A new design

A NEW DESIGN

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Abstract. We construct the first example of a quasi-residual design with $k < v/2$ for which the corresponding symmetric design cannot exist by the Bruck-Ryser-Chowla theorem. We also construct two new group-divisible designs.

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1. Introduction. The main result of this paper is the construction of a block design 2-(28,10,5). Among the designs for which the existence was unknown until now, only the elusive parameter set 2-(22,8,4) has a smaller value of v. The construction answers a question (that was open) concerning quasi-residual designs. This design is the first one that has been constructed with the properties : (i) $k < v/2$ and (ii) the design is quasi-residual and the corresponding symmetric design (in this case a 2-(43,15,5) design) does not exist by the Bruck-Ryser-Chowla theorem.

The idea of the construction is as follows. Assume that the design has a sufficiently nice automorphism group that fixes many blocks. This makes it possible to analyse the structure of the design and to find related designs by computer search. Once the existence of the design had been established we were able to give a completely computer-free description. In fact we believe that infinitely many designs with the same structure exist. A known 2-(10,4,2) design is of the same type as our design. Together they would be the cases $m = 1$ and $m = 3$ of a sequence (with $m = 3^s$). We remark that the method of Section 3 also works for $m = 5$, producing a third example of the group-divisible designs of Theorem 2.1. However, this does not lead to a quasi-symmetric design although a design with the corresponding parameters is known (see [2]).

2. Designs with an automorphism of order 3. It is not very difficult, although quite tedious, to show that the only primes which can be orders of an automorphism of a 2-(28,10,5) design are 2 and 3. The following lemma has played a crucial role in our construction of a design with an automorphism of order 3 (also cf. [3]).

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LEMMA 2.1. *An automorphism of order 3 of a* $2-(v,k,\lambda)$ *design fixes at most* $b-3r+3\lambda$ *blocks.*

Proof. Consider a 3-cycle of points and count the blocks that contain all three points or none of them. □

We shall now consider the extremal case, first for the parameters 2-(28,10,5).

LEMMA 2.2. *Suppose that there exists a 2-(28,10,5) design with an automorphism* f *of order 3 fixing exactly one point and the maximum number of blocks, i.e.* $b-3r+3\lambda=12$ *blocks. Then the cycles of* f *considered as "points" and the fixed blocks form a 2-(9,3,1) design, i.e. an affine plane of order 3, while the orbit matrix of the non-fixed points and blocks is a 9 by 10 matrix of the following form:*

$$\begin{pmatrix} 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 2 \end{pmatrix}. \tag{2.1}$$

In the above matrix each entry "1" has to be replaced by an appropriate power of the matrix

$$C=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

and each entry "2" by $J-I$ to obtain the incidence matrix of non-fixed points and non-fixed blocks of the design. In fact, Lemma 2.2 is a particular case of a more general construction described in the next theorem.

THEOREM 2.1. *Suppose that there exists a* $2-(9m+1,3m+1,(3m+1)/2)$ *design (*m *odd), with an automorphism* f *of order 3 fixing one point and the maximum number of blocks, i.e.* $3(3m-1)/2$ *blocks. Then the cycles of* f *considered as "points" and the fixed blocks form a* $2-(3m,m,(m-1)/2)$ *design. The orbit matrix of the non-fixed points and blocks is a* $3m$ *by* $3m+1$ *matrix of the form (2.1), and the incidence structure of the non-fixed points and blocks is a partially balanced group divisible design with group size 3,* $v=9m$, $b=3(3m+1)$, *three blocks of size* $3m$ *and* $9m$ *blocks of size* $3m+1$, $\lambda_1=1$, $\lambda_2=m+1$.

Proof. Consider the unique point, say P, fixed by f. Since $r=3(3m+1)/2$, P is contained in all $3(3m-1)/2$ fixed blocks plus 3 non-fixed blocks. Since there are $3m$ block orbits of length 3 not containing P and $r-\lambda=k=3m+1$, for each 3-cycle of f there is a non-fixed block not containing P and containing at least two points from that cycle. Consequently, since $\lambda=(3m+1)/2$, a 3-cycle can be contained in at most $\lambda-1=(3m-1)/2$ fixed blocks. Therefore, the unique block orbit of length 3 containing the fixed point must meet each cycle of f in at least one point. Since there are $3m$ 3-cycles and $k=3m+1$, each non-fixed point occurs together with P in exactly one non-fixed block. Therefore, each 3-cycle is contained in exactly $\lambda - 1 = (3m-1)/2$ fixed blocks, and each pair of points belonging to one and the same cycle occur together in exactly one non-fixed block not containing P, while each block from the remaining $3m-1$ block orbits of length 3 must contain exactly one point from the same cycle. Hence, a row of the orbit matrix of the non-fixed points and blocks consists of one 2 and $3m$ 1's. Since $k = 3m+1$, the same holds for the columns of the orbit matrix except the column corresponding to the orbit containing P which consists entirely of ones. Consequently, the orbit matrix is of the form (2.1). The scalar product of two rows of (2.1) is $3m+3 = 3(m+1)$. This implies that a pair of non-fixed points belonging to different cycles must occur together in exactly $m+1$ blocks not containing P. Therefore, each two 3-cycles occur together in exactly $\lambda - m + 1 = (m-1)/2$ fixed blocks. Hence the 3-cycles and the fixed blocks form a $2-(3m, m, (m-1)/2)$ design, while the non-fixed points and blocks form a partially balanced design such that a pair of points belonging to one and the same cycle occur in one block, while pairs of points from different cycles occur in $m+1$ blocks. This completes the proof. □

COROLLARY. *A sufficient condition for the existence of a* $2-(9m+1, 3m+1, (3m+1)/2)$ *design is the existence of a* $2-(3m, m, (m-1)/2)$ *design and a symmetric group divisible design with group size 3,* $v = 9m+3$, $k = 3m+2$, $\lambda_1 = 1$, $\lambda_2 = m+1$.

A class of $2-(3m, m, (m-1)/2)$ designs is provided by the affine geometry over $GF(3)$. Namely, if $m = 3^s$, then a $2-(3^{s+1}, 3^s, (3^s-1)/2)$ design is formed by the hyperplanes in $AG(s+1, 3)$. The corresponding symmetric group divisible design has the following parameters :

$$v = 3^{s+2} + 3, \quad k = 3^{s+1} + 2, \quad \lambda_1 = 1, \quad \lambda_2 = 3^s + 1.$$

A symmetric group divisible (12,5,1,2) design corresponding to $s = 0$ is given in Bose, Clatworthy and Shrikhande [1]. Together with the trivial 2-(3,1,0) design this leads to a 2-(10,4,2) design.

3. A symmetric group divisible design (30,11,1,4). We shall construct the required group divisible design D represented as a 10 by 10 matrix A^* in which the entries on the diagonal are $C+C^2$ and all the others are I,C,or C^2 (where C is as in (2.2)). Then $A^*A^{*\mathrm{T}}$ has diagonal entries $10I+J$ and all the other entries are $4J$. We now replace C by $\zeta = e^{2\pi i/3}$ (since $C^3 = I$, $\zeta^3 = 1$) noting that $J = I + C + C^2$ should be replaced by $0 = 1 + \zeta + \zeta^2$. The matrix A^* is then replaced by a 10 by 10 matrix A with -1 on the diagonal, cube roots of unity elsewhere and we must have $A\tilde{A} = 10I$. Note that a matrix of this kind of order 4 is $J - 2I$, providing an independent solution for the case $s = 0$ in Section 2.

We construct A by assuming even more regularity. Let C_5 denote the 5 by 5 circulant analogous to (2.2), so $C_5^5 = I_5$. We shall assume that there is a solution A with

$$A = I_5 \otimes P + (C_5 + C_5^4) \otimes Q + (C_5^2 + C_5^3) \otimes \overline{Q} = M(P, Q), \tag{3.1}$$

where P is a symmetric real matrix (with -1 on the diagonal) and Q is a symmetric matrix with cube roots of unity as entries, both of size 2 by 2. As usual $\otimes$ denotes the Kronecker product. We have

$$M(P,Q)\cdot \widetilde{M(P,Q)} = M(R,S), \tag{3.2}$$

where

$$\begin{aligned} R &= PP^\top + 2Q\overline{Q} + 2\overline{Q}Q, \\ S &= P\overline{Q} + QP + Q^2 + \overline{Q}Q + \overline{Q}^2. \end{aligned}$$

Substitution of $P = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} \zeta & \zeta \\ \zeta & 1 \end{pmatrix}$ yields $R = 10I$, $S = O$ and hence $A = M(P,Q)$ solves our problem!

It has been checked by Kapralov (private communication) using a computer, that up to isomorphism the group divisible (30,11,1,4) design D is unique under the assumption of an automorphism of order 3 without fixed points. The full automorphism group of this design is a semi-direct product of a cyclic group of order 15 with the cyclic group of order 4, splitting the 30 points into two orbits of length 15.

We could have constructed our design in another way, again assuming high regularity, by using the orbits of length 15. One proceeds as follows. Let P be the permutation matrix of order 15 of the type of (2.1), i.e. $P_{i,j} = 1$ if and only if $j = i+1 \pmod{15}$. We define

$$P_1 = P+P^2+P^4+P^8, P_2 = P^3+P^6+P^9+P^{12}, P_3 = P^5+P^{10}, P_4 = P^7+P^{11}+P^{13}+P^{14}$$

It is well known that I and P_1, P_2, P_3, P_4 form a 5-dimensional algebra of matrices of order 15. We consider a 30 by 30 matrix $A = \begin{pmatrix} X & Y \\ Y & Z \end{pmatrix}$ where each entry is of the form $\epsilon_0 I + \epsilon_1 P_1 + \epsilon_2 P_2 + \epsilon_3 P_3 + \epsilon_4 P_4$, ($\epsilon_i = 0$ or 1). This reduces the number of possibilities to a reasonable number. A multiplication table for $P_i P_j$ is easily found. We now calculate $AA^\top$. If this is to be the incidence matrix of D, then we should have

$$AA^\top = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \text{with} \quad \begin{aligned} A_2 &= A_3 &= 4J, \\ A_1 &= A_4 &= 7I + 4J - 3P_3 \end{aligned}$$

(where the $-3P_3$ accounts for the inner products 1 of rows from the same 3-cycle). These equations are easily solved. We find the (essentially unique) solution : $X = P_1 + P_3$, $Y = I + P_1$, $Z = P_2 + P_3$. This method may be easier to generalize to other orders than the first one.

4. On group divisible designs with $v = 9m$, $b = 3(3m+1)$, $\lambda_1 = 1$, $\lambda_2 = m+1$.

We consider the group divisible designs of Theorem 2.1, i.e. we assume that they have the form of (2.1) where each 1 represents an appropriate power of C and each 2 denotes $C + C^2$.

THEOREM 4.1. *If a group divisible design of type (2.1) with $v = 9m$, $b = 3(3m+1)$ exists, then a symmetric group divisible design of size $9m+3$ with "structure" $I + J$ exists.*

Proof. We use the method of Section 3 and replace C by ζ. So we now have a $3m$ by $3m+1$ matrix G of type (2.1), where each 2 is replaced by -1 and each 1 by a power of ζ and furthermore $G\tilde{G} = (3m+1)I$. Let $\mathbf{x}_i = (x_{i0}, x_{i1}, \ldots, x_{i,3m})$ denote the i-th row of G $(1 \leq i \leq 3m)$. Then $(\mathbf{x}_i, \overline{\mathbf{x}}_i) = 3m+1$ and $(\mathbf{x}_i, \overline{\mathbf{x}}_j) = 0$ if $i \neq j$. Define

$$\mathbf{x}_0 = \sum_{i=1}^{3m} \mathbf{x}_i - (3m+1)(1,0,0,\ldots,0). \tag{4.1}$$

(Note that $x_{00} = -1$.) By (4.1) we have

$$(\mathbf{x}_0, \overline{\mathbf{x}}_0) = 3m(3m+1) + (3m+1)^2 - 2(3m+1)3m = 3m+1, \tag{4.2}$$

$$(\mathbf{x}_0, \overline{\mathbf{x}}_i) = (\mathbf{x}_i, \overline{\mathbf{x}}_i) - (3m+1) = 0 \qquad \text{for} \qquad i > 0. \tag{4.3}$$

It follows that if we form G' by adjoining $\mathbf{x}_0$ as top row to G, then $G'\tilde{G}' = (3m+1)I$. From (4.2) we know that $\sum_{j=1}^{3m} |x_{ij}|^2 = 3m$. Each entry is a sum of $3m+1$ cube roots of unity (using $-1 = \zeta + \zeta^2$). But a sum of this number of cube roots of unity is either again a cube root of unity or it has absolute value greater than 1. It follows that all x_{ij} $(j = 1, 2, \ldots, 3m)$ are cube roots of unity and we are done. □

In the introduction we remarked that the method of Section 3 also works for $m = 5$. We shall now give the details.

THEOREM 4.2. *There exists a group divisible design (48,17,1,6).*

Proof. Consider the matrix of (3.1) but now substitute

$$P = -I + 2J, \quad Q = (1-\zeta)I + \zeta J, \qquad \text{where } I \text{ and } J \text{ are 3 by 3.}$$

We find

$$PP^{\top} = 4I - J, \qquad Q^2 = -3\zeta I + (-1+\zeta)J,$$
$$P\overline{Q} = (-2+2\zeta)I + J, \qquad Q\overline{Q} = \overline{Q}Q = 3I$$

and then by substitution (from (3.2)) : $R = 16I - J, S = -J$. Now define

$$A = \begin{pmatrix} -1 & \mathbf{j}^{\top} \\ \mathbf{j} & M(P,Q) \end{pmatrix}. \tag{4.4}$$

Then $A\tilde{A} = 16I$. Just as in Section 3 this now yields the desired group divisible design (48,17,1,6). □

We have checked that the methods of Section 3 and Theorem 4.2 do not work for other "Paley type" matrices than (3.1) and also only for the two choices of P and Q that we have used.

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