Applications of polynomials to spherical codes and designs

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Chapter 1

Introduction to spherical codes and designs

In this chapter we give some notations and properties of the spherical codes and designs.

1.1 Euclidean sphere

The unit sphere $\mathbf{S}^{n-1}$ in the n-dimensional Euclidean space $\mathbf{R}^n$ is the set of all unit norm vectors:

$$\mathbf{S}^{n-1} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : \|x\| = x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

The standard metric is defined through the equation

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \tag{1.1.1}$$

and the standard inner product is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \tag{1.1.2}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are arbitrary points in $\mathbf{R}^n$. When the points x and y lie on $\mathbf{S}^{n-1}$, the inner product $\langle x, y \rangle$ equals the cosine of the angle (in their usual sense) between the vectors x and y.

The distance between points on $\mathbf{S}^{n-1}$ and their inner product are in close connection. Indeed, they are connected by the equations

$$d(x,y) = \sqrt{2(1 - \langle x, y \rangle)} \tag{1.1.3}$$

and

$$\langle x, y \rangle = 1 - \frac{d^2(x,y)}{2}. \tag{1.1.4}$$

This observation implies that investigations on distances give the same information as investigations on inner products. Following the tradition (see [31, 46]) and for some reasons which will become clearer later on we prefer to work with the inner products.

1.2 Spherical codes

In this section we describe some basic parameters of spherical codes.

Definition 1.2.1. *Any finite nonempty subset C of the Euclidean sphere $\mathbf{S}^{n-1}$ is called spherical code.*

The most important parameters which characterize a spherical code $C \subset \mathbf{S}^{n-1}$ are as follows:

- **cardinality** or **size** $M = |C|$. This is the cardinality of the nonempty finite set C;
- **dimension** $n = \dim(\mathbf{S}^{n-1})$. This is the smallest dimension of any Euclidean space which contains C;
- **maximal cosine** $s = s(C)$. This is defined as the maximal possible inner product of any two different points of C, i.e.

$$s = s(C) = \max\{\langle x, y\rangle : x, y \in C, x \neq y\}.$$

- **minimum distance** $d = d(C)$. This is the minimum possible distance between any two different points of C, i.e.

$$d = d(C) = \min\{d(x, y) : x, y \in C, x \neq y\}.$$

It follows from (1.1.3) and (1.1.4) that the maximal cosine and the minimum distance are related. For this reason we will work with the maximal cosine only.

Definition 1.2.2. *A spherical code $C \subset \mathbf{S}^{n-1}$ is said to be an (n, M, s)-code if it has dimension n, cardinality $M = |C|$ and maximal cosine s.*

Generally speaking we are interested in finding codes with small dimension, large size and small maximal cosine. Obviously these three ambitions are in conflict.
For given dimension n one investigates the relations between the size M and the maximal cosine s (the minimum distance d respectively). For given n and s, one wishes to find the maximal possible cardinality of an (n, M, s)-code. Similarly, for fixed n and M, one wishes to find the minimum possible maximal cosine of an (n, M, s)-code.

1.3 Spherical designs

Spherical designs were introduced by Delsarte, Goethals and Seidel in 1977 (see [31]) as analogs on $\mathbf{S}^{n-1}$ of the classical combinatorial designs. They wrote "Thus Ω, Sym(v), and the classical t-designs, correspond to Ω_d, $O(d)$, and the spherical t-designs, respectively". (Here Ω is the set of the d-subsets of $\{1, 2, \ldots, v\}$, $1 \leq d \leq v/2$, Sym(v) is the symmetric group on v elements, $\Omega_d = \mathbf{S}^d$, and $O(d)$ is the orthogonal group.)

The spherical designs are special class of spherical codes. The original motivation for studying these objects came from the numerical evaluation of multi-dimensional integrals. The integral of a polynomial function over the sphere may be approximated by its average value at the code points. Thus, among all equivalent definitions for a spherical design, the following one gives a nice intuitive idea for this notion. Namely, the average value of any polynomial f of degree at most τ over the whole sphere is equal to the average value of this polynomial over the code.

Definition 1.3.1. *A spherical code C is a spherical τ-design ($\tau \geq 0$ is an integer) if and only if the equality*

$$\int_{\mathbf{S}^{n-1}} f(x)d\mu(x) = \frac{1}{|C|}\sum_{x\in C} f(x) \tag{1.3.1}$$

holds for any real n-variable polynomial $f(x) = f(x_1, x_2, \ldots, x_n)$ of total degree at most τ. Here $\mu(.)$ is the normalized Lebesgue measure, i.e. $\mu(\mathbf{S}^{n-1}) = 1$.

The number τ is called *strength* of the design. We will always assume that the strength is the maximum value of τ for which C is a spherical τ-design.

Of course, every spherical code is a spherical 0-design. Spherical 1-designs are nothing else than the codes which have their centre of mass in the origin. A spherical 2-design C in three dimensions is what Schläfli called a *eutactic star*, essentially the projection onto $\mathbf{S}^{n-1}$ of $M = |C|$ mutually orthogonal vectors (cf. Coxeter [28]).

Definition 1.3.2. *A spherical design $C \subset \mathbf{S}^{n-1}$ is said to be antipodal if and only if C equals $-C = \{-x : x \in C\}$.*

1.4 Two main problems

In this section we formulate the two main problems we are interested in. In some sense, the problems for codes and for designs are dual to each other.

1.4.1 Maximal size of a spherical code

Here, one wants to maximize the size of a spherical code provided the dimension and the maximal cosine are fixed.

Definition 1.4.1. *The maximal possible cardinality of a spherical code on $\mathbf{S}^{n-1}$ with prescribed maximal inner product s is denoted by $A(n, s)$, i.e.*

$$A(n, s) = \max\{M : C \textit{ is an-}(n, M, s)\textit{-code}\}.$$

Problem 1 Determine the exact value of $A(n, s)$ or find upper and lower bounds on this number.

The problem to find $A(n, s)$ comes from classical geometry, but is of interest for combinatorics, information theory, coding theory, etc. In particular, bounds for $A(n, s)$ can be used to obtain estimations on the maximal possible density of a sphere packing in $\mathbf{R}^n$ [27, 49] and on the error exponent for the Gaussian channel [1]. For further discussions on this theme we refer to the books by Conway-Sloane [27], Ericson-Zinoviev [36], Levenshtein [49] (Chapter 6 of Pless-Huffman [54]), and Zong [64].

Lower bounds for $A(n,s)$ are given in terms of explicit constructions. Our main interest is in obtaining better upper bounds.

Upper bounds for $A(n,s)$ are general and indicate limits beyond which codes do not exist. As usually in coding theory, the best upper bounds are those obtained from linear programming techniques. At present, the best universal (here "universal" means that the bound can be written for all n and s) bound is the linear programming bound due to Levenshtein [46, 47, 48, 49]. We explain this in detail in Chapter 2.

Chapter 3 is based on the paper [17]. There we prove necessary and sufficient conditions for the existence of particular improvements of the Levenshtein bounds on $A(n,s)$. In that chapter we also investigate these conditions further and show that better bounds do exist quite often.

Some problems require estimations on the quantity $D(n,M)$ – the maximal possible minimum distance of a spherical code in n dimensions of fixed cardinality M. We study lower bounds on $S(n,M)$ – the minimum possible maximal cosine of a spherical code in n dimensions of fixed cardinality M. These two quantities are related by

$$S(n,M) = 1 - \frac{D^2(n,M)}{2} \text{ and } D(n,M) = \sqrt{2(1 - S(n,M))}.$$

Therefore, upper/lower bounds for $S(n,M)$ lead to lower/upper bounds for $D(n,M)$. In Chapter 3 we show how new bounds on $A(n,s)$ can be used for obtaining new bounds on $D(n,M)$.

The values of $A(n,s)$ are known for $-1 \le s \le 0$ in all dimensions (for example, cf. [27]). Thus we can assume that $s \in (0,1)$ further. Apart from one infinite sequence of $(n, A(n,s), s)$ codes with $s > 0$, finitely many such codes are known.

1.4.2 Minimum size of a spherical design

One wants to minimize the size of a spherical design provided the dimension and the strength are fixed.

Definition 1.4.2. *The minimum possible cardinality of a τ-design in n dimensions is denoted by $B(n,\tau)$, i.e.*

$$B(n,\tau) = \min\{|C| : C \in \mathbf{S}^{n-1} \textit{ is a } \tau\textit{-design}\}.$$

Problem 2 Determine the exact value of $B(n,\tau)$ or find upper and lower bounds on this number.

Upper bounds for $B(n,\tau)$ are given by explicit constructions. For every fixed n and τ, there exist τ-designs of large enough cardinality (Seymour-Zaslavsky [58]). Examples, which became classical of spherical designs were described by Delsarte-Goethals-Seidel

[31], and further constructions can be found in Goethals-Seidel [39], Bannai [6, 7], Bajnok [3, 4, 5], Hardin-Sloane [40, 41], Reznick [55], etc.

Following the analogy with the classical designs, Delsarte-Goethals-Seidel [31] obtained the following Fisher-type (DGS) bound

$$B(n,\tau) \geq R(n,\tau) = \begin{cases} 2\binom{n+k-2}{n-1}, & \text{if } \tau = 2k-1, \\ \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}, & \text{if } \tau = 2k. \end{cases} \tag{1.4.1}$$

Despite of its combinatorial appearance, this bound can be easily obtained by linear programming as Delsarte-Goethals-Seidel did in [31, Section 5]. The linear programming approach to Problem 2 will be explained in Chapter 2.

The possibilities for attaining the DGS bound were investigated by Bannai-Damerell [8, 9]. Later, some linear programming improvements on the DGS bound were obtained by Boyvalenkov-Nikova [23, 24, 52] and Yudin [63].

Chapter 4 is based on [12, 13, 15]. We develop methods for proving nonexistence results for spherical designs which use ideas beyond the pure linear programming approach. We first derive restrictions on the structure of designs of relatively small cardinalities using linear programming techniques. Then we apply some geometric argument to strengthen these restrictions. This allows us to prove nonexistence results in the first open parameters of spherical designs as well as in some asymptotic processes.

Chapter 2

The linear programming bounds for spherical codes and designs

The best non-constructive bounds in coding theory are usually those obtained from linear programming techniques. The basic ideas go back to MacWilliams (cf. [25, 50]) and were developed by Delsarte [30]. The particular case of spherical codes was studied firstly by Delsarte-Goethals-Seidel [31] and Kabatianskii-Levenshtein [43] (cf. [32, 49]).

In this chapter we describe the linear programming techniques which are used for upperbounding $A(n,s)$ and for lowerbounding $B(n,\tau)$. The best upper bound on $A(n,s)$ was obtained by Levenshtein. We explain the logic of the Levenshtein's bound together with some properties of the parameters involved.

2.1 Gegenbauer polynomials

The linear programming bound is largely based on the theory of orthogonal polynomials. The situation on the Euclidean sphere is expressed in terms of the Gegenbauer polynomials (also called ultraspherical polynomials). These polynomials are orthogonal with respect to the inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)(1-t^2)^{(n-3)/2}dt,$$

i.e. we have

$$c_n\int_{-1}^{1} P_i^{(n)}(t)P_j^{(n)}(t)(1-t^2)^{(n-3)/2}dt = \delta_{ij},$$

where

$$c_n = \left(\int_{-1}^{1}(1-t^2)^{\frac{n-3}{2}}dt\right)^{-1} = \frac{\Gamma(n-1)}{2^{n-2}(\Gamma(\frac{n-1}{2}))^2}, \tag{2.1.1}$$

$\Gamma(z)$ is the Gamma function and δ_{ij} is the Kroneker symbol.

For fixed dimension $n\geq 3$, we consider the corresponding family of Gegenbauer polynomials $\{P_i^{(n)}(t)\}_{i=0}^{\infty}$. We use the recurrence relation to define the Gegenbauer polynomials in the following way. Let $P_0^{(n)}(t) = 1$ and $P_1^{(n)}(t) = t$. Then one has

$$(i+n-2)P_{i+1}^{(n)}(t) = (2i+n-2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t) \tag{2.1.2}$$

for $i \geq 1$.

The Gegenbauer polynomials can be also introduced as a particular case of Jacobi polynomials $\{P_i^{(\alpha,\beta)}(t)\}_{i=0}^{\infty}$ where one needs to set $\alpha = \beta = (n-3)/2$ and to normalize with $P_i^{\alpha,\beta}(1) = 1$ in order to obtain the Gegenbauer polynomials (cf. [61]).

The first few Gegenbauer polynomials are:

$$\begin{aligned}
P_2^{(n)}(t) &= \frac{nt^2-1}{n-1}, \\
P_3^{(n)}(t) &= \frac{(n+2)t^3-3t}{n-1}, \\
P_4^{(n)}(t) &= \frac{(n+2)(n+4)t^4-6(n+2)t^2+3}{(n-1)(n+1)}, \\
P_5^{(n)}(t) &= \frac{(n+4)(n+6)t^5-10(n+4)t^3+15t}{(n-1)(n+1)}.
\end{aligned}$$

It easily follows from (2.1.2) by induction that $P_i^{(n)}(1) = 1$. Another obvious property of the Gegenbauer polynomials is that the even (respectively odd) degree polynomials are even (respectively odd) functions, i.e. $P_i^{(n)}(t) = (-1)^i P_i^{(n)}(-t)$ for all integers $i \geq 0$ and all real t. Note also that the leading coefficient of the polynomial $P_i^{(n)}(t)$ is positive and that $\mathrm{sign}(P_i^{(n)}(-1)) = (-1)^i$ for $i \geq 0$.

Let us denote

$$P_i^{(n)}(t) = \sum_{j=0}^{i} a_{i,j}t^j = a_{i,0} + a_{i,1}t + \cdots + a_{i,i-1}t^{i-1} + a_{i,i}t^i.$$

Since $a_{i,j} = 0$ when $i+j$ is odd, we actually have

$$P_i^{(n)}(t) = a_{i,i}t^i + a_{i,i-2}t^{i-2} + a_{i,i-4}t^{i-4} + \cdots.$$

Let $f(t) = a_0 + a_1t + a_2t^2 + \cdots + a_kt^k$ be a real polynomial. It is well known that $f(t)$ can be uniquely expanded in terms of any series of orthogonal polynomials. In particular, let us consider the expansion

$$\begin{aligned}
f(t) &= f_0P_0^{(n)}(t) + f_1P_1^{(n)}(t) + \cdots + f_kP_k^{(n)}(t) \\
&= f_0 + f_1t + \cdots + f_kP_k^{(n)}(t).
\end{aligned}$$

in terms of Gegenbauer polynomials. We are interested in the coefficients f_0, f_1, $\ldots$, f_k. They can be found in different ways.

Since the Gegenbauer polynomials are orthogonal on the interval $[-1,1]$ with respect to weight $(1-t^2)^{(n-3)/2}$, the classical formulas for f_i $(0 \leq i \leq k)$ as Fourier coefficients of $f(t)$ give

$$f_i = c_n \int_{-1}^{1} f(t)P_i^{(n)}(t)(1-t^2)^{(n-3)/2}dt, \tag{2.1.3}$$

where the constant c_n is given by (2.1.1).

In particular, the coefficient f_0 can be calculated by the formula

$$\begin{aligned} f_0 &= c_n \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt \qquad (2.1.4) \\ &= a_0 + \sum_{i=1}^{[k/2]} \frac{a_{2i}(2i-1)!!}{n(n+2)\cdots(n+2i-2)} \\ &= a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)} + \cdots . \qquad (2.1.5) \end{aligned}$$

It follows from this formula that $f_0 = 0$ for polynomials which are odd functions.

Another way to calculate the coefficients f_i turns out to be more convenient when the indices i are close to the degree k. Indeed, in this case we may solve the (beginning of) linear system which is obtained by comparing the coefficient of the same degrees of t in the equality

$$f(t) = f_0 P_0^{(n)}(t) + f_1 P_1^{(n)}(t) + \cdots + f_k P_k^{(n)}(t).$$

We have the equations

$$\begin{aligned} a_k &= f_k a_{k,k}, \\ a_{k-1} &= f_{k-1} a_{k-1,k-1}, \\ a_{k-2} &= f_{k-2} a_{k-2,k-2} + f_k a_{k,k-2}, \\ a_{k-3} &= f_{k-3} a_{k-3,k-3} + f_k a_{k-1,k-3}, \end{aligned}$$

etc. Therefore we find

$$\begin{aligned} f_k &= \frac{a_k}{a_{k,k}}, \\ f_{k-1} &= \frac{a_{k-1}}{a_{k-1,k-1}}, \\ f_{k-2} &= \frac{a_{k-2}}{a_{k-2,k-2}} - \frac{f_k a_{k,k-2}}{a_{k-2,k-2}} \\ &= \frac{a_{k-2}}{a_{k-2,k-2}} - \frac{a_k a_{k,k-2}}{a_{k,k} a_{k-2,k-2}}, \\ f_{k-3} &= \frac{a_{k-3}}{a_{k-3,k-3}} - \frac{f_{k-1} a_{k-1,k-3}}{a_{k-3,k-3}} \\ &= \frac{a_{k-3}}{a_{k-3,k-3}} - \frac{a_{k-1} a_{k-1,k-3}}{a_{k-1,k-1} a_{k-3,k-3}}, \end{aligned}$$

etc.

Yet another (third) way to calculate f_0, the coefficient of special interest to us, will be given later.

We give the coefficients f_0 for the power polynomials $f(t) = t^k$.

Lemma 2.1.1. *Let b_k be a real number such that $t^k = b_k + \sum_{i=1}^{k} f_i P_i^{(n)}(t)$ then*

$$b_k = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \dfrac{(2j-1)!!}{n(n+2)\cdots(n+2j-2)}, & \text{if } k = 2j. \end{cases} \tag{2.1.6}$$

The first few nonzero values of the constants b_k are

$$b_0 = 1,\ b_2 = \frac{1}{n},\ b_4 = \frac{3}{n(n+2)},\ b_6 = \frac{15}{n(n+2)(n+4)}.$$

With this notation, formula (2.1.5) becomes

$$f_0 = a_0 + \sum_{i=1}^{[k/2]} \frac{a_{2i}(2i-1)!!}{n(n+2)\cdots(n+2i-2)} = \sum_{i=0}^{[k/2]} a_{2i}b_{2i}.$$

2.2 Harmonic polynomials and the addition formula

The relevance and the importance of the Gegenbauer polynomials for investigations on the Euclidean sphere are justified by the so called addition formula. This property is the bridge between the Gegenbauer polynomials and the harmonic analysis on the sphere [31, 42, 62].

Definition 2.2.1. *An n-variable polynomial $f(x) = f(x_1, x_2, \ldots, x_n)$ is called harmonic if it satisfies the Laplace equation*

$$\Delta(f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

The set of all harmonic polynomials of degree i forms a linear space which is denoted by $\mathrm{Harm}(i)$.

For example, $\mathrm{Harm}(0) = \langle 1 \rangle$ consists of all constants and $\mathrm{Harm}(1) = \langle x_1, x_2, ..., x_n \rangle$ consists of all linear polynomials. If

$$r_i = \dim(\mathrm{Harm}(i)),$$

then $r_0 = 1$, $r_1 = n$ and it can be shown in general that

$$r_i = \binom{n+i-1}{i} - \binom{n+i-3}{i-2} = \frac{n+2i-2}{i}\binom{n+i-3}{i-1}. \tag{2.2.1}$$

Definition 2.2.2. *For any integer $i \geq 1$ let $\{v_{ij}(x) : j = 1, 2, \ldots, r_i\}$ be an orthonormal basis of the space* $\mathrm{Harm}(i)$ *with respect to the inner product $\langle f, g \rangle = \int_{\mathbf{S}^{n-1}} f(x)g(x)d\mu(x)$.*

The connection between the harmonic polynomials and the Gegenbauer polynomials is given by the following relation which is widely known as the *addition formula* [2, 31, 45]:

$$P_i^{(n)}(\langle x, y \rangle) = \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)v_{ij}(y). \tag{2.2.2}$$

In this formula, both sides do not depend on the particular choices of the points x and y but only on their inner product. The Gegenbauer polynomials are what are called *zonal spherical functions* for the Euclidean sphere $\mathbf{S}^{n-1}$ (cf. [45]). The addition formula is connected also with the concept of positive definite functions on $\mathbf{S}^{n-1}$.

2.3 The linear programming bound (LPB) for spherical codes and designs

2.3.1 Main identity

The addition formula (2.2.2) allows the derivation of an identity which seems to be the main source of inequalities for spherical codes and designs (cf. Delsarte-Goethals-Seidel [31] and Levenshtein [47, 49]). In particular, we use this identity to prove the linear programming theorems.

Theorem 2.3.1 (The main identity; [31, 47]). *Let $C \subset \mathbf{S}^{n-1}$ be arbitrary spherical code (possibly a τ-design of strength $\tau \geq 1$) and $f(t)$ be an arbitrary real polynomial. Then the following identity holds*

$$|C|f(1) + \sum_{x,y \in C, x \neq y} f(\langle x, y \rangle) = |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} v_{ij}(x) \right)^2, \tag{2.3.1}$$

where $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$.

Proof. We calculate in two different ways the sum

$$\sum_{x,y \in C} f(\langle x, y \rangle).$$

For the left hand side we simply extract the $|C|$ members with $x = y$ thus giving number $f(\langle x, x \rangle) = f(1)$ exactly $|C|$ times.

For the right hand side we use the expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ and then the addition formula:

$$\begin{aligned}
\sum_{x,y\in C} f(\langle x,y\rangle) &= \sum_{x,y\in C}\sum_{i=0}^{k} f_i P_i^{(n)}(\langle x,y\rangle) \\
&= \sum_{x,y\in C} f_0 P_0^{(n)}(\langle x,y\rangle) + \sum_{i=1}^{k}\sum_{x,y\in C} P_i^{(n)}(\langle x,y\rangle) \\
&= |C|^2 f_0 + \sum_{i=1}^{k} f_i \sum_{x,y\in C} \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)v_{ij}(y) \\
&= |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x,y\in C} v_{ij}(x)v_{ij}(y)\right) \\
&= |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x\in C} v_{ij}(x)\right)\left(\sum_{y\in C} v_{ij}(y)\right) \\
&= |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x\in C} v_{ij}(x)\right)^2
\end{aligned}$$

which completes the proof. □

2.3.2 LPB for spherical codes

We wish to maximize size $M = |C|$ over all spherical codes of fixed dimension n and maximal inner product s. The linear programing bound relates this maximization problem to a minimization problem for certain real polynomials as follows.

Theorem 2.3.2 (LPB for spherical codes [31, 43]). *Let $n \geq 3$ and $f(t)$ be a real polynomial such that*

(A1) $f(t) \leq 0$ *for* $-1 \leq t \leq s$, *and*

(A2) *The coefficients in the Gegenbauer expansion* $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ *satisfy* $f_0 > 0$, $f_i \geq 0$ *for* $i = 1, \ldots, k$.

Then $A(n,s) \leq f(1)/f_0$.

Proof. The assertion follows by the main identity (2.3.1). Let C be an (n, M, s) code and let $f(t)$ satisfy the conditions of the theorem.

For the left hand side of (2.3.1) we apply condition (A1) to see that it does not exceed $Mf(1)$. Then for the right hand side we use condition (A2) to establish that it is not less than or equal to $M^2 f_0$. Hence we have

$$Mf(1) \geq M^2 f_0.$$

Since this inequality must be satisfied by all (n, M, s) codes, we conclude that

$$A(n,s) \leq \frac{f(1)}{f_0},$$

which completes the proof. $\square$

It will become clear later that for any pair of values of $n \geq 3$ and $s \in [-1, 1)$ the set of polynomials which satisfy the conditions **(A1)** and **(A2)** is nonempty.

2.3.3 LPB for spherical designs

For fixed dimension n and strength τ we wish to find the minimum possible cardinality of a τ-design on $\mathbf{S}^{n-1}$. Similarly to the case of spherical codes, the linear programing bound leads to a maximization problem for certain real polynomials.

Theorem 2.3.3 (LPB for spherical designs [31]). *Let $n \geq 3$ and $f(t)$ be a real polynomial such that*

(B1) *$f(t) \geq 0$ for $-1 \leq t \leq 1$, and*

(B2) *The coefficients in the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ satisfy $f_i \leq 0$ for $i = \tau + 1, \ldots, k$.*

Then $B(n, \tau) \geq f(1)/f_0$.

Proof. Let C be a τ-design on $\mathbf{S}^{n-1}$ and let $f(t)$ satisfy the conditions of the theorem. At the left hand side of (2.3.1) we apply condition **(B1)** to see that it is not less than or equal to $|C|f(1)$. Then for the right hand side we use condition **(B2)** to establish that it does not exceed $|C|^2 f_0$. Hence we have

$$|C|f(1) \leq |C|^2 f_0.$$

Note that $f_0 > 0$ by (A1). Since this inequality must be satisfied for an arbitrary choice of C, we conclude that

$$B(n,\tau) \geq \frac{f(1)}{f_0},$$

which completes the proof. $\square$

Condition **(B2)** is satisfied for the nonnegative polynomials of degree at most τ. Therefore for any pair of values of $n \geq 3$ and $\tau \geq 1$ the set of polynomials which satisfy the conditions **(B1)** and **(B2)** is nonempty.

2.4 Further properties of the Gegenbauer polynomials. Adjacent polynomials

We shall need some properties of the Gegenbauer polynomials. Many of them are valid in the general case (i.e. for any series of orthogonal polynomials). The proofs can be found in Szegö [61], Levenshtein [47].

Denote by

$$m_k = \frac{a_{k+1,k+1}}{a_{k,k}} = \frac{n+2k-2}{n+k-2} \tag{2.4.1}$$

the ratio of the coefficients of the highest degrees the Gegenbauer polynomials $P_{k+1}^{(n)}(t)$ and $P_k^{(n)}(t)$. Then recurrence relation (2.1.2) can be written in the following way:

$$m_k P_{k+1}^{(n)}(t) = \left(t + m_k + \frac{m_{k-1}r_{k-1}}{r_k} - 1\right) P_k^{(n)}(t) - \frac{m_{k-1}r_{k-1}}{r_k} P_{k-1}^{(n)}(t)$$

for $k \geq 0$, where $r_{-1} = m_{-1} = 0$ and $P_{-1}^{(n)}(t) \equiv 0$.

Denote

$$T_k(x,y) = \sum_{i=0}^{k} r_i P_i^{(n)}(x) P_i^{(n)}(y).$$

Lemma 2.4.1 (Christoffel-Darboux formula). *We have*

$$T_k(x,y) = \begin{cases} \dfrac{r_k m_k \left(P_{k+1}^{(n)}(x)P_k^{(n)}(y) - P_k^{(n)}(x)P_{k+1}^{(n)}(y)\right)}{x-y}, & \text{if } x \neq y, \\ r_k m_k \left(P_k^{(n)}(x)\dfrac{d}{dx}P_{k+1}^{(n)}(x) - P_{k+1}^{(n)}(x)\dfrac{d}{dx}P_k^{(n)}(x)\right), & \text{if } x = y. \end{cases} \tag{2.4.2}$$

We use the Christoffel-Darboux formulas to simplify the functions $T_k(x,y)$ for some special values of x and y.

Lemma 2.4.2. *Any polynomial $P_k^{(n)}(t)$ has exactly k different simple zeros inside the interval* $[-1,1]$

$$-1 < t_{k,1} < t_{k,2} < \ldots < t_{k,k} = t_k < 1.$$

We shall also need the so-called *adjacent* polynomials $\{P_i^{a,b}(t)\}_{i=0}^{\infty}$, where $a, b \in \{0,1\}$.

Definition 2.4.3. *The adjacent polynomials are (normalized by $P_i^{a,b}(1) = 1$) Jacobi polynomials of parameters $\alpha = a + (n-3)/2$ and $\beta = b + (n-3)/2$, i.e.*

$$P_i^{a,b}(t) = \frac{P_i^{a+(n-3)/2,b+(n-3)/2}(t)}{P_i^{a+(n-3)/2,b+(n-3)/2}(1)}.$$

Therefore, the adjacent polynomials $\{P_i^{a,b}(t)\}_{i=0}^{\infty}$ are orthogonal in $[-1,1]$ with respect to the weight function

$$(1-t)^{a+(n-3)/2}(1+t)^{b+(n-3)/2}.$$

This means that

$$c_n^{a,b}\int_{-1}^{1} P_i^{a,b}(t)P_j^{a,b}(t)(1-t)^a(1-t)^b(1-t^2)^{(n-3)/2}dt = \delta_{ij} \tag{2.4.3}$$

where $c_n^{a,b}$ is a positive constant.
Since $\{P_i^{1,1}(t)\}_{i=0}^{\infty}$ are Jacobi polynomials of parameters $\alpha = \beta = (n-1)/2$, we see that

$$P_i^{1,1}(t) = \frac{P_i^{1+(n-3)/2,1+(n-3)/2}(t)}{P_i^{1+(n-3)/2,1+(n-3)/2}(1)} = P_i^{(n+2)}(t).$$

This fact will be used often in what follows.

Lemma 2.4.4 (Levenshtein [47]). *We have*

$$P_k^{1,0}(t) = \frac{T_k(t,1)}{T_k(1,1)} = \frac{\binom{k+n-2}{k}\left(P_k^{(n)}(t) - P_{k+1}^{(n)}(t)\right)}{(1-t)T_k(1,1)} \tag{2.4.4}$$

and

$$P_{k-1}^{1,1}(t) = \frac{2k\left(P_{k-1}^{(n)}(t) - P_{k+1}^{(n)}(t)\right)}{(n+2k-2)(1-t^2)}. \tag{2.4.5}$$

We shall also need the functions

$$T_k^{a,b}(x,y) = \sum_{i=0}^{k} r_i^{a,b} P_i^{a,b}(x)P_i^{a,b}(y),$$

where $(a,b) = (1,0)$ or $(1,1)$ and $r_i^{a,b}$ are positive integers.
For the adjacent polynomials $\{P_i^{a,b}(t)\}_{i=0}^{\infty}$ the same Christoffel-Darboux formulas (Lemma 2.4.1) hold as for $\{P_i^n(t)\}_{i=0}^{\infty}$. As above, one needs these formulas to express $T_k^{a,b}(x,y)$ in a simpler form for some values of x and y.
Denote by $t_k^{a,b}$ (again $a,b \in \{0,1\}$) the largest zero of the adjacent polynomial $P_k^{a,b}(t)$.

Lemma 2.4.5 ([49]). *The largest zeros of the adjacent polynomials satisfy the following separation conditions:*

$$t_k^{1,0} < t_k^{1,1} < t_{k+1}^{1,0},$$
$$t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1},$$
$$t_k^{1,0} < t_k,$$
$$t_k^{1,1} < t_k^{0,1}$$

for any $k \geq 1$ ($t_0^{1,1} = -1$ by definition).

Many ratios of orthogonal polynomials are monotonic in intervals where the denominator does not vanish. Such properties can be proved by using separation rules as those in Lemma 2.4.5. We need only the following two facts.

Lemma 2.4.6 ([49]). a) *The rational function $P_k^{1,0}(t)/P_{k-1}^{1,0}(t)$ is increasing in t in every interval which does not contain zeros of its denominator.*

b) *The rational function $P_k^{1,1}(t)/P_{k-1}^{1,1}(t)$ is increasing in t in every interval which does not contain zeros of its denominator.*

Lemma 2.4.7 ([49]). *The equality*

$$\frac{P_k^{1,0}\left(t_{k-1}^{1,1}\right)}{P_{k-1}^{1,0}\left(t_{k-1}^{1,1}\right)} = \frac{P_k^{1,0}(-1)}{P_{k-1}^{1,0}(-1)}$$

holds.

2.5 Universal bounds for spherical codes and designs

2.5.1 Levenshtein bound for spherical codes

We are ready to describe the Levenshtein bound. Let us define the closed intervals

$$\mathcal{I}_m = \begin{cases} \left[t_{k-1}^{1,1}, t_k^{1,0}\right], & \text{if } m = 2k-1, \\ \left[t_k^{1,0}, t_k^{1,1}\right], & \text{if } m = 2k, \end{cases} \tag{2.5.1}$$

for $k = 1, 2, ...$ and $\mathcal{I}_0 = [-1, t_1^{1,0})$.

It follows from Lemma 2.4.5 that the intervals $\mathcal{I}_m$ are consecutive and non-overlapping. Therefore, they constitute a partition of the half-open interval $\mathcal{I} = [-1, 1)$. For $s \in \mathcal{I}_m$, Levenshtein uses the polynomial

$$f_m^{(n,s)}(t) = \begin{cases} (t-s)\left(T_{k-1}^{1,0}(t,s)\right)^2, & \text{if } m = 2k-1, \\ (t+1)(t-s)\left(T_{k-1}^{1,1}(t,s)\right)^2, & \text{if } m = 2k, \end{cases} \tag{2.5.2}$$

in order to obtain a linear programming bound from Theorem 2.3.2. It can be proved that the polynomials $f_m^{(n,s)}(t)$ satisfy the conditions of Theorem 2.3.2 and imply (after some calculations) the following universal bound.

Lemma 2.5.1 (Levenshtein [47]). *The polynomials $f_m^{(n,s)}(t)$ satisfy the conditions **(A1)** and **(A2)** for all $s \in \mathcal{I}_m$. Moreover, all coefficients f_i, $0 \leq i \leq m$, in the Gegenbauer expansion of $f_m^{(n,s)}(t)$ are strictly positive for $s \in \mathcal{I}_m$.*

Theorem 2.5.2 (Levenshtein bound for spherical codes [46, 47]). *Let $n \geq 3$ and $s \in [-1, 1)$. Then*

$$A(n,s) \leq \begin{cases} L_{2k-1}(n,s) = \binom{k+n-3}{k-1}\left[\dfrac{2k+n-3}{n-1} - \dfrac{P_{k-1}^{(n)}(s) - P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)}\right] \\ \qquad\qquad \text{for } s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s) = \binom{k+n-2}{k}\left[\dfrac{2k+n-1}{n-1} - \dfrac{(1+s)\left(P_k^{(n)}(s) - P_{k+1}^{(n)}(s)\right)}{(1-s)\left(P_k^{(n)}(s) + P_{k+1}^{(n)}(s)\right)}\right] \\ \qquad\qquad \text{for } s \in \mathcal{I}_{2k}. \end{cases}$$

For example, the third bound

$$A(n,s) \leq L_3(n,s) = \frac{n(1-s)\,[2+(n+1)s]}{1-ns^2}$$

is valid in the interval

$$\mathcal{I}_3 = \left[t_1^{1,1}, t_2^{1,0}\right) = \left[0, \frac{\sqrt{n+3}-1}{n+2}\right).$$

The graphs of the bounds $L_3(3,s)$, $L_4(3,s)$, $L_5(3,s)$ and $L_6(3,s)$ are shown in Figure 2.1.

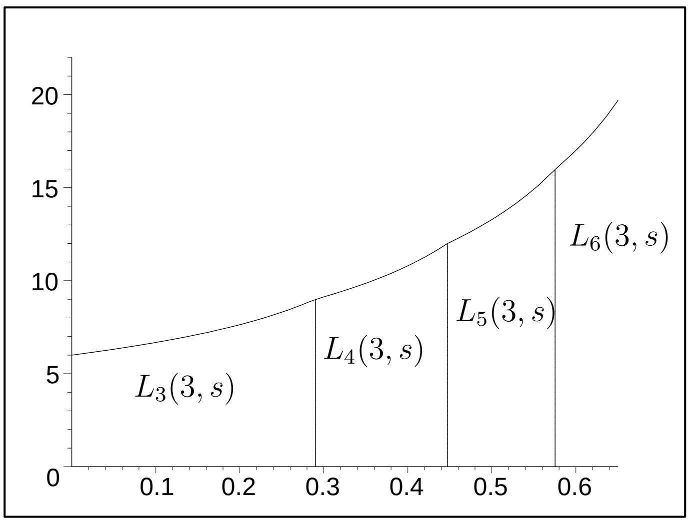

Figure 2.1: Four Levensthein bounds in three dimensions – $L_3(3,s)$, $L_4(3,s)$, $L_5(3,s)$ and $L_6(3,s)$

All known codes which attain the Levenshtein bound are listed in Table 3 [49]. The possibilities for existence of such codes were investigated in [19, 21, 44].

2.5.2 Delsarte-Goethals-Seidel bound for spherical designs

For fixed n and τ, Delsarte-Goethals-Seidel use the polynomial

$$f_\tau^{(n)}(t) = \begin{cases} (t+1)\left(P_{k-1}^{1,1}(t)\right)^2, & \text{if } \tau = 2k-1, \\ \left(P_k^{1,0}(t)\right)^2, & \text{if } \tau = 2k, \end{cases} \tag{2.5.3}$$

in order to obtain LPB by Theorem 2.3.3. It is obvious that the polynomials $f_\tau^{(n)}(t)$ satisfy the conditions of Theorem 2.3.3 (condition **(B1)** follows by the choice of $f_\tau^{(n)}(t)$

and condition **(B2)** is not relevant for these polynomials). The explicit form of Delsarte-Goethals-Seidel bound was given in (1.4.1) (see also (2.5.4) below). We have

$$\begin{aligned} R(n,2k-1) &= \left(1-\frac{P_{k-1}^{1,0}(-1)}{P_k(-1)}\right)\sum_{i=0}^{k-1} r_i = 2\binom{n+k-2}{n-1}, \\ R(n,2k) &= \sum_{i=0}^{k} r_i = \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}. \end{aligned}$$

The duality in the linear programming approach to spherical codes and designs implies some relations between the Levenshtein bound and the Delsarte-Goethals-Seidel bound. At the end points of the intervals $\mathcal{I}_m$ one has

$$L_{2k-2}(n,t_{k-1}^{1,1}) = L_{2k-1}(n,t_{k-1}^{1,1}) = R(n,2k-1) = 2\binom{n+k-2}{n-1}, \tag{2.5.4}$$

$$L_{2k-1}(n,t_{k}^{1,0}) = L_{2k}(n,t_{k}^{1,0}) = R(n,2k) = \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}. \tag{2.5.5}$$

In particular, this implies that the function

$$L(n,s) = \begin{cases} L_{2k-1}(n,s), & \text{if} \quad s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s), & \text{if} \quad s \in \mathcal{I}_{2k}, \end{cases}$$

is continuous in $s \in [-1,1)$.

2.6 Properties of the Levensthein polynomials

Our investigations in the next chapters are based on some observations on the connections between the parameters involved in the explanation of the Levenshtein bound.

2.6.1 Extremal polynomials

The best choice of polynomials for application in Theorem 2.3.2 is still unknown. Thus it makes sense to study some extremality properties of the polynomials already used.

Definition 2.6.1. *The set of suitable polynomials for applying in Theorem 2.3.2 is denoted by $A_{n,s}$, i.e. $f(t) = f_0 + f_1 P_1^{(n)}(t) + \cdots + f_k P_k^{(n)}(t)$ belongs to $A_{n,s}$ if and only if it satisfies $f(t) \le 0$ for $-1 \le t \le s$, and $f_0 > 0$, $f_1 \ge 0$, $\ldots$, $f_k \ge 0$.*

Among all polynomials in $A_{n,s}$, we wish to find the best one to estimate $A(n,s)$.

Definition 2.6.2. *A polynomial $f(t) \in A_{n,s}$ is called $A_{n,s}$-**extremal** (resp. $A_{n,s}$-**global extremal**) if it gives the best bound on $A(n,s)$ among the polynomials of the same or lower degree (resp. all polynomials) from $A_{n,s}$.*

Sidel'nikov [60] proved that the Levenshtein polynomials $f_m^{(n,s)}(t)$ are $A_{n,s}$-extremal. Other proofs were given later by Levenshtein [48, Section 4] and Boyvalenkov [10, Theorem 5.2]. Boyvalenkov [10] also introduces the notion of $A_{n,s}$-local extremality and proves that this is in fact the same as $A_{n,s}$-extremality.

In Chapter 3 we shall find necessary and sufficient conditions for the Levenshtein polynomials $f_m^{(n,s)}(t)$ to be $A_{n,s}$-global extremal.

2.6.2 Roots of $f_m^{(n,s)}(t)$ and another formula for f_0

The polynomial $(t-s)T_{k-1}^{1,0}(t,s)$ (see (2.5.2)) has k simple real zeros $\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1}$, all of them belonging to the interval $[-1,1)$. We take them in the following order:

$$-1 < \alpha_0 < \alpha_1 < \cdots < \alpha_{k-2} < \alpha_{k-1} = s.$$

Analogously, the polynomial $(t+1)(t-s)T_{k-1}^{1,1}(t,s)$ has $k+1$ simple zeros

$$-1 = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_{k-1} < \beta_k = s.$$

The numbers $\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{k-1}$ (respectively $\beta_0,\ \beta_1,\ \ldots,\ \beta_k$) can be considered as nodes for the Gauss-Jacobi-type formula for numerical integration.

Lemma 2.6.3 ([49]). a) *For every fixed $s \in \mathcal{I}_{2k-1}$ there exist positive numbers (weights) $\rho_0,\ \rho_1,\ \ldots,\ \rho_k$ such that the equality*

$$f_0 = \rho_k f(1) + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \tag{2.6.1}$$

holds for every real polynomial $f(t) = \sum_{i=0}^{2k-1} f_i P_i^{(n)}(t)$ of degree at most $2k-1$. Moreover, the numbers $\rho_0,\ \rho_1,\ \ldots,\ \rho_k$ are uniquely determined by n and s and the equality $\rho_k = 1/L_{2k-1}(n,s)$ holds.

b) *For every fixed $s \in \mathcal{I}_{2k}$ there exist positive numbers (weights) $\gamma_0,\ \gamma_1,\ \ldots,\ \gamma_{k+1}$ such that the equality*

$$f_0 = \gamma_{k+1} f(1) + \sum_{i=0}^{k} \gamma_i f(\beta_i) \tag{2.6.2}$$

holds for every real polynomial $f(t) = \sum_{i=0}^{2k} f_i P_i^{(n)}(t)$ of degree at most $2k$. Moreover, the numbers $\gamma_0,\ \gamma_1,\ \ldots,\ \gamma_{k+1}$ are uniquely determined by n and s and the equality $\gamma_{k+1} = 1/L_{2k}(n,s)$ holds.

Some formulas for the weights ρ_i $(0 \le i \le k-1)$ and γ_i $(0 \le i \le k)$ can be found in [49] (see also [19]).

Lemma 2.6.4. a)([49]) *The numbers $\alpha_0, \alpha_1, ..., \alpha_{k-1}$ are roots of the equation*

$$P_k^{1,0}(t)P_{k-1}^{1,0}(s) - P_k^{1,0}(s)P_{k-1}^{1,0}(t) = 0.$$

b)([49]) *The numbers $\beta_0, \beta_1, ..., \beta_k$ are roots of the equation*

$$P_k^{1,1}(t)P_{k-1}^{1,1}(s) - P_k^{1,1}(s)P_{k-1}^{1,1}(t) = 0.$$

c)([19]) *If $s \in \left[t_{k-1}^{1,1}, t_k^{1,0}\right]$ then α_i, $i = 0, 1, ..., k-1$, are strictly increasing the functions in s.*

d) *If $s = t_k^{1,0}$ then $\gamma_0 = 0, \rho_i = \gamma_{i+1}$ and $\alpha_i = \beta_{i+1}$ for $i = 0, 1, ..., k-1$. If $s = t_k^{1,1}$ then $\rho_i = \gamma_{i+1}$ and $\beta_{i+1} = \alpha_i$ for $i = 0, 1, ..., k$ and $\alpha_0 = -1$.*

Proof. **a)** The numbers $\alpha_0, \alpha_1, ..., \alpha_{k-1}$ are roots of the polynomial $(t-s)T_{k-1}^{1,0}(t, s)$. Using the Cristoffel-Darboux formulas we obtain that equation $(t-s)T_{k-1}^{1,0}(t, s) = 0$ is equivalent to $P_k^{1,0}(t)P_{k-1}^{1,0}(s) - P_k^{1,0}(s)P_{k-1}^{1,0}(t) = 0$.
b) Again using the Cristoffel-Darboux formulas we get the equivalence between equations $(t+1)(t-s)T_{k-1}^{1,1}(t, s) = 0$ and $P_k^{1,1}(t)P_{k-1}^{1,1}(s) - P_k^{1,1}(s)P_{k-1}^{1,1}(t) = 0$.
c) We write the equations **a)** and **b)** in the following form:

$$\frac{P_k^{1,0}(\alpha_i)}{P_{k-1}^{1,0}(\alpha_i)} = \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)}, \quad i = 0, 1, ..., k-1,$$

and

$$\frac{P_k^{1,1}(\beta_i)}{P_{k-1}^{1,1}(\beta_i)} = \frac{P_k^{1,1}(s)}{P_{k-1}^{1,1}(s)}, \quad i = 1, 2, ..., k.$$

The functions $\frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)}$ and $\frac{P_k^{1,1}(s)}{P_{k-1}^{1,1}(s)}$ are strictly increasing with respect to s in intervals where they are defined. Therefore α_i and β_i are strictly increasing in the same intervals (see Lemma2.4.6).
d) When $s = t_k^{1,0}$ polynomial $(t-s)T_{k-1}^{1,0}(t, s)$ divides $(t+1)(t-s)\left(T_{k-1}^{1,1}(t, s)\right)^2$ and when $s = t_k^{1,1}$ polynomial $(t+1)(t-s)T_{k-1}^{1,1}(t, s)$ divides $(t-s)\left(T_{k-1}^{1,0}(t, s)\right)^2$. Since the numbers α_i and β_i are ordered we obtain the desired relations.
The relations between the weights follow from the fact that they are solutions of the same system of linear equations with nonzero determinants. □

We shall need formulas for ρ_0.

Lemma 2.6.5 (Theorem 3.8, [16]). *We have*

$$\rho_0 = -\frac{(1-\alpha_1^2)(1-\alpha_2^2)\cdots(1-\alpha_{k-1}^2)}{\alpha_0(\alpha_0^2-\alpha_1^2)(\alpha_0^2-\alpha_2^2)\cdots(\alpha_0^2-\alpha_{k-1}^2)L_{2k-1}(n, s)}.$$

We also need some relations between the parameters under consideration.
For fixed $k \geq 2$, we consider the power sums

$$S_l = \rho_k + \sum_{i=0}^{k-1} \rho_i \alpha_i^l$$

and

$$R_l = \gamma_{k+1} + \sum_{i=0}^{k} \gamma_i \beta_i^l,$$

where l is a non-negative integer. It follows from Lemma 2.6.3 that

$$S_i = b_i$$

for $0 \le i \le 2k-1$ and that

$$R_i = b_i$$

for $0 \le i \le 2k$, respectively. We interpret these relations as a system of equations with respect to the numbers ρ_i, $i = 0, 1, \ldots, k$ (respectively for γ_i, $i = 0, 1, \ldots, k+1$).

Lemma 2.6.6. *The numbers ρ_i and α_i, $i = 0, 1, \ldots, k-1$ (respectively γ_i and β_i, $i = 0, 1, \ldots, k$) satisfy the following system of $2k$ (respectively $2k+1$) equations*

$$S_l = b_l,\ l = 0, 1, \ldots, 2k-1 \ (\ \textit{resp.}\ R_l = b_l,\ l = 0, 1, \ldots, 2k\ ). \tag{2.6.3}$$

Proof. Plug consecutively $f(t) = 1, t, \ldots, t^{2k-1}$ (respectively $1, t, \ldots, t^{2k}$) in the equality (2.6.1) (resp. in (2.6.2)) to obtain the system (2.6.3) by Lemma 2.1.1. □

Lemma 2.6.7 ([19]). a) *For $s \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right]$, the numbers α_i, $i = 0, 1, ..., k-1$, satisfy the following inequalities*

$$1 > |\alpha_0| > |\alpha_{k-1}| > |\alpha_1| > |\alpha_{k-2}| > \cdots > |\alpha_{[k/2]}| > 0.$$

b) *For $s \in \left(t_k^{1,0}, t_k^{1,1}\right]$ the numbers α_i, $i = 0, 1, ..., k-1$, satisfy the following inequalities*

$$1 = |\beta_0| > |\beta_1| > |\beta_k| > |\beta_2| > |\beta_{k-1}| > \cdots > |\beta_{[k/2]+1}| > 0.$$

Proof. **a)** To prove this assertion, we need the following fact: $|\alpha_i| \neq |\alpha_j|$ for any $s \in \mathcal{I}_{2k-1}$. Let us suppose that for $s = s_0$ we have $\alpha_i = -\alpha_j$. There exists a small enough $\varepsilon > 0$ such that $(s_0 - \varepsilon, s_0 + \varepsilon) \subset \mathcal{I}_{2k-1}$ and $\alpha_{i_1} \neq -\alpha_{i_2}$ for every $i_1, i_2 \in \{0, 1, \ldots, k-1\}$ whenever $s \in (s_0 - \varepsilon, s_0 + \varepsilon) \backslash \{s_0\}$. We set $f(t) = t, t^3, \ldots, t^{2k-1}$ in (2.6.1) and obtain a Vandermonde system with respect to the ratios $\rho_l/\rho_k, l = 0, 1, \ldots, k-1$. For $s \in (s_0 - \varepsilon, s_0 + \varepsilon) \backslash \{s_0\}$ this system has a unique solution. In particular, we have

$$\frac{\rho_i}{\rho_k} = -\frac{(1-\alpha_0^2)(1-\alpha_1^2)\ldots(1-\alpha_{i-1}^2)(1-\alpha_{i+1}^2)\ldots(1-\alpha_{k-1}^2)}{\alpha_i(\alpha_i^2-\alpha_0^2)(\alpha_i^2-\alpha_1^2)\ldots(\alpha_i^2-\alpha_{i-1}^2)(\alpha_i^2-\alpha_{i+1}^2)\ldots(\alpha_i^2-\alpha_{k-1}^2)}. \tag{2.6.4}$$

The ratio ρ_i/ρ_k has different signs when $s \in (s_0 - \varepsilon, s_0)$ and $s \in (s_0, s_0 + \varepsilon)$, because the numbers α_i are strictly increasing with respect to s (see Lemma 2.6.4c)). This is a contradiction with $\rho_i > 0$ for $i = 0, 1, \ldots, k$.

Now, using (2.6.4) and the fact that $\rho_i > 0$ for $i = 0, 1, \ldots, k$, we obtain the desired inequalities.

b) Analogously. □

At the end of this section we give an identity which relates the coefficients of even degree in the Gegenbauer polynomials.

Lemma 2.6.8. *For any integer $m > 0$ we have*

$$\sum_{i=0}^{m} \frac{(2i-1)!!}{n(n+2)\dots(n+2i-2)} a_{2m,2i} = \sum_{i=0}^{m} b_{2i} a_{2m,2i} = 0. \tag{2.6.5}$$

Proof. According to (2.1.5), the left-hand side is the coefficient f_0 in the expansion in terms of the Gegenbauer polynomials of $P_{2m}^{(n)}(t)$. Of course, this coefficient equals zero. □

2.7 Equivalent definitions for spherical designs

In this section we will give two other equivalent definitions for spherical designs.

Definition 2.7.1. *A spherical code $C \subset \mathbf{S}^{n-1}$ is a spherical τ-design ($n \geq 3$, $\tau \geq 1$) if and only if for any homogeneous non-constant harmonic polynomial $v(x)$ of degree at most τ the equality*

$$\sum_{x \in C} v(x) = 0$$

holds.

The equivalence between Definition 1.3.1 and Definition 2.7.1 follows from the fact that the integral over the sphere $\mathbf{S}^{n-1}$ vanishes for harmonic polynomials. We use Definition 2.7.1 to prove a further equivalence, which was observed by Fazekas-Levenshtein [37].

Theorem 2.7.2 ([37]). *A spherical code $C \subset \mathbf{S}^{n-1}$ is a spherical τ-design ($n \geq 3$, $\tau \geq 1$) if and only if the equality*

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0 |C| \tag{2.7.1}$$

holds for every point $y \in \mathbf{S}^{n-1}$ and for every real polynomial $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ of degree $k \leq \tau$.

Proof. We follow the proof of Nikova [52]. Let C be a spherical τ-design and $y \in \mathbf{S}^{n-1}$. Then the sum $\sum_{x \in C} P_i^{(n)}(\langle x, y \rangle)$ equals zero for $i \geq 1$. Indeed, using (2.2.2) for $1 \leq i \leq \tau$, we obtain

$$\begin{aligned} \sum_{x \in C} P_i^{(n)}(\langle x, y \rangle) &= \frac{1}{r_i} \sum_{x \in C} \sum_{j=1}^{r_i} v_{ij}(x) v_{ij}(y) \\ &= \frac{1}{r_i} \left(\sum_{j=1}^{r_i} v_{ij}(x) \right) \left(\sum_{x \in C} v_{ij}(y) \right) = 0, \end{aligned}$$

since $v_{ij}(x)$, $j = 1, 2, \ldots, r_i$, are harmonic polynomials from $\mathrm{Harm}(i)$, $i = 1, 2, \ldots, \tau$. Any real polynomial $f(t)$ can be written as $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$. Therefore,

$$\begin{aligned}\sum_{x \in C} f(\langle x, y\rangle) &= \sum_{x \in C} \sum_{i=0}^{k} f_i P_i^{(n)}(\langle x, y\rangle) \\ &= \sum_{x \in C} f_0 P_0^{(n)}(\langle x, y\rangle) + \sum_{x \in C} \sum_{i=1}^{k} f_i P_i^{(n)}(\langle x, y\rangle) \\ &= f_0|C| + \sum_{i=1}^{k} f_i \left(\sum_{x \in C} P_i^{(n)}(\langle x, y\rangle) \right) \\ &= f_0|C|.\end{aligned}$$

To prove the necessity, we may use (2.3.1) for C and for some real polynomial $f(t)$ of degree τ such that $f_i > 0$ for every $i = 0, 1, ..., \tau$. For example, $f(t) = \sum_{i=0}^{\tau} P_i^{(n)}(t)$ is such a polynomial ($f_i = 1$ for all $i = 0, 1, \ldots, \tau$).

On the left hand side of (2.3.1), the sum

$$\sum_{x, y \in C, x \neq y} f(\langle x, y\rangle)$$

decomposes into $|C|$ sums of the form (2.7.1), each of them therefore is equal to $f_0|C| - f(1)$. Therefore (2.3.1) becomes

$$\sum_{i=1}^{\tau} \frac{1}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} v_{ij}(x) \right)^2 = 0,$$

whence $\sum_{x \in C} v_{ij}(x) = 0$ for all $i = 1, ..., \tau$ and $j = 1, ..r_i$. Since $\{v_{ij}(x) : j = 1, 2, \ldots, r_i\}$ is a basis of $\mathrm{Harm}(i)$, this completes the proof. □

Equality (2.7.1) from Theorem 2.7.2 will be crucial for our investigations in Chapter 4.

We usually apply Theorem 2.7.2 when we investigate the structure of a design of which the existence is yet undecided. For such a putative τ-design $C \subset \mathbf{S}^{n-1}$, we use (2.7.1) for points $y \in C$. In this case (2.7.1) becomes

$$\sum_{x \in C \setminus \{y\}} f(\langle x, y\rangle) = f_0|C| - f(1). \tag{2.7.2}$$

To conclude this chapter we give the definition for the *indices* of spherical codes.

Definition 2.7.3. *Let $k \geq 1$ be an integer. A spherical code $C \subset \mathbf{S}^{n-1}$ is said to have index k if for any homogeneous harmonic polynomial $v(x)$ of degree k the equality*

$$\sum_{x \in C} v(x) = 0$$

holds.

It is clear that a τ-design has indices $1, 2, \ldots, \tau$ and vise versa – if $C \subset \mathbf{S}^{n-1}$ has indices $1, 2, \ldots, \tau$ then C is a τ-design. It is also obvious that for antipodal codes every odd integer is an index.

It follows as in the proof of Theorem 2.7.2 that the equality

$$\sum_{x \in C} P_k^{(n)}(\langle x, y \rangle) = 0$$

holds for every $y \in C$. Indices of some spherical codes were studied by Boyvalenkov-Danev-Kazakov [18]. In Chapter 5 we introduce so-called moments of spherical codes and find some connection between indices and moments.

Chapter 3

Conditions for possible improvements of the Levenshtein bound

This chapter is based on the paper [17]. After giving some preliminary results, we prove necessary and sufficient conditions for the existence of improvements of the Levenshtein bounds. Then we investigate these conditions to prove that better bounds are often possible. Examples of new better bounds are presented as well.

3.1 Some preliminaries

We did not find the next identity in the standard books on orthogonal (Gegenbauer) polynomials. This is why we give a detailed proof.

Lemma 3.1.1. *For any integer $k \geq 0$ we have*

$$(k+1)(t^2-1)P_k^{(n+2)}(t) = (n-1)[P_{k+2}^{(n)}(t) - tP_{k+1}^{(n)}(t)]. \tag{3.1.1}$$

Proof. We will use induction on k. If $k = 0$, then the equality

$$(t^2-1)P_0^{(n+2)}(t) = (n-1)[P_2^{(n)}(t) - tP_1^{(n)}(t)]$$

holds since $P_0^{(n)}(t) = 1$, $P_1^{(n)}(t) = t$ and $P_2^{(n)}(t) = (nt^2-1)/(n-1)$.
Analogously, for $k = 1$

$$2(t^2-1)P_1^{(n+2)}(t) = (n-1)\left[P_3^{(n)}(t) - tP_2^{(n)}(t)\right]$$

holds since $P_3^{(n)}(t) = ((n+2)t^3-3t)/(n-1)$.
Let $k \geq 1$ and let (3.1.1) holds for all positive integers smaller than $k+1$. Then we have

$$k(t^2-1)P_{k-1}^{(n+2)}(t) = (n-1)\left[P_{k+1}^{(n)}(t) - tP_k^{(n)}(t)\right]$$

and

$$(k+1)(t^2-1)P_k^{(n+2)}(t) = (n-1)\left[P_{k+2}^{(n)}(t) - tP_{k+1}^{(n)}(t)\right].$$

We calculate $(k+2)(t^2-1)P_{k+1}^{(n+2)}(t)$ using the recurrence relation for $P_{k+1}^{(n+2)}(t)$ and the induction assumptions. We consecutively have

$$\begin{aligned}
(k+2)(t^2-1)P_{k+1}^{(n+2)}(t) \;&=\; \frac{(k+2)(t^2-1)}{n+k}\,\Big[\,(n+2k)\;t\;P_k^{(n+2)}(t)\;-\;k\;P_{k-1}^{(n+2)}(t)\,\Big]\\
&= \frac{(k+2)(t^2-1)}{k+n}\,\Big[\frac{(n+2k)\;(n-1)\;t}{(k+1)\;(t^2-1)}\Big(P_{k+2}^{(n)}(t)\;-\;t\;P_{k+1}^{(n)}(t)\Big)\\
&\qquad -\;\frac{n-1}{t^2-1}\;\Big(\;P_{k+1}^{(n)}(t)-t\;P_k^{(n)}(t)\;\Big)\Big]\\
&= (n-1)\,\Big[\frac{1}{n+k}\;\Big(\;\frac{(n+2k)\;(k+2)}{k+1}\;t\;P_{k+2}^{(n)}(t)\;-\;(k+2)\;P_{k+1}^{(n)}(t)\;\Big)\\
&\qquad -\;\frac{(k+2)\;(n+2k)}{(n+k)\;(k+1)}\;t^2\;P_{k+1}^{(n)}(t)\;+\;\frac{k+2}{n+k}\;t\;P_k^{(n)}(t)\Big]\\
&= (n-1)\,\Big[\frac{1}{n+k}\Big(\;\Big(\;n+2k+2+\frac{n-2}{k+1}\;\Big)\;t\;P_{k+2}^{(n)}(t)\;-\;(k+2)\;P_{k+1}^{(n)}(t)\;\Big)\\
&\qquad -\;\frac{(k+2)\;t}{(n+k)\;(k+1)}\;\Big(\;(n+2k)\;t\;P_{k+1}^{(n)}(t)\;-\;(k+1)\;P_k^{(n)}(t)\;\Big)\Big]\\
&= (n-1)\,\Big[P_{k+3}^{(n)}(t)\;+\;\frac{(n-2)\;t}{(n+k)\;(k+1)}\;P_{k+2}^{(n)}(t)\;-\;\frac{(k+2)\;(n+k-1)\;t}{(n+k)\;(k+1)}\;P_{k+2}^{(n)}(t)\Big]\\
&= (n-1)\;\Big[\;P_{k+3}^{(n)}(t)\;-\;t\;P_{k+2}^{(n)}(t)\;\Big].
\end{aligned}$$

□

Lemma 3.1.2. *For the ratio of the first two nonzero coefficients of the Gegenbauer polynomial $P_k^{(n)}(t)$, $k \geq 2$, we have*

$$\frac{a_{k,k-2}}{a_{k,k}} = -\frac{k^2-k}{2(n+2k-4)}.$$

Proof. We have

$$\frac{a_{2,0}}{a_{2,2}} = -\frac{1}{n}$$

as an induction basis. Let

$$\frac{a_{k,k-2}}{a_{k,k}} = -\frac{k^2-k}{2(n+2k-4)}$$

be the induction assumption. In the recurrence relation

$$(k+n-2)P_{k+1}^{(n)}(t) = (2k+n-2)tP_k^{(n)}(t) - kP_{k-1}^{(n)}(t)$$

we compare the coefficients of t^{k-1} and obtain

$$a_{k+1,k-1} = \frac{(n+2k-2)a_{k,k-2} - ka_{k-1,k-1}}{n+k-2} = m_k a_{k,k-2} - \frac{ka_{k-1,k-1}}{n+k-2}.$$

Therefore

$$\begin{aligned}
\frac{a_{k+1,k-1}}{a_{k+1,k+1}} &= \frac{1}{m_k a_{k,k}}\left(m_k a_{k,k-2} - \frac{ka_{k-1,k-1}}{n+k-2}\right) \\
&= \frac{a_{k,k-2}}{a_{k,k}} - \frac{ka_{k-1,k-1}}{(n+2k-2)a_{k,k}} \\
&= -\frac{k^2-k}{2(n+2k-4)} - \frac{k}{(n+2k-2)m_{k-1}} \\
&= -\frac{k}{n+2k-4}\left(\frac{k-1}{2} + \frac{n+k-3}{n+2k-2}\right) \\
&= -\frac{k^2+k}{2(n+2k-2)},
\end{aligned}$$

which completes the induction. □

In particular, it follows from Lemma 3.1.2 that

$$\frac{a_{2k+3,2k+1}}{a_{2k+3,2k+3}} = -\frac{2k^2+5k+3}{n+4k+2}.$$

Now we give some more specific identities.

Lemma 3.1.3. a) *For every $k \geq 1$*

$$\frac{P_k^{1,0}(t_{k-1}^{1,1})}{P_{k-1}^{1,0}(t_{k-1}^{1,1})} = -\frac{n+2k-3}{n+2k-1}.$$

b) *For every $k \geq 1$*

$$\frac{P_k^{(n+2)}(t_k^{1,0})}{P_{k-1}^{(n+2)}(t_k^{1,0})} = -\frac{k}{n+k-1}.$$

Proof. **a)** From Lemma 2.4.7 we have the equality

$$\frac{P_k^{1,0}(t_{k-1}^{1,1})}{P_{k-1}^{1,0}(t_{k-1}^{1,1})} = \frac{P_k^{1,0}(-1)}{P_{k-1}^{1,0}(-1)}. \tag{3.1.2}$$

So we have to calculate $P_k^{1,0}(-1)$. From equality (2.4.4) we derive

$$P_k^{1,0}(-1) = \frac{T_k(-1,1)}{T_k(1,1)} = \frac{\sum_{i=0}^k r_i P_i(-1)P_i(1)}{\sum_{i=0}^k r_i (P_i(1))^2} = \frac{\sum_{i=0}^k (-1)^i r_i}{\sum_{i=0}^k r_i}. \tag{3.1.3}$$

We therefore need to find the sums $\sum_{p=0}^{l} r_{2p}$ and $\sum_{p=0}^{l} r_{2p+1}$, where $r_0 = 1$, $r_1 = n$ and $r_i = \binom{n+i-1}{i} - \binom{n+i-3}{i-2}$ from (2.2.1). We have

$$\begin{aligned}
\sum_{p=0}^{l} r_{2p} &= 1 + \sum_{p=1}^{l} r_{2p} = 1 + \sum_{p=1}^{l} \left[\binom{n+2p-1}{2p} - \binom{n+2p-3}{2p-2} \right] \\
&= 1 + \sum_{p=1}^{l} \binom{n+2p-1}{2p} - \sum_{p=1}^{l} \binom{n+2p-3}{2p-2} \\
&= 1 + \sum_{p=1}^{l} \binom{n+2p-1}{2p} - 1 - \sum_{p=2}^{l} \binom{n+2p-3}{2p-2} \\
&= \sum_{p=1}^{l} \binom{n+2p-1}{2p} - \sum_{p=1}^{l-1} \binom{n+2p-1}{2p} \\
&= \binom{n+2l-1}{2l}.
\end{aligned}$$

Since $\sum_{i=0}^{k} r_i$ is equal to the Delsarte-Goethals-Seidel bound

$$R(n,2k) = \binom{n+k-1}{k} + \binom{n+k-2}{k-1}$$

we obtain

$$\sum_{p=0}^{l} r_{2p+1} = \binom{n+2l}{2l+1}$$

(this also can be proved as above).
Hence it follows from (3.1.3) that

$$P_k^{1,0}(-1) = (-1)^k \frac{\binom{n+k-1}{k} - \binom{n+k-2}{k-1}}{\binom{n+k-1}{k} + \binom{n+k-2}{k-1}} = (-1)^k \frac{n-1}{n+2k-1}.$$

We plug this and the corresponding identity for $P_{k-1}^{1,0}(-1)$ in (3.1.2) to obtain

$$\frac{P_k^{1,0}(t_{k-1}^{1,1})}{P_{k-1}^{1,0}(t_{k-1}^{1,1})} = -\frac{n+2k-3}{n+2k-1}.$$

b) It follows from (2.4.4) that $t_k^{1,0}$ is a root of the equation $P_k^{(n)}(t) = P_{k+1}^{(n)}(t)$. Therefore we have the identity $P_k^{(n)}(t_k^{1,0}) = P_{k+1}^{(n)}(t_k^{1,0})$.
Using twice Lemma 3.1.1, we can derive the equalities:

$$k(t^2-1)P_{k-1}^{(n+2)}(t) = (n-1)\left[P_{k+1}^{(n)}(t) - tP_k^{(n)}(t)\right] \tag{3.1.4}$$

and

$$(k+1)(t^2-1)P_k^{(n+2)}(t) = (n-1)\left[P_{k+2}^{(n)}(t) - tP_{k+1}^{(n)}(t)\right]. \tag{3.1.5}$$

We combine (3.1.5) with the recurrence relation for the Gegenbauer polynomials for $P_{k+2}^{(n)}(t)$ and obtain

$$(t^2-1)P_k^{(n+2)}(t) = \frac{n-1}{k+n-1}\left[tP_{k+1}^{(n)}(t) - P_k^{(n)}(t)\right]. \tag{3.1.6}$$

We now plug $t = t_k^{1,0}$ in (3.1.4) and (3.1.6) and divide the first by the second to derive the desired ratio. $\square$

3.2 Test functions

For every integer $j \geq 1$ we introduce the following functions in n and s

$$Q_j(n,s) = \begin{cases} \dfrac{1}{L_{2k-1}(n,s)} + \displaystyle\sum_{i=0}^{k-1} \rho_i P_j^{(n)}(\alpha_i), & \text{for } s \in \mathcal{I}_{2k-1}, \\ \dfrac{1}{L_{2k}(n,s)} + \displaystyle\sum_{i=0}^{k} \gamma_i P_j^{(n)}(\beta_i), & \text{for } s \in \mathcal{I}_{2k}, \end{cases} \tag{3.2.1}$$

(We recall that $\rho_k = 1/L_{2k-1}(n,s)$ and $\gamma_{k+1} = 1/L_{2k}(n,s)$.)
It follows from Lemma 2.4.5 (see also the comments of the beginning of Subsection 2.5.1) that the functions $Q_j(n,s)$ are defined for all values of $s \in [-1,1)$ and for all dimensions $n \geq 3$.

Lemma 3.2.1. *For fixed n and j the function $Q_j(n,s)$ is continuous in s.*

Proof. Since $\gamma_0 = 0$, $\beta_i = \alpha_{i-1}$ for $i = 1,2,...,k$ and $\rho_i = \gamma_{k-1}$ for $i = 1,2,...,k$ (see Lemma 2.6.4) the values of $Q_j(n,s)$ in right end points of $\mathcal{I}_{2k-1}$ (from (3.2.1) for $\mathcal{I}_{2k-1}$) are equal to values of $Q_j(n,s)$ in left end point of $\mathcal{I}_{2k}$ (from (3.2.1) for $\mathcal{I}_{2k}$).
Analogously, for the right end points of $\mathcal{I}_{2k}$. $\square$

The next lemma shows that $Q_j(n,s)$ vanishes for some initial values for j.

Lemma 3.2.2. *We have*

$$Q_j(n,s) \equiv 0 \quad \textit{for} \quad \begin{cases} 1 \leq j \leq 2k-1, & \textit{when } s \in \mathcal{I}_{2k-1}, \\ 1 \leq j \leq 2k, & \textit{when } s \in \mathcal{I}_{2k}. \end{cases}$$

Proof. By Lemma 2.6.3, the right hand side in the definition of $Q_j(n,s)$ equals the coefficient f_0 in the Gegenbauer expansion of the polynomials $P_j^{(n)}(t)$ for $j \leq 2k-1$, when $t_{k-1}^{1,1} \leq s \leq t_k^{1,0}$, and for $j \leq 2k$, when $t_k^{1,0} \leq s \leq t_k^{1,1}$. Since this coefficient is actually zero, the assertion follows. $\square$

Therefore in the sequel we may assume that

$$j \geq \begin{cases} 2k, & \text{when } s \in \mathcal{I}_{2k-1}, \\ 2k+1, & \text{when } s \in \mathcal{I}_{2k}. \end{cases}$$

The functions $Q_j(n,s)$ were called "test functions" in [17]. The reason for this name will become clear below.

3.3 The main theorem

The next theorem is the main result in this chapter. It shows the importance of the test functions for investigations on the linear programming bound for spherical codes.

Theorem 3.3.1. *The bound $L_m(n,s)$ can be improved by means of a polynomial from $A_{n,s}$ of degree at least $m+1$ if and only if $Q_j(n,s)<0$ for some $j \geq m+1$. Moreover, if $Q_j(n,s)<0$ for some $j \geq m+1$, then $L_m(n,s)$ can be improved by a polynomial from $A_{n,s}$ of degree j.*

Proof. We give a proof for $m = 2k-1$. The proof for $m = 2k$ follows by the same arguments.

$\Rightarrow$ (necessity) We use Lemma 2.6.3a) several times. Let us assume that $Q_j(n,s) \geq 0$ for all integers $j \geq 2k$. For an arbitrary polynomial $f(t) \in A_{n,s}$ of degree $r \geq 2k$ we write

$$f(t) = g(t) + \sum_{i=2k}^{r} f_i P_i^{(n)}(t), \tag{3.3.1}$$

where $\deg(g) \leq 2k-1$. Then the first coefficients f_0 and g_0 in the Gegenbauer expansion of $f(t)$ and $g(t)$, respectively, are the same. For the calculation of g_0 we use (2.6.1) to obtain

$$f_0 = g_0 = \rho_k g(1) + \sum_{i=0}^{k-1} \rho_i g(\alpha_i). \tag{3.3.2}$$

We use (3.3.1) to substitute $g(\alpha_i)$, $i = 0,1,\ldots,k-1$, and $g(1)$ in (3.3.2) and obtain

$$\begin{aligned} f_0 &= \rho_k \left(f(1) - \sum_{j=2k}^{r} f_i \right) + \sum_{i=0}^{k-1} \rho_i \left[f(\alpha_i) - \sum_{j=2k}^{r} f_j P_j^{(n)}(\alpha_i) \right] \\ &= \rho_k f(1) + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) - \sum_{j=2k}^{r} f_j Q_j(n,s) \\ &\leq \rho_k f(1). \end{aligned}$$

For the last inequality we have made use of $f(t) \in A_{n,s}$ (i.e. $f(\alpha_i) \leq 0$ for $i = 0,1,\ldots,k-1$ and $f_i \geq 0$ for $i = 2k, 2k+1, \ldots, r$), $\rho_i > 0$ for $i = 0,1,\ldots,k$, and $Q_j(n,s) \geq 0$. We conclude that

$$\frac{f(1)}{f_0} \geq \frac{1}{\rho_k} = L_{2k-1}(n,s),$$

(see Lemma 2.6.3a)) i.e. the polynomial $f(t)$ does not improve the Levenshtein bound. Since we chose an arbitrary polynomial in $A_{n,s}$, it follows that no polynomial from $A_{n,s}$ can be used for improving the Levenshtein bound. This completes the proof of the necessity.

$\Leftarrow$ (sufficiency) Conversely, let us assume that $Q_j(n,s) < 0$ for some $j \geq 2k$. We shall construct a certain polynomial from $A_{n,s}$ of degree j which improves the Levenshtein bound.

We consider polynomials which can be simultaneously represented in the following two ways:

$$f(t) = g(t) + P_j^{(n)}(t) \tag{3.3.3}$$

$$= h(t) f_{2k-1}^{(n,s)}(t), \tag{3.3.4}$$

where $\deg(g) \leq 2k-1$ and $f_{2k-1}^{(n,s)}(t)$ is the corresponding Levenshtein polynomial.
We show that it is possible to construct $f(t)$ in such a way that the conditions **(A1)** and **(A2)** are satisfied. Let

$$f(t) = \sum_{i=0}^{j} f_i P_i^{(n)}(t).$$

It follows from (3.3.3) that $f_j = 1$ and $f_{2k} = f_{2k+1} = \cdots = f_{j-1} = 0$.
Denote

$$h(t) = a_0 t^{j-2k+1} + a_1 t^{j-2k} + \cdots + a_{j-2k} t + a_{j-2k+1}.$$

Then the coefficients $a_0, a_1, \ldots, a_{j-2k}$ can be uniquely determined by the triangular system of equations which can be obtained by equating the coefficients of the same degree of t in (3.3.3) and (3.3.4). Indeed, by $f_j = 1$ we find a_0, then by $f_{j-1} = 0$ we calculate a_1 and so on, finally computing a_{j-2k} by the equation $f_{2k} = 0$.
Therefore we have found the polynomial

$$h_1(t) = a_0 t^{j-2k+1} + a_1 t^{j-2k} + \cdots + a_{j-2k} t = h(t) - a_{j-2k+1}.$$

To find $h(t)$ itself, it remains to choose a_{j-2k+1} in such a way that $f(t) \in A_{n,s}$.
We already know that $f_i \geq 0$ for $i \geq 2k$. Let us consider the remaining coefficients f_i, $0 \leq i \leq 2k-1$. The polynomial

$$g_1(t) = P_j^{(n)}(t) - f_{2k-1}^{(n,s)}(t) h_1(t) = a_{j-2k+1} f_{2k-1}^{(n,s)}(t) - g(t) \tag{3.3.5}$$

has degree at most $2k-1$. Let consider the Gegenbauer expansion of $g_1(t)$ and $f_{2k-1}^{(n,s)}(t)$

$$g_1(t) = \sum_{i=0}^{2k-1} f_i' P_i^{(n)}(t)$$

and

$$f_{2k-1}^{(n,s)}(t) = \sum_{i=0}^{2k-1} f_i'' P_i^{(n)}(t).$$

Since $f(t) = P_j^{(n)}(t) + a_{j-2k+1} f_{2k-1}^{(n,s)}(t) - g_1(t)$ by (3.3.3) and (3.3.5), we obtain the equalities

$$f_i = a_{j-2k+1} f_i'' - f_i'$$

for $i = 0, 1, \ldots, 2k-1$. We need to choose a_{j-2k+1} to have $f_i \geq 0$ for all $i = 0, 1, \ldots, 2k-1$. This is possible because $f_i'' > 0$ for every $0 \leq i \leq 2k-1$ by Lemma 2.5.1. We therefore obtain that if

$$a_{j-2k+1} > \frac{f_i'}{f_i''}$$

for all $i = 0, 1, 2, \ldots, 2k-1$ then $f(t)$ satisfies **(A2)**. In particular, we have $f_0 > 0$ for this choice of a_{j-2k+1}.

Since $f_{2k-1}^{(n,s)}(t) \le 0$ for all $t \in [-1, s]$, it follows from the representation (3.3.4) that we must ensure $h(t) \ge 0$ for all $t \in [-1, s]$ in order to have $f(t) \le 0$ for all $t \in [-1, s]$. By the equality $h(t) = a_{j-2k+1} + h_1(t)$ we conclude that this aim will be achieved if we choose a_{j-2k+1} in such a way that

$$a_{j-2k+1} \ge \varepsilon = -\min\{h_1(t) : t \in [-1, s]\}$$

(ε exists and is uniquely determined).

Finally, we derive that if

$$a_{j-2k+1} > \max\left\{\varepsilon, \frac{f_0'}{f_0''}, \frac{f_1'}{f_1''}, \ldots, \frac{f_{2k-1}'}{f_{2k-1}''}\right\}$$

then we have $f(t) \in A_{n,s}$.

The above construction gives infinitely many polynomials from $A_{n,s}$. For each of them, as in the proof of necessity, we conclude that

$$\frac{f(1)}{f_0} < L_{2k-1}(n, s). \tag{3.3.6}$$

Indeed, we use the representation (3.3.3) to obtain as above

$$f_0 = \rho_k f(1) + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) - Q_j(n, s).$$

Since $f(\alpha_i) = h(\alpha_i) f_{2k-1}^{(n,s)}(\alpha_i) = 0$ by (3.3.4) (we recall that the α_i's are zeros of the Levenshtein's polynomial $f_{2k-1}^{(n,s)}(t)$) and $Q_j(n, s) < 0$, we obtain $f_0 > \rho_k f(1)$, which is equivalent to (3.3.6) because $f_0 > 0$ and $\rho_k = 1/L_{2k-1}(n, s)$. This completes the proof of the sufficiency and the whole theorem. □

Theorem 3.3.1 may be formulated as a necessary and sufficient condition for the global extremality of the Levenshtein polynomials. In this form it was included (in a more general setting) in "Handbook of Coding Theory", Chapter 6, Theorem 5.47 (reference [49]).

Corollary 3.3.2. *The Levenshtein polynomial $f_m^{(s)}(t)$ is $A_{n,s}$-global extremal if and only if $Q_j(n, s) \ge 0$ for all $j \ge m + 1$.*

The following "restricted" version of Theorem 3.3.1 (the proof is essentially the same) will be used (for $l = 2$) in the next section to derive a proof that the Levenshtein's polynomials are the best not only up to their degrees but to degree $m + 2$ as well.

Corollary 3.3.3. *The Levenshtein polynomial $f_m^{(s)}(t)$ gives the best upper bound on $A_{n,s}$ among all polynomials from $A_{n,s}$ of degree at most $m + l$ if and only if $Q_j(n, s) \ge 0$ for all $j = m + 1, \ldots, m + l$.*

Using a similar argument, if $Q_j(n,s) > 0$, one can construct polynomials

$$f(t) = g(t) - P_j^{(n)}(t) = h(t) f_m^{(n,s)}(t)$$

for which the only reason that they do not belong to $A_{n,s}$ is that $f_j = -1 < 0$. However, as before, we see that

$$\frac{f(1)}{f_0} < L_m(n,s).$$

Similar polynomials are shown to be useful (see [10, Theorems 3.1,3.2]) to prove that $f_j = 0$ for some $A_{n,s}$-extremal polynomials. In fact, better results in this direction are usually obtained by polynomials

$$f(t) = g(t) - f_{j_1} P_{j_1}^{(n)}(t) + P_{j_2}^{(n)}(t) = h(t) f_m^{(n,s)}(t) \notin A_{n,s},$$

where $f_{j_1} > 0$, $Q_{j_1}(n,s) > 0$, $Q_{j_2}(n,s) < 0$, and $m < j_1 < j_2$.

3.4 Extending the extremality of the Levenshtein's polynomials

We begin this section with a formula for the test functions $Q_j(n,s)$ in terms of some power sums S_l and R_l, the numbers b_{2l} (see Subsection 2.6.2), and some coefficients of the Gegenbauer polynomials.

Theorem 3.4.1. a) *For $s \in \mathcal{I}_{2k-1}$ and $r \geq k$*

$$Q_{2r}(n,s) = \sum_{l=k}^{r} (S_{2l} - b_{2l})\, a_{2r,2l} \tag{3.4.1}$$

and

$$Q_{2r+1}(n,s) = \sum_{l=k}^{r} S_{2l+1} a_{2r+1,2l+1}. \tag{3.4.2}$$

b) *For $s \in \mathcal{I}_{2k}$ and $r \geq k+1$*

$$Q_{2r}(n,s) = \sum_{l=k+1}^{r} (R_{2l} - b_{2l})\, a_{2r,2l} \tag{3.4.3}$$

and

$$Q_{2r-1}(n,s) = \sum_{l=k+1}^{r} R_{2l-1} a_{2r-1,2l-1}. \tag{3.4.4}$$

Proof. **a)** We use the defining formula (3.2.1) to obtain

$$Q_j(n,s) = \sum_{i=0}^{j} a_{j,i} S_i.$$

Then we subtract from this the equality $0 = \sum_{i=0}^{j} a_{j,i} b_i$ (see Lemma 2.6.8; for j odd this is simply "$0 = 0$") and take into account Lemma 2.6.6 to cancel the terms of indices at most $2k-1$.

b) This can be proved analogously. □

Using the next two assertions we prove that the Levenshtein bound $L_m(n,s)$ can not be improved by using polynomials of degree at most $m+2$. This strengthens the result of Sidelnikov [59] and shows that any improving polynomial would have degree at most $m+3$.

Lemma 3.4.2. a) *If $s \in \mathcal{I}_{2k-1}$, then $Q_{2k+1}(n,s) \geq 0$.*

b) *If $s \in \mathcal{I}_{2k}$, then $Q_{2k+1}(n,s) \geq 0$.*

Proof. **a)** It follows from Theorem 3.4.1a) that

$$Q_{2k+1}(n,s) = S_{2k+1} a_{2k+1,2k+1}$$

for $s \in \mathcal{I}_{2k-1}$. Since $a_{2k+1,2k+1} > 0$, it is enough to prove that $S_{2k+1} \geq 0$. We notice that $S_{2k+1} = 0$ for $s = t_{k-1}^{1,1}$ and prove that $S_{2k+1} > 0$ for $s \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right]$.

We consider (see Lemma 2.6.6) the equations

$$\begin{array}{l} \rho_0\alpha_0 + \rho_1\alpha_1 + \cdots + \rho_k = 0, \\ \rho_0\alpha_0^3 + \rho_1\alpha_1^3 + \cdots + \rho_k = 0, \\ \qquad\vdots \\ \rho_0\alpha_0^{2k-1} + \rho_1\alpha_1^{2k-1} + \cdots + \rho_k = 0, \\ \rho_0\alpha_0^{2k+1} + \rho_1\alpha_1^{2k+1} + \cdots + \rho_k = S_{2k+1}, \end{array} \tag{3.4.5}$$

as a linear system with respect to the weights $\rho_0, \rho_1, \ldots, \rho_k$. The number of equations is $k+1$ thus equal to the number of unknowns.

The determinant of (3.4.5) equals Vandermonde[1] like determinant

$$\Delta = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{k-1} & 1 \\ \alpha_0^3 & \alpha_1^3 & \cdots & \alpha_{k-1}^3 & 1 \\ & & \cdots & & \\ \alpha_0^{2k-1} & \alpha_1^{2k-1} & \cdots & \alpha_{k-1}^{2k-1} & 1 \\ \alpha_0^{2k+1} & \alpha_1^{2k+1} & \cdots & \alpha_{k-1}^{2k+1} & 1 \end{vmatrix} = V(\alpha_0^2, \alpha_1^2, \ldots, \alpha_{k-1}^2, 1) \prod_{i=0}^{k-1} \alpha_i. \tag{3.4.6}$$

[1] A Vandermonde matrix is a square matrix whose columns form a geometric progression. Consider the determinant

$$V(a_1, a_2, ..., a_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ & & \cdots & \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

Then the following is true.

(i) Determinant $V(a_1, a_2, \cdots, a_n)$ is a homogeneous polynomial in a_i of degree $n(n-1)/2$.

(ii) Determinant $V(a_1, a_2, \cdots, a_n)$ is divisible by $(a_j - a_i)$. It follows that $V(a_1, a_2, \cdots, a_n)$ is divisible by the product $\prod_{n \geq j > i \geq 1}(a_j - a_i)$.

(iii) Determinant $V(a_1, a_2, \cdots, a_n) = \prod_{n \geq j > i \geq 1}(a_j - a_i)$.

Since $\alpha_i \neq 0$ for all $i = 0, 1, \ldots, k$ and $|\alpha_i| \neq |\alpha_j|$ for $i \neq j$, we have $\Delta \neq 0$. Therefore (3.4.5) has a unique solution with respect to $\rho_0, \rho_1, \ldots, \rho_k$ and this solution must coincide with the weights $\rho_0, \rho_1, \ldots, \rho_k$ as defined by Levenshtein.

We calculate ρ_k by simple linear algebra to find

$$\rho_k = \frac{\Delta_{k+1}}{\Delta},$$

where

$$\Delta_{k+1} = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{k-1} & 0 \\ \alpha_0^3 & \alpha_1^3 & \cdots & \alpha_{k-1}^3 & 0 \\ & & \cdots & & \\ \alpha_0^{2k-1} & \alpha_1^{2k-1} & \cdots & \alpha_{k-1}^{2k-1} & 0 \\ \alpha_0^{2k+1} & \alpha_1^{2k+1} & \cdots & \alpha_{k-1}^{2k+1} & S_{2k+1} \end{vmatrix} = S_{2k+1} V(\alpha_0^2, \alpha_1^2, \ldots, \alpha_{k-1}^2) \prod_{i=0}^{k-1} \alpha_i.$$

Hence, we have

$$\rho_k = \frac{S_{2k+1} V(\alpha_0^2, \alpha_1^2, \ldots, \alpha_{k-1}^2)}{V(\alpha_0^2, \alpha_1^2, \ldots, \alpha_{k-1}^2, 1)} = \frac{S_{2k+1}}{\prod_{i=0}^{k-1}(1-\alpha_i^2)}. \tag{3.4.7}$$

Then $S_{2k+1} = \rho_k \prod_{i=0}^{k-1}(1-\alpha_i^2) > 0$ because $\rho_k > 0$ and $|\alpha_i| < 1$ for all $i = 0, 1, \ldots, k$.

b) Analogously to **a)**. □

Lemma 3.4.3. **a)** *If $s \in \mathcal{I}_{2k-1}$, then $Q_{2k}(n, s) \geq 0$.*

b) *If $s \in \mathcal{I}_{2k}$, then $Q_{2k+2}(n, s) \geq 0$.*

Proof. **a)** Denote

$$\begin{aligned} h(t) &= (1-t) \prod_{i=0}^{k-1} (t - \alpha_i) \\ &= (1-t)(t-s) T_{k-1}^{1,0}(t, s) \\ &= r_k m_{k-1} (1-t) \left(P_k^{1,0}(t) P_{k-1}^{1,0}(s) - P_{k-1}^{1,0}(t) P_k^{1,0}(s) \right), \end{aligned}$$

(cf. the Christofel-Darboux formula from Lemma 2.4.1).

We divide the polynomial $P_{2k}^{(n)}(t)$ by $h(t)$ to obtain

$$P_{2k}^{(n)}(t) = h(t) q(t) + r(t), \tag{3.4.8}$$

where $\deg r(t) < \deg h(t) = k + 1 \leq 2k - 1$. Then we have

$$Q_{2k}(n, s) = \rho_k + \sum_{i=0}^{k-1} \rho_i P_{2k}^{(n)}(\alpha_i) = \rho_k r(1) + \sum_{i=0}^{k-1} \rho_i r(\alpha_i).$$

It follows from the last observations and from Lemma 2.6.3a) that the test function $Q_{2k}(n, s)$ equals the coefficient f_0' in the Gegenbauer expansion of

$$r(t) = \sum_{i=0}^{\deg r(t)} f_i' P_i^{(n)}(t)$$

for every $s \in \mathcal{I}_{2k-1}$. Thus we need to prove that $f_0' \geq 0$ for $s \in \mathcal{I}_{2k-1}$.
If

$$h(t)q(t) = \sum_{i=0}^{2k} f_i'' P_i^{(n)}(t)$$

then (3.4.8) shows that $f_0' = -f_0''$. Now we prove that $f_0'' \leq 0$ for $s \in \mathcal{I}_{2k-1}$.
We write the polynomial $h(t)q(t)$ in the following form:

$$h(t)q(t) = r_k m_{k-1}(1-t)\left(P_k^{1,0}(t)P_{k-1}^{1,0}(s) - P_{k-1}^{1,0}(t)P_k^{1,0}(s)\right)\left(\sum_{i=0}^{k-1} q_i P_i^{1,0}(t)\right),$$

where we have expanded $q(t)$ in terms of the adjacent polynomials $\{P_i^{1,0}(t)\}_{i=0}^{\infty}$. Comparing the signs of the highest coefficient on both sides, we see that $q_{k-1} < 0$ (note that $P_{k-1}(s) > 0$ for $s \in \mathcal{I}_{2k-1}$).
Using consecutively (2.1.4) and the orthogonality relation (2.4.3) we obtain

$$\begin{aligned}
f_0'' &= c_n \int_{-1}^{1} h(t)q(t)(1-t^2)^{(n-3)/2}dt \\
&= r_{k-1}m_{k-1}c_n \int_{-1}^{1} \left(P_k^{1,0}(t)P_{k-1}^{1,0}(s) - P_{k-1}^{1,0}(t)P_k^{1,0}(s)\right) \cdot \\
&\qquad \left(\sum_{i=0}^{k-1} q_i P_i^{1,0}(t)\right)(1-t)(1-t^2)^{(n-3)/2}dt \\
&= r_{k-1}m_{k-1}c_n P_{k-1}^{1,0}(s) \int_{-1}^{1} P_k^{1,0}(t)\left(\sum_{i=0}^{k-1} q_i P_i^{1,0}(t)\right)(1-t)(1-t^2)^{(n-3)/2}dt \\
&\quad - r_{k-1}m_{k-1}c_n P_k^{1,0}(s) \int_{-1}^{1} P_{k-1}^{1,0}(t)\left(\sum_{i=0}^{k-1} q_i P_i^{1,0}(t)\right)(1-t)(1-t^2)^{(n-3)/2}dt \\
&= r_{k-1}m_{k-1}c_n P_{k-1}^{1,0}(s) \sum_{i=0}^{k-1} q_i \left(\int_{-1}^{1} P_k^{1,0}(t)P_i^{1,0}(t)(1-t)(1-t^2)^{(n-3)/2}dt\right) \\
&\quad - r_{k-1}m_{k-1}c_n P_k^{1,0}(s) \sum_{i=0}^{k-1} q_i \left(\int_{-1}^{1} P_{k-1}^{1,0}(t)P_i^{1,0}(t)(1-t)(1-t^2)^{(n-3)/2}dt\right) \\
&= -\frac{r_{k-1}m_{k-1}c_n}{c_n^{1,0}}\, P_k^{1,0}(s)\, q_{k-1}.
\end{aligned}$$

(Notice that $c_n = c_n^{1,0}$). Since $r_{k-1}m_{k-1}$, $P_k^{1,0}(s)$ and q_{k-1} in the last expression are positive for $s \in \mathcal{I}_{2k-1}$ we conclude that $f_0'' \leq 0$ whence $f_0' = Q_{2k}(n,s) \geq 0$ for $s \in \mathcal{I}_{2k-1}$. This completes the proof.

b) Analogously to **a)**. □

The main result in this section follows from the last two Lemmas and by Corollary 3.3.3 applied for $l = 2$.

Theorem 3.4.4. *The Levenshtein bound $L_m(n,s)$ can not be improved by polynomials from $A_{n,s}$ of degree at most $m+2$.*

We shall see in the next section that there exist values of n and s such that $Q_{m+3}(n,s) < 0$. As a by-product of the formulas in Theorem 3.4.1 and Lemma 3.4.3, we obtain the following inequalities.

Corollary 3.4.5. a) *For every $s \in \mathcal{I}_{2k-1}$*

$$S_{2k} = \rho_k + \sum_{i=0}^{k-1} \rho_i \alpha_i^{2k} \geq b_{2k}.$$

b) *For every $s \in \mathcal{I}_{2k}$*

$$S_{2k+2} = \gamma_k + \sum_{i=0}^{k} \gamma_i \beta_i^{2k+2} \geq b_{2k+2}.$$

Proof. a) It follows from Theorem 3.4.1b) that

$$Q_{2k}(n,s) = (S_{2k} - b_{2k})\, a_{2k,2k}$$

for $s \in \mathcal{I}_{2k-1}$. Since $a_{2k,2k} > 0$, Lemma 3.4.3 implies that $S_{2k} \geq b_{2k}$.
b) Analogously to **a)**. □

3.5 Some conditions for improving the Levenshtein bounds

It follows from the previous section that the first two test functions that are relevant for the Levenshtein bound $L_m(n,s)$, namely $Q_{m+1}(n,s)$ and $Q_{m+2}(n,s)$, are nonnegative. In this section we consider the function $Q_{2k+3}(n,s)$ which is either $Q_{m+4}(n,s)$ for $m = 2k-1$ or $Q_{m+3}(n,s)$ for $m = 2k$.
The next theorem gives formulas for $Q_{2k+3}(n,s)$ which turn out to be useful for the purposes of this section.

Theorem 3.5.1. a) *We have*

$$Q_{2k+3}(n,s) = S_{2k+1}\left[a_{2k+3,2k+3}(\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_{k-1}^2 + 1) + a_{2k+3,2k+1}\right]$$

for $s \in \mathcal{I}_{2k-1}$.

b) *We have*

$$Q_{2k+3}(n,s) = R_{2k+1}\left[a_{2k+3,2k+3}(1 + \beta_1^2 + \cdots + \beta_k^2) + a_{2k+3,2k+1}\right]$$

for $s \in \mathcal{I}_{2k}$.

Proof. **a)** It follows from Theorem 3.4.1 that

$$Q_{2k+3}(n,s) = S_{2k+1}a_{2k+3,2k+1} + S_{2k+3}a_{2k+3,2k+3}.$$

Therefore it is enough to prove that

$$S_{2k+3} = (\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_{k-1}^2 + 1)S_{2k+1}.$$

We consider the system (3.4.5) from the previous section together with the system

$$\left| \begin{array}{l} \rho_0\alpha_0 + \rho_1\alpha_1 + \cdots + \rho_k\alpha_k = 0, \\ \rho_0\alpha_0^3 + \rho_1\alpha_1^3 + \cdots + \rho_k\alpha_k^3 = 0, \\ \qquad\qquad \vdots \\ \rho_0\alpha_0^{2k-1} + \rho_1\alpha_1^{2k-1} + \cdots + \rho_k\alpha_k^{2k-1} = 0, \\ \rho_0\alpha_0^{2k+3} + \rho_1\alpha_1^{2k+3} + \cdots + \rho_k\alpha_k^{2k+3} = S_{2k+3}. \end{array} \right. \tag{3.5.1}$$

The system (3.5.1) is also linear with respect to ρ_0, ρ_1, …, ρ_k and has as many equations (namely $k+1$) as unknowns. We prove that it has a unique solution which therefore coincides with the weights ρ_0, ρ_1, …, ρ_k as defined by Levenshtein and the solution of (3.4.5). The determinant of (3.5.1) is

$$\Delta' = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{k-1} & 1 \\ \alpha_0^3 & \alpha_1^3 & \cdots & \alpha_{k-1}^3 & 1 \\ & & \ddots & & \\ \alpha_0^{2k-1} & \alpha_1^{2k-1} & \cdots & \alpha_{k-1}^{2k-1} & 1 \\ \alpha_0^{2k+3} & \alpha_1^{2k+3} & \cdots & \alpha_{k-1}^{2k+3} & 1 \end{vmatrix} = \prod_{i=1}^{k-1} \alpha_i \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha_0^2 & \alpha_1^2 & \cdots & \alpha_{k-1}^2 & 1 \\ & & \ddots & & \\ \alpha_0^{2k-2} & \alpha_1^{2k-2} & \cdots & \alpha_{k-1}^{2k-2} & 1 \\ \alpha_0^{2k+2} & \alpha_1^{2k+2} & \cdots & \alpha_{k-1}^{2k+2} & 1 \end{vmatrix}.$$

To calculate the last determinant (Δ'' say) we use the same tricks as in the calculation of a Vandermonde determinant.
We finally obtain

$$\Delta'' = V(\alpha_0^2, \alpha_1^2, \ldots, \alpha_{k-1}^2, 1)\left(1 + \sum_{i=0}^{k-1} \alpha_i^2\right).$$

In particular, we see that

$$\Delta' = \Delta'' \prod_{i=0}^{k-1} \alpha_i \neq 0.$$

Therefore, if we solve (3.5.1), then we should obtain the same answer as the solution of (3.4.5) gives. Thus our approach is to solve these two systems with respect to ρ_k and to equate the results.
From (3.5.1) we obtain

$$\rho_k = \frac{\Delta'_{k+1}}{\Delta'},$$

where

$$\Delta'_{k+1} = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{k-1} & 0 \\ \alpha_0^3 & \alpha_1^3 & \cdots & \alpha_{k-1}^3 & 0 \\ & & \ddots & & \\ \alpha_0^{2k-1} & \alpha_1^{2k-1} & \cdots & \alpha_{k-1}^{2k-1} & 0 \\ \alpha_0^{2k+3} & \alpha_1^{2k+3} & \cdots & \alpha_{k-1}^{2k+3} & S_{2k+3} \end{vmatrix} = S_{2k+3} V(\alpha_0^2, \alpha_1^2, \ldots, \alpha_{k-1}^2) \prod_{i=0}^{k-1} \alpha_i.$$

Hence we have

$$\rho_k = \frac{S_{2k+3}}{\left(1 + \sum_{i=0}^{k-1} \alpha_i^2\right) \prod_{i=0}^{k-1} (1 - \alpha_i^2)}.$$

We compare this to the expression (3.4.7) for ρ_k to obtain

$$S_{2k+3} = (\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_{k-1}^2 + 1) S_{2k+1}.$$

This completes the proof. Notice that $S_{2k+3} > 0$ follows in the same way from the last equality and $S_{2k+1} > 0$ what we proved in Lemma 3.4.2a).
b) This can be proved analogously. It follows from Theorem 3.4.1 that

$$Q_{2k+3}(n, s) = R_{2k+1} a_{2k+3,2k+1} + R_{2k+3} a_{2k+3,2k+3}.$$

Thus we have to prove that

$$R_{2k+3} = (1 + \beta_1^2 + \cdots + \beta_k^2) R_{2k+1}$$

(note that $\beta_0 = -1$).
When s belongs to the interval $\mathcal{I}_{2k}$, then we can derive the following two systems

$$\left| \begin{array}{l} \gamma_0\beta_0 + \gamma_1\beta_1 + \cdots + \gamma_{k+1} = 0 \\ \gamma_0\beta_0^3 + \gamma_1\beta_1^3 + \cdots + \gamma_{k+1} = 0 \\ \qquad\qquad \vdots \\ \gamma_0\beta_0^{2k-1} + \gamma_1\beta_1^{2k-1} + \cdots + \gamma_{k+1} = 0 \\ \gamma_0\beta_0^{2k+1} + \gamma_1\beta_1^{2k+1} + \cdots + \gamma_{k+1} = R_{2k+1} \end{array} \right. \qquad (3.5.2)$$

and

$$\left| \begin{array}{l} \gamma_0\beta_0 + \gamma_1\beta_1 + \cdots + \gamma_{k+1} = 0 \\ \gamma_0\beta_0^3 + \gamma_1\beta_1^3 + \cdots + \gamma_{k+1} = 0 \\ \qquad\qquad \vdots \\ \gamma_0\beta_0^{2k-1} + \gamma_1\beta_1^{2k-1} + \cdots + \gamma_{k+1} = 0 \\ \gamma_0\beta_0^{2k+3} + \gamma_1\beta_1^{2k+3} + \cdots + \gamma_{k+1} = R_{2k+3} \end{array} \right. \qquad (3.5.3)$$

Now we resolve both systems with respect to γ_{k+1} and equate the results:

$$\gamma_{k+1} = \frac{\begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_k & 0 \\ \beta_0^3 & \beta_1^3 & \cdots & \beta_k^3 & 0 \\ & & \ddots & & \\ \beta_0^{2k-1} & \beta_1^{2k-1} & \cdots & \beta_k^{2k-1} & 0 \\ \beta_0^{2k+1} & \beta_1^{2k+1} & \cdots & \beta_k^{2k+1} & R_{2k+1} \end{vmatrix}}{\begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_k & 1 \\ \beta_0^3 & \beta_1^3 & \cdots & \beta_k^3 & 1 \\ & & \ddots & & \\ \beta_0^{2k-1} & \beta_1^{2k-1} & \cdots & \beta_k^{2k-1} & 1 \\ \beta_0^{2k+1} & \beta_1^{2k+1} & \cdots & \beta_k^{2k+1} & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_k & 0 \\ \beta_0^3 & \beta_1^3 & \cdots & \beta_k^3 & 0 \\ & & \ddots & & \\ \beta_0^{2k-1} & \beta_1^{2k-1} & \cdots & \beta_k^{2k-1} & 0 \\ \beta_0^{2k+3} & \beta_1^{2k+3} & \cdots & \beta_k^{2k+3} & R_{2k+3} \end{vmatrix}}{\begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_k & 1 \\ \beta_0^3 & \beta_1^3 & \cdots & \beta_k^3 & 1 \\ & & \ddots & & \\ \beta_0^{2k-1} & \beta_1^{2k-1} & \cdots & \beta_k^{2k-1} & 1 \\ \beta_0^{2k+3} & \beta_1^{2k+3} & \cdots & \beta_k^{2k+3} & 1 \end{vmatrix}}.$$

□

We continue the investigation of the formulas from Theorem 3.5.1.

Corollary 3.5.2. **a)** *Let $s \in \mathcal{I}_{2k-1}$. Then $Q_{2k+3}(n,s) < 0$ if and only if*

$$\sum_{i=0}^{k-1} \alpha_i^2 - \frac{2k^2+k+1-n}{n+4k+2} < 0.$$

b) *Let $s \in \mathcal{I}_{2k}$. Then $Q_{2k+3}(n,s) < 0$ if and only if*

$$\sum_{i=1}^{k} \beta_i^2 - \frac{2k^2+k+1-n}{n+4k+2} < 0.$$

Proof. **a)** We have

$$Q_{2k+3}(n,s) = S_{2k+1} a_{2k+3,2k+3} \left(\sum_{i=0}^{k-1} \alpha_i^2 + \frac{a_{2k+3,2k+1}}{a_{2k+3,2k+3}} - 1 \right).$$

To obtain the desired inequality, we replace the ratio $a_{2k+3,2k+1}/a_{2k+3,2k+3}$ by $-(2k^2+5k+3)/(n+4k+2)$ (see the remark that follows Lemma 3.1.2) and subtract 1.
b) Similar to **a)**. □

For small values of k, we are already able to deal with the conditions from Corollary 3.5.2. Later on we consider the general case.

Example 3.5.3. *($k=2$; improving the bound $L_3(n,s)$)*
For $s \in \mathcal{I}_3 = \left[t_1^{1,1}, t_2^{1,0}\right] = \left[0, \frac{\sqrt{n+3}-1}{n+2}\right]$ the inequality $Q_7(n,s) < 0$ is equivalent to

$$\left(\frac{1+s}{1+ns}\right)^2 + s^2 < \frac{11-n}{n+10}. \tag{3.5.4}$$

Indeed, in this case we have $k = 2$,

$$f_3^{(n,s)}(t) = (t-s)\left(t + \frac{1+s}{1+ns}\right)^2,$$

whence $\alpha_0 = -(1+s)/(1+ns)$ *and* $\alpha_1 = s$. *Then, by Corollary 3.5.2a), we obtain (3.5.4). It follows that the Levenshtein bound* $L_3(n,s)$ *can be improved for these values (but possibly not only for them) of* n *and* s *for which (3.5.4) is satisfied.*

It is obvious that (3.5.4) can not be satisfied for $n \geq 11$. *Therefore we have to consider dimensions* $3 \leq n \leq 10$. *A little algebra shows that in this case (3.5.4) is equivalent to the inequality*

$$n^2(n+10)s^4+2n(n+10)s^3+(n^3-11n^2+2n+20)s^2+2(n^2-10n+10)s+2n-1 < 0, \quad (3.5.5)$$

where we have made use of the conditions $3 \leq n \leq 10$ *and* $s \in \mathcal{I}_3 = \left[0, \frac{\sqrt{n+3}-1}{n+2}\right]$.

We avoid the analytical solution but, instead, explain the MAPLE results. They show that (3.5.5) does not have a solution for $7 \leq n \leq 10$. *No solutions of (3.5.5) for* $n = 6$ *belong to* $\mathcal{I}_3$. *The results for* $3 \leq n \leq 5$ *are given in Table 3.1.*

n	$s_0(n)$	$t_2^{1,0}$
3	0.1845211	0.2898979
4	0.1830127	0.2742918
5	0.2	0.2612038

Table 3.1: Solutions of (3.5.4) and (3.5.5) in $\mathcal{I}_3$ *for* $3 \leq n \leq 5$

Therefore, the bound $L_3(n,s)$ *can be improved for* $3 \leq n \leq 5$ *(as given in Table 3.1) in the intervals* $\left(s_0(n), \frac{\sqrt{n+3}-1}{n+2}\right]$.

Example 3.5.4. *($k = 2$; improving the bound $L_4(n,s)$)*

For $s \in \mathcal{I}_4 = \left[t_2^{1,0}, t_2^{1,1}\right] = \left[\frac{\sqrt{n+3}-1}{n+2}, \frac{1}{\sqrt{n+2}}\right]$ *the inequality* $Q_7(n,s) < 0$ *is equivalent to*

$$\frac{1}{s^2(1+n)^2} + s^2 < \frac{11-n}{n+10}. \quad (3.5.6)$$

In this case we have

$$f_4^{(n,s)}(t) = (t+1)(t-s)\left(t + \frac{1}{s(n+2)}\right)^2,$$

whence $\beta_1 = -1/s(n+2)$ *and* $\beta_2 = s$. *Then, by Corollary 3.5.2b), we obtain (3.5.6). It follows that the Levenshtein bound* $L_4(n,s)$ *can be improved for these values (but possibly not only for them) of* n *and* s *for which (3.5.6) is satisfied.*

It is obvious that (3.5.6) can not be satisfied for $n \geq 11$. *It is equivalent to the following bi-quadratic inequality*

$$(n^3+14n^2+44n+40)s^4+(n^3-7n^2-40n-44)s^2+n+10<0. \tag{3.5.7}$$

Here, an analytical solution is easier. However, we firstly notice that it follows from the general case (see Corollary 3.5.16 below) that $L_4(n,s)$ *can be improved in the whole interval* $\mathcal{I}_4$ *for* $n \leq 2^2+2=6$. *Since (3.5.6) does not have solutions for* $n \geq 8$ *it remains to consider it for* $n=7$ *only, i.e. we have to find all solutions of* $1377s^4-324s^2+17<0$ *which belong to* $\mathcal{I}_4=\left[\frac{\sqrt{10}-1}{9},\frac{1}{3}\right] \approx [0.24025, 0.33333]$.
We conclude that the bound $L_4(7,s)$ *can be improved for*

$$s \in \left(\frac{\sqrt{1190}-\sqrt{34}}{102}, \frac{1}{3}\right] \approx (0.28103, 0.33333] \subset \mathcal{I}_4.$$

Example 3.5.5. *(*$k=3$*; improving the bound* $L_5(n,s)$*) For* $s \in \mathcal{I}_5=\left[t_2^{1,1}, t_3^{1,0}\right]=\left[\frac{1}{\sqrt{n+2}}, t_3^{1,0}\right]$ *the inequality* $Q_9(n,s)<0$ *is equivalent to*

$$\frac{(2s(1+s))^2}{[(n+2)s^2+2s-1]^2}-\frac{2(3-(n+2)s^2)}{(n+2)[(n+2)s^2+2s-1]}+s^2<\frac{22-n}{n+14}. \tag{3.5.8}$$

Here $\alpha_2=s$ *and the numbers* α_0 *and* α_1 *are the roots of the quadratic equation*

$$(n+2)[(n+2)s^2+2s-1]t^2+2s(s+1)(n+2)t+3-(n+2)s^2=0.$$

We use the Viète formulas

$$\alpha_0+\alpha_1=-\frac{2s(s+1)}{(n+2)s^2+2s-1}$$

and

$$\alpha_0\alpha_1=\frac{3-(n+2)s^2}{(n+2)[(n+2)s^2+2s-1]}$$

to compute

$$\alpha_0^2+\alpha_1^2=(\alpha_0+\alpha_1)^2-2\alpha_0\alpha_1=\frac{4s^2(1+s)^2}{[(n+2)s^2+2s-1]^2}-\frac{2(3-(n+2)s^2)}{(n+2)[(n+2)s^2+2s-1]}.$$

Then Corollary 3.5.2a) leads to (3.5.8). It follows that the Levenshtein bound $L_5(n,s)$ *can be improved for these values (but possibly not only for them) of* n *and* s *for which (3.5.8) is satisfied.*
As in Example 3.5.3 we obtain by MAPLE a sixth degree inequality for s *which must be solved in dimensions* $3 \leq n \leq 21$ *and for* $s \in \mathcal{I}_5=\left[\frac{1}{\sqrt{n+2}}, t_3^{1,0}\right)$. *This inequality does not have solutions for* $12 \leq n \leq 21$. *No solutions belonging to* $\mathcal{I}_5$ *appear in dimension* $n=11$. *The results for* $3 \leq n \leq 10$ *are given in Table 3.2.*

n	$s_0(n)$	$t_3^{1,0}$
3	0.5048054373	0.5753189235
4	0.4605005478	0.5379862044
5	0.4308531120	0.5077876296
6	0.4094610855	0.4826149646
7	0.3938450834	0.4611587038
8	0.3829844486	0.4425522091
9	0.3768786881	0.4261936288
10	0.3772571414	0.4116488702

Table 3.2: Solutions of (3.5.8) in $\mathcal{I}_5$ for $3 \leq n \leq 10$

The last example suggests what we have to do in the general case. We proceed with investigations of the sign of $Q_{2k+3}(n,s)$ for $s \in \mathcal{I}_{2k-1}$. Thus we study the possibilities for improving the bounds $L_{2k-1}(n,s)$ with polynomials of degree $2k+3$.
To find the sum $\sum_{i=0}^{k-1} \alpha_k^2$, which is required in Corollary 3.5.2a), we need to express the sums $\sum_{i=0}^{k-1} \alpha_i$ and $\sum_{0 \leq i<j \leq k} \alpha_i \alpha_j$ as functions of n, k and s.

Lemma 3.5.6. *For every $s \in \mathcal{I}_{2k-1}$, $k \geq 2$, the numbers $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ satisfy the equalities*

$$\sum_{i=0}^{k-1} \alpha_i = -\frac{k}{n+2k-2} X, \tag{3.5.9}$$

$$\sum_{0 \leq i<j \leq k-1} \alpha_i \alpha_j = -\frac{k^2-k}{2(n+2k-4)} + \frac{k(k-1)}{(n+2k-2)(n+2k-4)} X, \tag{3.5.10}$$

where

$$X = 1 - \frac{(n+2k-1)(n+k-2)}{k(n+2k-3)} \cdot \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)}.$$

Proof. The numbers α_1, α_2, $\ldots$, α_{k-1} are defined (see (2.5.2) and Subsection 2.6.2) as the roots of the equation

$$(t-s)T_{k-1}^{1,0}(t,s) = 0.$$

From the Christofel-Darboux formula, the left hand side equals the polynomial

$$q(t) = P_k^{1,0}(t) P_{k-1}^{1,0}(s) - P_k^{1,0}(s) P_{k-1}^{1,0}(t)$$

up to multiplication by a nonzero constant. We expand the polynomial $q(t)$ with respect to t and obtain

$$
\begin{aligned}
q(t) &= \frac{a_{k,k} r_k P_{k-1}^{1,0}(s)}{\sum_{i=0}^{k} r_i} t^k \\
&+ a_{k-1,k-1} r_{k-1} \left(\frac{P_{k-1}^{1,0}(s)}{\sum_{i=0}^{k} r_i} - \frac{P_k^{1,0}(s)}{\sum_{i=0}^{k-1} r_i} \right) t^{k-1} \\
&+ \left[\frac{(a_{k,k-2} r_k + a_{k-2,k-2} r_{k-2}) P_{k-1}^{1,0}(s)}{\sum_{i=0}^{k} r_i} - \frac{a_{k-2,k-2} r_{k-2} P_k^{1,0}(s)}{\sum_{i=0}^{k-1} r_i} \right] t^{k-2} + \ldots
\end{aligned}
$$

We actually need the first three coefficients of $q(t)$ to apply the Viète formulas. Therefore, we have

$$
\begin{aligned}
\sum_{i=0}^{k-1} \alpha_i &= - \frac{a_{k-1,k-1} r_{k-1} \left(\frac{P_{k-1}^{1,0}(s)}{\sum_{i=0}^{k} r_i} - \frac{P_k^{1,0}(s)}{\sum_{i=0}^{k-1} r_i} \right)}{\frac{a_{k,k} r_k P_{k-1}^{1,0}(s)}{\sum_{i=0}^{k} r_i}} \\
&= - \frac{a_{k-1,k-1} r_{k-1}}{a_{k,k} r_k} \left(1 - \frac{\sum_{i=0}^{k} r_i}{\sum_{i=0}^{k-1} r_i} \cdot \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)} \right) \\
&= - \frac{k}{n+2k-2} \left(1 - \frac{(n+2k-1)(n+k-2)}{k(n+2k-3)} \cdot \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)} \right).
\end{aligned}
$$

Here, we use that the ratio $a_{k-1,k-1}/a_{k,k}$ is in fact the constant $1/m_{k-1}$ defined through equality (2.4.1). The constants r_i are taken from (2.2.1) and $\sum_{i=0}^{k} r_i$ is equal to the Delsarte-Goethals-Seidel bound $R(n, 2k)$.

Similarly, we obtain

$$
\begin{aligned}
\sum_{0 \le i < j \le k-1} \alpha_i \alpha_j &= \frac{\frac{(a_{k,k-2} r_k + a_{k-2,k-2} r_{k-2}) P_{k-1}^{1,0}(s)}{\sum_{i=0}^{k} r_i} - \frac{a_{k-2,k-2} r_{k-2} P_k^{1,0}(s)}{\sum_{i=0}^{k-1} r_i}}{\frac{a_{k,k} r_k P_{k-1}^{1,0}(s)}{\sum_{i=0}^{k} r_i}} \\
&= \frac{a_{k,k-2}}{a_{k,k}} + \frac{a_{k-2,k-2} r_{k-2}}{a_{k,k} r_k} \left(1 - \frac{\sum_{i=0}^{k} r_i}{\sum_{i=0}^{k-1} r_i} \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)} \right) \\
&= -\frac{k^2 - k}{2(n+2k-4)} \\
&\quad + \frac{k(k-1)}{(n+2k-2)(n+2k-4)} \left(1 - \frac{(n+2k-1)(n+k-2)}{k(n+2k-3)} \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)} \right).
\end{aligned}
$$

□

It follows from Lemma 3.5.2a) and equalities (3.5.9) and (3.5.10), that we have to investigate the sign of the function

$$\begin{aligned} G(n,k,s) &= \sum_{i=0}^{k-1} \alpha_i^2 - \frac{2k^2+k+1-n}{n+4k+2} \\ &= \left(\sum_{i=0}^{k-1} \alpha_i\right)^2 - 2\left(\sum_{0\le i<j\le k-1} \alpha_i\alpha_j\right) - \frac{2k^2+k+1-n}{n+4k+2} \\ &= \frac{k^2}{(n+2k-2)^2}X^2 - 2\frac{k(k-1)}{(n+2k-2)(n+2k-4)}X \\ &\quad + \frac{k(k-1)}{n+2k-4} - \frac{2k^2+k+1-n}{n+4k+2}, \end{aligned}$$

where

$$X = 1 - \frac{(n+2k-1)(n+k-2)}{k(n+2k-3)} \cdot \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)},$$

and s belongs to the interval $\mathcal{I}_{2k-1} = \left[t_{k-1}^{1,1}, t_k^{1,0}\right]$.

Lemma 3.5.7. *For fixed n and k, the function $G(n,k,s)$ is decreasing in s in the interval $\mathcal{I}_{2k-1} = \left[t_{k-1}^{1,1}, t_k^{1,0}\right]$.*

Proof. The function $G(n,k,s)$ is quadratic with respect to X. Since $s \in \left[t_{k-1}^{1,1}, t_k^{1,0}\right] \subset \left(t_{k-1}^{1,0}, t_k^{1,0}\right]$, Lemma 2.4.6a) says that the ratio $P_k^{1,0}(s)/P_{k-1}^{1,0}(s)$ increases in $\mathcal{I}_{2k-1}$. Therefore X decreases in s in the same interval and we need to determine the numbers

$$X_1 = 1 - \frac{(n+2k-1)(n+k-2)}{k(n+2k-3)} \cdot \frac{P_k^{1,0}\left(t_{k-1}^{1,1}\right)}{P_{k-1}^{1,0}\left(t_{k-1}^{1,1}\right)}$$

and

$$X_2 = 1 - \frac{(n+2k-1)(n+k-2)}{k(n+2k-3)} \cdot \frac{P_k^{1,0}\left(t_k^{1,0}\right)}{P_{k-1}^{1,0}\left(t_k^{1,0}\right)}$$

(the end points of the interval of variation of X).

We now calculate the numbers X_1 and X_2 as functions of n and k. We have

$$\begin{aligned} X_1 &= 1 - \frac{(n+2k-1)(n+k-2)}{k(n+2k-3)} \cdot \frac{P_k^{1,0}\left(t_{k-1}^{1,1}\right)}{P_{k-1}^{1,0}\left(t_{k-1}^{1,1}\right)} \\ &= 1 + \frac{(n+2k-1)(n+k-2)(n+2k-3)}{k(n+2k-3)(n+2k-1)} \\ &= \frac{n+2k-2}{k}, \end{aligned}$$

using Lemma 3.1.3a), and

$$X_2 = 1,$$

because $P_k^{1,0}\left(t_k^{1,0}\right) = 0$.

We can already locate the numbers X_1 and X_2 with respect to the minimum of the graph of the quadratic function

$$\begin{aligned} g(X) &= G(n,k,s) \\ &= \frac{k^2}{(n+2k-2)^2}X^2 - \frac{2k(k-1)}{(n+2k-2)(n+2k-4)}X \\ &\quad + \frac{k(k-1)}{n+2k-4} - \frac{2k^2+k-n+1}{n+4k+2}. \end{aligned}$$

The minimum of $g(X)$ is attained at the point

$$X_0 = \frac{(k-1)(n+2k-2)}{k(n+2k-4)}.$$

We have

$$X_0 - X_2 = X_0 - 1 = -\frac{n-2}{k(n+2k-4)} < 0$$

for every $n \geq 3$ and $k \geq 2$. This shows that $X_0 < 1 = X_2 < X_1$ i.e. X_2 and X_1 lie on the left side of X_0. Hence $g(X)$ decreases from $g(X_1)$ to $g(X_2)$ when X decreases from X_1 to X_2. This means that $G(n,k,s)$ decreases in s in the whole interval $\mathcal{I}_{2k-1}$. This completes the proof. $\square$

Thus we need to consider the sign of the function $G(n,k,s)$ in the end points of the interval $\mathcal{I}_{2k-1} = [t_{k-1}^{1,1}, t_k^{1,0}]$. Define the functions

$$\varphi_1(n,k) = G\left(n,k,t_{k-1}^{1,1}\right) = g(X_1)$$

and

$$\varphi_2(n,k) = G\left(n,k,t_k^{1,0}\right) = g(X_2).$$

From the above we have

$$\varphi_1(n,k) > G(n,k,s) > \varphi_2(n,k)$$

for all $s \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right)$. We calculate $\varphi_1(n,k)$ and $\varphi_2(n,k)$.

Lemma 3.5.8. *For every $n \geq 3$ and $k \geq 2$ we have*

$$\varphi_1(n,k) = \frac{(4-n)k^2 + 4(n-2)k + 2n^2 - 5n}{(n+2k-4)(n+4k+2)}, \tag{3.5.11}$$

$$\varphi_2(n,k) = \frac{(n-2)(n+2k-1)(n-k^2-2)}{(n+4k+2)(n+2k-2)^2}. \tag{3.5.12}$$

Proof. Plug $X_1 = \frac{n+2k-2}{k}$ and $X_2 = 1$ in $g(X)$. $\square$

We can already describe the behaviour of the test function $Q_{2k+3}(n,s)$ for the odd bounds $L_{2k-1}(n,s)$.

Theorem 3.5.9. *Let $n \geq 3$, $k \geq 2$ and $s \in \left[t_{k-1}^{1,1}, t_k^{1,0}\right]$. Then the function $Q_{2k+3}(n,s)$ has the following properties:*

a) *If $k \geq 9$ and*

$$3 \leq n \leq \frac{k^2 - 4k + 5 + \sqrt{k^4 - 8k^3 - 6k^2 + 24k + 25}}{4},$$

then $Q_{2k+3}(n,s) < 0$ for all $s \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right]$.

b) *If $k \geq 9$ and*

$$\frac{k^2 - 4k + 5 + \sqrt{k^4 - 8k^3 - 6k^2 + 24k + 25}}{4} \leq n \leq k^2 + 1$$

or if $2 \leq k \leq 8$ and $3 \leq n \leq k^2 + 1$ then there exists $s_0 = s_0(n,k) \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right]$ such that

$$\begin{aligned} &Q_{2k+3}(n,s) > 0, \qquad \textit{for all } s \in \left[t_{k-1}^{1,1}, s_0\right), \\ &Q_{2k+3}(n,s_0) = 0, \\ &Q_{2k+3}(n,s) < 0, \qquad \textit{for all } s \in \left(s_0, t_k^{1,0}\right]. \end{aligned} \tag{3.5.13}$$

c) *If $n \geq k^2 + 2$ then $Q_{2k+3}(n,s) \geq 0$ for all $s \in \left(t_k^{1,0}, t_k^{1,1}\right]$.*

Proof. **a)** For $k \geq 9$, all values of n such that

$$3 \leq n \leq \frac{k^2 - 4k + 5 + \sqrt{k^4 - 8k^3 - 6k^2 + 24k + 25}}{4}$$

are solutions of the inequality $\varphi_1(n,k) < 0$ (see (3.5.11)). Therefore $G(n,k,s) < 0$ for all $s \in \left[t_k^{1,0}, t_k^{1,1}\right]$ in this case. This means that $Q_{2k+3}(n,s) < 0$ for $s \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right]$.
b) Inequality $\varphi_1(n,k) < 0$ does not have any solutions $n \geq 3$ for $2 \leq k \leq 8$. For $k \geq 9$ and

$$\frac{k^2 - 4k + 5 + \sqrt{k^4 - 8k^3 - 6k^2 + 24k + 25}}{4} \leq n \leq k^2 + 1,$$

we have $\varphi_1(n,k) \geq 0$. In both cases $n < k^2 + 2$ and we have

$$\varphi_1(n,k) > 0 > \varphi_2(n,k).$$

This means that the function $G(n,k,s)$ decreases from the positive value $\varphi_1(n,k)$ to the negative value $\varphi_2(n,k)$. Since $G(n,k,s)$ is continuous, there exists a value s_0 with the required properties. Therefore $Q_{2k+3}(n,s)$ behaves as described in Theorem 3.5.9.
c) In this case we have $\varphi_2(n,k) \geq 0$. Therefore

$$\varphi_1(n,k) \geq G(n,k,s) \geq \varphi_2(n,k) \geq 0,$$

which means that $Q_{2k+3}(n,s) \geq 0$. □

We are now in position to state the main theorem concerning the impact of the test functions $Q_{2k+3}(n,s)$ on the possibilities for improving the odd bounds $L_{2k-1}(n,s)$.

Corollary 3.5.10. *Let $n \geq 3$ and $k \geq 2$.*

a) *If $k \geq 9$ and*

$$3 \leq n \leq \frac{k^2-4k+5+\sqrt{k^4-8k^3-6k^2+24k+25}}{4},$$

then the Levenshtein bound $L_{2k-1}(n,s)$ can be improved in the interval $\left(t_{k-1}^{1,1}, t_k^{1,0}\right] = \mathcal{I}_{2k-1} \backslash \{t_{k-1}^{1,1}\}$.

b) *If $k \geq 9$ and*

$$\frac{k^2-4k+5+\sqrt{k^4-8k^3-6k^2+24k+25}}{4} \leq n \leq k^2+1,$$

then there exists a number $s_0 = s_0(n,k) \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right)$ such that the Levenshtein bound $L_{2k-1}(n,s)$ can be improved in the interval $\left(s_0, t_k^{1,0}\right]$.

Proof. The proof follows by the results in the counterparts of **a)** and **b)** in Theorem 3.5.9. □

Remark 3.5.11. *Since the test functions are continuous we conclude that $Q_{2k+3}(n,t_k^{1,0}) < 0$ for $n < k^2+2$.*

We proceed by investigating the sign of the test functions $Q_{2k+3}(n,s)$ for $s \in \mathcal{I}_{2k}$. Thus, we study the possibilities for improving the bounds $L_{2k}(n,s)$ by means of polynomials of degree $2k+3$. The situation is somewhat simpler.
To find the sum $\sum_{i=0}^{k} \beta_k^2 = 1 + \sum_{i=1}^{k} \beta_k^2$ which is required in Corollary 3.5.2b) we need to express $\sum_{i=1}^{k} \beta_i$ and $\sum_{1 \leq i < j \leq k} \beta_i \beta_j$ as functions of n, k and s.

Lemma 3.5.12. *For every $s \in \mathcal{I}_{2k}$, $k \geq 2$, the numbers $\beta_1, \beta_2, \ldots, \beta_k$ satisfy the equalities*

$$\sum_{i=1}^{k} \beta_i = \frac{(n+k-1)P_k^{1,1}(s)}{(n+2k-2)P_{k-1}^{1,1}(s)}, \tag{3.5.14}$$

$$\sum_{1 \leq i < j \leq k} \beta_i \beta_j = -\frac{k(k-1)}{2(n+2k-2)}. \tag{3.5.15}$$

Proof. The numbers β_1, β_2, $\ldots$, β_k are defined (see (2.5.2) and Subsection 2.6.2) as the roots of the equation

$$(t-s)T_{k-1}^{1,1}(t) = 0.$$

From the Christofel-Darboux formula, up to multiplication by a nonzero constant, the left hand side equals

$$P_k^{1,1}(t)P_{k-1}^{1,1}(s) - P_k^{1,1}(s)P_{k-1}^{1,1}(t).$$

We expand the last polynomial with respect to t and obtain

$$r(t) = a_{k,k}P_{k-1}^{1,1}(s)t^k + a_{k-1,k-1}P_k^{1,1}(s)t^{k-1} + a_{k,k-2}P_{k-1}^{1,1}(s)t^{k-2} + \cdots$$

(note that $P_i^{1,1}(t) = P_i^{(n+2)}(t)$ and that the $a_{i,j}$'s are the coefficients of Gegenbauer polynomial of degree j and dimension $n+2$, namely $P_i^{(n+2)}(t)$).
Therefore, we have by the Viète formulas

$$\sum_{i=1}^{k} \beta_i = -\frac{a_{k-1,k-1}P_k^{1,1}(s)}{a_{k,k}P_{k-1}^{1,1}(s)} = \frac{(n+k-1)P_k^{(n+2)}(s)}{(n+2k-2)P_{k-1}^{(n+2)}(s)}$$

(the ratio $a_{k-1,k-1}/a_{k,k}$ is in fact the constant $1/m_{k-1}$ calculated for dimension $n+2$, i.e. $m_k = (n+2k-2)/(n+k-1)$) and

$$\sum_{1 \le i < j \le k} \beta_i\beta_j = \frac{a_{k,k-2}}{a_{k,k}} = -\frac{k(k-1)}{2(n+2k-2)}$$

(see Lemma 3.1.2 which also must be recalculated for dimension $n+2$). □

It follows from Lemma 3.5.2b), (3.5.14) and (3.5.15) that we have to investigate the sign of the function

$$\begin{aligned} H(n,k,s) &= \sum_{i=1}^{k} \beta_i^2 - \frac{2k^2+k+1-n}{n+4k+2} \\ &= \left[\frac{(n+k-1)P_k^{1,1}(s)}{(n+2k-2)P_{k-1}^{1,1}(s)}\right]^2 + \frac{k(k-1)}{n+2k-2} - \frac{2k^2+k+1-n}{n+4k+2}. \end{aligned} \tag{3.5.16}$$

Lemma 3.5.13. *For fixed n and k, the function $H(n,k,s)$ is decreasing in s in the interval $\left[t_k^{1,0}, t_k^{1,1}\right]$.*

Proof. The function $H(n,k,s)$ is quadratic with respect to $Y = P_k^{1,1}(s)/P_{k-1}^{1,1}(s)$ and only Y in the definition of $H(n,k,s)$ depends on s. It follows from Lemma 2.4.6b) that Y increases with s. Since

$$\frac{P_k^{1,1}(t_k^{1,1})}{P_{k-1}^{1,1}(t_k^{1,1})} < 0,$$

we conclude that Y is negative in the whole interval under consideration. Thus after squaring it becomes decreasing in s and so does the function $H(n,k,s)$ with respect to s. This completes the proof. □

Thus we need to consider the sign of the function $H(n,k,s)$ in the end points of the interval $\left[t_k^{1,0}, t_k^{1,1}\right]$. Define

$$\psi_1(n,k) = H(n, t_k^{1,0})$$

and

$$\psi_2(n,k) = H(n, t_k^{1,1}),$$

(note that $\psi_1(n,k) = \varphi_2(n,k)$ because of the continuity of the test functions). Then we have

$$\psi_1(n,k) > H(n,k,s) > \psi_2(n,k)$$

for all $s \in (t_k^{1,0}, t_k^{1,1})$. We now calculate $\psi_1(n,k)$ and $\psi_2(n,k)$.

Lemma 3.5.14. *For every $n \geq 3$ and $k \geq 2$ we have*

$$\psi_1(n,k) = \frac{(n-2)(n+2k-1)(n-k^2-2)}{(n+4k+2)(n+2k-2)^2}, \tag{3.5.17}$$

$$\psi_2(n,k) = \frac{n^2-(k^2+3)n+2-2k}{(n+4k+2)(n+2k-2)}. \tag{3.5.18}$$

Proof. The value of $\psi_2(n,k)$ follows easily since $P_k^{1,1}(t_k^{1,1}) = 0$. For $\psi_1(n,k)$ we may use Lemma 3.1.3b) to replace the ratio $P_k^{1,1}(t_k^{1,0})/P_{k-1}^{1,1}(t_k^{1,0})$ by $-n/(n+k-1)$. After some calculations we obtain (3.5.17). □

We are now in a position to state the main theorem concerning $Q_{2k+3}(n,s)$ for the bounds $L_{2k}(n,s)$.

Theorem 3.5.15. *Let $n \geq 3$, $k \geq 2$ and $s \in \left[t_k^{1,0}, t_k^{1,1}\right]$. Then the function $Q_{2k+3}(n,s)$ has the following properties:*

a) *If $3 \leq n \leq k^2+1$ then $Q_{2k+3}(n,s) < 0$ for all $s \in \left[t_k^{1,0}, t_k^{1,1}\right)$.*

b) *If $n = k^2+2$ then $Q_{2k+3}\left(k^2+2, t_k^{1,0}\right) = 0$ and $Q_{2k+3}(k^2+2,s) < 0$ for all $s \in \left(t_k^{1,0}, t_k^{1,1}\right)$.*

c) *If $n = k^2+3$ then there exists $s_0 = s_0(n,k) \in \left(t_k^{1,0}, t_k^{1,1}\right)$ such that*

$$\begin{aligned} &Q_{2k+3}(k^2+3,s) > 0, && \text{for all } s \in \left[t_{k-1}^{1,1}, s_0\right), \\ &Q_{2k+3}(k^2+3,s_0) = 0, && \\ &Q_{2k+3}(k^2+3,s) < 0, && \text{for all } s \in \left(s_0, t_k^{1,0}\right). \end{aligned} \tag{3.5.19}$$

d) *If $n \geq n^2+4$ then $Q_{2k+3}(n,s) > 0$ for all $s \in \left(t_k^{1,0}, t_k^{1,1}\right)$.*

Proof. **a)** For $3 \leq n \leq k^2+1$ we have $\psi_1(n,k) < 0$ by (3.5.17) and therefore $H(n,k,s) < 0$ for all $s \in \left[t_k^{1,0}, t_k^{1,1}\right]$. This means that $Q_{2k+3}(n,s) < 0$ for $s \in \left[t_k^{1,0}, t_k^{1,1}\right)$.
b) For $n = k^2+2$ we use the same argument as in a) but now $\psi_1(k^2+2,k) = 0$ implies $Q_{2k+3}\left(n, t_k^{1,0}\right) = 0$.
c) In this case ($n = k^2+3$), we have

$$\psi_1(n,k) > 0 > \psi_2(n,k).$$

This means that the function $H(n,k,s)$ decreases from the positive value $\psi_1(n,k)$ to the negative value $\psi_2(n,k)$. Since $H(n,k,s)$ is continuous, there exists s_0 with the required properties.
d) Now $n \geq k^2+4$ and we have $\psi_2(n,k) > 0$ which shows that $H(n,k,s)$ is positive in the whole interval $\left(t_k^{1,0}, t_k^{1,1}\right)$. □

Corollary 3.5.16. *Let $n \geq 3$ and $k \geq 2$.*

a) *If $n \leq k^2+1$, the Levenshtein bound $L_{2k}(n,s)$ can be improved in the whole half-open interval $\left[t_k^{1,0}, t_k^{1,1}\right) = \mathcal{I}_{2k} \backslash \{t_k^{1,1}\}$.*

b) *If $n = k^2+2$, the Levenshtein bound $L_{2k}(n,s)$ can be improved in the whole open interval $\left(t_k^{1,0}, t_k^{1,1}\right)$.*

c) *If $n = k^2+3$, there exists a number $s_0 = s_0(k) \in \left(t_k^{1,0}, t_k^{1,1}\right)$ such that the Levenshtein bound $L_{2k}(n,s)$ can be improved in the interval $\left(s_0, t_k^{1,1}\right)$.*

d) *If $n > k^2+3$, the Levenshtein bound $L_{2k}(n,s)$ can not be improved by using polynomials of degree at most $2k+3$.*

Proof. The proof follows by the results in the counterparts of **a)**, **b)**, **c)** and **d)** in Theorem 3.5.15. For **d)** we recall that by Corollary 3.4.4 the bound $L_{2k}(n,s)$ can not be improved by using polynomials of degree at most $2k+2$. □

At the end of this section we consider the situation when the dimension n is small with respect to k. We combine Corollaries 3.5.10 and 3.5.16 to see which Levenshtein bounds $L_m(n,s)$ can be improved in the whole interval $\mathcal{I}_m$.

For $n \geq 5$, denote

$$k_0(n) = \frac{2n-4+\sqrt{2n^3-9n^2+4n+16}}{n-4}. \tag{3.5.20}$$

The first few values of $k(n)$ are $k(5) = 14$, $k(6) = 11$, $k(7) = 10$ and $k(8) = 9$.

Theorem 3.5.17. *If $n \geq 5$ and $m \geq 2k(n)-1$, then the Levenshtein bound $L_m(n,s)$ can be improved in the whole interval of its validity.*

Proof. We solve the inequality $\varphi_1(n,k) < 0$ with respect to k. Its positive solutions can be found from $k \geq k_0(n)$. It is clear that $n \geq 5$ is necessary.

If $k \geq k_0(n)$ then we have

$$0 > \varphi_1(n,k) > G(n,k,s) > \varphi_2(n,k)$$

for all $s \in \left(t_{k-1}^{1,1}, t_k^{1,0}\right)$.

Moreover, for $n \geq 5$ we have $k_0(n) \geq \sqrt{n-2}$ which implies that $n \leq k^2+2$ is a consequence of $k \geq k_0(n)$. Therefore $k \geq k_0(n)$ means that

$$0 > \psi_1(n,k) > H(n,k,s) > \psi_2(n,k)$$

for all $s \in \left(t_k^{1,0}, t_k^{1,1}\right)$.

Combining the last two observations we conclude that the test function $Q_{2k+3}(n,s)$ is negative in both intervals $\left(t_{k-1}^{1,1}, t_k^{1,0}\right)$ and $\left(t_{k-1}^{1,1}, t_k^{1,0}\right)$ provided $n \geq 5$ and $k \geq k_0(n)$. This completes the proof. □

Theorem 3.5.17 shows that, for every fixed dimension $n \geq 5$, there exists $m_0 = m_0(n)$ such that all Levenshtein bounds $L_m(n,s)$ with $m \geq m_0$ can be improved by using linear programming.

3.6 Algorithm for computing the test functions

The investigations on the test functions $Q_{2k+3}(n,s)$ in the previous section suggest that in general the analytical computation of the functions $Q_j(n,s)$ tends to be very difficult. In this section we give an algorithm for computer calculations of $Q_j(n,s)$ for given n and s.

Let us assume that the dimension n is fixed and some $s \in (0,1)$ is given. One wishes to calculate some test functions $Q_j(n,s)$ in order to decide if the Levenshtein bound

$$A(n,s) \leq L_m(n,s)$$

can be improved by linear programming. Recall that $Q_j(n,s) \equiv 0$ for $j \leq m$ and $Q_j(n,s) \geq 0$ for $j = m+1$ and $j = m+2$. Therefore the first "interesting" test functions are $Q_{m+3}(n,s)$ and $Q_{m+4}(n,s)$.

The whole procedure should be started by computing some Gegenbauer polynomials. The computer systems MAPLE and MATHEMATICA have many orthogonal series in their memory including the Jacobi polynomials. Therefore one can just take them paying attention for the normalization. Otherwise the Gegenbauer polynomials may be generated by the recurrence relation (2.1.2).

The first thing one needs is the number m. It can be found by finding the largest zeros of the adjacent polynomials $P_k^{1,0}(t)$ and $P_k^{1,1}(t)$. This procedure determines the intervals $\mathcal{I}_3$, $\mathcal{I}_4$, $\mathcal{I}_5$, etc. When one has $s \in \mathcal{I}_m$ the number m is found.

We present an algorithm for finding test functions $Q_j(n,s)$, $j \geq m+1$, which uses the formulas (3.4.1–3.4.4) from Theorem 3.4.1. We describe the cases $m = 2k-1$ and $m = 2k$ simultaneously.

1. Take Gegenbauer polynomials $P_i^{(n)}(t)$ for $i = 0, 1, ..., k$, from MAPLE's libraries or calculate them from the recurrence relation (2.1.2).

2. Find the adjacent polynomials

$$P_l^{1,0}(t) = \sum_{i=0}^{l} r_i P_i^{(n)}(t),$$

where $l = k-1$ and $l = k$ for $L_{2k-1}(n,s)$ and

$$P_l^{1,1}(t) = P_l^{(n+2)}(t),$$

where $l = k-1$ and $l = k$ for $L_{2k}(n,s)$.

Since $P_k^{1,0}(t)$ and $P_k^{1,1}(t)$ are Jacobi polynomials, this step can be also reduced to the use of MAPLE's libraries.

3. Find the polynomials

$$\begin{aligned} h_1(t) &= P_k^{1,0}(t)P_{k-1}^{1,0}(s) - P_k^{1,0}(s)P_{k-1}^{1,0}(t), && \text{for } L_{2k-1}(n,s), \\ h_2(t) &= (t+1)[P_k^{1,1}(t)P_{k-1}^{1,1}(s) - P_k^{1,1}(s)P_{k-1}^{1,1}(t)], && \text{for } L_{2k}(n,s). \end{aligned}$$

Define $h_i(t) = \sum_{l=0}^{k} a_l^{(i)} t^l$ for $i = 1, 2$.

4. Calculate (see Lemma 2.1.1)

$$b_i = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } i \geq 1 \text{ odd}, \\ \frac{(2p-1)!!}{n(n+2)\dots(n+2p-2)}, & \text{for } i = 2p. \end{cases}$$

5. Calculate

$$\rho_k = \frac{\sum_{i=0}^{k} a_i^{(1)} b_i}{h_1(1)} = \frac{1}{L_{2k-1}(n,s)},$$

$$\gamma_{k+1} = \frac{\sum_{i=0}^{k+1} a_i^{(2)} b_i}{h_2(1)} = \frac{1}{L_{2k}(n,s)}.$$

(these formulas follow from Lemma 2.6.3 (equalities (2.6.1) and (2.6.2)) and (2.1.5). This step can be reduced to $\rho_k = 1/L_{2k-1}(n,s)$ or $\gamma_{k+1} = 1/L_{2k}(n,s)$, respectively, if the bound $L_m(n,s)$ is already calculated.

6. Calculate

$$\sigma_l^{(1)} = \begin{cases} b_l - \rho_k, & 0 \leq l \leq 2k-1, \\ -\frac{\sum_{p=0}^{k-1} a_p^{(1)} \sigma_{p-k+l}^{(1)}}{a_k}, & l \geq 2k, \end{cases}$$

for $L_{2k-1}(n,s)$, and

$$\sigma_l^{(2)} = \begin{cases} b_l - \gamma_{k+1}, & 0 \leq l \leq 2k, \\ -\frac{\sum_{p=0}^{k} a_p^{(2)} \sigma_{p-k+l-1}^{(2)}}{a_{k+1}}, & l \geq 2k+1, \end{cases}$$

for $L_{2k}(n,s)$. The $\sigma_l^{(1)}$'s and $\sigma_l^{(2)}$'s are analogs of the power sums S_l and R_l which are computed in the next step.

7. Calculate (see Lemma 2.6.6)

$$S_l = \begin{cases} b_l, & 0 \leq l \leq 2k-1, \\ \sigma_l^{(1)} + \rho_k, & l \geq 2k, \end{cases}$$

for $L_{2k-1}(n,s)$, and

$$R_l = \begin{cases} b_l, & 0 \leq l \leq 2k, \\ \sigma_l^{(2)} + \gamma_{k+1}, & l \geq 2k+1, \end{cases}$$

for $L_{2k}(n,s)$.

8. Compute $Q_j(n,S)$ by the formulas in Theorem 3.4.1. Here, one needs some further coefficients of the Gegenbauer polynomials. These can be extracted by means of MAPLE or MATHEMATICA after having calculated the Gegenbauer polynomials.

We give examples of the test functions in dimensions 3 and 4. In Figures 3.1 and 3.2, vertical lines mark the limits between the intervals $\mathcal{I}_m$ for $3 \leq m \leq 10$. In every interval we plot the first four nonzero test functions, i.e. $Q_{m+i}(n,s)$ for $i = 1,2,3,4$, $n = 3$ in Fig. 3.1 and $n = 4$ in Fig. 3.1, and $s \in \mathcal{I}_m$. Thus one test function disappears in each end (vanishing afterwards) and one new starts from the same vertical line (starting from positive values for m odd and from zero for m even).

Figures 3.1 and 3.2 provide some justification of our efforts in Section 3.5 to investigate the test function $Q_{2k+3}(n,s)$. They also show that the test functions $Q_{2k+2}(n,s)$ for $m = 2k-1$ and $Q_{2k+4}(n,s)$ for $m = 2k$ provide further values of s for which improvement of the corresponding Levenshtein bounds are possible.

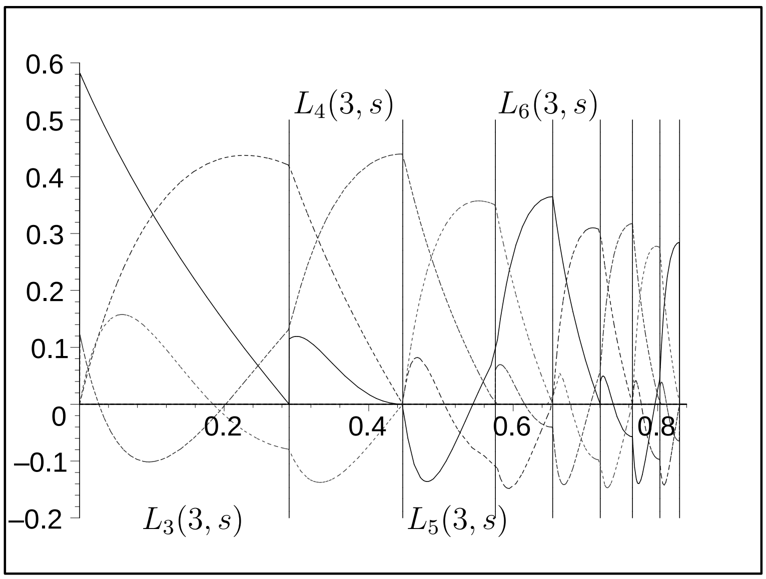

Figure 3.1: Some test functions in three dimensions – $Q_{m+1}(3,s)$, $Q_{m+2}(3,s)$, $Q_{m+3}(3,s)$ and $Q_{m+4}(3,s)$ in $\mathcal{I}_m$, where m is the number of the corresponding Levenshtein bound $L_m(3,s)$

Figure 3.1 suggests that the Levenshtein bound in three dimensions is not best possible for all $s \in [0.06, 1)$ with one exception – the case

$$L_4\left(3, t_2^{1,1}\right) = L_5\left(3, t_2^{1,1}\right) = 12.$$

This bound is attained by the icosahedron. In particular, the vanishing of the test function $Q_8(n,s)$ at the point $t_2^{1,1} = 1/\sqrt{5}$ means that the icosahedron has an index 8 (cf. [18]).

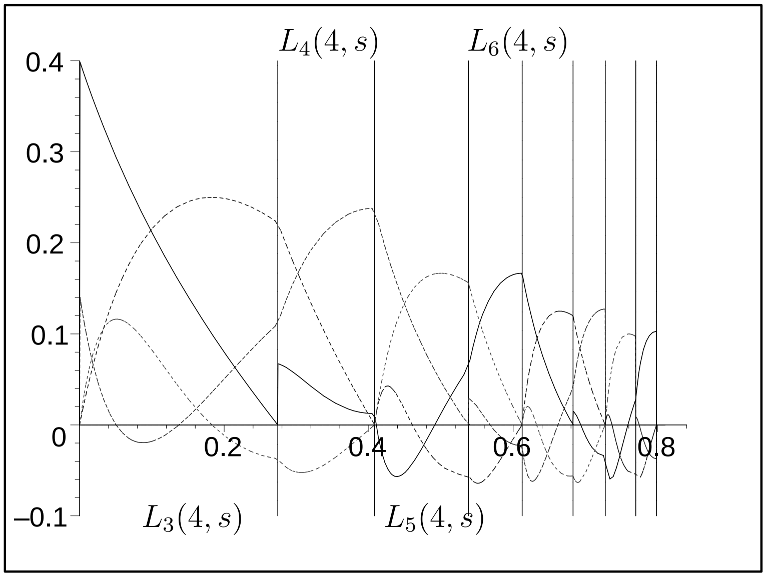

Figure 3.2: Some test functions in four dimensions – $Q_{m+1}(4,s)$, $Q_{m+2}(4,s)$, $Q_{m+3}(4,s)$ and $Q_{m+4}(4,s)$ in $\mathcal{I}_m$, where m is the number of the corresponding Levenshtein bound $L_m(4,s)$

3.7 Examples of new bounds

3.7.1 Some new bounds on $A(n,s)$

In [10], Boyvalenkov proposed a method for finding improvements of the Levenshtein bounds on $A(n,s)$ by using linear programming. Afterwards, the role of the test functions was explained as to show if the corresponding bound can be improved. This gave reason to design a computer program called SCOD which tests whether improvements are possible and, if so, finds some better bounds.

The program was announced in [14] and developed later by Kazakov [44] who is the principal author of SCOD. Since the exploitation of SCOD, some databases with new bounds were developed. We give a few examples.

Example 3.7.1. *($s = 1/\sqrt{5} = 0.44721359$, $3 \le n \le 25$) The icosahedron is an antipodal $(3, 12, 1/\sqrt{5})$ code which attains $L_5(n,s)$. Thus there are no negative test functions $Q_j(3, 1/\sqrt{5})$. In Table 3.3, all possible improvements of Levenshtein bounds in higher dimensions for $s = 1/\sqrt{5} \approx 0.44721359$ are shown. In the fourth column we give the indices of the negative test functions $Q_j(n, 1/\sqrt{5})$ with $m+1 \le j \le m+20$.*

Example 3.7.2. *($s = 0.5$, $3 \le n \le 25$) The number $A(n, 0.5)$ equals the maximal number of n-dimensional non-overlapping unit spheres that can touch $\mathbf{S}^{n-1}$. It is widely known*

n	m	$L_m(n, 1/\sqrt{5})$	j: $Q_j(n, 1/\sqrt{5}) < 0$	*New bound*
3	5	12.00	*No*	–
4	5	22.15	$8, 15$	21.97
5	5	38.09	$8, 9$	37.69
6	5	62.80	9	61.21
7	5	101.30	9	97.71
8	6	160.68	9	156.28
9	6	245.17	9	244.07
10	6	372.83	9	372.14
11	7	572.00	*No*	–
12	7	835.08	10	833.55
13	7	1204.48	10	1203.66
14	7	1724.06	11	1718.52
15	7	2460.26	11	2433.83
16	7	3518.16	11	3472.50
17	7	5073.74	11	5024.81
18	8	7352.23	*No*	–
19	8	10337.97	*No*	–
20	8	14683.06	*No*	–
21	9	21252.00	*No*	–
22	9	29314.66	12	29250.09
23	9	40134.06	12	40101.72
24	9	54713.19	*No*	–
25	9	74509.49	*No*	–

Table 3.3: Some improvements of $L_m\left(n, 1/\sqrt{5}\right)$

as kissing number and usually denoted by τ_n. *For* $n \geq 3$, *only three kissing numbers are known –* $\tau_3 = 12$ *(an object of a famous dispute between Newton and Gregory),* $\tau_8 = 240 = L_7(8, 0.5)$ *and* $\tau_{24} = 196560 = L_{11}(24, 0.5)$ *(the last two found by linear programming independently by Levenshtein [46] and Odlyzko-Sloane [53]; see also [27, Chapters 9,13]). Apart from* $n = 8$ *and* $n = 24$ *SCOD is able to improve* $L_n(n, 0.5)$ *in all dimensions* $4 \leq n \leq 24$.

Table 3.4 almost coincide with the table from [53] (cf. also Table 1.5 in [27, Chapter 1]). The small improvements for $n = 19, 21, 22$ *and* 23 *are explained by the slightly better accuracy of SCOD compared with the method of Odlyzko-Sloane [53] from 1978. The worse bound for* $n = 17$ *is because of the additional restrictions used in [53].*

Example 3.7.3. *(*$s = 0.55$, $3 \leq n \leq 30$*) Improvements of* $L_m(n, 0.55)$ *are possible in all dimensions* $3 \leq n \leq 25$. *The results are presented in Table 3.5.*

If the dimension n is fixed, SCOD can be applied to find all possible improvements starting from the first $s \in (0, 1)$ where a new bound is possible. We consider the situation in three dimensions improving $L_6(3, s)$.

n	m	$L_m(n,0.5)$	j: $Q_j(n,0.5)<0$	*New bound*
3	5	13.28	$8,15,22$	13.17
4	5	26.00	$9,16$	25.55
5	5	48.00	9	46.34
6	6	84.00	9	82.63
7	6	142.15	$9,10$	140.16
8	7	240.00	*No*	–
9	7	384.24	10	380.09
10	7	605.00	11	595.82
11	7	945.04	11	915.38
12	7	1478.75	11	1416.09
13	8	2328.18	11	2234.37
14	8	3546.66	$11,12$	3537.76
15	8	5460.92	11	5431.02
16	9	8364.00	12	8313.78
17	9	12373.30	12	12218.67
18	9	18199.29	13	17877.06
19	9	26771.00	13	25900.78
20	9	39655.00	13	37974.00
21	9	59693.12	13	56851.68
22	10	88391.88	$13,14$	86886.91
23	10	130340.04	$13,14$	128095.85
24	10	196560.00	*No*	–
25	11	282687.64	14	278364.37

Table 3.4: SCOD's results on kissing numbers

Example 3.7.4. *($n=3$, $0.5753189235 \le s \le 0.6546536707$, improving $L_6(3,s)$) We calculate new bounds on $A(3,s)$ for $s = 0.58+0.05i$, where $i = 0,1,\ldots,14$ (see Table 3.6).*

3.7.2 Some new bounds on $D(n,M)$

The problem of finding $D(3,M)$ (the minimum possible distance between M distinct points in the three dimensional sphere) mainly belongs to classical geometry. The optimal configurations for $M=3$, 4, 6 and 12 were described by Fejes Tóth [38] and are the expected ones. Solutions for $M=5$, 7, 8 and 9 were given by Schütte-van der Waerden [57], for $M=10$ and 11 by Danzer [29], and for $M=24$ by Robinson [56].) Therefore, the exact values of $D(3,M)$ are known only for $M \le 12$ and $M=24$.

The classical Fejes Tóth bound [38] gives

$$D(3,M) \le d_{FT} = \left(4 - \frac{1}{\sin^2 \frac{\pi M}{6(M-2)}}\right)^{\frac{1}{2}}$$

n	m	$L_m(n, 0.55)$	j: $Q_j(n, 0.55) < 0$	*New bound*
3	5	14.91	9	14.75
4	6	31.21	9, 17	30.61
5	6	60.33	9, 10, 17, 21	59.40
6	7	113.42	10	111.73
7	7	200.90	10, 11	197.64
8	7	348.20	11	337.96
9	7	600.47	11	572.34
10	8	1021.00	11, 12	1009.27
11	8	1703.87	11, 12	1652.10
12	9	2855.63	12	2773.05
13	9	4592.96	12, 13	4476.00
14	9	7339.02	13	7080.00
15	9	11772.36	13	11141.58
16	9	19216.84	13, 14	17826.35
17	10	30386.59	13, 14	28870.90
18	10	48757.53	13, 14	45558.64
19	11	76897.43	14	73248.87
20	11	118400.22	14, 15	113776.87
21	11	182197.92	15	173383.13
22	11	282837.90	15	264014.93
23	11	448535.05	15, 16	412536.81
24	12	691763.23	15, 16	640677.94
25	12	1082745.59	15, 16	976107.24

Table 3.5: Some improvements of $L_m(n, 0.55)$

in three dimensions.

The Levenshtein bound on $D(n, M)$ can be calculated by solving the equation $M = L_m(n, s)$. To do this we first determine the number m by comparing M with the values of the Levenshtein bound in the end points of the intervals $\mathcal{I}_m$ (this is a comparison between integers because of (2.5.4)). After finding s_L by this procedure we derive the Levenshtein bound on $D(n, M)$ as

$$D(n, M) \leq d_L = \sqrt{2(1 - s_L)}.$$

Improvements on the Levenshtein bounds on $A(n, s_L)$ lead to new bounds on $D(n, M)$ as well. The program SCOD has a module for calculating such bounds. It works by consecutive applications of the main module of SCOD to check how much s can be increased from s_L while keeping $A(n, s) \leq M$. At the last step the new bound $\sqrt{2(1 - s)}$ is calculated.

Example 3.7.5. *In Table 3.7 we show the situation in three dimensions. In all cases* $13 \leq M \leq 36$, *SCOD derives improvements on* d_L *and for* $13 \leq M \leq 27$ *on* d_{FT}. *Interestingly, in a few cases the Fejes Tóth bound lies between the two linear programming*

s	$L_6(3,s)$	j: $Q_j(n,s)<0$	*New bound*
$t_3^{1,0}=0.5753189235$	16.000	$9,16,23$	15.760
0.580	16.178	$9,16,17,23,24$	15.976
0.585	16.374	$9,16,17,23,24$	16.219
0.590	16.576	$9,16,17,24$	16.476
0.595	16.784	$9,17,24$	16.748
0.600	17.000	$9,17,24$	16.892
0.605	17.222	$9,17$	17.107
0.610	17.453	$9,17$	17.332
0.615	17.692	$9,17$	17.567
0.620	17.941	$9,10,17$	17.815
0.625	18.200	$9,10,17$	18.074
0.630	18.470	$9,10,17,18$	18.347
0.635	18.752	$9,10,17,18$	18.635
0.640	19.047	$9,10,17,18$	18.938
0.645	19.356	$9,10,17,18$	19.256
0.650	19.682	$9,10,17,18$	19.589
$t_3^{1,1}=0.6546536707$	20.000	10	19.905

Table 3.6: Some improvements on $L_6(3,s)$

bounds, which are extremal in the sense of subsection 2.6.1 [10, 48, 59]. The first column presents lower bounds on $D(3,M)$ *which are obtained by constructions [26, 36] and http://www.research.att.com/∼njas/.*

Examples of good spherical codes in higher dimensions are rare.

Sloane (http://www.research.att.com/∼njas/) maintains database of good codes which are obtained by computer, constructions based on block codes, lattices or polytopes.

Ericson-Zinoviev [36] publish tables of good spherical codes (some examples appeared earlier in Dodunekov-Ericson-Zinoviev [33] and in [34, 35]). In their tables there is a column of upper bounds on the squared minimum distance (i.e. on $D^2(n,M)$). With a few exceptions, these bounds are either Levenshtein bounds or bounds from SCOD (the last one is applied whenever the improvement is possible).

Example 3.7.6. *We select some cases from Tables VI and VII in Ericson-Zinoviev [35] and some tables from [34]. Nine examples are shown in Table 3.8.*

3.8 Concluding remarks

In some cases we obtain negative test functions $Q_j(n,s)<0$ for $j>m+4$. One interesting example is the case $n=4$, $s=0.5$ (the first unknown kissing number) where we have $m=5$, $Q_9(4,0.5)<0$ but also $Q_{16}(4,0.5)<0$. This suggests that a better bound could

M	*Lower bound [26]*	*New upper bound*	*Levenshtein bound*	d_{FT}
13	0.9564	1.0067	1.0105	1.0138
14	0.9338	0.9719	0.9756	0.9799
15	0.9026	0.9415	0.9463	0.9492
16	0.8805	0.9159	0.9216	0.9212
17	0.8624	0.8915	0.8944	0.8955
18	0.8382	0.8676	0.8704	0.8718
19	0.8085	0.8473	0.8494	0.8499
20	0.8043	0.8294	0.8310	0.8296
21	0.7752	0.8092	0.8111	0.8106
22	0.7611	0.7920	0.7928	0.7929
23	0.7445	0.7744	0.7762	0.7763
24	0.7442	0.7589	0.7612	0.7607
25	0.7107	0.7451	0.7476	0.7460
26	0.7010	0.7318	0.7332	0.7321
27	0.6951	0.7183	0.7198	0.7190
28	0.6734	0.7066	0.7074	0.7065
29	0.6629	0.6949	0.6959	0.6947
30	0.6609	0.6839	0.6854	0.6834
31	0.6463	0.6734	0.6743	0.6727
32	0.6424	0.6627	0.6638	0.6625
33	0.6222	0.6528	0.6539	0.6527
34	0.6148	0.6435	0.6446	0.6433
35	0.6067	0.6347	0.6359	0.6343
36	0.6045	0.6262	0.6278	0.6257

Table 3.7: Bounds on $D(3, M)$ for $13 \leq M \leq 36$. The lower bounds are taken from [26]

possibly be derived by using a 16-th degree polynomial. This is out of the range of SCOD for the parameters $n = 4$ and $s = 0.5$.

On the other hand, the investigations in the previous section and our computer calculations suggest that the principal test functions are the first two which could become negative: $Q_{m+3}(n, s)$ and $Q_{m+4}(n, s)$. This leads to the following conjecture.

Conjecture 3.8.1. *If $Q_j(n, s) \geq 0$ for $j = m + 3$ and $j = m + 4$, then $Q_j(n, s) \geq 0$ for all $j \geq m + 1$.*

Another important observation is the fact that for fixed n and s and when j tends to infinity, the sum in the defining formula (3.2.1) tends to $1/L_m(n, s) > 0$. Thus, we have the following assertion.

Theorem 3.8.2. *For fixed n and s there exists a constant $j_0 = j_0(n, s)$ such that test function $Q_j(n, s) > 0$ for all $j \geq j_0$.*

n	M	*Lower bound*	*New upper bound*	*Levenshtein bound*
4	30	0.84498	0.95418	0.96126
4	24	1.00000	1.02212	1.02474
5	42	0.89442	1.02238	1.02822
5	40	1.00000	1.03487	1.03956
6	72	1.00000	1.02126	1.02773
7	78	1.06911	1.09103	1.09257
8	120	1.01341	1.08535	1.08822

Table 3.8: Some new bounds on $D(n,M)$ by improvements on $L_5(n,s)$ by a polynomial of degree 9

On the other hand, the behaviour of the test functions suggest that the Levenshtein bounds $L_m(n,s)$ are good when the dimension n is large with respect to the bound m. We conjecture that this is the case when n tends to infinity and m is fixed.

Conjecture 3.8.3. *For fixed m there exists a constant $n_0 = n_0(m)$ such that for every $n \geq n_0$ the bound $L_m(n,s)$ can not be improved by using pure linear programming.*

Theorem 3.8.2 shows that we can not expect large degrees to work better in improving the Levenshtein bound. Therefore, the global extremality of the linear programming should be expected to be close to what is obtained by using degree $m+3$ or $m+4$ polynomials.

Chapter 4

Necessary conditions for the existence of spherical designs

This chapter is based on the paper [13]. We consider spherical designs with relatively small cardinalities, which means near to the Delsarte-Goethals-Seidel bound. We develop methods for obtaining restrictions on the structure of such designs. To do this, we use suitable polynomials in Definition 2.7.2. Our results can be considered as necessary conditions for the existence of designs. In many cases they imply nonexistence of designs of odd strengths and odd cardinalities.

4.1 Some preliminaries

We need a deeper explanation of the duality in the linear programming approaches for spherical codes and designs. The parameters which we used in the definition and the investigations of the test functions (see subsection 2.6.2) are useful for the description of the results on the structure of spherical designs.

We recall that the Levenshtein's polynomial $f_{2k-1}^{(n,s)}(t)$ has exactly k different zeros

$$\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} = s.$$

All α_i's belong to the interval $[-1, s]$. Furthermore, there exist positive weights ρ_i, $i = 0, 1, \ldots, k-1$, and a number ρ_k, which is positive for $s < t_k$, such that equality

$$f_0 = \rho_k f(1) + \sum_{i=0}^{k-1} \rho_i f(\alpha_i)$$

holds for any polynomial $f(t) = \sum_{i=0}^{m} f_i P_i^{(n)}(t)$ of degree $m \leq 2k-1$.

The equality $L_{2k-1}(n,s) = 1/\rho_k$ is valid not only for $t_{k-1}^{1,1} \leq s \leq t_k^{1,0}$, as we used so far, but also for $t_{k-1}^{1,1} \leq s \leq t_k$. The function $L_{2k-1}(n,s)$ is continuous and strictly increasing in the later interval. Hence, for any integer (in our case – cardinality) $M \in [R(n, 2k-1), +\infty)$ there exists a unique $s \in \left[t_{k-1}^{1,1}, t_k\right)$ such that

$$M = L_{2k-1}(n,s) = \frac{1}{\rho_k}.$$

Analogously, the polynomial $f_{2k}^{(n,s)}(t)$ has exactly $k+1$ different zeros

$$-1 = \beta_0 < \beta_1 < \cdots < \beta_k = s.$$

All β_i's belong to the interval $[-1, s]$. Furthermore, there exist positive weights γ_i, $i = 0, 1, \ldots, k$, and a number γ_{k+1}, which is positive for $s < t_k^{0,1}$, such that equality

$$f_0 = \gamma_{k+1} f(1) + \sum_{i=0}^{k} \gamma_i f(\beta_i)$$

holds for any polynomial $f(t) = \sum_{i=0}^{m} f_i P_i^{(n)}(t)$ of degree $m \le 2k$.
We have $L_{2k}(n,s) = 1/\gamma_k$ for $t_k^{1,0} \le s \le t_k^{0,1}$. The function $L_{2k}(n,s)$ is continuous and strictly increasing in the later interval. Hence, for any cardinality $M \in [R(n,2k), +\infty)$ there exists a unique $s \in \left[t_k^{1,0}, t_k^{0,1}\right)$ such that

$$M = L_{2k}(n,s) = \frac{1}{\gamma_{k+1}}.$$

In what follows we first assume the existence of a τ-design $C \subset \mathbf{S}^{n-1}$. Then we always associate C with the unique $s \in \left[t_{k-1}^{1,1}, t_k\right)$ for $\tau = 2k-1$ or the unique $s \in \left[t_k^{1,0}, t_k^{0,1}\right)$ for $\tau = 2k$ such that $|C| = L_{2k-1}(n,s)$ or $|C| = L_{2k}(n,s)$. Then all parameters defined above (the α_i's, β_i's, ρ_i's and γ_i's) come with this s in a unique way.
Let $C \subset \mathbf{S}^{n-1}$ be a spherical τ-design. The investigation of the structure of C with respect to its points is a useful tool in combinatorics and coding theory.

Definition 4.1.1. *For every point $x \in C$, we denote*

$$I(x) = \{\langle x, y\rangle : y \in C \setminus \{x\}\} = \{t_1(x), t_2(x), \ldots, t_{|C|-1}(x)\},$$

where we assume the following order

$$-1 \le t_1(x) \le t_2(x) \le \cdots \le t_{|C|-1}(x) < 1.$$

It is clear that $I(x)$ is a multiset (because $t_i(x) = t_{i+1}(x)$ is possible) of cardinality $|C|-1$. It may be different for distinct points of C. We shall prove some facts which are common for all sets $I(x)$. We shall discuss the sets $I(x)$ for some points in detail.
Equality (2.7.2) from Definition 2.7.2 will be our main tool. In the above notation it becomes

$$\sum_{i=1}^{|C|-1} f(t_i(x)) = |C| f_0 - f(1). \tag{4.1.1}$$

We use (4.1.1) for polynomials which have zeros at almost all points α_i, $i = 0, 1, \ldots, k-1$ (respectively β_i, $i = 0, 1, \ldots, k$). Then we apply (2.6.1) or (2.6.2). As a result, the right hand side of (4.1.1) becomes relatively simple. This allows us to obtain some estimations on the numbers $t_i(x)$ (for $i = 1, 2, |C|-2$ and $|C|-1$).

4.2 Constructions of spherical designs and nonexistence results

In this subsection we describe dimensions, strengths and cardinalities for the constructions of spherical designs that are available.

A spherical design is called *tight* if it attains the Delsarte-Goethals-Seidel bound (1.4.1). In Table 4.1 we present all known tight designs. Notice that tight 4- and 5-designs coexist.

τ	n	$\|C\|$	References
1	n	any pair of antipodal points on $\mathbf{S}^{n-1}$	
2	n	$(n+1)$-vertices of regular simplex in $\mathbf{R}^n$	Delsarte-Goethals-Seidel[31]
3	n	$2n$-vertices of cross polytope on $\mathbf{S}^{n-1}$	Delsarte-Goethals-Seidel[31]
4 4 4	6 22 m^2-3	27 275 if a tight spherical 4-design exists then $n = m^2-3$, m must be odd and $m \geq 3$	Delsarte-Goethals-Seidel[31]
5 5 5 5	3 7 23 m^2-2	12 56 552 if a tight spherical 5-design exists then $n = m^2-2$, m must be odd and $m \geq 3$	Delsarte-Goethals-Seidel[31]
7 7 7	8 23 $3m^2-4$	240 4 600 if a tight spherical 7-design exists then $n = 3m^2-4$ and $m \geq 2$	Delsarte-Goethals-Seidel[31]
11	24	196 560 (Leech lattice)	Bannai-Damerell [6, 7]

Table 4.1: Tight designs

Bannai and Damerell [3, 4] proved that spherical τ-designs on $\mathbf{S}^{n-1}$ do not exist if

$$\begin{aligned} \tau = \quad & 2e, \qquad e \geq 3, \quad (e = 3 \text{ was considered in [31]}) \\ \tau = \quad & 2e+1, \quad e \geq 4, \quad (\text{exept for the case } \tau = 11, n = 24) \end{aligned}$$

The existence of spherical designs for every τ, n and large enough cardinality C was first proved by Seymour-Zaslavsky in 1984 [58] and general constructions were first given by Bajnok in 1992 [3] .

Much work has been done for dimension three. We summarize the known results in Table 4.2.

Mimura [51] settled the case $\tau = 2$ in 1990. He gave constructions for $\tau = 2$ in all dimensions and all cardinalities $|C| \geq n_2$ for some positive integer n_2.

τ	$\|C\|$	References
1	2 antipodal points	
2	4 vertices of a regular tetrahedron	
3	the regular octahedron	
4	12, 14, $\geq$ 16	Hardin, Sloane [40]
5	the icosahedron	Delsarte-Goethals-Seidel [31]
5	12, 16, 18, 20, $\geq$ 22	Hardin-Sloane [41]
5	and conjecture that this list is complete	Reznick [55]
6	24, 26, $\geq$ 28	Hardin-Sloane [41]
7	24, 30, 32, 34, $\geq$ 36	Hardin-Sloane [41]
8	36, 40, 42, $\geq$ 44	Hardin-Sloane [41]
9	48, 50, 52, $\geq$ 54	Hardin-Sloane [41]
10	60, 62, $\geq$ 64	Hardin-Sloane [41]
11	70, 72, $\geq$ 74	Hardin-Sloane [41]
12	84, $\geq$ 86	Hardin-Sloane [41]

Table 4.2: Spherical designs in three dimension

For $\tau = 3$, the Delsarte-Goethals-Seidel bound gives $B(n,3) \geq 2n$. It is attained in all dimensions by the so-called bi-orthogonal code (an orthonormal basis together with the opposites). Moreover, it was shown by Bajnok [3, 4], that all even cardinalities are feasible, i.e. in every dimension $n \geq 3$ and for every even integer $m \geq 2n$, there exists a spherical 3-design on $\mathbf{S}^{n-1}$ of cardinality m. The odd cardinalities turn out to be more difficult to construct (see Table 4.3).

On one hand, Bajnok gives constructions of 3-designs in all dimensions $n \geq 7$ for all odd cardinalities greater than or equal to $5n/2$. In lower dimensions, he constructs 3-designs of all odd sizes beginning with 11 for $n = 3$ and $n = 4$ and with 15 for $n = 5$ and $n = 6$.

On the other hand, Boyvalenkov-Danev-Nikova [20] show that 3-designs on $\mathbf{S}^{n-1}$ of odd cardinality $2n+k$, where k is an odd positive integer, do not exist for $k \leq (\sqrt[3]{2}-1)n+0.3$. This completes the description of the possible sizes of 3-designs in dimensions four and six. Just one open case remains in all other dimensions below 14, and two open cases remain in dimensions $15 \leq n \leq 24$.

Much less is known for larger strengths. Some database on existing spherical designs can be found in Neil Sloane's home page *http://www.research.att.com/~njas/* (mainly in dimensions three and four).

Constructions of spherical 4-designs were given by Hardin-Sloane [40]. In particular, they show that 4-designs of size m on $\mathbf{S}^{n-1}$ exist precisely when $m = 12, 14$ and $m \geq 12$ for $n = 3$, $m \geq 20$ for $n = 4$, $m \geq 29$ for $n = 5$, $m = 27, 36$ and $m \geq 39$ for $n = 6$, etc. They conjecture that all remaining cardinalities are impossible. We collect their results in Table 4.4.

Apart from investigations (cf. [8, 9, 22]) on the existence of spherical 4-designs of the smallest possible cardinality $R(n,4) = n(n+3)/2$, no nonexistence results for 4-designs

n	$\lvert C\rvert = 2n+1$	$\lvert C\rvert = 2n+3$	$\lvert C\rvert = 2n+5$		
3	7^*	9	$\underline{\mid 11}$		
4	9^*	$\underline{\mid 11}$			
5	11^*	13	$\underline{\mid 15}$		
6	13^*	$\underline{\mid 15}$			
7	15^*	17	$\underline{\mid 19}$		
8	17^*	19	$\underline{\mid 21}$		
	$\cdots$				
n	$(2n+1)^*$	$(2n+3)^*$	$\underline{(\sqrt[3]{2}+1)n+0.3\ \mid}$	$\cdots$	$\underline{\mid \frac{5n}{2}}$

Table 4.3: Spherical 3-designs

Key to Table 4.3:

$\underline{\mid m}$	all designs of size $\geq m$ exist (Bajnok [3, 4])
$*$	nonexistence proved in [20] (Boyvalenkov-Danev-Nikova)
$\underline{(\sqrt[3]{2}+1)n+0.3\ \mid}$	nonexistence proved in [20] for $\lvert C\rvert \leq \underline{(\sqrt[3]{2}+1)n+0.3\ \mid}$.

were proved.

n	$\lvert C\rvert$
3	12, 14, ≥ 16
4	≥ 20
5	≥ 29
6	27, 36, ≥ 39
7	≥ 53
8	≥ 69

Table 4.4: Spherical 4-designs, Hardin-Sloane [40]

Spherical 5-designs were constructed (mainly in three dimensions) by Reznick [55] and Hardin-Sloane [40, 41]. Their results show that 5-designs exist in three dimensions for cardinalities 12, 16, 18 and ≥ 20 and conjectured that all remaining cardinalities are impossible.

Nonexistence results for 5-designs were obtained by Boyvalenkov-Danev-Nikova [20]. In particular, it was shown that there exist no 5-designs with 13 points in three dimensions.

4.3 Bounds on the smallest and largest inner products

Let $C \subset \mathbf{S}^{n-1}$ be a spherical $(2k-1)$-design and $s \geq t_{k-1}^{1,1}$ be such that $|C| = L_{2k-1}(n,s)$. As we know, this uniquely determines the parameters α_i, $i = 0,1,\ldots,k-1$, and ρ_i, $i = 0,1,\ldots,k$, where $\rho_k = 1/L_{2k-1}(n,s) = 1/|C|$.

The first step in our approach is to use suitable polynomials in (4.1.1) for obtaining some restrictions on the inner products in $I(x)$, $x \in C$. Let $x \in C$ be arbitrarily chosen. We derive an upper bound on the smallest inner product $t_1(x)$ and a lower bound on the greatest one $t_{|C|-1}(x)$. Both bounds do not depend on the choice of x.

Theorem 4.3.1. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical $(2k-1)$-design. Then for every point $x \in C$ we have*

$$t_1(x) \leq \alpha_0 \tag{4.3.1}$$

and

$$t_{|C|-1}(x) \geq s = \alpha_{k-1}. \tag{4.3.2}$$

If equality holds in one of these two cases then all elements of the multiset $I(x)$ belong to the set $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$.

Proof. Consider the polynomial

$$f(t) = \frac{(t - t_1(x))(t-s)f_{2k-1}^{(n,s)}(t)}{(t-\alpha_0)^2} = (t - t_1(x))\prod_{i=1}^{k-1}(t-\alpha_i)^2.$$

Since $f(t)$ has degree $2k-1$, we can apply (4.1.1) for C, x and $f(t)$.

The left hand side is equal to

$$\sum_{i=1}^{|C|-1} f(t_i(x)) = \sum_{i=2}^{|C|-1}\left((t_i(x) - t_1(x))\prod_{i=1}^{k-1}(t_i(x)-\alpha_i)^2\right)$$

(i.e. its first term is zero and all remaining terms are nonnegative). Therefore, the whole sum is nonnegative.

To calculate the right hand side we use (2.6.1) and the equality $\rho_k = 1/|C|$. Since $f(\alpha_i) = 0$ for $i = 1,2,\ldots,k-1$, we obtain

$$\begin{aligned} f_0|C| - f(1) &= |C|\left(\rho_k f(1) - \sum_{i=0}^{k-1}\rho_i f(\alpha_i)\right) - f(1) \\ &= (|C|\rho_k - 1)f(1) + \rho_0 f(\alpha_0)|C| \\ &= \rho_0 f(\alpha_0)|C|. \end{aligned}$$

Therefore, we have

$$0 \leq f(\alpha_0) = (\alpha_0 - t_1(x))\prod_{i=1}^{k-1}(\alpha_0 - \alpha_i)^2,$$

which implies $t_1(x) \le \alpha_0$.

If equality holds in (4.3.1) for some point $x \in C$, i.e. $t_1(x) = \alpha_0$, then the right hand side of (2.7.2) is zero. Thus we have $t_i(x) \in \{t_1(x), \alpha_1, \alpha_2, \dots, \alpha_{k-1}\}$ for all $i = 2, 3, \dots, |C|-1$. Therefore all elements of the multiset $I(x)$ belong to the set $\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}\}$.

To prove the the inequality $t_{|C|-1}(x) \ge s$, we use the polynomial

$$f(t) = \frac{(t - t_{|C|-1}(x)) f_{2k-1}^{(n,s)}(t)}{t - s} = (t - t_{|C|-1}(x)) \prod_{i=0}^{k-2} (t - \alpha_i)^2.$$

In this case, $f(t)$ also has degree $2k-1$ and we again can apply (4.1.1) for C, x and $f(t)$. The arguments for obtaining $t_{|C|-1}(x) \ge s$ and for the investigation of the case of equality are as above. □

Inequality $t_1(x) \le \alpha_0$ is new both in its appearance and in its nature, while inequality $t_{|C|-1}(x) \ge s$ can be considered as an extension of the inequality $s(C) \ge s$ which was proved by Fazekas-Levenshtein [37].

As a by-product we obtain a corollary that describes codes which attain the bounds in Theorem 4.3.1 for any point. These codes are maximal spherical codes.

Corollary 4.3.2. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical $(2k-1)$-design such that for every point $x \in C$ equality holds either in (4.3.1) or (4.3.2). Then C is an $(n, L_{2k-1}(n,s), s)$ code.*

Proof. It follows that all inner products of the points of C belong to $\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}\}$. In particular, we obtain $s(C) = s$. We apply the main identity (2.3.1) for C and the Levenshtein's polynomial $f_{2k-1}^{(n,s)}(t)$. Then the sums on both sides vanish (for the right hand side see Definition 2.7.1) and we get $f_{2k-1}^{(n,s)}(1)|C| = f_0|C|^2$, i.e. $|C| = f_{2k-1}^{(n,s)}(1)/f_0 = L_{2k-1}(n,s)$. □

Theorem 4.3.1 gives $t_1(x) \le \alpha_0$ for any point $x \in C$. In some sense this means that good $(2k-1)$-designs (with relatively small cardinality) are close to antipodal designs – each point $x \in C$ has (at least one) corresponding point which is close to $-x$.

For small cardinalities this has significant consequences. Indeed, we prove below that in such cases (to be described in terms of the dimensions, strengths and cardinalities) the points of the hypotetical design pair-off. In particular, this implies nonexistence when the cardinality is odd.

We continue with the counterpart of Theorem 4.3.1 for $(2k)$-designs. Let $C \subset \mathbf{S}^{n-1}$ be a $(2k)$-design and $s \ge t_k^{1,0}$ be such that $|C| = L_{2k}(n,s)$. This uniquely determines the parameters β_i, $i = 0, 1, \dots, k$, and γ_i, $i = 0, 1, \dots, k+1$, where $\gamma_{k+1} = 1/L_{2k}(n,s) = 1/|C|$. For every point $x \in C$ we derive an upper bound on $t_1(x)$ and a lower bound on $t_{|C|-1}(x)$.

Theorem 4.3.3. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical $(2k)$-design. Then for every point $x \in C$ we have*

$$t_1(x) \le \beta_1 \tag{4.3.3}$$

and

$$t_{|C|-1}(x) \ge s = \beta_k. \tag{4.3.4}$$

If equality holds in one of these two cases then all elements of the multiset $I(x)$ belong to the set $\{\beta_0, \beta_1, \ldots, \beta_k\}$.

Proof. The proof is analogous to the proof of Theorem 4.3.1 by making use of the polynomials of degree $2k$

$$\frac{(t-t_1(x))(t-s)f_{2k}^{(n,s)}(t)}{(t-\beta_1)^2} = (t+1)(t-t_1(x))\prod_{i=2}^{k}(t-\beta_i)^2$$

for (4.3.3) and

$$\frac{(t-t_{|C|-1}(x))f_{2k}^{(n,s)}(t)}{t-s} = (t+1)(t-t_{|C|-1}(x))\prod_{i=1}^{k-1}(t-\beta_i)^2$$

for (4.3.4). □

Inequality (4.3.3) seems to be rather weak while (4.3.4) can be viewed as a generalization of the Fazekas-Levenshtein inequality $s(C) \geq s$ [37].

When C is a $2k$-design the bounds (4.3.3) and (4.3.4) can not be achieved simultaneously by all points of the design.

Corollary 4.3.4. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical $(2k)$-design of cardinality $|C| > R(n, 2k)$. Then there exists a point $x \in C$ such that the bounds (4.3.3) and (4.3.4) are both strict.*

Proof. If for every point $x \in C$ equality holds either in (4.3.3) or (4.3.4) then it can be proved as in Corollary 4.3.2 that C is an $(n, L_{2k}(n, s), s)$ code. However, it was proved in [19] that such codes do not exist for $s > t_k^{1,0}$, which is equivalent to $|C| > R(n, 2k)$. Therefore both bounds (4.3.3) and (4.3.4) are strict for at least one point $x \in C$. □

4.4 Nonexistence results for $(2k-1)$-designs of odd cardinalities

4.4.1 A necessary condition

Let $C \subset \mathbf{S}^{n-1}$ be a $(2k-1)$-design. Inequality (4.3.1) must be valid for all points of C. It was observed (first by Boyvalenkov-Danev-Nikova [20]) that a similar property leads to nonexistence results for designs of odd cardinality. In this section we generalize the results from [20] and give some examples. As in [20], our approach is based on Theorem 4.3.1. The improvement is then caused by using more suitable polynomials in later investigations.

First we conclude that, in the case of odd cardinalities, there exists some special point $x \in C$.

Theorem 4.4.1. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical $(2k-1)$-design of odd cardinality. Then there exists a point $x \in C$ such that*

$$t_2(x) \leq \alpha_0. \tag{4.4.1}$$

Proof. Let us assume that $t_2(x) > \alpha_0$ for all points $x \in C$. Then the inequalities

$$t_1(x) \leq \alpha_0 < t_2(x)$$

mean that for point x there exists a unique point $y \in C$ such that the inner product $\langle y, x\rangle$ belongs to the interval $[-1, \alpha_0]$. In fact, y is nothing but the farthest point of C to x.
Since $\alpha_0 \geq \langle x, y\rangle \in I(y) = \{t_1(y), t_2(y), \ldots, t_{|C|-1}(y)\}$ and $t_1(y) \leq \alpha_0 < t_2(y)$, we obtain $t_1(y) = \langle x, y\rangle = t_1(x)$. Therefore, the point x is the farthest point of C to y.
The above argument implies that the points of C pair-off, i.e. they can be divided into disjoint pairs. This is a contradiction because such a division is impossible when the cardinality of C is odd. □

The next theorem is the main result of this section. It gives the second step in our approach – to use already obtained restrictions and a new polynomial in (4.1.1).

Theorem 4.4.2. *Let $C \subset \mathbf{S}^{n-1}$ be a $(2k-1)$-design with odd cardinality. Then*

$$\rho_0|C| \geq 2. \tag{4.4.2}$$

If equality holds then there exists a point x in C such that all elements of the multiset $I(x)$ belong to the set $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$.

Proof. By Theorem 4.4.1 there exists a point $x \in C$ such that $t_1(x) \leq t_2(x) \leq \alpha_0$. Consider the polynomial

$$f(t) = \frac{f_{2k-1}^{(n,s)}(t)(t-s)}{(t-\alpha_0)^2} = \prod_{i=1}^{k-1}(t-\alpha_i)^2.$$

Since $f(t)$ has degree $2k-2$ (i.e. smaller than $2k-1$), we can apply (4.1.1) for C, x and $f(t)$.
Polynomial $f(t)$ is decreasing in the interval $(-\infty, \alpha_1]$ which contains $t_1(x) \leq t_2(x) \leq \alpha_0$. This fact and the inequalities $f(t_i(x)) \geq 0$ for every $i = 3, 4, \ldots, |C|-1$ imply that on the left hand side of (4.1.1) we have

$$\sum_{i=1}^{|C|-1} f(t_i(x)) \geq f(t_1(x)) + f(t_2(x)) \geq 2f(t_2(x)) \geq 2f(\alpha_0). \tag{4.4.3}$$

We calculate the right hand side of (4.1.1) as in the proof of Theorem 4.3.1. Since

$$f_0 = \rho_0 f(\alpha_0) + \rho_k f(1)$$

by (2.6.1) and $|C| = 1/\rho_k$, we obtain

$$f_0|C| - f(1) = |C|\rho_0 f(\alpha_0). \tag{4.4.4}$$

The relations (4.4.3) and (4.4.4) show that

$$|C|\rho_0 f(\alpha_0) \geq 2f(\alpha_0)$$

which is equivalent to (4.4.2) because $f(\alpha_0) > 0$.
If equality holds then $t_1(x) = \alpha_0$ and the case of equality in Theorem 4.3.1 is applied. □

Theorem 4.4.2 reduces the existence problem for C to the calculation of the number $\rho_0|C|$. In concrete cases this can be done easily by computer. In fact, in Chapter 3 we developed methods for calculating all weights ρ_i, $i = 0, 1, \ldots, k$.

Using the formula in Lemma 2.6.5 we express condition (4.4.2) in terms of the numbers $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$.

Corollary 4.4.3. *Let $C \subset \mathbf{S}^{n-1}$ be a $(2k-1)$-design of odd cardinality. Then*

$$-\frac{(1-\alpha_1^2)(1-\alpha_2^2)\cdots(1-\alpha_{k-1}^2)}{\alpha_0(\alpha_0^2-\alpha_1^2)(\alpha_0^2-\alpha_2^2)\cdots(\alpha_0^2-\alpha_{k-1}^2)} \geq 2. \tag{4.4.5}$$

4.4.2 Nonexistence results in small dimensions

Condition (4.4.2) of Theorem 4.4.2 works well in small dimensions. In the numerical calculations we made use of (parts of) the results of Chapter 3. Indeed, for given n and s we can find all weights ρ_i, $i = 0, 1, \ldots, k$ and all numbers α_i, $i = 0, 1, \ldots, k-1$. Thus the only problem we have to solve here is to pass from the cardinality of C to the corresponding value of s. This can be done, for example, by solving the equation

$$|C| = L_{2k-1}(n, s) \tag{4.4.6}$$

with respect to s. In the general case, this is a k-degree equation which can be easily solved numerically (by MAPLE, e.g.). Notice that the MAPLE answer is some reordering of the array $[\alpha_0, \alpha_1, \ldots, \alpha_{k-1}]$ and we have to take $s = \alpha_{k-1}$ as the largest number of this array.

An even easier way to investigate the numerical consequences of Theorem 4.4.2 is given by Corollary 4.4.3. We find all parameters we need for Corollary 4.4.3 by solving equation (4.4.6).

We proceed with concrete nonexistence results. For $\tau = 3$ our results are the same as those of Boyvalenkov-Danev-Nikova [20] (see Table 4.5). This is because in the second step they have used optimization techniques to find the best polynomial of the form $f(t) = (t-a)^2$ (a is the parameter to be optimized). This has led to our $f(t) = (t-\alpha_1)^2 = (t-s)^2$.

We describe in detail the results for 3-designs despite the fact that they are not new. We do this for the following reason. In Section 4.6 below we refine our $\tau = 3$ approach by adding some further geometric arguments. We obtain nonexistence results which are already better than those from [20].

Boyvalenkov-Danev-Nikova [20] prove that no spherical 3-design on $\mathbf{S}^{n-1}$ with $2n+k$, $k \geq 1$ is odd, exists for $k = 1$ in all dimensions $n \geq 3$, nor for $k = 3$ in all dimensions $n \geq 11$, for $k = 5$ in all dimensions $n \geq 19$, for $k = 7$ in all dimensions $n \geq 25$, etc. Combined with the Bajnok's constructions [4], this leaves twenty-two open cases in dimensions $n \leq 20$. Namely, existence remains undecided for $(n, |C|) = (3, 9)$, $(5, 13)$, $(7, 17)$, $(8, 19)$, $(9, 21)$, $(10, 23)$, $(11, 27)$, $(12, 29)$, $(13, 31)$, $(14, 33)$, $(15, 35)$, $(15, 37)$, $(16, 37)$, $(16, 39)$, $(17, 39)$, $(17, 41)$, $(18, 41)$, $(18, 43)$, $(19, 45)$, $(19, 47)$, $(20, 47)$ and $(20, 49)$. It follows that the existence is completely decided in two cases – for dimensions 4 and 6, only one open case remains in any of the dimensions 3, 5, and $7 \leq n \leq 14$ and two cases remain in each dimension $15 \leq n \leq 20$.

n	$\lvert C\rvert = 2n+1$	$\lvert C\rvert = 2n+3$	$\lvert C\rvert = 2n+5$	$\lvert C\rvert = 2n+7$	$\lvert C\rvert = 2n+9$
3	7^*	9°	$\underline{\mid 11}$		
4	9^*	$\underline{\mid 11}$			
5	11^*	13°	$\underline{\mid 15}$		
6	13^*	$\underline{\mid 15}$			
7	15^*	17°	$\underline{\mid 19}$		
8	17^*	19°	$\underline{\mid 21}$		
9	19^*	21°	$\underline{\mid 23}$		
10	21^*	23°	$\underline{\mid 25}$		
11	23^*	25^*	27°	$\underline{\mid 29}$	
12	25^*	27^*	29°	$\underline{\mid 31}$	
13	27^*	29^*	31°	$\underline{\mid 33}$	
14	29^*	31^*	33°	$\underline{\mid 35}$	
15	31^*	33^*	35°	37°	$\underline{\mid 39}$
16	33^*	35^*	37°	39°	$\underline{\mid 41}$
17	35^*	37^*	39°	41°	$\underline{\mid 43}$
18	37^*	39^*	41°	43°	$\underline{\mid 45}$
19	39^*	41^*	43^*	45°	47°
20	41^*	43^*	45^*	47°	49°

Table 4.5: Spherical 3-designs
Key to Table 4.5:

$\underline{\mid m}$	all designs of size $\geq m$ exist (Bajnok [3, 4])
$*$	nonexistence proved in [20] (Boyvalenkov-Danev-Nikova) and in Theorem 4.4.2
$\circ$	open cases

For $\tau = 5$, Boyvalenkov-Danev-Nikova [20] describe a computer method for finding good polynomials for the second step and apply it in some examples. Our approach (Theorem 4.4.2) gives an analytical answer in all dimensions. For $\tau \geq 7$ the only available numerical results are the bounds from [24] and [52, Chapter 2]. Our bounds on minimum odd cardinalities are the same (in a few cases) or better than all of these examples.

Some new bounds for the minimum possible odd cardinalities of $(2k-1)$-designs are shown in Tables 4.6, 4.7 and 4.8 below. Further numerical consequences of (4.4.2) are available upon request.

The second and the fifth columns of Tables 4.6, 4.7 and 4.8 represent the values of $2\binom{n+k-2}{n-1}+1$ given by the Delsarte-Goethals-Seidel bound $R(n, 2k-1)$ plus one.

n	DGS bound	New bound	n	DGS Bound	New bound
3	12	15	12	156	169
4	20	23	13	189	197
5	30	33	14	210	227
6	42	47	15	240	259
7	56	61	16	272	293
8	72	79	17	306	329
9	90	97	18	342	369
10	110	119	19	380	409
11	132	143	20	420	451

Table 4.6: Some lower bounds on the minimum possible odd cardinality of spherical 5-designs ensured by (4.4.2), for $3 \leq n \leq 20$

n	DGS bound	New bound	n	DGS Bound	New bound
3	20	23	12	728	765
4	40	43	13	910	957
5	70	75	14	1120	1175
6	112	119	15	1360	1427
7	168	177	16	1632	1713
8	240	253	17	1938	2031
9	330	347	18	2280	2393
10	440	463	19	2660	2791
11	572	601	20	3080	3233

Table 4.7: Some lower bounds on the minimum possible odd cardinality of spherical 7-designs ensured by (4.4.2), for $3 \leq n \leq 20$

4.4.3 Asymptotic consequences of Theorem 4.4.2

The behaviour of the improvements suggests that an asymptotic improvement could be possible. We consider the condition (4.4.2) of Theorem 4.4.2 in the following asymptotic process. Let $\tau = 2k-1$ be fixed and n tend to infinity. We investigate the impact of Theorem 4.4.2 on $(2k-1)$-designs of cardinality approximately equal to n^{k-1} .

Denote

$$B_{odd}(n,\tau) = \min\{|C| : C \subset \mathbf{S}^{n-1} \text{ is a } \tau\text{-design, } |C| \text{ is odd}\}.$$

The Delsarte-Goethals-Seidel bound implies that

$$B_{odd}(n,2k-1) \geq R(n,2k-1)+1 = 2\binom{n+k-1}{n-1} \gtrsim \frac{2n^{k-1}}{(k-1)!},$$

n	DGS bound	New bound	n	DGS Bound	New bound
3	30	33	12	2730	2825
4	70	73	13	3640	3769
5	140	145	14	4760	4929
6	252	261	15	6120	6339
7	420	435	16	7752	8029
8	660	683	17	9690	10039
9	990	1025	18	11970	12403
10	1430	1479	19	14630	15159
11	2002	2071	20	17710	18355

Table 4.8: Some lower bounds on the minimum possible odd cardinality of spherical 9-designs ensured by (4.4.2), for $3 \leq n \leq 20$

where the inequality $\gtrsim$ should be interpreted as

$$\lim_{n\to\infty} \frac{B_{odd}(n, 2k-1)}{n^{k-1}} \geq \frac{2}{(k-1)!}.$$

Boyvalenkov-Danev-Nikova improve this to

$$B_{odd}(n, 2k-1) \gtrsim \frac{(1+2^{1/\tau})n^{k-1}}{(k-1)!}.$$

Although our bounds are better than those of Boyvalenkov-Danev-Nikova [20] for $\tau \geq 5$ in concrete cases they are the same asymptotically. The reason for this phenomenon is that the asymptotic behaviour of both bounds depends only on the asymptotics of α_0. The calculation below is missing in [13].

Lemma 4.4.4. *Let $n \to +\infty$ and k be fixed. Then all roots of the equation*

$$P_k^{1,0}(t)P_{k-1}^{1,0}(s) - P_k^{1,0}(s)P_{k-1}^{1,0}(t) = 0$$

tend to zero except for α_0 and

$$\alpha_0 \sim \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)}.$$

Proof. The first assertion follows from

$$|\alpha_0| > |\alpha_{k-1}| > |\alpha_1| > |\alpha_{k-2}| > \cdots$$

(cf. [19, Appendix], Theorem 2.6.7) and

$$s = \alpha_{k-1} \leq t_k,$$

which tends to zero when $n \to +\infty$ and k are fixed.

Now the behaviour of α_0 can be derived by the Viète formula (see (3.5.9))

$$\begin{aligned}
\sum_{i=1}^{k-1} \alpha_i &= -\frac{k}{n+2k-2}\left(1-\frac{(n+2k-1)(n+k-2)}{k(n+2k-3)}\cdot\frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)}\right)\\
&= -\frac{k}{n+2k-2}+\frac{(n+2k-1)(n+k-2)}{(n+2k-2)(n+2k-3)}\cdot\frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)}\\
&\sim \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)}
\end{aligned}$$

as n tends to infinity and k is fixed. □

It follows from Lemma 4.4.4 and Corollary 4.4.3 that it is enough to find the asymptotic behaviour of the ratio $P_k^{1,0}(s)/P_{k-1}^{1,0}(s)$. This can be done, for example, by using the following identity due to Levenshtein.

Lemma 4.4.5 ([49] equality (5.86)). *For $s \in \left[t_{k-1}^{1,1}, t_k^{0,0}\right)$ we have*

$$L_{2k-1}(n,s) = \left(1-\frac{P_{k-1}^{1,0}(s)}{P_k^{(n)}(s)}\right)R(n,2k) = \left(1-\frac{P_k^{1,0}(s)}{P_k^{(n)}(s)}\right)R(n,2k+2).$$

Theorem 4.4.6. *Let $n \to +\infty$, k be fixed and $C \subset \mathbf{S}^{n-1}$ be a $(2k-1)$-design of cardinality*

$$|C| \sim R(n,2k+1)+\gamma n^{k-1} \sim \left(\gamma+\frac{2}{(k-1)!}\right)n^{k-1},$$

where γ is some constant. Then

$$\alpha_0 \sim -\frac{1}{1+\gamma(k-1)!}.$$

Proof. We apply twice the asymptotic process under consideration in the identities from Lemma 4.4.5. Thus

$$|C| = L_{2k-1}(n,s) = \left(1-\frac{P_{k-1}^{1,0}(s)}{P_k^{(n)}(s)}\right)R(n,2k)$$

implies

$$1-\frac{P_{k-1}^{1,0}(s)}{P_k^{(n)}(s)} \sim \gamma(k-1)!+2,$$

whence

$$\frac{P_{k-1}^{1,0}(s)}{P_k^{(n)}(s)} \sim -1-\gamma(k-1)!.$$

Similarly, we have

$$\frac{P_k^{1,0}(s)}{P_k^{(n)}(s)} \sim 1.$$

Hence

$$\alpha_0 \sim \frac{P_k^{1,0}(s)}{P_{k-1}^{1,0}(s)} = \frac{P_k^{1,0}(s)}{P_k^{(n)}(s)} \cdot \frac{P_k^{(n)}(s)}{P_{k-1}^{1,0}(s)} \sim -\frac{1}{1+\gamma(k-1)!},$$

which completes the proof. □

We are now in a position to describe the asymptotic consequence of Corollary 4.4.3.

Theorem 4.4.7. *We have*

$$B_{odd}(n, 2k-1) \gtrsim \frac{1+2^{1/(2k-1)}}{(k-1)!} \cdot n^{k-1}.$$

Proof. Let us assume that $C \subset \mathbf{S}^{n-1}$ is a $(2k-1)$-design of cardinality

$$|C| < \left(\gamma + \frac{2}{(k-1)!}\right) n^{k-1},$$

where

$$\gamma = \frac{2^{1/(2k-1)}-1}{(k-1)!}.$$

Then it follows from Theorem 4.4.6 and Corollary 4.4.3 that

$$\begin{aligned} 2 \quad &< \quad -\frac{(1-\alpha_1^2)(1-\alpha_2^2)\cdots(1-\alpha_{k-1}^2)}{\alpha_0(\alpha_0^2-\alpha_1^2)(\alpha_0^2-\alpha_2^2)\cdots(\alpha_0^2-\alpha_{k-1}^2)} \\ &\sim \quad -\frac{1}{\alpha_0^{2k-1}} \\ &\sim \quad [1+\gamma(k-1)!]^{2k-1}. \end{aligned}$$

This implies that $\gamma > \left(2^{1/(2k-1)}-1\right)/(k-1)!$, which is a contradiction with the assumption that $\gamma = \left(2^{1/(2k-1)}-1\right)/(k-1)!$. This completes the proof. □

The first three cases are

$$B_{odd}(n,3) \gtrsim 2.2599n, \quad B_{odd}(n,5) \gtrsim 1.0743n^2, \quad \text{and} \quad B_{odd}(n,7) \gtrsim 0.3506n^3,$$

compared to the bounds

$$B_{odd}(n,3) \gtrsim 2n, \quad B_{odd}(n,5) \gtrsim n^2, \quad \text{and} \quad B_{odd}(n,7) \gtrsim \frac{n^3}{3},$$

which are ensured by the Delsarte-Goethals-Seidel bound.

4.5 Other inequalities for inner products

4.5.1 Inequalities for $(2k-1)$-designs of even cardinalities

Let $C \subset \mathbf{S}^{n-1}$ be a $(2k-1)$-design. In this subsection we assume that $\rho_0|C| < 2$, i.e. $|C|$ is even by Theorem 4.4.2. For such designs we obtain upper and lower bounds on their minimum and maximal inner products. In particular, we improve the upper bound $t_1(x) \le \alpha_0$ for every point $x \in C$.

We start with a lower bound on the second smallest inner product $t_2(x)$ and an upper bound on the largest one $t_{|C|-1}(x)$. Both bounds are valid for all points $x \in C$.

Lemma 4.5.1. *Let δ_1 and μ_1 be the smallest respectively the greatest root of the equation*

$$f(t) = A,$$

where $f(t) = \prod_{i=1}^{k-1}(t-\alpha_i)^2$ and $A = f(\alpha_0)(\rho_0|C|-1)$. Then $t_2(x) \ge \delta_1$ and $t_{|C|-1}(x) \le \mu_1$ for every point $x \in C$.

Proof. The polynomial $f(t)$ has degree $2k-2$. We apply (4.1.1) for C, x and $f(t)$. We first see that

$$\begin{aligned}\sum_{i=2}^{|C|-1} f(t_i(x)) &= f_0|C| - f(1) - f(t_1(x)) \\ &= |C|\rho_0 f(\alpha_0) - f(t_1(x))\end{aligned}$$

because $f(\alpha_i) = 0$ for $i = 1, 2, \ldots, k-1$. Since $f(t)$ is decreasing in $(-\infty, \alpha_1]$ and $t_1(x) \le \alpha_0$ (from Theorem 4.3.1), we have $f(t_1(x)) \ge f(\alpha_0)$ whence

$$f(\alpha_0)\rho_0|C| - f(t_1(x)) \le f(\alpha_0)(\rho_0|C|-1) = A.$$

Since $f(t)$ is nonnegative, we have

$$\sum_{i=2}^{|C|-1} f(t_i(x)) \ge f(t_2(x)) + f(t_{|C|-1}(x)).$$

The last two inequalities imply that

$$f(t_2(x)) + f(t_{|C|-1}(x)) \le f(\alpha_0)(\rho_0|C|-1) = A.$$

Moreover, both numbers $f(t_2(x))$ and $f(t_{|C|-1}(x))$ must be less than A since they are nonnegative.

The inequality $t_{|C|-1}(x) \le \mu_1$ follows since $f(t_{|C|-1}(x)) \le A$ and $f(t)$ is increasing in $[s, \infty)$. It is clear that $\delta_1 < \alpha_1$. This completes the proof if $t_2(x) \ge \alpha_1$. If $t_2(x) < \alpha_1$, the inequality $t_2(x) \ge \delta_1$ follows since $f(t_2(x)) \le A$ and $f(t)$ is decreasing in $(-\infty, \alpha_1]$. □

The bound $t_2(x) \ge \delta_1$ allows us to improve the bound $t_1(x) \le \alpha_0$.

Lemma 4.5.2. *Let λ_1 is the smallest root of the equation*

$$f(t) = B,$$

where $f(t) = (t - \delta_1)\prod_{i=1}^{k-1}(t - \alpha_i)^2$ and $B = f(\alpha_0)\rho_0|C|$. Then $t_1(x) \leq \lambda_1 < \alpha_0$ for every point $x \in C$.

Proof. Notice that $\rho_0|C| > 1$ from the proof of Lemma 4.5.1. Inequality $\lambda_1 < \alpha_0$ then follows from the definition of λ_1 as the smallest root of $f(t) = B$.

Polynomial $f(t)$ has degree $2k - 1$. To prove $t_1(x) \leq \lambda_1$ we use (4.1.1) for C, x and $f(t)$. The left hand side is at least $f(t_1(x))$ since $t_2(x) \geq \delta_1$ and $f(t)$ is nonnegative for $t \geq \delta_1$. On the other hand, the right hand side equals $f(\alpha_0)\rho_0|C|$. This already implies our assertion since $f(t)$ is increasing in $(-\infty, \delta_1]$. □

Since $\lambda_1 < \alpha_0$ we obtain improvements of both bounds of Lemma 4.5.1. Indeed, one can repeat the proof of Lemma 4.5.1 by replacing $f(t_1(x))$ by $f(\lambda_1)$ instead of by $f(\alpha_0)$. Let δ_2 and μ_2 be the smallest and respectively the greatest root of the equation of the Lemma 4.5.1 where $A = f(\alpha_0)\rho_0|C| - f(\lambda_1)$. If $t_2(x) \geq \delta_2$ is the new bound, we can use it in Lemma 4.5.2 (make use of polynomial $f(t) = (t - \delta_2))\prod_{j=1}^{k-1}(t - \alpha_j))$ for obtaining the better bound $t_1(x) \leq \lambda_2$ (it easily follows that $\lambda_2 < \lambda_1$).

It is clear that this process can be continued. We obtain bounds $t_2 \geq \delta_k > \delta_{k-1} > \cdots > \delta_1$, $t_{|C|-1} \leq \mu_k < \mu_{k-1} < \cdots < \mu_1$, and $t_1 \leq \lambda_k < \lambda_{k-1} < \cdots < \lambda_1$ for any integer k. (We get δ_i and μ_i as the smallest and respectively the greatest root of the equation of the Lemma 4.5.1 where $A = f(\alpha_0)\rho_0|C| - f(\lambda_{i-1})$ and λ_i is the smallest root of the equation from Lemma 4.5.2 by using polynomial $f(t) = (t - \delta_i))\prod_{j=1}^{k-1}(t - \alpha_j))$. Of course, it is not difficult to prove that the sequences $\{\delta_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$, and $\{\lambda_k\}_{k=1}^{\infty}$ are convergent. Therefore, the following theorem holds.

Theorem 4.5.3. *We have $t_2(x) \geq \delta = \lim_{k\to\infty}\delta_k$, $t_{|C|-1}(x) \leq \mu = \lim_{k\to\infty}\mu_k$, and $t_1(x) \leq \lambda = \lim_{k\to\infty}\lambda_k$.*

This implies new upper bounds on the maximal inner product of $(2k-1)$-designs under consideration.

Corollary 4.5.4. *For any $(2k-1)$-design $C \subset \mathbf{S}^{n-1}$ with $\rho_0|C| < 2$ we have*

$$s \leq s(C) \leq \mu.$$

Upper bounds on the maximal inner product of spherical designs of given dimension, strength and cardinality have not been found by us in the literature. Such bounds could not be obtained for codes of fixed dimension and cardinality since these codes could have points which are arbitrarily close to each other.

Example 4.5.5. *Since all 3-designs of feasible even cardinalities exist, the first possibility to apply Theorem 4.5.3 is for 5-designs and 7-designs. If $C \subset \mathbf{S}^{n-1}$ is a 5-designs of $n^2 + n + k$ points, k is even and $\rho_0|C| < 2$, we compute some approximations of the limits δ, μ and λ. The third column of Table 4.9 and 4.10 shows the number of iterative applications of Lemmas 4.5.1 and 4.5.2.*

n	$\lvert C\rvert$	Iterations	λ	δ	μ
5	32	29	-0.9424004212	-0.806032224	0.870495133
6	44	19	-0.9697717297	-0.727024557	0.764542217
7	58	16	-0.9810194629	-0.671955713	0.695576773
7	60	45	-0.9185988547	-0.838754089	0.885451652
8	74	15	-0.9870639819	-0.628551111	0.644353867
8	76	23	-0.9580261272	-0.755606053	0.787010751
9	92	12	-0.9907046805	-0.592756595	0.603837520
9	94	20	-0.9724610484	-0.703533136	0.725611999
9	96	32	-0.9331941108	-0.809120048	0.842083472
10	112	13	-0.9930555315	-0.562484986	0.570551665
10	114	16	-0.9804084660	-0.662841788	0.678937405
10	116	21	-0.9590125740	-0.747010127	0.771083685

Table 4.9: Some upper bounds of $t_1(x)$ and $t_{|C|-1}(x)$ (resp. λ and μ) and lower bounds of $t_2(x)$ (resp. δ) for spherical 5-designs

4.5.2 Inequalities for $(2k)$-designs

Let $C \subset \mathbf{S}^{n-1}$ be a $(2k)$-design. We know that $t_1(x) \le \beta_1$ and $t_{|C|-1}(x) \ge s$ for every point $x \in C$. In this subsection we obtain lower bounds on $t_1(x)$ and upper bounds on $t_{|C|-1}(x)$ which are valid for all points of C.

Lemma 4.5.6. *Let ξ_1 and η_1 be the least resp. the greatest root of the equation*

$$f(t) = D_1,$$

where $f(t) = \prod_{i=1}^{k}(t-\beta_i)^2$ and $D_1 = \gamma_0 f(-1)|C|$. Then we have $t_1(x) \ge \xi_1$ and $t_{|C|-1}(x) \le \eta_1$ for every point $x \in C$ (i.e. all elements of $I(x)$ belong to the interval $[\xi_1, \eta_1]$).

Proof. Polynomial $f(t)$ has degree $2k$. We use (4.1.1) for C, x and $f(t)$. The right hand side is equal to

$$|C|\left(\gamma_{k+1}f(1) + \sum_{i=0}^{k}\gamma_i f(\beta_i)\right) - f(1) = \gamma_0 f(-1)|C| = D_1$$

because $f(\beta_i) = 0$ for $i = 1, \ldots, k$ and $\gamma_{k+1} = 1/|C|$.
All terms in the sum on the left hand side are nonnegative. The assertion now follows since outside the interval $[\xi_1, \eta_1]$ we have $f(t) > D_1$ (i.e. if we assume that some elements of $I(x)$ do not belong to $[\xi_1, \eta_1]$ we obtain a contradiction). □

Using similar argument we obtain bounds on the inner products $t_2(x)$ and $t_{|C|-2}(x)$.

Lemma 4.5.7. *Let ξ_2 and η_2 be the least and the greatest root, respectively, of the equation*

$$f(t) = D_2,$$

n	$\|C\|$	Iterations	λ	δ	μ
5	72	17	-0.9809266013	-0.8256833155	0.8540162771
6	114	15	-0.9903837657	-0.7688111463	0.7830779317
6	116	22	-0.9711219849	-0.8491097310	0.8775152828
7	170	12	-0.9944928504	-0.7233980069	0.7313369340
7	172	17	-0.9854436087	-0.7900008816	0.8058653534
7	174	23	-0.9707742564	-0.8442191129	0.8679702660
8	242	12	-0.9965835090	-0.6855229067	0.6902881134
8	244	13	-0.9914203505	-0.7448522856	0.7543830233
8	246	19	-0.9843503982	-0.7892673722	0.8035590693
10	332	10	-0.9977538491	-0.6531888294	0.6562210113
10	334	12	-0.9945187925	-0.7074756538	0.7135421944
10	336	15	-0.9903790982	-0.7469050636	0.7560062102

Table 4.10: Some upper bounds of $t_1(x)$ and $t_{|C|-1}(x)$ (resp. λ and μ) and lower bounds of $t_2(x)$ (resp. δ) for spherical 7-designs

where $f(t) = \prod_{i=1}^{k}(t-\beta_i)^2$ *and* $D_2 = \gamma_0 f(-1)|C|/2 = D_1/2$. *Then we have* $t_2(x) \geq \xi_2$ *and* $t_{|C|-2}(x) \leq \eta_2$ *for every* $x \in C$ *(i.e. all elements of* $I(x) \setminus \{t_1(x), t_{|C|-1}(x)\}$ *belong to the interval* $[\xi_2, \eta_2]$*).*

Proof. Let us assume that $t_2(x) < \xi_2$. Polynomial $f(t)$ has degree $2k$. We use (4.1.1) for C, x and $f(t)$. As in Lemma 4.5.6 the right hand side is equal to $D_1 = 2D_2$ (the polynomial $f(t)$ is the same).

Since $f(t)$ is decreasing in $(-\infty, \beta_1]$ and $t_1(x) \leq t_2(x) < \xi_2 < \beta_1$, the left hand side is at least

$$f(t_1(x)) + f(t_2(x)) \geq 2f(t_2(x)) > 2f(\xi_2) = 2D_2.$$

This is a contradiction. We conclude that $t_2(x) \geq \xi_2$.

Inequality $t_{|C|-2}(x) \leq \eta_2$ can be proved in a similar way. □

For odd cardinalities $|C|$, we prove stronger restrictions for at least one point $x \in C$.

Lemma 4.5.8. *If* $|C|$ *is odd then there exists a point* $x \in C$ *such that simultaneously* $t_1(x) \geq \xi_2$ *and* $t_{|C|-1}(x) \leq \eta_2$ *(i.e. all elements of* $I(x)$ *belong to the interval* $[\xi_2, \eta_2]$*).*

Proof. Let $A = \{x \in C : t_1 \geq \xi_2\}$ and $B = \{x \in C : t_{|C|-1} \leq \eta_2\}$. We firstly prove that the sets A and B are nonempty.

Let us assume that $t_1(x) < \xi_2$ for every point $x \in C$. Then we can use the same argument as in the proof of Lemma 4.4.1 to see that the points of C can be divided into disjoint pairs. This is impossible since $|C|$ is odd. Therefore, inequality $t_1(x) \geq \xi_2$ is satisfied for at least one point $x \in C$.

Using a similar argument, we prove that $t_{|C|-1}(x) \leq \eta_2$ for at least one point $x \in C$, i.e. the set B is nonempty.

To complete the proof we need to show that the intersection $A \cap B$ is nonempty. Let us assume that $A \cap B = \phi$ and consider the sets $C \setminus A$ and $C \setminus B$. Again, as in the proof of Lemma 4.4.1, we conclude that the points in these two sets can be divided into disjoint pairs.

This implies that the cardinalities $|C \setminus A|$ and $|C \setminus B|$ are even. Since $|C|$ is odd, this shows that $|A|$ and $|B|$ are odd as well. Then $A \cup B = C$ is impossible since $|A \cap B| = 0$ by our assumption. Therefore there exists $x \in C$ which belongs neither to A nor B. This means that $t_1(x) < \xi_2$ and $t_{|C|-1}(x) > \eta_2$ for this point.

We now complete the proof by obtaining a contradiction in (4.1.1) for C, x and the polynomial $f(t) = \prod_{i=1}^{k}(t-\beta_i)^2$ from Lemmas 4.5.6 and 4.5.7. The right hand side equals $D_1 = \gamma_0 f(-1)|C|$, while the left hand side is at least

$$f(t_1(x)) + f(t_{|C|-1}(x)) > f(\xi_2) + f(\eta_2) = D_2 + D_2 = D_1,$$

a contradiction. □

For arbitrary cardinality $|C|$, Lemma 4.5.6 can be extended in the following way.

Lemma 4.5.9. *For every point $x \in C$ at least one of the following inequalities is true: $t_1(x) \geq \xi_2$ or $t_{|C|-1}(x) \leq \eta_2$ (i.e. all elements of $I(x)$ belong either to $[\xi_2, \eta_1]$ or to $[\xi_1, \eta_2]$).*

Proof. Let us suppose that $t_1(x) < \xi_2$ and $t_{|C|-1}(x) > \eta_2$ are simultaneously true for some point $x \in C$. Then we arrive to a contradiction in (4.1.1) for C, x and $f(t) = \prod_{i=1}^{k}(t-\beta_i)^2$ in the same way as at the end of the proof of Lemma 4.5.8. □

We combine inequality (4.3.4) from Theorem 4.3.3 with Lemma 4.5.6 to obtain bounds on the maximal inner product of $(2k)$-designs of fixed dimension, strength and cardinality.

Corollary 4.5.10. *For any $(2k)$-design $C \subset \mathbf{S}^{n-1}$ we have*

$$s \leq s(C) \leq \eta_1.$$

Corollary 4.5.11. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical $(2k)$-design which possesses a pair of antipodal points. Then*

$$|C| \geq R(n, 2k+1) = 2\binom{n+k-1}{n-1}.$$

Proof. Let x and $-x$ be pair of antipodal points of C. It follows from Lemma 4.5.6 that $t_1(x) = -1 \geq \xi_1$. Therefore $D_1 = f(\xi_1) \geq f(-1)$ for $f(t) = \prod_{i=1}^{k}(t-\beta_i)^2$. This inequality is equivalent to $\gamma_0|C| \geq 1$. The assertion now follows from a result of Boyvalenkov-Danev [16] which shows that $\gamma_0|C|$ belongs to the interval $[0, 1)$ when

$$R(n, 2k) \leq |C| < R(n, 2k+1).$$

□

Example 4.5.12. *We computed by means of a MAPLE program some values of ξ_1, ξ_2, η_1 and η_2 for 4-designs and 6-designs. The results are shown in Tables 4.11 and 4.12.*

n	$\|C\|$	ξ_1	ξ_2	η_2	η_1
3	10	-0.8340193108	-0.7651942262	0.5652072441	0.6340323288
3	11	-0.8555413134	-0.7675641087	0.6875632401	0.7755404449
3	13	-0.8984324723	-0.7856275790	0.8663183685	0.9791232617
4	15	-0.8060837800	-0.7409275407	0.5028293448	0.5679855840
4	16	-0.8306682339	-0.7509980960	0.5843259535	0.6639960914
4	17	-0.8511640007	-0.7607593767	0.6496520324	0.7400566564
4	18	-0.8693407551	-0.7704523520	0.7037818408	0.8026702438
5	21	-0.7764635392	-0.7114141682	0.4776500198	0.5426993909
5	22	-0.8020607702	-0.7259261485	0.5354527109	0.6115873325
5	23	-0.8227350536	-0.7379019065	0.5840549101	0.6688880572
5	24	-0.8405440992	-0.7485331783	0.6260849162	0.7180958371
6	28	-0.7500615327	-0.6841516342	0.4654014520	0.5313113505
6	29	-0.7747890720	-0.7001428755	0.5089640197	0.5836102162
6	30	-0.7949884051	-0.7132237930	0.5465568022	0.6283214143
6	31	-0.8123447615	-0.7245437900	0.5798055564	0.6676065279
7	36	-0.7275363961	-0.6605601496	0.4585408793	0.5255171258
7	37	-0.7504805628	-0.6764314780	0.4928549050	0.5669039898
7	38	-0.7696375349	-0.6896618641	0.5229965310	0.6029722018
7	39	-0.7862481808	-0.7011466384	0.5500344065	0.6351359489

Table 4.11: Some lower bounds of $t_1(x)$ and $t_2(x)$ (resp. ξ_1 and ξ_2) and upper bound of $t_{|C|-2}(x)$ and $t_{|C|-1}(x)$ (resp. η_2 and η_1) for spherical 4-designs

4.6 Refining the approach from Sections 4.3 and 4.4

In this section we use additional geometric arguments to strengthen the results from Section 4.4 for $(2k-1)$-designs of odd cardinality.

4.6.1 Method for investigation

Let $C \in \mathbf{S}^{n-1}$ be a $(2k-1)$-design. Our approach now is the following. First, we show that for odd cardinalities $|C|$ some special triples (x, y, z) of points of C appear. Then we use suitable polynomials in (4.1.1) to derive bounds on the smallest and the largest inner products in the sets $I(x)$, $I(y)$ and $I(z)$. In the third step, we organize an iterative process by using the new bounds and (other) suitable polynomials in (4.1.1). The final results are new bounds on inner products from $I(x), I(y)$ and $I(z)$ and sometimes the nonexistence of the designs under target.

Step 1. Our first step is based on Lemma 4.3.1 and the following simple observation.

Lemma 4.6.1. *Let $C \subset \mathbf{S}^{n-1}$ be a τ-design of odd cardinality $|C|$. Then there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$ and $t_2(x) = t_1(z)$.*

n	$\|C\|$	ξ_1	ξ_2	η_2	η_1
3	17	-0.9025321313	-0.8643157503	0.7119036818	0.7479734994
3	18	-0.8985034759	-0.8527163383	0.7678410465	0.8123008964
3	19	-0.8992541739	-0.8484584177	0.8118928781	0.8621076449
3	20	-0.9019510197	-0.8475003216	0.8474975957	0.9019482511
4	31	-0.9002148254	-0.8597854229	0.6709967013	0.7095199954
4	32	-0.8993570568	-0.8543269126	0.7007220944	0.7440536522
4	33	-0.9000232517	-0.8512837844	0.7272151355	0.7745136522
4	34	-0.9015825851	-0.8497751356	0.7509385955	0.8015692923
5	51	-0.8916787522	-0.8470134805	0.6572662722	0.7004309411
5	52	-0.8928338411	-0.8455425895	0.6741390433	0.7199992604
5	53	-0.8943313735	-0.8447052158	0.6899584579	0.7382413261
5	54	-0.8960676048	-0.8443492950	0.7048097569	0.7552826042
6	78	-0.8822286033	-0.8334412564	0.6526730399	0.7002991185
6	79	-0.8840453907	-0.8336774576	0.6630707075	0.7123074633
6	80	-0.8859016778	-0.8340609484	0.6730507912	0.7237960368
6	81	-0.8877841067	-0.8345668029	0.6826372152	0.7347987821
7	113	-0.8731696276	-0.8208596120	0.6513868604	0.7027942087
7	114	-0.8750510995	-0.8217236130	0.6582542615	0.7106942045
7	115	-0.8769070701	-0.8226082064	0.6649281963	0.7183564649
7	116	-0.8787380127	-0.8235103011	0.6714180974	0.7257936463

Table 4.12: Some lower bounds of $t_1(x)$ and $t_2(x)$ (resp. ξ_1 and ξ_2) and upper bound of $t_{|C|-2}(x)$ and $t_{|C|-1}(x)$ (resp. η_2 and η_1) for spherical 6-designs

Proof. Let Γ be the directed graph with as vertices the points of C and edges

$$\begin{aligned} x \to y &\quad \text{if and only if} \quad t_1(x) = \langle x, y\rangle, \\ x \leftarrow y &\quad \text{if and only if} \quad t_1(y) = \langle y, x\rangle, \\ x \leftrightarrow y &\quad \text{if and only if} \quad t_1(x) = t_1(y) = \langle x, y\rangle. \end{aligned}$$

For the cycles with length at least two of the following type $x \leftrightarrow y \leftrightarrow z \leftrightarrow x$ our wishes are satisfied.

Let us consider the cycles with at least one edge of the type $x \to y$ or $x \leftarrow y$. It is easy to see that induced cycles (of this type) in Γ are possible only if their length is two. Since $|\Gamma| = |C|$ is odd, it is impossible to divide Γ into disjoint cycles. Therefore, we must have the following situation $y \leftrightarrow x \leftarrow z$. This completes the proof. $\square$

It follows from Lemmas 4.3.1 and 4.6.1 that there exist distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y) \le \alpha_0$ and $t_2(x) = t_1(z) \le \alpha_0$. In the next steps we carry our a more detailed investigation of the triple (x, y, z).

Step 2. Inequalities of the type $t_1(x) = \langle x, y\rangle \le t_2(x) = \langle x, z\rangle \le a < 0$ may mean that the points y and z are close to each other.

Denote by φ the acute angle with cosine a. Then φ is greater than or equal to the angles between $-x$ and y and between $-x$ and z. This means that the angle between y and z does not exceed 2φ. Therefore

$$\langle y, z\rangle = \cos\angle(y,z) \geq \cos 2\varphi = 2a^2 - 1.$$

If $2a^2 - 1 > s$ in this argument, we actually have obtained better lower bounds on the inner products $t_{|C|-1}(y)$ and $t_{|C|-1}(z)$ because

$$t_{|C|-1}(y) \geq \langle y, z\rangle$$

and

$$t_{|C|-1}(z) \geq \langle y, z\rangle.$$

Remark. The improvement of the lower bounds (4.3.2) for y and z leads to an improvement of the Fazekas-Levenshtein bound on $s(C)$.

In turn, these two new bounds give better estimations on $t_1(y)$ and $t_1(z)$ respectively. In particular, this leads to the improvement

$$t_1(x) \leq t_2(x) \leq a' < a.$$

As we shall see below, the last fact follows by using suitable polynomials in (4.1.1) for C and z.

Step 3. If $2\alpha_0^2 - 1 > \alpha_{e-1}$, we can start an iterative process, by applying **Step 2** as many times as necessary.

Set $\delta_0 = \alpha_0$ and let $\delta_1 = a'$ be obtained by applying **Step 2** for $a = \alpha_0$. Now $2\delta_1^2 - 1 > 2\delta_0^2 - 1$ is a better lower bound for $t_{|C|-1}(y)$ and $t_{|C|-1}(z)$. In turn this calls for a second application of **Step 2** which gives better upper bounds $t_2(x) \leq \delta_2$. We can continue this process, checking (at each iteration) the possible existence of C by some polynomial in (4.1.1).

Theorem 4.6.2. *If there exists a real nonnegative polynomial $f(t)$ of degree at most $2k-1$ which decreases in the interval $[-1, \alpha_0)$ and if*

$$2f(\delta_i) > f_0|C| - f(1) \tag{4.6.1}$$

for some $i \geq 0$ then C does not exist.

Proof. We apply (4.1.1) to C, x and the polynomial $f(t)$. Then, the right hand side $f_0|C| - f(1)$ is at least $2f(t_2(x)) \geq 2f(\delta_i)$ for all $i \geq 0$. □

If we consider only $i = 0$ and take polynomial $f(x) = \prod_{i=1}^{k-1}(t - \alpha_i)^2$, we obtain Theorem 4.4.2.

In the next two subsections we apply the above method to investigate the existence of 3- and 5-designs of small odd cardinalities. Because of the results in Section 4, we shall always require $\rho_0|C| < 2$.

4.6.2 Some results for 3-designs

Let $\tau = 3$ and $C \subset \mathbf{S}^{n-1}$ be a 3-design of cardinality $|C| = R(n,3) + k = 2n + k$, where $k \geq 3$ is an odd integer. The numbers α_0 and α_1 are the roots of the quadratic equation

$$n(n+k-1)X^2 + n(n-1)X - k = 0 \tag{4.6.2}$$

(see [20, Eq. (8)]).

All 22 open cases in dimensions $3 \leq n \leq 20$ were listed in Subsection 4.4.1. Working on this list we may assume that k is odd, $k = 3$ for $n = 3, 5, 7, 8, 9, 10$, $k = 5$ for $11 \leq n \leq 18$, $k = 7$ for $15 \leq n \leq 20$. We actually applied our method to all open cases in dimensions $3 \leq n \leq 50$.

Let x, y, z be points in C obtained in **Step 1**, i.e. x, y and z are such that $t_1(x) = t_1(y) \leq t_2(x) = t_1(z) \leq \alpha_0$. In **Step 2**, we shall assume that

$$\mu_0 = 2\alpha_0^2 - 1 > s \tag{4.6.3}$$

(fortunately this is true in many cases we have to deal with, in other cases we can not apply this method). We use (4.6.3) for obtaining better upper bound on $t_1(z)$.

Lemma 4.6.3. *For any real $a \in [\alpha_0, s]$, we have*

$$t_1(z) \leq F(a) = -2\frac{n\alpha_0^2 a^2 + [2n(2\alpha_0^2 - 2\alpha_0^4 - 1) + |C|]\, a + n\alpha_0^2(4\alpha_0^4 - 6\alpha_0^2 + 3)}{(|C| - 2)na^2 + 4n\alpha_0^2 a + 2n(2\alpha_0^2 - 2\alpha_0^4 - 1) + |C|}.$$

Proof. We apply (4.1.1) to C, z and the polynomial

$$f(t) = (t - t_1(z))(t - a)^2,$$

where $a \leq s$. We calculate f_0 from (2.1.5)

$$f_0 = \frac{t_1(z) + 2a}{n} - a^2 t_1(z).$$

Therefore, the right hand side in (4.1.1) equals

$$\begin{aligned} f_0|C| - f(1) &= |C|\left(\frac{t_1(z) + 2a}{n} - a^2 t_1(z)\right) - (1 - t_1(z))(1 - a)^2 \\ &= t_1(z)\left[\frac{|C|}{n} - a^2|C| + (1 - a)^2\right] + \frac{2a|C|}{n} - (1 - a)^2. \end{aligned}$$

The left hand side is at least $f(t_{|C|-1}(z))$. Since the polynomial $f(t)$ is increasing in $(s, +\infty)$ and $t_{|C|-1}(z) \geq \mu_0 > s$, this is bounded from below by

$$\begin{aligned} f(\mu_0) &= (2\alpha_0^2 - 1 - t_1(z))(2\alpha_0^2 - 1 - a)^2 \\ &= -t_1(z)(2\alpha_0^2 - 1 - a)^2 + (2\alpha_0^2 - 1)(2\alpha_0^2 - 1 - a)^2. \end{aligned}$$

We obtain the inequality

$$t_1(z)\left[\frac{|C|}{n} - a^2|C| + (1-a)^2\right] \;+\; \frac{2a|C|}{n} \;-\; (1-a)^2$$
$$\geq \;-\; t_1(z)(2\alpha_0^2-1-a)^2 \;+\; (2\alpha_0^2-1)(2\alpha_0^2-1-a)^2$$

which is equivalent to

$$t_1(z) \leq -\frac{2(n\alpha_0^2a^2 + (2n(2\alpha_0^2-2\alpha_0^4-1)+|C|)a + n\alpha_0^2(4\alpha_0^4-6\alpha_0^2+3))}{(|C|-2)na^2 + 4n\alpha_0^2a + 2n(2\alpha_0^2-2\alpha_0^4-1)+|C|}$$

because the denominator of the last fraction is negative in $[\alpha_0, s]$. □

We have to find the value of $a \in (-\infty, s]$ that minimizes the function $F(a)$. In concrete cases (i.e. for given n and k) this can be easily performed numerically by MAPLE.
According to **Step 2**, we denote

$$\delta_1 = \min\{F(a) : a \in \mathbf{R}\}.$$

Then we have

$$t_1(y) \leq t_1(z) \leq \delta_1$$

whence

$$t_{|C|-1}(z) \geq \langle y, z\rangle \geq 2\delta_1^2 - 1 = \mu_1.$$

For the next implementations of **Step 2** (in the iterative process of **Step 3**) we use the analog of Lemma 4.6.3 by setting μ_1 instead of $\mu_0 = 2\alpha_0^2 - 1$ there. We obtain

$$t_1(y) \leq t_1(z) \leq \delta_2$$

and so on. To apply Theorem 4.6.2 we need to compute δ_i's until nonexistence of the code can be proved by (4.6.1) or their values remain the same.
To check the existence of C, we use Theorem 4.6.2 with the polynomial $f(t) = t^2$. Since $f_0 = 1/n$, we have to check if inequality

$$2\delta_i^2 > \frac{|C|}{n} - 1$$

is true for some $i \geq 1$. If so then we have proved the nonexistence of C.
The whole iteration process was realized by a simple MAPLE program which is available upon request from the author.

Example 4.6.4. *Let us consider the cases $n = 9$ and $n = 10$, with $k = 3$ in both dimensions. Let us assume that $C \subset \mathbf{S}^9$ is a $2n + k = 23$-point 3-design. For these parameters we have $\alpha_0 = 0.78197$, $\alpha_1 = 0.03197$ (all decimals are truncated after the fifth digit). Thus $2\alpha_0^2 - 1 > \alpha_1$ and we can start the iterative process. Already at the first iteration we obtain $\delta_1 = -0.81202$ whence*

$$2\delta_1^2 = 1.31875 > 1.3 = \frac{23}{10} - 1.$$

Therefore C does not exist.

Analogously, for a putative 21-point 3-design on $C \subset \mathbf{S}^8$, we obtain

$$2\delta_4^2 = 1.35909 > \frac{4}{3} = \frac{21}{9} - 1.$$

Therefore such a design does not exist.

There were 144 open cases in dimensions $3 \leq n \leq 50$. We ruled out 50 of them. The first nonexistence results show that there are no 3-designs of 21 points in nine dimensions ($k = 3$), 23 points in ten dimensions ($k = 3$), 35 points in fifteen dimensions ($k = 5$), etc. Therefore the problem for finding all possible cardinalities of 3-designs is completely solved in dimensions $n = 4, 6, 9$ and 10 (we consider two more dimensions than [20]) and only one open case remains in each dimension $n = 3, 5, 7, 8, 21, 22$ and for $11 \leq n \leq 18$ (considering six more dimensions than in [20]).

The present situation of the problem for finding all possible cardinalities of 3-designs in dimensions $3 \leq n \leq 24$ is presented on Table 4.13.

We now examine the asymptotic consequences of the refined approach. Table 4.13 and our observations in dimensions $25 \leq n \leq 50$ suggest that an asymptotic improvement could be possible.

We recall that Boyvalenkov-Danev-Nikova [20] prove that

$$B_{odd}(n,3) \gtrsim (1 + 2^{1/3})n \approx 2.2599n \tag{4.6.4}$$

as n tends to infinity (see also the end of Section 4.4). On the other hand, Bajnok's construction [3, 4] shows that $B_{odd}(n,3) \leq 2.5n$.

Therefore we have to consider designs with $2n + k$ points where

$$\frac{k}{n} = \gamma \in [2^{1/3} - 1, 0.5).$$

We cannot apply the iterative process from **Step 3** as many times as we like. Fortunately, already the first applications give better asymptotic results than (4.6.4).

Since α_0 and $s = \alpha_1$ are roots of (4.6.2), we have asymptotically

$$\alpha_0 \approx -\frac{1}{1+\gamma}$$

and $\alpha_1 \approx 0$. Now Lemma 4.6.3 with $a = 0$ gives

$$t_2(x) = t_1(z) \leq \delta_1 \approx -\frac{2(\gamma^5 + 8\gamma^4 + 19\gamma^3 + 13\gamma^2 - 2\gamma + 1)}{\gamma(\gamma^2 + 4\gamma + 5)^2(\gamma + 1)^4}.$$

We use MAPLE to solve numerically the corresponding equation $2\delta_1^2 = 1 + \gamma$ to obtain that

$$B_{odd}(n,3) \gtrsim 2.2949n.$$

We were able to implement four iterations to obtain the following assertion.

Theorem 4.6.5. *We have*

$$B_{odd}(n,3) \gtrsim 2.3227n. \tag{4.6.5}$$

Therefore, $2.3227n \lesssim B_{odd}(n,3) \lesssim 2.5n$ asymptotically. Our conjecture is that the upper bound gives the exact behaviour of $B_{odd}(n,3)$ both for small dimensions and as n tends to infinity.

4.6.3 Some results for 5-designs

Let $\tau = 5$ and $C \subset \mathbf{S}^{n-1}$ be a 5-design of cardinality $|C| = R(n,5) + k = n^2 + n + k$, where $k \geq 3$ is an odd integer.

In this case, α_0 and α_1 are the roots of the quadratic equation

$$(n+2)[(n+2)s^2 + 2s - 1]t^2 + 2s(s+1)(n+2)t + 3 - (n+2)s^2 = 0.$$

Let $x, y, z \in C$ be the points from Lemma 4.6.1. We assume that $\mu_0 = 2\alpha_0^2 - 1 > s$ as **Step 2** requires. Similarly to the case of 3-designs we obtain some bound on $t_2(x) = t_1(z)$ which now depends on two parameters. This is given by the following lemma.

Lemma 4.6.6. *For real a and b, we have*

$$t_1(z) \leq F(a,b) = \frac{2a|C|\left[(n+2)b+3\right] - n(n+2)\left[(1+a+b)^2 + (2\alpha_0^2-1)K\right]}{|C|\left[n(n+2)b^2 + (n+2)(a^2+2b) + 3\right] - n(n+2)\left[(1+a+b)^2 + K\right]},$$

where

$$K = \left[(2\alpha_0^2 - 1)^2 + a(2\alpha_0^2 - 1) + b\right]^2,$$

provided that the denominator in the last fraction is positive and that polynomial $f(t) = t^2 + at + b$ is increasing in $(s, 1)$.

Proof. This is similar to the proof of Lemma 4.6.3. We apply (4.1.1) to C, z and the fifth degree polynomial $f(t) = (t - t_1(z))(t^2 + at + b)^2$. □

We organize an iterative process as in the case $\tau = 3$. At each step the function $F(a,b)$ is minimized to give better bounds on $t_1(z) = t_2(x)$, which in turn improve the bounds on $t_{|C|-1}(z)$. To check the existence of C we apply Theorem 4.6.2 with the fourth degree polynomial $f(t) = (t - \alpha_1)^2(t - s)^2$ which is obviously decreasing in $[-1, \alpha_0)$.

Since $f_0|C| - f(1) = \rho_0 f(\alpha_0)|C|$, existence condition (4.6.1) from Theorem 4.6.2 becomes

$$2f(\delta_i) > \rho_0 f(\alpha_0)|C|.$$

Therefore, C could exist only if

$$\rho_0|C| \geq \frac{2f(\delta_i)}{f(\alpha_0)}. \tag{4.6.6}$$

The last condition could be considered as an improvement over (4.4.2) from Theorem 4.4.2.

The results in small dimensions are as follows. For each dimension n, $3 \le n \le 20$, we examine the first six open cases (i.e. those with $\rho_0|C| \ge 2$). Thus there are $108 = 18 \times 6$ designs under consideration. The above procedure rules out 53 of them. The new bounds are presented in Table 4.13. Some of the entries in this table improve the corresponding bounds in Table 4.5 in Subsection 4.4.2.

Let n tend to infinity and $|C| = R(n,s) + \gamma n^2 \sim (1+\gamma)n^2$ where $\gamma > 0$. Then α_0 tends to $-1/(1+2\gamma)$ while α_1 and s tend to zero (here $|\alpha_1| < s \sim 1/\sqrt{n}$).

Asymptotic results from [13, 20] show that

$$B_{odd}(n,5) \gtrsim \frac{1+2^{1/5}}{2} n^2 \approx 1.0743 n^2$$

as n tends to infinity. Using the same argument as in the previous subsection we are able to improve this. We apply the first iteration only.

Theorem 4.6.7. *We have*

$$B_{odd}(n,5) \ge 1.0930 n^2. \tag{4.6.7}$$

Proof. Let $n \to +\infty$ and $C \subset \mathbf{S}^{n-1}$ be a 5-design of cardinality $|C| = (1+\gamma)n^2$ where γ is some constant.

Since α_0 tends to $-1/(1+2\gamma)$ by Lemma 4.4.6, it follows from (4.6.6) that

$$\frac{1}{(1+2\gamma)^5} \ge \frac{2f(\delta_1)}{f(-1/(1+2\gamma))}.$$

We complete the proof by solving this numerically (with MAPLE) with respect to γ. □

4.7 Better bounds on the maximal inner product

In this section, we show how Theorem 2.7.2 can be used to obtain better upper bounds on the maximal inner product of designs of relatively small cardinalities. We apply (2.7.1) for a point $y \notin C$.

Lemma 4.7.1. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical τ-design and $x_1, x_2 \in C$ be such that $\langle x_1, x_2 \rangle = s(C)$. Then for every real polynomial $f(t)$ of degree at most τ we have*

$$2f\left(\sqrt{\frac{1+s(C)}{2}}\right) \le f_0|C| - (|C|-2)\varepsilon, \tag{4.7.1}$$

where $\varepsilon = \min\{f(t) : t \in [-1,1]\}$.

Proof. Let $\angle x_1 O x_2 = \varphi$ (O is the origin) and $y \in \mathbf{S}^{n-1}$ be such that the line Oy bisects $\angle x_1 O x_2$. Then $\cos\varphi = s(C)$, $\angle x_1 O y = \angle x_2 O y = \varphi/2$ and

$$\langle x_1, y \rangle = \langle x_2, y \rangle = \cos\frac{\varphi}{2} = \sqrt{\frac{1+\cos\varphi}{2}} = \sqrt{\frac{1+s(C)}{2}}.$$

We apply (2.7.1) to C, y and an arbitrary real polynomial $f(t)$ and obtain

$$\begin{aligned} f_0|C| &= \sum_{x\in C} f(\langle x,y\rangle) \\ &= f(\langle x_1,y\rangle)+f(\langle x_2,y\rangle)+\sum_{x\in C\setminus\{x_1,x_2\}} f(\langle x,y\rangle) \\ &\geq 2f\left(\sqrt{\frac{1+s(C)}{2}}\right)+(|C|-2)\varepsilon, \end{aligned}$$

where $\varepsilon=\min\{f(t):t\in[-1,1]\}$. This is equivalent to (4.7.1). □

Lemma 4.7.1 implies the following upper bound on $s(C)$.

Lemma 4.7.2. *Let $C\subset \mathbf{S}^{n-1}$ be a spherical τ-design and let $f(t)$ be a real polynomial of degree at most τ which is increasing in $[s,+\infty)$. Let ν denote the largest root of the equation*

$$2f(t)=A,$$

where $A=f_0|C|-(|C|-2)\varepsilon$ and $\varepsilon=\min\{f(t):t\in[-1,1]\}$. Then

$$s(C)\leq 2\nu^2-1. \tag{4.7.2}$$

Proof. It follows from Lemma 4.7.1 that

$$\sqrt{\frac{1+s(C)}{2}}\leq\nu,$$

which is equivalent to (4.7.2). □

We consider applications of Lemma 4.7.2 with polynomials $f(t)$ of maximal admissible degree τ which vanish at the zeros of the corresponding Levenshtein polynomial $f_\tau^{(n,s)}(t)$. This means that $f(\alpha_i)=0$ for $i=0,1\ldots,k-1$ and $\tau=2k-1$ or $f(\beta_i)=0$ for $i=0,1\ldots,k$ and $\tau=2k$ where all parameters are determined by $|C|=1/L_\tau(n,s)$. Then $f_0|C|=f(1)$ and the constant A from Lemma 4.7.2 becomes equal to $f(1)-(|C|-2)\varepsilon$, i.e. we have to find the largest root of the equation

$$2f(t)=f(1)-(|C|-2)\varepsilon. \tag{4.7.3}$$

In this situation $f(t)$ should be chosen to have ε close to zero. We have $k-1$ free parameters to choose.

Let us consider polynomials $f(t)$ having the $k-1$ remaining zeros (i.e. different from the α_i's or β_i's) in the interval $[-1,s)$. In this case $f(t)$ is increasing in the interval $[s,+\infty)$ and Lemma 4.7.2 can be applied.

Example 4.7.3. *One can find the upper bounds on maximal inner product $s(C)$ for 3-designs in $\mathbf{S}^{n-1}$ of relatively small cardinalities (i.e. $|C|=R(n,3)+k=2n+k$). The results are presented in Table 4.14.*

Example 4.7.4. *We have found the upper bounds on maximal inner product $s(C)$ for spherical 4-designs of relatively small cardinalities (i.e. $|C|=R(n,4)+k=n(n+3)/2+k$). The results are presented in Table 4.15.*

n	$\|C\| = 2n+1$	$\|C\| = 2n+3$	$\|C\| = 2n+5$	$\|C\| = 2n+7$	$\|C\| = 2n+9$
3	7^*	9	$\underline{\mid 11}$		
4	9^*	$\underline{\mid 11}$			
5	11^*	13	$\underline{\mid 15}$		
6	13^*	$\underline{\mid 15}$			
7	15^*	17	$\underline{\mid 19}$		
8	17^*	19	$\underline{\mid 21}$		
9	19^*	$21^\bullet$	$\underline{\mid 23}$		
10	21^*	$23^\bullet$	$\underline{\mid 25}$		
11	23^*	25^*	27	$\underline{\mid 29}$	
12	25^*	27^*	29	$\underline{\mid 31}$	
13	27^*	29^*	31	$\underline{\mid 33}$	
14	29^*	31^*	$33^\bullet$	$\underline{\mid 35}$	
15	31^*	33^*	$35^\bullet$	37	$\underline{\mid 39}$
16	33^*	35^*	$37^\bullet$	39	$\underline{\mid 41}$
17	35^*	37^*	$39^\bullet$	41	$\underline{\mid 43}$
18	37^*	39^*	$41^\bullet$	43	$\underline{\mid 45}$
19	39^*	41^*	43^*	45	47
20	41^*	43^*	45^*	47	49
21	43^*	45^*	47^*	$49^\bullet$	51
22	45^*	47^*	49^*	$51^\bullet$	53
23	47^*	49^*	51^*	$53^\bullet$	55
24	49^*	51^*	53^*	$55^\bullet$	57

Table 4.13: On 3-designs of odd cardinalities
Key to Table 4.13:

$\underline{\mid m}$	all designs of size $\geq m$ exist (Bajnok [3, 4])
$*$	nonexistence proved in [20] (Boyvalenkov-Danev-Nikova)
$\bullet$	nonexistence follows from Theorem 4.6.2 for $f(t) = t^2$ and some $i \geq 1$.

n	C	$s(C)$
5	13	0.436624441
7	17	0.473873164
8	19	0.516178363
9	21	0.567928898
10	23	0.626508894

Table 4.14: Bounds on $s(C)$ for spherical 3-design

n	C	$s(C)$	n	C	$s(C)$
3	9	0.454040229	7	35	0.761804242
3	10	0.554044149	7	36	0.782804931
3	11	0.652241258	7	37	0.804097726
3	13	0.864179325	7	38	0.825624795
4	14	0.546306244	8	44	0.821492583
4	15	0.602594523	8	45	0.838448975
4	16	0.659847911	8	46	0.855584935
4	17	0.717303128	8	47	0.872871621
5	20	0.626086394	9	54	0.877235198
5	21	0.663330983	9	55	0.891343646
5	22	0.701357438	9	56	0.905565258
5	23	0.739844530	9	57	0.919885083
6	28	0.724281874	10	65	0.929750218
6	29	0.751888019	10	66	0.941757004
6	30	0.779846135	10	67	0.953836741
6	31	0.808045757	10	68	0.965981019

Table 4.15: Bounds on $s(C)$ for spherical 4-design

Chapter 5

Moments of spherical codes and designs

This chapter is based on [11]. We introduce and investigate certain invariants of spherical codes which we call moments. Such investigations could give information about the structure of spherical codes and designs and therefore they are useful in dealing with linear programming bounds.

5.1 Definitions and main properties

In the proofs of the linear programming bounds for spherical codes and designs some terms on the right hand side of (2.3.1) were neglected. We now consider these terms.

Definition 5.1.1. *For a spherical code $C \subset \mathbf{S}^{n-1}$ and any integer $i \geq 1$, the number*

$$M_i = \frac{1}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} v_{ij}(x) \right)^2$$

is called i-th moment of C.

Some basic properties of the moments are described in the following theorem.

Theorem 5.1.2. a) *We have $M_i \geq 0$ for every $i \geq 1$.*

b) *$M_1 = M_2 = \cdots = M_\tau = 0$ if and only if C is a τ-design.*

c) *C has index i if and only if $M_i = 0$.*

d) *The formula*

$$M_i = |C| + \sum_{x,y \in C, x \neq y} P_i^{(n)}(\langle x, y \rangle) \tag{5.1.1}$$

holds.

Proof. **a)** This is obvious.

b) This follows immediately from Definitions 2.7.1 and 5.1.1.

c) This follows from Definitions 2.7.3 and 5.1.1.

d) Apply (2.3.1) to C and $f(t) = P_i(t)$. Since $f_i = 1$ and $f_j = 0$ for $j \neq i$, the right hand side equals M_i which proves the assertion. □

Since the moments are nonnegative they are usually neglected in (2.3.1). However, very often good codes have small strengths as spherical designs. This was our motivation to study the moments.

It follows by Theorem 5.1.2d) that the moments do not depend on the choice of the bases $\{v_{ij}(x) : 1 \leq j \leq r_i\}$. The moments are in close relation with the distance distribution of codes.

Definition 5.1.3. *Let C be a spherical code and $x \in C$. Then the system of nonnegative integers $\{A_t(x) : t \in [-1, 1)\}$ given by*

$$A_t(x) = |\{y \in C, \langle x, y\rangle = t\}|$$

defines the distance distribution of C with respect to x. The system of nonnegative rational numbers $\{A_t : t \in [-1, 1)\}$, where

$$A_t = \frac{1}{|C|} \sum_{x \in C} A_t(x)$$

is called distance distribution of C.

Notice that antipodal codes have $A_{-1} = A_{-1}(x) = 1$ for every x, $A_t(x) = A_{-t}(x)$ and $A_t = A_{-t}$ for every t and x.

Corollary 5.1.4. *A spherical code C is antipodal if and only if $M_i = 0$ for every odd i.*

Proof. The necessity is obvious by Definition 5.1.1. For the sufficiency, let us assume that code $C \subset \mathbf{S}^{n-1}$ is such that $M_i = 0$ for every odd i. Therefore $\sum_{x \in C} f(x) = 0$, for every odd polynomial $f(x)$. (This is because each real polynomial decomposes into homogeneous harmonic polynomials - cf. [27, 31]). Let us fix $d \in C$ and consider the polynomial $f(x) = \langle x, d\rangle^n$ where n is odd. So,

$$0 = \sum_{c \in C} f(c) = \sum_{c \in C} \langle c, d\rangle^n = 1 + \sum_{c \in C, c \neq d} \langle c, d\rangle^n \tag{5.1.2}$$

If $-c \notin C$ then

$$\lim_{n \to \infty} 1 + \sum_{c \in C, c \neq d} \langle c, d\rangle^n = 1 + 0.$$

This is a contradiction. □

Using Theorem 5.1.2d), one easily can calculate moments of known codes. In fact, it is enough to know the inner products and the distance distribution of the code under target.

Example 5.1.5. *The icosahedron is an antipodal $(3,12,1/\sqrt{5})$ code which is a spherical 5-design. Therefore we have $M_1 = M_2 = M_3 = M_4 = M_5 = 0$ and $M_{2i+1} = 0$ for every integer $i \geq 0$. It was proved in [18] that the icosahedron has indices 8 and 14, i.e. $M_8 = M_{14} = 0$. We shall determine the remaining moments M_i with $i \leq 20$.*

The distance distribution of the icosahedron is the system $\{A_{-1}, A_{-1/\sqrt{5}}, A_{1/\sqrt{5}}\}$, where

$$A_{-1} = A_{-1}(x) = 1$$

and

$$A_{-1/\sqrt{5}} = A_{-1/\sqrt{5}}(x) = A_{1/\sqrt{5}} = A_{-1/\sqrt{5}}(x) = 5.$$

As in the proof of Corollary 5.1.4 we obtain

$$\begin{aligned} M_{2k} &= 12\left[1 + 1.P_{2k}^{(n)}(-1) + 5.P_{2k}^{(n)}\left(-\frac{1}{\sqrt{5}}\right) + 5.P_{2k}^{(n)}\left(\frac{1}{\sqrt{5}}\right)\right] \\ &= 24\left[1 + 5P_{2k}^{(n)}\left(\frac{1}{\sqrt{5}}\right)\right]. \end{aligned}$$

Thus we have (for example by using MAPLE) that $M_6 = 1584/25 = 63.36$, $M_8 = 0$ confirmed, $M_{10} = 11856/625 = 18.9696$, $M_{12} = 154224/3125 = 49.35168$, $M_{14} = 0$ confirmed, $M_{16} = 452352/15625 = 28.950528$, $M_{18} = 619344/15625 = 39.638016$ and $M_{20} = 672336/3900625 = 1.72118016$.

Example 5.1.6. *We calculate some moments of the famous regular polytope in four dimensions known as the 600-cell [28]. It is an antipodal spherical 11-design with indices 14, 16, 18, 22, 26, 28, 34, 38, 46, 58.*

It has 120 vertices and its maximal inner product is equal to $\cos\pi/5 = (1+\sqrt{5})/4 \approx 0.80902$. This means that it is a $(4,120,(1+\sqrt{5})/4)$ code. The remaining inner products are -1, $-(1+\sqrt{5})/4$, $\pm 1/2$, $\pm 1/4$, $\pm(\sqrt{5}-1)/4$ and 0.

The distance distribution of the 600-cell is given by

$$A_{-1} = 1,$$

$$A_{-(1+\sqrt{5})/4} = A_{(1+\sqrt{5})/4} = A_{-(\sqrt{5}-1)/4} = A_{(\sqrt{5}-1)/4} = 12,$$
$$A_{-1/2} = A_{1/2} = 20$$
$$A_0 = 30.$$

Therefore we have as in the proof of Corollary 5.1.4 and in the previous example

$$M_{2k} = 240\left[1 + 12P_{2k}^{(n)}\left(\frac{1+\sqrt{5}}{4}\right) + 12P_{2k}^{(n)}\left(\frac{\sqrt{5}-1}{4}\right) + 20P_{2k}^{(n)}\left(\frac{1}{2}\right) + 15P_{2k}^{(n)}(0)\right].$$

This implies that the first four nonzero moments of the 600-cell are $M_{12} = 14400/13 \approx 1107.692$, $M_{20} = 4800/7 \approx 685.714$, $M_{24} = 576$ and $M_{30} = 14400/31 \approx 464.516$.

One can also calculate moments of many feasible (i.e. when the existence is undecided) classes of good codes and designs.

5.2 Modified linear programming bounds

In this section we formulate four modifications of the linear programming bounds for spherical codes and designs. As for the standard linear programming theorems the proofs follow immediately from the main identity (2.3.1). Notice that all four theorems below require preliminary information about moments of feasible codes (designs).

Theorem 5.2.1. *Let $f(t)$ be a real polynomial such that*

(A1) $f(t) \leq 0$ *for* $t \in [-1, s]$.

(A2) *In the Gegenbauer expansion* $f(t) = \sum_{i=0}^{k} f_i P_i(t)$, *all coefficients* f_i *satisfy* $f_i \geq 0$ *for all* $i \in A = \{0, 1, \dots, k\}$.

Assume that for some (n, M, s) *code the numbers* M_k *satisfy* $M_k \geq \alpha_k > 0$ *for all* $k \in B \subset A$. *Then*

$$Mf(1) \geq M^2 f_0 + \sum_{k \in B} f_k \alpha_k.$$

Proof. Apply the main identity (2.3.1) to C and $f(t)$. Then the left hand side is at most $Mf(1)$ as in Theorem 2.3.2 and the right hand side is at least $M^2 f_0 + \sum_{k \in B} f_k \alpha_k$. This completes the proof. □

Theorem 5.2.2. *Let $f(t)$ be a real polynomial such that*

(B1) $f(t) \geq 0$ *for* $t \in [-1, s]$.

(B2) *In the Gegenbauer expansion* $f(t) = \sum_{i=0}^{k} f_i P_i(t)$, *all coefficients* f_i *satisfy* $f_i \geq 0$ *for all* $i \in A = \{0, 1, \dots, k\}$.

Assume that for some (n, M, s) *code the numbers* M_k *satisfy* $M_k \leq \beta_k$ *for all* $k \in B \subset A$. *Then*

$$Mf(1) \leq M^2 f_0 + \sum_{k \in B} f_k \beta_k.$$

Proof. As of Theorem 5.2.1. □

Theorem 5.2.3. *Let $f(t)$ be a real polynomial such that*

(C1) $f(t) \leq 0$ *for* $t \in [-1, 1]$.

(C2) *In the Gegenbauer expansion* $f(t) = \sum_{i=0}^{k} f_i P_i(t)$, *all coefficients* f_i *satisfy* $f_i \geq 0$ *for all* $i \in A = \{\tau + 1, \tau + 2, \dots, k\}$.

Suppose also that for a τ*-design* $C \subset \mathbf{S}^{n-1}$ *of cardinality* M *we have* $M_k \geq \alpha_k > 0$ *for all* $k \in B \subset A$. *Then*

$$Mf(1) \leq M^2 f_0 + \sum_{k \in B} f_k \alpha_k.$$

Proof. As in Theorem 5.2.1. □

Theorem 5.2.4. *Let $f(t)$ be a real polynomial such that*

(D1) $f(t) \geq 0$ *for* $t \in [-1, 1]$.

(D2) *In the Gegenbauer expansion* $f(t) = \sum_{i=0}^{k} f_i P_i(t)$, *we have* $f_i \geq 0$ *for all* $i \in A = \{\tau + 1, \tau + 2, \ldots, k\}$.

Suppose also that for a τ*-design* $C \subset \mathbf{S}^{n-1}$ *of cardinality* M *we have* $M_k \leq \beta_k$ *for all* $k \in B \subset A$. *Then*

$$Mf(1) \geq M^2 f_0 + \sum_{k \in B} f_k \beta_k.$$

Proof. As of Theorem 5.2.1. □

It is clear by Theorem 5.1.2d) that upper bounds β_k exist for every k (these bounds could be used in Theorems 5.2.2 and 5.2.4). Some good bounds can be obtained by using suitable polynomials in (2.3.1).

More general, any polynomial which does not change in sign in the interval $[-1, s]$ for codes (respectively $[-1, 1]$ for designs) gives by (2.3.1) a linear inequality for the relevant moments. A set of such inequalities can be used as input for a conventional linear programming problem (i.e. it can be investigated by the simplex method).

Example 5.2.5. *Let us consider a hypothetical* $(4, 25, 0.5)$ *code which existence or nonexistence would determine the fourth kissing number to be 25 or 24 respectively. We write the main identity (2.3.1) for* C *and some polynomial* $f(t)$ *as*

$$25(f(1) - 25f_0) + \sum_{x,y \in C, x \neq y} f(\langle x, y \rangle) = \sum_{i=1}^{k} f_i M_i.$$

We assume that $f(t)$ *does not change in sign on* $[-1, 0.5]$. *Then we neglect the sum on the left hand side to obtain*

$$\sum_{i=1}^{k} f_i M_i \geq 25(f(1) - 25f_0)$$

when $f(t) \geq 0$ *for every* $t \in [-1, 0.5]$ *or*

$$\sum_{i=1}^{k} f_i M_i \leq 25(f(1) - 25f_0)$$

when $f(t) \leq 0$ *for every* $t \in [-1, 0.5]$.

In this way we obtain a set of linear inequalities with respect to the moments M_1, M_2, $\ldots$, M_k. *We collect such inequalities together to use them as restrictions in the simplex method. The objective function can be each of the moments* M_i, $i = 1, 2, \ldots, k$ *either for a maximization or a minimization problem.*

The first two candidates are the Levenshtein polynomial

$$f_5^{(4,0.5)}(t) = \left(t^2 + t + \frac{1}{6}\right)^2 \left(t - \frac{1}{2}\right)$$

and the ninth degree polynomial which is produced by SCOD for improving $L_5(4, 0.5)$, *namely*

$$t^9 - 2t^7 + 1.844953t^5 + 0.6933373t^4 - 0.2373005t^3 - 0.1680599t^2 - 0.02829665t - 0.00149061.$$

As a third polynomial we take $(t+1)(t-1/2)(t^2+5/7t+1/14)^2$. *Using the simplex method we obtain the following inequalities for the moments :*

$$0 \le M_0 \le 13.96103005$$

$$0 \le M_1 \le 3.858650822$$

$$0 \le M_2 \le 2.282869486$$

$$0 \le M_3 \le 1.983304866$$

$$0 \le M_4 \le 2.781122920$$

$$0 \le M_5 \le 6.035763582$$

$$0 \le M_6 \le 63.77551058$$

$$0 \le M_9 \le 30.85210010.$$

5.3 Moments of spherical designs

As usual, the design problem allows more detailed investigation. This is because conditions **(C1)** and **(D1)** in Theorems 5.2.3 and 5.2.4 are in fact stronger than necessary. Indeed, for designs of small cardinalities one usually knows that all inner products belong to some intervals $[a, b] \subset [-1, 1]$. This helps to obtain better bounds on the moments of spherical designs.

Let $C \subset \mathbf{S}^{n-1}$ be a spherical τ-design. Denote

$$\ell(C) = \min\{\langle x, y\rangle : x, y \in C\}.$$

Then $\ell(C)$ equals -1 if and only if C possesses a pair of antipodal points. Since this does not occur for $\tau = 2k$ and $R(n, 2k) < |C| < R(n, 2k + 1)$ the parameter $\ell(C)$ is nontrivial (i.e. $\ell(C) > -1$) in such cases. This has an impact on moments.

Theorem 5.3.1. *Let* $C \subset \mathbf{S}^{n-1}$ *be a spherical* $(2k)$*-design.*

a) *For every polynomial*

$$f(t) = (t - \ell(C))\, A^2(t),$$

where $A(t) = t^k + \cdots$ *is a* k *degree polynomial, we have*

$$M_{2k+1} \ge a_{2k+1,2k+1}|C|\,(f_0|C| - f(1))\,.$$

b) *For every polynomial*

$$f(t) = (t - s(C))A^2(t),$$

where $A(t) = t^k + \cdots$ *is a* k *degree polynomial, we have*

$$M_{2k+1} \le a_{2k+1,2k+1}|C|\left(f_0|C| - f(1)\right).$$

Proof. **a)** We apply the main identity (2.3.1) to C and $f(t) = \sum_{i=0}^{2k+1} f_i P_i^{(n)}(t)$. Since C is a $(2k)$-design, the right hand side reduces to

$$f_0|C|^2 + f_{2k+1}M_{2k+1} = f_0|C|^2 + M_{2k+1}/a_{2k+1,2k+1}.$$

On the left hand side we have

$$f(1)|C| + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) \ge f(1)|C|$$

because the polynomial $f(t)$ is nonnegative in the interval $[\ell(C), s(C)]$ which contains all inner products $\langle x,y\rangle$, $x,y \in C$. We combine the last two relations to obtain the inequality

$$M_{2k+1} \ge a_{2k+1,2k+1}|C|\left(f_0|C| - f(1)\right).$$

b) This is analogous to **a)**. □

Example 5.3.2. *Let us consider bounds for moments of some 4-designs of relatively small cardinalities which existence is undecided. Let* $C \subset \mathbf{S}^{n-1}$ *be a spherical 4-design. Then* $M_i = 0$ *for* $1 \le i \le 4$ *and the first "interesting" moment is* M_5. *Consider the polynomial*

$$f(t) = (t-\alpha)(t^2 + at + b)^2,$$

where a *and* b *are parameters to be optimized later and* α *is either* $\ell(C)$ *or* $s(C)$. *Then, by Theorem 5.3.1, we obtain*

$$M_5 \ge a_{5,5}|C|(f(1) - f_0|C|) = |C|F(\ell(C), a, b) \tag{5.3.1}$$

or

$$M_5 \le a_{5,5}|C|(f(1) - f_0|C|) = |C|F(s(C), a, b), \tag{5.3.2}$$

respectively. Here $a_{5,5} = (n+2)(n+4)/(n^2-1)$ *does not depend on* C, α, a *and* b, *and*

$$F(\alpha, a, b) = (1-\alpha)(1+a+b)^2 - |C|\left(-\alpha b^2 + \frac{2ab - \alpha(a^2+2b)}{n} + \frac{3(2a-\alpha)}{n(n+2)}\right)$$

(the coefficient f_0 *is calculated by (2.1.5)).*

For particular values of $\alpha = \ell(C)$ *or* $s(C)$, *we have to optimize function* $F(\alpha, a, b)$ *with respect to the parameters* a *and* b. *The optimization means maximization for* $\alpha = \ell(C)$ *and minimization for* $\alpha = s(C)$. *Since* $F(a,b)$ *is a quadratic form this can be carried out easily by MAPLE.*

The first open case is $n = 3$, $|C| = 10$ *(it is still unknown if there exists a 10-point 4-design in three dimensions). Since all inner products of such a design must belong to* $[-\sqrt{23/27}, 0.466)$, *we obtain that* $22.1 \le M_5 \le 33.6$. *(see Examples 4.7.3 and 4.7.4)*

Theorem 5.3.1 calls for better lower bounds on $\ell(C)$ and better upper bounds on $s(C)$. General results follow from the investigations in Subsection 4.5.2. Some even better bounds can be obtained in particular cases by using methods from Section 4.7. This will be investigated in the future.

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Samenvatting

In dit proefschrift wordt onderzoek gedaan naar een aantal problemen die verwantschap hebben met sferische codes en designs.

In het eerste hoofdstuk wordt een inleiding gegeven tot sferische codes en designs. Er zijn twee belangrijke problemen te onderscheiden. Enerzijds willen we de precieze waarde (of een boven- en ondergrens) van de grootst mogelijke kardinaliteit (i.e. $A(n,s)$) van een sferische code vaststellen, indien de dimensie n en de maximale cosinus s zijn gegeven.

Aan de andere kant willen we de grootte van een sferisch design minimaliseren voor vaste dimensie n en sterkte τ. De kleinst mogelijke kardinaliteit van een τ-design in n dimensies wordt aangegeven met $B(n,\tau)$. Het probleem is boven- en ondergrenzen voor $B(n,\tau)$ te vinden (of de precieze waarde).

Het tweede hoofdstuk behandelt de lineaire programmeer technieken die gebruikt worden voor het vinden van een bovengrens voor $A(n,s)$ en een ondergrens voor $B(n,\tau)$. De beste bovengrens voor $A(n,s)$ werd ontdekt door Levenshtein. Een uitleg van de logica van deze bound, samen met de eigenschappen van de betrokkene parameters wordt gegeven.

In het derde hoofdstuk worden noodzakelijke en voldoende voorwaarden gegeven voor het bestaan van verbeteringen van de Levenshtein bounds voor $A(n,s)$. Verder wordt er onderzoek gedaan naar deze voorwaarden en wordt er aangetoond dat betere grenzen vrij vaak bestaan.

In het vierde hoofdstuk worden beperkingen afgeleid op de distributie van de optredende inprodukten van een spferisch design met een relatief kleine kardinaliteit (i.e. dicht bij de klassieke grenzen). Deze condities blijken voldoende te zijn voor non-existentie in veel gevallen. Onze methode werkt efficient zowel in kleine dimensies als asymptotisch voor grote n. Voor $\tau = 3$ en $\tau = 5$ worden nieuwe asymptotische grenzen op de kleinst mogelijke oneven grootte van τ-designs afgeleid.

Het vijfde en laatste hoofdstuk introduceert en bestudeert bepaalde invarianten van sferische codes die momenten genoemd worden. Zulk onderzoek zou informatie kunnen geven over de structuur van sferische codes en designs.

Curriculum Vitae

Silvia Boumova was born on September 28, 1973 in Sandanski, Bulgaria. From October 1991 till June 1996 she was studying Mathematics in Sofia University "St. Kliment Ochridski", where she obtaned her Master of Science degree in Algebra. After her graduation she got a position as a researcher at the Institute of Mathematics and Informatics in Sofia. In November 1998 she started her PhD studies at Institute of Mathematics and Informatics. From March 2001 till February 2002 she was a PhD student at Eindhoven University of Technology at Discrete Mathematics Group.