Another geometric method for determining all positive semi-definite solutions of the algebraic Riccati equation

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ANOTHER GEOMETRIC APPROACH FOR DETERMINING ALL POSITIVE SEMI-DEFINITE SOLUTIONS OF THE ALGEBRAIC RICCATI EQUATION.

ABSTRACT

In this article it is shown that it is possible to determine all positive semi-definite solutions of the algebraic Riccati equation under weaker assumptions than the ones usually made in the literature. These solutions are of interest because they are the only possible candidates for representing optimal costs of non-negative definite linear-quadratic control problems. It will turn out that under only some stabilizability assumption all positive semi-definite solutions can be described in terms of the two extremal ones, the *smallest* and *largest* positive semi-definite solutions. The possible presence of invariant zeros on the imaginary axis does not matter, and can be left out as an assumption in order to prove the result.

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1. Introduction.

It is generally known ([3] - [4]) that the optimal cost for any infinite horizon non-negative definite regular LQCP (linear-quadratic control problem) is characterized by a positive semi-definite solution of a certain matrix quadratic equation (the algebraic Riccati equation; abbreviated ARE). The notion of "regularity" stands for positive definiteness of the cost criterion w.r.t. the control, as usual ([4], [7], [12]). Indeed, the optimal cost for the LQCP without stability (the free end-point problem) is represented by K^-, the *smallest* positive semi-definite solution of the ARE, and the optimal cost for the LQCP with stability (where the state trajectory is required to vanish as time goes to infinity) is characterized by K^+, the *largest* positive semi-definite solution (e.g. [12]). The remaining non-negative definite solutions of the ARE are and have been a topic of interest and several researchers have established bijective relations between these matrices and certain invariant subspaces ([1] - [3]). These subspaces, then, are related to the restrictions that have been imposed in the various LQCP's on the state trajectory (as time goes to infinity).

In this paper we will not discuss the LQCP's with "intermediate" stability requirements (for these, see e.g. [14] and [17]), but we will, once more, study the set of positive semi-definite solutions of the ARE. In [1] - [2] the above-mentioned bijective relations have been derived under two assumptions. Here, we will prove that one of them can be left out. In fact, the condition that can be left out is strongly tied up to the existence of inputs that achieve the optimal cost for a LQCP, whereas here we are interested in the mere existence of this optimal cost (read: the existence of a positive semi-definite solution of the ARE). Our result resembles assertions of the same kind as in [4] - [5], but our K^- and theirs are totally different (the K^- in [4] - [5] is the <u>overall</u> smallest solution of the ARE). Moreover, we only require some stabilizability assumption to hold instead of the (stronger) controllability assumption in [4] - [5] (see also [6]). In [1] - [2] the same stabilizability assumption is made, and we will demonstrate that this condition is sufficient for obtaining our results as well as the results known before on the positive semi-definite Riccati solutions.

Furthermore, we will present several by-results of interest. Probably the most relevant of these statements is a surprising proof of a claim concerning a certain subspace of points believed to be in the kernel of K^+ ([12, p. 334]).

Section 2 sums up all notions that are needed here, Section 3 provides our results and the last Section contemplates on a few related aspects.

2. Preliminaries.

Any infinite horizon non-negative definite regular LQCP can be stated as follows ([7]). Consider the finite-dimensional linear time-invariant system Σ:

$$\dot{x} = Ax + Bu \ , \ x(0) = x_0 \ , \tag{2.1a}$$

$$y = Cx + Du \ , \tag{2.1b}$$

and the quadratic cost functional

$$J(x_0, u) = \int_0^\infty y'y \, dt \ . \tag{2.2}$$

The state vector x(t) is assumed to be in $\mathbb{R}^n$, the input u(t) is in $\mathbb{R}^m$ and the output y(t) in $\mathbb{R}^r$ for all $t \geq 0$. The matrix D is left invertible and $u \in \mathcal{L}^m_{2,loc}(\mathbb{R}^+)$, the space of m-vectors whose components are locally square-integrable over $\mathbb{R}^+$ (in [7] it is noted that for studying LQCP's even the space of <u>smooth</u> functons over $\mathbb{R}^+$ is large enough). Now let $\mathcal{T}$ be a linear subspace, then we state the linear-quadratic control problem with <u>stability modulo</u> $\mathcal{T}$ ($(LQCP)_{\mathcal{T}}$) as follows: Find for all x_0,

$$J_{\mathcal{T}}(x_0) := \inf\{J(x_0, u) \mid u \in \mathcal{L}^m_{2,loc}(\mathbb{R}^+) \text{ such that } (x/_{\mathcal{T}})(\infty) = 0\} \tag{2.3}$$

and compute, if it exists, an input u^* such that $J(x_0, u^*) = J_{\mathcal{T}}(x_0)$ (i.e. u^* is optimal). Here, $(x/_{\mathcal{T}})(t) = P(x(t))$ where P denotes the canonical projection of $\mathbb{R}^n$ on $\mathbb{R}^n/_{\mathcal{T}}$ ([13, Ch. 0]), and $(x/_{\mathcal{T}})(\infty) := \lim_{t\to\infty} (x/_{\mathcal{T}})(t)$.

In the sequel the geometric concept of <u>weakly unobservable</u> subspace (also called output nulling subspace) is of importance ([7, Def. 3.8], [12], [13]). The weakly unobservable subspace $\mathcal{V} = \mathcal{V}(\Sigma)$ is the space of all initial conditions for which there exists an input u such that the resulting output $y = y(x_0, u) \equiv 0$. It is easy to show that for every $x_0 \in \mathcal{V}$ the "output nulling" control actually is smooth and can be described as the state feedback $u = -(D'D)^{-1}D'Cx$. Therefore

<u>Lemma 2.1</u>.

$$\mathcal{V} = \langle \ker(C_0) | A_0 \rangle$$

with $A_0 = A - B(D'D)^{-1}D'C$, $C_0 = (I - D(D'D)^{-1}D')C$.

Also, we will use the notion of the set of *invariant zeros* $\sigma^*(\Sigma)$: One establishes easily that in our situation ([13]), $\sigma^*(\Sigma) = \sigma(A_0|\mathcal{V}) = \sigma(A_0|\langle \ker(C_0)|A_0 \rangle)$, the set of weakly unobservable eigenvalues of A_0.

The subject of investigation in this paper is the quadratic matrix equation

$$C'C + A'K + KA - (KB + C'D)(D'D)^{-1}(B'K + D'C) = 0 \qquad (2.4a)$$

or, equivalently,

$$C_0'C_0 + A_0'K + KA_0 - KB(D'D)^{-1}B'K = 0 \quad , \qquad (2.4b)$$

where $K = K'$ is a real, symmetric matrix of dimension n. It has been shown in [3] that, preassuming that for all x_0, $J_0(x_0) < \infty$ ((2.3)), it turns out that $J_{\mathcal{T}}(x_0) = x_0'K_{\mathcal{T}}x_0$ with $K_{\mathcal{T}} \geq 0$ a solution of the ARE (2.4). In particular ([1] - [2], [8], [11] - [12]), $J_{\mathbb{R}^n}(x_0) = x_0'K^-x_0$ and $J_0(x_0) = x_0'K^+x_0$ with K^-, K^+ the *smallest* and *largest* positive semi-definite solution of (2.4), respectively, if (A, B) is stabilizable (w.r.t. $\mathbb{C}^- := \{s \in \mathbb{C} | \mathrm{Re}(s) < 0\}$). From now on, we will take this as a <u>standing assumption</u>. In the next Section the remaining positive semi-definite solutions are studied.

Remark 2.2.

Observe that the solutions of the ARE actually are those solutions of the linear matrix inequality ($K = K'$)

$$F(K) := \begin{bmatrix} C'C + A'K + KA & KB + C'D \\ B'K + D'C & D'D \end{bmatrix} \geq 0 \qquad (2.5)$$

for which the rank of the dissipation matrix ([8]) is minimal (i.e. equals normal rank $T(s) = m$, with $T(s) = D + C(sI - A)^{-1}B$). This inequality is in [4] called the dissipation inequality and it can be proven ([15]) that also for *singular* LQCP's the real symmetric matrices that determine the optimal costs for these LQCP's should be searched among the rank minimizing solutions of the dissipation inequality, a conjecture as old as 1971 ([4], see also [8]). In [9] and [16] two methods are proposed for calculating these solutions.

3. The positive semi-definite solutions of the Algebraic Riccati Equation as combinations of the smallest and the largest positive semi-definite ones.

In the present Section we will show that every positive semi-definite solution of the ARE (2.4) can be characterized uniquely in terms of K^- and K^+ (Sec. 2). The proof of our result will basicly follow the lines of the work done in [5], with here and there some adjustments. We will use the most pleasant form of the ARE, the one in (2.4b). One remark with respect to our notation: $\mathcal{L}^{-,0,+}(A)$ stand for the (A-invariant) subspaces spanned by the (generalized) eigenvectors corresponding to eigenvalues of A in $\mathbb{C}^{-,0,+}$, respectively, and, analogously, $\mathcal{L}^+(\sigma^*(\Sigma))$ stands for the subspace spanned by the (generalized) eigenvectors in $\langle \ker(C_0)|A_0 \rangle$ corresponding to eigenvalues of A_0 in $\mathbb{C}^+$ (Sec. 2).

Before presenting our main result we will repeat a few known by-results ([5]) and prove several other ones.

<u>Lemma 3.1</u>.

Let K_1, K_2 be any two solutions of (2.4) and set $M = K_2 - K_1$, $A_1 = A_0 - B(D'D)^{-1}B'K_1$, $A_2 = A_0 - B(D'D)^{-1}B'K_2$. Let $\mathcal{N} := \ker(M)$ and $q_0 = \dim(\mathcal{N})$; assume that M has q_+ positive and q_- negative eigenvalues (thus $q_0 + q_+ + q_- = n$). Now it holds that $A_1(\mathcal{N}) \subset \mathcal{N}$ and the restriction of A_1 w.r.t. $\mathcal{N}$, $A_1|\mathcal{N}$, equals $A_2|\mathcal{N}$. Next, if $\sigma(A_1|\mathcal{N}) = \{\lambda_1, \lambda_2, \ldots \lambda_{q_0}\}$ and $\sigma(A_1|\mathbb{R}^n/_{\mathcal{N}}) = \{\lambda_{q_0+1}, \ldots \lambda_n\}$, then $\sigma(A_2) = \{\lambda_1, \lambda_2, \ldots \lambda_{q_0}, -\lambda_{q_0+1}, \ldots -\lambda_n\}$. Also q_+ eigenvalues of $\sigma(A_1|\mathbb{R}^n/_{\mathcal{N}})$ have positive and q_- eigenvalues of $\sigma(A_2|\mathbb{R}^n/_{\mathcal{N}})$ have negative real parts.

Proof. See [5, Th. 2] and note that (A, B)-controllability is not necessary for the proof given there; stabilizability is already sufficient.

Corollary 3.2.

Let $A_0^- = A_0 - B(D'D)^{-1}B'K^-$, $A_0^+ = A_0 - B(D'D)^{-1}B'K^+$, $\Delta = K^+ - K^-$ and $\mathcal{V}_0 = \ker(\Delta)$. If K is any real symmetric solution K of (2.4) such that $K^- \leq K \leq K^+$ and $A_{o_K} = A_0 - B(D'D)^{-1}B'K$, then

$$A_0^-|\mathcal{V}_0 = A_{o_K}|\mathcal{V}_0 = A_0^+|\mathcal{V}_0 .$$

It holds that $\sigma(A_{o_K}|\mathcal{V}_0) \subset \bar{\mathbb{C}}^-$ (the closed left half-plane) and $\sigma(A_{o_K}|\mathbb{R}^n/\mathcal{V}_0) \cap \mathbb{C}^0 = \emptyset$, where $\mathbb{C}^0 = \{s \in \mathbb{C}|\mathrm{Re}(s) = 0\}$.

Proof. See [5, Corollary (ii)]. Observe that we only know that $\sigma(A_{o_K}|\mathcal{V}_0)$ is in $\bar{\mathbb{C}}^-$ (and not necessarily in $\mathbb{C}^0$).

Corollary 3.3.

It holds that $\mathcal{L}^+(A_{o_K}) \subset \ker(K)$ and that $(\sigma(A_{o_K}) \cap \mathbb{C}^0) \subset (\sigma(A_0^-) \cap \mathbb{C}^0)$ (where $A_{o_K} = A_0 - B(D'D)^{-1}B'K$).

Proof. Let $A_{o_K}v_1 = \lambda v_1$, $\mathrm{Re}(\lambda) \geq 0$. Pre- and postmultiplication of (2.4b) by $\bar{v}_1'$ and v_1, respectively, yields that $C_0v_1 = 0$, $B'Kv_1 = 0$, whence $A_0v_1 = \lambda v_1$, and thus $C_0A_0v_1 = 0$. We deduce that $v_1 \in \langle \ker(C_0)|A_0 \rangle = \ker(K^-)$ (e.g. [11, Remark 2]) and therefore $A_0^-v_1 = \lambda v_1$. We establish that $\lambda \in \sigma(A_0^-)$. Now also, necessarily, $\mathrm{Re}(\lambda)\bar{v}_1'Kv_1 = 0$ and hence, if $\mathrm{Re}(\lambda) > 0$, $v_1 \in \ker(K)$. Then, let $A_{o_K}v_2 = \lambda v_2 + v_1$, v_2 and v_1 independent (i.e.

v_2 is a generalized eigenvector corresponding to λ). We find again that $\bar{v}_2'Kv_2 = 0$, i.e., that $v_2 \in \ker(K)$. Thus $\mathcal{L}^+(A_{o_K}) \subset \ker(K)$.

Lemma 3.4.

Let $K \geq 0$ be as in Corollary 3.2. Then $A_0(\ker(K)) \subset \ker(K)$ and $\sigma(A_{o_K}|\mathbb{R}^n/_{\ker(K)}) \subset \bar{\mathbb{C}}^-$, $\sigma(A_0|\mathcal{V}/_{\ker(K)}) \subset \bar{\mathbb{C}}^+$, $\sigma(A_{o_K}|\mathbb{R}^n/_{\ker(K)}) \cap \mathbb{C}^0 = \sigma(A_0|\mathcal{V}/_{\ker(K)}) \cap \mathbb{C}^0 = \sigma(A_0^-|\mathbb{R}^n/_{\ker(K)}) \cap \mathbb{C}^0$.

Proof. The fact that $\ker(K)$ is A_0-invariant is widely known and easily re-established. Then, let $A_{o_K}v - \lambda v \in \ker(K)$, $v \notin \ker(K)$. Pre- and postmultiplication of (2.4b) by $\bar{v}'$ and v, respectively, yields that $2(\mathrm{Re}(\lambda))\bar{v}'Kv = -\bar{v}'[C_0'C_0 + KB(D'D)^{-1}B'K]v \leq 0$. If $\mathrm{Re}(\lambda) = 0$, then $C_0v = 0$ and $B'Kv = 0$, and thus $A_0v - \lambda v \in \ker(K)$, $C_0A_0v = 0$ ($\ker(K) \subset \ker(K^-) \subset \ker(C_0)$). We find that $v \in \ker(K^-)$. Next, if $A_0v - \lambda v \in \ker(K)$ with $v \in \ker(K^-)$ and $\bar{v}'Kv > 0$, then, analogously, we get that $2(\mathrm{Re}(\lambda))\bar{v}'Kv = \bar{v}'KB(D'D)^{-1}B'Kv \geq 0$ and, if $\mathrm{Re}(\lambda) = 0$, then $B'Kv = 0$ and $A_{o_K}v - \lambda v \in \ker(K)$. Finally, let $A_0^-v = i\omega v + p$ ($\omega \in \mathbb{R}$, $Kp = 0$, $Kv \neq 0$), then, from (2.4b) with $K = K^-$, $v \in \ker(K^-)$ (note that $K^-p = 0$) and thus $A_0v = i\omega v + p$. The converse is trivial.

Corollary 3.5.

The subspaces $\mathcal{V}_0$ and $\mathcal{L}^+(A_0^-)$ are independent and span the entire state space. Both are A_0^--invariant and $\sigma(A_0^-|\mathcal{V}_0) \subset \bar{\mathbb{C}}^-$, $\sigma(A_0^-|\mathcal{L}^+(A_0^-)) \subset \mathbb{C}^+$. Moreover, $\mathcal{L}^+(A_0^-) = \mathcal{L}^+(\sigma^*(\Sigma)) \subset \ker(K^-) = \langle \ker(C_0)|A_0 \rangle$, $\mathcal{L}^+(A_0^-)$ is A_0-invariant.

Proof. From Lemma 3.1 we learn that $\sigma(A_0^-|\mathbb{R}^n/_{\mathcal{V}_0}) \subset \mathbb{C}^+$ and that $\mathcal{V}_0$ is A_0^--invariant; from Corollary 3.2 it follows that $\sigma(A_0^-|\mathcal{V}_0) \subset \bar{\mathbb{C}}^-$. Next, let $A_0^- v_1 = \lambda v_1$ with $\mathrm{Re}(\lambda) \geq 0$. Pre- and postmultiplying (2.4b) with $K = K^-$ by $\bar{v}_1'$ and v_1, respectively, yields $C_0 v_1 = 0$, $B'K^- v_1 = 0$ and thus $A_0 v_1 = \lambda v_1$. But then also $C_0 A_0 v_1 = 0$, and hence $v_1 \in \langle \ker(C_0)|A_0 \rangle = \ker(K^-)$. If v_2 is a generalized eigenvector corresponding to λ (that is, $A_0^- v_2 = \lambda v_2 + v_1$, v_1 and v_2 independent), then again $v_2 \in \langle \ker(C_0)|A_0 \rangle$ and therefore $\mathcal{L}^+(A_0^-) \subset \mathcal{L}^+(A_0|\langle \ker(C_0)|A_0 \rangle)$ as well as $\mathcal{L}^0(A_0^-) \subset \mathcal{L}^0(A_0|\langle \ker(C_0)|A_0 \rangle)$. Now, trivially, $\mathcal{L}^+(A_0|\langle \ker(C_0)|A_0 \rangle) \subset \mathcal{L}^+(A_0^-)$, $\mathcal{L}^0(A_0|\langle \ker(C_0)|A_0 \rangle) \subset \mathcal{L}^0(A_0^-)$ and hence, in particular, $\mathcal{L}^+(A_0^-) = \mathcal{L}^+(A_0|\langle \ker(C_0)|A_0 \rangle)$.

<u>Corollary 3.6</u>.

It holds that $\mathcal{L}^0(A_0^-) = \mathcal{L}^0(A_0|\langle \ker(C_0)|A_0 \rangle)$ and $\mathcal{L}^+(A_0^-) = \mathcal{L}^+(A_0|\langle \ker(C_0|A_0 \rangle)$. Thus, $\ker(K^-) = \langle \ker(C_0)|A_0 \rangle = \mathcal{L}^-(A_0|\langle \ker(C_0)|A_0 \rangle) \oplus \mathcal{L}^0(A_0^-) \oplus \mathcal{L}^+(A_0^-)$.

Proof. The first two claims follow from the proof of the previous Corollary and then the third statement is immediate from the observation that $\langle \ker(C_0)|A_0 \rangle = \mathcal{L}^-(A_0|\langle \ker(C_0)|A_0 \rangle) \oplus \mathcal{L}^0(A_0|\langle \ker(C_0)|A_0 \rangle) \oplus \mathcal{L}^+(A_0|\langle \ker(C_0)|A_0 \rangle)$.

<u>Lemma 3.7</u>.

For $x_0 \in \mathcal{L}^-(A_{0_K})$ it holds that $(K^+ - K)x_0 = 0$, if $x_0 \in \mathcal{L}^+(A_{0_K})$ then $(K - K^-)x_0 = 0$ $(K^- \leq K \leq K^+)$.

Proof. See e.g. [4].

Lemma 3.8.

Let $\tilde{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and assume that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathbb{R}^n$, $\mathcal{V}_1$ and $\mathcal{V}_2$ are $\tilde{A}$-invariant and $\sigma(\tilde{A}|\mathcal{V}_1) \subset \mathbb{C}_1 \subset \mathbb{C}$, $\sigma(\tilde{A}|\mathcal{V}_2) \subset \mathbb{C}_2 := \mathbb{C}\backslash\mathbb{C}_1$. If $\tilde{\mathcal{V}}$ is an $\tilde{A}$-invariant subspace such that $\sigma(\tilde{A}|\tilde{\mathcal{V}}) \subset \mathbb{C}_1$, then $\tilde{\mathcal{V}} \subset \mathcal{V}_1$.

Proof. Let $0 \neq x \in \tilde{\mathcal{V}}$. Decompose $x = x_1 + x_2$ with $x_1 \in \mathcal{V}_1$ and $x_2 \in \mathcal{V}_2$. Then $x_1 \neq 0$ since $\mathcal{V}_2 \cap \tilde{\mathcal{V}} = 0$. Let $p(s)$ be the polynomial of least positive degree such that $p(\tilde{A})x_1 = 0$. Then $p(s)$ only has roots in $\mathbb{C}_1$. But from $p(\tilde{A})x = p(\tilde{A})x_2$, $p(\tilde{A})x \in \tilde{\mathcal{V}}$, $p(\tilde{A})x_2 \in \mathcal{V}_2$, we conclude that necessarily $p(\tilde{A})x = p(\tilde{A})x_2 = 0$ and hence, because $\sigma(\tilde{A}|\mathcal{V}_2) \subset \mathbb{C}_2$, we have $x_2 = 0$. Thus $x \in \mathcal{V}_1$.

Remark. Lemma 3.8 is a generalization of [5, Lemma 3].

Proposition 3.9.

$$\mathcal{L}^0(A_0^-) \subset \ker(K^+),\ \mathcal{L}^+(A_0^-) \cap \ker(K^+) = 0.$$

Proof. First, applying Lemma 3.8 ($\mathcal{V}_1 = \mathcal{V}_0$, $\mathcal{V}_2 = \mathcal{L}^+(A_0^-)$, $\mathbb{C}_1 = \overline{\mathbb{C}^-}$, $\mathbb{C}_2 = \mathbb{C}^+$ and Corollary 3.5 yields that $\mathcal{L}^0(A_0^-) \subset \mathcal{V}_0$. From Corollary 3.6 we also deduce that $\mathcal{L}^0(A_0^-) \subset \ker(K^-)$. But then $\mathcal{L}^0(A_0^-) \subset \mathcal{V}_0 \cap \ker(K^-) = \ker(K^+)$. Next, $\mathcal{L}^+(A_0^-) \cap \ker(K^+) = \mathcal{L}^+(A_0^-) \cap \mathcal{V}_0$ (Corr. 3.5) $= 0$.

Remark.

Proposition 3.9 expresses that for every $x_0 \in \mathcal{L}^0(A_0^-)$, $J_0(x_0) = 0$ (Sec. 2). This is also stated on page 334 of [12]. An optimal control, however, does not exist unless $x_0 = 0$.

Corollary 3.10.

$$\ker(K^+) = \mathcal{L}^0(A_0|\langle \ker(C_0)|A_0\rangle) \oplus \mathcal{L}^-(A_0|\langle \ker(C_0)|A_0\rangle).$$

Proof. We have $\ker(K^+) = \ker(K^+) \cap \ker(K^-) =$ (Corr. 3.6) $\ker(K^+) \cap [\mathcal{L}^0(A_0^-) \oplus \mathcal{L}^+(A_0^-) \oplus \mathcal{L}^-(A_0|\langle \ker(C_0)|A_0\rangle)] =$ (Corr 3.6, Prop. 3.9) $\mathcal{L}^0(A_0^-) \oplus \mathcal{L}^-(A_0|\langle \ker(C_0)|A_0\rangle) + [\ker(K^+) \cap \mathcal{L}^+(A_0^-)] = \mathcal{L}^0(A_0|\langle \ker(C_0)|A_0\rangle) \oplus \mathcal{L}^-(A_0|\langle \ker(C_0)|A_0\rangle)$.

Proposition 3.11.

Let K be any positive semi-definite solution of (2.4) ($K^- \leq K \leq K^+$). As earlier, set $A_{0K} = A_0 - B(D'D)^{-1}B'K$. Then it holds that $\sigma(A_{0K}|\mathbb{R}^n/_{\ker(K)}) \subset \mathbb{C}^-$ and $\sigma(A_0|\mathcal{V}/_{\ker(K)}) \subset \mathbb{C}^+$. Thus, $\mathcal{L}^0(A_{0K}) + \mathcal{L}^+(A_{0K}) \subset \ker(K)$.

Proof. First, it is easily found that $\sigma(A_0^-|\mathbb{R}^n/_{\mathcal{L}^0(A_0^-)}) \cap \mathbb{C}^0 = \emptyset$. Thus (Proposition 3.9) $\sigma(A_0^-|\mathbb{R}^n/_{\ker(K^+)}) \cap \mathbb{C}^0 = \emptyset$ and hence, with Lemma 3.4, $\sigma(A_0|\mathcal{V}/_{\ker(K^+)}) \cap \mathbb{C}^0 = \emptyset$. Therefore ($\ker(K^+) \subset \ker(K)$) $\sigma(A_0|\mathcal{V}/_{\ker(K)}) \cap \mathbb{C}^0 = \emptyset$ and, again with Lemma 3.4 and Corr. 3.3, this yields our claims.

Theorem 3.12.

Let $\mathcal{V}_1$ be an arbitrary subspace of $\mathcal{L}^+(\sigma^*(\Sigma))$ and $A_0(\mathcal{V}_1) \subset \mathcal{V}_1$. If $\mathcal{V}_2$ is the subspace such that for all $x_2 \in \mathcal{V}_2$, Δx_2 is orthogonal to $\mathcal{V}_1$ (i.e. $\mathcal{V}_2 = \Delta^{-1}\mathcal{V}_1^{\perp}$), then $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathbb{R}^n$. If P denotes the projection matrix onto $\mathcal{V}_1$ and along $\mathcal{V}_2$, then

$$K = K^-P + K^+(I - P) = K^+(I - P) \tag{3.1}$$

is a positive semi-definite solution of (2.4). Moreover, all positive semi-definite solutions are obtained in this way. Hence the correspondence between $\mathcal{V}_1$ and K is one-to-one.

Proof.

First part. We start with proving that $\mathcal{V}_1 \cap \mathcal{V}_2 = 0$. Let $x_0 \in \mathcal{V}_1 \cap \mathcal{V}_2$, then $x_0'\Delta x_0 = 0$ and therefore $(\Delta \geq 0)$ $\Delta x_0 = 0$, i.e. $x_0 \in \mathcal{V}_0$. But $\mathcal{V}_0 \cap \mathcal{V}_1 = 0$ ($\mathcal{V}_1 \subset \mathcal{L}^+(\sigma^*(\Sigma))$ and Corollary 3.5), hence $x_0 = 0$. Then we have

$$\dim(\Delta\mathbb{R}^n) = n - \dim(\ker(\Delta)) = n - \dim(\mathcal{V}_0),$$

$$\dim(\mathcal{V}_1{}^\perp) = n - \dim(\mathcal{V}_1),$$

hence

$$\dim(\Delta\mathbb{R}^n \cap \mathcal{V}_1{}^\perp) \geq 2n - \dim(\mathcal{V}_0) - \dim(\mathcal{V}_1) - n.$$

Now it is easily found that $(\Delta\mathbb{R}^n \cap \mathcal{V}_1{}^\perp) = (\Delta\mathcal{V}_2 \cap \mathcal{V}_1{}^\perp)$ (for, if $x = \Delta p$ and $\forall_{v_1 \in \mathcal{V}_1}: v_1'\Delta p = 0$, then $p \in \mathcal{V}_2$) and that $(\Delta\mathcal{V}_2 \cap \mathcal{V}_1{}^\perp) = \Delta\mathcal{V}_2$. But then $\dim(\Delta\mathbb{R}^n \cap \mathcal{V}_1{}^\perp) = \dim(\Delta\mathcal{V}_2) = \dim(\mathcal{V}_2) - \dim(\mathcal{V}_2 \cap \ker(\Delta)) = \dim(\mathcal{V}_2) - \dim(\mathcal{V}_0)$ ($\mathcal{V}_0 \subset \mathcal{V}_2$) and therefore $\dim(\mathcal{V}_2) = \dim(\mathcal{V}_0) + \dim(\Delta\mathbb{R}^n \cap \mathcal{V}_1{}^\perp) \geq n - \dim(\mathcal{V}_1)$. We conclude that $\mathcal{V}_1 + \mathcal{V}_2 = \mathbb{R}^n$. Thus $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathbb{R}^n$.

Next, it is easy to show that

$$\Delta A_0{}^- + (A_0{}^+)'\Delta = 0 \tag{3.2}$$

and thus for $x_1 \in \mathcal{V}_1$, $x_2 \in \mathcal{V}_2$ we establish that $x_2'(A_0{}^+)'\Delta x_1 = 0$ (recall that $\mathcal{V}_1$ is $A_0{}^-$-invariant) which means that $\mathcal{V}_2$ is $A_0{}^+$-invariant. Hence the projection P satisfies

$$PA_0{}^-P = A_0{}^-P \tag{3.3a}$$

and $(I - P)A_0{}^+(I - P) = A_0{}^+(I - P)$, (3.3b)

i.e. $PA_0{}^+P = PA_0{}^+$. (3.3b')

Since $\Delta\mathcal{V}_2$ is orthogonal to $\mathcal{V}_1$, also

$$P'\Delta(I - P) = 0, \tag{3.4}$$

i.e. $P'\Delta = P'\Delta P$ and thus, by symmetry,

$$P'\Delta = \Delta P \quad . \tag{3.5}$$

Combining (3.3a), (3.3b') yields

$$\begin{aligned}(I - P)A_0{}^- &= (I - P)A_0{}^-(I - P)\\ &= (I - P)(A_0{}^+ + B(D'D)^{-1}B'\Delta)(I - P)\\ &= A_0{}^+(I - P) + (I - P)B(D'D)^{-1}B'\Delta(I - P)\end{aligned}$$

and therefore, by (3.2),

$$\Delta(I - P)A_0^- + (A_0^-)'\Delta(I - P) = \Delta(I - P)B(D'D)^{-1}B'\Delta(I - P). \tag{3.6}$$

Hence if we define K by (3.1) then $K = K^- + \Delta(I - P)$ and we establish that K is symmetric ((3.5)) and positive semi-definite and $(K - K^-)$ satisfies (3.6). But then K satisfies (2.4). Note that $K^+ - K = \Delta P$. In addition, since $K^-\mathcal{V}_1 = 0$ (Corr. 3.5), we have that $K = K^+(I - P)$.

Moreover, for $x_1 \in \mathcal{V}_1$, $Kx_1 = K^-x_1$, and for $x_2 \in \mathcal{V}_2$, $Kx_2 = K^+x_2$. Hence

$$A_{0_K}(\mathcal{V}_1) = A_0^-(\mathcal{V}_1) \subset \mathcal{V}_1,\ A_{0_K}(\mathcal{V}_2) = A_0^+(\mathcal{V}_2) \subset \mathcal{V}_2$$

and

$$\sigma(A_{0_K}|\mathcal{V}_1) \subset \mathbb{C}^+,\ \sigma(A_{0_K}|\mathcal{V}_2) \subset \bar{\mathbb{C}}^-,$$

which shows that $\mathcal{V}_1$ is uniquely determined by K.

Second part. Let K be any positive semi-definite solution of (2.4). Then $\sigma(A_{0_K}|\mathcal{V}_0) \subset \bar{\mathbb{C}}^-$ and no other eigenvalue of A_{0_K} is on the imaginary axis (Corollary 3.2) and thus we establish that $\mathcal{V}_1 := \mathcal{L}^+(A_{0_K})$ and $\mathcal{V}_2 := (\mathcal{L}^-(A_{0_K}) + \mathcal{V}_0)$ are two independent subspaces that span $\mathbb{R}^n$ and are both A_{0_K}-invariant (observe that $A_{0_K}(\mathcal{V}_0) \subset \mathcal{V}_0$ and that $\mathcal{L}^0(A_{0_K}) \subset \ker(K)$ (Prop. 3.11) $\subset \ker(K^-)$ (hence $A_0^-(\mathcal{L}^0(A_{0_K})) = A_{0_K}(\mathcal{L}^0(A_{0_K})) \subset \mathcal{L}^0(A_{0_K})$)). In addition (Lemma 3.7), $Kx_1 = K^-x_1$ if $x_1 \in \mathcal{V}_1$ and $Kx_2 = K^+x_2$ if $x_2 \in \mathcal{V}_2$. Thus, if P is the projection onto $\mathcal{V}_1$ and along $\mathcal{V}_2$, then $K = K^-P + K^+(I - P)$. Moreover, $A_0^-(\mathcal{V}_1) \subset \mathcal{V}_1$ as well as $A_0^+(\mathcal{V}_2) \subset \mathcal{V}_2$ (Lemma 3.7). Now apply Lemma 3.8 with $\tilde{A} = A_0^-$, $\tilde{\mathcal{V}}_1 = \mathcal{L}^+(\sigma^*(\Sigma))$, $\mathcal{V}_2 = \mathcal{V}_0$ (recall Corollary 3.5), $\mathbb{C}_1 = \mathbb{C}^+$, $\mathbb{C}_2 = \bar{\mathbb{C}}^-$, $\mathcal{V} = \mathcal{V}_1$ in order to conclude that $\mathcal{V}_1 \subset \mathcal{L}^+(\sigma^*(\Sigma))$ and thus $A_0(\mathcal{V}_1) \subset \mathcal{V}_1$. But then $K^-\mathcal{V}_1 = 0$ and $K = K^+(I - P)$. Since $K - K^- = \Delta(I - P)$, (3.5) follows and therefore (P projection) $P'\Delta(I - P) = 0$, i.e. $\Delta\mathcal{V}_2$ is orthogonal to $\mathcal{V}_1$. From the fact that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathbb{R}^n$ it finally follows (see the first part of the proof) that actually $\mathcal{V}_2 = \Delta^{-1}\mathcal{V}_1^{\perp}$. This completes the proof.

Theorem 3.12 describes our main result. It links every positive semi-definite solution of (2.4) bijectively to a certain subspace. As can be expected, it now holds that the set of all positive semi-definite solutions of (2.4) forms a complete lattice (compare [1] - [6]). This is shown in the next Theorem.

Theorem 3.13.

Let K, $\tilde{K}$ be positive semi-definite solutions of (2.4) corresponding to the A_0-invariant subspaces $\mathscr{V}_1$, $\tilde{\mathscr{V}}_1$ (both in $\mathcal{L}^+(\sigma^*(\Sigma))$). Then $K \geq \tilde{K}$ if and only if $\mathscr{V}_1 \subset \tilde{\mathscr{V}}_1$.

Proof. $\Leftarrow$ Let $\tilde{v}_2 \in \tilde{\mathscr{V}}_2$, then for all $\tilde{v}_1 \in \tilde{\mathscr{V}}_1$, $\tilde{v}_2{}'\Delta\tilde{v}_1 = 0$, hence also for all $v_1 \in \mathscr{V}_1$, $v_2{}'\Delta v_1 = 0$ i.e. $\tilde{v}_2 \in \mathscr{V}_2$. Then

$$Kx_1 = K^- x_1 = \tilde{K}x_1 \text{ on } \mathscr{V}_1 \cap \tilde{\mathscr{V}}_1 = \mathscr{V}_1 ,$$
$$Kx_2 = K^+ x_2 = \tilde{K}x_2 \text{ on } \mathscr{V}_2 \cap \tilde{\mathscr{V}}_2 = \tilde{\mathscr{V}}_2 ,$$

and we have that $\mathscr{V}_1 \oplus \tilde{\mathscr{V}}_2 \subset \mathscr{N} := \ker(K - \tilde{K})$. Now $A_{o_K}(\mathscr{V}_1) \subset \mathscr{V}_1$ and for any $\tilde{v}_2 \in \tilde{\mathscr{V}}_2$, $A_{o_K}\tilde{v}_2 = A_{o_{\tilde{K}}}\tilde{v}_2 \in \tilde{\mathscr{V}}_2$, hence $A_{o_K}(\mathscr{V}_1 \oplus \tilde{\mathscr{V}}_2) \subset \mathscr{V}_1 \oplus \tilde{\mathscr{V}}_2$. Analogously, $A_{o_{\tilde{K}}}(\mathscr{V}_1 \oplus \tilde{\mathscr{V}}_2) \subset \mathscr{V}_1 \oplus \tilde{\mathscr{V}}_2$. It holds that $\sigma(A_{o_K}|\mathbb{R}^n/_{\mathscr{V}_1}) = \sigma(A_{o_K}|\mathscr{V}_2) \subset \bar{\mathbb{C}}^-$ and $\sigma(A_{o_{\tilde{K}}}|\mathbb{R}^n/_{\tilde{\mathscr{V}}_2}) = \sigma(A_{o_{\tilde{K}}}|\tilde{\mathscr{V}}_1) \subset \mathbb{C}^+$, from which we establish that $\sigma(A_{o_K}|\mathbb{R}^n/_{\mathscr{N}}) \subset \bar{\mathbb{C}}^-$, $\sigma(A_{o_{\tilde{K}}}|\mathbb{R}^n/_{\mathscr{N}}) \subset \mathbb{C}^+$. Applying Lemma 3.1, yields $\sigma(A_{o_K}|\mathbb{R}^n/_{\mathscr{N}}) \subset \mathbb{C}^-$, but then necessarily $\mathscr{N} = \mathscr{V}_1 \oplus \tilde{\mathscr{V}}_2$, because $\sigma(A_{o_{\tilde{K}}}|\mathscr{N}) = \sigma(A_{o_K}|\mathscr{N})$. Thus, also $A_{o_{\tilde{K}}}(\mathscr{N}) \subset \mathscr{N}$ and (Lemma 3.1) $K - \tilde{K} \geq 0$.

$\Rightarrow$ Suppose that $K \geq \tilde{K}$. Then $\mathscr{N} := \ker(K - \tilde{K})$ is $A_{o_{\tilde{K}}}$-invariant and hence has a unique decomposition $\mathscr{N} = \mathscr{N}_+ \oplus \mathscr{N}_-$, where $\mathscr{N}_+$, $\mathscr{N}_-$ are $A_{o_{\tilde{K}}}$-invariant and $\sigma(A_{o_{\tilde{K}}}|\mathscr{N}_+) \subset \mathbb{C}^+$, $\sigma(A_{o_{\tilde{K}}}|\mathscr{N}_-) \subset \bar{\mathbb{C}}^-$. Observe that $A_{o_K}(\mathscr{N}_+) \subset \mathscr{N}_+$. Now apply Lemma 3.8 twice: First to show that $\mathscr{N}_+ \subset \mathscr{V}_1$ (with $\tilde{A} = A_{o_K}$, $\mathscr{V}_{1,2}$ are the $\mathscr{V}_{1,2}$ corresponding to K, $\mathbb{C}_1 = \mathbb{C}^+$,

$\mathbb{C}_2 = \bar{\mathbb{C}}^-$) and then to prove that also $\mathcal{N}_+ \subset \tilde{\mathcal{V}}_1$ ($\tilde{A} = A_{0\tilde{K}}$, $\mathcal{V}_{1,2} = \tilde{\mathcal{V}}_{1,2}$). We establish from $\sigma(A_{0K}|\mathcal{V}_1) \subset \mathbb{C}^+$ that $\mathcal{V}_1 \cap \mathcal{N}_- = 0$ and that $\sigma(A_{0K}|\mathbb{R}^n/_{\mathcal{V}_1}) \subset \bar{\mathbb{C}}^-$. Thus, if $\mathcal{V}_1$ would be a real subspace of $\mathcal{N}_+$ (that is, not $\mathcal{N}_+ = \mathcal{V}_1$), then $\mathbb{C}^+ \supset \sigma(A_{0\tilde{K}}|\mathcal{N}_+) = \sigma(A_{0K}|\mathcal{N}_+) \subset \mathbb{C}^+ \cup \bar{\mathbb{C}}^-$. Therefore $\mathcal{N}_+ = \mathcal{V}_1$ and hence $\mathcal{V}_1 \subset \tilde{\mathcal{V}}_1$.

<u>Corollary 3.14</u>.

There exists a bijection η: $\mathcal{L}^+(\sigma^*(\Sigma)) \to \Gamma := \{K = K' \mid K \geq 0$, K satisfies (2.4)$\}$ and $\eta(0) = K^+$, $\eta(\mathcal{L}^+(\sigma^*(\Sigma))) = K^-$.

Combination of certain by-results and Theorem 3.12 yields our final statement, a generalization of Corr. 3.10.

<u>Corollary 3.15</u>.

Let K be as in Proposition 3.11. Then

$$\ker(K) = \mathcal{L}^0(A_0|\mathcal{V}) \oplus \mathcal{L}^-(A_0|\mathcal{V}) \oplus \mathcal{V}_1$$

where $\mathcal{V}_1$ is (uniquely) determined by Theorem 3.12.

Proof. First, we have (Corr. 3.10) $\mathcal{L}^0(A_0|\mathcal{V}) \oplus \mathcal{L}^-(A_0|\mathcal{V}) \subset \ker(K^+) \subset \ker(K)$. Thus, $\ker(K) = \ker(K) \cap \ker(K^-) = \ker(K) \cap \{\mathcal{L}^0(A_0|\mathcal{V}) \oplus \mathcal{L}^-(A_0|\mathcal{V}) \oplus \mathcal{L}^+(A_0|\mathcal{V})\} = \mathcal{L}^0(A_0|\mathcal{V}) \oplus \mathcal{L}^-(A_0|\mathcal{V}) + \{\ker(K) \cap \mathcal{L}^+(A_0|\mathcal{V})\}$ and the latter subspace equals $\mathcal{V}_1$ since $\mathcal{V}_1 \subset \mathcal{L}^+(A_0|\mathcal{V})$, $\mathcal{V}_1 \subset \ker(K)$ and $\mathcal{V}_2 \cap \ker(K) = \mathcal{V}_2 \cap \ker(K^+)$ (apply Prop. 3.9).

<u>Remark</u>.

Observe that the <u>only</u> assumption that we have used in this paper in order to obtain our results is: (A, B) is stabilizable. In [1] - [2] results of the same kind as Theorem 3.12 have been established under the same assumption and the (superfluous) additional assumption $\sigma(A_0|\langle \ker(C_0)|A_0 \rangle) \cap \mathbb{C}^0 = \emptyset$.

4. Discussion.

From the foregoing it is clear that both the smallest and the largest positive semi-definite solutions of the ARE exist if (A, B) is stabilizable. To be more accurate, it is shown in [11] that K^- exists if and only if $(\bar{A}_0^-, \bar{B})$ is stabilizable, where $\bar{A}_0^-$ and $\bar{B}$ are the induced maps of A_0 and B w.r.t. $\mathbb{R}^n/_{\mathcal{V}}$ (indeed $A_0(\mathcal{V}) \subset \mathcal{V}$!). This condition is easily seen to be equivalent to: $\mathcal{V} + \mathcal{L}^-(A) + \langle A|\mathrm{im}(B)\rangle = \mathbb{R}^n$ (see e.g. [9, Lemma 5.6] and [10], [13]). Thus, if (A, B) is stabilizable (i.e. $\mathcal{L}^-(A) + \langle A|\mathrm{im}(B)\rangle = \mathbb{R}^n$), then K^- exists, and also K^+ (e.g. [12]). The importance of the matrix K^+ is, that the spectrum of $A_0(K) := A_0 - B(D'D)^{-1}B'K$ is contained in $\bar{\mathbb{C}}^-$ if $K = K^+$. In other words, for all $x_0 \in \mathcal{L}^-(A_0(K^+))$ there exists an optimal control for $(LQCP)_0$, the problem with stability ([2] - [4], [12]). Now let us ask ourselves the question: When does there exist a solution $\tilde{K} \geq 0$ of the ARE such that $\sigma(A_0(\tilde{K})) \subset \bar{\mathbb{C}}^-$? If (A, B) is stabilizable, such a $\tilde{K}$ exists (K^+). Conversely, is it necessary for such a $\tilde{K}$ to exist that (A, B) is stabilizable? No. A simple counterexample: $A = 0$, $C = 0$, $D = I$, $m < n$. The ARE is: $0 = -KBB'K$ and $K^- = 0$, $\sigma(A_0(K^-)) \subset \bar{\mathbb{C}}^-$. However, (A, B) is not stabilizable ([10]). Of course, it is trivial that <u>necessary</u> for the existence of a solution $\tilde{K} \geq 0$ of the ARE such that $\sigma(A_0(\tilde{K})) \subset \bar{\mathbb{C}}^-$ is: $\exists_F$: $\sigma(A + BF) \subset \bar{\mathbb{C}}^-$ (or, equivalently, $\mathcal{L}^-(A) + \mathcal{L}^0(A) + \langle A|\mathrm{im}(B)\rangle = \mathbb{R}^n$) <u>and</u> $\exists_{K = K' \geq 0}$: K satisfies the ARE (equivalently, the <u>smallest</u> positive semi-definite solution K^- exists ([11])). Now we will demonstrate that these two conditions are also <u>sufficient</u> for the existence of such a $\tilde{K}$. The proof runs as follows. Since ([11]) K^- (the smallest positive semi-definite solution of (2.4)) exists, we have $0 = C_0'C_0 + (A_0^-)'K^- + K^-A_0^- + K^-B(D'D)^{-1}B'K^-$ and if K is any other positive semi-definite solution and $\Delta K := K - K^-$, then it holds that $0 = (A_0^-)'\Delta K + \Delta K A_0^- - \Delta K B(D'D)^{-1}B'\Delta K$. But also the converse is true: If we have a positive semi-definite ΔK satisfying the latter equation, then $K = K^- + \Delta K$ satisfies (2.4). Next, we decompose $\mathbb{R}^n = \mathcal{L}^+(A_0^-) \oplus \mathcal{L}^0(A_0^-) \oplus \mathcal{L}^-(A_0^-)$. The matrices A_0^- and B then look like

$$\begin{bmatrix} A_{011} & 0 & 0 \\ 0 & A_{022} & 0 \\ 0 & 0 & A_{033} \end{bmatrix} \text{ and } \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},$$

with $\sigma(A_{011}) \subset \mathbb{C}^+$, $\sigma(A_{022}) \subset \mathbb{C}^0$ and $\sigma(A_{033}) \subset \mathbb{C}^-$. It holds that (A_{011}, B_1) is controllable ([10]). Hence there exists a (unique) positive definite solution ΔK_{11} of the algebraic Riccati equation $0 = A_{011}'\Delta K_1 + \Delta K_1 A_{011} - \Delta K_1 B_1 (D'D)^{-1} B_1' \Delta K_1$ and $A_{011} - B_1 (D'D)^{-1} B_1 \Delta K_{11}$ is asymptotically stable (see e.g. [12, p. 334]). Thus

$$\begin{bmatrix} \Delta K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

represents a positive semi-definite solution ΔK of $0 = (A_0^-)'\Delta K + \Delta K A_0^- - \Delta K B (D'D)^{-1} B' \Delta K$ and $\sigma(A_0^- - B(D'D)^{-1}B'\Delta K) \subset \bar{\mathbb{C}}^-$. Therefore we have proven the existence of a solution $\tilde{K}$ ($= K^- + \Delta K$) of (2.4) such that $\sigma(A_0(\tilde{K})) \subset \bar{\mathbb{C}}^-$ (note that $A_0(\tilde{K}) = A_0^- - B(D'D)^{-1}B'\Delta K$). However, note that $\tilde{K}$ needs <u>not</u> to be the <u>largest</u> solution of (2.4)! For instance, in the above-mentioned example it is clear that $\tilde{K} = 0$ is such that $\sigma(A_0(\tilde{K})) \subset \bar{\mathbb{C}}^-$, but every K that satisfies $KB = 0$ is also a solution of the ARE $0 = -KBB'K$. The explanation for this phenomenon is hidden in the fact that there are points $x_0 \in \mathbb{R}^n$ for which $J_0(x_0)$ does <u>not</u> exist (observe that "(A, B) stabilizable" is equivalent to "$\forall_{x_0} \in \mathbb{R}^n$: $J_0(x_0) < \infty$" (see [10])).

Hence we conclude that the two conditions "$\exists_{K = K' \geq 0}$: K satisfies the ARE (2.4)" and "$\exists_F$: $\sigma(A + BF) \subset \bar{\mathbb{C}}^-$" are *necessary* and *sufficient* for the existence of a solution $\tilde{K}$ of (2.4) such that $\sigma(A_0(\tilde{K})) \subset \bar{\mathbb{C}}^-$. Let us give an interpretation of these conditions by means of a Kalman decomposition of (A_0, B, C_0).

We have

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{bmatrix}, \quad [0 \quad 0 \quad C_3 \quad C_4],$$

with the pairs $\left(\begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}\right)$, $\left([C_3, \; C_4], \begin{bmatrix} A_{33} & A_{34} \\ 0 & A_{44} \end{bmatrix}\right)$ controllable and observable, respectively (and note that $\langle A_0 | \mathrm{im}(B) \rangle = \langle A | \mathrm{im}(B) \rangle$). Since the first two subspaces that divide $\mathbb{R}^n$ span $\mathcal{V} = \langle \ker(C_0) | A_0 \rangle$, it is readily found that K^- exists if and only if $\sigma(A_{44}) \subset \mathbb{C}^-$ ($\ker(K^-) = \mathcal{V}$). The second condition corresponds to the condition $\sigma(A_{22}) \subset \bar{\bar{\mathbb{C}}}^-$; since the eigenvalues of A_{22} cannot be transformed to $\bar{\bar{\mathbb{C}}}^-$, we have to require that $\sigma(A_{22}) \subset \bar{\bar{\mathbb{C}}}^-$ for the possible existence of a $\tilde{K} \geq 0$ such that $\sigma(A_0(\tilde{K})) \subset \bar{\bar{\mathbb{C}}}^-$. Hence we have established that our two conditions "$\langle A | \mathrm{im}(B) \rangle + \mathcal{L}^-(A) + \mathcal{V} = \mathbb{R}^n$" and "$\mathcal{L}^0(A) + \mathcal{L}^-(A) + \langle A | \mathrm{im}(B) \rangle = \mathbb{R}^n$" are equivalent to: "$\sigma(A | \mathbb{R}^n /_{(\mathcal{V} + \langle A | \mathrm{im}(B) \rangle)}) \subset \mathbb{C}^-$" and "$\sigma(A | \mathcal{V} /_{(\mathcal{V} \cap \langle A | \mathrm{im}(B) \rangle)}) \subset \bar{\bar{\mathbb{C}}}^-$".

It is stated in Remark 2.2 that also for <u>singular</u> LQCP's the real symmetric matrices that determine the optimal costs for these problems are rank minimizing solutions of the dissipation inequality ((2.5)). Indeed, the optimal cost for the problem with stability is represented by K^+, the <u>largest</u> of these solutions ([8]), and the cost for the free end-point problem is characterized by K^-, the <u>smallest</u> positive semi-definite rank minimizing solution ([9]). In [16] we will specify in a one-to-one manner the relations between the remaining positive semi-definite rank minimizing solutions and certain subspaces. Since, in case of left invertibility of D, the rank minimizing solutions <u>are</u> the solutions of the ARE, we may consider the case $\ker(D) = \{0\}$ to be a special situation of the general case. Also for the lattice of the rank minimizing solutions such a generalization will be found: If in [16] $\ker(D)$ is assumed to be zero, then the results there transform into ours of Sec. 3.

References.

[1] V. Kucera, "On non-negative definite solutions to matrix quadratic equations", *Automatica*, vol. 8, pp. 413 - 423, 1972.

[2] J.L. Willems & F.M. Callier, "Large finite horizon and infinite horizon LQ-optimal control problems", *Optimal Control Applic. & Meth.*, vol. 4, pp. 31 - 45, 1983.

[3] B.P. Molinari, "The time-invariant linear-quadratic optimal control problem", *Automatica*, vol. 13, pp. 347 - 357, 1977.

[4] J.C. Willems, "Least squares stationary optimal control and the algebraic Riccati equation", *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 621 - 634, 1971.

[5] W.A. Coppel, "Matrix quadratic equations", *Bull. Austral. Math. Soc.*, vol 10, pp. 377 - 401, 1974.

[6] M. Shayman, "Geometry of the algebraic Riccati equation - part I", *SIAM J. Contr. & Opt.*, vol. 21, pp. 375 - 393, 1983.

[7] M.L.J. Hautus & L.M. Silverman, "System structure and singular control", *Lin. Alg. & Appl.*, vol 50, pp. 369 - 402, 1983.

[8] J.M. Schumacher, "The role of the dissipation matrix in singular optimal control", *Syst. & Contr. Lett.*, vol. 2, pp 262 - 266, 1983.

[9] Ton Geerts, "All optimal controls for the singular linear-quadratic problem without stability; a new interpretation of the optimal cost", to appear in *Lin. Alg. & Appl.*.

[10] M.L.J. Hautus, "Stabilization, controllability and observability of linear autonomous systems", *Nederl. Akad. Wetensch. Proc.*, Ser. A, 73, pp. 448 - 455, 1970.

[11] Ton Geerts, "A necessary and sufficient condition for solvability of the linear-quadratic control problem without stability", *Syst. & Contr. Lett.*, vol. 11, pp. 47 - 51, 1988.

[12] J.C. Willems, A. Kitapçi & L.M. Silverman, "Singular optimal control: a geometric approach", *SIAM J. Contr. & Opt.*, vol 24, pp. 323 - 337, 1986.

[13] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, sec. edition, Springer Verlag, New York, 1979.

[14] J.M. Soethoudt & H.L. Trentelman, "The regular indefinite linear-quadratic problem with linear endpoint constraints", Memorandum COSOR 88-15, Eindhoven University of Technology, 1988.

[15] A.H.W. Geerts, "All optimal controls for the singular linear-quadratic problem with stability; related algebraic and geometric results", EUT Report 88-WSK-04, Eindhoven University of Technology, 1988.

[16] Ton Geerts, "The algebraic Riccati equation and singular optimal control", Memorandum COSOR 88-31, Eindhoven University of Technology, 1988.

[17] Ton Geerts, PhD Thesis, in preparation.