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Improved Structured Least Squares for the Application of Unitary ESPRIT to Cross Arrays

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Abstract—A key problem in high-resolution multidimensional parameter estimation via unitary ESPRIT is to jointly solve a set of invariance equations by means of least-squares minimization. It has been shown previously that existing least-squares techniques fail when applied to the category of cross arrays, which consist of perpendicular uniform linear arrays crossing at the center of the array. Cross array geometries are of special interest because they provide a larger aperture and, hence, better resolution for a given number of array elements than other multidimensional uniform array geometries. This letter proposes an improved structured least-squares method that enables successful application of unitary ESPRIT to cross arrays. Results of simulated direction-of-arrival estimation experiments using a three-dimensional cross array indicate that considerable performance improvements can be achieved if the new method is used.

Index Terms—Array processing, cross arrays, structured least squares, superresolution, unitary ESPRIT.

I. INTRODUCTION

UNITARY ESPRIT is an efficient and popular technique for multidimensional harmonic retrieval with superresolution, an area that includes the problem of high-resolution direction-of-arrival (DOA) estimation from the outputs of a multidimensional array of antennas. A key step in this technique is solving an overdetermined set of equations, which are referred to as invariance equations, by means of least-squares minimization methods. It was pointed out in [1] that the most straightforward of these methods, namely, the least-squares (LS) and total least-squares (TLS) algorithms, are not optimal because they do not take into account the full structure of the invariance equations. The author of [1] therefore proposed the use of a multidimensional structured least-squares (SLS) method, which exploits the inherent relationships between the entries on both sides of the invariance equations. The SLS method has been shown to provide better performance than LS and TLS. Like LS and TLS, however, the SLS method breaks down if there is no unique solution to one or more of the invariance equations. This causes rank deficiency in the estimated signal subspace and

occurs in array configurations where two or more wavefronts with different DOAs have the same projection on one of the two subarrays corresponding to any one of the invariance equations. This can be the case for cross arrays or other geometries for which each invariance equation is associated with a linear array, e.g., uniform rectangular frame arrays (URFAs) or L-shaped geometries. Cross arrays are a specifically important category of multidimensional array geometries because they provide a larger maximum aperture and, hence, better resolution for a given number of array elements than any other array geometry with the same antenna spacing and number of dimensions [3].

The problem of rank deficiency in the estimated signal subspace was first pointed out in [2]. Here, it was shown that the problem can be solved by solving a set of nonlinear equations; however, this requires the use of additional and more complex numerical techniques.

Although the SLS method preserves the structure in each of the invariance equations individually, it does not exploit the fact that the solutions of the invariance equations must share the same set of eigenvectors. This letter proposes a modification to the SLS method that takes into account this additional constraint. In the improved SLS method, the invariance equations are solved jointly while forcing their solutions to span the same subspace. This enhancement resolves the rank deficiency problem mentioned above in a cost-effective manner without the need of solving additional nonlinear equations and therefore enables unitary ESPRIT to be applied to cross arrays.

II. METHOD

Consider the problem of estimating R parameters for each of the d waves incident on a centro-symmetric array consisting of M elements. In R-dimensional unitary ESPRIT, these parameters are estimated from the eigenvalues of $\mathbf{\Upsilon}^{(r)} \in \mathbb{R}^{d \times d}$, $r = 1, \ldots, R$. These matrices are given by the solutions to the real-valued invariance equations

$$\mathbf{K}_1^{(r)} \hat{\mathbf{E}}_s \mathbf{\Upsilon}^{(r)} \approx \mathbf{K}_2^{(r)} \hat{\mathbf{E}}_s, \quad r = 1, \ldots, R \qquad (1)$$

in which $\mathbf{K}_1^{(r)} \in \mathbb{R}^{m_r \times M}$ and $\mathbf{K}_2^{(r)} \in \mathbb{R}^{m_r \times M}$, where $m_r < M$ are known, real-valued matrices obtained from a transformation of the $m_r \times M$ selection matrices that assign the array elements to R pairs of overlapping subarrays [4]. The columns of the real-valued matrix $\hat{\mathbf{E}}_s \in \mathbb{R}^{M \times d}$ are the eigenvectors of the estimated signal subspace.

The SLS method is a popular technique for obtaining an approximate solution to (1). Its improvement over the LS and TLS methods is based on the explicit acknowledgment that $\hat{\mathbf{E}}_s$ is an

imperfect approximation of the true signal subspace and that an improved estimate can be obtained as

$$\mathbf{E}_{\mathrm{s}} = \hat{\mathbf{E}}_{\mathrm{s}} + \boldsymbol{\epsilon} \tag{2}$$

where $\boldsymbol{\epsilon}$ is an error matrix whose Frobenius norm is generally small compared to that of $\mathbf{E}_{\mathrm{s}}$. The method therefore proceeds by jointly minimizing the Frobenius norms of the residual matrices

$$\mathbf{R}^{(r)}\left(\mathbf{E}_s, \boldsymbol{\Upsilon}^{(r)}\right) = \mathbf{K}_1^{(r)}\mathbf{E}_s\boldsymbol{\Upsilon}^{(r)} - \mathbf{K}_2^{(r)}\mathbf{E}_s, \quad r = 1, \cdots, R \tag{3}$$

and the Frobenius norm of $\boldsymbol{\epsilon}$.

If the subarrays are linear, as is the case for cross arrays, it is possible that two or more wavefronts with different DOAs give rise to the same response on one pair of subarrays. As a result, $\mathbf{K}_1^{(r)}\mathbf{E}_s$ and $\mathbf{K}_2^{(r)}\mathbf{E}_s$ will be rank-deficient for that pair of subarrays, and the corresponding invariance equations individually will not have d unique solutions. Consequentially, the LS, TLS, and SLS methods will provide incorrect results.

The SLS method can be improved by exploiting the observation that the matrices $\boldsymbol{\Upsilon}^{(r)}$, $r = 1, \ldots, R$, share the same set of eigenvectors if $\hat{\mathbf{E}}_{\mathrm{s}}$ is a perfect estimate of the true signal subspace [5]. As a consequence, the matrices defined by

$$\mathbf{F}^{(r,r')}\left(\boldsymbol{\Upsilon}^{(r)}, \boldsymbol{\Upsilon}^{(r')}\right) = \boldsymbol{\Upsilon}^{(r)}\boldsymbol{\Upsilon}^{(r')} - \boldsymbol{\Upsilon}^{(r')}\boldsymbol{\Upsilon}^{(r)}$$
$$r = 1, \cdots, R, \ \ r' > r \tag{4}$$

must be null matrices [2].

Taking into account this additional constraint in solving the invariance equations solves the rank-deficiency problem and leads to more accurate estimates, as will be discussed in the next section.

Following an approach similar to that in [1], the improved SLS method proposed herein computes an approximate joint solution to the invariance equations in an iterative procedure, which simultaneously minimizes the Frobenius norms of the R matrices defined by (3), the $R(R-1)/2$ matrices defined by (4), and $\boldsymbol{\epsilon}$. Appropriate weighting factors are used in order to control the expected magnitudes of the entries of the different matrices, as will be discussed in the example below. The keys to the minimization procedure are the vectorization and linearization of (3) and (4), which result in

$$\mathrm{vec}\left\{\mathbf{R}^{(r)}\left(\mathbf{E}_{s,k+1}, \boldsymbol{\Upsilon}_{k+1}^{(r)}\right)\right\}$$
$$\approx \mathrm{vec}\left\{\mathbf{R}^{(r)}\left(\mathbf{E}_{s,k}, \boldsymbol{\Upsilon}_k^{(r)}\right)\right\} + \left[\mathbf{I}_d \otimes \left(\mathbf{K}_1^{(r)}\mathbf{E}_{s,k}\right)\right]$$
$$\times \mathrm{vec}\left\{\Delta\boldsymbol{\Upsilon}_k^{(r)}\right\} + \left[\left(\boldsymbol{\Upsilon}_k^{(r)}\right)^T \otimes \mathbf{K}_1^{(r)} - \mathbf{I}_d \otimes \mathbf{K}_2^{(r)}\right]$$
$$\times \mathrm{vec}\{\Delta\mathbf{E}_{s,k}\} \tag{5}$$

$$\mathrm{vec}\left\{\mathbf{F}^{(r,r')}\left(\boldsymbol{\Upsilon}_{k+1}^{(r)}, \boldsymbol{\Upsilon}_{k+1}^{(r')}\right)\right\}$$
$$\approx \mathrm{vec}\left\{\mathbf{F}^{(r,r')}\left(\boldsymbol{\Upsilon}_k^{(r)}, \boldsymbol{\Upsilon}_k^{(r')}\right)\right\}$$
$$+ \left[\boldsymbol{\Upsilon}_k^{(r')T} \otimes \mathbf{I}_d - \mathbf{I}_d \otimes \boldsymbol{\Upsilon}_k^{(r')}\right] \mathrm{vec}\left\{\Delta\boldsymbol{\Upsilon}_k^{(r)}\right\}$$
$$+ \left[\mathbf{I}_d \otimes \boldsymbol{\Upsilon}_k^{(r)} - \left(\boldsymbol{\Upsilon}_k^{(r)}\right)^T \otimes \mathbf{I}_d\right] \mathrm{vec}\left\{\Delta\boldsymbol{\Upsilon}_k^{(r')}\right\} \tag{6}$$

where $\mathbf{E}_{\mathrm{s},k+1} = \mathbf{E}_{\mathrm{s},k} + \Delta\mathbf{E}_{\mathrm{s},k}$ and $\boldsymbol{\Upsilon}_{k+1}^{(r)} = \boldsymbol{\Upsilon}_k^{(r)} + \Delta\boldsymbol{\Upsilon}_k^{(r)}$ are approximations of $\mathbf{E}_{\mathrm{s}}$ and $\boldsymbol{\Upsilon}^{(r)}$, respectively, at the $(k+1)$th iteration step, $\otimes$ denotes the Kronecker product, and $\mathrm{vec}\{\mathbf{R}\}$ denotes a vector-valued function that maps an $m_r \times d$ matrix $\mathbf{R}$ into an $m_r d$-dimensional column vector. Further, the vectorized signal subspace error matrix at step $k+1$ is equal to

$$\mathrm{vec}\{\boldsymbol{\epsilon}_{k+1}\} = \mathrm{vec}\{\mathbf{E}_{\mathrm{s},k} - \hat{\mathbf{E}}_{\mathrm{s}}\} + \mathrm{vec}\{\Delta\mathbf{E}_{\mathrm{s},k}\}. \tag{7}$$

At each step of the iterative procedure, $\mathrm{vec}\{\Delta\mathbf{E}_{\mathrm{s},k}\}$ and $\mathrm{vec}\{\Delta\boldsymbol{\Upsilon}_k^{(r)}\}$, $r = 1, \ldots, R$ are obtained by setting the left-hand sides of (5)–(7) to zero and computing the LS solution to the resulting overdetermined set of linear equations.

For example, if $R = 2$, the updates at iteration k can be computed by solving

$$\begin{bmatrix} \mathrm{vec}\left\{\mathbf{R}^{(1)}\left(\mathbf{E}_{\mathrm{s},k}, \boldsymbol{\Upsilon}_k^{(1)}\right)\right\} \\ \mathrm{vec}\left\{\mathbf{R}^{(2)}\left(\mathbf{E}_{\mathrm{s},k}, \boldsymbol{\Upsilon}_k^{(2)}\right)\right\} \\ \mathrm{vec}\left\{\mathbf{F}^{(1,2)}\left(\boldsymbol{\Upsilon}_k^{(1)}, \boldsymbol{\Upsilon}_k^{(2)}\right)\right\} \\ \kappa \cdot \mathrm{vec}\left\{\mathbf{E}_{\mathrm{s},k} - \hat{\mathbf{E}}_{\mathrm{s}}\right\} \end{bmatrix} + \mathbf{Z} \cdot \begin{bmatrix} \mathrm{vec}\left\{\Delta\boldsymbol{\Upsilon}_k^{(1)}\right\} \\ \mathrm{vec}\left\{\Delta\boldsymbol{\Upsilon}_k^{(2)}\right\} \\ \mathrm{vec}\left\{\Delta\mathbf{E}_{\mathrm{s},k}\right\} \end{bmatrix} = \mathbf{0}. \tag{8}$$

in which we get (9), shown at the bottom of the page. Here, the factor

$$\kappa = \sqrt{\frac{\sum\limits_{r=1}^{R} m_r + \frac{R(R-1)d}{2}}{\alpha M}} \tag{10}$$

provides a normalization that makes the minimization of $\boldsymbol{\epsilon}$ independent of the dimensions of the other matrices. Further, setting $\alpha > 1$ allows the entries of $\boldsymbol{\epsilon}$ to have larger magnitudes, on average, than the elements of the other matrices and therefore the change in signal subspace $\Delta\mathbf{E}_{\mathrm{s},k}$ is kept small. In practice, the performance of unitary ESPRIT is not very sensitive to the exact value of α, and good results are obtained using $\alpha = 10$ [1].

The iterative procedure can be initialized by setting $\mathbf{E}_{\mathrm{s},0} = \hat{\mathbf{E}}_{\mathrm{s}}$ and equating $\boldsymbol{\Upsilon}_0^{(r)}$, $r = 1, \ldots, R$ to the LS solutions of (1). It was observed from computer simulations that both the

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I}_d \otimes \left(\mathbf{K}_1^{(1)}\mathbf{E}_{s,k}\right) & \mathbf{0} & \boldsymbol{\Upsilon}_k^{(1)T} \otimes \mathbf{K}_1^{(1)} - \mathbf{I}_d \otimes \mathbf{K}_2^{(1)} \\ \mathbf{0} & \mathbf{I}_d \otimes \left(\mathbf{K}_1^{(2)}\mathbf{E}_{s,k}\right) & \boldsymbol{\Upsilon}_k^{(2)T} \otimes \mathbf{K}_1^{(2)} - \mathbf{I}_d \otimes \mathbf{K}_2^{(2)} \\ \boldsymbol{\Upsilon}_k^{(2)T} \otimes \mathbf{I}_d - \mathbf{I}_d \otimes \boldsymbol{\Upsilon}_k^{(2)} & \mathbf{I}_d \otimes \boldsymbol{\Upsilon}_k^{(1)} - \boldsymbol{\Upsilon}_k^{(1)T} \otimes \mathbf{I}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{Md} \end{bmatrix} \tag{9}$$
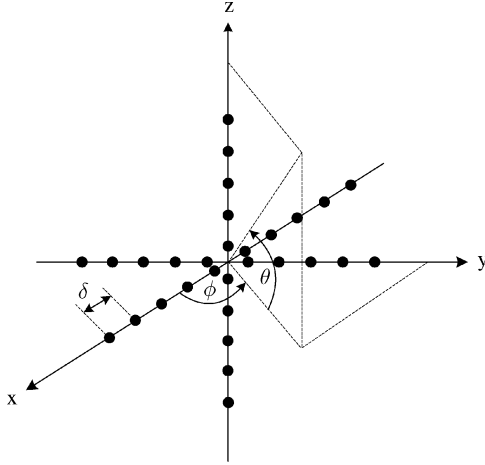
Fig. 1.   Sketch of three-axis crossed array.

Fig. 2.   RMS estimation error for source 1 and 2 as a function of SNR for two equipowered wavefronts impinging from ± 4 degrees azimuth and zero elevation.

multidimensional SLS method and the improved SLS method presented herein require several iterations to converge and that both methods fail to converge at all in some instances. This latter problem was circumvented by scaling the updated values vectors $\text{vec}\{\Delta\boldsymbol{\Upsilon}_k^{(r)}\}, r = 1, \ldots, R$, and $\text{vec}\{\Delta\mathbf{E}_{\text{s},k}\}$ by a constant factor $0 < \beta < 1$, chosen sufficiently small to keep the iterative procedure from "overshooting" the local optimum nearest to the initial solution.
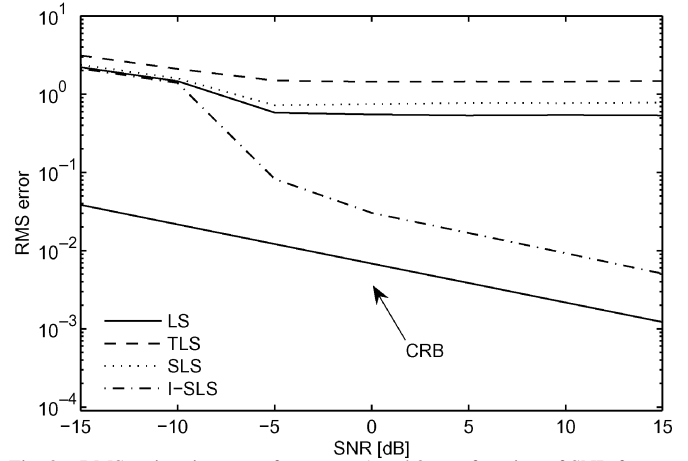
III. SIMULATION RESULTS

Simulations were performed in order to verify the performance of the modified SLS method in conjunction with multidimensional unitary ESPRIT. Consider the problem of estimating azimuth and elevation with the aid of the three-axis cross array of [2]. The array consists of $M = 3$ m identical elements located along three perpendicular linear arrays of m elements each, which are aligned with the x, y, and z-axes, as illustrated in Fig. 1. The inter-element spacings on the three subarrays are uniform and equal to δ. Incident on the array are d uncorrelated, narrowband planar wavefronts with azimuth angle $0 < \phi_i < 2\pi$, elevation angle $-\pi/2 < \theta_i < \pi/2$, $i = 1, \ldots, d$, and a common wavelength λ. The number of data snapshots is denoted by q. The real-valued data matrix $\mathbf{Z} \in \mathbb{R}^{M \times 2q}$ is obtained by individually transforming the $m \times q$ complex-valued data matrices corresponding to the signals at the elements of each of the linear arrays, according to [4, Eq. (7)] and stacking the results. The signal parameters to be estimated are uniquely related to the spatial frequencies $\mu_i^{(1)}$, $\mu_i^{(2)}$, and $\mu_i^{(3)}$, paired estimates of which can be determined from $\boldsymbol{\Upsilon}^{(1)}$, $\boldsymbol{\Upsilon}^{(2)}$, and $\boldsymbol{\Upsilon}^{(3)}$ with the aid of the simultaneous Schur decomposition [5]. The spatial frequencies to be estimated are related to ϕ_i and θ_i as

$$\mu_i^{(1)} = \frac{2\pi}{\lambda}\delta\cos\theta_i\cos\phi_i \tag{11}$$

$$\mu_i^{(2)} = \frac{2\pi}{\lambda}\delta\cos\theta_i\sin\phi_i \tag{12}$$

$$\mu_i^{(3)} = \frac{2\pi}{\lambda}\delta\sin\theta_i. \tag{13}$$

The root-mean-square (RMS) estimation error was computed from $K = 1000$ independent trials as

$$\text{RMSE}_i = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\sum_{r=1}^{3}\left|\hat{\mu}_{i,k}^{(r)} - \mu_i^{(r)}\right|^2}, \quad i = 1, \ldots, d \tag{14}$$

where $\hat{\mu}_{i,k}^{(r)}$, $r = 1,2,3$ are the estimated frequencies of the ith signal obtained in the kth trial.

Fig. 2 shows the RMS estimation error as a function of signal-to-noise ratio (SNR) for $d = 2$ equipowered waves impinging from $+4$ and -4 degrees azimuth resp., i.e., half the Rayleigh resolution limit, and zero elevation. The SNR was varied from -15 to 15 dB, the number of array elements was set to $M = 30$, with a separation of $\Delta = 0.45\lambda$, and the number of snapshots was $q = 100$. The Cramer–Rao bound (CRB) is plotted as a reference in Fig. 2, according to formulas presented in [6, Eq. (33)]. Since the two sources are located symmetrically compared to the array geometry, their performance is identical, and the result of only one source is presented here. The results of the LS, TLS, and SLS methods show a large constant RMS error that increases below a SNR value of -5 dB. This large error is caused by the inability of the algorithms to resolve the two sources, due to rank deficiencies occurring in the invariance equations associated with the x- and z-axes of the array, as discussed in the previous section. The improved SLS method presented herein, referred to as I-SLS in Fig. 2, leads to successful resolution, which results in a lower RMS error that follows the CRB for SNR values higher than -5 dB.

Fig. 3 shows the RMS estimation error as a function of SNR for $d = 2$ equipowered waves impinging from $+4$ and -4 degrees azimuth resp. and from 0 and 10 degrees elevation. It is interesting to observe that, although theoretically rank deficiency does not occur in this case, the new method still outperforms the other methods.

IV. CONCLUSION

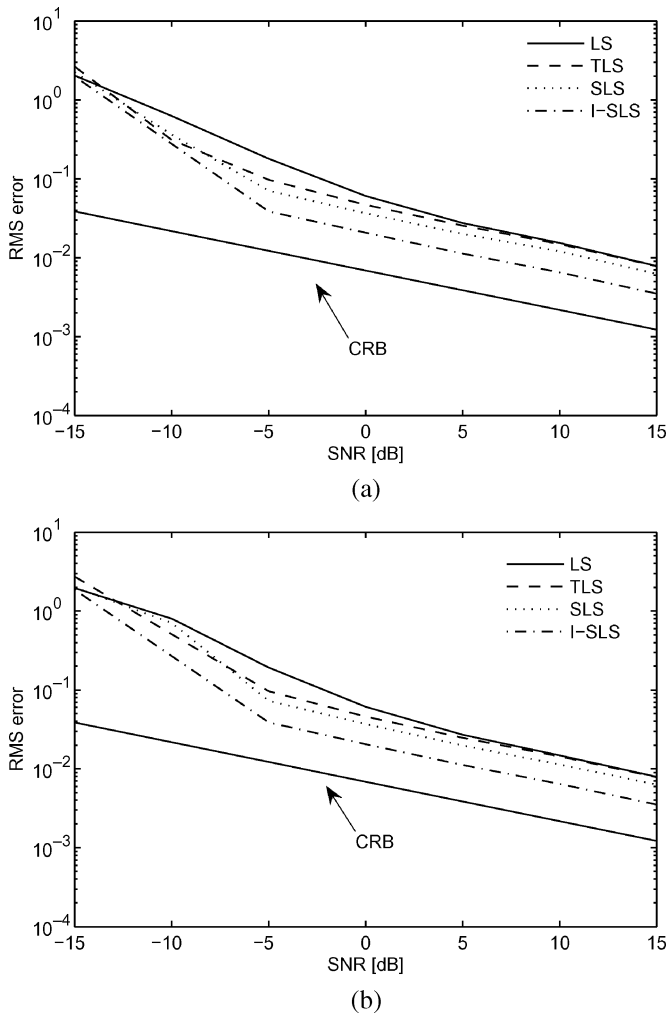An improved SLS method has been described that enables the unitary ESPRIT algorithm to be applied to the important

(a)

(b)

Fig. 3.   RMS estimation error for (a) source 1 and (b) source 2 as a function of SNR for two equipowered wavefronts impinging from ±4 degrees azimuth and, respectively, 0 and 10 degrees elevation.

category of cross arrays. The new method is based on an iterative minimization procedure similar to that of the SLS method but improves upon the SLS method by requiring the solutions of the invariance equations to share the same set of eigenvectors. From simulation results reported in this letter, it can be concluded that the improved SLS method will lead to successful resolution of multiple waves impinging on a three-axis cross array in cases where other methods fail, namely, when two or more waves with different DOAs give rise to the same response on one pair of subarrays.

Additionally, it can be concluded that the new method will result in significant performance improvement, even when there are no rank deficiencies as a result of the above-cited scenario.

Although the improved SLS technique described herein is derived for cross arrays with a common phase center, the same technique can also be applied to cross arrays in general when using standard ESPRIT (e.g., L-shaped geometries) or to any array geometry for which each invariance equation is associated with a linear array (e.g., URFAs).

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