Distribution planning for a divergent 2-echelon network without intermediate stocks under service restrictions

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Abstract

In this paper we discuss a distribution planning procedure for a system consisting of one central depot supplying a number of end stockpoints. The central depot is not allowed to hold stock and allocates all incoming goods immediately to these end stockpoints. An ordering and allocation policy is presented which is based on a decomposition method. The emphasis lies on the realization of pre-determined target service levels in the various stockpoints. In this paper we present two adjustment methods which improve the service performance considerably in certain cases. Another important contribution of this paper is the generalization of the concept of imbalance. An analytical approximation of the probability of imbalance is presented. An extensive simulation study validates the analytical results.

Keywords: imbalance, inventory, multi-echelon, service, simulation

1. Introduction

In this paper we consider a divergent 2-echelon inventory model that operates according to a periodic review policy without batch size restrictions. This model applies to a distribution network consisting of a central depot which supplies a number of end stockpoints. No intermediate stocks are held at the central depot, i.e. only stocks are held at the most downstream stockpoints of the network at which customer demand is satisfied. Every order that arrives at the depot is immediately allocated to the end stockpoints. The depot serves as a pure distribution centre. The model also applies to situations where according to a hierarchical product structure two successive decisions are made in time concerning planned production for an aggregate production volume (family or group of products) and for individual products. The planning of the production for individual products obeys the aggregate production volume constraint defined at the preceding level. We restrict ourselves to the application of the model to a distribution environment. For hierarchical production planning application we refer to De Kok(1990).

The emphasis in this paper is on the determination of the echelon order-up-to-level at the central depot for a known review period in order to ensure that the demand satisfied from stock on hand at the end stockpoints equals a pre-determined target level. This approach differs from most approaches reported in the literature. Usually one defines a cost structure and the aim is to find cost-optimal policies. See for example Clark and Scarf(1960), Eppen and Schrage(1981), Zipkin(1984), Federgruen and Zipkin(1984), Rosling(1989), Langenhoff and Zijm(1990) ,Van Houtum and Zijm(1991) and Svoronos and Zipkin(1991). In other cases one assumes that the stockout probability, i.e. the probability of negative stock immediately before order arrival of a replenishment order, is equal for all end stockpoints (cf. Eppen and Schrage(1981)).

It should be noted that in practice neither of the approaches is applicable. The cost-optimal policies cannot be used since in most cases penalty costs for shortages are unknown. The equal stockout probability assumption is not valid, since in most practical cases one tends to differentiate service levels and more important, one uses the service criterion mentioned above, i.e. the fraction of demand satisfied from stock on hand (cf. Tijms and Groenevelt(1984), Silver and Peterson(1985), De Kok(1990) and Lagodimos(1992)).

The contribution of this paper to the literature is the following. First of all the concept of imbalance is generalized. Instead of assuming equal stockout probabilities we define imbalance as the occurrence of negative allocation quantities after application of a straightforward allocation policy. Through this generalization we can in principle determine echelon policies that satisfy target service levels under any service criterion. Furthermore we derive analytical approximations for the probability of a negative allocation to a particular stockpoint as an indication for the probability of imbalance. The approach given in this paper is based on the approach used in De Kok(1990). In this paper we also present improvement methods to correct deficiencies found in applying the logic proposed by De Kok(1990).

The paper is organized as follows. In section 2 we introduce some important assumptions and definitions. In section 3 we analyze the echelon policy for the 2-echelon model as described above. This analysis relies heavily on De Kok(1990). Two improvement methods are presented in section 4 to correct deficiencies. In section 5 attention is given to the important phenomenon of imbalance. The generalization of this concept is one of the major findings. Finally in section 6 we present some conclusions and recommendations for further research. Throughout the paper simulation results are presented to validate the analytical results.

2. Assumptions and definitions

Throughout this paper we make use of a number of general assumptions. Similar assumptions have been made in most previous work in the area.

1) External demand is imposed at end stockpoints (i.e. stockpoints without successor).
2) Demand at end stockpoints are independent stochastic variables, uncorrelated in time, with known mean (μ) and standard deviation (σ).
3) All demand not satisfied from stock on hand is backlogged.
4) Lead times are constant.
5) There are no fixed order quantities nor capacity constraints.

We now introduce some important definitions. The **net inventory** of an end stockpoint is defined as the physical stock at this stockpoint minus backorders. The **echelon inventory position** of an end stockpoint is defined as the net inventory of this stockpoint plus all stock that has been allocated to this stockpoint but has not yet arrived. The echelon inventory position of the central depot is defined as the sum of the inventory positions of all end stockpoints plus the physical stock at the depot plus outstanding orders that have not yet arrived at the depot.

In this paper the central depot is not allowed to hold stock. Every order that arrives at the depot is immediately allocated to end stockpoints according to some allocation policy. There is however one case where the central depot is allowed to hold stock. That is when the central depot delivers directly to an external customer and therefore assumption (1) no longer holds. The stock at the depot is then exclusively reserved for this particular customer and may not be used for replenishment of end stockpoints. This situation can be modelled as an extra end stockpoint (representing the external customer) with zero lead time.

The **service criterion** considered in this paper is the fraction of demand satisfied directly from stock on hand. This definition of service is considered to be the most widely used in practice (cf. Tijms and Groenevelt(1984), Silver and Peterson(1985), Lagodimos(1993)).

The lead times are assumed to be constant. However, it can be shown that stochastic lead times for the end stockpoints can be implemented easily. A stochastic lead time for the central depot on the other hand complicates the analysis considerably.

3. System dynamics

In this section we analyze the echelon policy for the 2-echelon model as reported by De Kok (1990). We apply the policy for a distribution environment where De Kok(1990) uses a hierarchical production planning structure. The echelon policy is derived by a combination of exact reasoning, approximation schemes and empirical findings.

3.1 The model

The network we consider is shown in figure 3.1. It consists of a central depot (CD) supplying N individual stockpoints where the external demand is realized. The central depot operates a periodic review (R,S)-ordering policy. Every shipment that arrives at the CD is immediately allocated and distributed to the end stockpoints, which may have different lead times. Management will try to realize specified service levels for every stockpoint. We refer to these desired service levels as **target levels**. These target levels may also differ for the various stockpoints. To describe the network operation we will use the following notation:

L : lead time for CD

L_i : lead time for stockpoint i

D_{it} : demand in stockpoint i during period [t-1,t)

μ_i : mean period demand in stockpoint i

σ_i : standard deviation in period demand in stockpoint i

$\hat{\beta}_i$: target level for stockpoint i

Z_t : echelon inventory position of CD just before an order is issued by the CD at time t

$Z_{t,i}$: echelon inventory position of stockpoint i just before the allocation decision at CD is taken at time t

$$D_0 := \sum_{i=1}^{N}\sum_{t=1}^{L} D_{it} \qquad : \textit{aggregate demand during } [0,L)$$

$$D_k^{(1)} := \sum_{t=L+1}^{L+L_k} D_{kt} \qquad : \textit{demand in stockpoint } k \textit{ during } [L,L+L_k)$$

$$D_k^{(2)} := \sum_{t=L+1}^{L+L_k+R} D_{kt} \qquad : \textit{demand in stockpoint } k \textit{ during } [L,L+L_k+R)$$

$$\nu_0 := E[\sum_{i=1}^{N}\sum_{t=L+1}^{L+L_i+R} D_{it}] \qquad : \textit{expected aggregate demand during } [L,L+L_i+R)$$

$$\nu_k := E[\sum_{t=L+1}^{L+L_k+R} D_{kt}] \qquad : \textit{expected demand in stockpoint } k \textit{ during } [L,L+L_k+R)$$

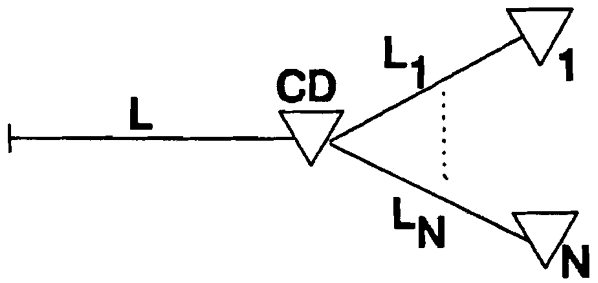

figure 1: 2-echelon model

We can now examine the system operation over time. Since the CD uses a periodic (R,S)-policy, at the beginning of every review period of length R its echelon inventory position is increased to an order-up-to-level S. So the quantity ordered by the CD at the beginning of a review period equals the aggregate realized demand in all stockpoints during the previous review period. Suppose that at time t=0 the CD orders a quantity Q. Then

$$Q = S - Z_0 \tag{1}$$

At the second decision level, after arrival of order Q at time t=L at the CD, we have to allocate the quantity Q to N different stockpoints. Let q_i be the quantity allocated to stockpoint i. Since the depot holds no inventory:

$$\sum_{i=1}^{N} q_i = Q \tag{2}$$

which implies that all arriving material at the CD is immediately allocated to the end stockpoints. Clearly, we need an allocation rule for determining these quantities q_i.

In order to achieve the pre-determined service performance, we need to specify the following parameters:

1) the order-up-to-level S for the CD, and

2) the allocation rule to determine the quantities q_i for the N separate stockpoints.

De Kok(1990) introduced the concept of allocation fractions p_i (i=1..N). The allocation fraction p_k for end stockpoint k represents the expected safety stock in end stockpoint k (as a result of the allocation at the CD at time t=L) as a fraction of the expected **aggregate** safety stock in all end stockpoints (as a result of that same allocation decision at the CD).

$$p_k = \frac{Z_{L,k} + q_k - v_k}{\sum_{i=1}^{N} (Z_{L,i} + q_i - v_i)} = \frac{Z_{L,k} + q_k - v_k}{S - D_0 - v_0} \tag{3}$$

where $Z_{L,k}+q_k$ represents the echelon inventory position of stockpoint k directly after the allocation decision at time t=L. The numerator represents the expected safety stock for stockpoint k, as a result of the allocation at time t=L. The denominator represents the expected aggregate safety stock in all stockpoints.

From expression (3) we have the following allocation rule:

$$q_k = p_k * \{ S - D_0 - v_0 \} + v_k - Z_{L,k} \tag{4}$$

where:

$$0 \le p_i \le 1 \quad and \quad \sum_{i=1}^{N} p_i = 1$$

Application of allocation rule (4) should result in a quantity q_k for stockpoint k that is sufficient to realize a service level equal to $\hat{\beta}_k$ in stockpoint k. Here we introduce an assumption whose importance is discussed in section 3.2.

Generalised Balance assumption: the allocation of the aggregate quantity Q at the CD is such that all allocation quantities q_i in (4) are positive (i=1..N).

It can easily be seen that the expected shortage in stockpoint k in the time-interval between two successive order arrivals equals

$$E[(D_k^{(2)} - (Z_{L,k} + q_k))^+] - E[(D_k^{(1)} - (Z_{L,k} + q_k))^+] \tag{5}$$

Expression (5) represents the expected shortage in stockpoint k at time $t=L+L_k+R$ (just before an order arrival) minus the expected shortage in stockpoint k at time $t=L+L_k$ (directly after an order arrival). Using the definition of service level we get the following service level equation for stockpoint k:

$$\beta_k = 1 - \frac{\mathrm{E}\,[(D_k^{(2)} - (Z_{L,k} + q_k))^+] - \mathrm{E}\,[(D_k^{(1)} - (Z_{L,k} + q_k))^+]}{R * \mu_k} \tag{6}$$

where $R^*\mu_k$ represents the expected demand in stockpoint k during a review period.
Applying allocation rule (4)

$$Z_{L,k} + q_k = p_k * \{ S - D_0 - \nu_0 \} + \nu_k$$

and substitution in expression (6), gives a general expression for β_k

$$\begin{aligned}\beta_k = 1 - \{ & E[((D_k^{(2)}+p_kD_0)-(p_kS-p_k\nu_0+\nu_k))^+] \\ & -E[((D_k^{(1)}+p_kD_0)-(p_kS-p_k\nu_0+\nu_k))^+] \} * \{R * \mu_k\}^{-1}\end{aligned} \tag{7}$$

3.2 The generalised balance assumption

The generalised balance assumption that is stated in the model description differs significantly from the balance assumption commonly used by other authors (Eppen and Schrage(1981), Donselaar and Wijngaard(1986), Jönsson and Silver(1986,1987), Lagodimos(1992)). They define the allocation assumption as the situation in which the allocation quantities are sufficient to ensure **equal** stockout probabilities for all stockpoints. In this paper we use a different definition of service level (fraction of demand delivered from stock on hand) and we allow for **different** target levels (service levels to be realized) for the various stockpoints. The generalised balance assumption states that these target levels can be realized with positive allocation quantities. Therefore the new generalised balance assumption can be seen as a generalization of the traditional one. In principle it is possible to find an allocation rule that yields any target value for any service criterion.

It is very well possible that due to high variation of demand application of the allocation rule results in some negative allocation quantities, i.e. imbalance occurs. In practice this would imply that goods that were allocated earlier on in the planning process have to be pulled back and allocated to other stockpoints. This is often impossible and therefore we assume that imbalance does not occur. The probability of imbalance and its effect on the service performance is discussed in section 5.

4. Solution methods

Given p_k and S, we can calculate the service level for stockpoint k, using expression (7). However, we need to solve the reverse problem. Given target levels $\hat{\beta}_k$ for each stockpoint k, calculate the required values of S and all fractions p_k. To calculate this we need to solve a multi-equation system with N+1 equations and N+1 unknowns (p_1,..,p_N and S):

$$\boxed{\begin{aligned} &\hat{\beta}_i = f(S,p_i) \quad ; \quad (i=1..N) \\ &\sum_{i=1}^{N} p_i = 1 \end{aligned}}$$

where $f(.)$ denotes service level equation (7) for stockpoint i.

In principle this algebraic system of equations can be solved exactly using numerical methods. For example, De Kok(1990) used a bisection scheme for the variable S and a nested bisection scheme for all p_i. Because of the time consuming nature of such exact methods, we propose an approximate decomposition method that is based on empirical findings. As we discuss later, this method is very fast and gives good results.

4.1 Decomposition method to evaluate p_i and S

Under the assumption of normally distributed demand, equal lead times and equal stockout probabilities for all stockpoints it can be shown that the allocation fractions p_k become

$$p_k = \frac{\sigma_k}{\sum_{i=1}^{N} \sigma_i}$$

It is interesting to note that these p_k are implied by Eppen and Schrage(1981) who used a different allocation rule than the one we used here. Clearly, in this special case the allocation fraction p_k can be interpreted as the fraction of the aggregate safety stock of N single-echelon models allocated to stockpoint k.

Numerical experiments reveal that this result approximately holds for the more general conditions in our model. It appears that the allocation fractions are insensitive to the common lead time L. Now we define the allocation fractions as follows

$$p_k = \frac{ss_k^{(1)}}{\sum_{i=1}^{N} ss_i^{(1)}} \qquad (1 \le k \le N) \tag{8}$$

where $ss_i^{(1)}$ denotes the expected safety stock for a single-echelon (R,S)-inventory system with lead time L_i, demand parameters μ_i and σ_i and target level $\hat{\beta}_i$.

A fast and accurate inversion-algorithm (see Appendix A) enables us to compute the order-up-to-level $S_i^{(1)}$ for such a single-echelon network. The safety stock $ss_i^{(1)}$ is then computed as follows

$$ss_i^{(1)} = S_i^{(1)} - (L_i + R) * \mu_i \tag{9}$$

Once we have computed the allocation fractions (applying the inversion-algorithm to N different single-echelon models), we are able to calculate the echelon order-up-to-level S for the CD. This order-up-to-level can be obtained by applying the inversion-algorithm to service level equation (7). As a result we obtain an order-up-to-level S_k for the CD, associated with stockpoint k. For every stockpoint i we find an echelon order-up-to-level S_i. The final order-up-to-level S for the CD is then simply computed by taking the mean of all these separate order-up-to-levels

$$S = \frac{1}{N} \sum_{i=1}^{N} S_i \tag{10}$$

In total we apply the inversion-algorithm 2*N times.

In general this method is justifiable because the differences between the values of S_i appear to be very small. However, when we are dealing with different target levels for the different stockpoints (ranging from e.g. 0.70 to 0.95), the values of S_i differ more than desirable. Averaging over these values implies that for certain stockpoints i the final value of S is too large (if $S_i < S$) and consequently the realized service performance too high, or the final value of S is too small (if $S_i > S$) and consequently the resulting service performance too low. This results in bad performance of the echelon policy.

We can improve the results in these situations by adjusting the allocation fractions. Increasing the value of allocation fraction p_i has to result in a decrease in the value of the matching S_i, in order to maintain the same service performance. Likewise a decrease of p_i results in an increase in the matching S_i. So by adjusting the allocation fractions we are able to bring the separate order-up-to-levels closer together. The adjusted values of the allocation fractions of course still have to sum up to one. The reason why we adjust the allocation fractions and then calculate the order-up-to-levels (and not vice versa) is the very fast inversion-algorithm that enables us to calculate these order-up-to-levels. Every single adjustment of the allocation fractions involves N applications of this inversion-algorithm. Adjusting S_i and next calculating p_i is very time consuming, as indicated above. We now describe two methods of adjusting the allocation fractions.

4.1.1 The group method

Divide the stockpoints i (i=1..N) into two groups A and B in the following way

- $i \in A \quad if \quad S_i < S$
- $i \in B \quad if \quad S_i \geq S$

The allocation fraction for a stockpoint from group A has to be decreased (in order to increase the matching order-up-to-level) and the allocation fraction for a stockpoint from group B has to be increased (in order to decrease the matching order-up-to-level). The adjusted values $\tilde{p}_i$ of the allocation fractions are determined in the following way

- $if \quad i \in A \quad then \quad \tilde{p}_i = \dfrac{(1-\delta)p_i}{1+\delta-2\delta\sum\limits_{i \in A} p_i}$
- $if \quad i \in B \quad then \quad \tilde{p}_i = \dfrac{(1+\delta)p_i}{1+\delta-2\delta\sum\limits_{i \in A} p_i}$

It is easy to see that the values of $\tilde{p}_i$ sum up to one. The parameter δ determines to what extent the allocation fractions are increased or decreased. The value of δ is determined by a local search method aimed at minimizing the following expression

$$\frac{S_{max} - S_{min}}{ASS} \tag{11}$$

$$\begin{aligned} \textit{where} \quad S_{max} &:= \max\{S_i | 1 \le i \le N\} \\ S_{min} &:= \min\{S_i | 1 \le i \le N\} \\ ASS &:= S - \sum_{i=1}^{N} (L+L_i+R) * \mu_i \end{aligned}$$

ASS represents the expected aggregate safety stock in all end stockpoints when using an order-up-to-level S for the CD. The exact solution would of course be reached when expression (11) equals zero ($S_{max}=S_{min}$) implying that all order-up-to-levels S_i are identical.

This procedure can be applied repeatedly until no further reduction of expression (11) is obtained. There is however no guarantee that we will obtain the exact solution.

4.1.2 The worst case method

This method selects the order-up-to-level that differs most from S (the worst case). Let this be S_k, the order-up-to-level resulting from the service level equation for stockpoint k (i.e. $S_k=S_{min}$ or $S_k=S_{max}$). The matching allocation fraction p_k is adjusted in the following way

- *if* $S_k<S$ *then* $\tilde{p}_k = p_k - \delta$
- *if* $S_k \ge S$ *then* $\tilde{p}_k = p_k + \delta$

The remaining allocation fractions p_i ($i \ne k$) are adjusted as follows

- *if* $S_k<S$ *then* $\tilde{p}_i = p_i + \delta * \frac{p_i}{p_{rest}}$
- *if* $S_k \ge S$ *then* $\tilde{p}_i = p_i - \delta * \frac{p_i}{p_{rest}}$

  where $p_{rest} := \sum_{i \ne k} p_i$

Again it is evident that the adjusted values $\tilde{p}_i$ sum up to one. The value of parameter δ is determined in the same way as in the group method. Again this method can be applied repeatedly until no further reduction of expression (11) is obtained. There is again no guarantee for optimality.

4.2 Numerical results

De Kok(1990) used the decomposition method with S given by (10) with very good results. This was due to the fact that the target levels considered varied in a limited range (0.90 and 0.95). If we broaden the target levels range the analytical results deteriorate. We considered some typical examples: R=1, N=6, lead time to all stockpoints is 3, expected period demand in all stockpoints is 100, the target levels vary from 0.70 to 0.95. The common lead time L is 5 resp. 9, the standard deviation (equal in all stockpoints) is 50 resp. 200, resulting in a coefficient of variation of 0.5 resp. 2.0. Using the results from the decomposition method (S and $\{p_i\}$), the service levels are analytically calculated by fitting the stochastic variables in (10) to a mixture of Erlang distributions (if the coefficient of variation is less than one) or a hyperexponential distribution (if the coefficient of variation is equal or greater than one). A detailed description of these calculations is given in Verrijdt(1992). Tables 1 and 2 show the original analytical results, the worst case method results and the group method results.

	cv = 0.5			cv = 2.0		
target	original	worst	group	original	worst	group
0.70	0.696	0.694	0.694	0.685	0.685	0.694
0.75	0.752	0.746	0.745	0.737	0.737	0.743
0.80	0.805	0.797	0.796	0.794	0.794	0.791
0.85	0.852	0.845	0.846	0.849	0.849	0.842
0.90	0.890	0.894	0.897	0.901	0.901	0.895
0.95	0.920	0.949	0.948	0.948	0.948	0.946

table 1: realized service in 2-echelon model with 6 stockpoints and common leadtime L=5

	cv = 0.5			cv = 2.0		
target	original	worst	group	original	worst	group
0.70	0.697	0.694	0.695	0.686	0.686	0.694
0.75	0.758	0.747	0.746	0.742	0.742	0.742
0.80	0.812	0.799	0.797	0.799	0.799	0.794
0.85	0.855	0.847	0.847	0.854	0.854	0.845
0.90	0.886	0.894	0.897	0.901	0.901	0.894
0.95	0.906	0.951	0.948	0.941	0.941	0.946

table 2: realized service in 2-echelon model with 6 stockpoints and common leadtime L=9

Observe that the service performance deterioration is stronger for lower coefficients of variation (cv=0.5) in combination with a larger CD lead time (L=9). Especially the stockpoint with the highest target level (0.95) is affected most. A realized service performance of 90.6% against a target level of 95% implies that the number of backorders is almost doubled! So realizing the target level is more important for stockpoints with high target levels (>90%) than for stockpoints with low target levels (<80%).

Both improvement methods (worst case and group method) adjust the allocation fractions such that the realized service levels are more in accordance with the target levels. Additional results are presented in Appendix B.

5. Exploring the modelling assumption

Imbalance is defined as the situation in which application of the allocation procedure at the CD results in one or more negative allocation quantities. In other words the generalised balance assumption is violated. In the preceding analysis we assumed that imbalance does not occur. In reality however imbalance does occur and disrupts our planning process.

When simulating the planning procedure we tackle the imbalance problem by adjusting the allocation quantities such that no negative quantities remain. In case of imbalance at the CD at allocation time t we adjust the allocation quantities $q_i(t)$ $(i=1..N)$ as follows

$$
\begin{aligned}
&\textit{if} \;\; q_i(t) < 0 \quad \textit{then} \quad \tilde{q}_i(t) := 0 \\
&\textit{if} \;\; q_i(t) \geq 0 \quad \textit{then} \quad \tilde{q}_i(t) := q_i(t) + \frac{q_i(t)}{q_{pos}} * q_{neg} \\
&\qquad\qquad \textit{with} \quad q_{pos} := \sum_{i:q_i \geq 0} q_i(t) \\
&\qquad\qquad\qquad\quad\;\; q_{neg} := \sum_{i:q_i < 0} q_i(t)
\end{aligned}
$$

Notice that expression (2) still holds for the adjusted quantities $\tilde{q}_i(t)$.

In order to quantify the impact of imbalance on our planning procedure and therefore on the realized service levels, we need some analytical measure of imbalance. An obvious such measure is the probability of imbalance at the CD:

$$P(\; \exists\, i : q_i(t) < 0 \;)$$

Because it is extremely difficult to derive an expression for this measure, we use a surrogate measure: the probability that the allocation quantity $q_k(t)$ **for a certain stockpoint k** is negative

$$P(\; q_k(t) < 0 \;) \qquad\qquad (1 \leq k \leq N)$$

A similar surrogate measure is modelled by Eppen and Schrage(1981) and Lagodimos(1992). We make the important restriction that the generalised balance assumption was not violated at the previous allocation period. In other words, at time t-R all allocation quantities $q_i(t-R)$ (i=1..N) are positive.

From the analysis of the 2-echelon model in section 3.1 we know

$$
\begin{aligned}
V_k(t) &= Z_k(t) + q_k(t) \\
&= p_k * \{\, S - D^{(t-L,t)} - \nu_0 \,\} + \nu_k
\end{aligned}
\tag{12}
$$

with:

$V_k(t)$: *echelon inventory position of stockpoint k directly after the allocation at time t*
$Z_k(t)$: *echelon inventory position of stockpoint k just before the allocation at time t*
p_k : *allocation fraction for stockpoint k*
S : *order-up-to-level for the CD*
$D^{(t-L,t)}$: *aggregate demand in all stockpoints during* $[t-L,t)$
$D_k^{(t,t+R)}$: *demand in stockpoint k during* $[t,t+R)$
ν_0 : *expected aggregate in all stockpoints demand during* $[t,t+L_i+R)$
ν_k : *expected demand in stockpoint k during* $[t,t+L_k+R)$

Assuming that R < L, we now have the following expression for the echelon inventory position of stockpoint k after allocation at time t+R:

$$V_k(t+R) = V_k(t) - D_k^{(t,t+R)} + q_k(t+R) \tag{13}$$

Under the condition that the generalised balance assumption holds at allocation time t (no imbalance at CD!) we can derive the following expression for $q_k(t+R)$

$$\begin{aligned} q_k(t+R) &= V_k(t+R) - V_k(t) + D_k^{(t,t+R)} \\ &= p_k \left(D^{(t-L,t)} - D^{(t+R-L,t+R)} \right) + D_k^{(t,t+R)} \\ &= p_k\, D^{(t-L,t-L+R)} - p_k \sum_{i\neq k} D_i^{(t,t+R)} + (1-p_k)\, D_k^{(t,t+R)} \\ &= Y - X \end{aligned} \tag{14}$$

with

$$\begin{aligned} Y &:= p_k\, D^{(t-L,t-L+R)} + (1-p_k)\, D_k^{(t,t+R)} \\ X &:= p_k \sum_{i\neq k} D_i^{(t,t+R)} \end{aligned}$$

The stochastic variables X and Y are independent. We can now calculate the probability π_k of a negative allocation quantity for stockpoint k

$$\begin{aligned} \pi_k &= P(\, q_k<0 \,) \\ &= P(\, Y-X<0 \,) \\ &= P(\, Y<X \,) \\ &= \int_0^\infty \left(\int_0^x f_Y(y)dy \right) f_X(x)dx \end{aligned} \tag{15}$$

In general, using two moments fits for X and Y we can evaluate π_k numerically for any demand distribution. If the coefficient of variation is less than one we use a mixture of Erlang distributions. Otherwise we apply a hyperexponential distribution (cf. Tijms (1986)).

5.1 Numerical results

We now take a look at some simulation results. For a more extensive numerical summary we refer to Appendix C. The simulation time is 30.000 time periods. The parameter setting is as follows: N=6, R=1, L=9, L_i=3, μ_i=100. The target levels are identical in all stockpoints (0.70 resp. 0.95). The standard deviation has three alternatives: σ_i=50, σ_i=200 (for all i) or $\sigma_1=\sigma_2=\sigma_3$=50 and $\sigma_4=\sigma_5=\sigma_6$=200. The tables presented below show the analytical results versus the simulation results. The analytical results are obtained after application of the worst case improvement method. The realized service levels β_k and the probabilities of imbalance π_k for each stockpoint k are tabulated.

	β_k			π_k		β_k			π_k	
k	target	analys.	sim.	analys.	sim.	target	analys.	sim.	analys.	sim.
1	0.70	0.699	0.698	0.01	0.01	0.70	0.697	0.708	0.32	0.34
2		0.699	0.702	0.01	0.01		0.697	0.714	0.32	0.34
3		0.699	0.700	0.01	0.01		0.697	0.703	0.32	0.34
4		0.699	0.698	0.01	0.01		0.697	0.700	0.32	0.33
5		0.699	0.697	0.01	0.01		0.697	0.699	0.32	0.34
6		0.699	0.703	0.01	0.01		0.697	0.699	0.32	0.33

table 3: realized service and imbalance with cv_k=0.5 (k=1..6)

table 4: realized service and imbalance with cv_k=2.0 (k=1..6)

	β_k			π_k		β_k			π_k	
k	target	analys.	sim.	analys.	sim.	target	analys.	sim.	analys.	sim.
1	0.95	0.948	0.936	0.01	0.01	0.95	0.947	0.925	0.32	0.34
2		0.948	0.942	0.01	0.01		0.947	0.925	0.32	0.34
3		0.948	0.942	0.01	0.01		0.947	0.923	0.32	0.34
4		0.948	0.940	0.01	0.01		0.947	0.919	0.32	0.33
5		0.948	0.939	0.01	0.01		0.947	0.922	0.32	0.34
6		0.948	0.942	0.01	0.01		0.947	0.920	0.32	0.33

table 5: realized service and imbalance with cv_k=0.5 (k=1..6)

table 6: realized service and imbalance with cv_k=2.0 (k=1..6)

	β_k			π_k		β_k			π_k	
k	target	analys.	sim.	analys.	sim.	target	analys.	sim.	analys.	sim.
1	0.70	0.697	0.596	0.00	0.00	0.95	0.942	0.887	0.01	0.00
2		0.697	0.577	0.00	0.00		0.942	0.894	0.01	0.00
3		0.697	0.565	0.00	0.00		0.942	0.887	0.01	0.00
4		0.671	0.700	0.32	0.36		0.954	0.931	0.32	0.35
5		0.671	0.710	0.32	0.37		0.954	0.934	0.32	0.35
6		0.678	0.723	0.33	0.38		0.954	0.939	0.32	0.35

table 7: realized service and imbalance with $cv_{1,2,3}$=0.5 and $cv_{4,5,6}$=2.0

table 8: realized service and imbalance with $cv_{1,2,3}$=0.5 and $cv_{4,5,6}$=2.0

It is clear from these results that π_k is strongly related to the coefficient of variation of demand processes. A high coefficient of variation at a stockpoint (cv=2.0) results in a high probability of imbalance at that stockpoint. The effect of imbalance on the service performance depends on the target levels and the coefficients of variation in the separate stockpoints. In case of low target levels the high probabilities of imbalance hardly affect the realized service performance (table 4). However, for high target levels in combination with high imbalance probabilities, the realized service levels are significantly lower than the target levels (table 6). The worst results are obtained in asymmetric configurations: different coefficients of variation in the various stockpoints (tables 7 and 8). The high imbalance probabilities for the stockpoints with high coefficients of variation affect the realized service performance in the stockpoints with low coefficients of variation enormously, resulting in very low service levels (compared to the target levels). This negative effect on the service performance can be noticed for situations with low target levels (table 7) as well as high target levels (table 8).

With respect to the probability of imbalance we can conclude that the simulation results are quite good. The small differences that occur (especially for high probabilities of imbalance in asymmetric configurations: tables 7 and 8) between analytical and simulation results can be partly explained by the assumption of balance at the previous time of allocation. In the simulation however it is very well possible that imbalance situations in a stockpoint occur at consecutive times of allocation. During the simulation the allocation quantities are adjusted when imbalance occurs, such that no negative quantities remain. This adjustment of the allocation procedure has a negative effect on the performance of the echelon policy and enlarges the probability of imbalance at the next time of allocation. This also explains the differences between analytical and simulation results.

When we look at configurations with a wide range of target levels (table C.2, appendix C) we can again observe deviations between analytical and simulation results for situations with a high coefficient of variation. Furthermore it is evident that the stockpoint with a high target level (0.95) and a low coefficient of variation (0.5) has a significant higher probability of imbalance than the other stockpoints. In situations with a high coefficient of variation (2.0) the analytically calculated probabilities of imbalance appear to be independent of the target levels. The simulation results however point out that there is a dependency.

6. Conclusions.

In this paper we developed a hierarchical planning procedure for a divergent 2-echelon distribution network. Given the lead times, the demand parameters and the desired service performance (i.e. target levels) for all end stockpoints, the decomposition algorithm developed here evaluates the required echelon order-up-to-level (defining the ordering policy) and the allocation fractions (defining the allocation policy). While the results of the algorithm are only approximations, these can be obtained very fast and yield excellent results. We can however identify two major problems.

First, when dealing with a wide range of target levels the analytically calculated service levels deviate from the target levels (especially the high service levels are affected!). Two improvement methods are presented which compensate these deviations by adjusting the allocation fractions. Both methods improve the analytical results considerably.

The second problem is the phenomenon of imbalance. High coefficients of variation at end stockpoints disrupt the allocation policy resulting in bad service performance. Our numerical experiments show that the algorithm defined in this paper yields excellent results with a negligible computation time if the probability of imbalance is small. Clearly, in case of imbalance the quality of the approximations deteriorate. One way of dealing with this problem is to hold stock at the central depot which can be used in situations of imbalance (cf. Van Donselaar(1990)). Another way is to smooth the highly variable market demand by satisfying large portions of demand directly from the central depot. As a result the coefficient of variation at the end stockpoints will get smaller. These suggestions will be subject of further research.

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Appendix A: inversion-algorithm

The algorithm described in this appendix enables us to determine the order-up-to-level S for an 1-echelon model such that a predetermined target level is realized. In this model an (R,S)-inventory strategy is applied: at the beginning of every review period of length R the echelon inventory position is increased to a level S. We need the following input data:

$\hat{\beta}$: target level

L : lead time

μ : mean period demand

σ : standard deviation in mean period demand

It can be easily shown that the service level can be written as a function of S.

$$\beta(S) = 1 - \frac{E[(D_{L+R}-S)^+] - E[(D_L-S)^+]}{E[D_R]} \qquad \text{(A1)}$$

where D_L = demand during a lead time

D_R = demand during a review period

D_{L+R} = demand during a lead time plus a review period

$\beta(S)$ is a monotone increasing function in S with $\beta(0)=0$ and $\beta(\infty)=1$ and can therefore be considered as a probability distribution function of a random variable X_β, i.e. $P(X_\beta \le S) = \beta(S)$. Next we make a two-moment gamma fit $\underline{\beta}(.)$ of $\beta(.)$. The first two moments of X_β can be determined as follows

$$E[X_\beta^k] = k\int_0^\infty y^{k-1}(1-\beta(y))dy \qquad \text{(A2)}$$

Given a target level $\hat{\beta}$ we now need to solve the following equation

$$\underline{\beta}(S) = \hat{\beta} \qquad \text{(A3)}$$

In order to solve (A3) for S we need to invert the gamma function $\underline{\beta}(.)$

$$S = \underline{\beta}^{-1}(\hat{\beta}) \qquad \text{(A4)}$$

For an exact description of this gamma inversion we refer to De Kok(1989). The final value of S follows from:

$$
\begin{aligned}
S &= (1+vc_\beta * k_\beta)E[X_\beta] \\
\textit{with} \quad vc_\beta &= \frac{\sqrt{(E[X_\beta^2]-E^2[X_\beta])}}{E[X_\beta]} \\
k_\beta &= (1-vc_\beta)*k_0+vc_\beta*k_1 \\
k_0 &= \Phi^{-1}(\hat{\beta}) \\
k_1 &= -1-\ln(1-\hat{\beta})
\end{aligned}
$$

$\Phi^{-1}(.)$ represents the inverted standardized normal probability distribution function, which is approximated polynomially (Abramowitz and Stegun(1965)).

Appendix B

Tables B.1 and B.2 show the analytically calculated service levels for a number of configurations.

A: realized service levels

B: realized service levels after applying the worst case method

C: realized service levels after applying the group method

The coefficient of variation (cv) and the end stockpoint lead times are equal in all stockpoints.
The expected period demand is 100. The review period (R) is 1.

		cv = 0.5			cv = 1.2			cv = 2.0		
N	target level	A	B	C	A	B	C	A	B	C
2	0.70	0.702	0.693	0.694	0.718	0.696	0.694	0.711	0.692	0.692
	0.95	0.912	0.952	0.948	0.927	0.945	0.947	0.935	0.947	0.947
2	0.80	0.819	0.793	0.794	0.813	0.793	0.793	0.809	0.797	0.795
	0.95	0.923	0.949	0.948	0.932	0.946	0.947	0.939	0.947	0.948
2	0.90	0.907	0.895	0.896	0.902	0.894	0.894	0.898	0.892	0.892
	0.95	0.939	0.948	0.948	0.940	0.946	0.946	0.945	0.949	0.949
4	0.70	0.697	0.694	0.694	0.699	0.694	0.695	0.693	0.692	0.696
	0.80	0.810	0.795	0.795	0.803	0.795	0.792	0.799	0.796	0.791
	0.90	0.893	0.896	0.897	0.895	0.892	0.895	0.899	0.898	0.896
	0.95	0.921	0.949	0.948	0.935	0.947	0.948	0.943	0.947	0.947
6	0.70	0.696	0.694	0.694	0.693	0.691	0.694	0.685	0.685	0.694
	0.75	0.752	0.746	0.745	0.745	0.743	0.744	0.737	0.737	0.743
	0.80	0.805	0.797	0.796	0.798	0.796	0.793	0.794	0.794	0.791
	0.85	0.852	0.845	0.846	0.849	0.847	0.844	0.849	0.849	0.842
	0.90	0.890	0.894	0.897	0.896	0.895	0.895	0.901	0.901	0.895
	0.95	0.920	0.949	0.948	0.939	0.947	0.946	0.948	0.948	0.946

Table B.1: $L=5$, $L_1=3$

		cv = 0.5			cv = 1.2			cv = 2.0		
N	target level	A	B	C	A	B	C	A	B	C
2	0.70	0.707	0.696	0.694	0.731	0.696	0.693	0.727	0.693	0.692
	0.95	0.900	0.939	0.948	0.916	0.945	0.947	0.927	0.947	0.947
2	0.80	0.832	0.797	0.795	0.826	0.793	0.793	0.818	0.792	0.792
	0.95	0.914	0.946	0.948	0.924	0.946	0.946	0.932	0.948	0.948
2	0.90	0.912	0.896	0.897	0.907	0.896	0.894	0.904	0.894	0.894
	0.95	0.935	0.948	0.948	0.938	0.946	0.947	0.941	0.948	0.948
4	0.70	0.700	0.694	0.695	0.705	0.693	0.695	0.695	0.696	0.695
	0.80	0.820	0.797	0.795	0.811	0.797	0.793	0.805	0.791	0.792
	0.90	0.890	0.897	0.897	0.894	0.895	0.895	0.899	0.893	0.894
	0.95	0.909	0.951	0.949	0.926	0.947	0.947	0.935	0.948	0.947
6	0.70	0.697	0.694	0.695	0.697	0.692	0.695	0.686	0.686	0.694
	0.75	0.758	0.747	0.746	0.751	0.746	0.745	0.742	0.742	0.742
	0.80	0.812	0.799	0.797	0.804	0.798	0.795	0.799	0.799	0.794
	0.85	0.855	0.847	0.847	0.852	0.848	0.844	0.854	0.854	0.845
	0.90	0.886	0.894	0.897	0.894	0.892	0.895	0.901	0.901	0.894
	0.95	0.906	0.951	0.948	0.929	0.946	0.948	0.941	0.941	0.946

Table B.2: L=9, L_1=3

Appendix C

				α_k		π_k	
k	cv_k	L_k	target level	anal.	simul.	anal.	simul.
1	0.5	3	0.90	0.897	0.886	0.01	0.01
2				0.897	0.892	0.01	0.01
3				0.897	0.892	0.01	0.01
4				0.897	0.889	0.01	0.01
5				0.897	0.889	0.01	0.01
6				0.897	0.893	0.01	0.01
1	2.0	3	0.90	0.895	0.878	0.32	0.33
2				0.895	0.862	0.32	0.34
3				0.895	0.872	0.32	0.33
4				0.895	0.881	0.32	0.34
5				0.895	0.873	0.32	0.33
6				0.895	0.876	0.32	0.34
1	0.5	3	0.90	0.894	0.825	0.01	0.00
2				0.894	0.816	0.01	0.00
3				0.894	0.820	0.01	0.00
4	2.0			0.897	0.883	0.32	0.36
5				0.897	0.878	0.32	0.36
6				0.897	0.867	0.32	0.35
1	0.5	3	0.99	0.989	0.982	0.01	0.01
2				0.989	0.985	0.01	0.01
3				0.989	0.986	0.01	0.01
4				0.989	0.984	0.01	0.01
5				0.989	0.984	0.01	0.01
6				0.989	0.986	0.01	0.01
1	2.0	3	0.99	0.989	0.975	0.32	0.34
2				0.989	0.975	0.32	0.34
3				0.989	0.973	0.32	0.34
4				0.989	0.972	0.32	0.33
5				0.989	0.973	0.32	0.34
6				0.989	0.973	0.32	0.33
1	0.5	3	0.99	0.985	0.959	0.01	0.00
2				0.985	0.961	0.01	0.00
3				0.985	0.959	0.01	0.00
4	2.0			0.994	0.983	0.32	0.35
5				0.994	0.980	0.32	0.35
6				0.994	0.984	0.32	0.35

Table C.1: L=9

				α_k		π_k	
k	cv_k	L_k	target level	anal.	simul.	anal.	simul.
1	0.5	3	0.70	0.694	0.681	0.00	0.00
2			0.75	0.747	0.742	0.00	0.00
3			0.80	0.799	0.792	0.00	0.00
4			0.85	0.847	0.836	0.01	0.00
5			0.90	0.894	0.882	0.02	0.01
6			0.95	0.951	0.954	0.15	0.19
1	2.0	3	0.70	0.686	0.663	0.31	0.19
2			0.75	0.742	0.706	0.32	0.23
3			0.80	0.799	0.769	0.32	0.27
4			0.85	0.854	0.842	0.32	0.33
5			0.90	0.901	0.889	0.33	0.39
6			0.95	0.941	0.950	0.34	0.48
1	0.5	3	0.70	0.694	0.562	0.00	0.00
2			0.75	0.744	0.606	0.00	0.00
3			0.80	0.795	0.675	0.00	0.00
4	2.0		0.85	0.840	0.817	0.33	0.29
5			0.90	0.892	0.869	0.32	0.34
6			0.95	0.947	0.936	0.34	0.45
1	2.0	3	0.70	0.694	0.691	0.34	0.24
2			0.75	0.752	0.750	0.33	0.29
3			0.80	0.798	0.802	0.32	0.34
4	0.5		0.85	0.850	0.752	0.00	0.00
5			0.90	0.893	0.818	0.01	0.00
6			0.95	0.943	0.970	0.23	0.29

Table C.2: L=9

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93-01	January	P. v.d. Laan C. v. Eeden	Subset selection for the best of two populations: Tables of the expected subset size
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93-03	February	Jan Beirlant John H.J. Einmahl	Asymptotic confidence intervals for the length of the shortt under random censoring.
93-04	February	E. Balas J. K. Lenstra A. Vazacopoulos	One machine scheduling with delayed precedence constraints
93-05	March	A.A. Stoorvogel J.H.A. Ludlage	The discrete time minimum entropy H_∞ control problem
93-06	March	H.J.C. Huijberts C.H. Moog	Controlled invariance of nonlinear systems: nonexact forms speak louder than exact forms
93-07	March	Marinus Veldhorst	A linear time algorithm to schedule trees with communication delays optimally on two machines
93-08	March	Stan van Hoesel Antoon Kolen	A class of strong valid inequalities for the discrete lot-sizing and scheduling problem
93-09	March	F.P.A. Coolen	Bayesian decision theory with imprecise prior probabilities applied to replacement problems
93-10	March	A.W.J. Kolen A.H.G. Rinnooy Kan C.P.M. van Hoesel A.P.M. Wagelmans	Sensitivity analysis of list scheduling heuristics
93-11	March	A.A. Stoorvogel J.H.A. Ludlage	Squaring-down and the problems of almost-zeros for continuous-time systems
93-12	April	Paul van der Laan	The efficiency of subset selection of an ε-best uniform population relative to selection of the best one
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93-14	May	L.C.G.J.M. Habets	On the Genericity of Stabilizability for Time-Day Systems
93-15	June	P. van der Laan C. van Eeden	Subset selection with a generalized selection goal based on a loss function
93-16	June	A.A. Stoorvogel A. Saberi B.M. Chen	The Discrete-time H_∞ Control Problem with Strictly Proper Measurement Feedback
93-17	June	J. Beirlant J.H.J. Einmahl	Maximal type test statistics based on conditional processes
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93-19	July	J.A. Hoogeveen J.K. Lenstra B. Veltman	Three, four, five, six or the Complexity of Scheduling with Communication Delays
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93-25	August	A.A. Stoorvogel	Stabilizing solutions of the H_∞ algebraic Riccati equation
93-26	August	R. Perelaer J.K. Lenstra M. Savelsbergh F. Soumis	The Bus Driver Scheduling Problem of the Amsterdam Transit Company
93-27	August	J.M. van den Akker C.P.M. van Hoesel M.W.P. Savelsbergh	Facet Inducing Inequalities for Single-Machine Scheduling Problems
93-28	September	H.J.C. Huijberts H. Nijmeijer	Dynamic disturbance decoupling of nonlinear systems and linearization

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93-29	September	L.C.G.J.M. Habets	Testing Reachability and Stabilizability of Systems over Polynomial Rings using Gröbner Bases
93-30	September	Z. Liu J.A.C. Resing	Duality and Equivalencies in Closed Tandem Queueing Networks
93-31	November	A.G. de Kok J. van der Wal	Assigning identical operators to different machines
93-32	November	M.J.A. van Eenige J.A.C. Resing J. van der Wal	A Matrix-Geometric Analysis of Queueing Systems with Periodic Service Interruptions
93-33	November	A.G. de Kok	Backorder lead time behaviour in (b, Q)-inventory models with compound renewal demand
93-34	November	J. van der Wal	The lead time shift theorem for the continuous review (s, Q) and (s, S) inventory systems
93-35	November	J.H.C.M. Verrijdt A.G. de Kok	Distribution Planning for a Divergent 2-echelon Network without Intermediate Stocks under Service Restrictions