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Instituut
Wiskundige Dienstverlening
Eindhoven

The Modeling of Polymer Flow Instabilities

Part II: Turbulence and Chaos

J. Molenaar

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J. MOLENAAR

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Contents

1. Introduction

The aim of this report is to provide the reader with a concise introduction into the recent literature on the use of concepts from chaos theory in the description of turbulence. The theory of turbulence is still far from being complete. As it is stated by Frisch and Orszag (1990, p. 32): *"Let us conclude by noting that less is known about the fine scale of turbulence – for example, the scale of 1 mm in the atmosphere – than about the structure of atomic nuclei"*.

The fascinating developments in chaos theory have clearly acted as a catalyzer for turbulence research. By Moon (1987, p. 3,7) this is expressed as follows: *"In the physical sciences, the paragon of chaotic phenomena is turbulence. Thus, a rising column of smoke or the eddies behind a boat or aircraft wing provide graphic examples of chaotic motion. The fluid mechanician, however, believes that the events are not random, because the governing equations can be written down. Also, at low velocities, the fluid patterns are quite regular and predictable. Beyond a critical velocity, however, the flow becomes turbulent. A great deal of the excitement in nonlinear dynamics today is centered around the hope that this transition from ordered to disordered flow may be explained or modeled with relatively simple mathematical equations. It is the recognition that chaotic dynamics are inherent in all of nonlinear physical phenomena that has created a sense of revolution in physics today (...). Perhaps the greatest hope lies in the possibility of understanding turbulence, which is one of the few remaining unsolved problems of classical physics"*.

This optimism, which usually accompanies break-throughs in science, has to be tempered also in this case. Turcotte (1988) remarks: *"Fractals are not going to 'solve' the problem of turbulence. Nevertheless, fractal concepts introduce new ways to treat data sets"*.

The report is organised as follows. In §2 we give a brief overview of classical concepts in turbulence theory. In §3 some aspects of chaos theory are dealt with. We focus on fractals and multifractals, because they play an essential role in recent turbulence theory. The application of multifractals in this context is presented in §4.

2. Turbulence

Here, we shall discuss the main issues of classical turbulence theory. "Classical" is here referring to the epoch before fractals entered the scene.

2a. Turbulence Models

In fluid mechanics the Navier Stokes (NS) equations are since long accepted as the fundamental equations of motion. See, e.g., Landau and Lifshitz (1959). They express the conservation of mass, momentum, and energy. The independent variables in these partial differential equations are time t and position x and the dependent variables are velocity $v(t, x)$, pressure $p(t, x)$, density $\rho(t, x)$ and entropy $S(t, x)$. The corpuscular character of matter is not taken into account; the NS equations govern macroscopic, i.e. averaged in space, quantities. The characteristics of the system under consideration come in by the so-called constitutive equation, which couples the stress tensor, representing the internal forces, to the strain tensor, representing the internal displacements. The model is completed by specifiying the initial conditions, i.e. the values of the dependent variables at one time, together with the boundary conditions, which contain information about the interaction of the fluid with its enclosing environment.

The NS equations greatly simplify in case of an incompressible, Newtonian flow. In an incompressible flow the density ρ has a constant value, say ρ_0. In a Newtonian flow the constitutive equation is linear. Despite of these restrictions, these flows exhibit turbulent behaviour. The NS equations describing this case are

$$\nabla \cdot v = 0$$

(2.1)
$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho_0} \nabla p + \nu \Delta v \ .$$

Here ν is the kinematic viscosity. External forces are ignored for the moment. In dimensionless form eq. (2.1b) reads:

(2.2)
$$S \frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p - \frac{1}{R} \Delta v \ .$$

S is the Strouhal number and R the Reynolds number. They are given by

$$S = L \, / \, VT$$

$$R = VL \, / \, \nu \ ,$$

with L, V, and T characteristic length, velocity, and time scales, respectively. For example, L may be the typical size of the equipment enclosing the fluid, and V the typical velocity of the main stream.

If R is not too large, the diffusive term Δv will be dominant. This term describes the influence of viscosity and tends to damp out velocity differences. Then, the flow is in the laminar

regime and, generally, accurate solutions can be obtained. The majority of these problems can be solved with the use of standard software packages. In some cases the presence of boundary layers may hamper a direct numerical approach, but a lot of energy has already been invested in the development of mathematical techniques appropriate to deal with these aspects in an analytical way.

With increasing values of R the nonlinearity of the system gains influence and is balanced by viscosity only in thin boundary layers. In the main stream instabilities occur, i.e. small variations in R may cause sudden changes of the global flow pattern. This phenomenon strongly resembles phase transitions observed in many other systems. This instable behaviour leads to unsurmountable difficulties in the numerical solution of the complete model. If R is increased more and more, the flow undergoes a series of transitions and, at some critical R value, becomes turbulent. With some phantasy one can recognize vortex-like structures in turbulent flow, which are referred to as *eddies*. Because the detailed, dynamical behaviour of a turbulent flow can by no means be calculated, already Osborne Reynolds (1842–1912) suggested to follow a statistical approach, in which the fast variations in time and space of relevant quantities, such as the velocity field, are ignored by averaging. The essential step is to split all dependent variables into a slowly and a fast varying part. If in equations (2.1) an averaging is performed over the latter parts, one obtains equations quite similar to eq. (2.1). The only difference is the presence of an extra (symmetric) tensor, the so-called Reynolds stress tensor. This tensor has dimensions 3×3 and contains 6 additional unknowns. So, to complete the model extra equations are required. The choice of these equations is known as the *closure problem of turbulence*.

There is a lot of freedom in selecting new equations, and a great variety of models have been studied. A review is given by Parchen et al. (1988). The so-called $k - \varepsilon$ model is most famous. It is relatively simple and does not need much computer time. However, its reliability is rather restricted. It contains a set of parameters, which has to be tuned at the flow under consideration.

The problems in this line of research are quite technical. No new concepts have been introduced since long, and no link with chaos theory can be appreciated.

2b. The Cascade Model

In §2a we argued that a solid mathematical model for turbulent flow in principle exists, but that at high R values the solution cannot be obtained. In terms of chaos theory we may state that the system is extremely sensitive to the initial conditions and any calculation of the solution at reasonably long time intervals is doomed to fail. Still, some aspects of turbulence have been surprisingly well captured by a qualitative approach, usually referred to as the *cascade model*. This model is proposed by Lewis Richardson (1922) and explored in more detail by Kolmogorov (1941). An extensive description is given by Tennekes and Lumley (1972). Concise introductions are given by Frisch and Orszag (1990) and Procaccia (1984). The quintessence of the cascade model is expressed by Richardson as follows:

> Big whirls have little whirls
> That feed on their velocity
> And little whirls have lesser whirls,
> And so on to viscosity.

The idea is that turbulent flow is in a dynamical equilibrium. At the largest scale, L say, i.e. the size of the apparatus, large eddies are generated, which extract their kinetic energy from the main flow. These eddies are not stable. While they decay, their energy is used to build up new eddies of smaller sizes, which on their turn have only finite life time. Eddies at large and intermediate sizes lose only an ignorable amount of their energy to direct viscous dissipation.
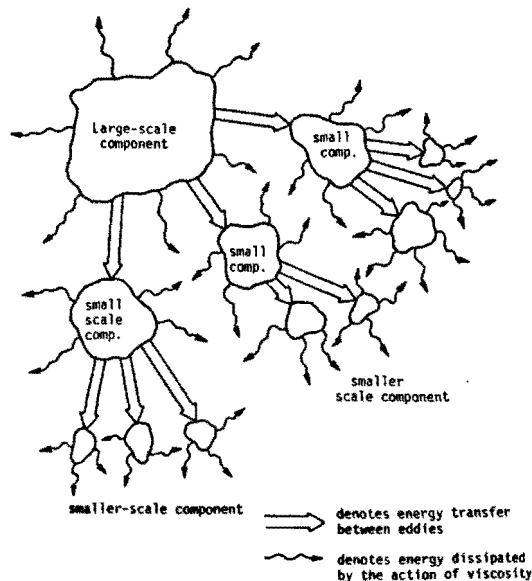

Fig. Distribution of energy between eddies. From Stanisic (1985).

It is assumed that at some smallest scale l_d all energy is dissipated, heating up the fluid a bit, and that below this scale the viscous effects are predominant. In the so-called *inertial range* between l_d and L a continuum of eddies exists. The estimation of the size distribution over this range is a quite subtle aspect of the model. We refer for this point to Tennekes and Lumley (1972). The estimation of the value of l_d is dealt with in §2c. A characteristic parameter of the model is the energy flux ε from large to small scales. This flux has the dimensions of energy per unit mass per unit time.

2c. Dimensional Analysis

Dimensional analysis has proven to be a powerful tool analyzing mathematical models. It should be applied before all other techniques, because it usually gives rise to considerable simplification of the problem. A clear introduction is given by Bender (1978). Kolmogorov has shown that dimensional analysis is especially fruitful if applied to the cascade model. Although the cascade model is based on rather vague, intuitive notions, it has led via dimensional analysis to remarkable scaling relations, which have been observed experimentally. The basic observation is that eqs. (2.1) are invariant under the scaling tranformation $(t, x) \rightarrow (\lambda t, \mu x)$ for arbitrary λ, μ, provided that viscosity is ignored, i.e. in the limit $R \rightarrow \infty$.

In the cascade model each eddy is characterized by its size l and a typical velocity v_l. One might interpret v_l as the characteristic, mean velocity of the fluid in the eddy, measured with

respect to the velocity of the main stream. The values of the energy flux ε and the eddy velocities v_l in the inertial range $l_d < l < L$ are typical for the system under consideration. E.g., the values of v_l in gases are much larger than those in fluids.

The smallest eddies with $l \approx l_d$ receive just as much energy from decaying larger eddies as they lose via dissipation. The parameters determining these smallest eddies are ε and the kinematic viscosity ν. From these parameters we may form the characteristic length l_d, time t_d, and velocity v_d:

$$(2.3) \qquad l_d = (\nu^3/\varepsilon)^{1/4} \ , \quad t_d = (\nu/\varepsilon)^{1/2} \ , \quad v_d = (\nu\varepsilon)^{1/4} \ .$$

We note, that the corresponding Reynolds number $l_d v_d/\nu$ equals unity.

The largest eddies with $l \approx L$ absorb the energy flux ε from the main flow. We denote their characteristic velocity by v_L. Their characteristic time scale is then given by $t_L = L/v_L$. These eddies supply their energy to smaller eddies and it is assumed that they do this at a rate reciprocally proportional to t_L. Dimensional analysis leads to

$$(2.4) \qquad \varepsilon \approx v_L^3/L \ .$$

Substituting (2.4) into (2.3) yields estimates for the ratio's of quantities at both ends of the inertial range:

$$l_d/L \approx (\nu/v_L L)^{3/4} = R^{-3/4}$$

$$(2.5) \qquad t_d/t_L \approx (\nu/v_L L)^{1/2} = R^{-1/2}$$

$$v_d/v_L \approx (v_L L/\nu)^{-1/4} = R^{-1/4} \ .$$

We observe that all characteristic scales of the smallest eddies are much smaller than those of the largest eddies. In particular, the width of the inertial range widens with increasing R values. Because $v_d \ll v_L$, we conclude that most of the kinetic energy is contained in the largest eddies. As for vorticity it is the other way around, because $t_d \ll t_L$. Vorticity is reciprocally proportional to the characteristic time scales of the eddies, so we find that the smallest eddies carry most of the vorticity.

An important feature of the cascade model is the distribution of energy over the inertial range of the eddies. The derivation of the scaling properties of this spectrum is quite subtle and given by Tennekes and Lumley. The eventual conclusion is that the kinetic energy E scales as

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

with the wave number k roughly equal to $2\pi/l$. This famous Kolmogorov-Obukhov law has experimentally been verified.

2d. Correlation Functions.

In experiments the velocity correlation function gets much attention, because velocity differences are relatively simple to measure and contain much information. This correlation function is defined by

$$(2d.1) \quad G_q(l) = < |u(\mathbf{x} + \mathbf{l}) - u(\mathbf{x})|^q > \ ,$$

with $l = |\mathbf{l}|$, u a component of the velocity (usually taken to be parallel to the main stream), and $< \ldots >$ denoting averaging over a region with fully developed turbulence. In practice this average in space is replaced by averaging in time. The experimental set up is then such that the velocity is measured at only one point in space but at many times. The *frozen-flow hypothesis* of Taylor probably justifies this replacement. It is based on the idea that turbulence patterns convect with the mean motion without distortion. Not all implications of this hypothesis are fully understood, but extensive literature on this topic exists. See for references, e.g., Sreenisavan (1991, p. 556). Van de Water et al. (1991) point out that the replacement has consequences for the interpretation of the data in terms of fractals. This point is also discussed in §4.

In §2c we applied dimensional analysis to the smallest and largest eddies. Here, we need the scaling properties at intermediate sizes. In the cascade model it is assumed that the scaling of the velocity v_l only depends on l and the energy flux ε. This implies that

$$(2d.2) \quad v_l \sim \varepsilon^{1/3} l^{1/3} \ .$$

The velocity difference in $G_q(l)$ will be mainly determined by eddies of size l, so this difference scales in the same manner:

$$(2d.3) \quad G_q(l) \sim l^{\zeta_q}$$

with scaling exponent $\zeta_q = q/3$. For small values of q ($q \ll 10$) this has indeed been measured. However, for larger q values substantial deviations in ζ_q are observed. This has motivated the search for modifications of the cascade model, which are discussed in §4.

6

3. Chaos Theory

Chaos theory deals with dynamic systems for which the solution exists if $t \to \infty$ and that do not approach a (quasi) periodic, a point, or any other smooth attractor in phase space. Dissipative, chaotic systems follow, for $t \to \infty$, trajectories which constitute a chaotic or strange attractor. The dimension of those attractors is smaller than the phase space dimension and generally fractal, i.e. non-integer. The (rare) chaotic attractors with integer dimension are called 'fatfractals'. The notion of fractality has fruitfully employed to modify the cascade model. These modifications are the subject of §4. Here, we shall present a concise introduction into fractality in §§3b,c. To provide a more complete view on the connections between chaos theory and fluid mechanics, we deal in §3a with the analogies between the behaviour of certain fluid flow experiments in the pre-turbulent stage and certain low-dimensional systems in the pre-chaotic phase.

3a. Period Doubling in Low Dimensional Systems

The logistic equation

$$x_{n+1} = 4\lambda x_n (1 - x_n)$$

is the most generic equation exhibiting chaos. This simple, scalar equation has extensively been studied by Feigenbaum. In his 1983 paper he gives an excellent overview of many aspects of the complex behaviour of this equation as a function of λ. Other quite informative references are Devaney (1986), and Guckenheimer and Holmes (1983). For $0 < \lambda < 3/4$ the system has one stable point attractor. At $\lambda = 3/4$ period doubling occurs for the first time: the solution is attracted to a stable 2-cycle, consisting of 2 points which are alternately visited. Increasing λ one finds that period doubling takes place at the bifurcation points λ_i, $i = 1, 2, \ldots$, where the system jumps from a 2^i-cycle mode to a 2^{i+1}-cycle mode. The ascending λ_i values geometrically converge, i.e. the ratio

$$\delta_i = \frac{\lambda_{i+1} - \lambda_i}{\lambda_{i+2} - \lambda_{i+1}}$$

converges to the constant

$$\delta = 4.6692016 .$$

Because δ is quite large, the λ_i-sequence rapidly converges to λ_∞, where the system enters the chaotic regime. For $\lambda > \lambda_\infty$ the system is chaotic except for some windows wherein period doubling cascades can be recognized again. This scenario is found for all iteration processes of the form

$$x_{n+1} = f(x_n) ,$$

where f has a maximum and is concave downward. Because the scenario is independent of the detailed form of f this *route to chaos* is called universal. For all f with a quadratic

7

maximum the value $\delta = 4.66\ldots$ is found. Other types of maxima yield other values of δ. Post (1991) has studied several modified logistic equations and paid special attention to the windows. The geometric convergence of the λ_i is not the only universal scaling property of these systems, but it is the one most easily detected in numerical experiments. Period doubling is not the only route to chaos, but alternative routes are not relevant here. We refer the reader to Eckmann (1981) and Gollub and Benson (1980) for more information on this point.

3b. Period Doubling in Fluid Flow

Fluid flow is, contrary to the low dimensional systems discussed in §3a, infinitely dimensional. But also in fluid flow experiments period doubling has been observed as a route to chaos, i.e., turbulence. Apparently, in these complicated systems the same mechanism is in force, but in a, until now, hidden fashion, because it is far from being transparent that something like a quadratic maximum is present.

By the way, we remark that in finite-dimensional systems one may find higher-dimensional maps, which also display transitions to higher-order periodic orbits. The corresponding bifurcations are related to the occurrence of *homoclinic chaos*.

A famous example of chaos in fluid flow is Taylor-Couette flow, in which fluid is enclosed by concentric cylinders. When the outer (or inner, or both, in opposite directions) cylinder rotates, the fluid may organize itself in a regular pattern. The speed of rotation acts here in the same way as the parameter λ in the logistic equation. At certain values of this speed the system jumps from one pattern to another. At increasing speeds the patterns get more and more detailed structure, and at a certain moment the flow loses its regular structure and becomes turbulent. See, e.g., Gollub and Swinney (1975) and Brandstäter et al. (1983).

Another well-known example is Rayleigh-Benard convection. In this system fluid is enclosed between horizontal plates, which differ in temperature. Here, the temperature difference is the steering parameter. In this experiment the regular flow patterns consist of a well-defined number of eddies. At the transitions the number of eddies doubles. See, e.g., Libchaber et al. (1983) and Jensen et al. (1985).

These examples concern closed-flow systems. Open flows behave quite differently in general. The differences are discussed by Sreenisavan (1991) and Brandstäter et al. (1983).

Why should closed-flow systems show such a strong analogy to simple systems like the logistic equation? We quote Feigenbaum (1983, page 38) on this point: "*The fluid equations make up a set of coupled field equations. They can be spatially Fourier-decomposed to an infinite set of coupled ordinary differential equations. Since a flow is viscous, there is some smallest spatial scale below which no significant excitation exists. Thus, the equations are effectively a finite coupled set of non-linear differential equations. The number of equations in the set is completely irrelevant. The universality theory is generic for such a dissipative system of equations. Thus it is possible that the flow exhibits period doubling. If it does, then our theory applies. However, to prove that a given flow (or any flow) actually should exhibit doubling is well beyond present understanding. All we can do is experiment.*"

3c. Fractals

The study of the detailed structure of some irregularly shaped physical objects like clouds and rocky coastlines (Mandelbrot, 1967) and of mathematical objects like Cantor sets and Koch curves (see, e.g., Turcotte, 1988) led to a reconsideration of the concept of dimension. Already in 1917 Hausdorff introduced a generalization of the classical notion of dimension. The famous book by Mandelbrot (1983) and the visualization of his ideas by Peitgen and Richter (1986) strongly pushed the common acceptance of the concept of fractal dimension. Here, we shall give a summary of the main aspects of the subject.

Let us start by remarking that many definitions of generalized dimension are in use. The one closest to intuition is called the *capacity*. The capacity d_{cap} of a bounded set $X \subset I\!\!R^n$ is given in terms of coverings with n-dimensional boxes with all edges of equal length. Let, for given $l > 0$, $N(l)$ be the minimal number of non-overlapping boxes of linear size l necessary to cover X. For $l \downarrow 0$ we have the scaling relation

$$(3c.1) \quad N(l) \sim l^{-d_{cap}}$$

where d_{cap} is defined as

$$(3c.2) \quad d_{cap} = -\lim_{l \downarrow 0} \frac{\ln N(l)}{\ln l} .$$

This definition is not very general, because the limit does not always exist. In the definition of the Hausdorff dimension d_H this flaw is circumvented. In that alternative X may also be unbounded, and the covering contains boxes of all sizes smaller than a given l. For each l the covering with the minimum number $N(l)$ of boxes is chosen. In such a minimal covering the i-th box $(1 \leq i \leq N(l))$ has size l_i. The Hausdorff dimension d_H is the unique number for which

$$\lim_{l \downarrow 0} \sum_{i=1}^{N(l)} l_i^{d_H}$$

takes on a value between 0 and ∞.

In most cases d_{cap} and d_H coincide and are then interpreted as the generalized dimension of X, which may attain non-integer values.

In other dimensions definition the contributions from different regions of X are not weighted equally. The weighting is introduced via a *measure* on X. Dynamical systems which organize themselves on an attractor in phase space induce a natural measure on that attractor. If the attractor, say X, is covered with boxes, the trajectory of the system on X will, for $t \to \infty$, not visit each box with the same frequency. The ratio's of these frequencies induce, after normalization, a measure or probability distribution on the covering and, in the limit $l \downarrow 0$, on X itself. Because one particular trajectory is used, the result will formally depend on the initial value of the trajectory. In most cases this dependence is not of influence. In the following we denote the measure of a box with index i by p_i. It gives the chance to find the system in box i if observed at an arbitrary time. If the measure is uniform over X and the boxes are of equal size, we have, of course,

$$p_i(l) \approx 1/N(l), \quad l \downarrow 0 .$$

A commonly used dimension definition based on a measure is the correlation dimension d_{cor} introduced by Grassberger and Procaccia (1983). We cover X with boxes of equal size l and choose that covering for which

$$(3c.5) \quad P(l) \equiv \sum_i p_i^2(l)$$

is minimal. The definition then reads

$$(3c.6) \quad d_{cor} = \lim_{l \downarrow 0} \frac{\ln P(l)}{\ln l} .$$

$P(l)$ may be interpreted as the chance that two arbitrary selected points of X have distance smaller than or equal to l. If the 2 in (3c.5) is replaced by an arbitrary $q \in I\!N$, this idea is generalized to sets consisting of q points. From (3c.6) we have the scaling relation

$$(3c.7) \quad P(l) \sim l^{d_{cor}} .$$

It is clear that regions with high probability mostly contribute to d_{cor}. For the uniform measure we, of course, have $d_{cor} = d_{cap}$.

Hentschel and Procaccia (1984) introduced a dimension definition, which comprises the definitions of d_{cap} and d_{cor}. Their definition generates an infinite number of dimensions D_q, $q \in I\!R$, $q \geq 0$, each of which provides its characteristic information about X. We define

$$(3c.8) \quad P_q(l) \equiv \sum_i p_i^q ,$$

where, just as above, the summation is taken over the covering with boxes of size l which minimizes the right hand side. Note, that $P_0(l) \equiv N(l)$ and $P_2(l) \equiv P(l)$. The D_q are defined by

$$(3c.9) \quad D_q = \lim_{l \downarrow 0} \frac{1}{(q-1)} \frac{\ln P_q(l)}{\ln l} .$$

It is clear that $D_0 \equiv d_{cap}$ and $D_2 \equiv d_{cor}$. For $q = 1$ we find from a careful limiting procedure that D_1 is equal to the so-called information dimension, which is defined by

$$(3c.10) \quad D_1 = \lim_{l \downarrow 0} \frac{S(l)}{\ln l} ,$$

where the entropy   is given by

$$(3c.11) \quad S(l) = \sum_i p_i \ln p_i .$$

We note, that the Hausdorff dimension, introduced already in 1917, does not coincide with one of the D_q.

We close this section with two remarks:

1. While d_{cap} and d_H are hard to handle in practice, the estimation of d_{cor} is quite convenient. The former two dimensions require box counting, i.e., the fractal object, which is in practice characterized by a finite number of points, is covered by boxes and the number of points in each box is counted as a function of box size. The box counting procedure for d_{cap} is easy to implement numerically. Its convergence, however, is very slow and the method requires in general very much data points. The box counting procedure is quite awkward in case of d_H, because it involves taking the minimum over coverings. In the procedure used to estimate d_{cor} (see Grassberger and Procaccia (1983a)) one calculates once and for all the distances between pairs. An efficient algorithm, which avoids unnecessary calculations, is given by Theiler (1987). $P_2(l)$ is then approximately given by the number of distances smaller than or equal to l divided by the total number of pairs. If enough points are available, a log-log plot of $P_2(l)$ as a function of l hopefully yields a straight line for some l-interval. Below this interval the procedure breaks down because of the finiteness of the sample and the noise in the data, while above this interval the limit $l \downarrow 0$ is not yet approximated well. The application of this procedure needs some care in special cases as pointed out by Grassberger and Procaccia (1983b).

2. Dimensions can sometimes be determined analytically if X has a self-similar structure. In those cases X is constructed through the application of a simple rule. Repeated application of such a rule yields a more and more refined structure. A well-known example is the Cantor set. If some part of a self-similar object is magnified a copy of X itself is obtained. Most fractal sets evoke the impression of being self-similar: upon magnifying of a part one obtains an object which resembles the whole object but is not an exact copy of it.

3d. Multifractals

In §3c the notion of dimension is introduced as a scaling property. In equations (3c1,7) d_{cap} and d_{cor} act as scaling exponents and these scaling relations are assumed to hold uniformly over X. Frisch and Parisi (1985), Halsey et al. (1986), and Jensen et al. (1985) proposed a generalization of this idea which appears to be very fruitful in turbulence theory. In this generalization different regions of X may scale differently. To make this explicit we introduce the concept of local scaling. Let $x \in X$ be arbitrarily chosen. We position a box of size l around x, such that x is an inner point of the box. The measure p of this box depends on x and l. We say that the measure on X scales with $\alpha(x)$ around x if for $l \downarrow 0$ it holds that

$$(3d.1) \quad p_x(l) \sim l^{\alpha(x)} .$$

The subset of X which contains all points scaling with the same α, has generally a fractal dimension (i.e. capacity), say $f(\alpha)$. The distribution of α over X is denoted by $F(\alpha, x)$. It

is the function which attains the value 1 at points x scaling with α and vanishes elsewhere. After normalization we have for all $x \in X$

$$(3d.2) \quad \int_X dp_x \int_{I\!\!R} d\alpha\, F(\alpha, x) = 1 \ .$$

Because X is now split up into fractal pieces, one refers to it as a *multifractal*. Frisch and Parisi coined this name. The function $F(\alpha)$ defined by

$$(3d.3) \quad F(\alpha) = \int_X F(\alpha, x)\, dp_x$$

could be interpreted as the volume fraction of the α-subset of X. The scaling properties of $F(\alpha)$ are then by definition given by

$$(3d.4) \quad F(\alpha) \sim l^{-f(\alpha)} \ , \quad l \downarrow 0 \ .$$

Mandelbrot (1984, 1989b) and Cates and Witten (1987) emphasize that the interpretation of $f(\alpha)$ as a scaling exponent is more essential than its interpretation as the dimension of the iso-α subset of X. For details we refer to these papers.

In practice, X is known through a finite number of points. The scaling of $F(\alpha)$ then involves the inclusion of logarithmic terms as pointed out by Van de Water and Schram (1988).

The characterizations of X via $f(\alpha)$, $\alpha \in I\!\!R$ and via D_q, $q \in I\!\!R^+$ contain the same information. This can be shown as follows. The scaling behaviour of $P_q(l)$ in (3c.8) for $l \downarrow 0$ is given by the scaling behaviours of the p_i in the different boxes. For $l \downarrow 0$ we may replace the summation in (3c.8) over the separate boxes by an integration over α, meanwhile taking together the boxes which scale with the same α. So,

$$(3d.5) \quad P_q(l) = \sum_i p_i^q \sim \int_{I\!\!R} d\alpha\, F(\alpha)\, l^{\alpha q} \sim \int_{I\!\!R} d\alpha\, l^{\alpha q - f(\alpha)} \ .$$

It is common to assume that $f(\alpha)$ is concave downward. In Fig. 2 we give a sketch of $f(\alpha)$. The nomenclature used is from Mandelbrot (1989b). Negative values of $f(\alpha)$ and even of α are hard to interpret, but still not excluded. Only one or two of the manifest, latent, and virtual parts can form the entire $f(\alpha)$ curve. The latent and virtual points, if they exist, contain information about rare events, which are hard to quantify experimentally. These events hardly contribute to $P_q(l)$ for small q values.
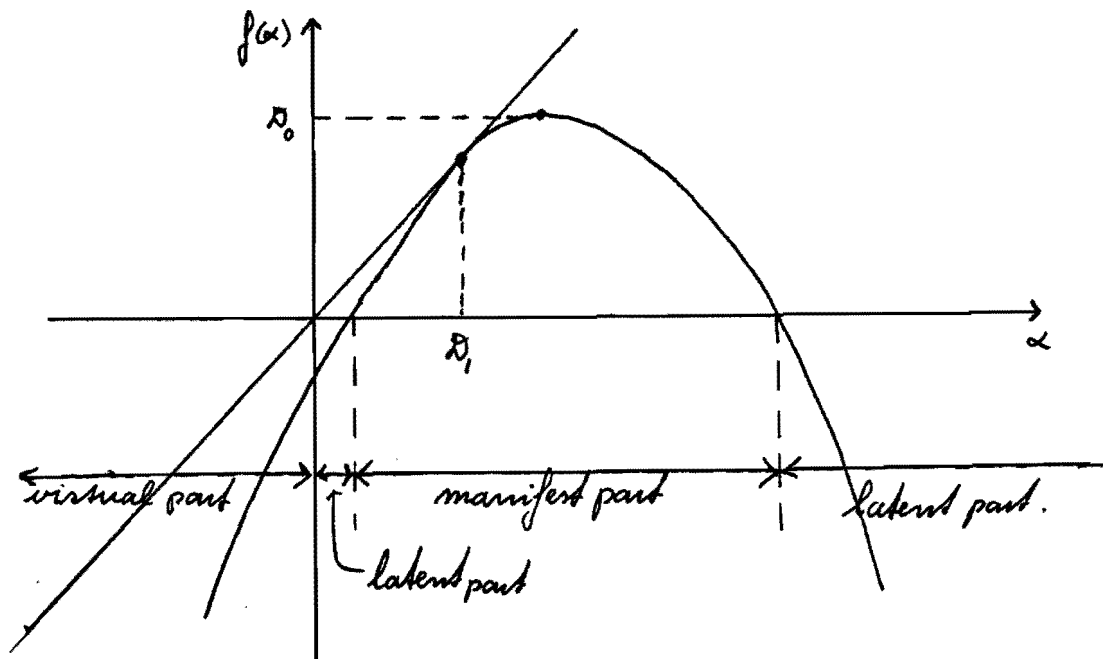
Fig. 2. General form of $f(\alpha)$

If $f(\alpha)$ has the shape sketched in Fig. 2, the function $\alpha q - f(\alpha)$ will, for fixed value of q, have a minimum, say at $\alpha = \alpha(q)$. In the Appendix we show that the integrand in (3d.5) only contributes around $\alpha(q)$ if $l \downarrow 0$. There we also show in (A.3) that $f(\alpha)$ and D_q are related by

$$(3d.6) \qquad D_q = \frac{1}{q-1}\left(q\alpha(q) - f(\alpha(q))\right).$$

So, D_q can be determined from $f(\alpha)$. The reverse also holds, because we have

$$(3d.7) \qquad \alpha(q) = \frac{d}{dq}\left[(q-1)\,D_q\right].$$

Here, we use that $\alpha q - f(\alpha)$ has a minimum at $\alpha(q)$, so that

$$(3d.8) \qquad q = f'(\alpha(q)).$$

Once $\alpha(q)$ is known, $f(\alpha)$ is found from inverting (3d.6). As stated above, the determination of D_q for all q is not simple, so that this way to calculate $f(\alpha)$ from D_q is not recommended. More direct methods to obtain $f(\alpha)$ are proposed by Chhabra et al. (1989a, 1989b, 1990a,b). From (3d.6) we understand that the functions $f(\alpha)$ and $(q-1)\,D_q$ are each other's Legendre transform.

Some special cases are of interest. Substituting $q = 0$ yields directly $D_0 = f(\alpha(0))$. Because $\alpha(0)$ is just the position of the maximum of $f(\alpha)$, we find that D_0 is equal to the maximum value of $f(\alpha)$. For $q = 1$ we have to consider

$$(3d.9) \qquad D_1 = \lim_{q \to 1} \frac{1}{q-1}\left(q\alpha(q) - f(\alpha(q))\right).$$

13

This limit yields a finite value only if the numerator $(\alpha(1) - f(\alpha(1)))$ vanishes. A careful expansion around $q = 1$ and application of (3d.8) yields $D_1 = \alpha(1)$. At $\alpha = \alpha(1)$ the $f(\alpha)$ curve touches the line $f = \alpha$. The limiting cases $q \to \pm\infty$ are also easily interpretable from geometry. We may immediately conclude from the concavity of $f(\alpha)$ that $\alpha(+\infty)$ is the lower bound and $\alpha(-\infty)$ the upper bound of the domain of $f(\alpha)$. If negative $f(\alpha)$ and α values are accepted, one might have that $\alpha(+\infty)$ or $\alpha(-\infty)$ are infinite. The corresponding D_q values are $D(\pm\infty) = \alpha(\pm\infty)$. In Jensen et al. (1985) $f(\alpha)$ curves are given, which are deduced from forced Rayleigh-Benard experiments.

4. Multifractals and Turbulence

Accurate measurements by Anselmet et al. (1984) revealed that the scaling exponent ζ_q of the correlation function G_q defined in (2d.1,3) deviates from the "cascade model" value $q/3$ for large q. Frisch and Parisi (1985) proposed to extend the theory by an adjustment of the scaling relation (2d.2). They aimed to include also the phenomenon of intermittency. The underlying considerations are extensively discussed by Meneveau and Sreenivasan (1991). The idea is to introduce the multifractal concept dealt with in §3d. The 3-dimensional flow in assumed to consist of fractal subsets. The scaling of the velocity v_l is made to depend on the subset under consideration. Instead of relation (cf. (2d.2))

$$(4.1) \qquad v_l \sim l^{1/3} \ ,$$

where the scaling exponent is taken to be uniformly distributed, one uses an α dependent scaling

$$(4.2) \qquad v_l \sim l^{(\alpha-2)/3} \ ,$$

where α may vary over the 3-dimensional flow. The parameter α is introduced such, that substitution of $\alpha = 3$ yields (4.1). The interpretation is that $\alpha < 3$ corresponds to singularities of the flow that are stronger than average. Frisch and Parisi originally used a slightly different notation. The iso-α sets are assumed to have fractal dimension $f(\alpha)$. The distribution $F(\alpha)$ defined in (3d.3) scales in this case as

$$(4.3) \qquad F(\alpha) \sim l^{3-f(\alpha)} \ .$$

$F(\alpha)$ is given by the fraction of an l-covering of the flow, that contains elements of the iso-α set. The total number of boxes scales, of course, with l^{-3}.

From assumptions (4.2) and (4.3) we find

$$
\begin{aligned}
G_q(l) \quad &= \ < |u(\mathbf{x}+\mathbf{l}) - u(\mathbf{x})|^q > \\
(4.4) \qquad &\sim \int F(\alpha) \, v_l^q \, d\alpha \\
&\sim \int l^{3-f(\alpha)+q(\alpha-2)/3} \, d\alpha
\end{aligned}
$$

We are interested in the behaviour of this integral for $l \downarrow 0$ and the derivations in the Appendix are directly applicable. We find that

$$(4.5) \qquad G_q(l) \sim l^{\zeta_q}$$

with

$$(4.6) \qquad \zeta_q = 3 - \tfrac{2}{3}q - f(\alpha(\tfrac{1}{3}q)) + \tfrac{1}{3}q\,\alpha(\tfrac{1}{3}q) \ ,$$

15

where $\alpha(\frac{1}{3}q)$ is that value of α which minimizes $\frac{1}{3}q\alpha - f(\alpha)$ as a function of α. Using expression (3d.6) for D_q we may relate ζ_q and $D_{q/3}$:

$$\zeta_q = 3 - \tfrac{2}{3}q + (\tfrac{1}{3}q - 1)D_{q/3} \ .$$

The dependence of ζ_q on q is quite diffuse now and in general non-linear. Measurement of the exponent ζ_q yields information about the multifractal scaling exponents $f(\alpha)$ of turbulent flow. As pointed out by Frisch and Vergassola (1991) the introduction of the multifractal concept makes a conceptual shift necessary. In stead of thinking in terms of decaying eddies – which are, indeed, not really observed – one should imagine the cascade process in terms of stretching and folding of smaller and smaller fluid fractions. The inertial range will not be the same for the different α-components of the flow, and the Kolmogorov inner scale is no longer . a fixed under bound. Van de Water et al. (1991) discuss several implications concerning the interpretation of experimental data. All measurements of correlation functions are based on the frozen flow hypothesis, i.e. spatial averaging at one moment is replaced by time averaging at one position. The common experimental set-up is to measure one or more components of fluid velocity using a point probe. The turbulent flow passes the probe thanks to the velocity of the main flow. So, the turbulent flow is scanned along a line. The intersection of this line and the iso-α set is non-empty only if, for the manifest part of $f(\alpha)$, $\alpha > 2$. This condition appears to be not always satisfied in the data. A possible explanation is that $f(\alpha)$ might possess a latent part where $f < 0$, but the corresponding events are such rare, that they are not yet detectable. Van de Water et al. (1991) announce experiments in which about 10^9 successive velocity measurement can be used at the same time for the time averaging. This might make possible the determination of G_q for q values up to $q \approx 15$. At the same time statistical aspects can be investigated more definitely.

5. Concluding Remarks

In §3d we introduced the concept of multifractality. It can quite generally be used as a language to describe objects with complicated scaling behaviour. A first application is in the field of the dynamics of finite dimensional, dissipative systems. The corresponding evolutions in the (virtual) phase space may contract on a so-called strange (or chaotic) attractor, which usually shows multifractal scaling behaviour. A second application is given in §4. Turbulent fluid flow in (real) $3 - d$ space exhibits scaling behaviour, which can also excellently be described in terms of multifractals. However, the fact that one and the same mathematical language can be used in these different fields does not imply that there is a direct relation between chaos theory and turbulence theory. Classical turbulence theory had a lot of phenomenological features; the results of recent research on non-linear dynamical systems has provided it with new mathematical tools.

At the moment many questions are still open. It is by no means possible to establish that multifractal scaling is also appropriate to describe rare events. The physics of small scales might be more complex than is assumed in the multifractal approach. To some extent it is clear that this approach is useful to explain measured deviations from the cascade model. However, it is not really understood yet what physical mechanisms are in force. It has to be noted that in this description one has lost track of the dynamical evolution of turbulence. But let us stress the positive aspects by quoting Sreenisavan (1991): '*Will the fractal approach survive and flourish? It is trite to say that fractals by themselves cannot solve the turbulence problem – whatever that may mean. To the extent that these are mere tools, the future depends on how intelligently and judiciously they are employed. But these tools have two advantages. First, they enable us to venture beyond the existing statistical tools, which are rather heavily based on central-limit-type arguments. Second, they have allowed us to enter a number of areas of non-linear science – thought to be beyond scientific description only a few years ago – making it easier to produce connections with those other fields. And this can only be beneficial in the long run.*'

Appendix

In multifractal theory (see §3d and, e.g., Halsey et al. (1986)) one often meet with integrals of the type

$$I(q,l) \equiv \int_{\mathbb{R}} l^{\alpha q - f(\alpha)} \, d\alpha \;,$$

which is studied for given q and $l \downarrow 0$. The function $f(\alpha) \in C^{\infty}$ is assumed to be concave downward. This implies that the exponent has a minimum, say at $\alpha = \alpha(q)$. For short we write
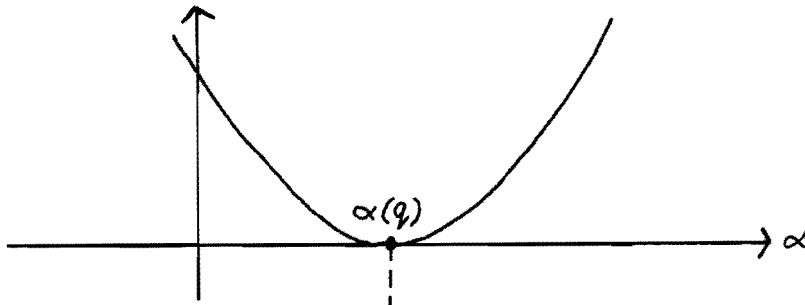
$$g(\alpha) \equiv \alpha q - f(\alpha) \;.$$

The minimum value of $g(\alpha)$ is denoted by
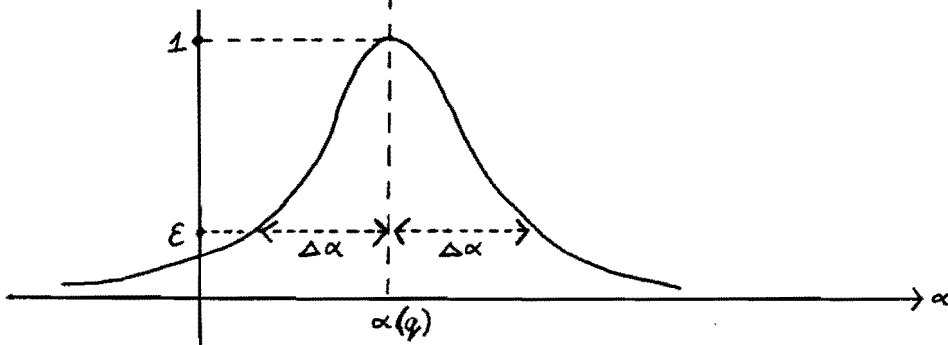
$$g_0 \equiv g(\alpha(q)) \;.$$

The sign of g_0 depends, of course, on q and f. We rewrite I in the form

$$I = l^{g_0} \int_{\mathbb{R}} l^{g(\alpha) - g_0} \, d\alpha \;.$$

The function $g(\alpha) - g_0$ has the form

The integrand has accordingly the form

The peak of the integrand, which has the fixed value one, becomes more and more narrow if $l \downarrow 0$. To approximate the area under the peak we introduce an effective peak width $\Delta\alpha(l)$ defined by

(A.1) $l^{g(\alpha(q)\pm\Delta(\alpha))-g_0} = \varepsilon$,

where ε is fixed and $\ll 1$. For small l we have that $g(\alpha) - g_0$ behaves quadratically around $\alpha(q)$:

$$g(\alpha) - g_0 \approx c(\alpha - \alpha(q))^2 , \quad c > 0 .$$

If we substitute this into condition (A.1) we find

$$\Delta\alpha(l) = \left(\frac{\ln \varepsilon}{c \ln l} \right)^{1/2} .$$

Because the peak has fixed height, the area under the peak scales as $\Delta\alpha(l)$. From this we conclude that

$$\frac{\ln I(q,l)}{\ln l} = g_0 + \frac{1}{\ln l} \ln \left(\frac{\ln \varepsilon}{c \ln l} \right)^{1/2} .$$

Because ε and c are constants and

$$\lim_{l\downarrow 0} \left(\frac{\ln \ln l}{\ln l} \right) = 0 ,$$

we have

(A.2) $\displaystyle \lim_{l\downarrow 0} \frac{\ln I(q,l)}{\ln l} = \alpha(q)q - f(\alpha(q))$.

This limit thus yields the value of the exponent $g(\alpha)$ at its minimum $\alpha(q)$.

The definition of D_q is given in (3c.9). If we substitute (3d.5) into (3c.9), we meet with the integral $I(q,l)$ dealt with in this Appendix. Application of the present result (A.2) yields a relation between $f(\alpha)$ and D_q:

(A.3) $\displaystyle D_q = \frac{1}{q-1} \left(q\alpha(q) - f(\alpha(q)) \right)$.

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