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The main lines of two methods for the determination of the buckling current of superconducting structures under prescribed current are presented. Applications of both methods will be given for systems of parallel rods and rings, and their results will be compared. Finally, the stability of a superconducting helix is investigated.

INTRODUCTION

Consider a superconducting slender body (beam-like), or a system of such bodies, carrying a prescribed current I_0, placed in a vacuum. Above a certain value of I_0 the system will become unstable and buckle. This buckling is solely due to the own fields of the conductors (there is no external magnetic field). In this paper we shortly present the main lines of two methods for the calculation of the buckling current (for the details we refer to [1], [2] and [3]). The first method is based upon a variational principle, whereas the second one starts from a formula for the (Lorentz-)force on a slender beam-like current carrier in the magnetic field of another electric circuit. This formula is derived from the Biot-Savart law (cf. [4], Sec. 2.6), and therefore we refer to this method as the Biot-Savart method.

Both methods will be applied in the calculation of the buckling current of superconducting structures which are systems of slender, straight or curved, beams. In the variational method it is assumed that the current runs over the surface of the body, while in the Biot-Savart-approach the distribution of the current over the cross-section of the beam-like structure is unspecified (and irrelevant). That the total current is prescribed can be expressed by means of Ampère's law, i.e.

$$\int_C (\mathbf{B},\tau)\, ds = \mu_0 I_0 \,, \tag{1}$$

where $\mathbf{B}$ is the magnetic induction in vacuum, C is a contour entirely in the vacuum and encircling the current carrier, and τ is the tangent vector at C. In the variational method this relation serves as an extra constraint.

VARIATIONAL PRINCIPLE

Our variational method is based upon the following Lagrangian for a superconducting body in vacuum, defined on the deformed configuration of the body, (cf. [3])

$$L = -\int_{G^-} \rho\, U\, dV + \frac{1}{2\mu_0} \int_{G^+} (\mathbf{B},\mathbf{B})\, dV\,, \tag{2}$$

where the first integral represents the elastic energy of the body and the second one is the magnetic energy of the field in the vacuum surrounding the body. In (2), G^- and G^+ are the deformed configuration (in $\mathbb{R}^3$) of the body and the vacuum, respectively, and $\rho\, U$ is the elastic energy density ($\mu_0 = 4\pi \times 10^{-7}$). In accordance with (1) we assume $\mathbf{B}$ proportional to I_0, implying that we may introduce a normalized **B**-field by

$$\hat{\mathbf{B}} = \frac{1}{I_0}\, \mathbf{B}\,, \tag{3}$$

such that $\hat{\mathbf{B}}$ is independent of I_0. Then, (2) can be rewritten as

$$L = -\int_{G^-} \rho\, U\, dV + \frac{I_0^2}{2\mu_0} \int_{G^+} (\hat{\mathbf{B}}, \hat{\mathbf{B}})\, dV\,, \tag{4}$$

where, now, the two integrals in this expression are independent of I_0.
The body deformes due to the action of the Lorentz-force generated by the prescribed current I_0. For stability considerations we have to distinguish between two different equilibrium states of the body, i.e. (i) the intermediate state, and (ii) the final (or buckled) state. The intermediate state may be approximated by the rigid-body state; the only unknown in this state then is the rigid-body magnetic field denoted by $\mathbf{B}_0$. In the final state the unknowns are the displacement $\mathbf{u}$ and the magnetic field $\mathbf{B}$, which is written as $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ (hence, $\mathbf{b}$ is the perturbation of the magnetic field due to the buckling of the structure).
The Lagrangian given by (4) is developed up to the second order in the perturbations $\mathbf{u}$ and $\mathbf{b}$. Variation of L with respect to these perturbations then yields

$$L = L^{(0)} + \delta L + \tfrac{1}{2}\, \delta^2 L + \cdots \tag{5}$$

Since the intermediate state is an equilibrium state the first variation of L must be zero, so

$$\delta L = 0. \tag{6}$$

Moreover, the zeroth order term $L^{(0)}$ is irrelevant, so we may write (5) as

$$L = \tfrac{1}{2}\, \delta^2 L =: J(\mathbf{b},\mathbf{u})\,, \tag{7}$$

where J is a homogeneously quadratic functional in the perturbations. Using the fact that also the final state is an equilibrium state, we find from (7) that

$$\delta J = 0\,, \tag{8}$$

too, but because J is homogeneously quadratic this implies

$$J = 0\,, \tag{9}$$

in this final state. In analogy with (4) we may write J in the simple form

$$J = -W + I_0^2\, K\,, \tag{10}$$

where the integrals W and K do not depend on I_0. Here, W is the total elastic energy in the final state, which only depends on $\mathbf{u}$ (or better $\nabla \mathbf{u}$), and K is the normalized magnetic energy, depending on $\mathbf{u}$, $\hat{\mathbf{B}}_0$

and $\hat{\mathbf{b}}$. From (9) and (10) we immediately conclude that either $W = K = 0$, implying that $\mathbf{u} = \mathbf{b} = \mathbf{0}$ too, or $(\mathbf{u},\mathbf{b}) \neq (\mathbf{0},\mathbf{0})$ and then I_0 is an eigenvalue of (9)-(10), obeying

$$I_0 = \sqrt{\frac{W}{K}} . \tag{11}$$

The lowest of these eigenvalues is the looked for buckling current.
The equation (11) gives an explicit relation for the buckling current. However, the main part of the work lies in the determination of K for which the fields $\mathbf{B}_0$ and $\mathbf{b}$ under an assumed displacement field $\mathbf{u}$ must be calculated. How this works in practice is shown in [2] and [3].

BIOT-SAVART METHOD

Two electrical circuits L_1 and L_2, to be considered as one-dimensional curves, carry currents I_1 and I_2, respectively. According to [4], Section 2.6, the Lorentz-force per unit of length in a point P_1 of L_1, having arc length s_1, is given by

$$\mathbf{F}_1(s_1) = \frac{\mu_0 I_1 I_2}{4\pi} \int_{L_2} \frac{(\mathbf{t}_1 \times (\mathbf{t}_2 \times \mathbf{R}))}{R^3} \, ds_2 , \tag{12}$$

where $\mathbf{R}$ is the position vector of P_1 with respect to the point P_2 on L_2, arc length s_2, and $\mathbf{t}_1 = \mathbf{t}_1(s_1)$ and $\mathbf{t}_2 = \mathbf{t}_2(s_2)$ are unit tangent vectors in P_1 on L_1 and in P_2 on L_2, respectively.
Let this formula refer to the deformed state of the system $\{L_1,L_2\}$. Introducing the (small) displacement $\mathbf{u}$, we can develop $\mathbf{F}_1$ with respect to $\mathbf{u}$, obtaining

$$\mathbf{F}_1(s_1) = \mathbf{F}_1^{(0)}(s_1) + \mathbf{f}_1(s_1) , \tag{13}$$

where $\mathbf{f}_1(s_1)$ is linear in $\mathbf{u}$ (it is only this part of the force that is of relevance in stability calculations). Once the force $\mathbf{f}_1$ is calculated it can serve as a load parameter in, for instance, a beam or ring equation (dependent on the form of the circuit). For detailed evaluations we again refer to [2] or [3].

Our ultimate aim is to investigate in how far one, or both, of these methods can be employed for the calculation of the buckling current in *one* electrical circuit in the shape of a helix. In principle, the Biot-Savart method can not be used for this, since it only applies to actions from one circuit on another circuit. The variational method can serve fairly well as a way to solution, but it then turns out that the (analytical) calculation of K becomes very difficult. Therefore, we suggest to follow a combined path, in the sense that we shall try to calculate (an approximation for) K from an expression for the electromagnetic force derived by means of the Biot-Savart formula (12) and then use (11) to obtain the desired buckling current I_0. However, before we can proceed along this path, we need some auxiliary results which will provide us with a more or less solid fundament for our approach. To this end we first report on some results derived by us for systems of n ($n \geq 2$) parallel rods or rings (for $n = 2$, see [2] and [3], for $n > 2$ we refer to the forthcoming paper [5]).

SYSTEMS OF N PARALLEL RODS OR RINGS

Consider an equidistant grid of n identical parallel rods L_i, $i = 1,2,...,n$, all carrying the same current I_0.

The rods are infinitely long, but periodically supported over finite distances l. Each rod has a circular cross-section, radius R, and the distance between the central lines of two adjacent rods is $2a$. The system is called a slender system in the sense that $R < a \ll l$.
For the variational method the unperturbed field $\mathbf{B}_0$ and the perturbed field $\mathbf{b}$ must be calculated. For $n = 2$ a completely analytical solution is found (see [2]), but for $n > 2$ this seems no longer possible. Therefore, for $n > 2$ we have developed a numerical procedure that will be presented in [5]. There, we obtained numerical results for the buckling current I_0 in case $n = 3$, 4 and 5 and for $a/R = 3$. These results will now be compared with those of the Biot-Savart method. Although the theoretical foundation of the latter method is less firm than that of the variational method, all results derived thus far by both methods show a satisfactory correspondence. For the system under consideration the Biot-Savart approach is very simple and straightforward. This is mainly due to the assumption of slenderness.
Let the z-axis be parallel to the rods and the x-axis perpendicular to it (all rods lying in the $x-z$-plane), and let $u_i(z)$ be the displacement in x-direction of a point of L_i with coordinate z. Moreover, let $f_i^{(j)}(z)$ be the force per unit of length normal to the rod L_i due to the current in L_{i+j} ($j \neq 0$). For a slender system, i.e. under the neglect of terms of $O(a^2/l^2)$, a rather simple analysis (cf. [2] or [5]) shows that

$$f_i^{(j)}(z) = \frac{k_1}{j^2}\left[u_i(z) - u_{i+j}(z)\right], \tag{14}$$

with

$$k_1 = \frac{\mu_0 I_0^2}{8\pi a^2}. \tag{15}$$

Hence, we see that in the perturbed force on L_i in z only the displacement of that point of L_{i+j} that has the same z-coordinate enters (note that this is the point on L_{i+j} that is nearest to the considered point z on L_i). The total force $f_i(z)$ per unit of length on L_i due to the remaining rods $L_{i+j} (j \neq 0)$ of the grid is equal to the sum of $f_i^{(j)}$ over all possible values of j, i.e.

$$f_i(z) = \sum_j f_i^{(j)}(z). \tag{16}$$

For L_i the beam equation (EI is the bending stiffness)

$$EI\, u_i^{iv}(z) = f_i(z), \tag{17}$$

holds, and similar relations hold for the other rods. Together with the support conditions these relations constitute a system of n homogeneous equations for the n unknown displacements u_i. In this way the problem is reduced to an eigenvalue problem: only for a set of discrete values of k_1 the system has a non-trivial solution and the lowest of these eigenvalues corresponds to the buckling current.
We have calculated buckling values for I_0 by both methods for $n = 2,3,4$ and 5 and for $a/R = 3$. The results, which can be found in [2] and [5], are listed in Table 1. Here, i_0 is defined by

$$I_0 = i_0 \frac{\pi^3 R^3}{l^2}\sqrt{\frac{E}{\mu_0}}, \tag{18}$$

and $(i_0)_{VP}$ and $(i_0)_{BS}$ are the values according to the variational and the Biot-Savart method, respectively, while $\Delta = [(i_0)_{VP} - (i_0)_{BS}] / (i_0)_{VP}$.

TABLE 1. Values of i_0 and Δ for $a/R = 3$

n	$(i_0)_{VP}$	$(i_0)_{BS}$	Δ in %
2	3.274	3.000	8,4
3	2.679	2.449	8,6
4	2.469	2.260	8,5
5	2.364	2.168	8,3

Table 1 shows us that the relative differences Δ are nearly the same for all values of n, and of the order of 8 % for $a/R = 3$. From [2] we know that this difference rapidly decreases with increasing a/R (e.g. $\Delta = 2$ % for $a/R = 6$, and $\Delta < 1$ % for $a/R = 10$). In our opinion these differences are small enough to justify the approximations that will be proposed below (at least as long as $(a/R) \geq 3$). To obtain a useful approximation for K we shall write the Biot-Savart method in variational form. To this end we multiply (17) by $u_i(z)$ (automatically summing over i from $i = 1$ to n) and then integrate the result from $z = 0$ to $2l$ (one full period). After two partial integrations we thus arrive at

$$W := EI \int_0^{2l} u_i''(z)\, u_i''(z)\, dz = \int_0^{2l} f_i(z)\, u_i(z)\, dz =: I_0^2 K \, . \tag{19}$$

The left-hand side of (19) is (two times) the elastic energy of the system. Comparing (19) with (9)-(10) we see that the right-hand side of (19) represents, apart from a for the moment irrelevant constant factor, the Biot-Savart version of the K-integral occurring in (10). Since the correspondence between the variational method and the Biot-Savart method is satisfactory (as is shown above) it seems acceptable to use the right-hand side of (19) as an, easily obtainable, approximation for K in (10) or (11).

As a next step we compare the value of the buckling current for a set of two rods (i.e. $(I_0)_{ro}$), as obtained in [2], with the corresponding result for a set of two (parallel or coaxial) rings (i.e. $(I_0)_{ri}$) from [3]. It turns out that the variational method and the Biot-Savart method yield the same value for the quotient $(I_0)_{ro} \,/\, (I_0)_{ri}$. Moreover, a detailed consideration of the variational method showed us that:

the value of K for a set of two rings was equal to the one for a set of two parallel rods.

Further calculations revealed that this also holds for $n > 2$. The consequence of this finding is that in calculating K for a set of n parallel rings we may replace the rings by an equivalent (i.e. equal a/R-ratio) set of parallel rods. Of course, this only holds for K; the values of the elastic energies are different.

In recapitulating the results of this section we conclude that (i) for slender systems (i.e. up to $O(a^2/R^2)$-terms) the values of K for equivalent systems of parallel rods and rings are equal, and (ii) the value of K obtained by a Biot-Savart approach (i.e. the right-hand side of (19)) is a reasonable approximation for the exact K associated with the variational method, at least as long as a/R is not too small (say $a/R \geq 3$) (see also the CONCLUSIONS at the end of this paper).

THE HELIX

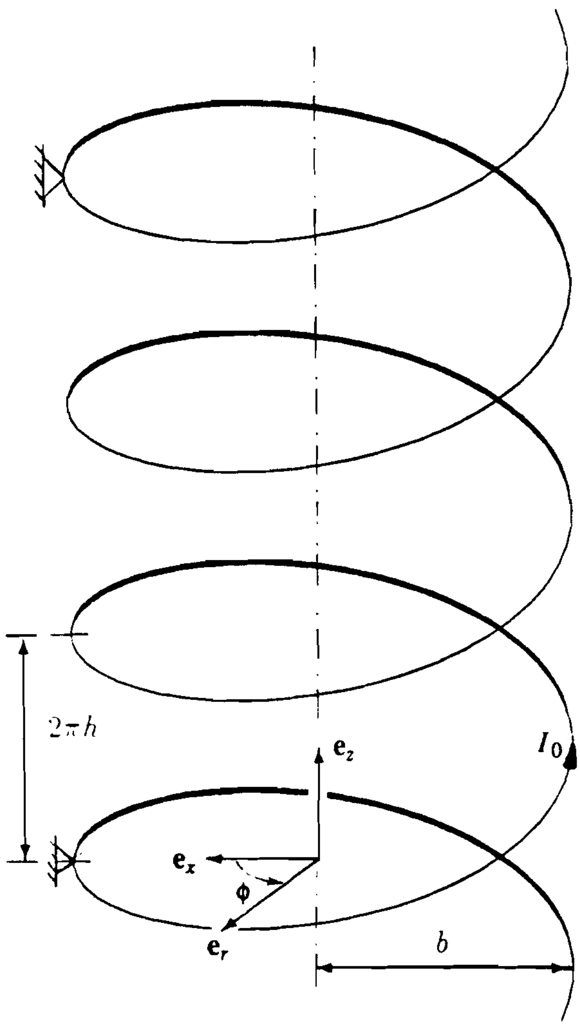

Fig. 1. The helical conductor

Consider an infinitely long superconductor in the form of a cylindrical helix. The radius of the helix is b and the (constant) pitch is h. For a slender helix one has $h \ll b$. In the undeformed configuration a point on the central line of the helix is given by its position vector (see Fig. 1)

$$\mathbf{X} = \mathbf{X}(\phi) = b \cos\phi\, \mathbf{e}_x + b \sin\phi\, \mathbf{e}_y + h\, \phi\, \mathbf{e}_z$$
$$= b\, \mathbf{e}_r + h\, \phi\, \mathbf{e}_z\,, \quad (-\infty < \phi < \infty)\,. \tag{20}$$

The cross-section of the conductor is circular, radius R ($R < \pi h \ll b$). The total current I_0 running through the conductor is prescribed. The helix is periodically supported in the points given by $\phi = 0$, $\phi = \pm 2n\pi$, $\phi = \pm 4n\pi$, etc.

In case the helix buckles the main component of the buckling displacement $\mathbf{u}$ will be binormal, or what is here nearly the same, in the $\mathbf{e}_z$-direction (this is due to the fact that the pitch angle $\alpha \approx b/h \ll 1$). The perturbed force density, which is caused by $\mathbf{u}$ and linear in $\mathbf{u}$, will then also be in that direction. Thus

$$\mathbf{f}(\phi) = f(\phi)\, \mathbf{e}_z\,, \quad \mathbf{u}(\phi) = w(\phi)\, \mathbf{e}_z\,. \tag{21}$$

Here, $w(\phi)$ is the displacement of the central line of the helical conductor which causes besides a bending also a torsion of the helix. However, this torsion does not affect the force $\mathbf{f}$; it will only enter the elastic energy which will be derived furtheron.

Our strategy for the solution of this buckling problem is as follows:

(i) We calculate K by using for K the expression on the right-hand side of (19) together with an appropriate expression for f following from (14)-(16). For the calculation of f for the present problem we have to replace in (14) $u_i(z)$ by a field $w(\phi)$, representing the expected buckling mode of the helix.

(ii) We calculate the elastic energy W, which is due to the bending and the torsion of the helix.

(iii) By variation of J according to (10) with respect to the torsion angle $\beta(\phi)$ we obtain a relation expressing $\beta(\phi)$ in $w(\phi)$.

(iv) Finally, from (11) the buckling current I_0 is calculated.

An appropriate choice for the buckling displacement, satisfying the periodic support conditions, is

$$w(\phi) = A \sin\left[\frac{\phi}{2n}\right]\,. \tag{22}$$

With the replacement of $u_{i+j}(z)$ by

$$w(\phi+2\pi j) = A \sin\left[\frac{\phi}{2n} + \frac{\pi}{n} j\right] , \tag{23}$$

for $j \in I\!N$, we find from (14)-(16) the following expression for f

$$f(\phi) = 2\, k_1 A \sum_{j=1}^{\infty} \frac{1}{j^2} \left[1 - \cos\left[\frac{\pi}{n} j\right]\right] \sin\left[\frac{\phi}{2n}\right] =$$

$$= \frac{k_1 \pi^2}{n} \left[1 - \frac{1}{2n^2}\right] A \sin\left[\frac{\phi}{2n}\right] , \tag{24}$$

(cf. [6], p. 92). Using for K a relation similar to the right-hand side of (19) we obtain

$$I_0^2 K = \int_0^{4\pi n} f(\phi)\, w(\phi)\, b\, d\phi = 2\, k_1 \pi^3 \left[1 - \frac{1}{2n^2}\right] A^2 b . \tag{25}$$

For the elastic energy of one period of the buckled helix we take the classical expression for a slender curved beam (of circular cross-section), i.e.

$$W = \frac{EI}{b^4} \int_0^{4\pi n} (w'' - b\,\beta)^2\, b\, d\phi + \frac{GI_p}{b^4} \int_0^{4\pi n} (w' + b\,\beta')^2\, b\, d\phi =$$

$$= \frac{\pi E R^4}{4 b^3} \int_0^{4\pi n} \{(w'' - b\,\beta)^2 + \frac{1}{1+\nu} (w' + b\,\beta')^2\}\, d\phi , \tag{26}$$

where we have used $I_p = 2I = \pi R^4 / 2$, and $G = E / 2(1+\nu)$, with ν denoting Poisson's modulus. In accordance with (22) we assume the torsion angle β of the form

$$\beta(\phi) = \frac{1}{b} B \sin\left[\frac{\phi}{2n}\right] . \tag{27}$$

Substitution of (22) and (27) into (26) yields

$$W = \frac{n\, \pi^2 E R^4}{2 b^3} \{ (\frac{A}{4n^2} + B)^2 + \frac{1}{4(1+\nu)\, n^2} (A + B)^2 \} . \tag{28}$$

Taking the first variation of W with respect to B, i.e. $\partial W / \partial B = 0$, we obtain

$$B = -\frac{(2+\nu)}{4(1+\nu)\, n^2 + 1} A . \tag{29}$$

Use of (29) in (28) yields

$$W = \frac{n\, \pi^2 E R^4}{2 b^3} \frac{(4n^2 - 1)^2}{16 n^4 [4(1+\nu)\, n^2 + 1]} A^2 . \tag{30}$$

Similarly to (19) we here take $W = I_0^2 K$, yielding with (25) and (30)

$$k_1 = \frac{E R^4}{4 \pi b^4} \frac{N}{4(1+\nu)\, n} , \tag{31}$$

where

$$N = N(n) = (1 - \frac{1}{2n})\,(1 + \frac{1}{2n})^2 \,/\, [1 + \frac{1}{4(1+\nu)\,n^2}]\,, \tag{32}$$

(note that $N(n) = 1 + O(n^{-1})$ for $n \to \infty$).
With k_1 given by (15) and with the replacement of $2a$ by $2\pi h$, we finally obtain for the buckling current I_0

$$I_0 = \frac{\pi h R^2}{b^2} \sqrt{\frac{E\,N}{2\mu_0(1+\nu)\,n}}\;. \tag{33}$$

CONCLUSIONS

In this paper we have derived a formula for the buckling current in a helical superconductor. The helix was assumed of infinite length, but periodically supported over n turns. If $n \gg 1$, the buckling current I_0 is proportional to $n^{-1/2}$ (see (33)). It is reasonable to assume that (33) also governs (in good approximation) the buckling of a finite helix of n turns in case n is large enough. In fact, for a finite helix (33) represents the result that remains after ignoring all end effects.
We have used a combination of two methods, knowing a variational method (VP) and a variational version of the so called Biot-Savart method (BS) to arrive at our result. For both methods the formula for the buckling current I_0 can be written in analogous form, i.e.

$$I_{0VP} = \sqrt{W / K_{VP}} \quad ; \quad I_{0BS} = \sqrt{W / K_{BS}}\;, \tag{34}$$

while we also have seen that

$$I_{0BS} \,/\, I_{0VP} = 1 - \Delta \quad , \quad (\Delta = \Delta(a/R))\,. \tag{35}$$

Hence,

$$\sqrt{K_{VP}} = (1 - \Delta)\,\sqrt{K_{BS}} \approx \sqrt{K_{BS}}\;, \tag{36}$$

if $\Delta \ll 1$, as is the case if $a/R > 3$. For smaller values of a/R, and, hence, larger values of Δ, we can use (36) and $(34)^1$ to get a better estimate for I_0.
Finally, we note that it is also possible to apply the method presented here to other forms of structural superconductors, such as for instance a superconductor wound in the shape of a flat spiral.

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