The correspondence between projective codes and 2-weight codes

The Correspondence between Projective Codes and 2-Weight Codes

A.E. Brouwer and M. van Eupen
Eindhoven University of Technology, Eindhoven, The Netherlands

Abstract — **The hyperplanes intersecting a 2-weight code in the same number of points obviously form the point set of a projective code. On the other hand, if we have a projective code C, then we can make a 2-weight code by taking the multiset of points $<c>\in PC$ with multiplicity $\gamma(w)$, where w is the weight of $c \in C$ and $\gamma(w) = \alpha w + \beta$ for some rational α and β depending on the weight enumerator of C. In this way we find a 1-1 correspondence between projective codes and 2-weight codes. The second construction can be generalized by taking for $\gamma(w)$ a polynomial of higher degree. In that case more information about the cosets of the dual of C is needed. Several new ternary codes will be constructed in this way.**

Let C be a projective q-ary $[n,k,d]$ code, with nonzero weights $w_1, ..., w_t$. Each subcode D of codimension 1 in C has nonzero weights $w_1, ..., w_t$ with respective frequencies $f_1, ..., f_t$, say, and these frequencies satisfy

$$\sum f_i = q^{k-1} - 1$$

(this follows by counting all nonzero vectors in D), and

$$\sum (n_D - w_i) f_i = n_D(q^{k-2} - 1),$$

where n_D is the effective length of D, that is, the number of coordinate positions where D is not identically zero (this follows by counting the zero entries of all vectors in D).
Since C is projective, we have $n_D = n - 1$ for n subcodes D, and $n_D = n$ for the remaining $(q^k - 1)/(q-1) - n$ subcodes of codimension 1.
It follows that for arbitrary choice of α, β the sum

$$\sum (\alpha w_i + \beta) f_i$$

does not depend on D but only on n_D, and hence only takes two values.
Fix α, β in such a way that all numbers $\alpha w_i + \beta$ are nonnegative integers, and consider the multiset X (in the projective space PC) consisting of the 1-spaces $\langle c \rangle$ with $c \in C$ taken $\alpha w + \beta$ times, if w is the weight of c. Then X is the point set of a 2-weight code.
For example from a ternary [16,5,9] code with weight enumerator $0^1\ 9^{116}\ 12^{114}\ 15^{12}$ we can construct a ternary [69,5,45] code with weight enumerator $0^1\ 45^{210}\ 54^{32}$ by taking $\alpha = \frac{1}{3}$ and $\beta = -3$.
Now, let C be a 2-weight linear q-ary $[n,k,d]$ code, with nonzero weights u and v. Let X be the corresponding multiset in $PG(k-1,q)$, so that $|X| = n$, and each hyperplane meets X in either $|X|-u$ or $|X|-v$ points. The hyperplanes meeting X in $|X|-u$ points ($|X|-v$ points, respectively) obviously form the point set of a projective code. This construction can be said to be the inverse of the first construction.
For example a ternary [149,5,99] code has been proved to have weight enumerator $0^1\ 99^{222}\ 108^{20}$ if it exists [2]. The second construction would then yield a projective ternary self-orthogonal [10,5,3] code, which cannot exist since 10 is not a multiple of 4.
The first construction can be generalized in the following way: Suppose we have an $[n,k,d]$ code C over $GF(q)$ with nonzero weights $w_1, \ldots, w_t$. Let D be a subcode of codimension 1 of C. Let the frequencies of $w_1, \ldots, w_t$ in D be denoted by $f_1, \ldots, f_t$. Then the Pless power moments [3] give us:

$$\sum_{i=1}^{t} v_i^r f_i = \sum_{j=0}^{n} B_j \left(\sum_{\nu=0}^{r} \nu! S(r,\nu) q^{k-1-\nu} \binom{n-j}{n-\nu} \right) - n^r,$$

where $v_i = n - w_i$, B_j is the number of codewords of weight j in the dual of D and $S(r,\nu)$ is a Stirling number of the second kind. Let $p_i(v) = \sum_{s=0}^{t-1} p_s^{(i)} v^s$ be the polynomial that is 0 for $v = v_h$, $h \neq i$ and is 1 for $v = v_i$, $(i = 1, \ldots, t)$. Consider the set X_i in the projective space PC consisting of 1-spaces $<c>$ $(c \in C)$ with multiplicity $p_i(v)$, where $w = n - v$ is the weight of c. Then X_i is a projective code that is intersected by D in $(\sum_{h=1}^{t} p_i(v_h) f_h)/(q-1)$ points. So if we can compute the weight enumerator up to weight $t-1$ of the dual of any codimension 1 subcode of C (which corresponds to a coset of the dual of C), then we can compute the weights in X_i $(i = 1, \ldots, t)$, using the Pless power moments. Once we know the weight in X_i corresponding to each coset of the dual for every i, we can construct codes by taking the union of some X_i's.
For example if we take for C the [12,6,6] extended ternary Golay code, then we find a ternary [220,6,144] and a [232,6,153] code, which both improve on the bounds in [1]. If we take for C the ternary [7,6,2] code, then we find a [140,6,90] and a [203,6,132] code, which also improve on the bounds in [1].

References

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