particular, $p_{l e v(a)}(a)=a$ for all $a \in A^{(n)}$, and $p_{l e v(a)+1}(a)=\operatorname{par}(a)$ for all $a \in A^{(n)} \backslash\{(0, \ldots, 0,1)\}$. It is easily verified now that for each $a \in A^{(n)} \backslash\{(0, \ldots, 0,1)\}$,

$$
\frac{\delta}{\delta y_{n}}\left\{\mathrm{E} B_{a}^{(n)}\right\}=-\mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=n-1, \ldots, \operatorname{lev}(a)\right\}
$$

i.e., in order for an increase in $y_{n}$ to have an effect on the backorder at a given $a$ below the root of the tree, there must be a path of positive backlogs in the tree between the root and $a$.

By substitution of this result in the formula given above for $g_{n}\left(y_{1}, \ldots, y_{n}\right)$, we obtain

$$
\begin{align*}
g_{n}\left(y_{1}, \ldots, y_{n}\right)=h_{n} & +\sum_{i=1}^{n-1} h_{i} \frac{R_{i}}{R_{n}} \sum_{a \in A_{i}^{(n)}} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=n-1, \ldots, i\right\} \\
& -\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=n-1, \ldots, 0\right\} . \tag{14}
\end{align*}
$$

For $n=1$, formula (14) reads as

$$
g_{1}\left(y_{1}\right)=h_{1}-\left(p+H_{1}\right) \frac{1}{R_{1}} \sum_{a \in A_{0}^{(1)}} \mathrm{P}\left\{B_{a}^{(1)}>0\right\}
$$

This expression is identical to the formula given by Lemma 4.
We now show that the formula given by the Lemma 4 is also correct for each $n=2, \ldots, N$. Consider Equation (14). For each $i=1, \ldots, n-1$ and $a \in A_{i}^{(n)}$, we find that

$$
\begin{aligned}
& \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=n-1, \ldots, i\right\} \\
& \quad=1-\mathrm{P}\left\{B_{p_{j}(a)}^{(n)}=0 \text { for some } j=n-1, \ldots, i\right\} \\
& \quad=1-\sum_{k=i}^{n-1} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0, B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, i\right\} \\
& \quad=1-\sum_{k=i}^{n-1} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, i \mid B_{p_{k}(a)}^{(n)}=0\right\} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0\right\}
\end{aligned}
$$

(with the convention that $\mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0\right.$ for $\left.j=k-1, \ldots, i \mid B_{p_{k}(a)}^{(n)}=0\right\}=1$ when $k=i$ ). Similarly, for each $a \in A_{0}^{(n)}$, we find

$$
\begin{aligned}
& \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=n-1, \ldots, 0\right\} \\
& \quad=\mathrm{P}\left\{B_{a}^{(n)}>0\right\}-\mathrm{P}\left\{B_{p_{j}(a)}^{(n)}=0 \text { for some } j=n-1, \ldots, 1, B_{a}^{(n)}>0\right\} \\
& =\mathrm{P}\left\{B_{a}^{(n)}>0\right\}-\sum_{k=1}^{n-1} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0, B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, 0\right\} \\
& =\mathrm{P}\left\{B_{a}^{(n)}>0\right\} \\
& \quad \quad-\sum_{k=1}^{n-1} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, 0 \mid B_{p_{k}(a)}^{(n)}=0\right\} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0\right\} .
\end{aligned}
$$

By substitution of these expressions into Equation (14), we obtain

$$
\begin{align*}
g_{n}\left(y_{1}, \ldots, y_{n}\right)= & h_{n}+\sum_{i=1}^{n-1} h_{i} \frac{R_{i}}{R_{n}} \sum_{a \in A_{i}^{(n)}}[1 \\
& \left.-\sum_{k=i}^{n-1} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, i \mid B_{p_{k}(a)}^{(n)}=0\right\} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0\right\}\right] \\
& -\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}}\left[\mathrm{P}\left\{B_{a}^{(n)}>0\right\}\right. \\
& \left.-\sum_{k=1}^{n-1} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, 0 \mid B_{p_{k}(a)}^{(n)}=0\right\} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0\right\}\right] \\
= & \sum_{i=1}^{n} h_{i}-\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}>0\right\} \\
& -\sum_{k=1}^{n-1} \sum_{i=1}^{k} h_{i} \frac{R_{i}}{R_{n}} \sum_{a \in A_{i}^{(n)}} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, i \mid B_{p_{k}(a)}^{(n)}=0\right\} \\
& \left.+\sum_{k=1}^{n-1}\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{p_{k}(a)}^{(n)}=0\right\} \\
& \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, 0 \mid B_{p_{k}(a)}^{(n)}=0\right\} \\
& \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0\right\} .
\end{align*}
$$

By using the relationships, for different values of $k$, between the sets $A_{i}^{(k)}$ and between the random variables $B_{a}^{(k)}($ cf. (11)), we find for all $k=1, \ldots, n-1$ and $i=1, \ldots, k$,

$$
\begin{aligned}
& \sum_{a \in A_{\hat{i}}^{(n)}} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, i \mid B_{p_{k}(a)}^{(n)}=0\right\} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0\right\} \\
& =\sum_{\hat{a} \in A_{k}^{(n)}} \sum_{a \in A_{i}^{(n)}, p_{k}(a)=\hat{a}} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, i \mid B_{\hat{a}}^{(n)}=0\right\} \mathrm{P}\left\{B_{\hat{a}}^{(n)}=0\right\} \\
& =\sum_{\hat{a} \in A_{k}^{(n)}} \mathrm{P}\left\{B_{\hat{a}}^{(n)}=0\right\} \sum_{\tilde{a} \in A_{i}^{(k)}} \mathrm{P}\left\{B_{p_{j}(\hat{a})}^{(k)}>0 \text { for } j=k-1, \ldots, i\right\}
\end{aligned}
$$

(with the convention that $\mathrm{P}\left\{B_{p_{j}(\tilde{a})}^{(k)}>0\right.$ for $\left.j=k-1, \ldots, i\right\}=1$ when $k=i$ ). Similarly, we find for all $k=1, \ldots, n-1$,

$$
\begin{aligned}
& \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{p_{j}(a)}^{(n)}>0 \text { for } j=k-1, \ldots, 0 \mid B_{p_{k}(a)}^{(n)}=0\right\} \mathrm{P}\left\{B_{p_{k}(a)}^{(n)}=0\right\} \\
& =\sum_{\hat{a} \in A_{k}^{(n)}} \mathrm{P}\left\{B_{\hat{a}}^{(n)}=0\right\} \sum_{\tilde{a} \in A_{0}^{(k)}} \mathrm{P}\left\{B_{p_{j}(\tilde{a})}^{(k)}>0 \text { for } j=k-1, \ldots, 0\right\}
\end{aligned}
$$

By substitution of these expressions into Equation (15), and by using (14), we find

$$
\begin{aligned}
& g_{n}\left(y_{1}, \ldots, y_{n}\right)=\sum_{i=1}^{n} h_{i}-\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}>0\right\} \\
& -\sum_{k=1}^{n-1} \frac{R_{k}}{R_{n}} \sum_{\hat{a} \in A_{k}^{(n)}} \mathrm{P}\left\{B_{\hat{a}}^{(n)}=0\right\}\left[\sum_{i=1}^{k} h_{i} \frac{R_{i}}{R_{k}} \sum_{\tilde{a} \in A_{i}^{(k)}} \mathrm{P}\left\{B_{p_{j}(\tilde{a})}^{(k)}>0 \text { for } j=k-1, \ldots, i\right\}\right. \\
& \left.-\left(p+H_{1}\right) \frac{1}{R_{k}} \sum_{\tilde{a} \in A_{0}^{(k)}} \mathrm{P}\left\{B_{p_{j}(\tilde{a})}^{(k)}>0 \text { for } j=k-1, \ldots, 0\right\}\right] \\
& =\sum_{i=1}^{n} h_{i}-\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}>0\right\}-\sum_{k=1}^{n-1} \frac{R_{k}}{R_{n}} \sum_{\hat{a} \in A_{k}^{(n)}} \mathrm{P}\left\{B_{\hat{a}}^{(n)}=0\right\} g_{k}\left(y_{1}, \ldots, y_{k}\right),
\end{aligned}
$$

(again with the convention that $\operatorname{P}\left\{B_{p_{j}(\tilde{a})}^{(n)}>0\right.$ for $\left.j=k-1, \ldots, i\right\}=1$ when $i=k$ and $\tilde{a}=(0, \ldots, 0,1)$ ). This completes the proof.

## Proof of Lemma 5

We prove Lemma 5 by induction to $n$.
Let $n=1$. Then, the formula in part (i) is identical to the formula in Lemma 4 with $n=1$, and hence correct. From this formula, it immediately follows that $g_{1}\left(y_{1}\right)$ is continuous and nondecreasing. In particular, $g_{1}\left(y_{1}\right)$ is equal to $-\left(p+H_{2}\right)<0$ for all $y_{1} \leq 0$ and it goes to $h_{1}$ for $y_{1} \rightarrow \infty$. If the demand distribution $F$ has finite support, then also the distributions of the $D_{t_{p a r(a)}, t_{a}}, a \in A_{0}^{(1)}$, have finite support and the derivative will be equal to $h_{1}$ for all values larger than or equal to a certain finite point. The behavior of $g_{1}\left(y_{1}\right)$ implies that $G_{1}\left(y_{1}\right)$ is convex. This completes the proof of part (i)-(iii).

Next, $S_{1}$ is defined as a point where $G_{1}\left(y_{1}\right)$ is minimized. If there are multiple points where $G_{1}\left(y_{1}\right)$ is minimized, then $S_{1}$ may be taken equal to any of these points. Obviously, since $g_{1}\left(y_{1}\right)<0$ for all $y_{1} \leq 0$ and $g_{1}\left(y_{1}\right)$ is continuous, we find that $S_{1}>0$. Further, if $h_{1}>0$, then $S_{1}<\infty$. If $h_{1}=0$ and the demand distribution $F$ has a finite support, then $S_{1}$ can be chosen equal to a finite value or equal to $S_{1}=\infty$. If $h_{1}=0$ and the demand distribution $F$ has an infinite support, then $S_{1}=\infty$. In all these cases, $S_{1}$ is such that $g_{1}\left(y_{1}\right)=0$ for $y_{1}=S_{1}$. This proves part (iv).

We can now show how the decision $a \in A_{1}$, i.e., the choices for the $z_{a}$ with $a \in A_{1}$, may be optimized for problem $\left(\operatorname{RP}\left(t_{0}\right)\right)$. Let $a \in A_{1}$. Decision $a$ is taken at the beginning of period $t_{a}$, and the choice for $z_{a}$ is bounded from above by $I L_{t_{a}, 2}$. This decision only affects the costs $\sum_{k=0}^{R_{1}-1} \mathrm{E} C_{t_{a}+l_{1}+k, 1}$, which, by (10), are
equal to $G_{1}\left(z_{a}\right)$. These costs are minimized by choosing $z_{a}$ equal to $z_{a}=S_{1}$ if $I L_{t_{a}, 2} \geq S_{1}$, and equal to $z_{a}=I L_{t_{a}, 2}$ if $I L_{t_{a}, 2}<S_{1}$. This completes the proof of part (v), and hence also the proof of the complete lemma for $n=1$.

Let us now suppose that the results above have been proved for all $m=1, \ldots, n-1$, where $n \in\{2, \ldots, N\}$. We shall prove that then the results also hold for $m=n$.

The derivative of $G_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$ as a function of $y_{n}$ is given by (cf. Lemma 4)

$$
\begin{aligned}
g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)= & \sum_{i=1}^{n} h_{i}-\left(p+H_{1}\right) \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}>0\right\} \\
& -\sum_{i=1}^{n-1} \frac{R_{i}}{R_{n}} \sum_{a \in A_{i}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}=0\right\} g_{i}\left(S_{1}, \ldots, S_{i}\right), \quad y_{n} \in \mathbb{R},
\end{aligned}
$$

where the $B_{a}^{(n)}$ are defined as in part (i) of Lemma 5. By the induction assumption, $g_{i}\left(S_{1}, \ldots, S_{i}\right)=0$ for each $i=1, \ldots, n-1$. Hence, all terms in the third sum in the formula for $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$ vanish, and we find the formula given in part (i) of Lemma 5.

It is easily verified that the distribution functions of the $B_{a}^{(n)}, a \in A^{(n)}$, behave continuously as a function of $y_{n}$ and that they are stochastically decreasing (= non-increasing) as a function of $y_{n}$. Hence, $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$ is continuous and nondecreasing as a function of $y_{n}$. In particular, this function appears to be equal to $-\left(p+H_{n+1}\right)<0$ for all $y_{n} \leq 0$. Note that if $y_{n}=0$, then it holds that $\mathrm{P}\left\{B_{a}^{(n)}>0\right\}=1$ for all $a \in A_{0}^{(n)}$. The number of elements of $A_{0}^{(n)}$ is equal to $r_{0} r_{1} \cdots r_{n-1}=R_{n}$. Hence,

$$
g_{n}\left(S_{1}, \ldots, S_{n-1}, 0\right)=\sum_{i=1}^{n} h_{i}-\left(p+H_{1}\right) \frac{1}{R_{n}} R_{n}=-\left(p+\sum_{i=n+1}^{N} h_{i}\right)=-\left(p+H_{n+1}\right)<0 .
$$

Further, $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right) \uparrow h_{n}$ when $y_{n} \rightarrow \infty$. To show the latter, we distinguish two cases. The first case is when $S_{i}=\infty$ for all $i=1, \ldots, n-1$. Then, first of all, it must hold that $h_{i}=0$ for all $i=1, \ldots, n-1$ (by the induction assumption). Further, the expressions for the $B_{a}^{(n)}, a \in A_{0}^{(n)}$, reduce to

$$
\begin{aligned}
& B_{a}^{(n)}=\left(D_{t_{(0, \ldots, 0,1)}, t_{a}}-y_{n}\right)^{+}\left(\text {take } S_{1}=\ldots=S_{n-1}=M \text { and let } M \rightarrow \infty\right): \\
& B_{a}^{(n)}=0 \quad \text { for } a=(0, \ldots, 0,1), \\
& B_{a}^{(n)}=\left(B_{p a r(a)}^{(n)}+D_{t_{\operatorname{par}(a)}, t_{a}-1}-\left(y_{n}-M\right)\right)^{+} \\
& =D_{t_{(0, \ldots, 0,1)}, t_{a}-1}+M-y_{n} \quad \text { for all } a \in A_{n-1}^{(n)}, \\
& B_{a}^{(n)}=\left(B_{\text {par }(a)}^{(n)}+D_{t_{\operatorname{par}(a)}, t_{a}-1}-(M-M)\right)^{+} \\
& =D_{t_{(0, \ldots, 0,1)}, t_{\operatorname{par}(a)}-1}+M-y_{n}+D_{t_{\operatorname{par}(a)}, t_{a}-1} \\
& =D_{t_{(0, \ldots, 0,1)}, t_{a}-1}+M-y_{n} \quad \text { for all } a \in A_{n-2}^{(n)}, \\
& B_{a}^{(n)}=\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{\operatorname{par}(a)}, t_{a}}-M\right)^{+} \\
& =\left(D_{t_{(0, \ldots, 0,1),}, t_{\operatorname{par}(a)}-1}+M-y_{n}+D_{\left.t_{\operatorname{par}(a), t_{a}}-M\right)^{+}}\right. \\
& =\left(D_{t_{(0, \ldots, 0,1)}, t_{a}}-y_{n}\right)^{+} \quad \text { for all } a \in A_{0}^{(n)} .
\end{aligned}
$$

Hence, if $y_{n} \rightarrow \infty, \mathrm{P}\left\{B_{a}^{(n)}>0\right\} \downarrow 0$ for all $a \in A_{0}$, and thus $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right) \rightarrow \sum_{i=1}^{n} h_{i}=h_{n}\left(\right.$ as $h_{i}=0$ for all $i \leq n-1$ ). The second case is when there is at least one $S_{i}$ which is finite. Then we define the index $k$ by $k:=\max \left\{i \mid S_{i}<\infty\right\}$. Then, first of all, it must hold that $h_{i}=0$ for all $i=k+1, \ldots, n-1$ (by the induction assumption). Further, the expressions for the $B_{a}^{(n)}, a \in \cup_{i=k}^{n} A_{i}^{(n)}$, reduce in a manner similar as above to:

$$
B_{a}^{(n)}=\left(D_{t_{(0, \ldots, 0,1)}, t_{a}-1}-\left(y_{n}-S_{k}\right)\right)^{+} \text {for all } a \in A_{k}^{(n)}
$$

(notice that this is only a real reduction when $k<n-1$ ). Further,

$$
\begin{aligned}
B_{a}^{(n)} & =\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{p a r(a)}, t_{a}-1}-\left(S_{i+1}-S_{i}\right)\right)^{+} \text {for all } 1 \leq i \leq k-1, a \in A_{i}^{(n)}, \\
B_{a}^{(n)} & =\left(B_{\operatorname{par}(a)}^{(n)}+D_{t_{p a r(a)}, t_{a}}-S_{1}\right)^{+} \text {for all } a \in A_{0}^{(n)}
\end{aligned}
$$

(notice that this is only a real reduction when $k<n-1$ ). Hence, if $y_{n} \rightarrow \infty$, then for each $a \in A_{k}^{(n)}, B_{a}^{(n)}$ becomes equal to 0 , and the $B_{a}^{(n)}, a \in \cup_{i=0}^{k-1} A_{i}^{(n)}$, become identical to the random variables $B_{a}^{(k)}, a \in A^{(k)}$,
given by

$$
\begin{aligned}
B_{a}^{(k)} & =0 \text { for } a=(0, \ldots, 0,1) \\
B_{a}^{(k)} & =\left(B_{\text {par }(a)}^{(k)}+D_{t_{p a r(a)}, t_{a}-1}-\left(S_{i+1}-S_{i}\right)\right)^{+} \text {for all } 1 \leq i \leq k-1, a \in A_{i}^{(k)}, \\
B_{a}^{(k)} & =\left(B_{p a r(a)}^{(k)}+D_{t_{p a r(a)}, t_{a}}-S_{1}\right)^{+} \quad \text { for all } a \in A_{0}^{(k)} .
\end{aligned}
$$

For each $a \in \cup_{i=0}^{k-1} A_{i}^{(n)}, B_{a}^{(n)}$ becomes identical to $B_{\operatorname{tr}(a)}^{(k)}$ with $\operatorname{tr}(a)=\left(a_{0}, \ldots, a_{k-1}, 1\right)$. By the induction assumption (in particular, part (i) and (iii)), $S_{k}$ has been chosen such that $g_{k}\left(S_{1}, \ldots, S_{k}\right)=0$, i.e., such that

$$
\frac{1}{R_{k}} \sum_{a \in A_{0}^{(k)}} \mathrm{P}\left\{B_{a}^{(k)}>0\right\}=\frac{\sum_{i=1}^{k} h_{i}}{p+H_{1}}
$$

So, if $y_{n} \rightarrow \infty$, then $\mathrm{P}\left\{B_{a}^{(n)}>0\right\} \uparrow \mathrm{P}\left\{B_{\operatorname{tr}(a)}^{(k)}>0\right\}$ for all $a \in A_{0}^{(n)}$, and hence

$$
\begin{aligned}
\frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{a}^{(n)}>0\right\} & \uparrow \frac{1}{R_{n}} \sum_{a \in A_{0}^{(n)}} \mathrm{P}\left\{B_{t r(a)}^{(k)}>0\right\} \\
= & \frac{1}{R_{n}} \sum_{\hat{a} \in A_{0}^{(k)}} \sum_{a \in A_{0}^{(n)}, \operatorname{tr}(a)=\hat{a}} \mathrm{P}\left\{B_{t r(a)}^{(k)}>0\right\} \\
& =\frac{1}{R_{k}} \sum_{\hat{a} \in A_{0}^{(k)}} \mathrm{P}\left\{B_{\hat{a}}^{(k)}>0\right\}=\frac{\sum_{i=1}^{k} h_{i}}{p+H_{1}}
\end{aligned}
$$

And thus $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right) \rightarrow \sum_{i=k+1}^{n} h_{i}=h_{n}\left(\right.$ as $h_{i}=0$ for $\left.k+1 \leq i<n\right)$. This completes the proof of part (ii).

Part (iii) follows immediately from the behavior of $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$.
We now define $S_{n}$ as a point where $G_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$ is minimized. If there are multiple points where this function is minimized, then $S_{n}$ may be taken equal to any of these points. Obviously, since $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)<0$ for all $y_{n} \leq 0$ and $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)$ is continuous as a function of $y_{n}$, we find that $S_{n}>0$. Further, if $h_{n}>0$, then $S_{n}<\infty$. If $h_{n}=0$ and the demand distribution $F$ has a finite support, then $S_{n}$ can be chosen equal to a finite value or equal to $S_{n}=\infty$. If $h_{n}=0$ and the demand distribution $F$ has an infinite support, then $S_{n}=\infty$. In all these cases, $S_{n}$ is such that $g_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)=0$ for $y_{n}=S_{n}$. This proves part (iv).

We can now show how the decisions $a \in A_{n}$, i.e., the choices for the $z_{a}, a \in A_{n}$, may be optimized for problem $\left(\operatorname{RP}\left(t_{0}\right)\right)$. Let $a \in A_{n}$. Decision $a$ is taken at the beginning of period $t_{a}$, and the choice for $z_{a}$ is bounded from above by $I L_{t_{a}, n+1}$. This decision only affects the costs $C_{t_{\hat{a}}+l_{\operatorname{lev}(\hat{a})}+k, l e v(\hat{a})}$, with $\hat{a} \in V(a)$ and
$k=0, \ldots, R_{\operatorname{lev}(\hat{a})}-1$. Here, $V(a)$ is the set of all successors of $a$ in the tree constituted by all $\hat{a} \in A$ ( $a$ itself is included in $V(a))$. Whatever choice is made for $z_{a}$, for the decisions $\hat{a} \in V(a) \backslash\{a\}$ it is optimal to take them according to a basestock policy with basestock levels $S_{n-1}, \ldots, S_{1}$ (by the induction assumption). So, let us assume that these decisions are taken in this way. Then, by (10), the expected value of the sum of the costs affected by decision $a$ is equal to

$$
\sum_{i=1}^{n} \sum_{\hat{a} \in A_{i} \cap V(a)} \sum_{k=0}^{R_{i}-1} \mathrm{E} C_{t_{\hat{a}}+l_{i}+k, i}=G_{n}\left(S_{1}, \ldots, S_{n-1}, y_{n}\right)
$$

These costs are minimized by choosing $z_{a}$ equal to $z_{a}=S_{n}$ if $I L_{t_{a}, n+1} \geq S_{n}$, and equal to $z_{a}=I L_{t_{a}, n+1}^{e c h}$ if $I L_{t_{a}, n+1}<S_{n}$. This completes the proof of part (v), and hence also the proof of the complete induction step.

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