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Product forms as a solution base for queueing systems

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Product forms as a solution base for queueing systems

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Abstract: A class of queueing networks has a product-form solution. It is interesting to investigate which queueing systems have solutions in the form of linear combinations of product forms. In this paper it is investigated when the equilibrium distribution of one or two-dimensional Markovian queueing systems can be written as linear combination of products of powers. Also some cases with extra supplementary variables are investigated.

1. Introduction

Many queueing problems may be modeled as random walks on multi-dimensional grids. Although, in principle, it is possible to study the transient behavior of such models, one usually concentrates on the analysis of the equilibrium behavior. Under certain ergodicity conditions, the equilibrium distribution is the unique normalized solution of the equilibrium equations. These equations may be viewed as partial difference equations. In the theory of partial differential equations, which are the continuous analogue of partial difference equations, a classical solution approach is the method of separation of variables. This method tries to solve partial differential equations by a linear combination of product forms. It seems natural to investigate whether it is also possible to solve equilibrium equations by a linear combination of product forms, and if so, under which conditions such solutions are feasible.

It is well-known that for a class of queueing systems the equilibrium equations can be solved by a product-form solution, see e.g. the paper of Baskett et al. [7]. Traditionally, this product-form solution is found by a sensible guess or by using balance arguments. It will be shown that this solution may also be found by directly trying to solve the equations by a product of powers. It then appears that the boundary conditions are crucial for the existence of a product-form solution. To look for solutions in the form of linear combinations of products seems to be a natural continuation of this research. In the last few years it has been found that there are several queueing problems which can be solved by a linear combination of product-form solutions. In some cases (like queueing systems of the type $E_k \,|\, E_r \,|\, c$) finitely many terms are needed, but in other cases (like the shortest queue problem) infinitely many terms are necessary. The approach used to find such solutions consists of first characterizing the product forms satisfying the conditions in the inner region and then using the products in this set to build up a linear combination that also satisfies the conditions at the boundaries of the state space. The present paper gives an overview of the results in this direction.

Queueing problems of the type $E_k \,|\, E_r \,|\, c$ can be described as a random walk on a multi-dimensional grid which is unbounded in one direction only. It turns out that the set of product forms satisfying the inner conditions is finite. However, this set is sufficiently rich in the sense that it is possible to construct a linear combination of products in this set, which also satisfies the boundary conditions. For queueing systems which can be described on a two-dimensional grid which is unbounded in both directions, it turns out that the set of products satisfying the inner conditions is infinite. It contains a countable subset, a linear combination of which also satisfies the boundary conditions. The construction of this linear combination is of a compensation type: after introducing the first term, new terms are subsequently added to compensate for the error of the previous term on one of the two boundaries. The conditions under which this linear combination provides a solution (i.e. it converges) can be formulated in terms of the random behavior in the interior points. For higher dimensional random walks on grids which are unbounded in more than two directions, the conditions under which this approach works, appear to be severe (as is shown in the paper by Van Houtum [12] in this volume).

This paper is organized as follows. Section 2 investigates product form networks as the solution of a set of partial difference equations with boundary conditions. Section 3 treats the queueing system $E_k \,|\, E_r \,|\, c$ and section 4 gives the general theory for two-dimensional systems. Section 5 gives some further results and conclusions.

2. Product form networks

The results in this paper may be viewed as an attempt to investigate under which conditions a linear combination of product forms provides a solution. The method used to construct such a linear combination essentially consists of first finding the product forms satisfying the inner conditions and then, by confronting these solutions with the boundary conditions, building a linear combination that also satisfies the boundary conditions. The most trivial outcome of this method is that there is a product form satisfying the inner, as well as the boundary conditions. In fact, it is well-known that for a class of queueing networks the equilibrium probabilities have a product form (see [7]). In this section we investigate the solution of some simple networks by using the approach mentioned above.

Let us consider an open queueing network with stations $1, \cdots, N$. In station i new jobs arrive according to a Poisson stream with rate λ_i and the service times are exponentially distributed with parameter μ_i, $i = 1, \cdots, N$. A job departing from station i is routed to station j with probability p_{ij} and leaves the network with probability p_{i0}, $i, j = 1, \cdots, N$. This network can be described by a continuous time Markov process with states $\underline{n} = (n_1, \cdots, n_N)$ where n_i denotes the number of jobs in station i, $i = 1, \cdots, N$. Let $\{p(\underline{n})\}$ be the equilibrium distribution. The equilibrium equations state that:

$$p(\underline{n})\left[\sum_{i=1}^{N} \mu_i \varepsilon_i(\underline{n}) + \sum_{i=1}^{N} \lambda_i\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} p(\underline{n}+\underline{e}_i-\underline{e}_j)\mu_i \varepsilon_j(\underline{n}) p_{ij} \tag{1}$$

$$+ \sum_{i=1}^{N} p(\underline{n}-\underline{e}_i)\varepsilon_i(\underline{n})\lambda_i + \sum_{i=1}^{N} p(\underline{n}+\underline{e}_i)\mu_i p_{i0} \qquad (\underline{n} \geq 0),$$

where $\varepsilon_i(\underline{n}) = 1$ if $n_i > 0$ and 0 otherwise, and $\underline{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with the 1 on place i. We try to solve these equations by a product $p(\underline{n}) = \alpha_1^{n_1} \cdots \alpha_N^{n_N}$. Insertion of this product in (1) and then dividing the equation by common factors yields

$$\sum_{i=1}^{N} \mu_i \varepsilon_i(\underline{n}) + \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\alpha_i}{\alpha_j}\mu_i \varepsilon_j(\underline{n}) p_{ij} + \sum_{i=1}^{N} \varepsilon_i(\underline{n})\frac{\lambda_i}{\alpha_i} + \sum_{i=1}^{N} \alpha_i \mu_i p_{i0} \qquad (\underline{n} \geq 0). \tag{2}$$

It appears that this set of equations is highly dependent. For $\underline{n} = \underline{0}$ equation (2) reduces to

$$\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \alpha_i \mu_i p_{i0} \tag{3}$$

and for the boundary state $\underline{n} = k\underline{e}_j$ with $k > 0$ we get

$$\mu_j + \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \frac{\alpha_i}{\alpha_j} \mu_i p_{ij} + \frac{\lambda_j}{\alpha_j} + \sum_{i=1}^{N} \alpha_i \mu_i p_{i0} \qquad (j = 1, \cdots, N) . \tag{4}$$

It is easily verified that each equation in the set (2) can be expressed as a linear combination of the equations (3)-(4). Subtraction of (3) from (4) and then multiplying the equation with α_j gives

$$\alpha_j \mu_j = \sum_{i=1}^{N} \alpha_i \mu_i p_{ij} + \lambda_j \qquad (j = 1, \cdots, N) . \tag{5}$$

The sum of the equations (5) yields equation (3). Hence the set of equations (2) can be produced by linear combinations of the equations (5). Under mild conditions the equations (5) have a unique solution (remark that $\alpha_j \mu_j$ can be interpreted as the throughput of station j).

For open Markovian networks we have found that the set of equations resulting from substitution of a product of powers in the equilibrium equations is spanned up by a small set of basic equations, which are related to a subset of the boundary conditions. The solution of the basic equations is straightforward. Closed Markovian networks appear to have the same property. Hence the boundary conditions seem to be crucial for the existence of product-form solutions. It would be interesting to investigate which boundary conditions imply a product-form solution. To illustrate such investigation, we consider the example of two independent $M \mid M \mid 1$ queues, where we modify the boundary behavior by assuming that server i works with rate γ_i whenever the other one is idle, $i = 1, 2$. For $\gamma_1 = \mu_1$ and $\gamma_2 = \mu_2$ we have seen that this problem can be solved by a product form. The case $\gamma_1 = \gamma_2 = \mu_1 + \mu_2$ corresponds to the situation where the idle server helps the busy one. This model is referred to as the coupled processor model, see e.g. Fayolle and Iasnogorodski [10] and Konheim et al. [14]. It is well-known that this model has no product-form solution, but in fact, constitutes a hard problem. So, clearly, perturbation of boundary conditions is subtle. To investigate for which values of γ_i the problem can be solved by a product $\alpha^m \beta^n$, we substitute this product into the equilibrium equations, yielding that

$$\lambda_1 + \lambda_2 = \alpha\gamma_1 + \beta\gamma_2 \qquad (n_1 = n_2 = 0) ; \tag{6}$$

$$\gamma_1 + \lambda_1 + \lambda_2 = \alpha\gamma_1 + \beta\mu_2 + \lambda_1/\alpha \qquad (n_2 = 0,\ n_1 > 0) ; \tag{7}$$

$$\gamma_2 + \lambda_1 + \lambda_2 = \beta\gamma_2 + \alpha\mu_1 + \lambda_2/\beta \qquad (n_1 = 0,\ n_2 > 0) ; \tag{8}$$

$$\mu_1 + \mu_2 + \lambda_1 + \lambda_2 = \alpha\mu_1 + \beta\mu_2 + \lambda_1/\alpha + \lambda_2/\beta \qquad (n_1,\ n_2 > 0) . \tag{9}$$

Equation (9) can be expressed as a linear combination of (6)-(8) iff $\gamma_1 + \gamma_2 = \mu_1 + \mu_2$. Subtraction of (6) from (7) and (8) gives

$$\alpha\gamma_1 = \alpha\beta(\mu_2 - \gamma_2) + \lambda_1 ; \tag{10}$$

$$\beta\gamma_2 = \alpha\beta(\mu_1 - \gamma_1) + \lambda_2 . \tag{11}$$

Equation (6) is the sum of (10) and (11). Hence, if $\gamma_1 + \gamma_2 = \mu_1 + \mu_2$, then the equations (6)-(9) can be expressed as a linear combination of the basic equations (10) and (11). The solution of (10)-(11) is straightforward. The finding that $\gamma_1 + \gamma_2 = \mu_1 + \mu_2$ implies a product-form solution has also been reported by Fayolle and Iasnogordski in [10].

It is well-known that the $M \mid M \mid c$ system has a geometric queue length distribution. In the next section we show that replacing the exponential distributions by sums of exponential distributions does not seriously affect the solution structure.

3. Queueing systems of the type $E_k \mid E_r \mid c$

The $E_k \mid E_r \mid c$ system is a typical example of a problem the analysis of which is more complex than the simple formulation suggests. The equilibrium equations become almost intractable if one attempts to solve them by using multi-dimensional generating function techniques, see e.g. Mayhugh and McCormick [15], Heffer [11] and Poyntz and Jackson [16]. It appears, however, that the $E_k \mid E_r \mid c$ system can be solved completely by the method which uses linear combinations of products. The analysis has been worked out in [6]. Below the solution method will be demonstrated for the $M \mid E_r \mid 1$ system. At the end of this section we summarize the results for the $E_k \mid E_r \mid c$ queue and comment on possible extensions.

Consider a single server system where jobs arrive according to a Poisson stream with rate λ. The service times are Erlang-r distributed with mean $r\mu^{-1}$. The state of this system can be described by the number of service stages in the system. Let p_n be the equilibrium probability for state $n = 0, 1, \cdots$. The equilibrium equations state that:

$$p_0 \lambda = p_1 \mu ; \tag{12}$$

$$p_n(\lambda + \mu) = p_{n+1} \mu \qquad (n = 1, \cdots, r-1) ; \tag{13}$$

$$p_n(\lambda + \mu) = p_{n-r} \lambda + p_{n+1} \mu \qquad (n \geq r) . \tag{14}$$

The equations (12)-(13) are the boundary conditions, the equations (14) form the inner conditions. We first try to find a sufficiently rich solution base of products α^n satisfying (14) and then use this base to construct a linear combination which also satisfies (12)-(13). Substituting $p_n = \alpha^n$ in (14) yields

$$\alpha^r (\lambda + \mu) = \lambda + \alpha^{r+1} \mu . \tag{15}$$

Only roots with $|\alpha| < 1$ are useful, since the sum of α^n over all n must be finite (necessary for normalization). Using Rouché's theorem it follows that equation (15) has r simple roots $\alpha_1, \cdots, \alpha_r$ inside the unit circle, provided $\lambda < r\mu^{-1}$, which, clearly, is an utilization condition. For each choice of c_i,

$$p_n = \sum_{i=1}^{r} c_i \alpha_i^n \qquad (n = 0, 1, \cdots) \tag{16}$$

satisfies (14). Substitution of (16) into (13) leads to $r-1$ homogeneous linear equations for c_i. These equations have a nonnull solution and can be solved explicitly by exploiting their VanderMonde-type structure. Due to the dependence of the equilibrium equations, equation (12) is automatically satisfied. Hence normalization of (16) produces the equilibrium distribution.

This method also works for the $E_k \mid E_r \mid c$ system. The waiting process for this system can be described by the vector $\underline{n} = (n_{-1}, n_0, n_1, \cdots, n_c)$ where n_{-1} is the number of remaining arrival stages,

n_0 the number of waiting jobs and n_i, $i = 1, \cdots, c$, the number of remaining service stages for server i. In [6] it has been shown that the equilibrium probability $p(\underline{n})$ can be expressed as

$$p(\underline{n}) = \sum_{j=1}^{r^c} c_j \alpha_{-1,j}^{n_{-1}} \alpha_{0,j}^{n_0} \cdots \alpha_{c,j}^{n_c} \qquad (n_i > 0\,,\, i = 1, \cdots, c) \tag{17}$$

for suitably chosen coefficients c_j and factors $\alpha_{-1,j}, \cdots, \alpha_{c,j}$. This result is also valid for nonidentical servers. In case of identical servers, the required number of terms in (17) decreases to $\binom{c+r-1}{r-1}$. To some extent, a similar approach is followed by Bertsimas [8] for the more general $C_k \,|\, C_r \,|\, c$ system. He proves that the probabilities for states with all $n_i > 0$, $i = 1, \cdots, c$, can be written as a linear combination of terms which are geometric in n_0. Expression (17) refines Bertsimas' result, namely (17) shows that the terms are geometric in $n_{-1}, n_1, \cdots, n_c$ as well.

The analysis can be extended to the model with feedback and multiple Erlang input streams. The method also works for some simple closed Erlang networks, and extensions to more general network structures may be feasible. So far, attempts to analyse open Erlang networks failed. These failures, however, are not quite understood yet.

In this section we studied the $E_k \,|\, E_r \,|\, c$ system which can be described as a random walk on a $c+1$-dimensional grid which is unbounded in the n_0-direction only. In the next section we concentrate on the analysis of random walk on a two-dimensional grid which is unbounded in both directions.

4. The compensation approach for two-dimensional random walks

Recently, it has been shown in the papers [1,5] and [9] that a countable linear combination of product forms provides a solution for different practical models, which can be described as a two-dimensional random walk. In each case the construction of the linear combination is of a compensation-type: after introducing the main term, products are added one by one so as to alternately compensate for the error on the two boundaries. In [3] the main conditions under which this approach works are investigated. The analysis will briefly be outlined below.

We consider a continuous time Markov process on the pairs (m, n) of nonnegative integers for which the transition rates are constant in the interior points and also on each of the two axes. Transitions are restricted to neighboring states. The transition-rate diagram is depicted in figure 1. Let $\{p_{m,n}\}$ be the equilibrium distribution, which we suppose to exist. This distribution is the unique normalized solution of the set of equilibrium equations. We attempt to construct a solution by combining products of the form $\alpha^m \beta^n$ satisfying the inner conditions. It is easily verified that $\alpha^m \beta^n$ satisfies the equilibrium equations in points (m, n) with $m, n > 1$ iff α and β are roots of the quadratic equation

$$\alpha\beta q = \alpha^2 q_{-1,1} + \alpha q_{0,1} + q_{1,1} + \beta q_{1,0} + \beta^2 q_{1,-1} + \alpha\beta^2 q_{0,-1} + \alpha^2\beta^2 q_{-1,-1} + \alpha^2 \beta q_{-1,0}\,. \tag{18}$$

We are going to use the products in this set to build a linear combination also satisfying the boundary conditions. Let us start with an arbitrary product $\alpha_0^m \beta_0^n$ with α_0, β_0 satisfying (18) and suppose that $\alpha_0^m \beta_0^n$ violates the equilibrium equations in the points $(0, n)$ and $(1, n)$ with $n > 1$. These equations form the vertical boundary conditions. To satisfy these conditions we try to find α, β, c_1 with α, β satisfying (18) such that the sum $\alpha_0^m \beta_0^n + c_1 \alpha^m \beta^n$ satisfies the vertical boundary conditions. By inserting this sum

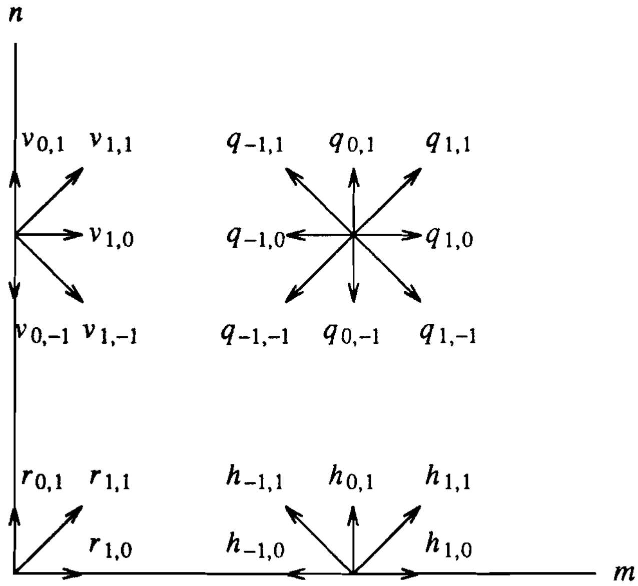

Figure 1: *Transition-rate diagram for a Markov process with constant rates and transitions restricted to neighboring states. $q_{i,j}$ is the transition rate from (m, n) to $(m+i, n+j)$ with m, $n > 0$ and a similar notation is used for the transition rates on each of the two axes.*

into the boundary conditions, it is immediately clear that we are forced to take $\beta = \beta_0$ and thus $\alpha = \alpha_1$ where α_1 is the other root of (18) with $\beta = \beta_0$. Then we can divide the conditions by the common factor β_0^{n-1} yielding two equations for c_1 which have, in general, no solution. Therefore we introduce an extra coefficient by considering $\alpha_0^m \beta_0^n + c_1 \alpha_1^m \beta_0^n$ for m, $n > 0$ and $e_0 \beta_0^n$ for $m = 0$, $n > 0$. Inserting this form into the vertical boundary conditions leads to two equations for c_1 and e_0 which can readily be solved. The addition of $c_1 \alpha_1^m \beta_0^n$ leads to a new error on the horizontal boundary, since this term violates the horizontal boundary conditions. To compensate for this error we add $d_1 c_1 \alpha_1^m \beta_1^n$ where β_1 is the other root of (18) with $\alpha = \alpha_1$. However $d_1 c_1 \alpha_1^m \beta_1^n$ violates the vertical boundary conditions, so we have to add again a term, and so on. Thus compensation of $\alpha_0^m \beta_0^n$ generates an infinite sequence of compensation terms. An analogous sequence is generated by starting the compensation of $\alpha_0^m \beta_0^n$ on the horizontal boundary. The resulting sum is depicted in figure 2.

$$\cdots + \underbrace{d_{-1} c_{-1} \alpha_{-1}^m \beta_{-1}^n + \overbrace{d_{-1} c_0 \alpha_0^m \beta_{-1}^n}^{H}}_{V} \overbrace{ + \underbrace{d_0 c_0 \alpha_0^m \beta_0^n + d_0 c_1 \alpha_1^m \beta_0^n}_{V}}^{} + d_1 c_1 \alpha_1^m \beta_1^n + \cdots$$

Figure 2: *The final sum of compensation terms. By definition $c_0 = d_0 = 1$. Sums of two terms with the same β-factor satisfy the vertical boundary conditions (V) and sums of two terms with the same α-factor satisfy the horizontal boundary conditions (H).*

Let $x_{m,n}(\alpha_0, \beta_0)$ be the infinite sum of compensation terms. Set

$$x_{m,n}(\alpha_0, \beta_0) = \sum_{i=-\infty}^{\infty} d_i (c_i \alpha_i^m + c_{i+1} \alpha_{i+1}^m) \beta_i^n \qquad (m,\ n > 0) .$$

The compensation on the boundaries requires to introduce new coefficients in the sums on the axes, so

$$x_{0,n}(\alpha_0, \beta_0) = \sum_{i=-\infty}^{\infty} d_i e_i \beta_i^n \qquad (n > 0); \qquad x_{m,0}(\alpha_0, \beta_0) = \sum_{i=-\infty}^{\infty} c_i f_i \alpha_i^m \qquad (m > 0).$$

The α_i and β_i are generated recursively from (18) and the coefficients c_i, d_i, e_i and f_i are generated recursively from the boundary conditions. The infinite sum $x_{m,n}(\alpha_0, \beta_0)$ is a formal solution of the equilibrium equations. The question is for what values of α_0, β_0 this sum converges. For convergence of $x_{m,n}(\alpha_0, \beta_0)$ for fixed m, n we require that α_i and β_i converge to zero as i tends to infinity. It can be shown that the condition

$$q_{0,1} = q_{1,1} = q_{1,0} \qquad (19)$$

is necessary and sufficient for convergence to zero of α_i and β_i. For normalization of $x_{m,n}(\alpha_0, \beta_0)$ we require that $|\alpha_i|$, $|\beta_i| < 1$ for all i. This requirement forces us to start the generation of compensation terms with a product $\alpha_0^m \beta_0^n$ satisfying the horizontal or vertical boundary conditions. Such pairs of α_0, β_0 are called feasible. The random behavior on the boundaries implies that there are at most four feasible pairs. Now our main result states that, if condition (19) holds, then for all points (m, n), except for a (possibly empty) subset of points near the origin,

$$p_{m,n} = \sum_{(\alpha_0, \beta_0)} x_{m,n}(\alpha_0, \beta_0),$$

where (α_0, β_0) runs through the set of at most four feasible pairs.

In this section we sketched the analysis of random walks on the first quadrant by using the compensation approach. It appears that the essential condition for the approach is constituted by the requirement that there may be no transitions to the North, North-East and East in the inner points (see (19)). In the final section we comment on possible extensions.

5. Conclusions and extensions

It has been shown that the concept of constructing solutions in the form of linear combinations of product forms is very useful for several queueing problems. This paper has presented recent results in this area. Clearly more research remains to be done here.

With respect to possible extensions, there are several interesting directions. One direction consists of replacing the E_k, E_r in the $E_k | E_r | c$ queue by more general distributions such as mixtures of Erlang distributions. Another interesting direction is to generalize the compensation approach to random walks on more general forms of the state space (see the analysis of the asymmetric shortest queue problem in [2] and an interesting variant of this problem in [13] and [4]) or with more complex random behavior. The extension to higher dimensional random walks is investigated in the paper of Van Houtum [12].

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List of COSOR-memoranda - 1992

Number	Month	Author	Title
92-01	January	F.W. Steutel	On the addition of log-convex functions and sequences
92-02	January	P. v.d. Laan	Selection constants for Uniform populations
92-03	February	E.E.M. v. Berkum H.N. Linssen D.A. Overdijk	Data reduction in statistical inference
92-04	February	H.J.C. Huijberts H. Nijmeijer	Strong dynamic input-output decoupling: from linearity to nonlinearity
92-05	March	S.J.L. v. Eijndhoven J.M. Soethoudt	Introduction to a behavioral approach of continuous-time systems
92-06	April	P.J. Zwietering E.H.L. Aarts J. Wessels	The minimal number of layers of a perceptron that sorts
92-07	April	F.P.A. Coolen	Maximum Imprecision Related to Intervals of Measures and Bayesian Inference with Conjugate Imprecise Prior Densities
92-08	May	I.J.B.F. Adan J. Wessels W.H.M. Zijm	A Note on "The effect of varying routing probability in two parallel queues with dynamic routing under a threshold-type scheduling"
92-09	May	I.J.B.F. Adan G.J.J.A.N. v. Houtum J. v.d. Wal	Upper and lower bounds for the waiting time in the symmetric shortest queue system
92-10	May	P. v.d. Laan	Subset Selection: Robustness and Imprecise Selection
92-11	May	R.J.M. Vaessens E.H.L. Aarts J.K. Lenstra	A Local Search Template (Extended Abstract)
92-12	May	F.P.A. Coolen	Elicitation of Expert Knowledge and Assessment of Imprecise Prior Densities for Lifetime Distributions
92-13	May	M.A. Peters A.A. Stoorvogel	Mixed H_2/H_∞ Control in a Stochastic Framework

Number	Month	Author	Title
92-14	June	P.J. Zwietering E.H.L. Aarts J. Wessels	The construction of minimal multi-layered perceptrons: a case study for sorting
92-15	June	P. van der Laan	Experiments: Design, Parametric and Nonparametric Analysis, and Selection
92-16	June	J.J.A.M. Brands F.W. Steutel R.J.G. Wilms	On the number of maxima in a discrete sample
92-17	June	S.J.L. v. Eijndhoven J.M. Soethoudt	Introduction to a behavioral approach of continuous-time systems part II
92-18	June	J.A. Hoogeveen H. Oosterhout S.L. van der Velde	New lower and upper bounds for scheduling around a small common due date
92-19	June	F.P.A. Coolen	On Bernoulli Experiments with Imprecise Prior Probabilities
92-20	June	J.A. Hoogeveen S.L. van de Velde	Minimizing Total Inventory Cost on a Single Machine in Just-in-Time Manufacturing
92-21	June	J.A. Hoogeveen S.L. van de Velde	Polynomial-time algorithms for single-machine bicriteria scheduling
92-22	June	P. van der Laan	The best variety or an almost best one? A comparison of subset selection procedures
92-23	June	T.J.A. Storcken P.H.M. Ruys	Extensions of choice behaviour
92-24	July	L.C.G.J.M. Habets	Characteristic Sets in Commutative Algebra: an overview
92-25	July	P.J. Zwietering E.H.L. Aarts J. Wessels	Exact Classification With Two-Layered Perceptrons
92-26	July	M.W.P. Savelsbergh	Preprocessing and Probing Techniques for Mixed Integer Programming Problems

Number	Month	Author	Title
92-27	July	I.J.B.F. Adan W.A. van de Waarsenburg J. Wessels	Analysing $E_k\|E_r\|c$ Queues
92-28	July	O.J. Boxma G.J. van Houtum	The compensation approach applied to a 2×2 switch
92-29	July	E.H.L. Aarts P.J.M. van Laarhoven J.K. Lenstra N.L.J. Ulder	Job Shop Scheduling by Local Search
92-30	August	G.A.P. Kindervater M.W.P. Savelsbergh	Local Search in Physical Distribution Management
92-31	August	M. Makowski M.W.P. Savelsbergh	MP-DIT Mathematical Program data Interchange Tool
92-32	August	J.A. Hoogeveen S.L. van de Velde B. Veltman	Complexity of scheduling multiprocessor tasks with prespecified processor allocations
92-33	August	O.J. Boxma J.A.C. Resing	Tandem queues with deterministic service times
92-34	September	J.H.J. Einmahl	A Bahadur-Kiefer theorem beyond the largest observation
92-35	September	F.P.A. Coolen	On non-informativeness in a classical Bayesian inference problem
92-36	September	M.A. Peters	A Mixed H_2/H_∞ Function for a Discrete Time System
92-37	September	I.J.B.F. Adan J. Wessels	Product forms as a solution base for queueing systems