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THE COMPENSATION APPROACH FOR THREE OR MORE DIMENSIONAL RANDOM WALKS

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Abstract. In this paper we investigate for which random walks with three or more dimensions the compensation approach can be used to determine the equilibrium distribution. As we will see, the compensation approach is not appropriate for the symmetric shortest queue system with three queues, but for the 2×3 buffered switch it is. By using this compensation approach, we show that for the 2×3 buffered switch the equilibrium distribution can be expressed as a linear combination of six series of binary trees of product-form (geometric) distributions.

1. Introduction

In another paper in this volume, Adan and Wessels [2] give an overview of the class of queueing systems for which the equilibrium distribution can be written as a linear combination of product forms. In some cases (like Jackson networks) the equilibrium distribution can be expressed as a linear combination of a finite number of product forms, while in other cases (like the two-dimensional shortest queue problem) an infinite number of product forms is needed. In this paper we focus on the latter problems.

In [1] Adan et al. develope a *compensation approach* for two-dimensional, ergodic random walks on the lattice of the first quadrant, with the following properties:

(i) only transitions to neighbouring states occur;

(ii) *semi-homogeneity*: the same transition rates occur for all interior points, and similarly for all points on the horizontal boundary and for all points on the vertical boundary;

(iii) transitions from interior points to the North, North-East and East are not permitted, i.e. for the transitions rates $q_{i,j}$ for the interior points ($q_{i,j}$ is the rate for a transition in the direction (i,j)), it is required that

$$q_{0,1} = q_{1,1} = q_{1,0} = 0 . \tag{1}$$

By using the compensation approach it is shown that for such a random walk the equilibrium distribution can be written as a linear combination of at most four series of product forms.

For a two-dimensional, semi-homogeneous, ergodic random walk with transitions to neighbouring states only, it appears that condition (iii) is sufficient for succesfully applying the compensation approach; if an infinite number of product forms is needed to construct the equilibrium distribution, then this condition (1) is also necessary. In this paper we try to find the analogue of condition (iii) for the three or more dimensional case. An idea about when the compensation approach works and when not, can be obtained by analyzing the ratios of the equilibrium probabilities for neighbouring states. Let us consider a three-dimensional problem, for example. If the compensation approach can be used to determine the equilibrium distribution $\{p_{m,n,r}\}$, then this equilibrium distribution is a linear combination of product forms, i.e.

$$p_{m,n,r} = \sum_i c_i \alpha_i^m \beta_i^n \gamma_i^r ,$$

and therefore the sequences of ratios $p_{m+1,n,r}/p_{m,n,r}$ have the same limit for all n and r:

$$p_{m+1,n,r}/p_{m,n,r} \rightarrow \alpha_{\max} := \max_i \alpha_i \ , \text{ if } m \rightarrow \infty;$$

a similar property holds for the ratios $p_{m,n+1,r}/p_{m,n,r}$ and $p_{m,n,r+1}/p_{m,n,r}$.

In the sections 2 and 3 we consider the three-dimensional versions of the symmetric shortest queue system with two queues and the 2×2 buffered switch, which both belong to the class of two-dimensional problems for which the equilibrium distribution can be determined by using the compensation approach. Analyzing the ratios of the equilibrium probabilities learns that the compensation approach is not appropriate for the symmetric shortest queue problem with three queues, but it may be appropriate for the 2×3 buffered switch. In section 4 we confirm our conjecture: by using the compensation approach we show that for the 2×3 buffered switch the equilibrium distribution can be written as a linear combination of six series of binary trees of product forms. Finally, in section 5 we present the analogue of condition (iii) for three or more dimensional random walks.

2. The symmetric shortest queue system with three queues

Consider the shortest queue system with three identical parallel servers, each with exponentionally distributed service times with parameter 1. At this system jobs arrive according to a Poisson stream with intensity 3ρ, $0 < \rho < 1$, (ρ is the average workload) and on arrival a job always joins the shortest queue. The behavior of this symmetric shortest queue system is described by a Markov process with states (m,n,r), where m denotes the number of jobs in the shortest queue (including the job being served), n denotes the difference between the second shortest and the shortest queue and r denotes the difference between the longest and the second shortest queue. This Markov process is a semi-homogeneous, ergodic random walk with transitions to neighbouring states only. For later reference, we list the transition rates $q_{i,j,k}$ for the interior points:

$$q_{1,-1,0} = 3\rho , \quad q_{-1,1,0} = q_{0,-1,1} = q_{0,0,-1} = 1 , \tag{2}$$

and $q_{i,j,k} = 0$ for all other directions (i,j,k), $i,j,k \in \{-1,0,1\}$.

Let $\{p_{m,n,r}\}$ be the equilibrium distribution of the Markov process. By truncating the state space and using successive approximations to compute the solution of the equilibrium equations of the truncated process, we get accurate approximations for the equilibrium probabilities $p_{m,n,r}$ with m, n and r not too large. For the case $\rho = 0.8$, we used this technique to compute the ratios of the equilibrium

ρ=0.8	r=0				r=1				r=2				r=3			
n	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
m=0	1.55	0.64	0.48	0.46	0.82	0.49	0.46	0.46	0.48	0.44	0.45	0.46	0.45	0.44	0.46	0.46
1	0.62	0.53	0.51	0.50	0.55	0.50	0.50	0.50	0.48	0.49	0.50	0.50	0.47	0.49	0.50	0.50
2	0.53	0.51	0.51	0.51	0.52	0.51	0.51	0.51	0.50	0.51	0.51	0.51	0.50	0.51	0.51	0.51
3	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51
4	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51
5	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51	0.51

Table 1. The ratios $p_{m+1,n,r}/p_{m,n,r}$ for the 3-dimensional symmetric shortest queue system.

ρ=0.8	r=0				r=1				r=2				r=3			
m	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
n=0	0.89	0.37	0.32	0.31	0.28	0.17	0.15	0.15	0.20	0.18	0.18	0.19	0.19	0.18	0.19	0.19
1	0.16	0.12	0.12	0.11	0.13	0.12	0.12	0.12	0.14	0.14	0.14	0.14	0.14	0.15	0.15	0.15
2	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14
3	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.13	0.13	0.13	0.13	0.14	0.14	0.14	0.14
4	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.13	0.13	0.13	0.13	0.14	0.14	0.14	0.14
5	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.13	0.13	0.13	0.13	0.14	0.14	0.14	0.14

Table 2. The ratios $p_{m,n+1,r}/p_{m,n,r}$ for the 3-dimensional symmetric shortest queue system.

ρ=0.8	n=0				n=1				n=2				n=3			
m	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
r=0	0.80	0.42	0.37	0.36	0.25	0.19	0.18	0.18	0.21	0.20	0.19	0.19	0.21	0.21	0.21	0.21
1	0.17	0.10	0.09	0.09	0.12	0.11	0.11	0.11	0.12	0.12	0.12	0.12	0.13	0.13	0.13	0.13
2	0.12	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12
3	0.11	0.11	0.11	0.11	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12
4	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12
5	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12	0.12

Table 3. The ratios $p_{m,n,r+1}/p_{m,n,r}$ for the 3-dimensional symmetric shortest queue system.

probabilities, which are listed in the tables 1, 2 and 3. As we see in table 1, the sequences $p_{m+1,n,r}/p_{m,n,r}$ have the same limit for all n and r; and similarly for the sequences $p_{m,n,r+1}/p_{m,n,r}$ (see table 3). However, for the sequences $p_{m,n+1,r}/p_{m,n,r}$ the limit seems to increase if r increases (see table 2). For other workloads ρ the ratios of the probabilities have the same behavior. This behavior suggests that for the three-dimensional symmetric shortest queue system the equilibrium distribution probably is not a linear combination of product forms and therefore the compensation approach is probably not appropriate for this system.

3. The 2×3 clocked buffered switch

Buffered switches are being used in parallel data processing networks to route messages and to resolve conflicts. A $k \times s$ clocked buffered switch is a switching system with k input and s output ports. Such a system is modeled as a discrete time queueing system with s parallel servers and k types of arriving jobs. The 2×2 buffered switch has been succesfully analyzed by using the compensation approach (see [3]). By using a uniformization technique, Jaffe [4] analyzed the *symmetric* 2×2 switch (i.e. a switch with identical arrival rates for the two types of jobs and identical routing probabilities). In this section and the next one, we consider the 2×3 switch. This system is not very interesting from a practical point of view, however, it surely is from a mathematical point of view, since the 2×3 switch is described by a three-dimensional random walk for which the compensation approach works. To shorten the analysis, we restrict ourselves to the *symmetric* case. All results can be extended to the asymmetric case.

The symmetric 2×3 clocked buffered switch is modeled as a discrete time queueing system with 3 parallel servers and 2 types of arriving jobs (see figure 1). Jobs of type i, $i = 1,2$, arrive according to a Bernoulli stream with rate p, $0 < p \leq 1$, i.e. every time unit (clock cycle) the number of arriving jobs of type i is one with probability p and zero with probability $1-p$. Jobs always arrive at the beginning of a time unit and once a job of type i has arrived, it joins the queue at server j with probability 1/3, $j = 1,2,3$. A server serves exactly one job per time unit, if one present. Jobs are served according to FCFS service discipline.

The behavior of the 2×3 switch is described by an irreducible, aperiodic Markov chain with states (m,n,r), $m,n,r \geq 0$, where m, n and r denote the numbers of waiting jobs at the servers 1, 2 and 3 at the end of a time unit. Again, we have a semi-homogeneous random walk with transitions to neighbouring states only. The fact that we have discrete instead of continuous time does not affect the analysis of the equilibrium distribution. The non-zero transition probabilities $q_{i,j,k}$ for the interior points are given by:

$$q_{1,-1,-1} = q_{-1,1,-1} = q_{-1,-1,1} = p^2/9, \quad q_{-1,0,0} = q_{0,-1,0} = q_{0,0,-1} = 2p^2/9,$$
$$q_{-1,-1,0} = q_{-1,0,-1} = q_{0,-1,-1} = 2p(1-p)/3, \quad q_{-1,-1,-1} = (1-p)^2. \tag{3}$$

For the other states each transition probability can be written as a sum of the probabilities $q_{i,j,k}$. The sets of transitions at the boundary planes, at the axes and at the origin are "projections" of the set of transitions in the interior. For example, for a state $(m,n,0)$, $m,n > 0$, at the boundary plane $r = 0$ the transition probability for each direction $(i,j,1)$ equals $q_{i,j,1}$ and the transition probability for each direction $(i,j,0)$ equals $q_{i,j,0} + q_{i,j,-1}$. This remarkable property is called the *projection property*. Due to this property we can easily derive formulae for the equilibrium (marginal) distributions of the component (projected) chains.

Let $\{p_{m,n,r}\}$, $\{p_m^{(1)}\}$, $\{p_n^{(2)}\}$ and $\{p_r^{(3)}\}$ be the equilibrium distributions of the full Markov chain and the three component chains, which describe the numbers of jobs present at one particular server. By symmetry we have

$$p_{m,n,r} = p_{m,r,n} = p_{n,m,r} = p_{n,r,m} = p_{r,m,n} = p_{r,n,m}, \quad m \geq 0,\ n \geq 0,\ r \geq 0, \tag{4}$$

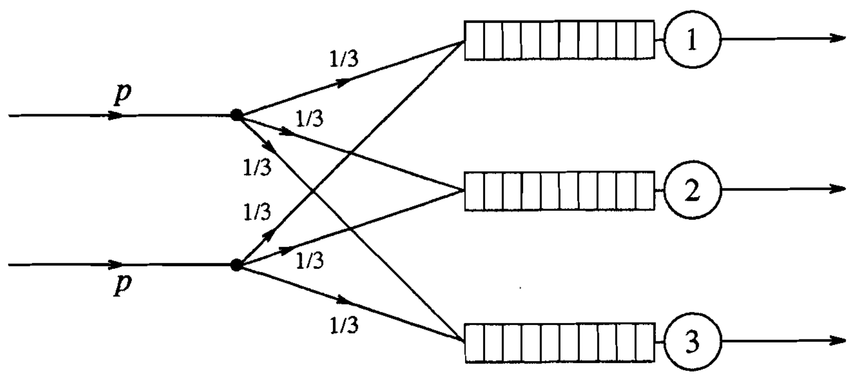

Figure 1. The 2×3 buffered switch.

$p=1$	$r=0$				$r=1$				$r=2$				$r=3$			
n	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
$m=0$	0.50	0.09	0.05	0.04	0.09	0.00	0.00	0.00	0.05	0.00	0.00	0.00	0.04	0.00	0.00	0.00
1	0.27	0.15	0.07	0.05	0.15	0.10	0.04	0.02	0.07	0.04	0.00	0.00	0.05	0.02	0.00	0.00
2	0.25	0.22	0.15	0.07	0.22	0.22	0.14	0.07	0.15	0.14	0.10	0.04	0.07	0.07	0.04	0.00
3	0.25	0.24	0.22	0.15	0.24	0.24	0.22	0.14	0.22	0.22	0.22	0.14	0.15	0.14	0.14	0.10
4	0.25	0.25	0.24	0.22	0.25	0.25	0.24	0.22	0.24	0.24	0.24	0.22	0.22	0.22	0.22	0.22
5	0.25	0.25	0.25	0.24	0.25	0.25	0.25	0.24	0.25	0.25	0.25	0.24	0.24	0.24	0.24	0.24
6	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
7	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25

Table 4. The ratios $p_{m+1,n,r}/p_{m,n,r}$ for the symmetric 2×3 buffered switch.

and the distributions $\{p_m^{(1)}\}$, $\{p_n^{(2)}\}$ and $\{p_r^{(3)}\}$, which are marginal distributions of $\{p_{m,n,r}\}$, are identical. Each component chain is a random walks on $\mathbb{Z}_+$ with a negative drift: for all $i \geq 0$ the transition probability for a transition from i to $i+1$ is equal to $(p/3)^2$, while the probability for the opposite transition is equal to $(1-p/3)^2$, which is larger than $(p/3)^2$. This implies that each component chain is ergodic and we obtain the following geometric distribution for $\{p_m^{(1)}\}$, $\{p_n^{(2)}\}$ and $\{p_r^{(3)}\}$:

$$p_m^{(1)} = p_m^{(2)} = p_m^{(3)} = \left[1 - \frac{(p/3)^2}{(1-p/3)^2}\right]\left[\frac{(p/3)^2}{(1-p/3)^2}\right]^m, \quad m \geq 0. \tag{5}$$

Since all component chains are ergodic, the full Markov chain is also ergodic and its equilibrium distribution $\{p_{m,n,r}\}$ is the unique solution of the equilibrium equations and the normalizing equation.

In the same way as for the symmetric shortest queue system, we computed the ratios of the equilibrium probabilities to check whether the compensation approach may be appropriate to find $\{p_{m,n,r}\}$. BY symmetry, we only have to analyze the ratios $p_{m+1,n,r}/p_{m,n,r}$. For the case $p=1$ these ratios are listed in table 4. As we see, for all fixed n and r the sequence $p_{m+1,n,r}/p_{m,n,r}$ has limit 0.25. This limit is reached very fast as soon as $m > \max[n,r]$. Notice that the compensation approach may be appropriate to determine the equilibrium distribution. Finally, we remark that the limit 0.25 is equal to the factor of the geometric distribution for the component chains (see (5)). The factor of the geometric distribution in (5) appears to be the maximum α-, β- and γ-factor in the linear combination of product forms which is found for the equilibrium distribution $\{p_{m,n,r}\}$ in the next section.

4. The compensation approach for the 2×3 buffered switch

By using the compensation approach for the 2×3 buffered switch, we generate binary trees of product forms, which are called *formal solutions*. The equilibrium distribution $\{p_{m,n,r}\}$ appears to consist of an infinite linear combination of such formal solutions. The main ideas with respect to the compensation approach are explained below.

A formal solution consists of an infinite linear combination of product forms $\alpha^m\beta^n\gamma^r$, which all are solutions of the equilibrium equations for the interior of the state space. We can restrict ourselves to product forms with real factors α, β and γ with $0<\alpha,\beta,\gamma<1$. Substituting a product form in the equilibrium equation for the interior shows that each product form is a solution of the following *quadratic equation*:

$$\alpha\beta\gamma = q_{1,-1,-1}\,\beta^2\gamma^2 + q_{-1,1,-1}\,\alpha^2\gamma^2 + q_{-1,-1,1}\,\alpha^2\beta^2 + q_{-1,0,0}\,\alpha^2\beta\gamma + q_{0,-1,0}\,\alpha\beta^2\gamma + q_{0,0,-1}\,\alpha\beta\gamma^2$$
$$+ q_{0,-1,-1}\,\alpha\beta^2\gamma^2 + q_{-1,0,-1}\,\alpha^2\beta\gamma^2 + q_{-1,-1,0}\,\alpha^2\beta^2\gamma + q_{-1,-1,-1}\,\alpha^2\beta^2\gamma^2\,. \qquad (6)$$

The first term of a formal solution, called the *start term*, is chosen such that it also satisfies the equilibrium equations for one of the boundaries $m=0$, $n=0$ and $r=0$. The other terms are called *compensation terms*, since they are added to compensate for an error of a previous term on of the three boundaries.

By symmetry we can restrict ourselves to formal solutions starting on the boundary $r=0$. Let us consider the construction of such a formal solution, denoted by $x_{m,n,r}(\alpha,\beta,\gamma)$. Here α, β and γ are the factors of the start term, which we want to satisfy the equilibrium equations for the interior and the boundary $r=0$. It appears that α and β have to be chosen such that (α,β) satisfies the quadratic equation (6) for fixed $\gamma=1$ (i.e. such that (α,β) satisfies the equilibrium equation for the interior points of the Markov chain that describes the joint behavior of the queues at the servers 1 and 2). Further, we have to set $\gamma=f(\alpha,\beta)$, where

$$f(x,y) := \frac{q_{1,-1,-1}\,x^2y^2}{q_{-1,1,-1}\,y^2 + q_{-1,-1,1}\,x^2 + q_{-1,0,0}\,xy + q_{-1,0,-1}\,xy^2 + q_{-1,-1,0}\,x^2y + q_{-1,-1,-1}\,x^2y^2}\,. \qquad (7)$$

For fixed α and β, $f(\alpha,\beta)$ represents the product of the two solutions for γ of the quadratic equation (6); similarly for fixed α and γ and for fixed β and γ (use the symmetry to prove this).

For the start term of the formal solution $x_{m,n,r}(\alpha,\beta,\gamma)$ we take $c_1\alpha_1^m\beta_1^n\gamma_1^r$ with $\alpha_1=\alpha$, $\beta_1=\beta$, $\gamma_1=\gamma$ and $c_1=(1-\alpha_1)(1-\beta_1)(1-\gamma_1)$. This start term satisfies the equilibrium equations for the interior and the boundary $r=0$, but it violates the equations for the boundaries $n=0$ and $r=0$. To compensate for the errors on these two boundaries, we add two compensation terms to the initial term. We add $c_2\alpha_2^m\beta_2^n\gamma_2^r$ such that this second term, together with the first term, satisfies the equilibrium equations for the interior and the boundary $m=0$. It appears that the choice $\alpha_2=f(\beta_1,\gamma_1)/\alpha_1$, $\beta_2=\beta_1$, $\gamma_2=\gamma_1$, $c_2=-c_1(1-\alpha_2)/(1-\alpha_1)$ (so $c_2=-(1-\alpha_2)(1-\beta_2)(1-\gamma_2)$) suffices. For the compensation on the boundary

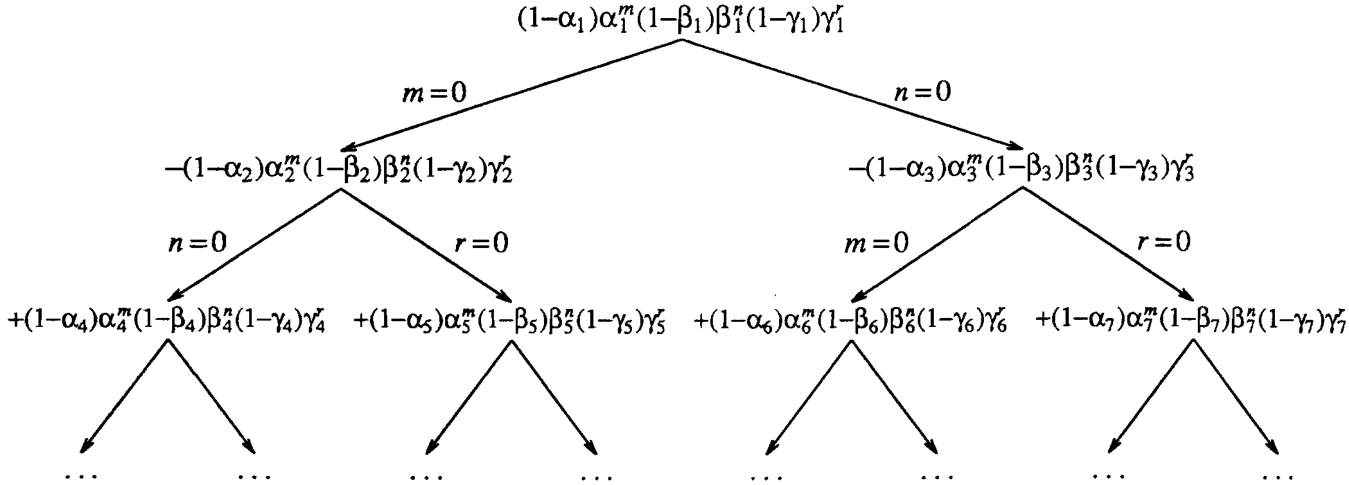

Figure 2. The construction of a binary tree of product forms starting on the boundary $r=0$.

$n=0$, we add the term $c_3\alpha_3^m\beta_3^n\gamma_3^r$ with $\alpha_3=\alpha_1$, $\beta_3=f(\alpha_1,\gamma_1)/\beta_1$, $\gamma_3=\gamma_1$ and $c_3=-c_1(1-\beta_2)/(1-\beta_1)$. The second term as well as the third term compensates an error of the first term, but they both introduce two new errors: the second term on the boundaries $n=0$ and $r=0$ and the third term on the boundaries $m=0$ and $r=0$. Just as for the first term, each compensation term needs two new compensation terms. Repeating the above procedure leads to the construction of the binary tree pictured in figure 2.

For all α, β and γ, the binary tree $x_{m,n,r}(\alpha,\beta,\gamma)$ appears to be absolutely convergent in all states (m,n,r) with $\delta(m)+\delta(n)+\delta(r)\geq 2$. Here, $\delta(i):=0$ for $i=0$ and $\delta(i):=1$ for all $i\geq 1$. The following theorem states that $\{p_{m,n,r}\}$ is the sum of six series of binary trees of product forms.

Theorem. *For all $m\geq 0$, $n\geq 0$, $r\geq 0$, $\delta(m)+\delta(n)+\delta(r)\geq 2$, define*

$$y_{m,n,r} := \sum_{i=1}^{\infty}(-1)^{i+1}\, x_{m,n,r}(\tilde{\alpha}_i,\tilde{\beta}_i,\tilde{\gamma}_i) \ , \tag{8}$$

where $\tilde{\alpha}_1=f(1,1)=(p/3)^2/(1-p/3)^2$, $\tilde{\beta}_1=f(\tilde{\alpha}_1,1)$, $\tilde{\gamma}_1=f(\tilde{\alpha}_1,\tilde{\beta}_1)$ *and for all $i\geq 2$,*

$$\tilde{\alpha}_i=f(\tilde{\beta}_{i-1},1)/\tilde{\alpha}_{i-1}\ ,\tilde{\beta}_i=\tilde{\beta}_{i-1}\ ,\tilde{\gamma}_i=f(\tilde{\alpha}_i,\tilde{\beta}_i)\ , \quad \text{if } i \text{ even},$$

$$\tilde{\alpha}_i=\tilde{\alpha}_{i-1}\ ,\tilde{\beta}_i=f(\tilde{\alpha}_{i-1},1)/\tilde{\beta}_{i-1}\ ,\tilde{\gamma}_i=f(\tilde{\alpha}_i,\tilde{\beta}_i)\ , \quad \text{if } i \text{ odd}.$$

Then

$$p_{m,n,r} = y_{m,n,r}+y_{m,r,n}+y_{n,m,r}+y_{n,r,m}+y_{r,m,n}+y_{r,n,m}\ , \quad m,n,r\geq 0,\ \delta(m)+\delta(n)+\delta(r)\geq 2, \tag{9}$$

$$p_{m,0,0} = p_{0,m,0} = p_{0,0,m} = (1-\tilde{\alpha}_1)\tilde{\alpha}_1^m - \sum_{\substack{n\geq 0,\ r\geq 0\\ n+r\geq 1}} p_{m,n,r}\ , \quad m\geq 1, \tag{10}$$

$$p_{0,0,0} = 1-\sum_{\substack{m\geq 0,\ n\geq 0,\ r\geq 0\\ m+n+r\geq 1}} p_{m,n,r}\ . \tag{11}$$

The theorem can be proved by showing that the solution defined by (9)-(11) satisfies all equilibrium equations, for which one has to use the fact that the solution in the theorem is defined in such a way that it satisfies the expressions for the two-dimensional marginal distributions (see also [3], in which the Main Theorem has been proved along the same lines).

5. Condition for applying the compensation approach to three or more dimensional random walks

Consider an ergodic, N-dimensional, semi-homogeneous random walk with transitions to neighbouring states only. Let the transition rates for the interior points be given by $q_{i_1,\ldots,i_N}$. We would like to know for which cases the compensation approach works. The compensation approach that we described for the 2×3 buffered switch can be extended to N-dimensional random walks which satisfy the projection property and the following property:

$$q_{i_1,\ldots,i_N} > 0 \;\Rightarrow\; i_k + i_l \le 0 \;\text{ for all } k,l \in \{1,\ldots,N\},\; k \ne l. \tag{12}$$

So, for an N-dimensional random walk with the projection property, condition (12) is sufficient. This condition appears to be also necessary if an infinite number of product forms is needed to construct the equilibrium distribution. Condition (12) originates from the requirement that the compensation approach also has to work for each two-dimensional marginal distribution. For the case $N=2$, condition (12) is equivalent with condition (1). Unfortunately, condition (12) is rather severe; if $q_{i_1,\ldots,i_N} > 0$, then $i_k = 1$ implies $i_l = -1$ for all $l \ne k$. For an N-dimensional random walk *without* the projection property, condition (12) will also be necessary (and maybe also sufficient), if an infinite number of product forms is needed. It follows from (2) that the three-dimensional symmetric shortest queue system violates condition (12), which confirms the suspicion stated at the end of section 3.

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3. BOXMA, O.J. AND VAN HOUTUM, G.J., "The compensation approach applied to a 2x2 switch ," Memorandum COSOR, Eindhoven University of Technology, Dept. of Math. and Comp. Sci., 1992. Submitted for publication.
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92-01	January	F.W. Steutel	On the addition of log-convex functions and sequences
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